DEPARTMENT OF THE ARMY

SIGNAL CORPS

Contract DA 36-039 SC-78130

Technical Report No. 12

AN APPROXIMATE METHOD IN SIGNAL DETECTION

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Abstract. A theorem from the theory of Toeplitz forms (ref. 1) is applied to the problem of estimating the best test statistic for the detection of Gaussian signals in Gaussian noise.

Let $x(t)$ be a stationary Gaussian process, mean zero, sampled at successive time points to provide an observer with a finite sample

$$\underline{x} = (x_1, x_2, \ldots, x_n). \tag{1}$$

The time points are to be closely spaced to produce almost unit correlation. It is known to the observer that (1) is either a sample from a Gaussian ensemble Ω_0 with mean zero and power-density spectrum $f_0(\lambda)$ or a sample from a Gaussian ensemble Ω_1 with mean zero and power-density spectrum $f_1(\lambda)$, and he is to decide whether (1) came from Ω_0 or Ω_1. We take Ω_0 to be noise alone and Ω_1 signal plus noise and write

$$f_0(\lambda) = f_n(\lambda) \tag{2}$$

$$f_1(\lambda) = f_n(\lambda) + f_s(\lambda) \tag{3}$$

where $f_n(\lambda)$ and $f_s(\lambda)$ are the power spectral densities of noise and signal, respectively; the noise and signal processes are here assumed independent and their spectial densities thus additive.

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Let H_o denote the hypothesis that noise only is present and H_1 that we observe signal plus noise. If $\Omega = \{\omega\} = \Omega_o \cup \Omega_1$ denotes the sample space of all possible realizations of the process x(t) in a finite interval of time, so that ω is a function of $t \in (0,T)$, then we define a critical region

$$W \subset \Omega \tag{4}$$

in the sense that

$$\text{if} \quad \omega \in W \quad , \quad H_o \text{ is rejected;}$$
$$\text{if} \quad \omega \notin W \quad , \quad H_o \text{ is accepted.}$$

The probability that H_o be rejected though true is denoted by $P_o(W)$, that it be accepted though false by $P_1(W^*)$, and the power of the test, i.e. the probability of rejection of H_o, consequently by

$$P_1(W) = 1 - P_1(W^*). \tag{5}$$

It is well known that although perfect detection is not possible in the case we are considering, the most powerful critical region W_{MP} is given by the Neyman-Pearson test

$$W_{MP} = \{\underline{x} | L(\underline{x}) > c\} \tag{6}$$

where the likelihood function L(x) is given by

$$L(x) = \frac{p_1(x_1,x_2,\ldots,x_n)}{p_o(x_1,x_2,\ldots,x_n)} \; . \tag{7}$$

Here, $p_o(\underline{x})$ and $p_1(\underline{x})$, the probability densities induced on Ω by H_o and H_1, respectively are given by

$$p_0(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det R_0}} \exp(-\frac{1}{2}\underline{x}^* R_0^{-1}\underline{x}) \tag{8}$$

$$p_1(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det R_1}} \exp(-\frac{1}{2}\underline{x}^* R_1^{-1}\underline{x}); \tag{9}$$

R_0 is the covariance matrix of the noise, R_1 that of the signal plus noise; these are given by

$$R_0 = \{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu-\mu)} f_0(\lambda)d\lambda;\ \nu,\mu=1,2,\ldots,n\} \tag{10}$$

and similarly for R_1. Hence

$$L(\underline{x}) = K \exp[\frac{1}{2}\underline{x}^*(R_0^{-1} - R_1^{-1})\underline{x}]. \tag{11}$$

The likelihood function is thus a monotonically increasing function of the quadratic form

$$\underline{x}^*(R_0^{-1} - R_1^{-1})\underline{x} = \underline{x}^* Q\underline{x}\ , \quad \text{say}; \tag{12}$$

hence

$$W_{MP} = \{\underline{x}|\underline{x}^*(R_0^{-1} - R_1^{-1})\underline{x} > c\}\ . \tag{13}$$

For practical purposes (12) is not a convenient expression since its use requires the inversion of large matrices. We shall now show how a theorem from the theory of Toeplitz forms (ref. 1) can be applied here to obtain a computationally feasible procedure. A similar approximation has previously (ref. 2) been found effective for the estimation of the spectral density of a random process.

The probability density of the quadratic form (12) can be looked on as being completely determined

(i) under hypothesis H_0, by the eigenvalues of the matrix

$$QR_0 = I - R_1^{-1}R_0; \tag{14}$$

(ii) under hypothesis H_1, by the eigenvalues of the matrix

$$QR_1 = R_0^{-1}R_1 - I. \tag{15}$$

Approximations to the distribution of quadratic forms such as (12) are considered in ref. 1. It is shown that the distribution is asymptotically normal, but since the matrices are nearly of Toeplitz character, closer approximations are suggested by Toeplitz theory. It can be shown that, if μ_0 denotes the trace of QR_0 and μ_1 the trace of QR_1,

$$\mu_0 \cong \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_1(\lambda) - f_0(\lambda)}{f_1(\lambda)}\, d\lambda, \tag{16}$$

$$\mu_1 \cong \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_1(\lambda) - f_0(\lambda)}{f_0(\lambda)}\, d\lambda. \tag{17}$$

This suggests that we should use for our computations the quadratic form

$$x^*Q_a x$$

where

$$Q_a = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu-\mu)\lambda}\left(\frac{1}{f_0(\lambda)} - \frac{1}{f_1(\lambda)}\right)d\lambda;\quad \nu,\mu = 1,2,\ldots,n\right\} \tag{18}$$

and construct what we might call the "approximately most powerful test"

$$W_{aMP} = \{\mathbf{x} \mid \mathbf{x}^*Q_a\mathbf{x} > c\} \tag{19}$$

where, with (2) and (3),

$$\frac{1}{n}\,\mathbf{x}^*Q_a\mathbf{x} = \frac{1}{2\pi n}\int_{-\pi}^{\pi}\sum_{\nu,\mu=1}^{n} x_\nu e^{i(\nu-\mu)\lambda} x_\mu \frac{f_s(\lambda)}{f_n(\lambda)[f_n(\lambda) + f_s(\lambda)]}\, d\lambda. \tag{20}$$

But we know (cf. ref. 3, p. 91) that

$$\frac{1}{2\pi n}\left|\sum_{\nu=1}^{n} x\, e^{i\nu\lambda}\right|^{2} = I(\lambda), \tag{21}$$

the periodogram, which is an unbiased and inconsistent estimate of the spectral density $f(\lambda)$ of the process $x(t)$. Thus,

$$\frac{1}{n}\, x^{*}Q_{a}x = \int_{-\pi}^{\pi} I(\lambda)\, \frac{f_s(\lambda)}{f_n(\lambda)[f_n(\lambda)+f_s(\lambda)]}\, d\lambda$$

$$= \text{an estimate of } \int_{-\pi}^{\pi} f(\lambda)\, \frac{f_s(\lambda)}{f_n(\lambda)[f_n(\lambda)+f_s(\lambda)]}\, d\lambda\,. \tag{22}$$

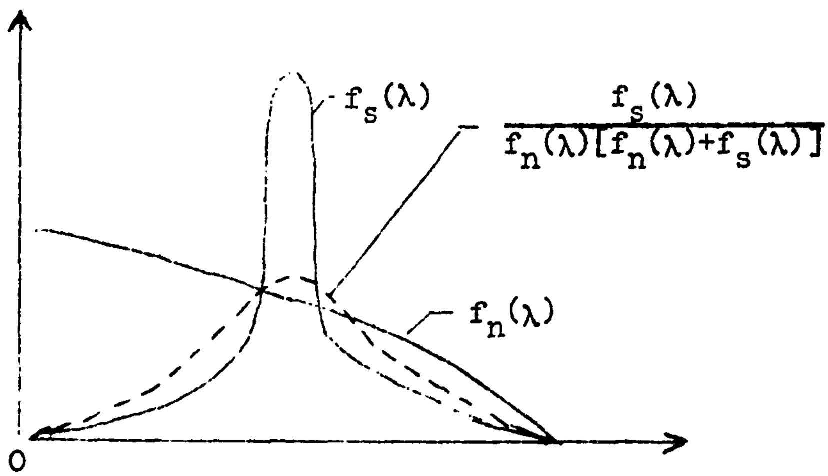
Hence, the approximately best test statistic is a weighted periodogram. The resulting signal detection method could therefore be represented schematically as follows:

$$x(t) \rightarrow \boxed{\text{filter}} \rightarrow \boxed{\text{power meter}} \rightarrow \boxed{\text{power}\ \begin{matrix} >c \\ <c \end{matrix}}\ \begin{matrix} \rightarrow \text{signal present} \\ \rightarrow \text{signal absent} \end{matrix}$$

The filter will have to have the relevant characteristic, viz

$$\sqrt{\frac{f_s(\lambda)}{f_n(\lambda)[f_n(\lambda)+f_s(\lambda)]}}\ ;$$

the power meter is a quadratic integrator. Usually, shapes of the spectral densities will be somewhat as follows:



With this procedure, then, we do not have to invert large matrices but can construct simple physical devices; it is also suitable for high-speed computation.

To test the accuracy of this method, the frequency functions corresponding to the most powerful test based on

$$x^*Qx$$

and the approximately most powerful test based on

$$x^*Q_ax$$

were computed and compared numerically for a simple case.

We chose for the covariance matrix of the noise the unit matrix

$$R_o = N = I$$

and for the covariance matrix of the signal

$$R_1 - R_o = S = \{s_{\nu\mu};\quad \nu,\mu = 1,2,\ldots,n\}$$

with

$$s_{\nu\mu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\rho^2}{|1-\rho e^{i\lambda}|^2}\, e^{i(\nu-\mu)\lambda} d\lambda = \rho^{|\nu-\mu|}$$

The spectral density of the noise is, hence, $f_n(\lambda) \equiv 1$ and that of the signal

$$f_s(\lambda) = \frac{1-\rho^2}{|1-\rho e^{i\lambda}|^2}$$

is the Poisson kernel which, for values of ρ fairly close to 1, is narrow-band. The parameters chosen were

$$n = 20\ , \qquad \rho = .7\ .$$

If we denote the eigenvalues of S by λ_ν, those of

$$Q = N^{-1} - (N+S)^{-1} = I - (I+S)^{-1}$$

will be

$$1 - \frac{1}{1+\lambda_\nu} .$$

We denote by $g_o(x)$ the frequency function corresponding to the eigenvalues

$$1 - \frac{1}{1+\lambda_\nu}$$

and by $g_1(x)$ that corresponding to the eigenvalues λ_ν. These are then the "exact" frequency functions.

For the approximate theory, we define the matrix Q_a with elements

$$\ell_{\nu\mu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_s(\lambda)}{f_n(n)[f_n(\lambda)+f_s(\lambda)]} e^{i(\nu-\mu)\lambda} d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\frac{1-\rho^2}{|1-\rho e^{i\lambda}|^2}}{1 + \frac{1-\rho^2}{|1-\rho e^{i\lambda}|^2}} e^{i(\nu-\mu)} d\lambda$$

$$= \frac{1-\rho^2}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i(\nu-\mu)\lambda}}{|1-\rho e^{i\lambda}|^2 + 1 - \rho^2} d\lambda .$$

By choosing the constant r as the solution of the equation

$$r[(1-\rho^2) + (1+\rho^2)] = \rho(1+r^2)$$

such that $0 < r < 1$, $\ell_{\nu\mu}$ can be brought to the form

$$\ell_{\nu\mu} = \frac{r}{2\pi\rho}\int_{-\pi}^{\pi} \frac{1-\rho^2}{|1-re^{i\lambda}|^2} e^{i(\nu-\mu)\lambda} d\lambda$$

$$= \frac{r}{\rho(1-\rho^2)} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{|1-re^{i\lambda}|^2} e^{i(\nu-\mu)\lambda} d\lambda$$

$$= \frac{r}{\rho(1-\rho^2)} \cdot r^{(\nu-\mu)} ,$$

since $r^{(\nu-\mu)}$ is the Fourier transform of $\frac{1-r^2}{|1-re^{i\lambda}|^2}$. The eigenvalues of

$$Q_a = \{\ell_{\nu\mu};\ \nu,\mu = 1,\ldots,n\}, \quad n = 20, \quad \rho = .7$$

are computed to give the frequency function $\gamma_o(x)$ which is to be an approximation to $g_o(x)$ as defined above.

Similarly, the eigenvalues of

$$Q_a R_1 = Q_a(R_o + S) = Q_a(I + M)$$

are found and used to determine the frequency function $\gamma_1(x)$ which serves as an approximation to $g_1(x)$ as defined above.

Figures 1 and 2 show the curves $g_o(x)$, $\gamma_o(x)$ and $g_1(x)$, $\gamma_1(x)$, and agreement is seen to be very good.

Acknowledgment. Thanks are due to Professor Ulf Grenander on whose work and suggestions this paper is based.

References

1. U. Grenander and G. Szegö, "Toeplitz Forms and their Applications", University of California Press, 1958.

2. W. Freiberger and U. Grenander, Approximate distributions of noise power measurements, Quart. Appl. Math. **17**, 272-283 (1959).

3. U. Grenander and M. Rosenblatt, "Statistical Analysis of Stationary Time Series", John Wiley & Sons, New York; Almquist & Wiksell, Stockholm, 1957.

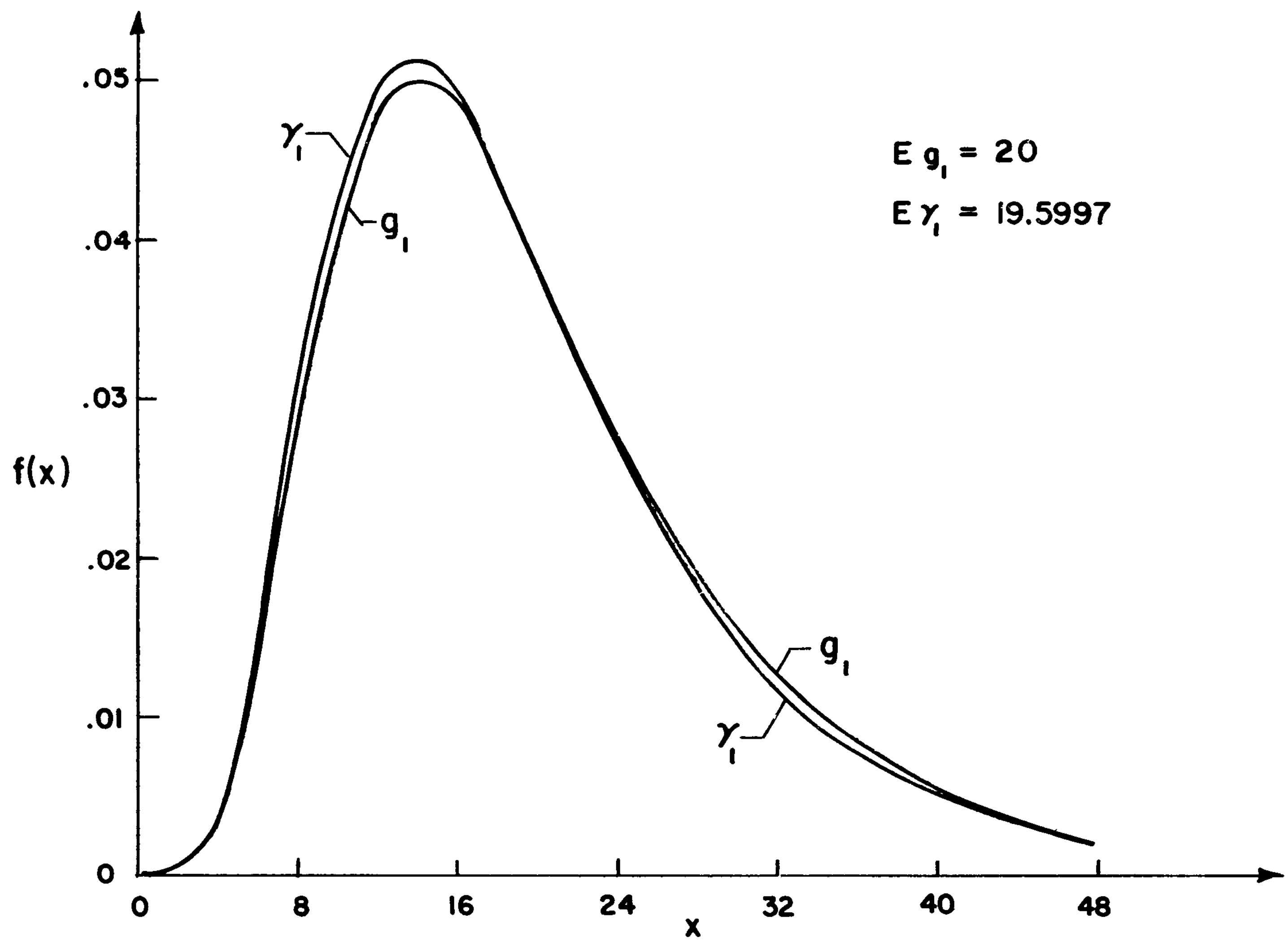

.05
.04
.03
.02
.01
0
f(x)
0
8
16
24
32
40
48
x
γ_1
g_1
g_1
γ_1
$E\,g_1 = 20$
$E\,\gamma_1 = 19.5997$

FIGURE 1

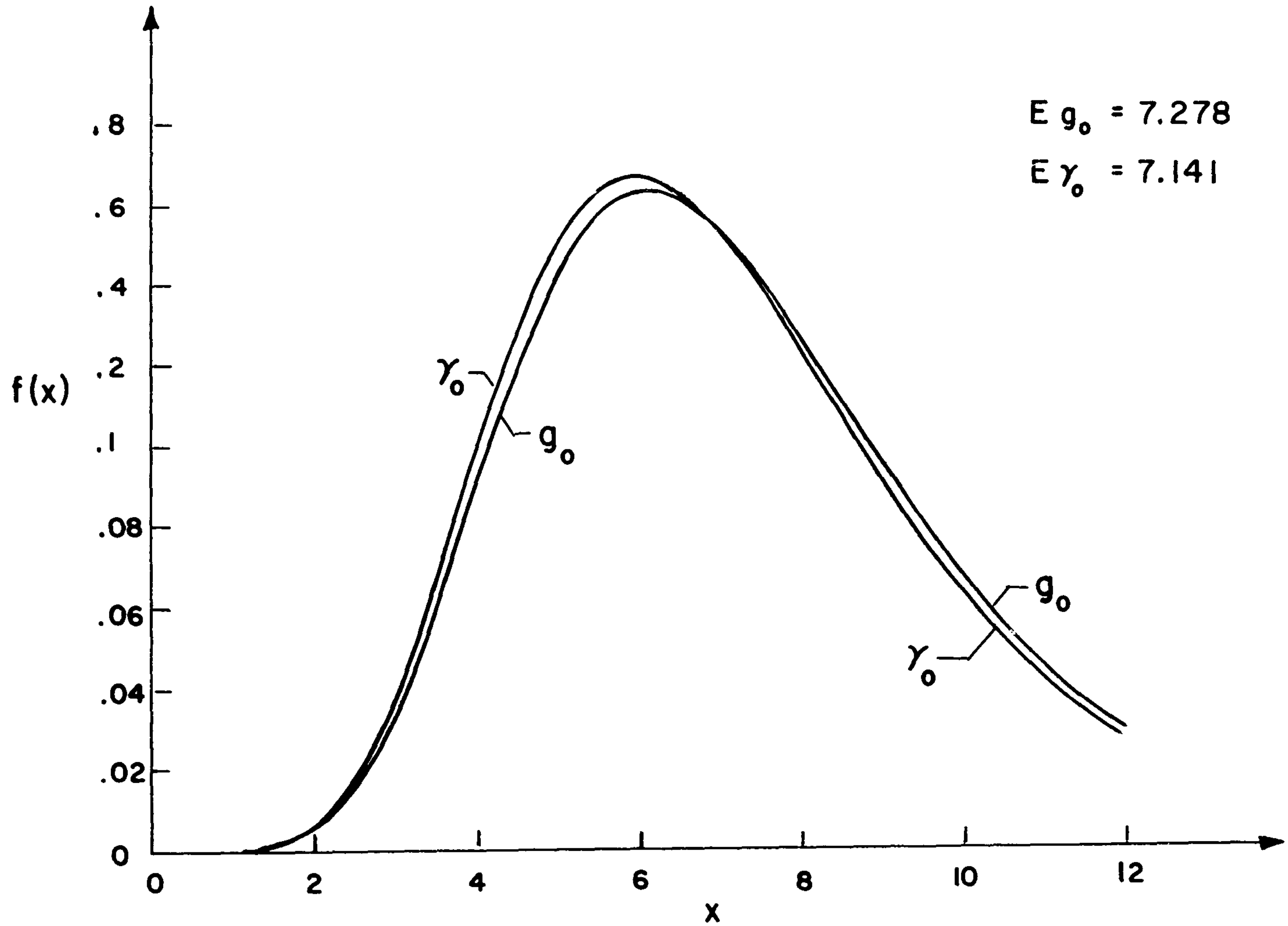

f(x)
.8
.6
.4
.2
.1
.08
.06
.04
.02
0
0
2
4
6
8
10
12
x
$E\,g_o = 7.278$
$E\,\gamma_o = 7.141$
γ_o
g_o
g_o
γ_o

FIGURE 2