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BIBLIOGRAPHY

- (1) Borel, A.: Seminar on transformation groups.
Annals Study 46, Princeton Univ. Press,
1960.
- (2) ———: Cohomologie des espaces localement
compacts, d'après J. Leray, Notes, Fed.
School of Techn. Zurich, Switzerland,
2nd Ed. 1957.
- (3) ———: Nouvelle démonstration d'un
théorème de P.A. Smith, Comment. Math.
Helv. 29 (1955), pp. 27-39.
- (4) Bourbaki, N.: Topologie générale, Chap. X,
Act. Sci. Ind. 1084, Paris, 1949.
- (5) Cartan, H. and Eilenberg, S.: Homological
Algebra, Princeton Univ. Press, 1956.
- (6) Chevalley, C.: Theory of Lie groups, Princeton
Univ. Press, 1946.
- (7) Cohen, H.: A cohomological definition of
dimension for locally compact Hausdorff
spaces, Duke Math. J. 21 (1954), pp. 209-224.
- (8) Conner, P.: On the action of the circle group,
Michigan Math. Jour. 4 (1957).
- (9) Eilenberg, S. and Steenrod, N.E.: Foundations
of Algebraic Topology, Princeton Univ.
Press, 1952.
- (10) Floyd, E.E. and Conner, P.: Orbit spaces of
circle groups of transformations, Annals
of Math. 67 (1958), pp. 90-98.
- (11) Gleason, A.: Spaces with a compact Lie group
of transformations, Proc. Amer. Math. Soc.
1 (1950), pp. 35-43.
- (12) Godement, R.: Théorie des faisceaux, Act.
Sci. Ind. 1252, Paris, 1958.
- (13) Hu, S.T.: Homotopy theory, Academic Press,
New York, 1959.

- (14) Mann, L.N.: Compact abelian transformation groups, Trans. Amer. Math. Soc. 99 (1961), pp. 41-59.
- (15) Seifert, H. and Threlfall, W.: Lehrbuch der Topologie, Teubner, Leipzig, 1934.
- (16) Smith, P.A.: New results and old problems in finite transformation groups, Bull. Amer. Math. Soc. 6 (1960), pp. 415.
- (17) ———: Fixed points of periodic transformations, Appendix B in Lesfchetz, Algebraic topology, Amer. Math. Soc. Colloquium Publications, 27, New York, 1942.
- (18) Steenrod, N.E.: The topology of fibre bundles, Princeton Univ. Press, 1951.
- (19) Stewart, T.E.: Lifting the action of a group in a fibre bundle, Bull. Amer. Math. Soc. 66 (1960), pp. 129-132.

Preface

Let G be a topological transformation group on a compact Hausdorff space Y and $F(G;Y)$ its fixed point set. The present paper is devoted to the study of the cohomology structure of $F(G;Y)$ in the following three cases:

- (1) G is the group Z_2 of integers modulo 2 and Y has the mod 2 cohomology ring of the real projective n -space.
- (2) G is the group Z_p of integers modulo p , where p is an odd prime number, and Y has the mod p cohomology structure of the lense $(2n+1)$ -space mod p .
- (3) G is the circle group S^1 and Y has the integral cohomology ring of the complex projective n -space.

For simplicity, we shall call Y a cohomology real projective n -space or a cohomology lense $(2n+1)$ -space mod p or a cohomology complex projective n -space if its cohomology structure is that described in (1) or (2) or (3). (Formal definitions of these notions will be given later.)

Our study of the problem proposed above is motivated by two recent theorems obtained separately by P.A. Smith and C.T. Yang. In (16), Smith proved

that if Z_2 acts effectively on the real projective n -space, then the fixed point set is either empty, or it has exactly two components c_1 and c_2 , where each c_i is a cohomology real projective n_i -space, $i = 1, 2$, and $n_1 + n_2 = n-1$. Later in an unpublished work, Yang proved that if S' acts differentiably on the complex projective n -space, then the fixed point set is non-empty, and has at most $n+1$ components, say $c_1, \dots, c_k$, $k \leq n+1$, where every c_i is a cohomology complex projective n_i -space, $i = 1, 2, \dots, k$, and $n_1 + n_2 + \dots + n_k = n-k+1$. Thus roughly speaking, their theorems start from the actual projective space, a real one in Smith's case and a complex one in Yang's case, and end up with asserting that the fixed point set is an union of a finite number of cohomology projective spaces. Naturally, as suggested by Smith, one would inquire what is the situation when, in their hypothesis, the actual projective spaces are replaced by the weaker notion of cohomology projective spaces. This is precisely the cases (1) and (3). Our main purpose is to show that, under the more general setting of (1) and (3), essentially the same conclusions obtained by Smith and Yang still hold true. We also include a study of case (2), which is the natural counterpart of case (1) when p is odd.

The key point for the proofs of Smith's and Yang's theorems is to make use of the close relation between the projective spaces and the spheres. There exists a free action of Z_2 on the n -sphere S^n with the real projective n -space RP^n as the orbit space. Similarly, there exists a canonical free action of S^1 on the $(2n+1)$ -sphere S^{2n+1} with the complex projective n -space CP^n as the orbit space. Thus RP^n can be viewed as the base space in a principal bundle (S^n, RP^n, Z_2, π) and CP^n as the base space in a principal bundle $(S^{2n+1}, CP^n, S^1, \pi)$, where in each case π denotes the projection from the total space to the base space. Now if RP^n is acted on by a group Z_2 , it is possible to lift this action to S^n in the sense that an action of Z_2 on S^n can be defined so that the projection $\pi : S^n \longrightarrow RP^n$ becomes equivariant. This is so because S^n is the space of all paths of RP^n ; hence any map of RP^n into itself induces a map of the space of all paths into itself in a natural fashion. Similarly, if S^1 acts on CP^n differentiably, a lifting can also be constructed through analytic means. In both cases, the idea is to shift the given action to an action on spheres. Once this is done, the theorems can be proved via a theorem of A. Borel (1). It is then clear from what has been said that our problem can be solved along the same line of thought if we can do

the following: First, to exhibit that cohomology projective spaces are covered by cohomology spheres as in the actual case; second, to show that this relation permits one to lift an action (of a suitable group) on the former to an action on the latter.

Our paper is divided into two parts. In Part 1, we treat the cases (1) and (2) where the acting group is a finite cyclic group of prime order. In part 2, we treat the case (3) where the acting group is the circle group. The schemes of development of these two parts are entirely parallel to each other and the division is made chiefly because of some technical differences between handling a finite and an infinite group. Each part begins with a preliminary section in which known results needed later and the likes are collected. In Section 2 we first prove that if Z_p acts freely on a cohomology sphere mod p , then the orbit space is a cohomology real projective space or a cohomology lense space mod p according to whether p is even or odd. This is, of course, more or less well-known. Much more interesting is the fact that the converse is also true; that is, if Z_p acts freely on a connected compact Hausdorff space X such that the orbit space X/Z_p is a cohomology real projective space or a cohomology lense space mod p , then X itself must be a cohomology sphere mod p . Similar results for the circle group are obtained

in Section 6. These may be termed as the uniqueness theorems which assert that cohomology spheres are essentially the only spaces on which Z_p or S' can act freely to give a cohomology projective space as the orbit space. The existence problem (i.e. to see if every cohomology projective space can actually be obtained as the orbit space of a suitable transformation group on a cohomology sphere.) is studied in Section 3 and Section 7. In Section 3, we start from a connected compact Hausdorff space Y and then describe how, for each non-zero element $\alpha \in H^1(Y; Z_p)$, a principal bundle (X, Y, Z_p, π) can be constructed for which $\pi^*: H^1(Y; Z_p) \longrightarrow H^1(X; Z_p)$ takes α into zero. Owing to this last property, we call X a cohomology covering space of Y with respect to α . In fact, it is constructed the same way as the classical covering space. This construction also takes care of the lifting problem automatically. Similarly, in Section 7, we assume that Y is just compact Hausdorff and prove, for each element a_0 of the integral cohomology group $H^2(Y)$, zero or not, the existence of a principal bundle (X, Y, S', π) such that $\pi^*: H^2(Y) \longrightarrow H^2(X)$ takes a_0 into zero. Unlike the previous case, this bundle is now obtained indirectly with the aid of the obstruction theory. Simple as this procedure is, the lack of an

explicit construction makes the lifting problem quite difficult. Fortunately this difficulty can be overcome by a recent result of T.E. Stewart (19) concerning problems of this nature. Section 4 and Section 8 contain the proofs of the main theorems of this paper.

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PART 1

Section 1. Preliminaries of Part 1

In this section we collect some algebraic and topological facts that will be used later. Most of them are well-known. Hence proofs will be omitted, although references will be given in each case.

1. 1. Let G be any finite group and A an Abelian group on which G operates as a group of automorphisms. Then the functor $H^*(G; A) = \sum_{s=0}^{\infty} H^s(G; A)$, called the cohomology of G with coefficients in A , can be defined. We shall be concerned only with the case when G is the group Z_p of integers modulo a prime number p . In this case, the groups $H^s(Z_p; A)$ can be calculated explicitly as

$$H^s(Z_p; A) = \begin{cases} A^T & \text{if } s = 0; \\ A^T / \tau A & \text{if } s = 2n, \quad n \geq 0; \\ \sigma A / \tau A & \text{if } s = 2n + 1, \quad n \geq 0, \end{cases}$$

where T is a generator of Z_p , $\tau = 1 - T$, $\sigma = \sum_{i=0}^{p-1} T^i$, $A^T = \text{Ker } \tau$ and $\sigma A = \text{Ker } \sigma$. If, in particular, $A = Z_p$, then Z_p can operate only trivially on it and the above formula will give $H^s(Z_p; Z_p) = Z_p$ for all $s \geq 0$. Moreover, there exists a product on $H^*(Z_p; Z_p)$ which makes it a ring. This ring can be described as follows: If $p = 2$, we have

$$H^*(Z_2; Z_2) = Z_2[x],$$

where $Z_2[x]$ is the polynomial ring with coefficients in

2. The grading of $H^*(Z_2; Z_2)$ is obtained by assigning $\deg x$ as the degree of x (abbreviated $\deg x$). If $p \neq 2$, the situation is a little more complicated. In this case we have

$$H^*(Z_p; Z_p) = \wedge[a] \otimes Z_p[x],$$

where $\wedge[a]$ is the exterior algebra generated by a with coefficients in Z_p and $Z_p[x]$ is the polynomial ring with coefficients in Z_p . The grading of $H^*(Z_p; Z_p)$ is obtained by assigning $\deg a = 1$ and $\deg x = 2$ [5; Chap. XII].

1. 2. In the ring $H^*(Z_p; Z_p)$, $p \neq 2$, we shall need another cohomology operation beside the product, namely the Bockstein operator β . This is the coboundary

$$\beta: H^s(Z_p; Z_p) \rightarrow H^{s+1}(Z_p; Z_p)$$

associated with the exact sequence of coefficient groups

$$0 \rightarrow Z_p \rightarrow Z_{p^2} \rightarrow Z_p \rightarrow 0.$$

β^s is an isomorphism if s is odd, and is trivial if s is even and $s \neq 0$. In particular, we may choose a , and x such that $\beta^s(a) = x$.

Suppose that A is a vector space over the field Z_p , that the group Z_p operates on A as a group of linear automorphisms and that σ is trivial.

1. 3. Lemma. If $A^T = 0$, then $A = 0$.

Proof. For any $\alpha \in A$, we have

$$0 = \sigma(\alpha) = \tau^{p-1}(\alpha) = \tau(\tau^{p-2}(\alpha)).$$

Since $A^T = 0$, it follows that $\tau^{p-2}(\alpha) = 0$. Repeating this argument, we have $\tau^{p-3}(\alpha) = 0$, $\tau^{p-4}(\alpha) = 0$, ..., $\alpha = 0$.

1.4. Lemma. If $\dim A^T$ (dimension of A^T) = 1, then
 $\dim A \leq p-1$.

Proof. The sequence

$$0 \rightarrow A^T \rightarrow A \rightarrow \tau A \rightarrow 0$$

is exact and hence splits since A is a vector space.

Choose $\alpha_0 \neq 0$ such that $A = Z_p \alpha_0 \oplus \tau A$. If $\alpha \in A$ is any element of A , there exist $\eta_0 \in Z_p$ and $\beta_1 \in A$ such that

$$(1) \quad \alpha = \eta_0 \alpha_0 + \tau(\beta_1)$$

Similarly there exists $\eta_1 \in Z_p$ and $\beta_2 \in A$ such that

$$(2) \quad \beta_1 = \eta_1 \alpha_0 + \tau(\beta_2)$$

Substituting (2) in (1), we obtain

$$(3) \quad \alpha = \eta_0 \alpha_0 + \eta_1 \tau(\alpha_0) + \tau^2(\beta_2)$$

Repeating this argument, we can find $\eta_0, \eta_1, \dots, \eta_{p-1} \in Z_p$
 and $\beta_{p-1} \in A$ such that

$$(4) \quad \alpha = \eta_0 \alpha_0 + \eta_1 \tau(\alpha_0) + \eta_2 \tau^2(\alpha_0) + \dots + \eta_{p-2} \tau^{p-2}(\alpha_0) + \eta_{p-1} \tau^{p-1}(\beta_{p-1})$$

The last term in the right-hand side of (4) vanishes because $\tau^{p-1} = \sigma = 0$ by assumption. Hence (4) reduces to

$$(5) \quad \alpha = \eta_0 \alpha_0 + \eta_1 \tau(\alpha_0) + \dots + \eta_{p-2} \tau^{p-2}(\alpha_0)$$

As α is arbitrary, this means that the set $\{\alpha_0, \tau \alpha_0, \dots, \tau^{p-2} \alpha_0\}$ is a set of generators of A , which proves the lemma.

Suppose that X is a compact Hausdorff space and that G acts on X as a transformation group. We denote the orbit space by X/G and the canonical projection by $\pi : X \rightarrow X/G$. Let $H^*(X; L)$ be the Alexander-Wallace-Spanier cohomology

of X with coefficient in a certain group L . Then G also operates on $H^*(X;L)$ as a group of automorphisms so that $H^*(G; H^*(X;L))$ would make sense. Let $H^*(X;L)^G$ be the subgroup of invariant elements of $H^*(X;L)$, i.e.

$$H^*(X;L)^G = \{ \alpha \in H^*(X;L) \mid g^*(\alpha) = \alpha \text{ for all } g \in G \}.$$

In case $G = Z_p$, this subgroup will be denoted by $H^*(X;L)^{T^*}$, where T is a generator of Z_p . The following well-known theorem of the Leray-Cartan spectral sequence is the main tool that will be used in Part 1 of this paper.

1. 5. Proposition. If a finite group G acts freely on a compact Hausdorff space X , then there exists a spectral sequence (E_r) whose E_2 - term is given by

$$E_2^{s,t} = H^s(G; H^t(X;L)),$$

and whose E_2 - term is associated with $H^*(X/G;L)$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} H^t(X/G;L) & \xrightarrow{\pi} & H^t(X;L) \\ \psi_* \searrow & & \nearrow i \\ E_2^{s,t} & = & H^t(X;L)^G \end{array},$$

Where ψ_* is the canonical "edge homomorphism" [5 ; P. 332], and i is the inclusion.

A proof can be found, for instance, in [3].

1. 6. Besides the Leray-Cartan spectral sequence, we shall also occasionally make use of the Smith special cohomology theory. Following [1], a modern version of this

theory can be described briefly as follows.

Let Z_p act freely on X . Consider the Leray sheaf $\underline{A}$ associated with the canonical projection $: X \rightarrow X/Z_p$. This is a sheaf over X/Z_p whose stalk on each $y \in X/Z_p$ is given by $H^*(\pi^{-1}(y)) = H^0(\pi^{-1}(y))$, where Z_p is used as the coefficient group. Now the group Z_p operates on $\underline{A}$ as a group of sheaf automorphisms so that we can consider the endomorphisms $\tau = 1 - T$ and $\sigma = \sum_{i=0}^{p-1} \tau^i$ of $\underline{A}$, where T is a generator of Z_p . Following usual convention, if one of them is denoted by ρ , the other will be denoted by $\bar{\rho}$. The cohomology groups $H^*(X/Z_p; \rho \underline{A}) = \sum_{s=0}^{\infty} H^s(X/Z_p; \rho \underline{A})$ of X/Z_p with coefficients in the sheaf of $\rho \underline{A}$ are called the Smith special cohomology groups which we will denote simply by $H^*(\rho)$.

It can be shown that

$$0 \longrightarrow \bar{\rho} \underline{A} \xrightarrow{i_{\bar{\rho}}} \underline{A} \xrightarrow{\rho} \rho \underline{A} \longrightarrow 0$$

form an exact sequence of sheaves where $i_{\bar{\rho}}$ is the inclusion and that $H^*(X/Z_p; \underline{A})$ can be identified canonically with $H^*(X)$. There is therefore an exact sequence

$$\dots \xrightarrow{\delta} H^s(\bar{\rho}) \xrightarrow{i_{\bar{\rho}}^*} H^s(X) \xrightarrow{\rho^*} H^s(\rho) \xrightarrow{\delta} H^{s+1}(\bar{\rho}) \xrightarrow{i_{\bar{\rho}}^*} \dots$$

of cohomology groups which is known as the Smith sequence.

Moreover, $H^s(\sigma)$ can be identified canonically with $H^s(X/Z_p)$ and this identification carries $i_{\sigma}^*: H^s(\pi) \rightarrow H^s(X)$ over to $\pi^*: H^s(X/Z_p) \rightarrow H^s(X)$. We have therefore two exact sequences.

$$\dots \xrightarrow{\delta} H^s(X/Z_p) \xrightarrow{\pi^*} H^s(X) \xrightarrow{\tau^*} H^s(\tau) \xrightarrow{\delta} H^{s+1}(X/Z_p) \xrightarrow{\pi^*} \dots$$

and

$$\dots \xrightarrow{\delta} H^s(\tau) \xrightarrow{i^*} H^s(X) \xrightarrow{\pi^*} H^s(X/Z_p) \xrightarrow{\delta} H^{s+1}(\tau) \xrightarrow{i^*} \dots$$

Finally, it is not difficult to see that

$$i^* \circ \tau^* = 1 - T^* \quad \text{and} \\ \pi^* \circ \delta^* = \sum_{i=0}^{p-1} \tau^{*i}$$

[1 ; pp. 40-42].

We call a compact Hausdorff space X a cohomology n -sphere over L if its cohomology $H^*(X; L)$ is the same as that of the n -sphere, i.e.

$$H^s(X; L) = \begin{cases} L & \text{if } s = 0, n; \\ 0 & \text{otherwise} \end{cases}.$$

We agree that the empty set is regarded as a cohomology (-1) -sphere over L . A cohomology n -sphere over Z_p will also be called a cohomology n -sphere mod p and a cohomology n -sphere over the group of integers Z will also be called an integral cohomology n -sphere.

1. 7. Proposition. If X is a cohomology n -sphere mod p on which the group Z_p acts, then the fixed point set $F(Z_p; X)$ is a cohomology r -sphere mod p for some $-1 \leq r \leq n$. Moreover, $n-r$ is even if p is odd.

This is a well-known theorem of P. A. Smith, originally formulated in terms of homology under the assumption that X has finite Lebesgue covering dimension [17]. It was subsequently shown by L. Mann [14], also in terms of homology, that the dimensionality condition can be removed. The above version is just the dual form of Mann's result.

1. 8. Finally, we shall need one more theorem concerning the action of $Z_p \times Z_p$ on a cohomology n -sphere mod p , X . The fixed point set $F(Z_p \times Z_p; X)$ is of course still a cohomology sphere mod p , say of dimension r . Besides, $Z_p \times Z_p$ contains $p-1$ non-trivial cyclic groups N^i , $i=0, 1, \dots, p$. By 1.7 each $F(N^i; X)$ is a cohomology n_i -sphere mod p for some n_i , $i=0, 1, \dots, p$.

1. 9. Proposition. Let $Z_p \times Z_p$ act on a cohomology n -sphere mod p , X . Let N^i , n_i , $i=0, 1, \dots, p$ and r be as in 1.8. Then

$$\sum_{i=0}^p (n_i - r) = n - r.$$

This proposition can be found in [1; p. 175] where X is assumed to have finite cohomology dimension over Z_p . Again this condition can be removed. Indeed, the finite dimensionality condition is used only to assure that $H^*(F(Z_p \times Z_p; X); Z_p)$, $H^*(X/Z_p \times Z_p; Z_p)$, $H^*(F(N^i; X); Z_p)$ and $H^*(X/N^i; Z_p)$, $i=0, 1, \dots, p$, have finite dimensions when $H^*(X; Z_p)$ does. By Mann's result [14;], this is true without such restriction.

Section 2. Cohomology real projective spaces and cohomology lense spaces.

Throughout this section, X is a compact Hausdorff space on which Z_p acts freely. Cohomology always has Z_p as the coefficient group. We shall distinguish the case $p = 2$ from other primes; so we shall write out the coefficient group in the cohomology for this particular case and $H^*(X)$ will denote $H^*(X; Z_p)$ only when $p \neq 2$.

2. 1. Definition. A compact Hausdorff space Y is said to be a cohomology real projective n-space if the cohomology ring $H^*(Y; Z_2)$ is given by

$$H^*(Y; Z_2) = Z_2[x] / (x^{n+1}), \text{ deg } x = 1,$$

where $Z_2[x]$ is the polynomial ring with coefficients in Z_2 and (x^{n+1}) is the ideal generated by x^{n+1} .

2. 2. Definition. A compact Hausdorff space Y is said to be a cohomology lense (2n+1)-space mod p if the cohomology ring $H^*(Y)$ is given by

$$H^*(Y) = \wedge[a] \otimes Z_p[x] / (x^{n+1}), \text{ deg } a = 1, \text{ deg } x = 2,$$

and if $\beta'(a) = x$ where β' is the Bockstein coboundary

$$\beta': H^1(Y) \rightarrow H^2(Y).$$

Notice that the anti-podal map (a map in this paper is always meant to be a continuous one) on the n -sphere S^n defines a free action of Z_2 on S^n for which the orbit space is the real projective n -space whose mod 2 cohomology is exactly given by 2.1. Similarly, if we let

$$S^{2n+1} = \{(z_0, \dots, z_n) \mid \sum_{i=0}^n |z_i|^2 = 1, z_i \text{ complex numbers}\},$$

then the map

$$S^{2n+1} \longrightarrow S^{2n+1}$$

$$(z_0, \dots, z_n) \longrightarrow (z_0 e^{2\pi i/p}, \dots, z_n e^{2\pi i/p}) \quad p \neq 2$$

defines a free action of Z_p on S^{2n+1} for which the orbit space is the lens $(2n+1)$ -space mod p [15] whose mod p cohomology is exactly given by 2.2. These motivate the definitions just given. More generally, we have the following

2.3. Proposition. If $p \neq 2$ and X is a cohomology $(2n+1)$ -sphere mod p , then X/Z_p is a cohomology lens $(2n+1)$ -space mod p .

Proof. We observe first that

$$H^s(X/Z_p) = 0 \quad \text{for all } s > 2n+1.$$

This follows from [14;].

Consider now the Leray-Cartan spectral sequence (E_r) of

1.5. As

$$E_2^{s,t} = H^s(Z_p; H^t(X)) = 0 \quad \text{for all } t \neq 0, 2n+1,$$

there exists an exact sequence

$$\dots \longrightarrow E_2^{s-2n-2, 2n+1} \longrightarrow E_2^{s,0} \xrightarrow{\phi_s} H^s(X/Z_p) \longrightarrow E^{s-2n-1, 2n+1} \longrightarrow \dots,$$

where ϕ_s is the canonical edge homomorphism [5; p. 326].

This gives immediately that

$$\phi_s: E_2^{s,0} \longrightarrow H^s(X/Z_p)$$

is an isomorphism for all $0 \leq s \leq 2n$. For $s = 2n+1$, we have

the exact sequence

$$\begin{aligned} 0 \longrightarrow E_2^{-1, 2n+1} \longrightarrow E_2^{2n+1, 0} \longrightarrow H^{2n+1}(X/Z_p) \longrightarrow E_2^{0, 2n+1} \longrightarrow E_2^{2n+2, 0} \\ \longrightarrow H^{2n+2}(X/Z_p) = 0. \end{aligned}$$

Since $E_2^{0, 2n+1} = E_2^{2n+2, 0} = Z_p$, $E_2^{0, 2n+1} \longrightarrow E_2^{2n+2, 0}$ must

necessarily be an isomorphism. Hence Φ_{2n+1} is also an isomorphism. The desired cohomology structure of X/Z_p then follows from 1.2.

In exactly the same way, one can prove

2. 4. Proposition. If $p = 2$ and X is a cohomology n -sphere mod 2, then X/Z_2 is a cohomology real projective n -space.

The main purpose of the present section is to prove the converses of the preceding two propositions.

2. 5. Theorem. If $p \neq 2$, X is connected and X/Z_p is a cohomology lens (2n+1) -space mod p , then X is a cohomology (2n+1)-sphere mod p .

Proof. In the Leray-Cartan spectral sequence (E_r) of 1.5, we first claim

(1) $\Phi_s: E_2^{s,0} \rightarrow H^s(X/Z_p)$ is an isomorphism for all $1 \leq s \leq 2n+1$.

(2) $\pi^*: H^s(X/Z_p) \rightarrow H^s(X)$ is trivial for all $s \geq 1$.

There exists an exact sequence of lower terms

(3) $0 \rightarrow E_2^{1,0} \xrightarrow{\Phi_1} H^1(X/Z_p) \xrightarrow{\pi^*} E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{\Phi_2} H^2(X/Z_p)$
 [5; P. 328]. If $n = 0$, (1) and (2) follow directly from the exactness of (3). Suppose that $n > 0$. Since X is connected,
 $E_2^{*,0} = \sum_{s=0}^{\infty} E_2^{s,0} = \sum_{s=0}^{\infty} H^s(Z_p; H^0(X)) = \sum_{s=0}^{\infty} H^s(Z_p; Z_p)$.
 Let $a \in E_2^{1,0}$ be a generator of $E_2^{1,0}$; $a' = \Phi_1(a)$ is a generator of $H^1(X/Z_p)$, since Φ_1 must necessarily be an isomorphism. According to 1.2, $x = \beta'(a)$ is a generator of $E_2^{2,0}$ and in view of definition 2.2 and the assumption that $n > 0$,

$$\phi_2(x) = \phi_2 \circ \beta'(a) = (\beta' \circ \phi_1)(a) = \beta'(a^1) \neq 0.$$

It follows that ϕ_2 is also an isomorphism. Since a and x generate the ring $H^*(Z_p; Z_p)$, (1) is proved. From the exactness of (3), we have

$$\pi^*(\phi_1(a)) = 0$$

so that

$$\pi^*(\phi_2(x)) = \pi^* \circ \phi_2 = \phi_1(a) = \beta' \circ \pi^* \circ \phi_1(a) = \beta'(\pi^*(\phi_1(a))) = 0.$$

Since $\phi_1(a)$ and $\phi_2(x)$ generate the ring $H^*(X/Z_p)$, (2) is proved.

Let us draw some conclusions from (1) and (2). By (2) and 1.6, we see that

$$(4) \quad \sum_{i=0}^{p-1} T^{*i}: H^s(X) \rightarrow H^s(X) \text{ is trivial for all } s \geq 1,$$

where, as we recall, T denotes a generator of Z_p and $T^*: H^s(X) \rightarrow H^s(X)$ is the homomorphism induced by T . From (1), we deduce that $E_{\infty}^{s,0} = E^{s,0}$ for all $1 \leq s \leq 2n+1$. This implies that none of $E_{\infty}^{s,0}$, $1 \leq s \leq 2n+1$, $r \geq 2$, can have any non-zero cobounding element. Hence

$$(5) \quad d_{s+1}: E_{s+1}^{0,s} \rightarrow E_{s+1}^{s+1,0} \text{ is trivial for all } 1 \leq s \leq 2n+1.$$

Moreover, since

$$\dim H^s(X/Z_p) = \sum_{t=0}^s \dim E_{\infty}^{s-t,t}$$

and, by (1),

$$H^s(X/Z_p) = \text{Im } \phi_s = E_{\infty}^{s,0}, \quad 1 \leq s \leq 2n+1$$

it follows that

$$(6) \quad E_{\infty}^{0,s} = 0 \text{ for all } 1 \leq s \leq 2n+1.$$

We now proceed to prove by induction that

(7) $H^s(X) = 0$ for all $1 \leq s < 2n+1$.

For $n = 0$, there is nothing to prove. Therefore we assume here that $n > 0$. By (1) and the exactness of (3), we have $E_2^{0,1} = 0$. By 1.1 and 1.5, this means $H^1(X)^{T^*} = 0$ so that by (4) and lemma 1.3 we have $H^1(X) = 0$. Suppose it has been proved that $H^1(X) = 0$ for all $1 \leq i < s < 2n+1$. Consider the differentials

$$d_r: E_r^{0,s} \rightarrow E_r^{r, s-r+1}.$$

Clearly $d_r = 0$ for all $r > s+1$. If $1 < r < s+1$, then $1 < s-r+1 < s$. By induction hypothesis we have

$$E_2^{r, s-r+1} = H^r(Z_p; H^{s-r+1}(X)) = 0$$

So that $E_2^{r, s-r+1} = 0$. Hence $d_r: E_r^{0,s} \rightarrow E_r^{r, s-r+1}$ is trivial for all $r \geq 2$ and $r \neq s+1$. But $d_{s+1}: E_{s+1}^{0,s} \rightarrow E_{s+1}^{s+1,0}$ is trivial by (5). We can therefore conclude that $d_r: E_r^{0,s} \rightarrow E_r^{r, s-r+1}$ is trivial for all $r \geq 2$, or equivalently that $E_2^{0,s} = E^{0,s} = E_\infty^{0,s}$. By 1.1, 1.5 and (6), this gives

$$H^s(X)^{T^*} = E_2^{0,s} = E_\infty^{0,s} = 0.$$

Hence again by (4) and lemma 1.3, we conclude that $H^s(X) = 0$.

Next, we take up the case when $s = 2n+1$. Again we consider the differentials.

$$d_r: E_r^{0,2n+1} \rightarrow E_r^{r, 2n+2-r}.$$

With the aid of (7), which has just been proved, it is easily seen that $d_r: E_r^{0,2n+1} \rightarrow E_r^{r, 2n+2-r}$ is trivial for all $r \geq 2$ and $r \neq 2n+2$. We have therefore $E_{2n+2}^{0,2n+1}$ and $E_{2n+3}^{0,2n+1} = E^{0,2n+1} = 0$, where the last equality holds because of (6). But by definition, $E_{2n+3}^{0,2n+1} = \text{Ker} (E_{2n+2}^{0,2n+1} \xrightarrow{d_{2n+2}} E_{2n+2}^{2n+2,0})$ so that $d_{2n+2}: E_{2n+2}^{0,2n+1} \rightarrow E_{2n+2}^{2n+2,0}$ is a monomorphism. Similarly consider the

differentials

$$d_r: E_{r, 2n+2-r, r-1}^{2n+2, 0} \rightarrow E_{r, 2n+2, 0}^{2n+2, 0}$$

Again it is easily seen, with the help of (7), that d_r is trivial for all $r \geq 2$ and $r \neq 2n+2$. We have therefore $E_{2, 2n+2, 0}^{2n+2, 0} = E_{2n+2, 2, 0}^{2n+2, 0}$ and $E_{2n+3, 2n+2, 0}^{2n+2, 0} = E_{\infty, 2n+2, 0}^{2n+2, 0} \subset H^{2n+2}(X/Z_p) = 0$.

But by definition,

$$E_{2n+3, 2n+2, 0}^{2n+2, 0} = E_{2n+2, 2n+2, 0}^{2n+2, 0} / \text{Im} (E_{2n+2, 2n+2}^{0, 2n+1} \xrightarrow{d_{2n+2}} E_{2n+2, 2n+2}^{2n+2, 0}),$$

it follows that $d_{2n+2}: E_{2n+2, 2n+2}^{0, 2n+1} \rightarrow E_{2n+2, 2n+2}^{2n+2, 0}$ is an epimorphism and hence an isomorphism, since it is already known to be a monomorphism. We have thus proved that $E_{2, 2n+2}^{0, 2n+1} = E_{2n+2, 2}^{0, 2n+1} \cong E_{2n+2, 2}^{2n+2, 0} = E_2^{2n+2, 0} = Z_p$.

By (4) and lemma 1.4, we deduce that

$$\dim H^{2n+1}(X) \leq p-1.$$

Of course this does not determine $H^{2n+1}(X)$ completely yet. We shall come back to it after determining $H^s(X)$ for $s > n+1$. Now we claim that $H^s(X) = 0$ for all $s > 2n+1$. Using the Smith sequence of 1.6 we have, for $s > 2n+1$, the exact sequences

$$\begin{aligned} 0 &= H^s(X/Z_p) \rightarrow H^s(X) \xrightarrow{\tau^*} H^s(\) \rightarrow H^{s+1}(X/Z_p) = 0 \text{ and} \\ &\dots \rightarrow H^s(\tau) \xrightarrow{i_{\tau}^*} H^s(X) \rightarrow H^s(X/Z_p) = 0. \end{aligned}$$

From these we see immediately that

$$i_{\tau^*} = \tau^* = 1 - T^*$$

is an epimorphism on $H^s(X)$ for $s > 2n+1$. Hence

$$\begin{aligned} H^s(X) &= (1-T^*) H^s(X) \\ &= (1-T^*)^2 H^s(X) \\ &= \dots \\ &= (1-T^*)^{p-1} H^s(X) \end{aligned}$$

$$= \left(\sum_{i=0}^{p-1} T^{*i} \right) H^s(X) \\ = 0 \quad \text{for all } s \geq 2n+1.$$

Finally let us return to $H^{2n+1}(X)$. Since now we have shown that $\dim H^*(X) < \infty$, the Euler-characteristic formula of Floyd (cf. 1; p. 40)

$$\sum_{s=0}^{\infty} (-1)^s \dim H^s(X) = p \sum_{s=0}^{\infty} (-1)^s \dim H^s(X/Z_p)$$

can be applied. In our case, this reduces to

$$1 - \dim H^{2n+1}(X) = 0 \quad \text{or} \\ H^{2n+1}(X) = Z_p.$$

This completes the proof of 2.5.

The next theorem is analogous to 2.5, but the proof is much simpler.

2.6. Theorem. If $p = 2$, X is connected and X/Z_2 is a cohomology real projective n -space, then X is a cohomology n -sphere mod 2.

Proof. Just as in 2.5, using the exact sequence (3) and definition 2.1, one deduces that

$$(1) \quad \pi^*: H^s(X/Z_2; Z_2) \rightarrow H^s(X; Z_2)$$

is trivial for all $s \geq 1$. Now consider the Smith sequence of 1.6. Owing to the fact that for Z_2 we have $\tau = \sigma$, the Smith sequence reduces in this case to

$$\begin{array}{ccccccc} \dots & \xrightarrow{\delta} & H^s(X/Z_2; Z_2) & \xrightarrow{\pi^*} & H^s(X; Z_2) & \xrightarrow{\tau^*} & H^s(X/Z_2, Z_2) \\ & & \xrightarrow{\delta} & H^{s-1}(X/Z_2; Z_2) & \xrightarrow{\pi^*} & \dots & \end{array}$$

From this and (1) we conclude immediately that

$$H^s(X; Z_2) = \begin{cases} Z_2 & \text{if } s = 0, n \\ 0 & \text{otherwise} \end{cases}$$

2.7. Remark. In the proof of 2.5 and 2.6, we have employed both the methods of spectral sequence and Smith special cohomology theory. It would be desirable to give a proof using a purely spectral sequence argument. This can be done for 2.6. One can first prove by induction that $H^s(X; Z_2) = 0$ for all $1 \leq s < n$ in the same way as 2.5. Next, still in the same way as 2.5, one deduces that $\dim H^n(X; Z_2) \leq p-1 = 2-1 = 1$. But this time it would determine $H^n(X; Z_2)$ without waiting for the information of $H^s(X; Z_2)$ for $s > n$. Then, one can prove inductively (by assuming that $H^{n+j}(X; Z_2) = 0$ for all $1 \leq j < i$) the existence of an exact sequence of the form

$$0 \rightarrow E_2^{1,n} \xrightarrow{\Theta_i} E_2^{n+1+1,0} \rightarrow E_2^{0,n+1} \xrightarrow{\Theta_{i+1}} E_2^{i+1,n} \xrightarrow{\Theta_{i+2}} E_2^{n+1+2,0}$$

(cf. 6.3 of Part 2). Since $E_2^{1,n} = E_2^{n+1+1,0} = Z_2$, we know Θ_i is an isomorphism. By the product structure of $H^*(Z_2; Z_2)$, one can deduce that Θ_{i+1} is also an isomorphism, which implies that $E_2^{0,n+1} = H^{n+1}(X; Z_2)^T = 0$ and hence $H^{n+1}(X; Z_2) = 0$.

For $p \neq 2$, a similar exact sequence can also be constructed. But we are not able to determine the product and Beckstein operations of $H^*(Z_p; H^{2n+1}(X))$, since our knowledge on $H^{2n+1}(X)$ comes later. This seems to be the reason for the breakdown of this method when $p \neq 2$.

Section 3. Cohomology covering spaces and lifting of actions

In the preceding section, we showed that if a free action of Z_p on a connected compact Hausdorff space X is given such that the orbit space X/Z_p is known to be a cohomology real projective space or a cohomology lense space mod p , then X is itself a cohomology sphere mod p . We wish to know now that given a cohomology real projective space or a cohomology lense space Y mod p , can we always find a free action of Z_p on a connected compact Hausdorff space X such that the orbit space is Y ? More generally, we wish to investigate the question that given a connected compact Hausdorff space Y , when could Y be homeomorphic to the orbit space of a free action of Z_p on a connected compact Hausdorff space X . We shall show in this section that under a rather mild condition such an action can actually be constructed. The method resembles very much to the construction of the universal covering space of a pathwise connected, locally pathwise connected and locally pathwise simply connected space with the dual of the first cohomology group playing the role of the fundamental group. Let us agree again that in this section all cohomology groups have Z_p as the coefficient group, but no distinction will be made for p to be even or odd.

3. 1. Suppose that Y is a connected compact Hausdorff space and that $\alpha \in H^1(Y)$ is a non-zero element. Let $f: Y^2 \rightarrow Z_p$ be a 1- cocycle representing α . Then there exists an open covering $\mathcal{U}$ of Y such that

(1) $(y_0, y_2) = f(y_0, y_1) + f(y_1, y_2)$ whenever $y_0, y_1, y_2 \in V \in \mathcal{V}$.

Fix a point $b \in Y$. By a $\mathcal{V}$ -chain we mean a finite sequence $(y_i)_{i=0}^n$ of points of Y such that $\{y_{i-1}, y_i\}$ is contained in some $V_i \in \mathcal{V}$ for all $i = 1, 2, \dots, n$. By a $\mathcal{V}$ -chain with base point b we mean a $\mathcal{V}$ -chain $(y_i)_{i=0}^n$ such that $y_0 = b$. The set of all $\mathcal{V}$ -chains with base point b is denoted by $\mathcal{X}$. Two elements $(y_i)_{i=0}^n$ and $(y'_j)_{j=0}^m$ of $\mathcal{X}$ are said to be equivalent if

$$(i) \quad y_n = y'_m \quad \text{and}$$

$$(ii) \quad \sum_{i=1}^n f(y_{i-1}, y_i) = \sum_{j=1}^m f(y'_{j-1}, y'_j).$$

This is obviously an equivalence relation. The quotient set of $\mathcal{X}$ under this equivalence relation is denoted by X and the equivalence class of $(y_i)_{i=0}^n \in \mathcal{X}$ is denoted by $[y_i]_{i=0}^n$. Also, the function $\pi: X \rightarrow Y$ given by $\pi([y_i]_{i=0}^n) = y_n$ is clearly well-defined.

Now we topologize X as follows. For each $x = [y_i]_{i=0}^n$ of X , the set

$\beta(\pi(x)) = \{B(\pi(x)) \mid \pi(x) \in B(\pi(x)), B(\pi(x)) \text{ open in } Y \text{ and } B(\pi(x)) \in V \text{ for some } V \in \mathcal{V}\}$ forms a base of neighborhoods of $\pi(x) \in Y$. To each $B(\pi(x)) \in \beta(\pi(x))$, define

$$(3) \quad B^*(x) = \left\{ [y'_j]_{j=0}^m \mid y'_m \in B(\pi(x)), \sum_{i=1}^n f(y_{i-1}, y_i) + f(y_n, y'_m) + \sum_{j=1}^m f(y'_j, y'_{j-1}) = 0 \right\}.$$

It is straightforward to verify that a topology $\mathcal{T}$ can be

defined on X for which

$$\mathcal{B}(x) = \{ B^*(x) \mid B(\pi(x)) \in \mathcal{B}(\pi(x)) \}$$

forms a base of open neighborhoods of x .

The topology $\mathcal{J}$ is Hausdorff. In fact, let $x = [y_i]_{i=0}^n$ and $x' = [y'_j]_{j=0}^m$ be two distinct points of X . If $\pi(x) \neq \pi(x')$, we may take $B(\pi(x)) \in \mathcal{B}(\pi(x))$ and $B'(\pi(x'))$ such that $B(\pi(x)) \cap B'(\pi(x')) = \emptyset$. It is then clear that $B^*(x) \cap B'^*(x') = \emptyset$. If $\pi(x) = \pi(x') = y$, we take any $B(y) \in \mathcal{B}(y)$. If $x'' = [y''_k]_{k=0}^l \in B^*(x) \cap B^*(x')$, we have by definition

$$\sum_{i=1}^n f(y_{i-1}, y_i) + f(y, y_1'') + \sum_{k=1}^l f(y''_{k-1}, y''_k) = 0 \text{ and}$$

$$\sum_{j=1}^m f(y'_{j-1}, y'_j) + f(y, y_1'') + \sum_{k=1}^l f(y''_{k-1}, y''_k) = 0.$$

This gives

$$\sum_{i=1}^n f(y_{i-1}, y_i) = \sum_{j=1}^m f(y'_{j-1}, y'_j),$$

contradicting the assumption that $x \neq x'$.

It is easily seen that under the topology $\mathcal{J}$, $\pi : X \rightarrow Y$ becomes a continuous map. In fact, for any $x \in X$ and $B(\pi(x)) \in \mathcal{B}(\pi(x))$, π maps $B^*(x)$ homeomorphically onto $B(\pi(x))$. Hence π is a local homeomorphism. Moreover, there exists an open covering $\mathcal{U}^*$ of X such that every $V^* \in \mathcal{U}^*$ is mapped by π homeomorphically onto some $V \in \mathcal{U}$.

3.2. Lemma. For each $y \in Y$, $\pi^{-1}(y)$ has exactly p points.

Proof. It is easily seen that the cardinal of $\pi^{-1}(y)$ is independent of y ; hence it suffices to consider the case when $y = b$. The function $\phi : \pi^{-1}(b) \rightarrow Z_p$ defined by $\phi([y_i]_{i=0}^n) = \sum_{i=1}^n f(y_{i-1}, y_i)$, where $[y_i]_{i=0}^n \in \pi^{-1}(b)$, is obviously injective. Moreover, the image of ϕ is a subgroup of Z_p . For $\phi([y_i]_{i=0}^n) + \phi([y'_j]_{j=0}^m) = \phi([y''_k]_{k=0}^{n+m})$, where

$$y_k'' = \begin{cases} y_n & \text{if } 0 \leq k \leq n \\ y_{n+m-k} & \text{if } n \leq k \leq n+m. \end{cases}$$

Furthermore, $\phi([b]) = f(b, b) = 0$. This subgroup must be either Z_p or 0. If it is zero, we define a 0-cochain $g: Y \rightarrow Z_p$ by $g(y) = \sum_{i=1}^n f(y_{i-1}, y_i)$, where $(y_i)_{i=0}^n$ is any $\mathcal{U}$ -chain with base point b such that $y_n = y$. Such a $\mathcal{U}$ -chain exists (since Y is connected) and g is well-defined. If $y, y' \in V \in \mathcal{U}$, we have

$$g(y') - g(y) = f(y, y'),$$

because we can represent $g(y')$ by $\sum_{i=1}^n f(y_{i-1}, y_i) + f(y, y') = g(y) + f(y, y')$. But this means f is cobounding, contradicting the assumption that $\alpha \neq 0$. Hence the lemma is proved.

Notice that 3.2 also implies that X is compact.

3.3. Lemma. The homomorphism $\pi^*: H^1(Y) \rightarrow H^1(X)$ induced by $\pi: X \rightarrow Y$ maps α into zero.

Proof. The function $h: X \rightarrow Z_p$ given by $h([y_i]_{i=0}^n) = \sum_{i=1}^n f(y_{i-1}, y_i)$ is clearly a well-defined 0-cochain on X . If $x = [y_i]_{i=0}^n$ and $x' = [y_j]_{j=0}^m$ are contained in some $B^*(x'')$, where $x'' = [y_k]_{k=0}^l$.

According to (3) we have

$$\begin{aligned} \sum_{k=1}^l f(y_{k-1}'', y_k'') + f(y_1'', y_n) + \sum_{i=1}^n f(y_i, y_{i-1}) &= 0 \text{ and} \\ \sum_{k=1}^l f(y_{k-1}'', y_k'') + f(y_1'', y_m) + \sum_{j=1}^m f(y_j', y_{j-1}') &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^n f(y_i, y_{i-1}) - \sum_{j=1}^m f(y_j', y_{j-1}') &= f(y_1'', y_m) - f(y_1'', y_n) \\ &= f(y_1'', y_m) + f(y_n, y_1'') \\ &= f(y_n, y_m), \end{aligned}$$

or

$$h(x) - h(x') = f(\pi(x), \pi(x')) .$$

This means precisely that $\pi^*(\alpha) = 0$; hence the lemma is proved.

By a $\mathcal{V}^*$ -chain we mean a finite sequence $(x_i)_{i=0}^n$ of points of X such that $\{x_{i-1}, x_i\}$ is contained in some $V_i^* \in \mathcal{V}^*$ for all $i=1, 2, \dots, n$. A $\mathcal{V}^*$ -chain $(x_i)_{i=0}^n$ is said to cover a $\mathcal{V}$ -chain $(y_i)_{i=0}^n$ if $\pi(x_i) = y_i$ for all $i=0, 1, \dots, n$. Notice that the function h defined in 3.3 has the following property:

- (4) If a $\mathcal{V}^*$ -chain $(x_i)_{i=0}^n$ covers a $\mathcal{V}$ -chain $(y_i)_{i=0}^n$, then
- $$\sum_{i=1}^n f(y_{i-1}, y_i) = h(x_0) - h(x_n).$$

3.4. Lemma. (Chain lifting property and monodromy property) Given a $\mathcal{V}$ -chain $(y_i)_{i=0}^n$ and a point $x \in \pi^{-1}(y_0)$, there exists a unique $\mathcal{V}^*$ -chain $(x_i)_{i=0}^n$ such that $(x_i)_{i=0}^n$ covers $(y_i)_{i=0}^n$ and $x_0 = x$. If $(y_i)_{i=0}^n$ and $(y_j)_{j=0}^m$ are two $\mathcal{V}$ -chains such that $y_0 = y'_0$ and $y_n = y'_m$, $(x_i)_{i=0}^n$ and $(x'_j)_{j=0}^m$ are two $\mathcal{V}^*$ -chains covering $(y_i)_{i=0}^n$ and $(y_j)_{j=0}^m$ respectively such that $x_0 = x'_0$. Then $x_n = x'_m$ if and only if

$$\sum_{i=1}^n f(y_{i-1}, y_i) = \sum_{j=1}^m f(y'_{j-1}, y'_j).$$

Proof. The first part of the lemma is an immediate consequence of the fact that π is a local homeomorphism of X onto Y . For the second part, if $x_n = x'_m$, we have

$$\sum_{i=1}^n f(y_{i-1}, y_i) = h(x_0) - h(x_n) = h(x'_0) - h(x'_m) = \sum_{j=1}^m f(y'_{j-1}, y'_j).$$

Conversely, if $\sum_{i=1}^n f(y_{i-1}, y_i) = \sum_{j=1}^m f(y'_{j-1}, y'_j)$, then by (4) we have $h(x_0) - h(x_n) = h(x'_0) - h(x'_m)$ and hence $h(x_n) = h(x'_m)$ since $x_0 = x'_0$. Since $\pi(x_n) = y_n = y'_m = \pi(x'_m)$, we have

$x_n = x'_n$ by the definition of h and (2).

3.5. Lemma. There exists a free action of Z_p on X such that $X/Z_p = Y$ and π coincides with the canonical projection.

Proof. Define a function S on $\pi^{-1}(b)$ by $S(x) = \phi^{-1}(\phi(x)+1)$, where $\phi: \pi^{-1}(b) \rightarrow Z_p$ is the function defined in 3.2. Let us extend S as follows. If $x = \{y_i\}_{i=0}^n$ is an arbitrary point of X , we let $(x_i)_{i=0}^n$ be a $\mathcal{U}^*$ -chain covering $(y_i)_{i=0}^n$ such that $x_0 = s([b])$ and then define $S(x) = x_{n+1}$. This is well-defined in view of lemma 3.4. Now let us prove that S is continuous. Take a neighborhood $B^*(x_n)$ of $x_n = S(x)$, where $B(y_n)$ is a neighborhood of $\pi(x_n) = \pi(x) = y_n$. If $x' \in B^*(x)$, then x' can be represented by $\{y_i\}_{i=0}^{n+1}$, where $y_{n+1} = \pi(x') \in B(y_n)$. Since π maps $B^*(x_n)$ homeomorphically onto $B(y_n)$, there exists $x_{n+1} \in B^*(x_n)$ such that $\pi(x_{n+1}) = y_{n+1}$. It follows that $(x_i)_{i=0}^{n+1}$ is a $\mathcal{U}^*$ -chain which covers $(y_i)_{i=0}^{n+1}$ with $x_0 = S([b])$. By definition we have then $S(x') = x_{n+1} \in B^*(x_n)$. It is clear from the definition of S that we have $h(S(x)) - h(x) = h(S[b])$ so that $h(x) - h(S^p(x)) = p h(S[b]) = 0$. Since $\pi(x) = \pi(S^p(x))$, we must have $S^p(x) = x$. Hence S is a periodic map on X with period p . Finally, S has no fixed point. In fact, for any $x \in X$ we have $h(x) - h(S(x)) \neq -h(S[b]) = 1 \neq 0$; hence $x \neq S(x)$ for all $x \in X$.

3.6. Lemma. X is connected.

Proof. Consider the exact sequence

$$0 \rightarrow E_2^{1,0} \rightarrow H^1(Y) \xrightarrow{\pi^*} E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow H^2(Y)$$

of the Leray-Cartan spectral sequence (of. (3) of 2.5).

By 1.5 and lemma 3.3, we have $\dim (\text{Ker } \pi^*) \geq 1$. Therefore by 1.1 and the exactness of the above sequence we have

$$\dim_{\sigma} H^0(X) / \tau H^0(X) = \dim E_2^{1,0} = \dim (\text{Ker } \pi^*) \geq 1.$$

On the other hand, we have obviously $E_2^{0,0} = E_{\infty}^{0,0} = H^0(Y)$.

By 1.1, 1.5 and the connectedness of Y , this gives

$$(5) \quad \dim H^*(X) S^* = \dim E_2^{0,0} = 1.$$

But (5) means that $\tau H^0(X)$ is a hyperplane of $H^0(X)$ so that

$$\dim H^0(X) / \tau H^0(X) = 1.$$

It follows that

$$\begin{aligned} \dim H^0(X) / \sigma H^0(X) &= \dim H^0(X) / \tau H^0(X) - \dim \\ &\sigma H^0(X) / \tau H^0(X) \leq 0 \text{ or } \sigma H^0(X) = H^0(X). \end{aligned}$$

That is to say, $\sum_{i=0}^{p-1} S^{*1}_i$ is trivial on $H^0(X)$. By (5) and lemma 1.4, we have

$$(6) \quad \dim H^0(X) \leq p-1.$$

Hence X has at most $p-1$ components.

Suppose that X is not connected. Let C be a component of X . We can not have $S(C) = C$. For if so, $\pi(C)$ would be a proper non-empty subset of Y . As X has only a finite number of components, C is both open and closed in X and so is $\pi(C)$ in Y since π is an open map and X is compact. This contradicts the fact that Y is connected. Therefore we must have $S(C) \neq C$ and hence X has at least p components, contradicting (6).

3.7. Remark. We notice that in the proof of 3.6, we have also shown that $\dim (\text{Ker } \pi^*) = 1$. In other words, $\text{Ker } \pi^*$ is precisely the 1-dimensional subspace of $H^1(Y)$ generated by α .

Collecting 3.1 - 3.7, we have proved the following theorem.

3.8. Theorem. If Y is a connected compact Hausdorff space, then for every non-zero element $\alpha \in H^1(Y)$ there exists a connected compact Hausdorff space X and a free action of Z_p on X such that $X/Z_p = Y$ and the kernel of the homomorphism $\pi^*: H^1(Y) \rightarrow H^1(X)$ induced by the canonical projection $\pi: X \rightarrow Y$ is precisely the 1-dimensional subspace generated by α .

The space X of 3.8 may be called a cohomology covering space of Y . This space can actually be characterized abstractly. The techniques involved consist largely of repetitions of the arguments used in 3.1 through 3.6. So we are content to give here a brief sketch.

3.9. Definition. Let Y be a connected compact Hausdorff space and α a non-zero element of $H^1(Y)$. By a cohomology covering space of Y with respect to α we mean a compact Hausdorff space X and a free action of Z_p on X such that $X/Z_p = Y$ and the kernel of the homomorphism $\pi^*: H^1(Y) \rightarrow H^1(X)$ induced by the canonical projection $\pi: X \rightarrow Y$ is precisely the 1-dimensional subspace generated by α .

3.10. Let Y, α be the same as in 3.9 and X a cohomology covering space of Y with respect to α . Just as 3.6, we can prove first that X is connected. Using the condition that $\pi^*(\alpha) = 0$ and the fact that π^* is a local homeomorphism of X onto Y , we can find an open covering $\mathcal{V}$ of Y , a 1-cocycle f on Y , an open covering $\mathcal{U}^*$ of X and a 0-cochain h on X such

that (1) and (4) holds and such that each $V \in \mathcal{V}^*$ is mapped homeomorphically onto some $V' \in \mathcal{V}'$ by θ . With these we can use the notions of $\mathcal{V}$ -chains, $\mathcal{V}'$ -chains and covering chains. Because of the condition that $\alpha \neq 0$ and the connectedness of X , we can prove the chain lifting property and the monodromy property of 3.4. After this is established, the following uniqueness theorem can be proved by standard argument.

3. 11. Theorem. Let Y be a connected compact Hausdorff space and α a non-zero element of $H^1(Y)$. If X and X' are two cohomology covering spaces of Y with respect to α , then there exists an equivariant homeomorphism $\theta: X \rightarrow X'$ of X onto X' . More explicitly, we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta} & X' \\ \pi \searrow & & \swarrow \pi' \\ & Y & \end{array},$$

Where π and π' are the canonical projections. Furthermore, if x_0 and x'_0 are any two preassigned points of X and X' such that $\pi(x_0) = \pi'(x'_0)$, then θ can be chosen in such a way that $\theta(x_0) = x'_0$, and it is completely determined by this condition.

3. 12. Again let Y be a connected compact Hausdorff space, α a non-zero element of $H^1(Y)$ and X a cohomology covering space of Y with respect to α . By definition there exists a free action of Z_p on X which we represent by a periodic map S (see 3.5) and call it the "deck-transformation".

Let us assume now that the space Y itself carries an action of Z_p ; in other words, we have a periodic transformation $T: Y \rightarrow Y$ of period p . A periodic map $\tilde{T}: X \rightarrow X$ of period p is said to be a lifting of T if it commutes with the canonical projection $\pi: X \rightarrow Y$, i.e. if $\pi \circ \tilde{T} = T \circ \pi$. We wish to study now when such a lifting exists. To do this, let us adjust the construction of 3.1 under the following assumptions:

1. There is an action of Z_p of Y , i.e. we have a periodic map $T: Y \rightarrow Y$ of period p .

2. α is invariant under T^* , i.e. $T^*(\alpha) = \alpha$, where $T^*: H^1(Y) \rightarrow H^1(Y)$ is the homomorphism induced by T .

3. The fixed point set of T is non-empty, i.e. there exists $b \in Y$ such that $T(b) = b$.

Just as in 3.1, let $f: Y^2 \rightarrow Z_p$ be a 1-cocycle representing α . Using condition 2, it is easily seen that there exists an open covering $\mathcal{V}$ of Y and a p -cochain $k: Y \rightarrow Z_p$ with the properties that (1) of 3.1 holds, that $T(V) \in \mathcal{V}$ for all $V \in \mathcal{V}$ and that

$$(1) \quad k(y) - k(y') = f(y, y') - f(T(y), T(y')) \text{ whenever } y, y' \in V \in \mathcal{V}.$$

Construct the set X and the space X in the same way as 3.1, but with the agreement that the base point b is taken to be a fixed point of T . Then X is a cohomology covering space of Y with respect to α . Now clearly the function $\tilde{T}: (y_i)_{i=0}^n \rightarrow (T(y_i))_{i=0}^n$ maps X into itself. Moreover, if $(y_i)_{i=0}^n = \left[y_j^* \right]_{j=0}^m$, then, by (1), we have

$$\begin{aligned}
\sum_{i=1}^n f(T(y_{i-1}), T(y_i)) &= \sum_{i=1}^n f(y_{i-1}, y_i) - (k(b) - k(y_n)) \\
&= \sum_{j=1}^m f(y'_{j-1}, y'_j) - (k(b) - k(y'_m)) \\
&= \sum_{j=1}^m f(T(y_{j-1}'), T(y'_j))
\end{aligned}$$

Hence $\tilde{T}([y_i]_{i=0}^n) = [T(y_i)]_{i=0}^n$ is a well-defined function of X into itself. It is easily seen that $\tilde{T} : X \rightarrow X$ is a periodic transformation of period p and that it is indeed a lifting of T .

We claim that $\tilde{T}$ commutes with S , where S is the deck-transformation defined in 3.5. Let $S([b]) = [y_i]_{i=0}^n$. Then $y_n = \pi(S[b]) = b$. By (1), we have

$\sum_{i=1}^n f(y_{i-1}, y_i) - \sum_{i=1}^n f(T(y_{i-1}), T(y_i)) = k(b) - k(b) = 0$; in other words, $\tilde{T}(S[b]) = S([b])$. Now if $x = [y'_j]_{j=0}^m$ is an arbitrary point of X and $(x'_j)_{j=0}^m$ is the $\mathcal{V}^*$ -chain covering $(y_j)_{j=0}^m$ such that $x'_0 = s([b])$, then $S(x) = x'_m$. But $(\tilde{T}(x'_j))_{j=0}^m$ is a $\mathcal{V}^*$ -chain covering $(T(y'_j))_{j=0}^m$ such that $\tilde{T}(x'_0) = \tilde{T}(S([b])) = S([b])$.

By the definition of S we have then $\tilde{T}(x'_m) = S([T(y'_j)]_{j=0}^m)$, that is $\tilde{T} \circ S(x) = S \circ \tilde{T}(x)$.

3.13. Proposition. Suppose that Y is a connected compact space, that T is a periodic transformation on Y of period p such that the fixed point set of T is non-empty, that α is a non-zero element of $H^1(Y)$ which is invariant under T^* and that X is a cohomology covering space of Y with respect to α . Then T has a lifting $\tilde{T}$ on X which commutes with the deck-transformation S on X .

Proof. 3.13 is just proved in 3.12 when X is the specific cohomology covering space constructed there. The case when X is an arbitrary cohomology covering space of Y with respect to α then follows from the uniqueness theorem of 3.11.

Section 4. Fixed point set of an action of Z_p on a cohomology real projective space or on a cohomology lense space mod p

In this section we discuss the fixed point set of an action of Z_p on a cohomology real projective space or on a cohomology lense space mod p . Our object is to show that each component of the fixed point set of such an action inherits the same cohomology structure of the space itself. All the machineries needed have already been built up in the previous two sections and it is now just a matter of patching them up.

4.1. Theorem. Suppose that Y is a cohomology lense $(2n+1)$ -space mod p on which Z_p acts, where $p \neq 2$. Then the fixed point set F has at most p components and every component of F is a cohomology lense space mod p . If F has K components $C_1, \dots, C_k$, $1 \leq k \leq p$, and C_1 is a cohomology lense $(2n_1+1)$ -space mod p , $i = 1, 2, \dots, k$, then

$$\sum_{i=1}^k n_i = n-k+1.$$

Proof. We may assume that F is non-empty, for otherwise 4.1 is trivial. Let T be a generator of Z_p and α a generator of $H^1(Y) = Z_p$. Let X be a cohomology covering space of Y with respect to α . Such an X exists by 3.8 and according to 2.5 X is a cohomology $(2n+1)$ -sphere mod p . Since α is necessarily invariant under T^* , by 3.13 T has a lifting $\tilde{T}$ on X which commutes with the deck-transformation S on X (cf. 3.12). It follows that $\tilde{T}$ and S together define an action of $Z_p \times Z_p$ on X . Let $\mathcal{N}$ be the set of all non-trivial

cyclic subgroups N of $Z_p \times Z_p$ such that the fixed point set $F(N)$ is non-empty. Since $Z_p \times Z_p$ contains only $p+1$ non-trivial cyclic subgroups and since the cyclic subgroup generated by $(S, 1)$, which we denote by N^0 , acts freely on X , $N^0 \notin \mathcal{N}$.

Hence $\mathcal{N}$ has at most p elements. On the other hand, we have $\bigcup_{N \in \mathcal{N}} (F(N)) = F$; hence $\mathcal{N}$ is non-empty. Let $N^1, \dots, N^k$, $1 \leq k \leq p$, be the elements of $\mathcal{N}$. By 1.7, $F(N^i)$ is a cohomology r_i -sphere mod p and r_i is odd, say $r_i = 2n_i + 1$, $i = 1, 2, \dots, k$. Let $C_i = \pi_0(F(N^i))$, $i = 1, 2, \dots, k$; then C_i is connected since $F(N^i)$ is. Moreover, it is easily seen that $C_i \cap C_j = \emptyset$ if $i \neq j$. Since $F = \bigcup_{N \in \mathcal{N}} F(N) = \bigcup_{i=1}^k C_i$, we conclude that $\{C_i\}_{i=1}^k$ is precisely the set of all components of F . Now the cyclic group N^0 acts on $F(N^i)$, freely of course, and C_i is just the orbit space of this action. Therefore, by 2.3, C_i is a cohomology lens $(2n_i + 1)$ -space mod p . Finally, by 1.9, we have

$$\sum_{i=1}^k [(2n_i + 1) - (-1)] = (2n + 1) - (-1),$$

which reduces exactly to

$$\sum_{i=1}^k n_i = n - k + 1.$$

In exactly the same way, one can also prove

4.2. Theorem. Suppose that Y is a cohomology real projective n -space on which Z_2 acts. Then the fixed point set F has at most 2 components and each component of F is a cohomology real projective space. If F has k components $C_1, \dots, C_k$, $1 \leq k \leq 2$, and C_i is a cohomology real projective n_i -space, $i = 1, 2, \dots, k$, then

$$\sum_{i=1}^k n_i = n - k + 1.$$

PART 2

Section 5. Preliminaries of Part 2

As in section 1, the present section is a collection of preliminary results to be used in the following sections.

5.1 Let G be a compact Lie group acting on a compact Hausdorff space X . For each $x \in X$, $G_x = \{g \in G \mid gx = x\}$ is a closed subgroup of G called the isotropic subgroup at x and the set $G(x) = \{gx \mid g \in G\} \subset X$ is called an orbit. The conjugate class $[G_x]$ of G_x in G depends only on the orbit $G(x)$; hence it is called an orbit type. We say that the action has finite orbit structure if the set $\{[G_x] \mid x \in X\}$ is finite. In case that G is abelian (such as the circle group S^1), it is the same to say that there are only a finite number of distinct isotropic subgroups, [1; p. 104].

5.2 Let X be a locally compact space and L a principal ideal domain. The cohomology dimension of X over L , denoted by $\dim_L X$ is defined by

$$\dim_L X \leq n \text{ if } H_C^{n+1}(U; L) = 0 \text{ for all open subset } U \subset X,$$

where $H_C^*(U; L) = \sum_{i=0}^{\infty} H_C^i(U; L)$ is the Alexander - Wallace - Spanier cohomology of U with coefficients in L and with compact support. X is said to have a finite cohomology dimension over L if $\dim_L X \leq n$ for some n . In symbols, we write $\dim_L X < \infty$.

The function $\dim_L X$ is introduced by H. Cohen [7]. It is known to have the following properties:

- (i) $\dim_L X \leq \dim_Z S$, where Z denotes the ring of integers,
- (ii) $\dim_L X = \max(\dim_L A, \dim_L X - A)$ for any closed subset $A \subset X$.

(iii) $\dim_L(X \times Y) \leq \dim_L X + \dim_L Y$, where Y is a locally compact space and $X \times Y$ is the product space of X and Y .

(iv) If X is a compact Hausdorff space of finite Lebesgue covering dimension n , then $\dim_L X = n$.

Let (X, Y, S^1, π) be a principal bundle in the sense of [18], where the fibre S^1 is the circle group, the base space Y is compact Hausdorff and $\dim_L Y < \infty$. By the properties (i) - (iv) listed above, it is easily seen that $\dim_L X < \infty$.

5.3. Proposition. Let G be a compact Lie group acting on a compact Hausdorff space X and X/G the orbit space. Then

$$\dim_L X/G \leq \dim_L X.$$

A proof of 5.3 can be found in [1; p. 111].

5.4. Let S^1 act on a connected compact Hausdorff space X . Consider the $(2N+1)$ - sphere

$$S^{2N+1} = \{(z_0, z_1, \dots, z_N) \mid \sum_{i=0}^N |z_i|^2 = 1, z_i = \text{complex number for all } i = 0, 1, \dots, N\}.$$

The map

$$S^1 \times S^{2N+1} \longrightarrow S^{2N+1}$$

$$(e^{2\pi\theta i}, (z_0, \dots, z_N)) \longrightarrow (e^{2\pi\theta i} z_0, \dots, e^{2\pi\theta i} z_N),$$

$0 \leq \theta \leq 1$, defines a free action of S^1 on S^{2N+1} for which the orbit space is the complex projective N -space CP^N . Let $P: S^{2N+1} \rightarrow CP^N$ denote the canonical projection. Now let S^1 act on the product space $X \times S^{2N+1}$ diagonally by $g(x, u) = (gx, gu)$, $g \in S^1$, $x \in X$ and $u \in S^{2N+1}$. Denote the resulting orbit space by X_S , and the canonical projection of $X \times S^{2N+1}$ into X_S , by q . There are two canonical maps $\pi_1: X_S \rightarrow X/S^1$ and $\pi_2: X_S \rightarrow CP^N$ such that the following diagram

$$\begin{array}{ccccccc}
 & & X_{S'} & & & & \\
 & \swarrow \pi_1 & & \searrow \pi_2 & & & \\
 X/S' & \xleftarrow{\pi} & X & \xleftarrow{\text{Pr}_1} & X \times S^{2N+1} & \xrightarrow{\text{Pr}_2} & S^{2N+1} \xrightarrow{p} \mathbb{CP}^N
 \end{array}$$

$\uparrow q$

is commutative, where Pr_1 and Pr_2 are the projections of the product space $X \times S^{2N+1}$ onto its first and second factor spaces X and S^{2N+1} respectively. The map π_2 is a fibring with fibre X for which the Leray sheaf is constant since $\mathbb{CP}^N$ is simply connected. There is then a spectral sequence (E_r) whose E_2 - term is given by

$$E_2^{s,t} = H^s(\mathbb{CP}^N; H^t(X; L))$$

and whose E_∞ - term is associated with $H^*(X_{S'}; L)$. The map π_1 is in general not a fibring. In fact, for each $y = \pi(x) \in X/S'$, we have $\pi_1^{-1}(y) = S^{2N+1}/S'_x$. Finally we remark that the cohomology ring of $\mathbb{CP}^N$ is given by

$$(1) \quad H^*(\mathbb{CP}^N; L) = L[x]/(x^{N+1}), \quad \deg x = 2,$$

where $L[x]$ is the polynomial ring with coefficients in L and (x^{N+1}) is the ideal generated by x^{N+1} ,

For details of the above, one may consult [1; Chap. iv, § 1, 2 and 3] where the general case of compact Lie groups is treated and with [10] and [8] where the specific case of S' is discussed.

In later applications, we shall always assume that X has finite cohomology dimension over L and that N is chosen so that $2N+1 \gg \dim_L X$. This convention will be adopted from now on without further explanation (cf. 1; p. 52),

Suppose S' acts on X freely. Then $\pi_1^{-1}(y) = S^{2N+1}$ for all $y \in X/S'$. Since $H^k(S^{2N+1}; L) = 0$ for all $1 \leq k < 2N+1$, by the Vietoris mapping theorem we have $\pi_1^*: H^k(X/S'; L) \rightarrow H^k(X_{S'}; L)$ is an isomorphism for all $0 \leq k < 2N+1$. We have therefore

5.5 Proposition. If S' acts freely on a connected compact Hausdorff space X such that $\dim X < \infty$, then for a sufficiently large N there exists a spectral sequence (E_r) whose E_2 - term is given by

$$E_2^{s,t} = H^s(CP^N; H^t(X; L))$$

and whose E_∞ - term is associated with $H^*(X/S'; L)$ up to $\dim X$.

More precisely, the last statement of 5.5 means that there exists a suitable filtration on $\sum_{i=0}^{2N} H^i(X/S'; L)$ whose associated graded group is given by $\sum_{s+t \leq 2N} E_\infty^{s,t}$. We notice furthermore that for $s < 2N+1$, the canonical edge homomorphisms

$$\phi_s: E_2^{s,0} \rightarrow H^s(X/S'; L) \text{ and}$$

$$\psi_s: H^s(X/S'; L) \rightarrow E_2^{0,s}$$

of (E_r) are given by π_1^* or π_2^* and π^* respectively.

In the next section, we shall apply 5.5 to the case where X is a cohomology k -sphere over L . There is then the Gysin exact sequence

$$\dots \xrightarrow{\gamma_a} E_2^{s,0} \longrightarrow H^s(X/S'; L) \longrightarrow E_2^{s-k,k} \xrightarrow{\gamma_a} E_2^{s+1,0} \longrightarrow \dots$$

$S < 2N+1.$

We remark that γ_a is the multiplication by an element $a \in H^{k+1}(\mathbb{CP}^N; L)$ [2; Exposé IX, théorème 8].

5.6 Let S' act on X with the action being not necessarily free and let F be the fixed point set. Considering F as a space acted on by S' , we can form $F_{S'}$, and consider the maps $\pi_1: F_{S'} \longrightarrow F/S' = F$ and $\pi_2: F_{S'} \longrightarrow \mathbb{CP}^N$. It is easily seen that the space $F_{S'}$ is just the product space $F \times \mathbb{CP}^N$, since π_1 and π_2 are just the projections of $F \times \mathbb{CP}^N$ onto its factor spaces, the spectral sequence of $\pi_2: F \times \mathbb{CP}^N \longrightarrow \mathbb{CP}^N$ is of no interest. On the other hand, the inclusion $i: F \longrightarrow X$ induces an inclusion $i_{S'}: F_{S'} \longrightarrow X_{S'}$, which in turn induces a homomorphism $i_{S'}^*: H^*(X_{S'}; L) \longrightarrow H^*(F_{S'}; L)$. It is known [1; p. 55] that if $\dim_L X < \infty$, then $i_{S'}^*: H^k(X_{S'}; L) \longrightarrow H^k(F_{S'}; L)$ is an isomorphism for all $\dim_L X < K \leq 2N$, provided that L is a field of characteristic zero.

5.7 Proposition. Let X be a cohomology n -sphere over L , where L is either $\mathbb{Z}$ or a field of characteristic zero. Let S' act on X . Assume moreover that $\dim_L X < \infty$ and that the orbit structure is finite. Then the fixed point set $F(S'; X)$ is a cohomology r -sphere over L for some $-1 \leq r \leq n$ and $n-r$ is even.

This result can be found in [1; p. 63].

5.8 Let X be a cohomology n -sphere over L , where L is a field of characteristic zero and let $S' \times S'$ act on X . Assume moreover that $\dim_L X < \infty$ and that the orbit structure is finite. By 5.7, the fixed point set $F(S' \times S'; X)$ is a cohomology r -sphere over L . Moreover, let $\mathcal{H} = \{H \mid H \subset S' \times S'\}$ be the set of all closed subgroups of $S' \times S'$ which are isomorphic to S' . Then each $F(H, X)$, $H \in \mathcal{H}$ is a cohomology $n(H) - r$ - sphere over L . We have the following theorem due to A. Borel [1; p. 175].

5.9. Proposition. Let the hypothesis and the notations be the same as in 5.8. Then

$$n - r = \sum_{H \in \mathcal{H}} (n(H) - r).$$

Section 6. Cohomology Complex Projective Spaces

Throughout this section, X is a compact Hausdorff space on which the circle group S^1 acts freely. As usual, X/S^1 denotes the orbit space and $\pi: X \rightarrow X/S^1$ denotes the canonical projection. We also agree that for the rest of this paper, $H^*(X)$ denotes the integral Alexander-Wallace-Spanier Cohomology of X . Whenever groups (rings or fields) other than $\mathbb{Z}$ are used as coefficient groups, they will be written out explicitly.

6.1 Definition. A compact Hausdorff space Y is said to be a cohomology complex projective n -space if its integral cohomology ring $H^*(Y)$ is given by

$$H^*(Y) = \mathbb{Z}[x]/(x^{n+1}), \quad \deg x = 2,$$

where $\mathbb{Z}[x]$ is the polynomial ring with coefficients in $\mathbb{Z}$ and (x^{n+1}) is the ideal generated by x^{n+1} .

6.2 Proposition. If X is an integral cohomology $(2n+1)$ -sphere such that $\dim_{\mathbb{Z}} X < \infty$, then X/S^1 is a cohomology complex projective n -space,

Proof. Consider the spectral sequence (E_r) of 5.5, where we have

$$E_2^{s,t} = H^s(\mathbb{C}P^N; H^t(X)).$$

Since X is an integral cohomology $(2n+1)$ -sphere, by 5.5 we have the Gysin sequence

$$\begin{aligned} \dots \rightarrow E_2^{s-2n-2, 2n+1} &\xrightarrow{\gamma_a} E_2^{s, 0} \xrightarrow{\phi_s} H^s(X/S^1) \rightarrow E_2^{s-2n-1, 2n+1} \\ &\xrightarrow{\gamma_a} \dots, \end{aligned}$$

where γa is the multiplication by a generator $a \in H^{2n+2}(CP^N)$. From this exact sequence, it is immediate that

$$\phi_s: E_2^{s,0} \longrightarrow H^s(X/S')$$

is an isomorphism for all $0 \leq s \leq 2n$. For $2n < s < 2N+1$, we obtain

$$H^s(X/S') = 0$$

by the fact that γa is an isomorphism, For $2N+1 \leq s$, we obtain

$$H^s(X/S') = 0$$

by the agreement that $2N+1 > \dim_{\mathbb{C}} X$ and 5.3. Finally, as $\phi_s = \pi_1^{*-1} \circ \pi_2^*$ commutes with cup-product, we obtain the desired cohomology structure of $H^*(X/S')$ in view of (1) of 5.4.

The next theorem is an analogue of 2.5.

6.3. Theorem. Suppose that S' acts freely on a compact Hausdorff space X such that X/S' is a cohomology complex projective n -space. Assume moreover that $\dim_{\mathbb{C}} X/S' < \infty$ and $\pi^*: H^2(X/S') \longrightarrow H^2(X)$ is trivial. Then X is an integral cohomology- $(2n+1)$ sphere and $\dim_{\mathbb{C}} X < \infty$.

Proof. By a theorem of Gleason, [11], the system $(X, X/S', S', \pi)$ forms a principal bundle; hence we have $\dim_{\mathbb{C}} X < \infty$ by 5.3. We can therefore consider the spectral sequence (E_r) of 5.5, where we have

$$E_2^{s,t} = H^s(CP^N; H^t(X))$$

for a sufficiently large N .

By the universal coefficient theorem, it is easily seen that for any coefficient group L ,

$$H^S(\mathbb{CP}^N, L) = \begin{cases} L, & \text{if } S=2k, 0 \leq k \leq N; \\ 0, & \text{otherwise.} \end{cases}$$

$$(1) \quad E_2^{s,t} = 0 \text{ for all odd } S.$$

We recall that E_∞ is the graded ring associated with a suitable filtration of $H^*(X/S')$ and we suppose that this filtration is given by a decreasing sequence $\{H^*(X/S')_s\}_{s=0}^\infty$ of subgroups $H^*(X/S')_s$ of $H^*(X/S')$; i.e. $H^*(X/S')_{s+1} \subset H^*(X/S')_s$ for all s , $H^*(X/S')_0 = H^*(X/S')$ and $H^*(X/S') = \bigcup_{s=0}^\infty H^*(X/S')_s$. Then we have

$$(2) \quad E_\infty^{s,t} = H^{s+t}(X/S')_s / H^{s+t}(X/S')_{s+1},$$

where $H^k(X/S')_s = H^k(X/S') \cap H^*(X/S')_s$. Moreover, in the sequence

$$E_2^{s,0} \xrightarrow{\phi_s} H^s(X/S') \xrightarrow{\psi_s} E_2^{0,s}$$

where ϕ_s and ψ_s are the edge homomorphisms, we have

$$(3) \quad \begin{aligned} \text{Im } \phi_s &= H^s(X/S')_s, \\ \text{Ker } \psi_s &= H^s(X/S')_1. \end{aligned}$$

We now claim that

$$(4) \quad \phi_s: E_2^{s,0} \longrightarrow H^s(X/S')$$

is an isomorphism for all $S = 2K$, $0 \leq k \leq n$. Clearly

$\phi_0: E_2^{0,0} \rightarrow H^0(X/S')$ is an isomorphism; hence (4) is trivial if $n=0$. Consider the exact sequence of lower terms (cf. (3) of 2.5)

$$0 \rightarrow E_2^{1,0} \xrightarrow{\phi_1} H^1(X/S') \xrightarrow{\psi_1} E_2^{0,1} \xrightarrow{d_1} E_2^{2,0} \xrightarrow{\phi_2} H^2(X/S').$$

Since $H^1(X/S') = 0$, this reduces to

$$0 \rightarrow E_2^{0,1} \xrightarrow{d_1} E_2^{2,0} \xrightarrow{\phi_2} H^2(X/S').$$

By (3), we have $\text{Im } \phi_2 = H^2(X/S')_2$ and $\text{Ker } \psi_2 = H^2(X/S')_1$. But $H^2(X/S')_1/H^2(X/S')_2 = E_\infty^{1,1} = 0$ because $E_2^{1,1} = 0$ by (1). Hence $\text{Im } \phi_2 = \text{Ker } \psi_2$. In other words, the following sequence

(5)

$$0 \rightarrow E_2^{0,1} \xrightarrow{d_1} E_2^{2,0} \xrightarrow{\phi_2} H^2(X/S') \xrightarrow{\psi_2} E_2^{0,2}$$

is exact. For $n > 0$, we have $H^2(X/S') = \mathbb{Z}$ by 6.1. By 5.5, the homomorphism ψ_2 is just $\pi^*: H^2(X/S') \rightarrow H^2(X)$ which is trivial by assumption. Hence (5) reduces to

$$0 \rightarrow E_2^{0,1} \rightarrow \mathbb{Z} \xrightarrow{\phi_2} \mathbb{Z} \rightarrow 0$$

and this implies that $\phi_2: E_2^{2,0} \rightarrow H^2(X/S')$ is an isomorphism. Since

(6)

$$E_2^{*,0} = \sum_{s=0}^{\infty} E_2^{s,0} = \sum_{s=0}^{\infty} H^s(\mathbb{CP}^N; H^0(X)) =$$

$$\sum_{s=0}^{\infty} H^s(\mathbb{CP}^N) = H^*(\mathbb{CP}^N)$$

and $\phi = \pi_1^{*-1} \circ \pi_2^*$ (see 5.5) commutes with cup-product, (4) follows from the ring structures of $H^*(\mathbb{CP}^N)$ and $H^*(X/S')$.

Relation (4) implies that none of the terms $E_r^{s,0}$, $s=2k$, $0 \leq k \leq n$ have any non-zero cobounding element for $r \geq 2$. In particular, we have

$$(7) \quad d_s: E_s^{0,s-1} \longrightarrow E_s^{s,0}$$

is trivial for all $s=2K$, $1 \leq K \leq n$. Moreover, we have $E_{\infty}^{0,s} = H^s(X/S')/H^s(X/S')_1$ by (2) and $\text{Im } \phi_s = H^s(X/S')_1$ by (3). Hence (4) implies that $H^s(X/S')_s = H^s(X/S')_1 = H^s(X/S)$ for $s=2K$, $1 \leq K \leq n$. In other words, $E_{\infty}^{0,s} = 0$ for all $s = 2K$, $1 \leq K \leq n$. Clearly we have also that $E_{\infty}^{0,s} = 0$ for all odd s simply because $H^s(X/S') = 0$ when s is odd. Thus we have

$$(8) \quad E_{\infty}^{0,s} = 0 \text{ for all } 1 \leq s \leq 2n.$$

Now we prove by induction that

$$(9) \quad H^s(X) = 0 \quad \text{for all } 0 < s < 2n+1.$$

If $n=0$, (9) is trivially true. So we assume that $n > 0$. By (4), ϕ_2 is an isomorphism; hence $E_2^{0,1} = 0$ from the exactness of (5); i.e.

$$H^1(X) = E_2^{0,1} = 0.$$

Suppose that it has been proved that $H^1(X) = 0$ for all $1 \leq i < s$, $s \leq 2n$. Consider the differentials

$$d_r: E_r^{0,s} \longrightarrow E_r^{r,s-r+1}.$$

Clearly $d_r = 0$ if $r > s+1$. If $1 < r < s+1$, we have $1 \leq s-r+1 < s$; hence $E_2^{r,s-r+1} = H^r(\mathbb{C}P^N; H^{s-r+1}(X)) = 0$ by the induction hypothesis. Therefore $d_r: E_r^{0,s} \longrightarrow E_r^{r,s-r+1}$ is trivial since $E_r^{r,s-r+1} = 0$. The only case left for consideration is $d_{s+1}: E_{s+1}^{0,s} \longrightarrow E_{s+1}^{s+1,0}$. If s is even, $d_{s+1}: E_{s+1}^{0,s} \longrightarrow E_{s+1}^{s+1,0}$ is trivial simply because $E_{s+1}^{s+1,0} = 0$; hence $E_{s+1}^{s+1,0} = 0$. If s is odd, then $s+1 \leq 2n$ and $d_{s+1}: E_{s+1}^{0,s} \longrightarrow E_{s+1}^{s+1,0}$ is trivial according to (7). It follows that $d_r: E_r^{0,s} \longrightarrow E_r^{r,s-r+1}$ is trivial for all $r \geq 2$. Hence $E_2^{0,s} = E_\infty^{0,s}$ and by (8) we have

$$H^s(X) = E_2^{0,s} = 0.$$

Next we shall prove that

$$(10) \quad H^{2n+1}(X) = \mathbb{Z}.$$

If $n=0$, then (5) reduces to $0 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow 0$.

Hence

$$H^{2n+1}(X) = H'(X) \cong E_2^{2,0} = \mathbb{Z}.$$

Suppose then that $n > 0$. Consider the differentials

$$d_r: E_r^{0,2n+1} \longrightarrow E_r^{r,2n+2-r}, \quad r \geq 2.$$

It is easily seen from (9), which has just been proved, that these differentials are all trivial except possibly when $r = 2n+2$. Hence we have

$$E_2^{0,2n+1} = E_{2n+2}^{0,2n+1} \quad \text{and} \quad E_{2n+3}^{0,2n+1} = E_{\infty}^{0,2n+1} = 0$$

where the last equation holds because $H^{2n+1}(X/S') = 0$.

Similarly, we also have

$$E_2^{2n+2,0} = E_{2n+2}^{2n+2,0} \quad \text{and} \quad E_{2n+3}^{2n+2,0} = E_{\infty}^{2n+2,0} = 0.$$

But by definition

$$E_{2n+3}^{0,2n+1} = \text{Ker}(E_{2n+2}^{0,2n+1} \xrightarrow{d_{2n+2}} E_{2n+2}^{2n+2,0}) \quad \text{and}$$

$$E_{2n+3}^{2n+2,0} = E_{2n+2}^{2n+2,0} / \text{Im}(E_{2n+2}^{0,2n+1} \xrightarrow{d_{2n+2}} E_{2n+2}^{2n+2,0}).$$

It follows that $d_{2n+2}: E_{2n+2}^{0,2n+1} \longrightarrow E_{2n+2}^{2n+2,0}$ is an isomorphism and hence

$$H^{2n+1}(X) = E_2^{0,2n+1} = E_{2n+2}^{0,2n+1} \cong E_{2n+2}^{2n+2,0} = E_2^{2n+2,0} = \mathbb{Z}.$$

Up to this stage, the argument is quite the same as

that used in 2.5. To determine the higher dimensional groups, we have of course no special cohomology theory available here. Instead, we propose to prove by induction that

$$(11) \quad H^{2n+K}(X) = 0 \text{ for all } 2 \leq K \leq \dim X - 2n.$$

Since $H^*(\mathbb{C}P^N)$ is torsion free, we can write $E_2 = H^*(\mathbb{C}P^N; H^*(X)) = H^*(\mathbb{C}P^N) \otimes H^*(X)$. We have just shown that $\theta_0 = d_{2n+2}: E_2^{0,2n+1} \rightarrow E_2^{2n+2,0}$ is an isomorphism (for all $n \geq 0$). This isomorphism can be described as follows: Let a denote the generator of $H^2(\mathbb{C}P^N)$ and 1 denote the generator of $H^0(\mathbb{C}P^N)$ as well as that of $H^0(X)$. Consider the element $a^{n+1} \otimes 1 \in E_2^{2n+2,0}$, regarded as in $E_{2n+2}^{2n+2,0}$. Then there is an element $b \in H^{2n+1}(X)$ such that the element $1 \otimes b \in E_2^{0,2n+1}$, considered as in $E_{2n+2}^{0,2n+1}$, has the property that $d_{2n+2}(1 \otimes b) = a^{n+1} \otimes 1$, and θ_0 is completely determined by $\theta_0(1 \otimes b) = a^{n+1} \otimes 1$.

$$\text{Let } Z(E_r^{s,t}) = \text{Ker}(E_r^{s,t} \xrightarrow{d_r} E_r^{s+r,t-r+1}), \mu_r^{s,t}:$$

$$Z(E_r^{s,t}) \rightarrow E_{r+1}^{s,t} \text{ be the canonical projection and } j_r^{s,t}:$$

$$Z(E_r^{s,t}) \rightarrow E_r^{s,t} \text{ the inclusion. We agree that if we write}$$

$$\mu_r^{s,t}: E_r^{s,t} \rightarrow E_{r+1}^{s,t}, \text{ then it is tacitly assumed that}$$

$$Z(E_r^{s,t}) = E_r^{s,t}. \text{ Similarly, if we write } j_r^{s,t}: E_{r+1}^{s,t} \rightarrow E_r^{s,t}, \text{ then it is tacitly assumed that } Z(E_r^{s-r,t+r-1}) =$$

$E_r^{s-r, t+r-1}$ and $E_{r+1}^{s, t}$ has been identified with $Z(E_r^{s, t})$.

We observe that from (2) and 6.1, we have always

$E_\infty^{s, t} = 0$ whenever $s+t > 2n$. All these will be used from now on without further explanation.

From (1), (9), and (10), it is easily verified that we have the following diagram when $n > 0$.

$$\begin{array}{ccccccc}
 & & j_2^{0, 2n+2} & & & & \\
 E_2^{0, 2n+2} & \xleftarrow{\quad} & E_3^{0, 2n+2} & = & E_{2n+3}^{0, 2n+2} & = & E_{2n+4}^{0, 2n+2} = E_\infty^{0, 2n+2} = 0 \\
 \downarrow d_2 & & & & & & \\
 E_2^{2, 2n+1} & \xrightarrow{\mu_2^{2, 2n+1}} & E_3^{2, 2n+1} & = & E_{2n+2}^{2, 2n+1} & \xleftarrow{j_{2n+2}^{2, 2n+1}} & E_{2n+3}^{2, 2n+1} = E_\infty^{2, 2n+1} = 0 \\
 & & \downarrow d_{2n+2} & & & & \\
 E_2^{2n+4, 0} & = & & & E_{2n+2}^{2n+4, 0} & &
 \end{array}$$

In this diagram, $\text{Ker} d_2 = \text{Im } j_2^{0, 2n+2} = 0$. Define θ_2 :

$E_2^{2, 2n+1} \rightarrow E_2^{2n+4, 0}$ by $\theta_2 = d_{2n+2} \circ \mu_2^{2, 2n+1}$. Then

because $\text{Ker } d_{2n+2} = \text{Im } j_{2n+2}^{2, 2n+1} = 0$, we have $\text{Ker } \theta_2 =$

$\text{Ker } \mu_2^{2, 2n+1} = \text{Im } d_2$. In other words, the sequence

$$0 \rightarrow E_2^{0, 2n+2} \xrightarrow{d_2} E_2^{2, 2n+1} \xrightarrow{\theta_2} E_2^{2n+4, 0}$$

is exact. When $n = 0$, we define $\theta_2: E_2^{2, 1} \rightarrow E_2^{4, 0}$ simply as $\theta_2 = d_2$. Then it is directly verified that the above sequence is also exact for $n = 0$. Now let us calculate

$\theta_2(a \otimes b)$. We first observe that $E_2^{2,0} = E_{2n+2}^{2,0}$. For $n=0$, this is trivial; for $n > 0$, this follows from (4). Hence we may consider the element $a \otimes 1 \in E_2^{2,0}$ as in $E_{2n+2}^{2,0}$. Clearly we have $d_{2n+2}(a \otimes 1) = 0$. Hence

$$\begin{aligned}
 \theta_2(a \otimes b) &= d_{2n+2} \circ \mu_2^{2,2n+1}((a \otimes 1)(1 \otimes b)) \\
 &= d_{2n+2}((a \otimes 1) \mu_2^{0,2n+1}(1 \otimes b)) \\
 &= d_{2n+2}((a \otimes 1)(1 \otimes b)) \\
 &= (a \otimes 1)d_{2n+2}(1 \otimes b) \\
 &\quad + (d_{2n+2}(a \otimes 1))(1 \otimes b) \\
 &= (a \otimes 1)(a^{n+1} \otimes 1) \\
 &= a^{n+2} \otimes 1
 \end{aligned}$$

if $n > 0$ and

$$\begin{aligned}
 \theta_2(a \otimes b) &= d_2(a \otimes b) = d_2((a \otimes 1)(1 \otimes b)) \\
 &= (a \otimes 1)d_2(1 \otimes b) + (d_2(a \otimes 1))(1 \otimes b) \\
 &= (a \otimes 1)(a \otimes 1) \\
 &= a^2 \otimes 1
 \end{aligned}$$

if $n=0$. In both cases, this means that we have the following commutative diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & E_2^{0,2n+2} & \xrightarrow{d_2} & E_2^{2,2n+1} & \xrightarrow{\theta_2} & E_2^{2n+4,0} \\
 & & \uparrow \gamma_a \otimes 1 & & \uparrow \gamma_a \otimes 1 & & \\
 & & E_2^{0,2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2,0} & &
 \end{array}$$

where $\gamma_a: H^K(\mathbb{CP}^N) \rightarrow H^{K+2}(\mathbb{CP}^N)$ is the multiplication by a . Since γ_a is an isomorphism, it follows that θ_2 is an isomorphism and hence, by the exactness of the upper row, we have

$$H^{2n+2}(X) = \bigoplus E_2^{0,2n+2} = 0.$$

Suppose it has been proved that $H^{2n+1}(X) = 0$, for all $2 \leq 1 < K$, $K \leq \dim_{\mathbb{Z}} X - 2n$. We consider two cases:

Case 1. K is even; say $K = 2m$. There are again three cases:

(A) $1 < m < n+1$. Then we have the following diagram

$$\begin{array}{ccccccc}
 E_2^{0,2n+2m} = E_{2m}^{0,2n+2m} & \xleftarrow{j_{2m}^{0,2n+2m}} & E_{2m+1}^{0,2n+2m} = E_{2n+2m+1}^{0,2n+2m} = E_{2n+2m+2}^{0,2n+2m} = E_{\infty}^{0,2n+2m} = 0 \\
 \downarrow d_{2m} & & & & & & \\
 E_2^{2m,2n+1} = E_{2m}^{2m,2n+1} & \xrightarrow{j_{2m}^{2m,2n+1}} & E_{2m+1}^{2m,2n+1} = E_{2n+2}^{2m,2n+1} & \xleftarrow{j_{2n+2}^{2m,2n+1}} & E_{2n+3}^{2m,2n+1} = E_{\infty}^{2m,2n+1} = 0 \\
 & & \downarrow d_{2n+2} & & & & \\
 E_2^{2n+2m+2,0} & = & E_{2n+2}^{2n+2m+2,0} & & & &
 \end{array}$$

In this diagram, define $\delta: E_2^{0,2n+2m} \rightarrow E_2^{2m,2n+1}$ by $\delta = d_{2m}$ and $\theta_m: E_2^{2m,2n+1} \rightarrow E_2^{2n+2m+2,0}$ by $\theta_m = a_{2n+2} \circ \mu_{2m}^{2m,2n+1}$. Then $\text{Ker } \delta = \text{Im } j_{2m}^{0,2n+2m} = 0$ and $\text{Ker } \theta_m = \text{Ker } \mu_{2m}^{2m,2n+1} = \text{Im } \delta$ because $\text{Ker } d_{2n+2} = \text{Im } j_{2n+2}^{2m,2n+1} = 0$. Moreover, we have

$$\begin{aligned}
 \theta_m(a^m \otimes b) &= d_{2n+2} \circ \mu_{2m}^{2m,2n+1} ((a^m \otimes 1)(1 \otimes b)) \\
 &= d_{2n+2} ((a^m \otimes 1) \mu_2^{0,2n+1}(1 \otimes b)) \\
 &= d_{2n+2} ((a^m \otimes 1)(1 \otimes b)) \\
 &= (a^m \otimes 1) d_{2n+2}(1 \otimes b) + m(d_{2n+2}(a \otimes 1))(1 \otimes b) \\
 &= (a^m \otimes 1)(a^{n+1} \otimes 1) \\
 &= a^{n+m+1} \otimes 1.
 \end{aligned}$$

It follows that we have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_2^{0,2n+2m} & \xrightarrow{\delta} & E_2^{2m,2n+1} & \xrightarrow{\theta_m} & E_2^{2n+2m+2,0} \\
 & & \uparrow \gamma_a^m \otimes 1 & & & & \uparrow \gamma_a^m \otimes 1 \\
 & & E_2^{0,2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2,0} & &
 \end{array}$$

where the upper row is exact. Hence θ_m must be an isomorphism and therefore we have

$$H^{2n+K}(X) = E_2^{0,2n+2m} = 0.$$

(B) $n+1 < m$, Then we have the following diagram

$$\begin{array}{c}
 E_2^{0, 2n+2m} = E_2^{0, 2n+2m} \xleftarrow{j_{2n+2m}^{0, 2n+2m}} E_2^{2m, 2n+1} \xleftarrow{j_{2n+1}^{2m, 2n+1}} E_2^{2m, 2n} \xleftarrow{j_{2n}^{2m, 2n}} E_2^{2m, 2n-1} \xleftarrow{j_{2n-1}^{2m, 2n-1}} E_2^{2m, 2n-2} \xleftarrow{j_{2n-2}^{2m, 2n-2}} \dots \xleftarrow{j_{2n+2}^{2m, 2n+2}} E_2^{2m, 2n+1} = 0 \\
 \downarrow d_{2m} \\
 E_2^{2m, 2n+1} \xleftarrow{j_{2n+1}^{2m, 2n+1}} E_2^{2m, 2n} \xleftarrow{j_{2n}^{2m, 2n}} E_2^{2m, 2n-1} \xleftarrow{j_{2n-1}^{2m, 2n-1}} E_2^{2m, 2n-2} \xleftarrow{j_{2n-2}^{2m, 2n-2}} \dots \xleftarrow{j_{2n+2}^{2m, 2n+2}} E_2^{2m, 2n+1} = 0 \\
 \downarrow d_{2n+2} \\
 E_2^{2n+2m+2, 0} \xleftarrow{j_{2n+2}^{2n+2m+2, 0}} E_2^{2n+2m+1, 0} \xleftarrow{j_{2n+1}^{2n+2m+1, 0}} E_2^{2n+2m, 0} \xleftarrow{j_{2n}^{2n+2m, 0}} E_2^{2n+2m-1, 0} \xleftarrow{j_{2n-1}^{2n+2m-1, 0}} E_2^{2n+2m-2, 0} \xleftarrow{j_{2n-2}^{2n+2m-2, 0}} \dots \xleftarrow{j_{2n+2}^{2n+2m+2, 0}} E_2^{2n+2m+1, 0} = 0
 \end{array}$$

In this diagram, define $\Delta: E_2^{0, 2n+2m} \rightarrow E_2^{2m, 2n+1}$
 by $\Delta = j_{2n+2}^{2m, 2n+1} \circ d_{2m}$ and $\theta_m: E_2^{2m, 2n+1} \rightarrow$
 $E_2^{2n+2m+2, 0}$ by $\theta_m = d_{2n+2}$. Just as in (A), we have the
 following diagram

$$\begin{array}{ccccc}
 0 \longrightarrow & E_2^{0, 2n+2m} & \xrightarrow{\Delta} & E_2^{2m, 2n+1} & \xrightarrow{\theta_m} & E_2^{2n+2m+2, 0} \\
 & \uparrow \gamma_a^m \circ 1 & & \uparrow \gamma_a^m \circ 1 & & \\
 & E_2^{0, 2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2, 0} & & ,
 \end{array}$$

where the upper row is exact, Hence we conclude again
 that

$$H^{2n+K}(X) = E_2^{0, 2n+2m} = 0.$$

(C) $n+1 = m$. Then we have the following diagram

$$\begin{array}{c}
 E_2^{0, 4n+2} = E_{2n+2}^{0, 4n+2} \xleftarrow{j_{2n+2}^{0, 4n+2}} E_{2n+3}^{0, 4n+2} = E_{4n+3}^{0, 4n+2} = E_{4n+4}^{0, 4n+2} = E_{\infty}^{0, 4n+2} = 0 \\
 \downarrow d_{2n+2} \\
 E_2^{2n+2, 2n+1} = E_{2n+2}^{2n+2, 2n+1} \\
 \downarrow d_{2n+2} \\
 E_2^{4n+4, 0} = E_{2n+2}^{4n+4, 0}
 \end{array}$$

In this diagram, define $\delta : E_2^{0, 4n+2} \rightarrow E_2^{2n+2, 2n+1}$ by $d_{2n+2} : E_{2n+2}^{0, 4n+2} \rightarrow E_{2n+2}^{2n+2, 2n+1}$ and $\theta_{n+1} : E_2^{2n+2, 2n+1} \rightarrow E_2^{4n+4, 0}$ by $d_{2n+2} : E_{2n+2}^{2n+2, 2n+1} \rightarrow E_{2n+2}^{4n+4, 0}$. Then we have $\text{Im } \delta \subset \text{Ker } \theta_{n+1}$ and $\text{Ker } \delta = \text{Im } j_{2n+2}^{0, 4n+2} = 0$. This time, we have the following commutative diagram

$$\begin{array}{ccccc}
 0 \rightarrow E_2^{0, 4n+2} & \xrightarrow{\delta} & E_2^{2n+2, 2n+1} & \xrightarrow{\theta_{n+1}} & E_2^{4n+4, 0} \\
 & & \uparrow \gamma_a^{n+1} \bullet 1 & & \uparrow \gamma_a^{n+1} \bullet 1 \\
 & & E_2^{0, 2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2, 0}
 \end{array}$$

where in the upper row, δ is a monomorphism and

$\text{Im } \delta \subset \text{Ker } \theta_{n+1}$. But the commutativity implies that θ_{n+1} is an isomorphism; hence again we have $H^{2n+K}(X) = E_2^{0,4n+2} = 0$.

Case 2, K is odd; say $K = 2m+1$. Then we have the following diagram

$$\begin{array}{c}
 E_2^{2m,2n+1} = E_{2n+2}^{2m,2n+1} \\
 \downarrow d_{2n+2} \\
 E_2^{2n+2m+2,0} = E_{2n+2}^{2n+2m+2,0} \xrightarrow{\mu_{2n+2}^{2n+2m+2,0}} E_{2n+3}^{2n+2m+2,0} = E_{2n+2m+2}^{2n+2m+2,0} \xrightarrow{\mu_{2n+2m+2}^{2n+2m+2,0}} E_{2n+2m+3}^{2n+2m+2,0} = E_{\infty}^{2n+2m+2,0} = 0 \\
 \uparrow d_{2n+2m+2} \\
 E_2^{0,2n+2m+1} = E_{2n+1}^{0,2n+2m+1} - E_{2m+2}^{0,2n+2m+1} = E_{2n+2m+2}^{0,2n+2m+1} \xleftarrow{j_{2n+2m+1}^{0,2n+2m+1}} E_{2n+2m+3}^{0,2n+2m+1} = E_{\infty}^{0,2n+2m+1} = 0
 \end{array}$$

We have $\text{Ker } d_{2n+2m+2} = \text{Im } j_{2n+2m+1}^{0,2n+2m+1} = 0$ and $\text{Im } d_{2n+2m+2} =$

$\text{Ker } \mu_{2n+2m+2}^{2n+2m+2,0} = E_{2n+2m+2}^{2n+2m+2,0}$. Hence $d_{2n+2m+2}$ is an isomorphism. Define $\omega : E_2^{2n+2m+2,0} \rightarrow E_2^{0,2n+2m+1}$ by

$$\omega = d_{2n+2m+2}^{-1} \circ \mu_{2n+2}^{2n+2m+2,0} \quad \text{and } \theta_m : E_2^{2m,2n+1} \rightarrow$$

$E_2^{2n+2m+2,0}$ by $\theta_m = d_{2n+2}$. Then we have $\text{Im } \theta_m =$

$\text{Im } d_{2n+2} = \text{Ker } \mu_{2n+2}^{2n+2m+2,0} = \text{Ker } \omega$. Clearly ω is an epimorphism. Hence

$$E_2^{2m,2n+1} \xrightarrow{\theta_m} E_2^{2n+2m+2,0} \xrightarrow{\omega} E_2^{0,2n+2m+1} \rightarrow 0$$

is exact. Moreover, it is easily verified that the following diagram

$$\begin{array}{ccccc} E_2^{2m,2n+1} & \xrightarrow{\theta_m} & E_2^{2n+2m+2,0} & \xrightarrow{\omega} & E_2^{0,2n+2m+1} \rightarrow 0 \\ \gamma_a^m \circ 1 \uparrow & & \uparrow \gamma_a^m \circ 1 & & \\ E_2^{0,2n+1} & \xrightarrow{\theta_0} & E_2^{2n+2,0} & & \end{array}$$

is commutative. Hence θ_m is an isomorphism and we have $H^{2n+K}(X) = E_2^{0,2n+2m+1} = 0$.

Finally we have

$$H^{2n+K}(X) = 0 \quad \text{for all } K > \dim_{\mathbb{Z}} X - 2n$$

by the definition of $\dim_{\mathbb{Z}} X$. This completes the proof of 6.3.

Section 7. Lifting of an action in a principal bundle

$$(X, Y, S', \pi)$$

Given a connected compact Hausdorff space Y and a non-zero element $\alpha \in H^1(Y; \mathbb{Z}_p)$, we have described in Section 3 how a cohomology covering space X with respect to α can be constructed and also how a prescribed action of $\mathbb{Z}_p$ on Y can be lifted to an action of $\mathbb{Z}_p$ on X in such a way that it commutes with the deck transformation on X . In this section we shall treat the corresponding problem when $\mathbb{Z}_p$ is replaced by S' . This can be formulated as follows. Given a compact Hausdorff space Y and an element $a_0 \in H^2(Y)$, does there exist a principal bundle (X, Y, S', π) such that $\pi^*: H^2(Y) \rightarrow H^2(X)$ maps a_0 into zero? Moreover, if a prescribed action of S' is given on Y , can this action be lifted to an action of S' on X in such a way that each $g \in S'$ gives a bundle automorphism of X ? The next proposition answers the first part of the question.

7.1. Proposition. Let Y be a compact Hausdorff space and a_0 an element of $H^2(Y)$. Then there exists a principal bundle (X, Y, S', π) with compact Hausdorff total space X such that $\pi^*: H^2(Y) \rightarrow H^2(X)$ maps a_0 into zero.

Proof. We can represent Y as the inverse limit of an inverse system $\{Y_m, \phi_m^{m'}\}$ of triangulable spaces [9; p.284], i.e. $Y = \varprojlim \{Y_m, \phi_m^{m'}\}$, where each Y_m is a finite simplicial complex. Let $\phi_m: Y \longrightarrow Y_m$ be the projection. Then we have a system $\phi_m^*: H^2(Y_m) \longrightarrow H^2(Y)$ of homomorphisms which defines a homomorphism $\phi^*: \varinjlim \{H^2(Y_m), \phi_m^{m'*}\} \longrightarrow H^2(Y)$ from the direct limit of the direct system $\{H^2(Y_m), \phi_m^{m'*}\}$ into $H^2(Y)$. By the continuity theorem, we know ϕ^* is an isomorphism. Hence there exists an index m and an element $a_m \in H^2(Y_m)$ such that $\phi_m^*(a_m) = a_0$. Consider the bundle (S^{2N+1}, CP^N, S^1, P) (cf. 5.4), where N is chosen so that $2N+1 > \dim Y_m$ ($\dim Y_m$ means the dimension of the simplicial complex Y_m). From the homotopy sequence of this bundle, [13; p. 152], it is easily seen that

$$\pi_n(CP^N) = \begin{cases} 0; & \text{if } 0 \leq n < 2 \\ \mathbb{Z}; & \text{if } n=2 \\ 0; & \text{if } n < 2 < 2N+1 \end{cases},$$

where $\pi_n(CP^N)$ denotes the n -th homotopy group of CP^N . In particular, CP^N is n -simple and $H^{n+1}(Y_m, \pi_n(CP^N)) = 0$ for all $2 < n < \dim Y_m$, [13; p. 132]. Hence according to the obstruction theory, [13; p. 192], there exists a

map $g: Y_m \longrightarrow \mathbb{C}P^N$ such that $g^*(a) = a_m$, where a denotes the generator of $H^2(\mathbb{C}P^N)$. Let $f: Y \longrightarrow \mathbb{C}P^N$ be defined as $f = g \circ \phi_m$. Then we have $f^*(a) = a_0$. Let (X, Y, S', π) be the principal bundle over Y induced by f ; that is, $X = \{(y, u) \mid y \in Y, u \in S^{2N+1}, f(y) = p(u)\} \subset Y \times S^{2N+1}$ and $\pi: X \longrightarrow Y$ is given by $\pi(y, u) = y$. Define $h: X \longrightarrow S^{2N+1}$ by $h(y, u) = u$. Then we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & S^{2N+1} \\ \pi \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & \mathbb{C}P^N \end{array}$$

which in turn gives the following commutative diagram for cohomology groups

$$\begin{array}{ccc} H^2(X) & \xleftarrow{h^*} & H^2(S^{2N+1}) = 0 \\ \pi^* \uparrow & & \uparrow p^* \\ H^2(Y) & \xleftarrow{f^*} & H^2(\mathbb{C}P^N) \end{array}$$

Therefore

$$\pi^*(a_0) = \pi^* \circ f^*(a) = h^* \circ p^*(a) = h^*(0) = 0.$$

7.2 Let (X, Y, S', π) be a principal bundle where the base space Y is compact Hausdorff. We may regard the structure group S' as a transformation group acting on X freely with Y as the orbit space and $\pi: X \longrightarrow Y$ as the canonical projection. Denote this action by β , i.e. $\beta: S' \times X \longrightarrow X$ is a map satisfying

$$\beta(g_1, \beta(g_2, x)) = \beta(g_1 g_2, x) \text{ for all } g_1, g_2 \in S', x \in X,$$

$$\beta(e, x) = x \text{ for all } x \in X, e = \text{identity of } G.$$

Suppose we have an action $\bar{\alpha}: S' \times Y \longrightarrow Y$ of S' on Y . Then by a bundle lifting of $\bar{\alpha}$, we mean an action $\alpha: S' \times X \longrightarrow X$ of S' on X such that

$$(1) \quad \pi \circ \alpha(g, x) = \bar{\alpha}(g, \pi(x)) \text{ for all } g \in S', x \in X,$$

$$(11) \quad \alpha(g_1, \beta(g_2, x)) = \beta(g_2, \alpha(g_1, x)).$$

In other words, α makes π an equivariant map and α commutes with the action β of the structure group. We shall now study the question of the existence of a bundle lifting for a given action $\bar{\alpha}$ of S' on the base space Y . This question has been studied in [19] under a more general situation by T.E. Stewart. The following lemma is a slight modification of his result and its proof is entirely the same as that

used in [19]. Hence we shall only sketch its main idea.

7.3. Lemma. Let (X, Y, S', π) , β and $\bar{\alpha}$ be the
same as in 7.2. Then there exists a neighborhood V of
the identity $e \in S'$ and a map $\alpha': V \times X \longrightarrow X$ satisfying

- (1) (i) $\alpha'(g_1, \alpha'(g_2, x)) = \alpha'(g_1 g_2, x)$ for all
 $g_1, g_2, g_1 g_2 \in V, x \in X,$
(ii) $\alpha'(e, x) = x$ for all $x \in X,$
(iii) $\alpha'(g_1, \beta(g_2, x)) = \beta(g_2, \alpha'(g_1, x))$ for all
 $g_1 \in V, g_2 \in S', x \in X,$
(iv) $\pi \circ \alpha'(g_1, x) = \bar{\alpha}(g_1, \pi(x))$ for all $g_1 \in V, x \in X.$

Proof. For convenience, denote $\bar{\alpha}(g, y)$ by $g \cdot y$ for $g \in S'$ and $y \in Y$. Choose a neighborhood U of e homeomorphic to the unit interval. The identity map $\alpha_e: X \longrightarrow X$ is clearly a bundle map and the restriction of $\bar{\alpha}$ on $U \times Y$ is a homotopy of the induced map of α_e . Since Y is compact Hausdorff, we can apply the first covering homotopy theorem, [18; p. 50]. This gives a map $\alpha: U \times X \longrightarrow X$ satisfying all conditions of (1) (with V replaced by U) except possibly (i). Choose V homeomorphic to the unit interval such that $V^3 \subset U$ and define the error function (cf, 19) $\bar{f}: V \times V \times Y \longrightarrow S'$ by the equation

$$(2) \quad \alpha(g_1 g_2, x) = \beta(\bar{f}(g_1, g_2, \pi(x)), \alpha(g_1, \alpha(g_2, x))).$$

By the associative law of the multiplication in S' ,
we have

$$(3) \quad \bar{f}(g_1, g_2, g_3 \cdot y)^{-1} f(g_2, g_3, y) \bar{f}(g_1, g_2 g_3, y) \bar{f}(g_1 g_2, g_3, y)^{-1} = e.$$

Let R be the group of real numbers and $\varphi: R \rightarrow S'$ be
the exponential map defined by $\varphi(t) = e^{2\pi t i}$, $t \in R$. The
map $\bar{f}$ is a homotopy of the map $\varphi \circ f_0: VxY \rightarrow S'$

where $f_0: VxY \rightarrow R$ is the map defined by $f_0(g, y) = 0$,

Hence we can apply the second covering homotopy theorem,
[18; p. 54]. This gives a map $f: VxVxY \rightarrow R$ satisfying

$$f(e, g, y) = 0$$

and f is unique since R is the universal covering group
of S' . The uniqueness of f implies that f satisfies the
equation

$$(4) \quad f(g_2, g_3, y) - f(g_1, g_2, g_3 \cdot y) + f(g_1, g_2 g_3, y) - f(g_1 g_2, g_3, y) = 0,$$

Let

$$(5) \quad P(g_1, g_2, g_3)(y) = f(g_1^{-1} g_2, g_2^{-1} g_3, g_3^{-1} \cdot y),$$

By (4), we have

$$(6) \quad P(g_2, g_3, g_4)(y) - P(g_1, g_3, g_4)(y) + P(g_1, g_2, g_4)(y) \\ - P(g_1, g_2, g_3)(y) = 0.$$

Let $C(Y, \mathbb{R})$ be the space of all continuous real valued functions on Y endowed with the compact open topology; or equivalently, $C(Y, \mathbb{R})$ has the usual norm topology since Y is compact. Then P can be regarded as a continuous function defined on some neighborhood of the diagonal of $S' \times S' \times S'$ taking values in $C(Y, \mathbb{R})$ and satisfying (6). Since $C(Y, \mathbb{R})$ is a metrizable absolute retract, [13, p. 20], we may assume that P is defined on all of $S' \times S' \times S'$, i.e., $P: S' \times S' \times S' \rightarrow C(Y, \mathbb{R})$ is a map. Let $\underline{A}^n(S', C(Y, \mathbb{R}))$ be the sheaf of germs of continuous Alexander-Spanier n -cochains with coefficients in $C(Y, \mathbb{R})$. Then $\underline{A}^*(S', C(Y, \mathbb{R})) = \{\underline{A}^n(S', C(Y, \mathbb{R}))\}$ is a soft resolution [12] of the constant sheaf $C(Y, \mathbb{R})$ over S' (cf. 19). Since $H^2(S', C(Y, \mathbb{R})) = 0$, there exists a continuous function $Q: S' \times S' \rightarrow C(Y, \mathbb{R})$ such that

$$(7) \quad P(g_1, g_2, g_3) = Q(g_2, g_3) - Q(g_1, g_3) + Q(g_1, g_2)$$

on a neighborhood of the diagonal of $S' \times S'$. Define $Q': S' \times S' \rightarrow C(Y, \mathbb{R})$ by

$$Q'(g_1, g_2)(y) = \int_S Q(gg_1, gg_2)(g \cdot y) dg,$$

where the integral is taken with respect to the normalized Haar measure of S' . Then we have

$$(8) \quad Q'(gg_1, gg_2)(g \cdot y) = Q'(g_1, g_2)(y)$$

and (7) is still true when Q is replaced by Q' . Now define

$$h(g, y) = Q'(e, g)(g \cdot y).$$

Since Y is compact, h is simultaneously continuous in both variables, [4; Chap. X, p. 24]. By (8), (7) and (5), it is easily verified that

$$(9) \quad f(g_1, g_2, y) = h(g_2, y) + h(g_1 g_2, y) - h(g_1 g_2, y).$$

Finally, define α' : $VxX \longrightarrow X$ by

$$\alpha'(g, x) = \beta(\varphi \circ h(g, x), \alpha(g, x)).$$

Then by (9) and (2), it is directly verified that α' satisfies (i) - (iv) of (1).

7.4. Proposition. Let (X, Y, S', τ) be a principal bundle and $\bar{\alpha}: S'xY \longrightarrow Y$ an action of S' on Y . Assume that Y is compact Hausdorff and $H'(Y) = 0$. Then there exists a bundle lifting $\alpha: S'xX \longrightarrow X$ of $\bar{\alpha}$.

Proof. As before, we let $\beta: S'xX \longrightarrow X$ denote the action of the structure group S' on X and $\varphi: R \longrightarrow S'$ denote the exponential map. We shall first define an action of R on X . Let $\mathcal{H}(X, X)$ be the group of all homeomorphisms of X onto itself endowed with the compact open topology. Since X is clearly compact Hausdorff, $\mathcal{H}(X, X)$ is a topological group [18; p. 20] and an action of R on X is

equivalent to a continuous homomorphism of R into $\mathcal{H}(X, X)$. By 7.3, there exists a neighborhood V of the identity $e \in S'$ and a map $\alpha': V \times X \longrightarrow X$ satisfying (1) of 7.3. We may assume that V is small enough so that there exists a neighborhood W of $0 \in R$ such that W is mapped homeomorphically onto V by ϕ . Hence α' may be considered as being defined on W . Moreover, equations (1) and (11) of (1) of 7.3 imply that α' is a local homomorphism of R into $\mathcal{H}(X, X)$. Since R is simply connected, α' can be extended to a continuous homomorphism of R into $\mathcal{H}(X, X)$, [6; p. 49]. In other words, we have a map $\alpha': R \times X \longrightarrow X$ satisfying

- (1) (i) $\alpha'(t_1, \alpha'(t_2, x)) = \alpha'(t_1 + t_2, x)$ for all $t_1, t_2 \in R, x \in X$,
(ii) $\alpha'(0, x) = x$ for all $x \in X$,
(11) $\pi \circ \alpha'(t, x) = \bar{\alpha}(\phi(t), \pi(x))$ for all $t \in R, x \in X$,
(iv) $\alpha(t, \beta(\phi(S), x)) = \beta(\phi(S), \alpha(t, x))$ for all $t, S \in R, x \in X$.

For every $y \in Y$ and $x \in \pi^{-1}(y)$, we have $\pi \circ \alpha'(0, x) = y = \pi \circ \alpha'(1, x)$. Hence there exists a unique element $g(y) \in S'$ such that

$$(2) \quad \alpha'(0, x) = \beta(g(y), \alpha'(1, x)), \quad y = \pi(x).$$

It is easily verified that $y \longrightarrow g(y)$ defines a map $g: Y \longrightarrow S'$ which satisfies

$$(3) \quad g(\alpha(\varphi(t), y)) = g(y) \quad \text{for all } t \in R, y \in Y.$$

Let $\pi'(Y)$ be the Bruschlinsky group [13; p. 48] of Y . It is known [13; p. 59] that $\pi'(Y)$ and $H'(Y)$ are isomorphic; hence $\pi'(Y) = 0$ by our hypothesis. Therefore g is homotopic to zero. Since Y is compact, by the second covering homotopy theorem (cf. 7.3) there exists a map $h: Y \longrightarrow R$ such that $g = \varphi \circ h$. Define $\alpha'': R \times X \longrightarrow X$ by

$$(4) \quad \alpha''(t, x) = \beta(\varphi(th(\pi(x))), \alpha'(t, x)).$$

Then it is easily verified that α'' satisfies (ii), (iii) and (iv) of (1). To verify (i) of (1), we let $y = \pi(x)$, and obtain

$$\begin{aligned} \alpha''(t_1, \alpha''(t_2, x)) &= \alpha'(t_1, \beta(\varphi(t_2 h(y)), \alpha'(t_2, x))) \\ &= \beta(\varphi(t_1 h(\alpha(\varphi(t_2), y))), \alpha'(t_1, \\ &\quad \beta(\varphi(t_2 h(y)), \alpha'(t_2, x)))) \\ &= \beta(\varphi(t_1 h(y)), \beta(\varphi(t_2 h(y)), \alpha'(t_1, \alpha'(t_2, x)))) \\ &= \beta(\varphi(t_1 + t_2) h(y), \alpha'(t_1 + t_2, x)) \\ &= \alpha''(t_1 + t_2, x). \end{aligned}$$

Moreover, by (2), we have

$$\begin{aligned}
\alpha''(o,x) &= \beta(\phi(oh(y)), \alpha'(o,x)) \quad \text{and} \\
\alpha''(1,x) &= \beta(\phi(h(y)), \alpha'(1,x)) = \beta(g(y), \alpha'(1,x)) \\
&= \alpha'(o,x) = \alpha''(o,x).
\end{aligned}$$

It follows that $\alpha: S' \times X \longrightarrow X$ given by

$$\alpha(\phi(t), x) = \alpha''(t, x)$$

is a well-defined action of S' on X which is a bundle lifting of $\bar{\alpha}$.

Section 8. Actions of S' on cohomology complex projective spaces.

We now turn to the action of the circle group on a cohomology complex projective space Y . As in Section 4, our final goal will be the determination of the cohomology structure of the fixed point set. The result we shall obtain here is however less general than that given in Section 4, in the sense that a stronger hypothesis will be imposed. In fact, we shall assume that Y is of finite cohomology dimension and that the action has finite orbit structure. Whether these unwelcome conditions can be removed is unknown to the author.

We first prove a proposition that will be used in the proof of the main theorem and which is also interesting by itself:

8.1. Proposition. Let S' act on a cohomology complex projective n -space Y such that $\dim_{\mathbb{Z}} Y < \infty$. Then the fixed point set F is non-empty and F has at most $n+1$ components.

Proof. Consider the spectral sequence (E_r) of the fibring $\pi_2: Y_{S'} \longrightarrow \mathbb{CP}^N$ of 5.4 with the field of rationals $\mathbb{Q}$ as coefficient group and where N is so chosen that $2N+1 > \dim_{\mathbb{Z}} Y$. We have

$$E_2^{s,t} = H^S(\mathbb{CP}^N; H^t(Y; \mathbb{Q}))$$

and the E_∞ -term is associated with $H^*(Y_S; Q)$. By the universal coefficient theorem, it is clear that

$$H^*(Y; Q) = Q[x] / (x^{n+1}), \quad \deg x = 2,$$

where $Q[x]$ is the polynomial ring with coefficients in Q and (x^{n+1}) is the ideal generated by x^{n+1} . It follows that we have $E_2^{s,t} = 0$ when either s or t is odd. If s and t are both even, we have $d_r: E_r^{s,t} \rightarrow E_r^{s+r, t+1-r}$ is trivial for all $r \geq 2$ since at least one of $s+r$ or $t+1-r$ is odd for any $r \geq 2$. It follows that the spectral sequence (E_r) is trivial and we have

$$\dim H^K(Y_S; Q) = \sum_{s=0}^K \dim E_\infty^{k-s, s} = \sum_{s=0}^K \dim E_2^{k-s, s}.$$

This relation determines $H^*(Y_S; Q)$ immediately as

$$(1) \quad \dim H^K(Y_S; Q) = \begin{cases} 0 & \text{if } K \text{ is odd,} \\ m+1 & \text{if } K = 2m, \quad 0 \leq m \leq n, \\ n+1 & \text{if } K = 2m, \quad n \leq m \leq N, \\ n-(m-N)+1 & \text{if } K = 2m, \\ & N \leq m \leq N+n, \\ 0 & \text{if } K = 2m, \quad N+n < m. \end{cases}$$

Suppose that $F = \emptyset$. Consider the map $\pi_1:$

$Y_S \rightarrow Y/S'$ of 5.4. For each $Z = \pi(y) \in Y/S'$, where

$\pi: Y \rightarrow Y/S'$ is the canonical projection, we have

$\pi_1^{-1}(Z) = S^{2N+1}/S'y$, where $S'y$ is the isotropic subgroup

at $y \in Y$. Since $F = \emptyset$, $S'y$ is a finite group and it is known that $H^k(S^{2N+1}/S'y; \mathbb{Q}) = 0$ for all $1 \leq K \leq 2N$, (cf. 1; p. 54). Hence by the Vietoris mapping theorem, $\pi_1^*: H^k(Y/S'; \mathbb{Q}) \rightarrow H^k(Y_{S'}; \mathbb{Q})$ is an isomorphism for all $0 \leq K \leq 2N$. In particular, take K even such that $\dim_{\mathbb{Z}} Y < K \leq 2N$. Then by (1) we have $H^k(Y/S'; \mathbb{Q}) \neq 0$. On the other hand, by 5.2 and 5.3, we have $\dim_{\mathbb{Q}} Y/S' \leq \dim_{\mathbb{Z}} Y/S' \leq \dim_{\mathbb{Z}} Y$. This gives a contradiction.

Since $\dim_{\mathbb{Q}} Y < \infty$, by 5.6 we have $i_{S'}^*: H^k(Y_{S'}; \mathbb{Q}) \rightarrow H^k(F_{S'}; \mathbb{Q}) = \sum_{s=0}^K H^{K-s}(CP^N) \otimes H^s(F, \mathbb{Q})$ is an isomorphism for all $\dim_{\mathbb{Q}} Y < K \leq 2N$. Take $K = 2m$ such that $\dim_{\mathbb{Q}} Y < K \leq 2N$. Then by (1) we obtain $\dim H^0(F; \mathbb{Q}) \leq n+1$, which proves our assertion.

Now we present the main theorem of Part 2 of this paper,

8.2. Theorem. Let S' act on a cohomology complex projective n -space Y . Suppose that $\dim_{\mathbb{Z}} Y < \infty$ and that the orbit structure is finite. Then the fixed point set F is non-empty and F has at most $n+1$ components $C_1, \dots, C_K$, $K \leq n+1$, where each C_i is a cohomology complex projective n_i -space for some n_i , $i = 1, 2, \dots$, K , and

$$(1) \quad \sum_{i=1}^K n_i = n-K+1.$$

Proof. We have already proved in 8.1 that $F \neq \emptyset$

and that F has at most $n+1$ components. Let $C_1, \dots, C_K$ ($K \leq n+1$) be the components on F . It remains to show that each C_1 is a cohomology complex projective n_1 -space for some n_1 and equation (1) holds.

Let $a_0 \in H^2(Y)$ be a generator of $H^2(Y)$. By 7.1 and 6.1, there exists a principal bundle (X, Y, S', π) with Y as base space such that $\pi^*: H^2(Y) \rightarrow H^2(X)$ is trivial. Let $\beta: S' \times X \rightarrow X$ denote the action of the structure group S' on X . Then the action β is free. Since X is obviously compact Hausdorff, by 6.3 we know that X is an integral cohomology $(2n+1)$ -sphere. Let $\bar{\alpha}: S' \times Y \rightarrow Y$ denote the given action of S' on Y . Since $H^1(Y) = 0$, $\bar{\alpha}$ has a bundle lifting α according to 7.4. Since α commutes with β , the map $\gamma: (S' \times S') \times X \rightarrow X$ given by

$$\gamma((g_1, g_2), x) = \alpha(g_1, \beta(g_2, x)), \quad g_1, g_2 \in S', \quad x \in X$$

defines an action of $S' \times S'$ on X . Clearly, γ has no fixed point.

We claim that the action γ has finite orbit structure. Take any $x \in X$ and let $y = \pi(x)$. Suppose that $(g_1, g_2) \in G_x \subset S' \times S'$. Then we have $\pi \circ \gamma((g_1, g_2), x) = \pi \circ \alpha(g_1, \beta(g_2, x)) = \bar{\alpha}(g_1, y) = y$. Hence $g_1 \in G_y$, where $G_y \subset S'$ is the isotropic subgroup at y under the action $\bar{\alpha}$. We consider two cases:

Case 1. $G_y \neq S'$. Then G_y is a finite cyclic group, say of order K . Let g_0 be a generator of G_y . We have $g_1 = g_0^\ell$ for some $0 \leq \ell < K$. Since $\bar{\alpha}(g_0, y) = y$, there exists a unique $g_0' \in S'$ such that $\beta(g_0', \alpha(g_0, x)) = y$. From this we get $\alpha(g_0', x)^K = e$. Hence there exists some $0 \leq m_0 < K$ such that $g_0' = g_0^{m_0}$. Now $x = \alpha(g_1, \beta(g_2, x)) = \alpha(g_0^\ell, \beta(g_2, x)) = \beta(g_2 g_0'^{-\ell}, x) = \beta(g_2 g_0^{-\ell m_0}, x)$. Hence $g_2 = g_0^{\ell m_0}$ so that every element of G_x is of the form $(g_0^\ell, g_0^{\ell m_0})$ for some $0 \leq \ell < K$. Clearly every element of this form is also in G_x . Thus we have shown that $G_x = G_y \times N_y$ where N_y is a subgroup of G_y . As there are only a finite number of G_y of finite order and of them they have only a finite number of subgroups, the number of G_x which are of the form $G_y \times N_y$ with G_y of finite order are finite.

Case 2. $G_y = S'$, $1, e, y \in F$. For each $g \in S'$, there exists a unique $\bar{g} \in S'$ such that $\beta(\bar{g}, \alpha(g, x)) = X$. It is easily seen that the correspondence $g \longrightarrow \bar{g}$ is a continuous homomorphism of S' into S' , i.e. an element of the dual group $S'^* = Z$ of S' . Hence it must be of the form $g \longrightarrow g^{K_y}$, where K_y is an integer depending only on $y = \pi(X)$. The set $\{(g, g^{K_y}) | g \in S'\}$ is a subgroup $H(K_y)$ of $S' \times S'$ and one can see that G_x is actually equal to it. The correspondence $y \longrightarrow K_y$ defines a function $K: F \longrightarrow Z$. Using the continuity of γ it is readily verified that K is

continuous on F . It must then be constant on each component C_1 of F . Hence we can define $H_1 = H(K_y)$, $y \in C_1$, $1 = 1, 2, \dots, K$. Each G_x which is not of the form given in case 1 must be one of the H_1 . Hence again there is only a finite number of G_x of this kind.

Now each H_1 is a circle group and the restriction of γ on $H_1 \times X$ defines an action of H_1 on X which obviously has finite orbit structure. Moreover, we have $\dim_{\mathbb{Z}} X < \infty$ by 6.3. Hence by 5.7, the fixed point set $F(H_1; X) = F_1 \subset X$ is an integral cohomology m_1 -sphere for some odd m_1 , say $m_1 = 2n_1 + 1$. It is easily verified that $F_1 = \pi^{-1}(C_1)$ and that $\beta: S' \times F_1 \rightarrow F_1$ defines a free action of S' on F_1 . By 5.2, we have $\dim_{\mathbb{Z}} F_1 \leq \dim_{\mathbb{Z}} X < \infty$. Hence by 6.2, C_1 is a cohomology complex projective n_1 -space.

We remark that $S' \times S'$ may contain subgroups N isomorphic to S' other than these H_1 's, but none of them can have a fixed point. In fact, if $x \in F(N; X)$, we have $G_x = N$ so that N would be one of the H_1 . We now apply the theorem of Borel given in 5.9. Since we know $\dim_{\mathbb{Q}} X < \infty$, γ has finite orbit structure and each F_1 is a cohomology $(2n_1 + 1)$ -sphere over $\mathbb{Q}$. By the remark just made and the fact that γ has no fixed point, this would give

$$(2n+1) - (-1) = \sum_{i=1}^K [(2n_i+1) - (-1)]$$

which reduces exactly to (1).