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POISSON'S EQUATION AND GENERALIZED
AXIALLY SYMMETRIC POTENTIAL THEORY*

by

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12. Abstract: In this paper a method is developed by which one may
    obtain solutions to the non-homogeneous equation of
    generalized axially symmetric potential theory
    (GASPT). Representation formulae are obtained and
    the method is shown to be extendable under certain
    symmetry conditions to the Poisson equation in three
    variables.

I. Introduction

Professor A. Weinstein[1][2][3][4] has referred
to the study of solutions of the partial differential equations

$$L^{(\kappa)}[\phi] \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\kappa}{y}\frac{\partial \phi}{\partial y} = 0, \quad \kappa \geqslant 0, \quad (1)$$

as generalized axially symmetric potential theory (GASPT). This
equation is initially arrived at by considering those solutions of the
n-dimensional Laplace equation which depend solely on

$$x = x_1, \quad y = (x_2^2 + \cdots + x_n^2)^{1/2} \quad \text{(in this case } \kappa = n - 2).$$

In previous works this author[5][6][7][8] *+
has studied properties of the GASPT equation by function theoretic
methods based on the use of an integral operator $a_\kappa[f]$, which
naps analytic functions $f(\sigma)$ onto solutions of (1).

$$\phi(z) = a_\kappa[f] \equiv \alpha_\kappa \int_{+1}^{-1} f(\sigma)(\zeta - \zeta^{-1})^{\kappa-1} \frac{d\zeta}{\zeta} , \quad (2)$$

$$\alpha_\kappa = \frac{4}{(4i)^\kappa \Gamma(\frac{\kappa}{2})^2} , \quad \sigma = x + \frac{iy}{2}(\zeta + \zeta^{-1}), \quad |z - z^0| < \epsilon ,$$

$$z = x + iy, \quad z^0 = x^0 + iy^0,$$

is an initial point of definition for $\phi(z)$, $\epsilon > 0$ is sufficiently

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* For similar investigations of the integral operator method as applied
to GASPT, see Henrici [9] and Mackie [10] .

+ The function-theoretic approach which we use here is primarilly due
to S. Bergman, see for instance [11] , [12] and [13] .

small, and integration path is the upper semi-circular arc connecting + 1 to - 1 .

In this paper we shall investigate the properties of

solutions to the non-homogeneous GASPT equation, namely

$$L^{(\kappa)}[\phi] = \rho(x,y) \tag{3}$$

by means of integral operator methods. One notes first, that when $K=0$, equation (3) becomes

$$\Delta\phi = \rho(x,y), \tag{4}$$

and that by replacing X, Y by $X = \dfrac{z+\bar{z}}{2}$, $Y = \dfrac{z-\bar{z}}{2i}$, we may rewrite (4) as

$$4\frac{\partial^2\phi}{\partial z\partial\bar{z}} = P(z,\bar{z}),$$

which integrates directly into

$$\phi = \int^{\bar{z}}\int^{z} P\,dz\,d\bar{z} + g(z) + \overline{h(z)} .$$

This method suggests that we consider the integral operator $A_\kappa[f]$:

$$\phi(z) = A_\kappa[f] \equiv \alpha_\kappa \int_{+1}^{-1} f(\sigma,\sigma^*)(\varsigma-\varsigma^{-1})^{\kappa-1}\frac{d\varsigma}{\varsigma}, \tag{5}$$

where $f(\sigma, \sigma^*)$ is an analytic function of the two complex variables σ, σ^*; σ is the same as before, $\sigma^* = x - \frac{iy}{2}(s + s^{-1})$, $|z - z^0| < \epsilon$, and $\epsilon > 0$ is sufficiently small etc. . It is clear, that if $|s| = 1$ and x, y are real, then $\sigma^* = \bar{\sigma}$. Also, if $\frac{\partial f}{\partial \sigma} \equiv 0$, or $\frac{\partial f}{\partial \sigma^*} \equiv 0$, then f is either a function of σ^* or σ respectively; in these cases $A_\kappa \to a_\kappa$, and $\phi(z)$ is a solution of (1).

In order to obtain a solution of (3) we apply the operator $L^{(\kappa)}$ to (5) and interchange the orders of integration and differentiation,

$$L^{(\kappa)}[\phi] = \alpha_\kappa \int_{+1}^{-1} L^{(\kappa)}\left[f(\sigma, \sigma^*)\right] (s - s^{-1})^{\kappa-1} \frac{ds}{s}$$

$$= \frac{\alpha_\kappa}{4} \int_{+1}^{-1} \left[-\frac{\partial^2 f}{\partial \sigma^2}(s - s^{-1})^2 + 2\frac{\partial^2 f}{\partial \sigma \partial \sigma^*}\left(4 + [s + s^{-1}]^2\right) \right.$$

$$\left. -\frac{\partial^2 f}{\partial \sigma^{*2}}(s - s^{-1})^2 + 2i\frac{\kappa}{y}\frac{\partial f}{\partial \sigma} - 2i\frac{\kappa}{y}\frac{\partial f}{\partial \sigma^*} \right] (s - s^{-1})^{\kappa-1}\frac{ds}{s} .$$

$$(6)$$

By rearranging terms, and integrating by parts one obtains

$$L^{(K)}[\phi] = -\frac{\alpha_K}{4} \int_{+1}^{-1} f_{\sigma\sigma} \, (\mathfrak{z}-\mathfrak{z}^{-1})^{K+1} \frac{d\mathfrak{z}}{\mathfrak{z}} \;+\; \frac{i\alpha_K}{2\gamma}\left[f_\sigma \, (\mathfrak{z}-\mathfrak{z}^{-1})^K \right]_{-1}^{+1}$$

$$-\frac{i\alpha_K}{2\gamma} \int_{+1}^{-1} \frac{d}{d\mathfrak{z}} f_\sigma \, (\mathfrak{z}-\mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}} \;-\; \frac{\alpha_K}{4} \int_{+1}^{-1} f_{\sigma^*\sigma^*} \, (\mathfrak{z}-\mathfrak{z}^{-1})^{K+1} \frac{d\mathfrak{z}}{\mathfrak{z}}$$

$$-\frac{i\alpha_K}{2\gamma} \left[f_{\sigma^*} \, (\mathfrak{z}-\mathfrak{z}^{-1})^K \right]_{-1}^{+1} \;+\; \frac{i\alpha_K}{2\gamma} \int_{+1}^{-1} \frac{d}{d\mathfrak{z}} f_{\sigma^*} \, (\mathfrak{z}-\mathfrak{z}^{-1})^K d\mathfrak{z}$$

$$+ \alpha_K \int_{+1}^{-1} f_{\sigma^*\sigma} \left[2 + \tfrac{1}{2}(\mathfrak{z}+\mathfrak{z}^{-1})^2 \right] (\mathfrak{z}-\mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}}$$

$$= \alpha_K \int_{+1}^{-1} f_{\sigma^*\sigma} \left[2 + \tfrac{1}{2}(\mathfrak{z}+\mathfrak{z}^{-1})^2 - \tfrac{1}{2}(\mathfrak{z}-\mathfrak{z}^{-1})^2 \right] (\mathfrak{z}-\mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}}$$

$$= 4\alpha_K \int_{+1}^{-1} f_{\sigma^*\sigma} \, (\mathfrak{z}-\mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}} = \rho(x,y). \qquad (7)$$

We realize from this, that if $f(\sigma,\sigma^*)$ satisfies the integral equation (7), then $\phi(z) = A_K[f]$ is a solution of equation (3).

II. Properties of the Operator $A_\kappa [f]$.

We first prove

Theorem 1: In a sufficiently small neighborhood, $N(z^0)$, of an initial point of definition, z^0 , the representation

$$\phi(z) = A_\kappa [f] \equiv \alpha_\kappa \int_{+1}^{-1} f(\sigma, \sigma^*) (\varsigma - \varsigma^{-1})^{\kappa-1} \frac{d\varsigma}{\varsigma} ,$$

yields the most general, analytic solution to $L^{(\kappa)} [\phi] = \rho(x,y),$ providing that f is contained in the class of analytic functions of two variables, $\mathcal{F}$, which satisfy the integral equation,

$$\rho(x,y) = 4 \int_{+1}^{-1} \frac{\partial^2 f}{\partial \sigma \partial \sigma^*} (\varsigma - \varsigma^{-1})^{\kappa-1} \frac{d\varsigma}{\varsigma} .$$

Proof: It is clear that if $f \varepsilon \mathcal{F}$, then $\phi(z) = A_\kappa [f]$ is a solution of (3) if $z \varepsilon N(z^0)$, and $N(z^0)$ is sufficiently small. Furthermore, if $F(\sigma, \sigma^*) \varepsilon \mathcal{F}$, then

$$F(\sigma, \sigma^*) = f(\sigma, \sigma^*) + g(\sigma) + h(\sigma^*)$$

, where g and h are arbitrary analytic functions of σ, σ^* respectively. One then has, that

$$A_\kappa [F(\sigma, \sigma^*)] = A_\kappa [f(\sigma, \sigma^*) + g(\sigma) + h(\sigma^*)]$$

$$= A_\kappa [f] + a_\kappa [g] + a_\kappa^* [h] ,$$

where $Q_\kappa^*[h] \overset{def}{\equiv} u(\bar{z})$, if $Q_\kappa[h] = u(z)$.

The difference $A_\kappa[F] - A_\kappa[f] = Q_\kappa[g] + Q_\kappa^*[h]$

represents the general analytic (in x, y) solution of (1) is some

neighborhood $N(z^0)$ of z^0, since either $Q_\kappa[g]$ or $Q_\kappa^*[h]$

represent general solutions of the GASPT equation. (See Gilbert [5]

[6][7]). It follows then that $A_\kappa[F]$ represents the most

general solution to (3) in a sufficiently small neighborhood $N(z^0)$.

The next problem we consider is the representation of

solutions of $L^{(\kappa)}[\phi] = \rho(x,y)$ by $A_\kappa[F]$. For instance, if

$F(\sigma, \sigma^*)$ has the series expansion

$$F(\sigma, \sigma^*) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm}\, \sigma^n \sigma^{*m}, \qquad (8)$$

what is the series representation in terms of x, y etc. for

$\phi(z) = A_\kappa[F]$? We begin by recalling[14]

$$\frac{n!}{\Gamma(\kappa+n)}\, r^n\, C_n^{\kappa/2}(\zeta) = Q_\kappa[\sigma^n] = \alpha_\kappa \int_{+1}^{-1} \sigma^n\, (\zeta - \zeta^{-1})^{\kappa-1} \frac{d\zeta}{\zeta}, \quad (9)$$

$$|\zeta| = 1$$

where $\zeta = \frac{x}{r}$, $r = |z|$, and $C_n^{\kappa/2}(\zeta)$ is a Gegenbauer

polynomial[14]. It follows then, that

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\alpha_n\, n!}{\Gamma(\kappa+n)}\, r^n\, C_n^{\kappa/2}(\zeta) = Q_\kappa[f], \text{ where } f(\sigma) = \sum_{n=0}^{\infty} \alpha_n \sigma^n, \quad (10)$$

for $|z| < \epsilon$ suffic.'ently small.

It is interesting to note that the class of functions $f(\sigma) \in C^2$, where $\sigma = x + \frac{iy}{2}(\zeta + \zeta^{-1})$ are not even locally $\left(z \in N(z^*) \right)$ solutions of the GASPT equation [6]. Consequently, the homogeneous polynomials, $A_{n,\nu}(z)$ of degree n in X, Y, which are defined as coefficients of ζ in the trinomial expansion of σ^n.

$$\sigma^n = \sum_{\nu = -n}^{+n} A_{n,\nu}(z) \, \zeta^\nu, \tag{11}$$

are not in general solutions of (1) either.*   It can be shown, nevertheless, that $\phi(z) = Q_\kappa \left[f(\sigma) \right]$ does generate solutions to (1), and that the set of functions $f_n(\sigma) \equiv \sigma^n$ generates a complete system of polynomials [14]. To show that

$$\phi = Q_\kappa \left[f(\sigma) \right], \quad f \in C^2 \qquad \text{is a solution of (1) we need}$$

only integrate by parts as before

$$L^{(\kappa)}\left[\phi \right] = \alpha_\kappa \int_{+1}^{-1} L^{(\kappa)}\left[f(\sigma) \right] \left(\zeta - \zeta^{-1} \right)^{\kappa - 1} \frac{d\zeta}{\zeta}$$

---

* When $\nu = 0$, the polynomial $A_{n\nu}(z)$ is a solution of the GASPT equation for $\kappa = 1$, since $A_{n0}(z) = 2 r^n P_n(\zeta)$. We shall see shortly, that the $A_{n\nu}(z)$ = constant $r^n P_n^\nu(\zeta)$ where the $P_n^\nu(\zeta)$ are associated Legendre functions.

$$= -\frac{\alpha_\kappa}{4} \int_{+1}^{-1} f'' \, (\varsigma-\varsigma^{-1})^{\kappa+1} \frac{d\varsigma}{\varsigma} + i \frac{\alpha_\kappa K}{2\gamma} \int_{+1}^{-1} f' \, (\varsigma+\varsigma^{-1})(\varsigma-\varsigma^{-1})^{\kappa-1} \frac{d\varsigma}{\varsigma} \quad .$$

$$= -\frac{\alpha_\kappa}{4} \int_{+1}^{-1} f'' \, (\varsigma-\varsigma^{-1})^{\kappa+1} \frac{d\varsigma}{\varsigma} + \frac{\alpha_\kappa i}{2\gamma} \, f'(\sigma)(\varsigma-\varsigma^{-1})^\kappa \Big|_{+1}^{-1}$$

$$- \frac{\alpha_\kappa i}{2\gamma} \int_{+1}^{-1} \frac{\partial f'(\sigma)}{\partial \varsigma} \, (\varsigma-\varsigma^{-1})^{\kappa-1} \frac{d\varsigma}{\varsigma}$$

(12)

It may be shown in the same way, that $\phi = a_\kappa \left[f(\sigma^*) \right]$ is also a solution of the GASPT equation.

There is a relation, however, between the polynomials $A_{n,\nu}(z)$ and the associated Legendre functions. This may be seen from Heine's integral relation [ ] ,

$$\frac{n!}{i^m \, (m+n)!} \; r^n \, P_n^m(\mathcal{I}) \;=\; \frac{1}{2\pi i} \int_{|\mathcal{I}|=1} \left(x + i\frac{y}{2}(\mathcal{I}+\mathcal{I}^{-1}) \right]^n \mathcal{I}^m \, \frac{d\mathcal{I}}{\mathcal{I}}$$

$$=\; \frac{1}{2\pi i} \int_{|\mathcal{I}|=1} \sigma^n \mathcal{I}^m \, \frac{d\mathcal{I}}{\mathcal{I}} \;=\; A_{n,-m}(z) .$$

$$(13)$$

Likewise, the coefficients $A_{n,\nu}^{*}(z)$ of the expansion of $\sigma^{*\,n}$ also may be related to the Legendre functions by

$$\frac{i^m \, n!}{(n-m)!} \; r^n \, P_n^{-m}(\mathcal{I}) \;=\; \frac{1}{2\pi i} \int_{|\mathcal{I}|=1} \left(x - i\frac{y}{2}(\mathcal{I}+\mathcal{I}^{-1}) \right)^n \mathcal{I}^m \, \frac{d\mathcal{I}}{\mathcal{I}}$$

$$=\; \frac{1}{2\pi i} \int_{|\mathcal{I}|=1} \sigma^{*\,n} \mathcal{I}^m \, \frac{d\mathcal{I}}{\mathcal{I}} \;=\; A_{n,-m}^{*}(z) .$$

$$(14)$$

We note in passing, that expressions (13), (14) may be used to give a representation for the Gegenbauer polynomials in terms of the Legendre functions, when $K > 0$.

$$r^n \, C_n^{\kappa/2}(\mathfrak{z}) \;=\; \frac{4}{(4i)^\kappa} \frac{\Gamma(\kappa+n)}{\Gamma(\frac{\kappa}{2})^2 \, n!} \int_{\substack{+1 \\ |\mathfrak{z}|=1}}^{-1} \sigma^n \, (\mathfrak{z}-\mathfrak{z}^{-1})^{\kappa-1} \frac{d\mathfrak{z}}{\mathfrak{z}} \;,$$

hence

$$C_n^{\kappa/2}(\mathfrak{z}) \;=\; \frac{4}{(4i)^\kappa} \frac{\Gamma(\kappa+n)}{\Gamma(\frac{\kappa}{2})} \sum_{m=-n}^{+n} i^m \, \frac{P_n^{-m}(\mathfrak{z})}{(n-m)!} \int_{\substack{+1 \\ |\mathfrak{z}|=1}}^{-1} \mathfrak{z}^m \, (\mathfrak{z}-\mathfrak{z}^{-1})^{\kappa-1} \frac{d\mathfrak{z}}{\mathfrak{z}} \;.$$

However, it is known[*] that for $\kappa > 0$,

$$\int_{\substack{+1 \\ |\mathfrak{z}|=1}}^{-1} \mathfrak{z}^m \, (\mathfrak{z}-\mathfrak{z}^{-1})^{\kappa-1} \frac{d\mathfrak{z}}{\mathfrak{z}} \;=\; \frac{\pi i^{\kappa+m}}{\kappa \, B\!\left(\frac{\kappa+1+m}{2}, \frac{\kappa+1-m}{2}\right)} \;, \text{ where } B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \;,$$

is the Beta-function. One then has

$$C_n^{\kappa/2}(\mathfrak{z}) \;=\; \frac{\pi \, 4^{1-\kappa}}{\kappa} \frac{\Gamma(\kappa+n)}{\Gamma(\frac{\kappa}{2})^2} \sum_{m=-n}^{+n} \frac{(-)^m}{(n+m)!} \frac{P_n^m(\mathfrak{z})}{B\!\left(\frac{\kappa+1+m}{2}, \frac{\kappa+1-m}{2}\right)} \qquad (15)$$

---

* See page 12 of Higher Transcendental Functions, Vol. I[   ] .

One may consider the integral

$$a_K \left[\sigma^n \sigma^{*m} \right] = \alpha_K \int_{+1}^{-1} \sigma^n \sigma^{*m} (\mathfrak{z} - \mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}}$$
$$|\mathfrak{z}|=1$$

$$= \alpha_K \, n! \, m! \, r^{m+n} \sum_{\nu=-n}^{+n} \sum_{\mu=-m}^{+m} i^{\nu-\mu} \frac{P_n^\nu(\mathfrak{z}) \, P_m^\mu(\mathfrak{z})}{(n+\nu)! \, (m+\mu)!} \int_{+1}^{-1} \mathfrak{z}^{\nu+\mu} (\mathfrak{z}-\mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}}$$

$$= \frac{\pi}{K} \alpha_K \, i^K \, n! \, m! \, r^{n+m} \sum_{\nu=-n}^{+n} \sum_{\mu=-m}^{+m} \frac{(-)^\nu}{(n+\nu)! \, (m+\mu)!} \frac{P_n^\nu(\mathfrak{z}) \, P_m^\mu(\mathfrak{z})}{B\left(\frac{K+1+\mu+\nu}{2}, \frac{K+1-\mu-\nu}{2}\right)} .$$
$$(16)$$

III. Particular Solutions Generated by $A_K[f]$.

In order to generate particular solutions of $L^K[\phi] = \rho$

where ρ is analytic about the origin in the real variables x, y

we consider the integral equation

$$\int_{+1}^{-1} g(\sigma, \sigma^*) (\mathfrak{z}-\mathfrak{z}^{-1})^{K-1} \frac{d\mathfrak{z}}{\mathfrak{z}} = \rho(x,y),$$
$$|\mathfrak{z}|=1$$

where $g(\sigma, \sigma^*) = 4 \alpha_K \frac{\partial^2 f}{\partial \sigma \partial \sigma^*}$. If $\rho(x,y)$ has the

Taylor expansion. $\rho(x,y) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu\nu} x^\mu y^\nu,$ $|z| < R$,

then one may solve formally for the Taylor coefficients, g_{MN} , of $g(\sigma, \sigma^*)$ as follows.

$$a_{mn} = \frac{\partial^{m+n}}{\partial x^m \partial y^n} \rho(0,0) = \int\limits_{\substack{+1 \\ |s|=1}}^{-1} \frac{\partial^{m+n}}{\partial x^m \partial y^n} g(\sigma,\sigma^*) \Bigg|_{x=y=0} (s-s^{-1})^{k-1} \frac{ds}{s}$$

$$= \int\limits_{\substack{+1 \\ |s|=1}}^{-1} \left(\frac{i}{2}\right)^n \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \binom{m}{\mu}\binom{n}{\nu} (-)^\nu \frac{\partial^{m+n} g(\sigma,\sigma^*)}{\partial\sigma^{\mu+\nu} \partial\sigma^{*(m+n)-(\mu+\nu)}} \Bigg|_{x=y=0}$$

$$\cdot (s+s^{-1})^n (s-s^{-1})^{k-1} \frac{ds}{s}$$

$$= \left(\frac{i}{2}\right)^n \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \binom{m}{\mu}\binom{n}{\nu} (-)^\nu g_{\mu+\nu, m+n-(\mu+\nu)} \int\limits_{\substack{+1 \\ |s|=1}}^{-1} (s+s^{-1})^n (s-s^{-1})^{k-1} \frac{ds}{s},$$

or

$$a_{mn} = \beta_{nk} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} (-)^\nu \binom{m}{\mu}\binom{n}{\nu} g_{\mu+\nu, (m+n)-(\mu+\nu)} , \tag{17}$$

where $\quad \beta_{n\kappa} = \left(\frac{-1}{2}\right)^{n} \kappa\pi\, i^{\kappa} \sum_{\nu=0}^{n} \frac{(-)^{\nu}\binom{n}{\nu}}{B\left(\frac{\kappa+1-n}{2}-\nu, \frac{\kappa+1+n}{2}-\nu\right)}$

Several special cases are suggested immediately, for instance, where ρ is a function of x or y alone. If $\rho(x,y) \equiv p(x)$, we may then choose $g(\sigma,\sigma^{*})$ to have the form $G(\sigma,\sigma^{*}) \equiv G(2x)$; one then has

$$g(\sigma,\sigma^{*}) = G(2x) = \frac{p(x)}{\beta_{0\kappa}} \quad , \quad \beta_{0\kappa} = \frac{i^{\kappa}\pi}{\kappa\, B\left(\frac{\kappa+1}{2}, \frac{\kappa+1}{2}\right)} ,$$

and

$$f(\sigma,\sigma^{*}) = \frac{p(x)}{4\alpha_{\kappa}\beta_{0\kappa}} \sigma\sigma^{*}$$
. A particular solution to
the Poisson equation is then given by

$$\phi(z) = \frac{p(x)}{4\beta_{0\kappa}} \int_{+1}^{-1} \sigma\sigma^{*}\,(\zeta-\zeta^{-1})^{\kappa-1}\frac{d\zeta}{\zeta} = \frac{\pi}{\kappa} r^{2} i^{\kappa} p(x) \sum_{\nu=-1}^{+1}\sum_{\mu=-1}^{+1}\frac{(-)^{\nu}}{(1+\nu)!\,(1+\mu)!}\frac{P_{1}^{\nu}(\zeta)\,P_{1}^{\mu}(\zeta)}{B\left(\frac{\kappa+1+\mu+\nu}{2}, \frac{\kappa+1-\mu-\nu}{2}\right)} .$$
$$|\zeta|=1 \tag{18}$$

In the case $\rho(x,y) \equiv p(y)$, $g(\sigma,\sigma^{*})$ may be represented as $g = G(\sigma-\sigma^{*}) = G\left(iy[\zeta+\zeta^{-1}]\right)$, and we consider then the integral equation,

$$\int_{+1}^{-1} G\left(i\gamma\left[\mathfrak{z}+\mathfrak{z}^{-1}\right]\right)\left(\mathfrak{z}-\mathfrak{z}^{-1}\right)^{\kappa-1}\frac{d\mathfrak{z}}{\mathfrak{z}} = \mathfrak{p}(\gamma) .$$
$$|\mathfrak{z}|=1$$

If $\quad G(\mathfrak{z}) = \sum_{\nu=0}^{\wedge} g_\nu\,\mathfrak{z}^\nu \quad$, and $\quad \mathfrak{p}(\gamma) = \sum_{\nu=0}^{\infty} \mathfrak{p}_\nu\,\gamma^\nu$

one may compute the Taylor coefficients g_ν formally as follows.

$$\int_{+1}^{-1} \sum_{n=0}^{\infty} g_n \sum_{r=0}^{n} (i\gamma)^n \binom{n}{\nu} \mathfrak{z}^{2\nu-n} \left(\mathfrak{z}-\mathfrak{z}^{-1}\right)^{\kappa-1}\frac{d\mathfrak{z}}{\mathfrak{z}}$$
$$|\mathfrak{z}|=1$$

$$= \sum_{n=0}^{\infty} (i\gamma)^n\, g_n \sum_{\nu=0}^{n} \binom{n}{\nu} \int_{+1}^{-1} \mathfrak{z}^{2\nu-n} \left(\mathfrak{z}-\mathfrak{z}^{-1}\right)^{\kappa-1}\frac{d\mathfrak{z}}{\mathfrak{z}}$$
$$|\mathfrak{z}|=1$$

$$= \pi\kappa\, i^\kappa \sum_{n=0}^{\infty} g_n\,(-\gamma)^n \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-)^\nu}{B\left(\frac{\kappa+1-n}{2}+\nu,\ \frac{\kappa+1+n}{2}-\nu\right)} ;$$

from which we have

$$g_n = \frac{(-)^n\, \mathfrak{p}_n}{\pi\kappa\, i^\kappa\, \gamma_{\kappa n}} \quad , \quad \text{where} \quad \gamma_{\kappa n} \equiv \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-)^\nu}{B\left(\frac{\kappa+1-n}{2}+\nu,\ \frac{\kappa+1+n}{2}-\nu\right)}$$

providing $|Y_{\kappa n}| \geqslant a > 0$,　　　　　for all n, and for $\kappa > 0.$

One may proceed as before and obtain a particular solution,

$$\phi = \frac{1}{4} \int_{+1}^{-1} \sigma\sigma^* \, G\left(i\gamma\left[\zeta+\zeta^{-1}\right]\right) \left(\zeta-\zeta^{-1}\right)^{\kappa-1} \frac{d\zeta}{\zeta} \ .$$
$$|\zeta|=1$$

IV.　The Poisson Equation in Three Variables

It is interesting to note that the methods developed in the previous sections may be extended to Poisson's equation in three variables, that is we consider the equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \rho\left(x_1, x_2, x_3\right) \ . \tag{20}$$

We may generate solutions to this equation by means of an integral representation similar to the Whittaker-Bergman operator which maps functions of two complex variables into solutions of Laplace's equation, $\Delta H(\mathbf{X}) = 0$.　　[11][12][13]

$$H(\mathbf{X}) = P_3(f) \equiv \frac{1}{2\pi i} \int_{\mathbf{L}} f(t, \zeta) \, \frac{d\zeta}{\zeta} , \tag{21}$$

$$t = \left[-(x_1 - ix_2)\frac{\zeta}{2} + x_3 + (x_1 + ix_2)\frac{1}{2\zeta} \right] \, , \quad \| \overline{X} - \overline{X}^0 \| < \epsilon, \, \overline{X} \equiv (x_1, x_2, x_3) \, .$$

$\overline{X}^0 \equiv (x_1^0, x_2^0, x_3^0)$, where $\mathcal{L}$ is a closed differentiable arc in ζ - plane, and $\epsilon > 0$ is sufficiently small.

Following our previous method we introduce the integral representation $\overline{\Phi}_3(F)$,

$$u(\overline{X}) = \overline{\Phi}_3(F) \equiv \frac{1}{2\pi i} \int_{\mathcal{L}} F(t, t^*, \zeta) \frac{d\zeta}{\zeta} \, , \tag{22}$$

where F is an analytic function of the three complex variables t, t^*, ζ ; t is the same as above,

$$t^* = \left[(x_1 - ix_2)\frac{\zeta}{2} + x_3 - (x_1 + ix_2)\frac{1}{2\zeta} \right] \, , \quad \| \overline{X} - \overline{X}^0 \| < \epsilon, \, \text{etc.} \, .$$

It is clear, that for $|\zeta| = 1$ and $\overline{X} \equiv (x_1, x_2, x_3)$ a real point, then $t^* = \bar{t}$. If $\frac{\partial F}{\partial t} = 0$, or $\frac{\partial F}{\partial t^*} = 0$, then F is a function of t or t^* respectively; in these cases $\overline{\Phi}(F)$ becomes the Whittaker-Bergman operator, and $u(\overline{X})$ is a harmonic function. If $\rho(\overline{X}) \in C^2$, then we have by interchanging orders of differentiation and integration

$$\rho(\overline{X}) = \triangle u = \frac{1}{2\pi i} \int_{|\zeta| = 1} \triangle F \frac{d\zeta}{\zeta} = \frac{1}{4\pi i} \int_{|\zeta| = 1} (\zeta - \zeta^{-1})^2 \frac{\partial^2 F}{\partial t \partial t^*} \frac{d\zeta}{\zeta} \, . \tag{23}$$

Hence, as before we may obtain a particular solution of $\Delta u = \rho$, with the representation (22) if F is a solution of integral equation (23). In order, to see how $\Phi(F)$ transforms analytic functions of three variables into solutions of (20), we first consider the integrals

$$u_{n,m,\lambda}(\underline{X}) = \Phi\left(t^n \, t^{*\,m} \, \mathfrak{z}^\lambda\right) \equiv \frac{1}{2\pi i} \int_{|\mathfrak{z}|=1} t^n \, t^{*\,m} \, \mathfrak{z}^\lambda \, \frac{d\mathfrak{z}}{\mathfrak{z}} \, , \quad -n-m \leq \lambda \leq n+m ,$$

$$= \frac{1}{2\pi i} \, r^{n+m} \, n! \, m! \int_{|\mathfrak{z}|=1} \left(\sum_{\nu=-n}^{+n} P_n^\nu(\mathfrak{z}) \frac{i^{-\nu} \, \mathfrak{z}^{-\nu} \, e^{i\nu\psi}}{(n+\nu)!} \right) \left(\sum_{\mu=-m}^{+m} (-1)^\mu \, P_m^\mu(\mathfrak{z}) \frac{i^{-\mu} \, \mathfrak{z}^{-\mu} \, e^{i\mu\psi}}{(m+\mu)!} \right) \mathfrak{z}^\lambda \frac{d\mathfrak{z}}{\mathfrak{z}}$$

$$= e^{i\lambda\varphi} \, i^{-\lambda} \, r^{n+m} \, n! \, m! \sum_{\nu \in N(m,n)} \frac{P_n^\nu(\mathfrak{z}) \, P_m^{\lambda-\nu}(\mathfrak{z})}{(n+\nu)! \, (m+\lambda-\nu)!} \, ,$$

$$(24)$$

where $N(m,n)$ is the set of indices $\left\{ \nu \, \middle| \, |\nu| \leq n \, , \, |\lambda-\nu| \leq m \right\}$. Consequently, if F is the analytic function

$$F(t, t^*, \mathfrak{z}) = \sum_{n,m=0}^{\infty} \sum_{\lambda=-n-m}^{n+m} a_{nm\lambda} \, t^n \, t^{*\,m} \, \mathfrak{z}^\lambda \, , \qquad (25)$$

defined in the region $|t| < R$, $|t^*| < R$, $1-\epsilon < |\mathfrak{z}| < 1+\epsilon$, the corresponding function $u(\underline{X})$, which is regular in a neighborhood of the origin is

$$u(\underline{X}) = \sum_{n,m=0}^{\infty} \sum_{\lambda=-n-m}^{n+m} a_{nm\lambda} \, u_{nm\lambda}(\underline{X}). \qquad (26)$$

Since, the integral (24) vanishes whenever $|\lambda| > m+n$ we need just consider the functions F defined by (25). We shall realize that this class of functions does not in general generate solutions to $\Delta u = \rho$, when ρ is an arbitrary real analytic function of X_1, X_2, X_3 .

Under certain conditions the integral equation (23) may be solved; to see when this is possible we consider the integral

$$\frac{1}{2\pi i} \int_{|\xi|=1} G(t, t^*, \xi)(\xi - \xi^{-1})^2 \, \frac{d\xi}{\xi} \quad , \text{ where } \quad G = \frac{1}{2} \frac{\partial^2 F}{\partial t \, \partial t^*} \quad ,$$

and G has a representation such as (25). The general term in the series expansion for G integrates as follows

$$V_{nm\lambda}(\underline{X}) = \frac{1}{2\pi i} \int_{|\xi|=1} t^n \, t^{*m} \, \xi^\lambda \, (\xi - \xi^{-1})^2 \, \frac{d\xi}{\xi}$$

$$= n! \, m! \, r^{n+m} \sum_{\nu=-n}^{+n} \sum_{\mu=-m}^{+m} \frac{P_\nu^n(\xi) \, P_\mu^m(\xi)}{i^{\mu+\nu} \, (n+\nu)! \, (m+\mu)!} e^{i(\nu+\mu)\varphi} \frac{1}{2\pi i} \int_{|\xi|=1} \xi^{\lambda-\mu-\nu} (\xi - \xi^{-1})^2 \frac{d\xi}{\xi}$$

$$= -i^\lambda \, n! \, m! \, r^{n+m} \sum_{\nu=-n}^{+n} \sum_{\mu=-m}^{+m} (-)^{\mu+\nu} \frac{P_\nu^n(\xi) \, P_\mu^m(\xi) \, e^{i(\nu+\mu)\varphi}}{(n+\nu)! \, (m+\mu)! \, \Gamma\left(2 + \frac{\lambda-\mu-\nu}{2}\right) \Gamma\left(2 - \frac{\lambda-\mu-\nu}{2}\right)} \left[1 + e^{i(\lambda-\nu-\mu)\pi}\right].$$

If $\rho(X_1, X_2, X_3)$ may be expanded in terms of the polynomials $\{V_{n\,m\,\lambda}\}$, we may find a function G which satisfies (23), and consequently generate all solutions of (20). This clearly is not the case in general; however, under certain criteria of symmetry on $\rho(X_1, X_2, X_3)$ such an expansion will be possible.

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