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THE LOSS OF MECHANICAL ENERGY IN  
THE FLOW OF A DISSOCIATING GAS

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## ABSTRACT

### Statement of problem

This is a general study of the problem of nonequilibrium drag on an arbitrary body in steady adiabatic subsonic dissociating gas flow, viscosity, heat conduction and diffusion being neglected.

### Formulation of problem

Adopting the Lagrange's approach we consider a moving material element along a streamline. Besides the condition of conservation of mass, the basic equations are

- (1) thermal and caloric equations of state
- (2) chemical relaxation equation
- (3) general equation of thermodynamic change
- (4) first law of thermodynamics

These basic equations describe completely the thermodynamic behavior of the moving material element subjected to the effects of dissociation and recombination. We linearize these equations and combine them to obtain

$$\frac{v_0^2}{a_f^2} \frac{d^2 p'}{dt^2} + \frac{d^2 v'}{dt^2} + \frac{1}{\tau} \left[ \frac{v_0^2}{a_e^2} \frac{dp'}{dt} + \frac{dv'}{dt} \right] = 0 \quad (1)$$

which provides the governing relation between the variations of  $p'$  and  $v'$  of the moving material element on a streamtube. We have adopted here the following symbols:

$p' = p - p_0 =$  local pressure - free stream pressure,

$v' = v - v_0 =$  local specific volume - free stream specific volume,

$\tau =$  relaxation time based on free stream value

$a_f, a_e =$  frozen and equilibrium speed of sound, respectively

$t =$  time variable

### General expression of drag on a body

On a streamtube, we take two points P and Q denoted by  $z = -\infty$  and  $z = +\infty$ , respectively) which are at the positions upstream and downstream far from the body where all perturbations almost vanish. We show that the body drag, D, can be computed as follows:

$$D = -\frac{1}{v_0} \iint d\sigma \int_{z=-\infty}^{z=+\infty} p dv \quad (2)$$

where  $\iint d\sigma$  denotes the integration with respect to all streamtubes, the streamtube cross section being  $d\sigma$ .

Results. Clearly, to compute the drag we must determine  $p', v'$  along a streamtube. In the subsonic flow problem the function  $v'$  between P and Q can be generally represented by a Fourier Series:

$$v' = \sum_{n=1}^{\infty} a_n \sin 2\pi n \frac{t}{v^*} + \sum_{n=1}^{\infty} b_n \cos 2\pi n \frac{t}{v^*} \quad (3)$$

where  $v^*$  denotes the time interval during which the material element moves from P to Q. The real Fourier coefficients  $a_n, b_n$  can be determined by consideration of the dynamic and kinematic conditions appropriate to the problem.

We now integrate Eq. (1) to obtain  $p'$  and then substitute  $p'$  and  $v'$  into Eq. (2) to obtain

$$D = \frac{1}{V_0} \sum_{n=1}^{\infty} \iint (A_n^2 + B_n^2) d\sigma \geq 0 \quad (4)$$

where

$$A_n^2 = \frac{\omega_n v^*}{2} \frac{a_n^2}{V_0^2} \frac{(\omega_n \tau) a_f^2 \left(\frac{1}{k^2} - 1\right)}{\frac{1}{k^2} + (\omega_n \tau)^2} \geq 0 \quad (5)$$

$$B_n^2 = \frac{\omega_n v^*}{2} \frac{b_n^2}{V_0^2} \frac{(\omega_n \tau) a_f^2 \left(\frac{1}{k^2} - 1\right)}{\frac{1}{k^2} + (\omega_n \tau)^2} \geq 0 \quad (6)$$

$$\omega_n = \frac{2\pi n}{v^*}$$

$$\frac{1}{k^2} = \frac{a_f^2}{a_e^2} > 1$$

In equilibrium flow,  $\omega_n \tau \rightarrow 0$ ,  $A_n^2 = B_n^2 = 0$  hence,  $D = 0$ . In frozen flow,  $\omega_n \tau \rightarrow \infty$ ,  $A_n^2 = B_n^2 = 0$  again,  $D = 0$ . In nonequilibrium flow, the subsonic drag must take the positive sign. Thus, we have shown:

- (1) The body will experience a positive nonequilibrium drag in subsonic dissociating gas flow.
- (2) Within the accuracy of Eq. (4), the nonequilibrium drag is of the second order in perturbation quantities  $a_n, b_n$ .

We show further that

- (3) The specific entropy increases along a streamline and is of second order in perturbation quantities.
- (4) The effects of entropy increase along the streamline is to contribute to the above drag formula a correction term which is of the third order in perturbation quantities.

Remarks

When we consider the problem of nonequilibrium supersonic flow, the drag would consist of supersonic wave drag and the nonequilibrium drag. In such a case, the disturbance propagating along the Mach waves are damped by the nonequilibrium effects. Consequently, the wave drag may decrease from the conventional value.

The Loss of Mechanical Energy in the Flow  
of a Dissociating Gas

Introduction

Clarke<sup>(1)</sup> and Li and Wang<sup>(2)</sup> recently discussed the problem of nonequilibrium dissociating gas flows past slender bodies. In the case of a slender body with pointed ends, these authors obtained subsonic nonequilibrium drag formulas which are formally the same. Clarke<sup>(1)</sup> stated that in the subsonic range a double-pointed slender body would experience a positive nonequilibrium drag. However, by applying the mean value theorem to the integral expression in the slender body's subsonic nonequilibrium drag formula, Li and Wang<sup>(2)</sup> demonstrated that the sign of the integral may depend on the slender body's cross sectional area distribution. Physically speaking, because of the chemical reaction in a dissociating gas, part of the mechanical energy in the flow field is transformed into internal energy of the gas. Thus, the effects of chemical relaxation are in some sense similar to the viscosity effects which, indeed, also transforms the flow mechanical energy into heat. By this physical argument, the subsonic drag on a body placed in nonequilibrium dissociating flow should take the positive sign. In the present paper, we shall examine the general problem of the loss of mechanical energy in the flow of a dissociating gas. We shall establish the physically interesting result that a finite body would experience a positive drag in subsonic nonequilibrium dissociating gas flows.

### Thermodynamic Considerations

We consider here the stationary subsonic flow of a dissociating gas past an arbitrary body. We neglect molecular transport phenomena such as viscosity, heat conduction and diffusion. Take a small material element moving with the local velocity  $g$  along a streamtube of cross section  $d\sigma$  (Fig. 1). During a small time interval  $\Delta t$ , the moving material element occupies a volume

$$dv^* = g \Delta t d\sigma \quad (1)$$

The work performed against the environment by this material element is

$$\delta W = p dv^* \quad (2)$$

where  $p$  denotes the local pressure in inviscid flow. The first law of thermodynamics states that

$$\delta Q = dE^* + \delta W \quad (3)$$

where  $\delta Q$  denotes the heat received by the material element and  $dE^*$  denotes the change in internal energy of the material element. In adiabatic flow,  $\delta Q = 0$ . Assuming local homogeneity, we define the local density as follows:

$$\rho = \frac{m}{v^*} = \frac{1}{v} \quad (4)$$

where  $m$  represents the mass in the volume  $V^*$  and  $V$  is specific volume i.e. volume per unit mass. The condition of conservation of mass along the streamtube, viz., the continuity equation, gives

$$m = \rho \int d\sigma \Delta t = \rho_0 U_0 d\sigma_0 \Delta t = m_0 \quad (5)$$

where the subscript 0 denotes quantities in the free stream.

From Eq. (3) we obtain

$$dE^* + p d(mV) = 0$$

$$\text{i.e.} \quad dE^* + \rho_0 U_0 d\sigma_0 \Delta t p dV = 0 \quad (3a)$$

Let  $E = \bar{E}/m$  denote the internal energy per unit mass. Eq. (3a) then becomes

$$dE + p dV = 0 \quad (6)$$

Consider the dissociation reaction of a symmetrical diatomic gas  $A_2$ :



where each  $A_2$  molecule is assumed to be made up from two  $A_1$  atoms.

The species  $A_3$  may be either  $A_2$  or  $A_1$ , either is assumed to be equally effective, and  $K_f$  and  $K_r$  are the overall reaction rate constants for forward and reverse reactions. The dissociating gas is assumed to be a mixture of perfect gases. The thermal and caloric equations of state can be written as follows:

$$pV = (1 + \alpha)RT, \quad R = \frac{\bar{R}}{M_2} \quad (8)$$

$$E = [C_{v_2} + \alpha(C_{v_1} - C_{v_2})]T + \alpha D \quad (9)$$

where  $\bar{R}$  denotes the universal gas constant,  $M_2$  the molecular weight of  $A_2$ ,  $D$  the dissociation energy per unit mass,  $\alpha$  the mass fraction of the atomic species, and  $C_{v_1}$  and  $C_{v_2}$  the specific heats of the atoms and molecule respectively. Simple calculations yield

$$C_{v_2} = \left[ \frac{5}{2} + f(\theta) \right] R \quad (10)$$

$$C_{v_1} - C_{v_2} = \left[ \frac{1}{2} - f(\theta) \right] R$$

where the translational and rotational energies are assumed the fully excited classical values and vibrational energies are assumed the equilibrium values based on simple harmonic oscillator model in quantum theory, i.e.,

$$f(\theta) = \frac{\theta}{e^\theta - 1} \quad (11)$$

$$\theta = \frac{h^* \nu}{kT}$$

$h^*$ ,  $k$  being Planck's constant and Boltzmann's constant respectively and  $\nu$  being vibrational frequency. For simplicity, Lighthill's ideal dissociating gas model<sup>(3)</sup> can be adopted here, thus  $f(\theta) = \frac{1}{2}$ . As in Refs. 4 and 5, the chemical relaxation equation for the

dissociating gas can be written as follows:

$$\frac{d\alpha}{dt} = \frac{1}{\tau^*} \{K^* (1-\alpha) - \alpha^2\} \quad (12)$$

where  $\tau^*$  and  $K^*$  are defined as

$$\tau^* = \frac{M_2^2}{4 K_v \rho^2 (1+\alpha)} \quad (13)$$

$$K^* = \frac{M_2}{4\rho} \frac{K_f}{K_v} \quad (14)$$

It is worthwhile to notice here that Eq. (12) is referred to the moving material element. The differentiation with respect to time represented in Eq. (12) should be realized in the Lagrange's sense. If we wish to have the Euler's derivative, Eq. (12) must be interpreted as

$$\frac{D\alpha}{Dt} = \bar{V} \cdot \nabla \alpha = \frac{1}{\tau^*} \{K^* (1-\alpha) - \alpha^2\} \quad (12a)$$

where  $\bar{V}$  denotes the local velocity vector. In the present paper,  $\frac{d}{dt}$  will denote Lagrange's derivative.

#### General Expression of Drag on a Body

In Eq. (3a), the work performed by the material element during the time interval  $\Delta t$  is given as

$$\delta W = \rho \cdot U_0 \cdot dV_0 \cdot \Delta t (p dv)$$

Summing up with respect to all material elements along a streamtube

we have

$$W_1 = \rho_0 U_0 d\sigma_0 \Delta t \int_{z=-\infty}^{z=+\infty} p dv$$

Integrating  $W_1$  expression with respect to all the streamtubes we obtain the work performed against the environment (in this case, the body) by the whole fluid during the time interval  $\Delta t$ . Thus we have

$$W = \rho_0 U_0 \Delta t \iiint_{z=-\infty}^{z=+\infty} p dv d\sigma_0 \quad (15)$$

The force acting on the body by the fluid along the uniform flow direction is the drag,  $D$ . The work performed against the body during the time interval  $\Delta t$  can be represented as

$$W = -DU_0 \Delta t \quad (16)$$

Comparing Eqs. (15) and (16) we obtain

$$D = -\rho_0 \iiint_{z=-\infty}^{z=+\infty} p dv d\sigma_0 \quad (17)$$

In applying Eq. (17) to calculate the body drag, we must have a knowledge of the complete flow field around the body. To ascertain the sign of the drag integral we shall study the sign of the integral

$$-\int_{z=-\infty}^{z=+\infty} p dv \quad (18)$$

along a streamtube. In the case of subsonic dissociating gas flow, we shall examine the equations of motion expressed in Lagrange's sense. We shall show that the integral in Eq. (18) always takes the positive sign and therefore we shall conclude mathematically that the nonequilibrium flow effects cause a positive subsonic drag on the body.

#### Entropy Change Along a Streamtube

In a nonequilibrium state, the general equation of thermodynamic change in the moving material element is expressible as<sup>(5)</sup>

$$\delta Q = T ds + (\mu_1 - \mu_2) d\alpha \quad (19)$$

where T, S denote temperature and entropy, and  $\mu_1$  and  $\mu_2$  denote the chemical potentials of atom and molecules, respectively. Following Clarke<sup>(4)</sup>, we have

$$\mu_1 - \mu_2 = RT \ln \left\{ \left( \frac{\alpha}{\alpha_e} \right)^2 \frac{1 - \alpha_e^2}{1 - \alpha^2} \right\} \quad (20)$$

where  $\alpha_e$  represents the equilibrium atomic mass fraction and is expressible as follows:

$$\alpha_e = f(p, s) \quad (21)$$

Combining Eqs. (19) and (20) we have for adiabatic flow

$$\frac{ds}{dt} = -R \ln \left\{ \left( \frac{\alpha}{\alpha_e} \right)^2 \frac{1 - \alpha_e^2}{1 - \alpha^2} \right\} \frac{d\alpha}{dt} \quad (22)$$

which gives the entropy change along a streamtube in nonequilibrium

dissociating gas flow.

### Linearization of Equations

If the body placed in the flow is very thin, the temperature, the pressure, the specific volume can be considered to be slightly different from the corresponding quantities at infinity upstream. The atomic mass fraction  $\alpha$  is assumed to be slightly different from the local equilibrium value  $\alpha_e$ . Furthermore, we assume  $\alpha_e$  differs only slightly from  $\alpha_0$  in the uniform free stream,  $\alpha_0$  being necessarily an equilibrium value. Then, we have

$$\begin{aligned} T &= T_0 + T' \\ P &= P_0 + P' \\ V &= V_0 + V' \\ \alpha &= \alpha_e + \alpha' \\ \alpha_e &= \alpha_0 + \alpha_e' \end{aligned} \tag{23}$$

where the subscript "0" represents the quantities in the uniform free stream and the superscript "'" the perturbation quantities.

From Eqs. (6) and (9), we obtain

$$[C_{v_2} + \alpha(C_{v_1} - C_{v_2})] \frac{dT}{dT} + [D + (C_{v_1} - C_{v_2})T] \frac{d\alpha}{dT} + P \frac{dV}{dT} = 0 \tag{24}$$

In obtaining this equation,  $C_{v_2}$  and  $C_{v_1} - C_{v_2}$  are assumed independent of T, i.e., we take  $f(\theta) = \text{constant}$ . As has been already pointed out, this simplification corresponds to Lighthill's ideal dissociating gas model.

Substituting Eq. (23) into Eq. (24) and neglecting the squares of perturbation quantities we obtain

$$[C_{v_2} + \alpha_0(C_{v_1} - C_{v_2})] \frac{dT'}{dt} + [D + (C_{v_1} - C_{v_2})T_0] \left( \frac{d\alpha_e'}{dt} + \frac{d\alpha'}{dt} \right) + P_0 \frac{dV'}{dt} = 0 \quad (25)$$

Similarly, neglecting the second order terms of perturbation quantities, we obtain from Eq. (8)

$$P_0 \frac{dV'}{dt} + V_0 \frac{dP'}{dt} = (1 + \alpha_0)R \frac{dT'}{dt} + RT_0 \left( \frac{d\alpha_e'}{dt} + \frac{d\alpha'}{dt} \right) \quad (26)$$

Eq. (12) can be linearized in a similar manner to yield

$$\frac{d\alpha}{dt} = \frac{d\alpha_e'}{dt} + \frac{d\alpha'}{dt} = - \frac{\alpha'}{\tau} \quad (27)$$

where  $\tau$  represents the chemical relaxation time which is a positive constant in the present approximate calculations<sup>(6)</sup>. From Eq. (22), we obtain

$$\frac{ds}{dt} = -2R \frac{\alpha'}{\alpha_0(1-\alpha_0^2)} \frac{d\alpha}{dt} \quad (28)$$

where only the linear term in the coefficient of the right hand side term in Eq. (22) has been retained. Combining Eqs. (27) and (28) we obtain

$$\frac{ds}{dt} = \frac{2R}{\tau \alpha_0 (1-\alpha_0^2)} \alpha'^2 + O(\alpha'^3) \quad (29)$$

This equation shows that the entropy increases along the streamtube and is of second order in the perturbation quantity  $\alpha'$ . By Eq. (21),

we have

$$\frac{d\alpha_e}{dt} = \left(\frac{\partial\alpha_e}{\partial P}\right)_s \frac{dP}{dt} + \left(\frac{\partial\alpha_e}{\partial S}\right)_P \frac{dS}{dt} \quad (30)$$

By Eq. (23), we then have

$$\frac{d\alpha_e'}{dt} = \lambda_0 \frac{dP'}{dt} + \mu_0 \frac{dS}{dt} \quad (31)$$

where  $\lambda_0$  and  $\mu_0$  denote  $\left(\frac{\partial\alpha_e}{\partial P}\right)_s$  and  $\left(\frac{\partial\alpha_e}{\partial S}\right)_P$  respectively in the uniform flow. Eq. (31) shows that along the streamtube  $\frac{d\alpha_e'}{dt}$ ,  $\frac{dP'}{dt}$  and  $\frac{dS}{dt}$  are closely related. By Eq. (29), we realize that  $\frac{dS}{dt}$  is of second order in the perturbation quantity  $\alpha'$  and can be neglected in a linear theory. Therefore, we have

$$\frac{d\alpha_e'}{dt} = \lambda_0 \frac{dP'}{dt} \quad (31a)$$

Eliminating  $\frac{dT'}{dt}$ ,  $\frac{d\alpha'}{dt}$  and  $\frac{d\alpha_e'}{dt}$  from Eqs. (25), (26), (27) and (31a), we obtain the following equation:

$$L \frac{d^2 P'}{dt^2} + \frac{M}{\tau} \frac{dP'}{dt} + N \frac{d^2 v'}{dt^2} + \frac{N}{\tau} \frac{dv'}{dt} = 0 \quad (32)$$

where  $L = [C_{v_2} + \alpha_0(C_{v_1} - C_{v_2})] V_0 \quad (33)$

$$N = [C_{v_2} + \alpha_0(C_{v_1} - C_{v_2}) + (1 + \alpha_0)R] p_0 \quad (34)$$

$$M = L + \lambda_0 [(1 + \alpha_0)RD + RT_0(C_{v_1} - 2C_{v_2})] \quad (35)$$

If the perturbation specific volume  $v'$  is given as a function of  $t$  along a streamtube, we can find by Eq. (32) the perturbation pressure

$p'$  as a function of  $t$ . We cannot have the actual functional form of  $v'$  unless the equations of motion for the flow problem are solved.

Now, we shall take two points P, Q on a streamline (Fig. 2). These points P, Q are at the positions upstream and downstream far from the body, where all the perturbations almost vanish. Between these two points, the function  $v'$  in a subsonic flow problem can be represented as a Fourier series

$$v' = \sum_{n=1}^{\infty} a_n \sin 2\pi n \frac{t}{\gamma^*} + \sum_{n=1}^{\infty} b_n \cos 2\pi n \frac{t}{\gamma^*} \quad (36)$$

where  $\gamma^*$  denotes the time interval during which the material element moves from P to Q. The Fourier coefficients  $a_n$  and  $b_n$ , in principle, can be determined for a given flow problem.

Solution of Eq. (32) for  $v' = a_n \sin \omega_n t$

We shall study the solution  $p'$  for a Fourier component

$$v' = a_n \sin \omega_n t \quad (37)$$

where

$$\omega_n = \frac{2\pi n}{\gamma^*}$$

Substituting Eq. (37) into Eq. (32) we have

$$L \frac{d^2 p'}{dt^2} + \frac{M}{t} \frac{dp'}{dt} - a_n \omega_n^2 N \sin \omega_n t + a_n \omega_n \frac{N}{t} \cos \omega_n t = 0 \quad (38)$$

A solution of Eq. (38) can be written as

$$p' = a_n (X \sin \omega_n t + Y \cos \omega_n t) \quad (39)$$

where X, Y denote the integration constants which can be determined from the following simultaneous algebraic equations

$$\begin{aligned} (\tau \omega_n L) X + M Y &= -\tau \omega_n N \\ M X - (\tau \omega_n L) Y &= -N \end{aligned} \quad (40)$$

Solving Eq. (40) with respect to X and Y, we obtain

$$\begin{aligned} X &= -\frac{(\tau \omega_n)^2 LN + MN}{(\tau \omega_n L)^2 + M^2} \\ Y &= -\frac{(\tau \omega_n)(MN - LN)}{(\tau \omega_n L)^2 + M^2} \end{aligned} \quad (41)$$

These results can be discussed in the following cases: (1) equilibrium flow (2) frozen flow and (3) nonequilibrium or relaxing flow.

(1) Equilibrium flow,  $\tau \omega_n \rightarrow 0$

From Eq. (41) we have

$$X = -\frac{N}{M} \quad Y = 0 \quad (42)$$

then we obtain by Eq. (39)

$$p' = -a_n \frac{N}{M} \sin \omega_n t \quad (43)$$

We easily find that

$$\frac{dP'}{dV'} = \frac{dP'/dt}{dV'/dt} = X = -\frac{N}{M} \quad (44)$$

Eq. (22) shows that the equilibrium flow is isentropic.

Thus, Eq. (44) can be restated as follows:

$$\frac{dP'}{dV'} = \frac{dP}{dV} = -\rho_0^2 \left( \frac{dP}{d\rho} \right)_s = -\rho_0^2 a_e^2 = -\frac{a_e^2}{V_0^2} \quad (45)$$

where  $a_e$  denotes the equilibrium speed of sound in the undisturbed free stream. Therefore we have

$$a_e^2 = \frac{N}{M} V_0^2 \quad (46)$$

(2) Frozen flow  $\tau W_n \longrightarrow \infty$

From Eq. (41) we have

$$X = -\frac{N}{L} \quad Y = 0 \quad (47)$$

After the same treatment as above, we obtain

$$a_f^2 = \frac{N}{L} V_0^2 \quad (48)$$

where  $a_f$  denotes the frozen speed of sound in the undisturbed free stream. From Eqs. (46) and (48), we have

$$\frac{a_f^2}{a_e^2} = \frac{M}{L} = 1 + \frac{\lambda_0}{L} [(1+\alpha_0)RD + RT_0(C_{v_1} - 2C_{v_2})] \quad (49)$$

This can be also written as

$$\frac{a_f^2}{a_e^2} = 1 + \left(\frac{\partial \alpha_e}{\partial P}\right)_{s_0} \left[ - \frac{(\partial h / \partial \alpha)_{P,P}}{(\partial h / \partial P)_{P,\alpha}} \right]_0 a_f^2 \quad (50)$$

where

$$\left(\frac{\partial \alpha_e}{\partial P}\right)_{s_0} = \lambda_0$$

$$\left(\frac{\partial h}{\partial \alpha}\right)_{P,P,0} = \frac{1}{1+\alpha_0} [(C_{v1} - 2C_{v2})T_0 + (1+\alpha_0)D]$$

$$\left(\frac{\partial h}{\partial P}\right)_{P,\alpha,0} = -\frac{1}{(1+\alpha_0)R} \frac{P_0}{P_0^2} [C_{v2} + \alpha_0(C_{v1} - C_{v2}) + (1+\alpha_0)R]$$

$h = E + \frac{P}{\rho}$  denotes the specific enthalpy and suffix "o" denotes free stream quantities. Eq. (50) which relates the quantities  $a_f^2$  and  $a_e^2$  has been previously obtained by Clarke<sup>(7)</sup> who also has shown that

$$\frac{1}{k^2} = \frac{a_f^2}{a_e^2} > 1 \quad (51)$$

(3) Nonequilibrium or relaxing flow,  $\tau \omega_n = \text{finite}$

Now we can rewrite the results in Eq. (41) as follows:

$$X = - \frac{[(\tau \omega_n)^2 + \frac{1}{k^2}] a_f^2}{[(\tau \omega_n)^2 + \frac{1}{k^4}] v_0^2} \quad (52)$$

$$Y = - \frac{\tau \omega_n \left[\frac{1}{k^2} - 1\right] a_f^2}{[(\tau \omega_n)^2 + \frac{1}{k^4}] v_0^2}$$

From Eq. (39) we obtain

$$P' = - \frac{a_n}{v_0^2} \frac{\left\{ [(\tau \omega_n)^2 + \frac{1}{k^2}]^2 a_f^4 + (\tau \omega_n)^2 a_f^4 \left(\frac{1}{k^2} - 1\right)^2 \right\}^{\frac{1}{2}}}{(\tau \omega_n)^2 + \frac{1}{k^4}} \sin(\omega_n t + \varphi) \quad (53)$$

where

$$\phi = \tan^{-1} \left\{ \frac{\tau \omega_n \left( \frac{1}{k^2} - 1 \right)}{(\tau \omega_n)^2 + \frac{1}{k^2}} \right\} \quad (54)$$

Eq. (54) also reveals the following facts:

$$\text{for } \tau \omega_n = \text{finite}, \quad 0 < \phi < \frac{\pi}{2}$$

$$\text{for } \tau \omega_n = 0 \text{ or } \tau \omega_n \rightarrow \infty, \quad \phi = 0$$

Comparing Eqs. (37) and (53), we may conclude  $v'$  and  $p'$  are out of phase due to relaxation effects. In equilibrium and frozen flows, however,  $v'$  and  $p'$  are in phase and  $\phi = 0$ . The consequence of the phase shift due to relaxation effects, of course, is to cause a relaxation drag as can be seen from application of the above results in Eq. (18).

#### Subsonic Drag due to Nonequilibrium Effects

In the simple case discussed in the previous section, the integral in Eq. (18) becomes

$$\begin{aligned} - \int_{z=-\infty}^{z=+\infty} p \, dv &= - \int_{z=-\infty}^{z=+\infty} (p_0 + p') \, d(v_0 + v') \\ &= - \int_{z=-\infty}^{z=+\infty} p_0 \, dv' - \int_{z=-\infty}^{z=+\infty} p' \, dv' = - p_0 \int_0^{v'} \frac{dv'}{a^2} dt - \int_0^{v'} p' \frac{dv'}{a^2} dt \\ &= \frac{(\omega_n \tau) a_f^2 \left( \frac{1}{k^2} - 1 \right)}{\frac{1}{k^2} + (\omega_n \tau)^2} \frac{a_n^2}{v_0^2} \frac{\omega_n v''}{2} \geq 0 \end{aligned} \quad (55)$$

In the general case,  $v'$  should be expressed as in Eq. (36). The corresponding solution,  $p'$ , can also be represented by Fourier series. Thus, the integral in Eq. (18) can be obtained as follows:

$$-\int_{z=-\infty}^{z=+\infty} p dv = \sum_{n=1}^{\infty} A_n^2 + \sum_{n=1}^{\infty} B_n^2 \quad (56)$$

where

$$A_n^2 = \frac{\omega_n v^*}{2} \frac{a_n^2}{V_0^2} \frac{(\omega_n \tau) a_f^2 \left(\frac{1}{k^2} - 1\right)}{\frac{1}{k^2} + (\omega_n \tau)^2} \geq 0$$

$$B_n^2 = \frac{\omega_n v^*}{2} \frac{b_n^2}{V_0^2} \frac{(\omega_n \tau) a_f^2 \left(\frac{1}{k^2} - 1\right)}{\frac{1}{k^2} + (\omega_n \tau)^2} \geq 0 \quad (57)$$

Thus, the integral in Eq. (18) always take the positive sign. By Eq. (17), the drag experienced by the body in subsonic flow is

$$D = \rho_0 \sum_{n=1}^{\infty} \iint (A_n^2 + B_n^2) d\sigma_0 \geq 0 \quad (58)$$

In equilibrium flow,  $\omega_n \tau \rightarrow 0$ ,  $A_n^2 = B_n^2 = 0$  hence,  $D = 0$ .

In frozen flow,  $\omega_n \tau \rightarrow \infty$ ,  $A_n^2 = B_n^2 = 0$  again,  $D = 0$ . In nonequilibrium flow, the subsonic drag must take the positive sign.

### Discussion

We have studied the general problem of subsonic drag due to a nonequilibrium effect. In subsonic adiabatic flow, the work performed against the drag is transformed into the internal energy of gas. Because of the irreversible chemical changes in the gas there is an increase in the entropy of the gas. In the present calculations,

the entropy change has been neglected in the first approximation. A higher approximation may be considered which includes the effects of entropy change. Thus, in Eq. (18), we write

$$\begin{aligned}
 -\int_{Z=-\infty}^{Z=+\infty} p dv &= -\int_{Z=-\infty}^{Z=+\infty} p_0 dv' - \int_{Z=-\infty}^{Z=+\infty} p' dv' \\
 &= -P_0 \int_0^{\nu^*} \frac{d\nu'}{dt} dt - \left[ \nu' p' \Big|_0^{\nu^*} - \int_0^{\nu^*} \nu' \frac{dp'}{dt} dt \right] \\
 &= -P_0 [\nu'(\nu^*) - \nu'(0)] - p'(\nu^*) \nu'(\nu^*) + p'(0) \nu'(0) \\
 &\quad + \int_0^{\nu^*} \nu' \frac{dp'}{dt} dt \\
 &= \int_0^{\nu^*} \nu' \frac{dp'}{dt} dt
 \end{aligned} \tag{59}$$

where we can take  $\nu'(\nu^*) = \nu'(0) = 0$  by having P, Q (Fig. 2) chosen to be such that  $\nu_p = \nu_q = \nu_0$ . By Eq. (31), we have

$$\int_0^{\nu^*} \nu' \frac{dp'}{dt} dt = \frac{1}{\lambda_0} \int_0^{\nu^*} \nu' \left[ \frac{d\alpha'}{dt} - \mu \cdot \frac{ds}{dt} \right] dt \tag{60}$$

In the linear theory in the preceding sections, the term containing  $\frac{ds}{dt}$  in Eq. (60) has been neglected because  $\frac{ds}{dt} = o(\alpha'^2)$ . Within the accuracy of this approximation, we have obtained the drag, the leading term of which is of the second order in perturbation quantities. If we take account of the increase in entropy of

the dissociating gas we may obtain a second approximate theory which would be correct to terms of the third order in perturbation quantities.

In the conventional supersonic flow, disturbances propagate along the Mach waves. Mechanical energy is carried away to infinity along these waves. A part of the conventional supersonic drag, i.e., wave drag, must be associated with the mechanical energy loss of these wave disturbances. When we consider the problem of nonequilibrium supersonic flow, the drag would consist of the supersonic wave drag and the nonequilibrium drag. In such a case, the disturbances propagating along the Mach waves are damped by the nonequilibrium effects. As a result, the wave drag may decrease from the conventional value. Unless the flow problem is completely solved, it would be rather difficult to forecast the total effects of nonequilibrium on the supersonic drag.

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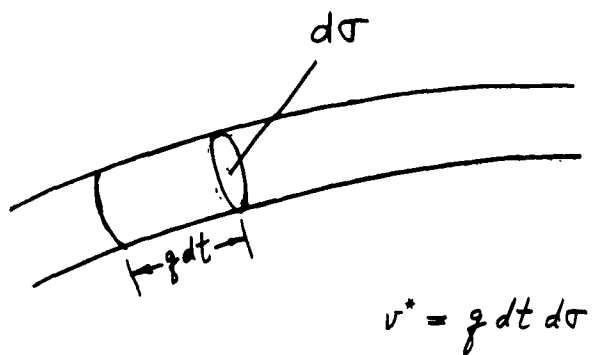


Fig. 1 A Stream tube and a small material element



Fig. 2 The geometrical configuration of the points P & Q