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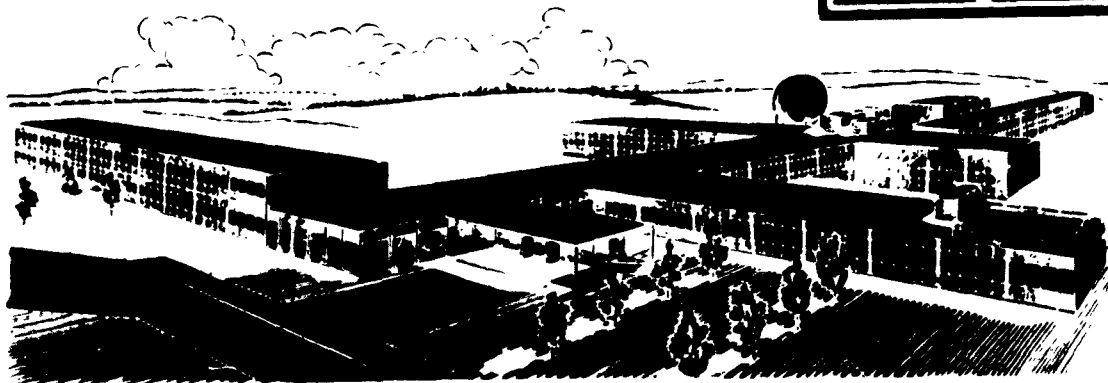
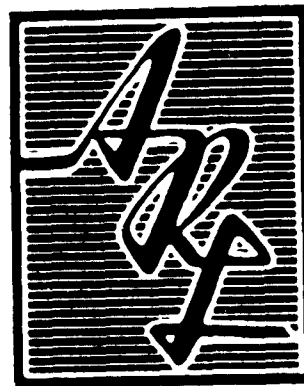
**A STUDY OF THE DIFFERENTIAL EQUATIONS OF
COUPLED VIBRATIONS AND FREE CONVECTION
FROM A HEATED HORIZONTAL CYLINDER**

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DECEMBER 1961

AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE



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In this report, the basic equations and
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**AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO**

FOREWORD

This final technical report was prepared by the Massachusetts Institute of Technology, Cambridge, Massachusetts, on Contract AF 33(616)-6076 for the Aeronautical Research Laboratory, Office of Aerospace Research, United States Air Force. The work reported herein was accomplished on Task 70138, "Research on a Novel Technique of Measuring Very High Temperature Gas" of Project 7064, "Research on Aerodynamic Fields", under the technical cognizance of Dr. Max G. Scherberg and Mr. Erich Soehngen of the Thermo-Mechanics Research Branch of ARL.

Part I is based on a Mechanical Engineering thesis entitled, "An Analytical Study of the Effect of Vibrations on Heat Transfer from a Heated Horizontal Cylinder" presented in 1960 by T. Chiang, Research Assistant, Research Laboratory for Heat Transfer in Electronics, MIT.

Appreciation is extended to Prof. R. J. Nickerson for his valuable original suggestions and for his critical reading of the final manuscript. Prof. Nickerson suggested the analytical approach by means of which the free-convection solution described in Mr. Chiang's thesis was obtained.

ABSTRACT

In this report, the basic equations and the boundary conditions which govern the problem of coupled transverse vibrations and free convection from a heated horizontal cylinder are presented. By applying a method developed by C. C. Lin, it is shown: 1) that the presence of harmonic oscillations modify the steady-flow solution only when pressure gradients are present; 2) that the modifying forces have their most pronounced effect on the fluid closest to the surface; and 3) that the product $a\omega$ is a measure of the magnitude of the modifying forces. The use of the quantity $a\omega$ as a measure of the magnitude of the influence of vibrations on free convection agrees with experimental correlations.

By transforming the differential equations into dimensionless form, it is shown that four dimensionless parameters are needed to fully describe the flow. Four sets of dimensionless parameters and their corresponding differential equations are presented; the advantages and disadvantages of each set are discussed. A perturbation method is applied to one set of equations, and the zeroth-order solution is obtained. This zeroth-order solution, which corresponds to free convection, agrees with Hermann's analysis for a heated horizontal cylinder.

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NOMENCLATURE

Latin Alphabet

a	sinusoidal displacement amplitude of vibration
c_p	specific heat at constant pressure
f	frequency in cycles per second
F	function defined by Eq. (59)
\mathcal{F}	function defined by Eq. (15)
g	gravitational acceleration
G	function defined by Eq. (60)
Gr	Grashoff number, $\frac{R^3 g \beta \Theta_w}{\nu^2}$
h	local coefficient of heat transfer
k	thermal conductivity
M	dimensionless stream function
Nu	Nusselt number, $\frac{hR}{k}$
p	pressure
\bar{p}	pressures defined by Eqs. (8)
p_1	
Pr	Prandtl number, $\frac{c_p \mu}{k}$
\mathcal{R}	function defined by Eq. (16)
R	radius of cylinder
$(Re)_{af}$	vibration Reynolds number, $\frac{a \omega R}{\nu}$
t	dimensional and dimensionless time
T	temperature
T_w	wall temperature

T_{∞}	ambient temperature
u	dimensional velocities
v	
\bar{u}	velocities defined by Eqs. (8)
\bar{v}	
u_1	
v_1	
U	free-stream velocity
\bar{U}	free-stream velocities defined by Eqs. (8)
U_1	
x	dimensional and dimensionless coordinates
y	

Greek Alphabet

α	quantity defined by Eqs. (32) and Table 1
β	coefficient of volumetric expansion
γ	quantity defined by Eqs. (32) and Table 1
δ	boundary-layer thickness
δ_0	"AC" boundary-layer thickness, $\sqrt{\frac{\nu}{\omega}}$
ξ	function defined by Eq. (22)
Θ	temperature potential, $T - T_{\infty}$
Θ_w	wall temperature potential, $T_w - T_{\infty}$
$\bar{\Theta}$	functions defined by Eqs. (8)
Θ_1	
λ	angle between vertical and direction of vibration
μ	dynamic viscosity

- ν kinematic viscosity
- χ function defined by Eq. (21)
- ρ density
- ϕ dimensionless temperature potential, $\frac{T - T_{\infty}}{T_w - T_{\infty}}$
- ψ stream function
- ω circular frequency of oscillation, radians per second
- Ω vibration parameter B defined in Table 2

Special symbols for Chapter II

Bars over quantities indicate time averages

Subscript 1 indicates the quantity is a function of time

Special symbols for Chapter III

Flexes indicate dimensionless quantities

Superscripts in parenthesis above the dimensionless stream function, M, refer to the Case Numbers in Table 1.

Subscripts

0 zero-order quantity

1 first-order quantity

2 second-order quantity

x } partial differentiation with respect to dimensionless coordinates
y }

I. INTRODUCTION

During the past few years, considerable interest has arisen with regard to the interaction between vibrations and convective heat transfer. These coupled phenomena have been studied experimentally by two methods. The first method utilizes a sound field in a fluid medium surrounding a heated object; in the second method, a heated object is caused to vibrate by mechanical means in an otherwise undisturbed fluid medium. Both methods result in a relative velocity vector between the heated surface and the fluid medium which changes periodically in magnitude and direction.

One of the earliest investigations of the influence of vibrations on convective heat transfer was performed by Martinelli and Boelter⁽¹⁾ using an apparatus in which a heated horizontal cylinder was vibrated vertically in a tank of water. For sufficiently intense vibrations, Martinelli and Boelter reported that the overall heat-transfer coefficient was increased by as much as four times its value without vibrations. More recently, Fand and Kaye^(2, 3) have made experimental investigations of the influence of acoustical and mechanical vibrations on heat transfer by free convection from horizontal cylinders. Their data show that when the intensity of vibrations, defined as the product of amplitude and frequency, af , exceeds a certain "critical value" (approximately 0.3 ft/sec), the coefficient of heat transfer increases significantly. The influence of vibrations on heat transfer has also been studied by Lemlich⁽⁵⁾, Tsui⁽⁶⁾, Shine⁽⁷⁾, and Sprott, Holman, and Durand⁽⁸⁾.

Fand and Kaye^(3, 4) have carried out flow-visualization studies which demonstrate that the boundary layer flow about a heated horizontal cylinder changes radically in the presence of super-critical vibrations. Their photographs of boundary-layer flow, using smoke as the indicating medium, show that vertical (transverse) vibrations affect the boundary layer in a different way than do horizontal (transverse) vibrations. In the case of horizontal super-critical vibrations, a vortex-type flow, called thermoacoustic streaming, was observed to occur about the cylinder; whereas, in the case of vertical super-critical vibrations, a wide zone of turbulence surrounded the test cylinder. Thermoacoustic streaming has also been observed by Sprott, Holman, and Durand⁽⁸⁾ by shadowgraph methods.

The analytical aspects of the coupling between vibrations and free convection constitute a very difficult problem—to date no theoretical analysis of this problem has appeared. However, analyses of several related problems have been successfully carried out. Stokes⁽⁹⁾, and later Rayleigh⁽¹⁰⁾, analyzed the case of an infinitely long plate executing harmonic oscillations parallel to its own plane in a viscous fluid otherwise at rest. This case was relatively simple because there was no temperature difference between the plate and the surrounding fluid, and also because the motion, being independent of the coordinate along the plate, satisfies a linear differential equation. Stokes' solution is known as "shear-wave" flow. Schlichting⁽¹¹⁾ considered analytically the case of a cylinder executing harmonic oscillations in a fluid at rest with no temperature difference between the cylinder and the fluid. He used boundary-layer approximations and then applied an iterative

approach against a steady mean flow, which is identically zero, in order to solve the governing equations. He found that the first approximation to the solution represents a periodic flow, and the second approximation to the solution contains a steady-flow term which is independent of time. Schlichting's analytical results explained the acoustic streaming phenomena observed earlier by Andrade⁽¹²⁾. Schlichting's original analysis of acoustic streaming has been refined and extended by Holtmark et al⁽¹³⁾, Raney et al⁽¹⁴⁾, and Skavlem and Tjøtta⁽¹⁵⁾. The effect of free-stream oscillations on laminar boundary layers have been considered by Ostrach⁽¹⁶⁾, Moore⁽¹⁷⁾, and Cheng and Elliott⁽¹⁸⁾.

Lighthill⁽¹⁹⁾ studied the laminar boundary-layer response to fluctuations in stream velocity by using an approximate boundary-layer method. Nickerson⁽²⁰⁾, in a combined theoretical and experimental investigation, considered the effect of free-stream oscillations on the laminar boundary layer from a flat plate and the associated forced-convective heat-transfer problem; the fluctuations in the free stream were assumed to be small compared to the steady mean velocity. Nickerson's analysis predicted a negligible change in the coefficient of heat-transfer from the plate surface; experimental data corroborated this analytical result. Lin⁽²¹⁾ made a theoretical analysis on the motion in boundary layers with a rapidly oscillating external flow. His analysis was not limited to small oscillations in the free stream. Lin's results indicated that the influence of oscillatory motion on the mean boundary-layer profile is contingent upon the existence of pressure gradients

outside the boundary layer and predicted the possibility of a periodic back flow.

The foregoing brief literature review shows that several experimental investigations of the interaction between vibrations and free convection have been performed, but no theoretical analysis of this interaction has been carried out. This report represents a first step toward the solution of this complex problem. More specifically, this report contains a formulation, partial solution, and discussion of the system of differential equations which describe the coupling between free convection and transverse vibrations of a heated horizontal cylinder.

II. BASIC EQUATIONS AND THEIR PROPERTIES

2.1 Coordinate System

The system of differential equations for the velocity and temperature distributions near a heated horizontal cylinder subjected to transverse harmonic vibrations will be derived in this report. It will be assumed that the cylinder is sufficiently long so that there is no flow in the axial direction; hence the problem is two-dimensional.

An oscillating reference frame which is fixed to the cylinder will be used for the analysis; thus, for an observer standing on the cylinder, the cylinder appears stationary, and the fluid at infinity executes harmonic oscillations. This oscillating frame of reference will give results which are equivalent to those using a reference frame fixed in space, due to the fact that any inertia force per unit volume is balanced by an equal pressure gradient. This point has been explained by Lighthill⁽¹⁹⁾.

A curvilinear orthogonal system of coordinates will be used to simplify the analysis: The x-axis of this coordinate system is along the wall of the cylinder, and the y-axis is perpendicular to it, as shown in Fig. 1.

2.2 Basic Equations

Tollmien⁽²²⁾ derived expressions for the complete Navier-Stokes equations in the coordinate system indicated in Fig. 1. These equations will be used in a modified form in the analysis which follows. In this modification, the density variation will be introduced exclusively in

the buoyancy term in the momentum equations, and the additional assumption will be made that the radius of the cylinder is large compared to the boundary-layer thickness.

By applying the usual boundary-layer assumptions and adding the buoyancy-force term in the momentum equations derived by Tollmien, the boundary-layer equations become:

$$\text{Continuity: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\text{Momentum: } \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + g\beta\theta \sin \frac{x}{R} \quad (2)$$

$$\text{Energy: } \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 \theta}{\partial y^2} \quad (3)$$

It should be noted that these equations do not apply near the upper portion of the cylinder where the normal component of velocity becomes of the same order of magnitude as the tangential component, because this is contrary to the boundary-layer assumptions.

Outside the boundary layer, the nonsteady Bernoulli equation holds.

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \quad (4)$$

2.3 Boundary Conditions

The most general form of the boundary conditions for Eqs. (1) through (4) are:

$$\begin{aligned} \text{when } y = 0, \quad u = v = 0, \quad \theta = \theta_w \\ \text{when } y \rightarrow \infty, \quad u \rightarrow U(x, t), \quad \theta \rightarrow 0 \end{aligned} \quad (5)$$

These boundary conditions can be made more specific by choosing various types of velocity conditions far from the cylinder. The class of boundary conditions considered in this report are those in which the fluid far from the cylinder executes harmonic vibrations in one direction. The mathematical expression for this class of boundary conditions is:

$$\text{when } y \rightarrow \infty, u \rightarrow U = 2 a \omega \sin \omega t \sin \left(\frac{x}{R} + \lambda \right) \quad (6)$$

This boundary condition corresponds to the cylinder being vibrated in an otherwise undisturbed fluid with the coordinate system fixed on the cylinder. It also approximates the case of a stationary cylinder in a strong transverse sound field when the wave length of the sound field is large compared to the diameter of the cylinder. The direction of these vibrations can be specified by an appropriate choice of λ in Eq. (6). Thus, for vertical vibrations, $\lambda = 0$; for horizontal vibrations, $\lambda = \pi/2$.

2.4 Method of Solution Developed by C. C. Lin

C. C. Lin⁽²¹⁾ has developed a method of attacking unsteady forced-flow boundary-layer problems which supplies a great amount of physical insight. This method applies to periodic flows when the "AC" boundary-layer thickness is small compared to the steady boundary-layer thickness; that is, when

$$\frac{\nu}{\delta_{2f}} \ll 1 \quad (7)$$

In Lin's method, the assumption is made that the flow can be divided into a steady part and an oscillating part in the following form:

$$\begin{aligned}
 u(x, y, t) &= \bar{u}(x, y) + u_1(x, y, t) & \bar{u}_1 &= 0 \\
 v(x, y, t) &= \bar{v}(x, y) + v_1(x, y, t) & \bar{v}_1 &= 0 \\
 U(x, t) &= \bar{U}(x) + U_1(x, t) & \bar{U}_1 &= 0 \\
 p(x, t) &= \bar{p}(x) + p_1(x, t) & \bar{p}_1 &= 0 \\
 \theta(x, y, t) &= \bar{\theta}(x, y) + \theta_1(x, y, t) & \bar{\theta}_1 &= 0
 \end{aligned} \tag{8}$$

With these assumptions, the boundary-layer equations are separated into two sets: one for the steady part of the flow and one for the oscillating part. The steady set of equations is obtained by taking the time average of the complete equations. The equations for the oscillating flow are obtained by subtracting the steady equations from the complete equations according to the following procedure.

First, Eq. (4) is used to eliminate the pressure term from Eq. (2). Then Eqs. (8) are substituted into Eqs. (1), (2), and (3). This manipulation results in the following set of equations:

$$\frac{\partial(\bar{u} + u_1)}{\partial x} + \frac{\partial(\bar{v} + v_1)}{\partial y} = 0 \tag{9}$$

$$\begin{aligned}
 \frac{\partial u_1}{\partial t} + (\bar{u} + u_1) \frac{\partial(\bar{u} + u_1)}{\partial x} + (\bar{v} + v_1) \frac{\partial(\bar{u} + u_1)}{\partial y} &= \frac{\partial U_1}{\partial t} + \\
 + (\bar{U} + U_1) \frac{\partial(\bar{U} + U_1)}{\partial x} + \nu \frac{\partial^2(\bar{u} + u_1)}{\partial y^2} + g\beta(\bar{\theta} + \theta_1) \sin \frac{x}{R} &
 \end{aligned} \tag{10}$$

$$\begin{aligned} & \frac{\partial \theta_1}{\partial t} + (\bar{u} + u_1) \frac{\partial (\bar{\theta} + \theta_1)}{\partial x} + (\bar{v} + v_1) \frac{\partial (\bar{\theta} + \theta_1)}{\partial y} = \\ & = \frac{k}{\rho c_p} \frac{\partial^2 (\bar{\theta} + \theta_1)}{\partial y^2} \end{aligned} \quad (11)$$

When the time averages of Eqs. (9), (10), and (11) are taken, the following expressions for the time independent part of the flow are obtained:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (12)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} + g \beta \bar{\theta} \sin \frac{x}{R} + \mathcal{F} \quad (13)$$

$$\bar{u} \frac{\partial \bar{\theta}}{\partial x} + \bar{v} \frac{\partial \bar{\theta}}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 \bar{\theta}}{\partial y^2} - \mathcal{Q} \quad (14)$$

$$\text{where } \mathcal{F} \equiv \overline{U_1 \frac{\partial U_1}{\partial x}} - \overline{u_1 \frac{\partial u_1}{\partial x}} - \overline{v_1 \frac{\partial u_1}{\partial y}} \quad (15)$$

$$\mathcal{Q} \equiv \overline{u_1 \frac{\partial \theta_1}{\partial x}} + \overline{v_1 \frac{\partial \theta_1}{\partial y}} \quad (16)$$

$$\text{B.C. } y = 0, \bar{u} = \bar{v} = 0, \bar{\theta} = \theta_w \quad (17)$$

$$y \rightarrow \infty, \bar{u} \rightarrow 0, \bar{\theta} \rightarrow 0$$

The equations describing the oscillating part of the flow are obtained by subtracting Eqs. (12), (13), and (14) from Eqs. (9), (10), and (11). These equations are:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \quad (18)$$

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial y^2} - \frac{\partial U_1}{\partial t} = \mathcal{F} \quad (19)$$

$$\frac{\partial \theta_1}{\partial t} - \frac{k}{\rho c_p} \frac{\partial^2 \theta_1}{\partial y^2} = \mathcal{G} \quad (20)$$

$$\begin{aligned} \text{where } \mathcal{F} \equiv & U_1 \frac{\partial U_1}{\partial x} - u_1 \frac{\partial u_1}{\partial x} - v_1 \frac{\partial u_1}{\partial y} - \mathcal{F} + \\ & - \bar{u} \frac{\partial u_1}{\partial x} - \bar{v} \frac{\partial u_1}{\partial y} + \\ & - u_1 \frac{\partial u_1}{\partial x} - v_1 \frac{\partial \bar{u}}{\partial y} \end{aligned} \quad (21)$$

$$\begin{aligned} \mathcal{G} \equiv & - u_1 \frac{\partial \theta_1}{\partial x} - v_1 \frac{\partial \theta_1}{\partial y} + \mathcal{G} + \\ & - \bar{u} \frac{\partial \theta_1}{\partial x} - \bar{v} \frac{\partial \theta_1}{\partial y} + \\ & - u_1 \frac{\partial \bar{\theta}}{\partial x} - v_1 \frac{\partial \bar{\theta}}{\partial y} \end{aligned} \quad (22)$$

$$\text{B.C. } y = 0, u_1 = v_1 = 0, \theta_1 = 0 \quad (23)$$

$$y \rightarrow \infty, u_1 \rightarrow 2a\omega \sin \omega t \sin \left(\frac{x}{R} + \lambda \right), \theta_1 \rightarrow 0$$

Lin suggests an iterative approach when his method is applied to forced flows. In forced flows, the coupling between the velocity and temperature fields can be considered negligible in many cases. Hence, for forced flows, the equations can be solved for the velocity distribution independently of the temperature distribution. However, the strong coupling between the velocity and the temperature fields which exists in

the case of free convection presents a serious obstacle to obtaining a solution by this approach; for, with free convection, the velocity and temperature distributions cannot be solved independently.

2.5 Characteristics of the Flow Shown by Lin's Method

Although Eqs. (12) through (23) are difficult to solve, they do show several interesting characteristics of unsteady flows. First, they show that the steady part of the flow, when oscillations are present, is different from the steady solution with no oscillations-- due to the terms \mathcal{F} and \mathcal{Z} which appear in the momentum and energy equations. These modifying terms depend upon the unsteady part of the solution.

Another characteristic of these equations is related to the way in which the quantity \mathcal{F} varies. For example, if the fluid far from the cylinder is vibrating in an arbitrary direction, Eq. (6) represents the unsteady component of the flow far from the cylinder. Thus, utilizing Lin's notation:

$$U_1 = 2 a \omega \sin \omega t \sin \left(\frac{x}{R} + \lambda \right) \quad (24)$$

$$\frac{\partial U_1}{\partial x} = 2 \frac{a\omega}{R} \sin \omega t \cos \left(\frac{x}{R} + \lambda \right) \quad (25)$$

Substituting Eqs. (24) and (25) into the defining Eq. (15) and taking the time averages gives:

$$\mathcal{F} = \frac{2(a\omega)^2}{R} \sin \left(\frac{x}{R} + \lambda \right) \cos \left(\frac{x}{R} + \lambda \right) - \overline{u_1 \frac{\partial u_1}{\partial x}} - \overline{v_1 \frac{\partial u_1}{\partial y}} \quad (26)$$

Inspection of Eq. (26) shows that:

$$\text{At } y = 0, u_1 = v_1 = 0, \mathcal{F} = 2 \frac{(a\omega)^2}{R} \sin\left(\frac{x}{R} + \lambda\right) \cos\left(\frac{x}{R} + \lambda\right) \quad (27)$$

$$\text{At } y \rightarrow \infty, u_1 \rightarrow U_1, \frac{\partial u_1}{\partial y} \rightarrow 0, \mathcal{F} \rightarrow 0$$

The preceding calculation points out the interesting fact that the quantity \mathcal{F} has its maximum effect at the wall. Also, this calculation shows that the quantity $a\omega$ is an important parameter in determining the amount by which the flow is modified due to the oscillations.

This conclusion agrees with experimental results.

Another important characteristic of the flow which Lin demonstrated by transforming Eq. (15) is that a pressure gradient outside the boundary layer is necessary in order that the quantity \mathcal{F} be different from zero.

Lin's method is an iterative method of solution and makes use of the basic equations in a dimensional form. For most other analytical methods of attack, the differential equations in dimensionless form are the most useful. These dimensionless equations also supply important hints as to which dimensionless groups best describe the problem. Such dimensionless equations will be discussed in the next chapter.

III. DIMENSIONLESS EQUATIONS AND SPECIFIC SOLUTIONS

3.1 Stream Function

A simplification of the differential equations and boundary conditions can be obtained by use of the stream function, ψ . The continuity equation is satisfied identically by ψ ; thus,

$$u \equiv \frac{\partial \psi}{\partial y}, \quad v \equiv -\frac{\partial \psi}{\partial x} \quad (28)$$

The energy and momentum equations take the following form:

$$\begin{aligned} \frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = 2 a \omega \cos \omega t \sin \left(\frac{x}{R} + \lambda \right) + \\ + 4 \frac{(a \omega)^2}{R} \sin^2 \omega t \sin \left(\frac{x}{R} + \lambda \right) \cos \left(\frac{x}{R} + \lambda \right) + \nu \frac{\partial^3 \psi}{\partial y^3} + g \beta \theta \sin \frac{x}{R} \end{aligned} \quad (29)$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 \theta}{\partial y^2} \quad (30)$$

$$\begin{aligned} \text{B.C. } y = 0, \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0, \quad \theta = \theta_w \\ y \rightarrow \infty, \quad \frac{\partial \psi}{\partial y} \rightarrow 2 a \omega \sin \omega t \sin \left(\frac{x}{R} + \lambda \right), \quad \theta \rightarrow 0 \end{aligned} \quad (31)$$

In deriving the above expression for the momentum equation, Eq. (25), the expression for the free-stream velocity, Eq. (6), was substituted into the non-steady Bernoulli equation, Eq. (4), and the result of this substitution was used to eliminate the pressure term from Eq. (2).

3.2 General Dimensionless Equations

Eqs. (29) and (30) and the boundary conditions (31) can be made dimensionless by the following substitutions:

$$\begin{aligned}\hat{t} &= \omega t & M &= \frac{1}{\gamma} \psi \\ \hat{x} &= \frac{x}{R} & \phi &= \frac{\theta}{\theta_w}\end{aligned}\quad (32)$$

$$\hat{y} = \alpha y$$

In terms of these dimensionless quantities, Eqs. (29), (30), and (31) take the following form:

$$\begin{aligned}\frac{R\omega}{\alpha\gamma} M_{\hat{y}\hat{t}} + M_{\hat{y}} M_{\hat{x}\hat{x}} - M_{\hat{x}} M_{\hat{y}\hat{y}} &= \frac{2a\omega^2 R}{(\alpha\gamma)^2} \cos \hat{t} \sin(\hat{x} + \lambda) + \\ + 4 \left(\frac{a\omega}{\alpha\gamma}\right)^2 \sin^2 \hat{t} \sin(\hat{x} + \lambda) \cos(\hat{x} + \lambda) &+ \frac{\partial \alpha R}{\gamma} M_{\hat{y}\hat{y}\hat{y}} +\end{aligned}\quad (33)$$

$$+ \frac{g\beta\theta_w R}{(\alpha\gamma)^2} \phi \sin \hat{x}$$

$$\frac{R\omega}{\alpha\gamma} \phi_{\hat{t}} + M_{\hat{y}} \phi_{\hat{x}} - M_{\hat{x}} \phi_{\hat{y}} = \frac{k\alpha R}{\rho c_p \gamma} \phi_{\hat{y}\hat{y}}\quad (34)$$

$$\text{B.C. } \hat{y} = 0, M_{\hat{x}} = M_{\hat{y}} = 0, \phi = 1\quad (35)$$

$$\hat{y} \rightarrow \infty, M_{\hat{y}} \rightarrow \frac{2a\omega}{\alpha\gamma} \sin \hat{t} \sin(\hat{x} + \lambda), \phi \rightarrow 0$$

where the subscripts denote partial differentiation with respect to dimensionless coordinates.

3.3 Specific Dimensionless Equations

From this point on, all coordinates in this report will be dimensionless, and no flexes will be used to designate them as such, in order to simplify the writing.

The quantities α and γ which appear in Eqs. (33) through (35) are as yet unspecified. These quantities can now be chosen so as to make as many of the coefficients in Eq. (33) equal to unity as possible. Two is the maximum number that can be made equal to unity simultaneously. Four possible choices for α and γ are indicated in Table 1.

When the pairs of values for α and γ in Table 1 are substituted into Eqs. (33), (34), and (35), the following four sets of equations are obtained:

Case I,

$$\begin{aligned} \left(\frac{\omega_R^2}{g\beta\Theta_w}\right)^{\frac{1}{2}} M_{yt}^{(1)} + M_y^{(1)} M_{yx}^{(1)} - M_x^{(1)} M_{yy}^{(1)} &= 2\left(\frac{a}{R}\right) \left(\frac{\omega_R^2}{g\beta\Theta_w}\right) \cos t \sin(x + \lambda) + \\ + 4\left(\frac{a}{R}\right)^2 \left(\frac{\omega_R^2}{g\beta\Theta_w}\right) \sin^2 t \sin(x + \lambda) \cos(x + \lambda) &+ M_{yyy}^{(1)} + \phi \sin x \end{aligned} \quad (36)$$

$$\left(\frac{\omega_R^2}{g\beta\Theta_w}\right)^{\frac{1}{2}} \phi_t + M_y^{(1)} \phi_x - M_x^{(1)} \phi_y = \frac{1}{Pr} \phi_{yy} \quad (37)$$

$$\begin{aligned} \text{B.C. } y = 0, \quad M_x^{(1)} = M_y^{(1)} = 0, \quad \phi = 1 \\ y \rightarrow \infty, \quad M_y^{(1)} \rightarrow 2\left(\frac{a}{R}\right) \left(\frac{\omega_R^2}{g\beta\Theta_w}\right) \sin t \sin(x + \lambda), \quad \phi \rightarrow 0 \end{aligned} \quad (38)$$

Case II,

$$\begin{aligned} M_{yt}^{(2)} + M_y^{(2)} M_{yx}^{(2)} - M_x^{(2)} M_{yy}^{(2)} &= 2\frac{a}{R} \cos t \sin(x + \lambda) + \\ + 4\left(\frac{a}{R}\right)^2 \sin^2 t \sin(x + \lambda) \cos(x + \lambda) &+ M_{yyy}^{(2)} + \\ + \left(\frac{g\beta\Theta_w R^3}{\nu^2}\right) \left(\frac{\nu}{\omega_R^2}\right)^2 \phi \sin x \end{aligned} \quad (39)$$

$$\phi_t + M_y^{(2)} \phi_x - M_x^{(2)} \phi_y = \frac{1}{Pr} \phi_{yy} \quad (40)$$

$$\text{B.C. } y = 0, M_y^{(2)} = M_x^{(2)} = 0, \phi = 1$$

$$y \rightarrow \infty, M_y^{(2)} \rightarrow 2 \frac{a}{R} \sin t \sin(x + \lambda), \phi \rightarrow 0 \quad (41)$$

Case III,

$$\begin{aligned} \left(\frac{R}{a}\right) M_{yt}^{(3)} + M_y^{(3)} M_{yx}^{(3)} - M_x^{(3)} M_{yy}^{(3)} &= 2 \left(\frac{R}{a}\right) \cos t \sin(x + \lambda) + \\ + 4 \sin^2 t \sin(x + \lambda) \cos(x + \lambda) &+ M_{yyy}^{(3)} + \\ + \left(\frac{g\beta\theta_w R^3}{\nu^2}\right) \left(\frac{\nu}{a\omega R}\right)^2 \phi \sin x \end{aligned} \quad (42)$$

$$\left(\frac{R}{a}\right) \phi_t + M_y^{(3)} \phi_x - M_x^{(3)} \phi_y = \frac{1}{Pr} \phi_{yy} \quad (43)$$

$$\text{B.C. } y = 0, M_x^{(3)} = M_y^{(3)} = 0, \phi = 1$$

$$y \rightarrow \infty, M_y^{(3)} \rightarrow 2 \sin t \sin(x + \lambda), \phi \rightarrow 0 \quad (44)$$

Case IV,

$$\begin{aligned} \sqrt{\frac{R}{a}} M_{yt}^{(4)} + M_y^{(4)} M_{yx}^{(4)} - M_x^{(4)} M_{yy}^{(4)} &= 2 \cos t \sin(x + \lambda) + \\ + 4 \left(\frac{a}{R}\right) \sin^2 t \sin(x + \lambda) \cos(x + \lambda) &+ M_{yyy}^{(4)} + \\ + \left(\frac{g\beta\theta_w R^3}{\nu^2}\right) \left(\frac{\nu}{\sqrt{aR}\omega R}\right)^2 \phi \sin x \end{aligned} \quad (45)$$

$$\sqrt{\frac{R}{a}} \phi_t + M_y^{(4)} \phi_x - M_x^{(4)} \phi_y = \frac{1}{Pr} \phi_{yy} \quad (46)$$

$$\text{B.C. } y = 0, \quad M_x^{(4)} = M_y^{(4)} = 0, \quad \phi = 1$$

$$y \rightarrow \infty, \quad M_y^{(4)} \rightarrow 2\sqrt{\frac{g}{R}} \sin t \sin(x + \lambda), \quad \phi \rightarrow 0 \quad (47)$$

The numbers in parentheses placed above the dimensionless stream function, M , in the preceding sets of equations, refers to the Case No. in Table 1.

3.4 Dimensionless Parameters

The preceding sets of equations show that there are four dimensionless parameters that fully describe the flow; namely, a fluid parameter represented by the Prandtl number, a temperature parameter represented by the Grashoff number, a vibration parameter (vibration parameter A) represented by the ratio of vibration amplitude to cylinder radius, and another vibration parameter (vibration parameter B) which may be represented by any of four different dimensionless groups. The four sets of dimensionless parameters corresponding to the four cases in Table 1 are listed in Table 2.

3.5 Solutions for Uncoupled Vibrations and Free Convection

Before discussing the properties of each of the four sets of equations given in Section 3.3, it will be helpful to mention here solutions obtained for boundary-layer flow for the case of vibrations in the absence of heat transfer and for free convection in the absence of vibrations. The first of these cases is that of a cylinder oscillating in a fluid otherwise at rest with no temperature difference between the cylinder and the fluid. The solution to this problem was obtained by

Schlichting⁽¹¹⁾ by the use of an iterative method. Schlichting's choice of a dimensionless y-coordinate was based on an "AC" boundary-layer thickness as indicated in Case II.

The problem of natural convection without vibrations was solved analytically by Hermann⁽²³⁾ for a heated horizontal cylinder. Hermann nondimensionalized his equations by the same substitutions as indicated in Case I. The equations were then transformed into ordinary differential equations by an approximate similarity variable and solved numerically.

3.6 Complete Equations, Case I

Using the substitution of variables for Case I, Table II, T. Chiang⁽²⁴⁾, following some ideas suggested by R. J. Nickerson, attacked the complete analytical problem for simultaneous heat transfer and vertical vibrations ($\lambda = 0$). By a perturbation method of solution, Chiang obtained numerical values that satisfied his zeroth-order equations; he also worked out the form for the higher-order equations. The numerical values for the zeroth-order equations, which are the same equations as for steady natural convection, agree very well with the values obtained by Hermann. T. Chiang used a different similarity variable than Hermann, in order to be able to use the same variable for the higher-order equations. A short summary of Chiang's perturbation method and results follows. For more complete details, the reader is referred to the original work⁽²⁴⁾.

In Chiang's approach, the dimensionless stream function, $M^{(1)}$, and temperature function, ϕ , are expanded in power series in terms of a/R :

$$M^{(1)} = M_0^{(1)}(x, y) + \left(\frac{a}{R}\right) M_1^{(1)}(x, y, t) + \left(\frac{a}{R}\right)^2 M_2^{(1)}(x, y, t) + \dots \quad (48)$$

$$\phi = \phi_0(x, y) + \left(\frac{a}{R}\right) \phi_1(x, y, t) + \left(\frac{a}{R}\right)^2 \phi_2(x, y, t) + \dots \quad (49)$$

These expressions are substituted into Eqs. (36), (37), and (38). Then, by equating coefficients of like powers of the perturbation parameter, a/R , the following set of equations and boundary conditions are obtained:

Zeroth-Order Equations

$$M_{oy}^{(1)} M_{oyx}^{(1)} - M_{ox}^{(1)} M_{oyy}^{(1)} = M_{oyyy}^{(1)} + \phi_0 \sin x \quad (50)$$

$$M_{oy}^{(1)} \phi_{ox} - M_{ox}^{(1)} \phi_{oy} = \frac{1}{Pr} \phi_{oyy} \quad (51)$$

$$\text{B.C. } y = 0, \quad M_{ox}^{(1)} = M_{oy}^{(1)} = 0, \quad \phi_0 = 1 \quad (52)$$

$$y \rightarrow \infty, \quad M_{oy}^{(1)} \rightarrow 0, \quad \phi_0 \rightarrow 0$$

First-Order Equations

$$\begin{aligned} & \Omega M_{lyt}^{(1)} + M_{oy}^{(1)} M_{lyx}^{(1)} + M_{ly}^{(1)} M_{oyx}^{(1)} - M_{ox}^{(1)} M_{lyy}^{(1)} - M_{lx}^{(1)} M_{oyy}^{(1)} = \\ & = 2 \Omega^2 \cos t \sin x + M_{lyyy}^{(1)} + \phi_1 \sin x \end{aligned} \quad (53)$$

$$\Omega \phi_{lt} + M_{oy}^{(1)} \phi_{lx} + M_{ly}^{(1)} \phi_{ox} - M_{ox}^{(1)} \phi_{ly} - M_{lx}^{(1)} \phi_{oy} = \frac{1}{Pr} \phi_{lyy} \quad (54)$$

$$\text{B.C. } y = 0, \quad M_{lx}^{(1)} = M_{ly}^{(1)} = 0, \quad \phi_1 = 0 \quad (55)$$

$$y \rightarrow \infty, \quad M_{ly}^{(1)} \rightarrow 2 \Omega \sin t \sin x, \quad \phi_1 \rightarrow 0$$

Second-Order Equations

$$\begin{aligned} & \Omega M_{2yt}^{(1)} + M_{oy}^{(1)} M_{2yx}^{(1)} + M_{ly}^{(1)} M_{lyx}^{(1)} + M_{2y}^{(1)} M_{oyx}^{(1)} - M_{ox}^{(1)} M_{2yy}^{(1)} - M_{lx}^{(1)} M_{lyy}^{(1)} + \\ & - M_{2x}^{(1)} M_{oyy}^{(1)} = 4 \Omega^2 \sin^2 t \sin x \cos x + M_{2yyy}^{(1)} + \phi_2 \sin x \end{aligned} \quad (56)$$

$$\Omega \phi_{2t} + M_{oy}^{(1)} \phi_{2x} + M_{ly}^{(1)} \phi_{1x} + M_{2y}^{(1)} \phi_{ox} - M_{ox}^{(1)} \phi_{2y} - M_{1x}^{(1)} \phi_{1y} +$$

$$-M_{2x}^{(1)} \phi_{oy} = \frac{1}{Pr} \phi_{2yy} \quad (57)$$

$$\text{B.C. } y = 0, \quad M_{2x}^{(1)} = M_{2y}^{(1)} = 0, \quad \phi_2 = 0$$

$$y \rightarrow \infty, \quad M_{2y}^{(1)} \rightarrow 0, \quad \phi_2 \rightarrow 0 \quad (58)$$

A reduction of the zeroth-order equations to ordinary differential equations can be achieved by the following substitution:

$$M_o^{(1)} = X F_o(y) + X^3 F_1(y) + X^5 F_2(y) \quad (59)$$

$$\phi_o = G_o(y) + X^2 G_1(y) + X^4 G_2(y) \quad (60)$$

The values for the functions $F_o, F_1, F_2, G_o, G_1, G_2$, and their derivatives have been calculated and are given in Tables 3, 4, and 5 of the Appendix.

From Eqs. (32), (59), (60) and Tables 3 to 5, it can be shown that the local Nusselt number and the dimensionless velocity and temperature are, respectively,

$$Nu = \frac{hR}{k} = - \left[-0.3702 + 0.0161 X^2 - 0.00007 X^4 \right] Gr^{\frac{1}{4}} \quad (61)$$

$$\frac{uR}{Gr^{\frac{1}{4}}} = X F_{oy} + X^3 F_{1y} + X^5 F_{2y} \quad (62)$$

$$\frac{T - T_{\infty}}{T_w - T_{\infty}} = G_o + X^2 G_1 + X^4 G_2 \quad (63)$$

By using expansions similar to Eqs. (59) and (60), the higher-order equations can also be reduced to ordinary differential equations.

An advantage of this perturbation method is that all dimensionless parameters, except the Prandtl number, have been eliminated from the zeroth-order equations, and hence these equations need to be solved only once for each Prandtl number. The higher-order equations are linear; this fact constitutes another advantage of the perturbation method.

Two important disadvantages are present in this method. The first concerns the question of how to relate the theoretical solution to the experimental results. Experiments indicate that the quantity $a\omega$ is an important factor for determining the nature of the flow. However, this combination does not appear anywhere among the dimensionless parameters of this case (Case I, Table 2). This difficulty might prove imaginary when the equations are solved and the numerical results are compared with the experimental results. However, the second disadvantage is very real and concerns the dimensionless parameter, Ω , which appears in the higher-order equations (in addition to the Prandtl number). Since this frequency parameter, Ω , contains the cylinder temperature and the frequency of vibration, it necessitates a completely new solution of the higher-order equations for almost every different value of cylinder temperature and frequency.

These disadvantages provided the motivation to investigate the possibility of using other sets of parameters than those in Case I.

These other sets of parameters—Cases II, III, and IV in Table 2—overcome some of the above disadvantages, but new disadvantages appear in their place. However, by studying these additional cases, a greater insight into the problem is obtained.

3.7 Complete Equations, Case II

This set of equations is very similar to those of Case I and can be expanded in a power series like Eqs. (48) and (49). When this is done, the following sets of equations are obtained:

Zeroth-Order Equations

$$M_{oy}^{(2)} M_{oyx}^{(2)} - M_{ox}^{(2)} M_{oyy}^{(2)} = M_{oyyy}^{(2)} + Gr \left(\frac{\delta_0}{R}\right)^2 \phi_0 \sin x \quad (64)$$

$$M_{oy}^{(2)} \phi_{ox} - M_{ox}^{(2)} \phi_{oy} = \frac{1}{Pr} \phi_{oyy} \quad (65)$$

$$\text{B.C. } y = 0, M_{oy}^{(2)} = M_{oy}^{(2)} = 0, \phi_0 = 1$$

$$y \rightarrow \infty, M_{oy}^{(2)} \rightarrow 0, \phi_0 \rightarrow 0 \quad (66)$$

First-Order Equations

$$M_{lyt}^{(2)} + M_{oy}^{(2)} M_{lyx}^{(2)} + M_{ly}^{(2)} M_{oyx}^{(2)} - M_{ox}^{(2)} M_{lyy}^{(2)} - M_{lx}^{(2)} M_{oyy}^{(2)} = 2 \cos t \sin(x + \lambda) + M_{lyyy}^{(2)} + Gr \left(\frac{\delta_0}{R}\right)^2 \phi_1 \sin x \quad (67)$$

$$\phi_{lt} + M_{oy}^{(2)} \phi_{lx} + M_{ly}^{(2)} \phi_{ox} - M_{ox}^{(2)} \phi_{ly} - M_{lx}^{(2)} \phi_{oy} = \frac{1}{Pr} \phi_{lyy} \quad (68)$$

$$\text{B.C. } y = 0, M_{lx}^{(2)} = M_{ly}^{(2)} = 0, \phi_1 = 0$$

$$y \rightarrow \infty, M_{ly}^{(2)} \rightarrow 2 \sin t \sin(x + \lambda), \phi_1 \rightarrow 0 \quad (69)$$

Second-Order Equations

$$\begin{aligned}
 & M_{2yt}^{(2)} + M_{oy}^{(2)} M_{2yx}^{(2)} + M_{ly}^{(2)} M_{lyx}^{(2)} + M_{2y}^{(2)} M_{oyx}^{(2)} - M_{ox}^{(2)} M_{2yy}^{(2)} - M_{lx}^{(2)} M_{lyy}^{(2)} + \\
 & -M_{2x}^{(2)} M_{oyy}^{(2)} = 4 \sin^2 t \cos(x + \lambda) \sin(x + \lambda) + M_{2yy}^{(2)} + \\
 & + Gr \left(\frac{\delta_0}{R}\right)^2 \phi_2 \sin x
 \end{aligned} \tag{70}$$

$$\begin{aligned}
 & \phi_{2t} + M_{oy}^{(2)} \phi_{2x} + M_{ly}^{(2)} \phi_{lx} + M_{2y}^{(2)} \phi_{ox} - M_{ox}^{(2)} \phi_{2y} - M_{lx}^{(2)} \phi_{ly} + \\
 & -M_{2x}^{(2)} \phi_{oy} = \frac{1}{Pr} \phi_{2yy}
 \end{aligned} \tag{71}$$

$$\begin{aligned}
 \text{B.C. } y = 0, \quad M_{2x}^{(2)} = M_{2y}^{(2)} = 0, \quad \phi_2 = 0 \\
 y \rightarrow \infty, \quad M_{2y}^{(2)} \rightarrow 0, \quad \phi_2 \rightarrow 0
 \end{aligned} \tag{72}$$

An interesting possibility presented by this set of equations is to try to eliminate the dimensionless parameters from the first and second-order equations. This could be done by absorbing the quantity $Gr \left(\frac{\delta_0}{R}\right)^2$ into a new definition of the temperature function. Since the energy equation is linear and the temperature boundary conditions are homogeneous, this step presents no great difficulty.

However, one difficulty still remains with this set of equations: Since the quantity $Gr \left(\frac{\delta_0}{R}\right)^2$ appears in the momentum equation, Eq. (64), the zeroth-order equations have to be solved for each new condition of temperature or frequency. Eliminating this quantity from Eq. (64), by defining a new temperature function, does not overcome this

difficulty because the temperature boundary conditions, Eq. (66), are not homogeneous. Hence, although the differential equation would remain unchanged, the boundary conditions at the cylinder wall would become $\phi_0 = Gr \left(\frac{\delta_s}{R}\right)^2$ instead of being equal to unity. This boundary condition would change for each new wall temperature or frequency, thereby necessitating a complete numerical solution for each new condition.

There is a possible way of circumventing the main disadvantages of Cases I and II. This would be to combine the zeroth-order equations of Case I with the higher-order equations of Case II. Then, the equations would have to be solved only once for each Prandtl number. This possibility should be investigated in the future.

The advantage of linear higher-order equations is present in Case II, as in Case I. Also, as in Case I, the quantity $a\omega$ does not appear anywhere among the dimensionless parameters of Case II—for reasons stated in section 3.6, the nonappearance of the quantity $a\omega$ may be a disadvantage in solving the problem.

3.8 Complete Equations, Case III

For Case III, expanding the equations by perturbation methods does not give physically meaningful equations, since the highest derivative terms do not appear in the zeroth-order solution. Therefore, other mathematical approaches must be tried. Integral approaches similar to those used by Lighthill⁽¹⁹⁾ might prove successful.

In this case, unlike the two previous cases, the product $a\omega$ does appear in the dimensionless parameters. It appears in vibration

parameter B , the vibrational Reynolds number. Experiments indicate that this product, $a\omega$, is an important characteristic of the flow. For this reason, this form of the equations might bring fruitful results with additional investigation.

3.9 Complete Equations, Case IV

Case IV appears to have no special advantages over Case III which it resembles. The appearance of fractional powers of the ratio a/R is an additional complication without compensating advantages.

IV. SUMMARY

In this report, the basic equations and the boundary conditions which govern the problem of coupled transverse vibrations and free convection from a heated horizontal cylinder are presented. By applying a method developed by C. C. Lin⁽²¹⁾ some important characteristics of these equations are demonstrated. It is shown that the steady-flow solution is modified by the presence of harmonic oscillations only when pressure gradients occur. These modifications have their most pronounced effect on the fluid closest to the surface. Furthermore, Lin's method indicates that the product $\alpha\omega$ is an important parameter for determining the magnitude of the changes produced in the steady-flow solution by the presence of harmonic oscillations; this fact agrees with experimental observations.

Lin's method is an iterative method of solution and makes use of the basic equations in a dimensional form. For most other analytical methods of attack, the differential equations in dimensionless form are the most useful. For this reason, the basic differential equations were converted into dimensionless form. An examination of these dimensionless equations shows that four dimensionless parameters are needed to fully describe the flow. Table 2 gives four possible sets of dimensionless parameters; the differential equations associated with these four sets of parameters are presented in Chapter III.

Considerable work has been done on the first set of parameters in Table 2 (Case I) by T. Chiang⁽²⁴⁾. The main advantage of formulating the problem in terms of the parameters in Case I is the simplicity of

the zeroth-order equations obtained when a perturbation expansion in terms of a/R is applied. These zeroth-order equations, which are identical with the equations for free convection without vibration, have been solved by Chiang, and the results of this solution agree very well with a solution obtained by Hermann⁽²³⁾ who used other methods.

For the dimensionless parameters in Case II, the higher-order equations, obtained by applying a perturbation expansion in terms of a/\bar{R} , are simpler than the corresponding higher-order equations for Case I. By combining the perturbation equations for Case I and Case II, the advantages of each might be retained for a complete solution. This possibility merits further investigation.

The dimensionless parameters for Case III contain the product $a\omega$, which experiment has shown to be a controlling variable; for this reason, Case III represents a promising line of attack. Case IV seems to present no particular advantages over the first three cases in obtaining a solution to the problem.

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TABLE 1

Values for α and γ used to nondimensionalize the basic differential equations

Case No.	α	γ
I	$\frac{Gr}{R} = \left(\frac{g \beta \theta_w}{R \nu^2} \right)^{\frac{1}{4}}$	$\nu Gr = (g \beta \theta_w \nu^2 R^3)^{\frac{1}{4}}$
II	$\frac{1}{\delta_0} = \sqrt{\frac{\omega}{\nu}}$	$R \sqrt{\nu \omega}$
III	$\frac{\sqrt{(Re)_{af}}}{R} = \sqrt{\frac{a \omega}{R \nu}}$	$\sqrt{(Re)_{af}} = \sqrt{a \omega R \nu}$
IV	$\frac{1}{R} \left(\frac{a \omega^2 R^3}{\nu^2} \right)^{\frac{1}{4}} = \left(\frac{a \omega^2}{\nu^2 R} \right)^{\frac{1}{4}}$	$(a \omega^2 \nu^2 R^3)^{\frac{1}{4}}$

TABLE 2

Sets of dimensionless parameters

Case No.	I	II	III	IV
Fluid Parameter	Pr	Pr	Pr	Pr
Temperature Parameter	Gr	Gr	Gr	Gr
Vibration Parameter (A)	$\frac{a}{R}$	$\frac{a}{R}$	$\frac{a}{R}$	$\frac{a}{R}$
Vibration Parameter (B)	$\Omega^2 = \frac{\omega^2 R}{g \beta \theta_w}$	$\left(\frac{\delta_0}{R} \right)^2 = \frac{\nu}{\omega R^2}$	$(Re)_{af} = \frac{a \omega R}{\nu}$	$\frac{\sqrt{a R} \omega R}{\nu}$

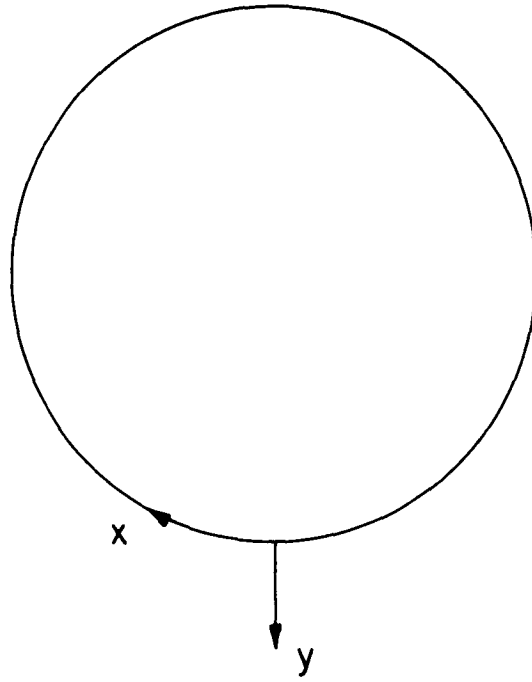


Figure 1. System of Coordinates

APPENDIX

TABLE 3

Tabulation of the functions F_o and G_o and their derivatives

(Pr = 0.70)

y	F_{oyy}	F_{oy}	F_o	G_{oy}	G_o
0.0	0.8595	0.0000	0.0000	-0.3702	1.0000
.1	.7614	.0810	.0041	-.3702	.9630
.2	.6677	.1524	.0159	-.3699	.9260
.3	.5789	.2147	.0343	-.3693	.8890
.4	.4953	.2684	.0585	-.3681	.8521
.5	.4171	.3140	.0877	-.3662	.8154
.6	.3446	.3520	.1211	-.3636	.7789
.7	.2778	.3831	.1579	-.3600	.7427
.8	.2166	.4078	.1975	-.3556	.7069
.9	.1611	.4266	.2393	-.3502	.6716
1.0	.1109	.4401	.2826	-.3439	.6369
1.1	.0661	.4490	.3271	-.3366	.6029
1.2	.0263	.4535	.3723	-.3285	.5696
1.3	-.0088	.4544	.4177	-.3195	.5372
1.4	-.0393	.4519	.4630	-.3098	.5057
1.5	-.0655	.4467	.5080	-.2995	.4753
1.6	-.0878	.4390	.5523	-.2886	.4459
1.7	-.1066	.4292	.5957	-.2772	.4176
1.8	-.1219	.4178	.6381	-.2655	.3904
1.9	-.1343	.4049	.6792	-.2535	.3645
2.0	-.1439	.3910	.7190	-.2414	.3397
2.1	-.1511	.3762	.7574	-.2292	.3162
2.2	-.1560	.3609	.7943	-.2171	.2939
2.3	-.1591	.3451	.8296	-.2051	.2728
2.4	-.1605	.3291	.8633	-.1933	.2529
2.5	-.1604	.3130	.8954	-.1818	.2341
2.6	-.1590	.2971	.9259	-.1703	.2165
2.7	-.1566	.2813	.9543	-.1597	.2000
2.8	-.1534	.2658	.9821	-.1492	.1846
2.9	-.1493	.2506	1.0080	-.1392	.1701
3.0	-.1448	.2359	1.0320	-.1296	.1567
3.2	-.1343	.2080	1.0770	-.1118	.1326
3.4	-.1229	.1822	1.1160	-.0959	.1119
3.6	-.1111	.1588	1.1500	-.0818	.0941
3.8	-.0995	.1378	1.1790	-.0695	.0790
4.0	-.0883	.1190	1.2050	-.0588	.0662
4.2	-.0778	.1024	1.2270	-.0496	.0554
4.4	-.0681	.0879	1.2460	-.0417	.0463
4.6	-.0593	.0751	1.2620	-.0350	.0386
4.8	-.0514	.0641	1.2760	-.0293	.0322
5.0	-.0443	.0545	1.2880	-.0245	.0269

TABLE 3 (cont'd)

(Pr = 0.70)

y	F _{oyy}	F _{oy}	F _o	G _{oy}	G _o
5.2	-.0381	.0463	1.2980	-.0204	.0224
5.4	-.0326	.0392	1.3070	-.0170	.0186
5.6	-.0278	.0332	1.3140	-.0142	.0155
5.8	-.0237	.0281	1.3200	-.0118	.0129
6.0	-.0202	.0237	1.3250	-.0098	.0108
6.5	-.0133	.0154	1.3350	-.0061	.0069
7.0	-.0087	.0100	1.3410	-.0038	.0044
7.5	-.0057	.0064	1.3450	-.0024	.0029
8.0	-.0037	.0041	1.3480	-.0015	.0019
8.5	-.0025	.0026	1.3490	-.0009	.0013
9.0	-.0017	.0016	1.3500	-.0006	.0009
9.5	-.0011	.0009	1.3510	-.0004	.0007
10.0	-.0008	.0004	1.3510	-.0002	.0006
10.5	-.0006	.0000	1.3510	-.0001	.0005

TABLE 4

Tabulation of the functions F_1 and G_1 and their derivatives

(Pr = 0.70)

y	F_{1yy}	F_{1y}	F_1	G_{1y}	G_1
0.0	-0.09342	0.0000	0.0000	0.01609	0.0000
.1	-.0772	-.0085	-.0004	.0161	.0016
.2	-.0620	-.0155	-.0017	.0160	.0032
.3	-.0481	-.0210	-.0035	.0159	.0048
.4	-.0354	-.0251	-.0058	.0157	.0064
.5	-.0242	-.0281	-.0085	.0154	.0080
.6	-.0143	-.0300	-.0114	.0149	.0095
.7	-.0058	-.0310	-.0144	.0143	.0109
.8	.0013	-.0312	-.0176	.0136	.0123
.9	.0073	-.0308	-.0207	.0127	.0136
1.0	.0122	-.0298	-.0237	.0118	.0149
1.1	.0160	-.0284	-.0266	.0107	.0160
1.2	.0189	-.0266	-.0294	.0095	.0170
1.3	.0210	-.0246	-.0319	.0083	.0179
1.4	.0224	-.0224	-.0343	.0070	.0187
1.5	.0232	-.0202	-.0364	.0057	.0193
1.6	.0234	-.0178	-.0383	.0045	.0198
1.7	.0233	-.0155	-.0400	.0032	.0202
1.8	.0227	-.0132	-.0414	.0020	.0205
1.9	.0219	-.0110	-.0426	.0008	.0206
2.0	.0208	-.0088	-.0436	-.0003	.0206
2.2	.0182	-.0049	-.0450	-.0023	.0204
2.4	.0153	-.0016	-.0456	-.0039	.0197
2.6	.0124	.0012	-.0456	-.0051	.0188
2.8	.0096	.0034	-.0452	-.0060	.0177
3.0	.0070	.0051	-.0443	-.0065	.0165
3.2	.0048	.0062	-.0432	-.0068	.0151
3.4	.0029	.0070	-.0418	-.0068	.0138
3.6	.0013	.0074	-.0404	-.0066	.0124
3.8	.0001	.0075	-.0389	-.0063	.0111
4.0	-.0009	.0074	-.0374	-.0060	.0099
4.2	-.0016	.0072	-.0359	-.0055	.0088
4.4	-.0021	.0068	-.0345	-.0051	.0077
4.6	-.0024	.0064	-.0332	-.0046	.0067
4.8	-.0026	.0058	-.0320	-.0041	.0059
5.0	-.0027	.0053	-.0308	-.0037	.0051
5.5	-.0025	.0040	-.0286	-.0027	.0035
6.0	-.0021	.0028	-.0269	-.0020	.0023
6.5	-.0016	.0019	-.0257	-.0014	.0015
7.0	-.0012	.0012	-.0249	-.0010	.0009
7.5	-.0008	.0007	-.0244	-.0007	.0005
8.0	-.0005	.0004	-.0242	-.0004	.0002
8.5	-.0003	.0002	-.0241	-.0003	.0001
8.7	-.0002	.0001	-.0240	-.0002	.0000

TABLE 5

Tabulation of the functions F_2 and G_2 and their derivatives

(Pr = 0.70)

y	F_{2yy}	F_{2y}	F_2	G_{2y}	G_2
0.0	0.00245	0.00000	0.00000	-0.00007	0.00000
.1	.00165	.00020	.00001	-.00007	-.00001
.2	.00094	.00033	.00004	-.00007	-.00001
.3	.00034	.00040	.00008	-.00007	-.00002
.4	-.00015	.00041	.00012	-.00006	-.00003
.5	-.00053	.00037	.00016	-.00005	-.00003
.6	-.00081	.00030	.00019	-.00003	-.00004
.7	-.00099	.00021	.00022	-.00001	-.00004
.8	-.00108	.00011	.00023	.00001	-.00004
.9	-.00111	-.00000	.00024	.00003	-.00004
1.0	-.00107	-.00011	.00023	.00005	-.00003
1.1	-.00099	-.00021	.00021	.00007	-.00003
1.2	-.00087	-.00031	.00019	.00008	-.00002
1.3	-.00072	-.00039	.00015	.00009	-.00001
1.4	-.00055	-.00045	.00011	.00009	-.00000
1.5	-.00036	-.00050	.00006	.00009	.00001
1.6	-.00017	-.00052	.00001	.00008	.00001
1.7	.00002	-.00053	-.00004	.00006	.00002
1.8	.00022	-.00052	-.00009	.00004	.00003
1.9	.00042	-.00049	-.00014	.00001	.00003
2.0	.00061	-.00044	-.00019	-.00003	.00003
2.1	.00080	-.00036	-.00023	-.00007	.00002
2.2	.00099	-.00027	-.00026	-.00012	.00001
2.3	.00118	-.00017	-.00028	-.00017	.00000