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Charles H. Murphy

Department of the Army Project No. 503-03-001
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BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

Although the effect of an exponentially varying air density on the angular motion of a re-entering missile has been intensively studied for linear aerodynamic moments, this effect for nonlinear moments has only been investigated by a quasi-linear technique and this work was limited to planar motion. In this report, various modifications of the quasi-linear analysis for a cubic static moment are compared with the exact solutions and the best of these used to determine the general effect of varying density on combined pitching and yawing motion. Next the perturbation method which makes use of the exact solution for a cubic static moment is considered and the necessary modifications introduced by the varying density are made. The predicted density-gradient-induced damping for planar motion is determined for both theories and compared with the numerical integration of the exact differential equation. The perturbation prediction is found to be superior.

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LIST OF SYMBOLS

C_1	twice the total energy, $C_1 = \xi' \bar{\xi}' + \hat{V}(\delta^2)$
C_2	twice the trajectory component of the angular momentum of the pitching and yawing motion, $C_2 = i(\bar{\xi}' \xi - \xi' \bar{\xi})$
C_D	drag coefficient
C_{L_α}	lift coefficient
C_{M_α}	static moment coefficient
$C_{M_{\dot{\alpha}}}$, C_{M_q}	damping moment coefficients
$C_{M_{p\alpha}}$	Magnus moment coefficient
C_m , C_n	coefficients of the transverse components of the aerodynamic moment
c^*	non-conservative part of C_{M_α}, $c^* = C_{M_\alpha} - c_o - c_2\delta^2$
D	$4\omega^2\bar{\omega}^2\hat{M}_o^{-2}$
E(k)	complete elliptic integral of the second kind
H	$\frac{\rho S \ell}{2m} \left[C_{L_\alpha} - C_D - k_t^{-2} (C_{M_q} + C_{M_{\dot{\alpha}}}) \right]$
I_x	axial moment of inertia
$I_y = I_z$	transverse moments of inertia
K(k)	complete elliptic integral of the first kind
K_1, K_2	amplitudes of two modes of linear oscillation
k	modulus of the elliptic integral

k_a	axial radius of gyration, $k_a = \sqrt{I_x/m\ell^2}$
k_t	transverse radius of gyration, $k_t = \sqrt{I_y/m\ell^2}$
ℓ	reference length
M_y, M_z	transverse components of aerodynamic moment
M	$\frac{\rho S \ell}{2m} \left[k_t^{-2} C_{M_\alpha} - C'_{L_\alpha} \right]$
$\hat{M}_o$	$M_o - \left[\frac{P}{2} \right]^2$
M^*	non-conservative part of M, $M^* = M - M_o - M_2\delta^2$
M_o, M_2	cubic static moment coefficients
m	mass
m	ratio of nonlinear portion of cubic moment to linear, $m = m_2\delta_2^2$
m_2	$M_2/\hat{M}_o$
P	gyroscopic spin, $P = \frac{I_x}{I_y} \frac{p\ell}{V}$
p, q, r	components of angular velocity
S	reference area
s	dimensionless distance along flight path
s_g	stability factor, $s_g = \frac{P^2}{4M_o}$
T	$\frac{\rho S \ell}{2m} \left[C_{L_\alpha} + k_a^{-2} C_{M_{p\alpha}} \right]$
T	temperature ($^\circ$K)

LIST OF SYMBOLS (Cont'd)

u, v, w	components of velocity
V	magnitude of velocity, $V = \sqrt{u^2 + v^2 + w^2}$
$\hat{V}$	the potential function associated with $\hat{M}$
x, y	squared amplitudes of the two modes of linear oscillation
z	altitude (ft.)
α	angle of attack
β	angle of sideslip
γ	cosine of total angle of attack
δ	$\lvert \xi \rvert$
δ_1	minimum value of δ
δ_2	maximum value of δ
θ	angle the flight path makes with respect to the vertical
θ	argument $\hat{\xi}$
λ_1, λ_2	aerodynamic damping doefficients
ξ	$\frac{v + iw}{V}$
$\tilde{\xi}$	$\xi\, e^{\cdot i \int \frac{p\ell}{V} ds}$
$\hat{\xi}$	$\tilde{\xi}\, e^{-i \left(\frac{1}{2}\right) Ps}$
ρ	air density
ρ_o	sea-level air density
σ	coefficient of exponential density variation

$\tilde{\sigma}$	$\sigma \ell \cos\theta$
ϕ_j	phase angle of the jth mode
ϕ	$\phi_1 - \phi_2$
ψ_j'	$\phi_j' \mp \sqrt{-\hat{M}_o}$
ω^2	$-\hat{M}_o\left[1 + \frac{m_2}{2} (\delta_1^2 + 2\delta_2^2)\right]$
ϖ^2	$-\hat{M}_o\left[1 + \frac{m_2}{2} (\delta_2^2 + 2\delta_1^2)\right]$

Subscript

a	average value over cycle of motion

Superscripts

'	derivative with respect to arclength, s
-	complex conjugate
~	quantity related to non-rotating coordinate system
^	quantity related to coordinate system which is rotating with velocity $\frac{P}{2}$

1. INTRODUCTION

The influence of an exponentially varying air density on the planar pitching of a re-entering missile with linear aerodynamic moments has been studied by a number of authors.[1,2] This work was extended to combined pitching and yawing motion of a spinning missile by Leon[3] for no aerodynamic damping and by Garber[4] for linear aerodynamic damping. Recently, Coakley, Laitone, and Mass[5] have made use of a quasi-linear technique to describe the planar motion of a missile with a cubic static moment flying through an exponential atmosphere.

In this report we will study the influence of such density variations on the general combined pitching and yawing motion of a missile acted on by nonlinear moments. The nonlinear analysis will make use of a perturbation technique[6] which is more accurate than the quasi-linear analysis employed by Coakley, Laitone, and Mass. The quasi-linear analysis will, however, be derived for combined pitching and yawing motion for comparison with the more exact method. Although the primary objective is the influence of varying density, the development will be so formulated that the effect of Mach number or Reynold's number variation of the aerodynamic coefficients themselves may be studied.

2. QUASI-LINEAR SOLUTION FOR CONSTANT DENSITY AND NO DAMPING

In this section we will consider various modifications of the quasi-linear technique for no aerodynamic damping or drag, constant density, and cubic static moment. For this case the aerodynamic moment may be written in the form:

$$M_y + iM_z = -(1/2)\rho V^2 S\ell(c_o + c_2\delta^2)\xi \tag{1}$$

where $\xi = \frac{v+iw}{V} \doteq \beta + i\alpha$

$\delta = \left| \xi \right|$ and the other symbols are defined in the List of Symbols.

In Equation (1) the complex moment and complex angle are in a missile-fixed coordinate system. If we derive the differential equation for the pitching and yawing motion for this moment* in a non-rotating coordinate system, it would have the form[6]

$$\tilde{\xi}'' - \left(\frac{\gamma'}{\gamma} + iP\right)\tilde{\xi}' - (M_o + M_2\delta^2)\tilde{\xi} = 0 \tag{2}$$

where $\tilde{\xi} = \xi \exp\left[i\int \frac{p\ell}{V}\, ds\right]$

$\gamma = (1 - \delta^2)^{1/2}$ cosine of total angle of attack

$P = \frac{I_x}{I_y} \frac{p\ell}{V}$

$M_o = \left(\frac{\rho S \ell^3}{2I_y}\right) c_o$

$M_2 = \left(\frac{\rho S \ell^3}{2I_y}\right) \left[c_2 - (1/2)c_o\right]$ and

$s = \int_o^t \frac{V}{\ell}\, dt$ is the independent variable.

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* Since the static moment coefficient is multiplied by γ, a cubic moment produces higher powers in the equation of motion. Only the cubic powers are retained in Equation (2).

For moderate geometrical angles $\gamma' \doteq 0$. For constant P, a transformation to a coordinate system which has an angular rate of P/2 reduces this equation to a simple form.

$$\hat{\xi}'' - \hat{M}_o(1 + m_2\delta^2)\,\hat{\xi} = 0 \qquad (3)$$

where

$$\hat{\xi} = \tilde{\xi} e^{-i(1/2)Ps}$$

$$\hat{M}_o = M_o - P^2/4$$

$$m_2 = M_2/\hat{M}_o \quad .$$

For a linear moment and negative $\hat{M}_o$, ξ moves along an ellipse with equation

$$\hat{\xi} = K_1 e^{i\hat{\phi}_1} + K_2 e^{i\hat{\phi}_2} \qquad (4)$$

where K_j are constants

$$\hat{\phi}_j = \hat{\phi}_{jo} + \hat{\phi}_j' s \quad \text{and}$$

$$\hat{\phi}_j' = \pm \sqrt{-\hat{M}_o} \quad .$$

The semi-major axis is $K_1 + K_2$ and the semi-minor axis is $|K_1 - K_2|$. When the spin is zerp the negative $\hat{M}_o$ requirement is that for static stability. In the case of a spinning missile, this requirement is essentially that for gyroscopic stability

$$\frac{1}{s_g} < 1 \qquad (5)$$

where $s_g = \dfrac{P^2}{4M_o}$.

The motion in the non-rotating frame of reference becomes epicyclic. The frequencies in these coordinates, $\hat{\phi}_j'$, for a statically stable missile ($M_o < 0$) are opposite in sign and for a statically unstable missile they have the same sign as that of P.

The quasi-linear method assumes a solution for Equation (5) of the form of Equation (4) with frequencies which depend on amplitude. The actual calculation of this dependence makes use of algebra used in the method of variation of parameters. If Equation (5) is written in the form

$$\hat{\xi}'' - \hat{M}_o \hat{\xi} = \hat{M}_o m_2 \delta^2 \hat{\xi}, \tag{6}$$

Equation (4) is the solution for $m_2 = 0$. We assume the solution of Equation (6) to be

$$\hat{\xi} = K_1 e^{i\hat{\phi}_1} + K_2 e^{i\hat{\phi}_2} \tag{7}$$

$$K_j = K_j(s)$$

$$\hat{\phi}_j = \hat{\phi}_{jo} + \hat{\phi}_j' s$$

$$\hat{\phi}_j' = \pm \sqrt{- \hat{M}_o} + \psi_j'(s) \quad .$$

Then $$\delta^2 = \hat{\xi}\bar{\hat{\xi}} = K_1^2 + K_2^2 + K_1 K_2 (e^{i\hat{\phi}} + e^{-i\hat{\phi}})$$

$$= K_1^2 + K_2^2 + 2\, K_1 K_2 \cos \hat{\phi} \tag{8}$$

where $\hat{\phi} = \hat{\phi}_1 - \hat{\phi}_2$

$$\hat{\xi}' = i\sqrt{- \hat{M}_o} \left[K_1 e^{i\hat{\phi}_1} - K_2 e^{i\hat{\phi}_2} \right] + (K_1' + i\psi_1' K_1) e^{i\hat{\phi}_1} + (K_2' + i\psi_2' K_2) e^{i\hat{\phi}_2} \tag{9}$$

Let $$(K_1' + i\psi_1' K_1) e^{i\hat{\phi}_1} + (K_2' + i\psi_2' K_2) e^{i\hat{\phi}_2} = 0 \tag{10}$$

and differentiate again

$$\hat{\xi}'' = \hat{M}_o \left[K_1 e^{i\hat{\phi}_1} + K_2 e^{i\hat{\phi}_2} \right] + i \sqrt{-\hat{M}_o} \left[(K_1' + i\psi_1' K_1) e^{i\hat{\phi}_1} - (K_2' + i\psi_2' K_2) e^{i\hat{\phi}_2} \right] \tag{11}$$

Equations (7 - 11) are now substituted in Equation (6).

$$\frac{K_1'}{K_1} + i\psi_1' = \frac{\hat{M}_o m_2 \delta^2}{2i\sqrt{-\hat{M}_o}} \left(1 + \frac{K_2}{K_1} e^{-i\hat{\phi}}\right) \tag{12}$$

Equation (12) and a similar equation for the other mode are exact but rendered quite complicated by the presence of $\hat{\phi}$. The basic assumption that damping and frequency shift over a cycle of $\hat{\phi}$ is small has to be made. If this is the case Equation (12) may be averaged over a cycle of $\hat{\phi}$ with the result

$$\frac{K_1'}{K_1} + i\psi_1' = i\, \frac{m_2(K_1^2 + 2K_2^2)}{2} \sqrt{-\hat{M}_o} \tag{13}$$

Thus the quasi linear solution has no damping but does have frequencies

$$\hat{\phi}_1' = \sqrt{-\hat{M}_o} \left[1 + \frac{m_2}{2} (K_1^2 + 2K_2^2)\right] \tag{14}$$

$$\hat{\phi}_2' = -\sqrt{-\hat{M}_o} \left[1 + \frac{m_2}{2} (2K_1^2 + K_2^2)\right] \tag{15}$$

This method can easily be extended to treat both linear and nonlinear damping[7].

Equations (14-15) can be substantially improved in accuracy if the average value of the coefficient of ξ is used on the left side of Equation (6) instead of its linear value. Since the average value of δ^2 is $K_1^2 + K_2^2$, Equation (6) would become

$$\hat{\xi}'' - \hat{M}_o \left[1 + m_2(K_1^2 + K_2^2)\right] \hat{\xi} = \hat{M}_o m_2(\delta^2 - K_1^2 - K_2^2)\hat{\xi} \tag{16}$$

From this equation, quasi-linear values of ϕ_j' may be derived in a manner similar to the above.

$$\hat{\phi}_1' = \sqrt{-\hat{M}_o\left[1 + m_2(K_1^2 + K_2^2)\right]}\left[1 + \frac{\left(\frac{1}{2}\right) m_2 K_2^2}{1 + m_2(K_1^2 + K_2^2)}\right] \quad (17)$$

$$\hat{\phi}_2' = -\sqrt{-\hat{M}_o\left[1 + m_2(K_1^2 + K_2^2)\right]}\left[1 + \frac{\left(\frac{1}{2}\right) m_2 K_1^2}{1 + m_2(K_1^2 + K_2^2)}\right] \quad (18)$$

This improved quasi linear technique was used with some success in Reference 6. One clear advantage of Equations (17-18) is that they may be applied to periodic motion of a missile which is gyroscopically unstable for small amplitudes ($\hat{M}_o > 0$, $m_2 < 0$). In this case infinite frequencies are predicted for $K_1^2 + K_2^2 = -\frac{1}{m_2}$. As we will see the correct answers are bounded. To avoid this difficulty and obtain slightly more accurate estimates for other cases, the substitution method of Reference 8 can be used. This method will be developed as another variant of quasi-linear method.

We assume a solution of the form

$$\hat{\xi} = K_1 e^{i\hat{\phi}_1} + K_2 e^{i\hat{\phi}_2} \quad (19)$$

where K_j and $\hat{\phi}_j$ are functions of s.

Then,

$$\hat{\xi}' = i\left[\hat{\phi}_1' K_1 e^{i\phi_1} + \hat{\phi}_2' K_2 e^{i\phi_2}\right] + K_1' e^{i\hat{\phi}_1} + K_2' e^{i\hat{\phi}_2} \quad (20)$$

Let $$K_1' e^{i\hat{\phi}_1} + K_2' e^{i\hat{\phi}_2} = 0 \quad (21)$$

This equation is essentially the neglection of damping in comparison with frequency. We now differentiate again.

$$\hat{\xi}'' = i\left[\hat{\phi}_1'(K_1' + i\hat{\phi}_1'K_1)e^{i\hat{\phi}_1} + \hat{\phi}_2'(K_2' + i\hat{\phi}_2'K_2)e^{i\hat{\phi}_2} + \hat{\phi}_1''K_1e^{i\hat{\phi}_1} + \hat{\phi}_2''K_2e^{i\hat{\phi}_2}\right] \tag{22}$$

Equations (19 - 22) can now be substituted in Equations (3) and the result manipulated into the form

$$(\hat{\phi}_1')^2 - i(\hat{\phi}_1' - \hat{\phi}_2')\frac{K_1'}{K_1} = -\hat{M}_o - \hat{M}_o m_2\delta^2\left(1 + \frac{K_2}{K_1}e^{-i\hat{\phi}}\right) - \left[(\hat{\phi}_2')^2 + \hat{M}_o - i\hat{\phi}_2''\right]\frac{K_2}{K_1}e^{-i\hat{\phi}} + i\hat{\phi}_1'' \tag{23}$$

For linear moments with constant coefficients the right side of Equation (23) is constant. When the moment is nonlinear or the coefficients vary, approximations for frequency and damping may be obtained by averaging the right side over a cycle of $\hat{\phi}$.

$$\hat{\phi}_1' = \sqrt{-\hat{M}_o\left[1 + m_2(K_1^2 + 2K_2^2)\right]} \tag{24}$$

$$\frac{K_1'}{K_1} = -\frac{\hat{\phi}_1''}{\hat{\phi}_1' - \hat{\phi}_2'} \tag{25}$$

Similar relations apply for the other mode

$$\hat{\phi}_2' = -\sqrt{-\hat{M}_o\left[1 + m_2(2K_1^2 + K_2^2)\right]} \tag{26}$$

$$\frac{K_2'}{K_2} = -\frac{\hat{\phi}_2''}{\hat{\phi}_2' - \hat{\phi}_1'} \tag{27}$$

When $\hat{M}_o$ or m_2 are functions of s, Equations (25) and (27) allow us to compute the effect of this on the damping. This will be done in Section 3. For constant $\hat{M}_o$ and m_2 damping is zero and the frequencies are given by Equations (24) and (26).

With these three sets of values for the frequencies derived it is quite important to determine their relative accuracy. To do this we will make use of the exact solution for δ. Four fundamentally different variations of static moment may be expressed by Equation (3) when different algebraic signs are assigned to the coefficient $\hat{M}_o$ and m_2. Only three of these moments, however, can cause periodic motion. These three are illustrated in Figure 1 and may be identified in the following way:

(a) Stable at small angles; more stable at larger angles ($\hat{M}_o < 0$, $m_2 > 0$)

(b) Stable at small angles; less stable at larger angles ($\hat{M}_o < 0$, $m_2 < 0$)

(c) Unstable at small angles; stable at larger angles ($\hat{M}_o > 0$, $m_2 < 0$)

The periodic solutions of Equation (3) are derived in References 6 and 9 and are sumarized below

type (a) moment

$$\delta^2 = \delta_2^2 - (\delta_2^2 - \delta_1^2)\, sn^2(\omega s, k) \tag{28}$$

type (b) moment

$$\delta^2 = \delta_1^2 + (\delta_2^2 - \delta_1^2)\, sn^2(\tilde{\omega} s, k) \tag{29}$$

$$\omega^2 > 0$$

type (c) moment

$$\delta^2 = \delta_2^2 - (\delta_2^2 - \delta_1^2)\, sn^2(\omega s, k) \tag{30}$$

$$\hat{M}_o \left[1 + m_2 \left(\frac{\delta_1^2 + \delta_2^2}{2} \right) \right] \leq 0$$

where δ_1 is minimum value of δ

δ_2 is maximum value of δ

$$\omega^2 = -\hat{M}_o \left[1 + \left(\frac{m_2}{2}\right) (\delta_1^2 + 2\delta_2^2)\right]$$

$$\tilde{\omega}^2 = -\hat{M}_o \left[1 + \left(\frac{m_2}{2}\right) (2\delta_1^2 + \delta_2^2)\right]$$

$$k^2 = \frac{-\hat{M}_o m_2(\delta_2^2 - \delta_1^2)}{2\omega^2} \qquad \text{types (a) and (c)}$$

$$= \frac{\hat{M}_o m_2(\delta_2^2 - \delta_1^2)}{2\tilde{\omega}^2} \qquad \text{type (b)}$$

The inequalities associated with Equations (29-30) are quite important in themselves. Three interesting observations may be made:

1. For type (a) static moment, periodic motion of any amplitude is possible.

2. For type (b) static moment, symmetric planar periodic motions ($P = 0$, $\delta_1 = 0$) are possible for all amplitudes for which the moment does not change sign; circular motion ($\delta_1 = \delta_2$), however, is possible only when the nonlinear part of the moment is not greater than two-thirds of the linear part.

3. For type (c) static moment, possible periodic motions are those for which the median value of δ^2 yields a stable moment.

The variables δ_1 and δ_2 may be approximately related to the modal amplitudes K_1 and K_2 by the equations

$$\delta_1^2 = (K_1 - K_2)^2 \qquad (31)$$

$$\delta_2^2 = (K_1 + K_2)^2 \qquad (32)$$

The period of the elliptic sine function in Equations (28 - 30) corresponds to half the period of $\emptyset$ in Equation (6) since that equation can be put in the form

$$\delta^2 = \delta_2^2 - (\delta_2^2 - \delta_1^2)\sin^2(\hat{\emptyset}/2)$$
$$= \delta_1^2 + (\delta_2^2 - \delta_1^2)\sin^2(\hat{\emptyset} + \pi)/2 \qquad (33)$$

$$\therefore \hat{\emptyset}' = \hat{\emptyset}_2' - \hat{\emptyset}_1' = \frac{\pi\omega}{K(k)} \qquad \text{types (a) and (c)}$$
$$= \frac{\pi\hat{\omega}}{K(k)} \qquad \text{type (b)} \qquad (34)$$

where K(k) is the complete elliptic integral of the first kind. (The period of sm (ws,k) is 4K/w.)

The various approximate values of $\hat{\emptyset}'$ may now be compared with the exact value of Equation (34). In Figures 2-3 this is done for the three different moment types and both planar and circular motion. As can be seen from these plots of $\hat{\emptyset}'/2\sqrt{|\hat{M}_o|}$ versus $m = m_2\delta_2^2$ the unaltered quasi-linear estimate (QL) is only good for small amplitudes while both the improved quasi-linear (IQL) and the substitution methods are quite good for the range of m considered with the exception of the interval $-1 < m < -1/2$. Since $\hat{M}_o$ is positive for a type (c) moment, a quasi-linear value of $\hat{\emptyset}'$ can not be computed. Although substitution method is only slightly better than the improved quasi-linear, its potentiality of describing the effect of varying coefficients M_o and m_2 make it more valuable than the improved quasi-linear. The good agreement of both methods encourages us to introduce varying coefficients as well as nonlinear damping.

3. QUASI-LINEAR SOLUTION FOR VARYING DENSITY

The coefficients in the differential equation for a missile's pitching and yawing motion may be functions of the independent variable for a number of reasons: varying air density, varying missile mass and/or moments of inertia, varying aerodynamic coefficients as a result of their dependence on Mach number or Reynold's number. At the present the greatest interest lies in the influence of varying density on a missile leaving or entering the earth's atmosphere. Although most of the equations of this report will apply to any cause of varying coefficients, the effect of density will be our primary objective.

The actual density variation can be reasonably well approximated* by an exponential up to altitudes of 300,000 ft.

$$\rho = \rho_o e^{-\sigma z} \tag{35}$$

ρ_o = sea level density

z = altitude in feet and

$$\sigma = \frac{1}{22{,}000 \text{ ft}}$$

If $\theta(s)$ is the angle the flight path makes with respect to the vertical, z can be related to our independent variable s by the equation:

$$z = -\int_0^s \ell \cos\theta(s_1)\, ds_1 \tag{36}$$

where $0 < \theta < 90^\circ \rightarrow$ entering the atmosphere

$90^\circ < \theta < 180^\circ \rightarrow$ leaving the atmosphere

* Dommett[12] has approximated the ARDC model atmosphere by a set of four exponentials. This more accurate description could be used in the theory of this report and is described in the appendix.

Equations (35-36) can now be used to obtain derivatives of $\hat{M}_o$ and m_2.

$$\frac{\hat{M}_o'}{\hat{M}_o} = \frac{\tilde{\sigma} M_o}{\hat{M}_o} = \frac{\tilde{\sigma}}{1 - s_g} \tag{37}$$

$$\frac{m_2'}{m_2} = \frac{M_2'}{M_2} - \frac{\hat{M}_o'}{\hat{M}_o} = - \tilde{\sigma} \left[\frac{s_g}{1 - s_g} \right] \tag{38}$$

where $\tilde{\sigma} = \sigma \ell \cos \Theta = \frac{\rho'}{\rho}$.

Note the simple form these derivatives assume for zero spin. $(s_g = 0)$

With the introduction of varying $\hat{M}_o$ and m_2 we can now return to Equations (24-27) which were derived by the quasi-linear theory and consider the effect of this variation. In order to increase the generality of the results both linear and nonlinear aerodynamic damping will be introduced. Thus Equation (3) will be replaced by

$$\hat{\xi}'' - \hat{M}_o(1 + m_2\delta^2)\,\hat{\xi} = - H\hat{\xi}' + \left[M^* + iP(T - \frac{H}{2}) \right] \hat{\xi} \tag{39}$$

where $H = \frac{\rho S \ell}{2m} \left[\gamma C_{L_\alpha} - C_D - k_t^{-2} \left(C_{M_q} + \gamma C_{M_{\dot{\alpha}}} \right) \right]$

$T = \frac{\rho S \ell}{2m} \gamma \left[C_{L_\alpha} + k_a^{-2} C_{M_{p\alpha}} \right]$

$M^* = \frac{\rho S \ell}{2m} \gamma \left[k_t^{-2} C^* - C_{L_\alpha}' - \frac{\rho'}{\rho} C_{L_\alpha} + \frac{\rho S \ell}{2m} C_{L_\alpha} \left(k_t^{-2} C_{M_q} - C_D \right) \right]$

$\doteq \frac{\rho S \ell}{2m} \gamma \left[k_t^{-2} C^* - C_{L_\alpha}' \right]$

$k_t = \sqrt{\frac{I_y}{m\ell^2}}$ is transverse radius of gyration

$k_a = \sqrt{\frac{I_x}{m\ell^2}}$ is axial radius of gyration and

$C^* = C_{M_\alpha} - c_o - c_2\delta^2 = C^*(\delta^2, (\delta^2)')$

It should be emphasized that the aerodynamic coefficients defined in Equation (39) are coefficients and not derivatives. They may be functions of δ^2 and $(\delta^2)'$. C^* is the nonpotential part of C_{M_α}. Since this is a rather strange quantity, we will consider the value of it and the other quantities in Equation (39) for a nonspinning missile with cubic moments and forces. (Example 3 of Reference 6). The force and moment were assumed to have the form

$$C_D = e_o + e_2\delta^2 \tag{40}$$

$$C_{L_\alpha}\xi = \left[a_o + a_2\delta^2\right]\xi \tag{41}$$

$$\begin{aligned} C_m + iC_n &= -i\left\{\left[c_o + c_2\xi\bar{\xi} + c_{11}\xi\bar{\xi}'\right]\xi + \left[d_o + d_2\xi\bar{\xi}\right]\xi'\right\} \\ &= -i\left\{\left[c_o + c_2\delta^2 + c_{11}(\delta^2)'\right]\xi + \left[d_o + (d_2 - c_{11})\delta^2\right]\xi'\right\} \end{aligned} \tag{42}$$

$$C_{M_q} + C_{M_{\dot\alpha}} = d_o + (d_2 - c_{11})\delta^2 \tag{43}$$

$$C_{M_\alpha} = c_o + c_2\delta^2 + c_{11}(\delta^2)' \tag{44}$$

$$C^* = (c_{11})\,(\delta^2)' \tag{45}$$

$$H = \frac{\rho S \ell}{2m}\left[(a_o - e_o - k_t^{-2}d_o) + \left(a_2 - e_2 - k_t^{-2}(d_2 - c_{11})\right)\delta^2\right] \tag{46}$$

$$M^* = \frac{\rho S \ell}{2m}\left[k_t^{-2}c_{11} - a_2\right](\delta^2)' \tag{47}$$

If the small effect of aerodynamic damping on frequency is neglected, the substitution quasi-linear method yields the following relations.

$$\hat{\phi}_1' = \sqrt{-\hat{M}_o \left[1 + m_2(K_1^2 + 2K_2^2)\right]} \tag{48}$$

$$\hat{\phi}_2' = -\sqrt{-\hat{M}_o \left[1 + m_2(2K_1^2 + K_2^2)\right]} \tag{49}$$

$$\frac{K_1'}{K_1} = -\frac{\hat{\phi}_1''}{\hat{\phi}_1' - \hat{\phi}_2'} + \lambda_1 \tag{50}$$

$$\frac{K_2'}{K_2} = -\frac{\hat{\phi}_2''}{\hat{\phi}_2' - \hat{\phi}_1'} + \lambda_2 \tag{51}$$

where

$$\lambda_1 = \frac{-1}{2\pi(\hat{\phi}_1' - \hat{\phi}_2')} \int_0^{2\pi} \left\{ H \left[\hat{\phi}_1' + \hat{\phi}_2' \frac{K_2}{K_1} \cos\hat{\phi}\right] + M^* \left(\frac{K_2}{K_1}\right) \sin\hat{\phi} - P(T - \frac{H}{2}) \left[1 + \frac{K_2}{K_1} \cos\hat{\phi}\right]\right\} d\hat{\phi}$$

$$\lambda_2 = \frac{-1}{2\pi(\hat{\phi}_2' - \hat{\phi}_1')} \int_0^{2\pi} \left\{ H \left[\hat{\phi}_2' + \hat{\phi}_1' \frac{K_1}{K_2} \cos\hat{\phi}\right] - M^* \left(\frac{K_1}{K_2}\right) \sin\hat{\phi} - P(T - \frac{H}{2}) \left[1 + \frac{K_1}{K_2} \cos\hat{\phi}\right]\right\} d\hat{\phi}$$

Equations (48-49) may be differentiated and solved simultaneously for $\hat{\phi}_1''/(\phi_1' - \phi_2')$ and $\hat{\phi}_2''/(\phi_2' - \phi_1')$.

$$\frac{\hat{\phi}_1''}{\hat{\phi}_1' - \hat{\phi}_2'} = \frac{\hat{M}_o'}{\hat{M}_o} a_{11} + 2\lambda_1 a_{12} + 2\lambda_2 a_{13} + \frac{m_2'}{m_2} (a_{12} + a_{13}) \tag{52}$$

$$\frac{\hat{\phi}_2''}{\hat{\phi}_2' - \hat{\phi}_1'} = \frac{\hat{M}_o'}{\hat{M}_o} a_{21} + 2\lambda_1 a_{22} + 2\lambda_2 a_{23} + \frac{m_2'}{m_2}(a_{22} + a_{23}) \qquad (53)$$

where a_{jk} are defined in Table I.

Equations (52-53) can be used with Equations (50-51) to predict the variations in amplitude but the algebraic expressions are quite cumbersome. For two special cases of some importance considerable simplification is possible. The first case is that of a linear moment ($m_2 = 0$) and the second is motion through zero amplitude ($\delta_1 = |K_1 - K_2| = 0$). For a linear moment Equations (50-51) reduce to the very simple form

$$\frac{K_1'}{K_1} = \lambda_1 - (1/4)\frac{\hat{M}_o'}{\hat{M}_o} = \lambda_1 - \frac{\tilde{\sigma}}{4(1 - s_g)} \qquad (54)$$

$$\frac{K_2'}{K_2} = \lambda_2 - (1/4)\left(\frac{\hat{M}_o'}{\hat{M}_o}\right) = \lambda_2 - \frac{\tilde{\sigma}}{4(1 - s_g)} \qquad (55)$$

According to Equations (54-55) the aerodynamic damping (λ_j) of a reentering missile will grow from zero to sea level values, while the influence of the density gradient ($\tilde{\sigma}$) depends on the stability factor. The motion of a statically stable nonspinning missile will be damped by this term. If it possesses constant nonzero spin* this density gradient damping will grow from zero as s_g varies from minus infinity to its sea level value! A statically unstable missile with constant spin will have density-gradient-induced undamping which grows from zero as s_g decreases from plus infinity.

* McShane, Kelley, and Reno[10] consider the effect of varying spin by means of the WKB method.

TABLE I

$$a_{11} = \frac{-1}{2d} \left\{ (\hat{\phi}_1')^2 \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + 2\hat{M}_o m_2 (K_1^2 + K_2^2) \right] - 2(\hat{\phi}_2')^2 \hat{M}_o m_2 K_2^2 \right\}$$

$$a_{12} = \frac{\hat{M}_o m_2 K_1^2}{2d} \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + 2\hat{M}_o m_2 (K_1^2 - K_2^2) \right]$$

$$a_{13} = \frac{\hat{M}_o m_2 K_2^2}{d} \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + \hat{M}_o m_2 (2K_1^2 + K_2^2) \right]$$

$$a_{21} = \frac{-1}{2d} \left\{ (\hat{\phi}_2')^2 \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + 2\hat{M}_o m_2 (K_1^2 + K_2^2) \right] - 2(\hat{\phi}_1')^2 \hat{M}_o m_2 K_1^2 \right\}$$

$$a_{22} = \frac{\hat{M}_o m_2 K_1^2}{d} \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + \hat{M}_o m_2 (K_1^2 + 2K_2^2) \right]$$

$$a_{23} = \frac{\hat{M}_o m_2 K_2^2}{2d} \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + 2\hat{M}_o m_2 (K_2^2 - K_1^2) \right]$$

$$d = \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + 2\hat{M}_o m_2 (K_1^2 - K_1 K_2 + K_2^2) \right] \left[\hat{\phi}_1' \hat{\phi}_2' + \hat{M}_o + 2\hat{M}_o m_2 (K_1^2 + K_1 K_2 + K_2^2) \right]$$

This strange state of affairs may be quickly resolved. The quasi-linear technique assumes that changes over a cycle are reasonably small, i.e., $\left(\frac{\sigma}{4}\right)\left(\frac{2\pi}{\sqrt{-\hat{M}_o}}\right)$ is small. This condition determines a maximum altitude for which the theory is applicable. For some missiles a maximum altitude of 300,000 ft. is appropriate. At this altitude the effect of increasing spin can be determined and as can be seen from Equations (54-55) the damping for a statically stable missile and the undamping for a statically unstable missile are both reduced.

When the pitching and yawing motion goes through zero amplitude $\hat{\phi}_1' = -\hat{\phi}_2'$ and $K_1 = K_2 = \frac{\delta_2}{2}$. These relations are very helpful in reducing Equations (52-53) to a reasonable degree of complexity. Equations (50-51), then, become

$$\frac{K_1'}{K_1} = \frac{(4 + 3m)}{(8 + 5m)(8 + 9m)}\left[2\lambda_1(8 + 7m) - 4\lambda_2 m - \left(\frac{\hat{M}_o'}{2\hat{M}_o}\right)(8 + 5m)\right] - \frac{3\left(\frac{m_2'}{2m_2}\right)m}{8 + 9m} \tag{56}$$

$$\frac{K_2'}{K_2} = \frac{(4 + 3m)}{(8 + 5m)(8 + 9m)}\left[2\lambda_2(8 + 7m) - 4\lambda_1 m - \left(\frac{\hat{M}_o'}{2\hat{M}_o}\right)(8 + 5m)\right] - \frac{3\left(\frac{m_2'}{2m_2}\right)m}{8 + 9m} \tag{57}$$

If $\lambda_1 \neq \lambda_2$, K_1 will not remain equal to K_2. Equations (56-57), therefore, are valid only when K_1 is nearly equal to K_2 or the minimum δ is near zero.

For planar motion, spin is zero and the aerodynamic damping rates are equal. Equations (56-57) collapse into a single equation.

$$\frac{\delta_2'}{\delta_2} = \frac{\left(2\lambda - \frac{M_o'}{2M_o}\right)(4 + 3m) - 3\left(\frac{m_2'}{2m_2}\right) m}{(8 + 9m)} \qquad (58)$$

$$= 2\left(\frac{4 + 3m}{8 + 9m}\right)\left(\lambda - \frac{\tilde{\sigma}}{4}\right)$$

Coakley, Laitone and Mass[5] derived Equation (58) for $\lambda = 0$ in a somewhat different manner. The pole at $m = -8/9$ has rather strange consequences. According to Equation (58) the angular motion of a missile with a type (b) moment, ascending in the earth's atmosphere, will grow in amplitude until $\delta_2 = -(8/9)\frac{M_o}{M_2}$. This upper bound does not exist for $\tilde{\sigma} = 0$ since Equation (29) has solutions for $-1 < m < 0$. This difficulty will be resolved in the next section.

4. PERTURBATION SOLUTION FOR VARYING DENSITY

In Reference 6, the nonlinear damping is treated by perturbing the exact solution of cubic static moment equation. As is shown in Section 2 this solution involves an elliptic function. The perturbation method makes use of two quasi-constants of the motion, the energy and angular momentum.

$$C_1 = \hat{\xi}' \bar{\hat{\xi}}' + \hat{V}(\delta^2) = (\delta')^2 + (\delta\hat{\theta}')^2 + \hat{V}(\delta^2) \tag{59}$$

$$C_2 = i(\bar{\hat{\xi}}' \hat{\xi} - \hat{\xi}' \bar{\hat{\xi}}) = 2\delta^2\hat{\theta}' \tag{60}$$

$$\text{where } \hat{V}(\delta^2) = - \hat{M}_o(\delta^2 + \frac{m_2}{2} \delta^4)$$

Equation (39) may now be rewritten in terms of derivatives of these quasi-constants

$$C_1' = - 2H(C_1 - \hat{V}) + (M^*)(\delta^2)' + P(T - \frac{H}{2}) C_2 + \frac{\hat{M}_o'}{\hat{M}_o} \hat{V} - \hat{M}_o m_2' \left(\frac{\delta^4}{2}\right) \tag{61}$$

$$C_2' = - HC_2 + 2P(T - \frac{H}{2}) \delta^2 \tag{62}$$

H, $\hat{V}$, M^* and T are functions of δ^2 and $(\delta^2)'$. These quantities are periodic functions given by Equations (28-30). The right sides of Equations (61-62) may be averaged over the period, P^*, to yield functions of C_1, C_2, δ_1^2, and δ_2^2. Generalized modal amplitudes may now be introduced.

$$\delta_1^2 = (\sqrt{x} - \sqrt{y})^2 \tag{63}$$

$$\delta_2^2 = (\sqrt{x} + \sqrt{y})^2 \tag{64}$$

The quasi-constants may be expressed in terms of the generalized modal amplitudes

$$C_1 = - \hat{M}_o \left[2(x + y) + \frac{m_2}{2} (3x^2 + 10xy + 3y^2)\right] \qquad (65)$$

$$C_2 = 2(x - y) \sqrt{- \hat{M}_o(1 + m_2(x + y))} \qquad (66)$$

Equations (65-66) can be differentiated and solved for x' and y'.

$$x' = \frac{\left[2 + m_2(x + 3y)\right] (C_1')^* + \left[2 + m_2(5x + 3y)\right] \sqrt{- \hat{M}_o(1 + m_2(x + y))}\ (C_2')^*}{- 2D\hat{M}_o} \qquad (67)$$

$$y' = \frac{\left[2 + m_2(3x + y)\right] (C_1')^* - \left[2 + m_2(3x + 5y)\right] \sqrt{- \hat{M}_o(1 + m_2(x + y))}\ (C_2')^*}{- 2D\hat{M}_o} \qquad (68)$$

where $D = 4\hat{M}_o^{-2} \omega^2 \hat{\omega}^2$

$$= \left[2 + m_2(3x + 2\sqrt{xy} + 3y)\right] \left[2 + m_2(3x - 2\sqrt{xy} + 3y)\right]$$

$$(C_1')^* = C_1' - \left(\frac{\hat{M}_o'}{\hat{M}_o} + \frac{m_2'}{m_2}\right) C_1 - 2\hat{M}_o \left(\frac{m_2'}{m_2}\right) (x + y)$$

$$(C_2')^* = C_2' - \frac{1}{2} \left[\frac{\hat{M}_o'}{\hat{M}_o} + \frac{m_2'}{m_2}\right] C_2 - \left(\frac{m_2'}{m_2}\right) \frac{\hat{M}_o(x - y)}{\sqrt{- \hat{M}_o(1 + m_2(x + y))}}$$

$(C_1')^*$ and $(C_2')^*$ may be evaluated by the use of Equations (61-62)

$$(c_1')^* = \left\{ -2 \left[H + \frac{\hat{M}_o'}{2\hat{M}_o} + \frac{m_2'}{2m_2} \right] (C_1 - \hat{V}) + (M^*) (\delta^2)' \right. \tag{69}$$

$$\left. + P(T - \frac{H}{2}) C_2 \right\}_a + \hat{M}_o \left(\frac{m_2'}{m_2} \right) \left[\delta_a^2 - 2(x + y) \right]$$

$$(c_2')^* = \left\{ - \left[H + \frac{\hat{M}_o'}{2\hat{M}_o} + \frac{m_2'}{2m_2} \right] C_2 + 2P(T - \frac{H}{2}) \delta^2 \right\}_a \tag{70}$$

$$- \left(\frac{m_2'}{m_2} \right) \frac{\hat{M}_o (x - y)}{\sqrt{- \hat{M}_o (1 + m_2(x + y))}}$$

$$\text{where } \{ \ \}_a = \frac{1}{P^*} \int_0^{P^*} \{ \ \} \, ds$$

For a nonspinning missile in an exponential atmosphere m_2' is zero and the effect of the density gradient is to replace H by $H + \frac{\sigma}{2}$. In fact an inspection of the derivation of preceeding equations indicates that this replacement of H by $H + \frac{\sigma}{2}$ for a nonspinning missile in an exponential atmosphere is true for any $\hat{V}$, i.e., any nonlinear static moment. Here again it must be emphasized that the results of the theory are valid only when the density-gradient-induced damping is small over a cycle of the basic periodic motion. The averages of δ^{2n} for circular motion are given in Reference 6 and for planar motion in Reference 11. For planar motion $(x = y = (1/4)\ \delta_2^2,\ C_2 = 0)$ and a constant H, Equations (67-68) collapse to

$$\frac{\delta_2'}{\delta_2} = \frac{-(2 + m - 2A_2 - mA_4)}{(1 + m)} \left[\frac{H}{2} + \frac{\tilde{\sigma}}{4}\right] . \qquad (71)$$

For a type (b) moment

$$A_2 = k_p^{-2} \left[1 - E_p/K_p\right]$$

$$A_4 = (1/3)\, k_p^{-2} \left[2(1 + k_p^2)A_2 - 1\right]$$

$K_p = K(k_p)$ complete elliptic integral of the first kind

$E_p = E(k_p)$ complete elliptic integral of the second kind

$k_p^2 = -\frac{m}{2 + m}$ modulus for planar motion

The coefficient of $\frac{\tilde{\sigma}}{4}$ in Equation (71) has a pole at $m = -1$. Since this is the upper bound for δ_2 in Equation (29), this is much more reasonable than the pole at $m = -8/9$ in Equation (58). The coefficient of $\frac{\tilde{\sigma}}{4}$ in these two Equations is plotted versus m in Figure 4. Equation (3) was numerically integrated for planar motion and an exponential density and the logarithmic derivative of δ_2 calculated. Numerical values of the coefficient of $\frac{\tilde{\sigma}}{4}$ could thus be determined and are plotted in Figure 4. The agreement with Equation (71) is very gratifying.

5. SUMMARY

1. The improved quasi-linear and substitution methods for a cubic static moment predict frequencies to very good accuracy.

2. These methods do not predict the effect of varying density to quite the same accuracy.

3. The effect of density gradient on the damping of missiles with linear static moment is to add $\tilde{\sigma}/4(1 - s_g)$ to each damping exponent.

4. The effect of density gradient on the damping of a nonspinning missile with a nonlinear static moment is to add $\tilde{\sigma}/2$ to H. This modification allows the use of the results of the perturbation theory.

Charles H. Murphy
CHARLES H. MURPHY

CUBIC STATIC MOMENTS

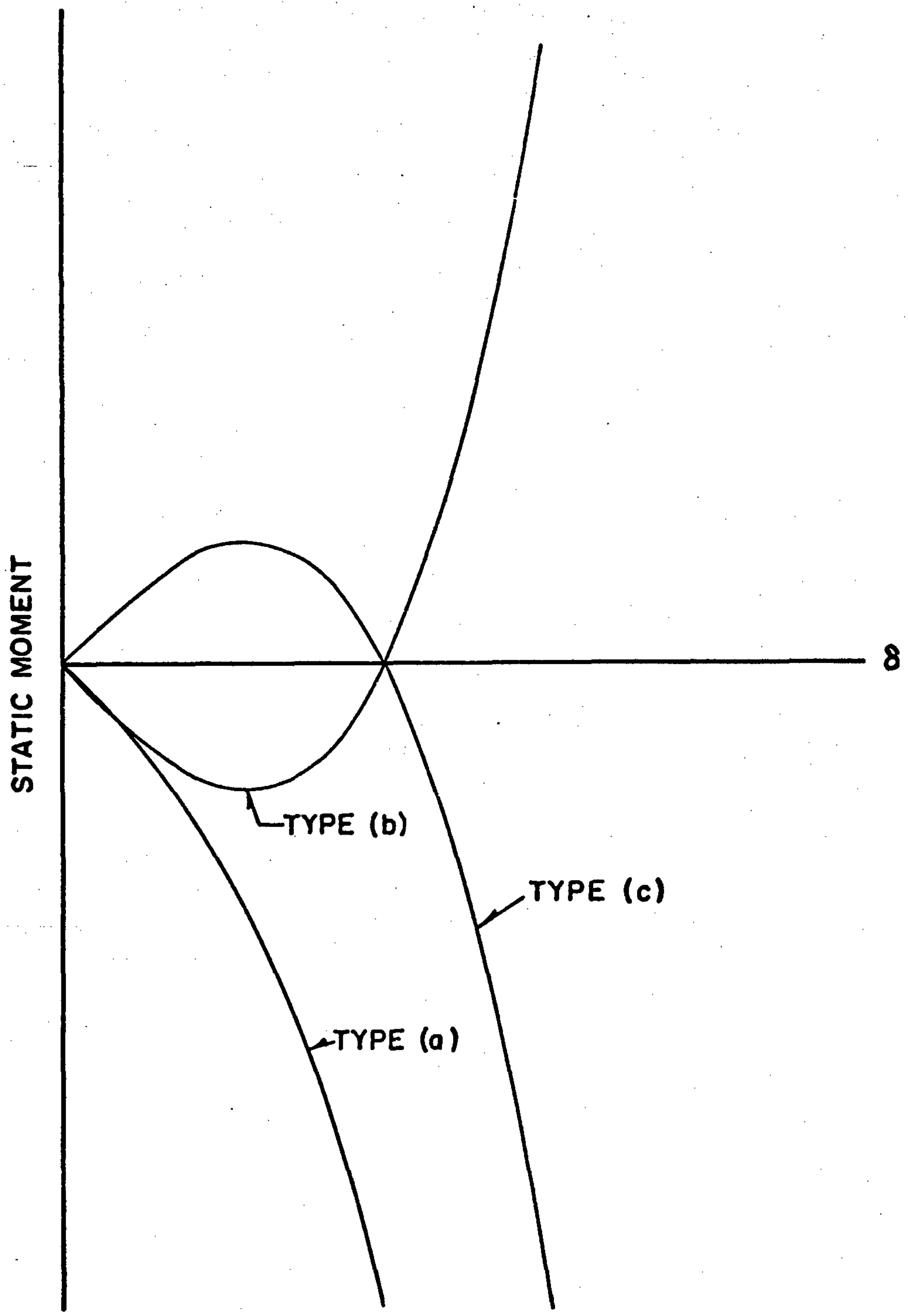
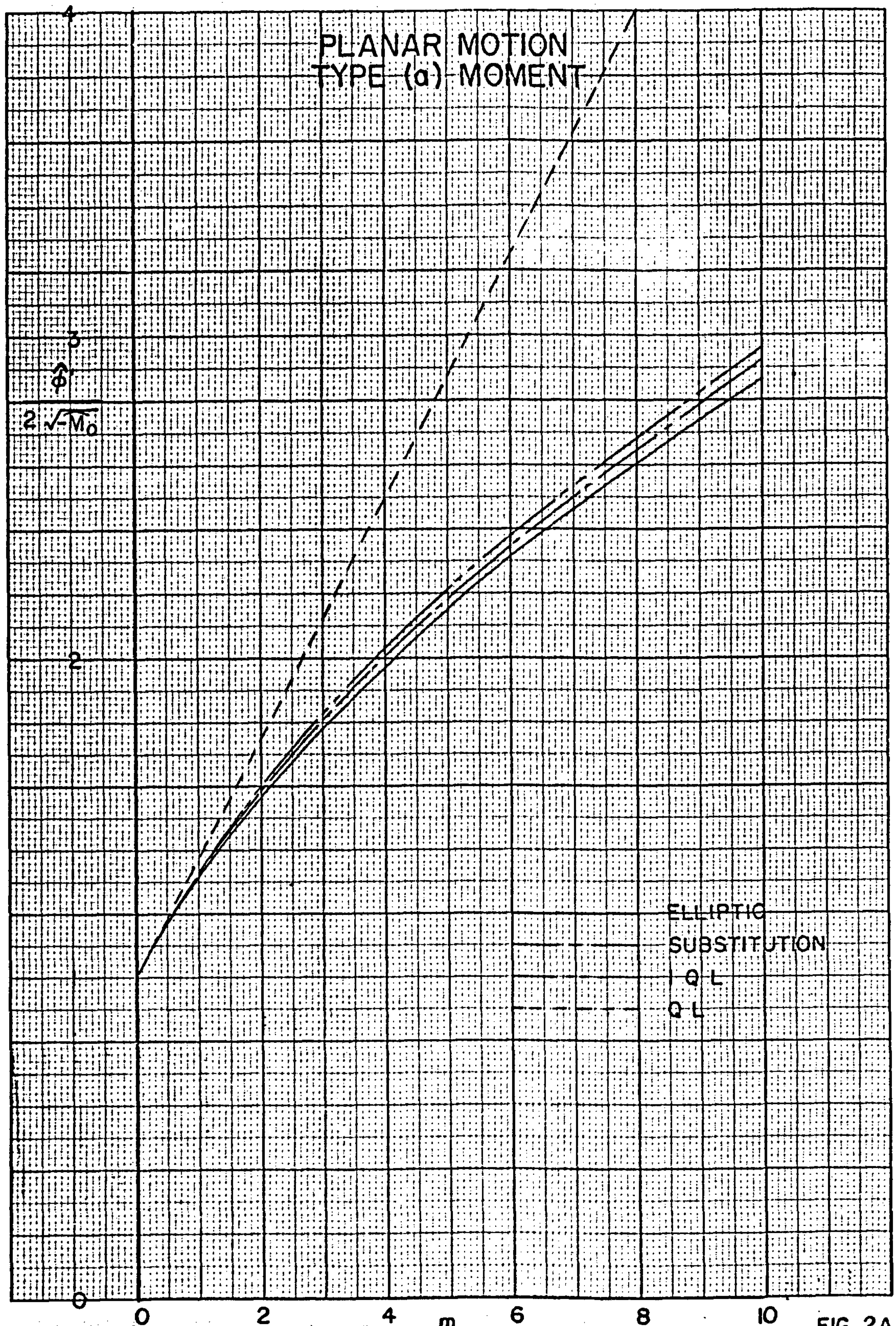


FIG. 1



PLANAR MOTION
TYPE (a) MOMENT
4
3
$\hat{\phi}'$
$2\sqrt{-M_0}$
2
0
ELLIPTIC
SUBSTITUTION
I Q L
Q L
0
2
4
m
6
8
10

FIG. 2A

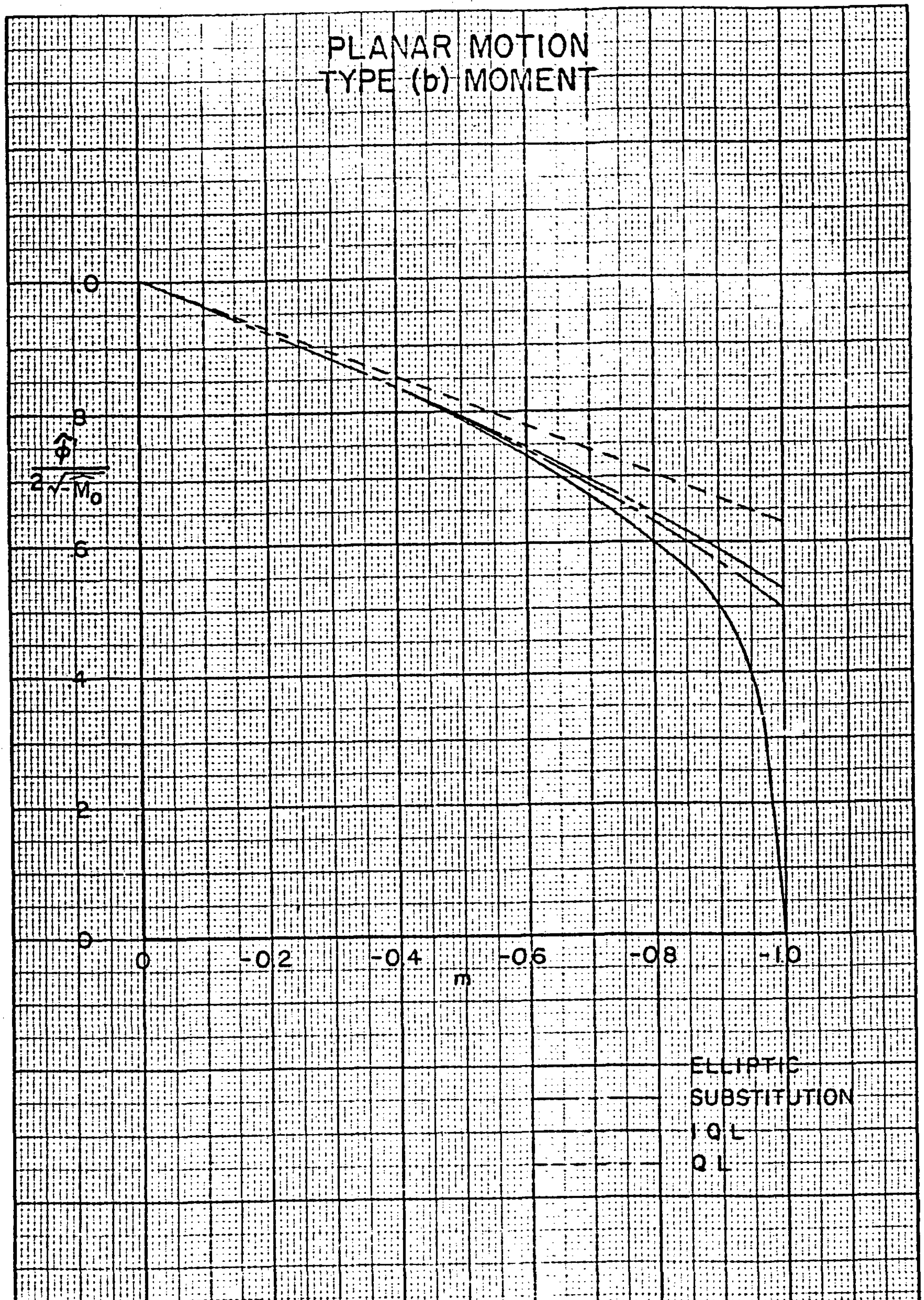

PLANAR MOTION
TYPE (b) MOMENT
$\frac{\hat{\Phi}}{2\sqrt{-M_0}}$
1.0
.8
.6
.4
.2
0
0
-0.2
-0.4
-0.6
-0.8
-1.0
m
ELLIPTIC
SUBSTITUTION
IQL
QL

FIG. 2B

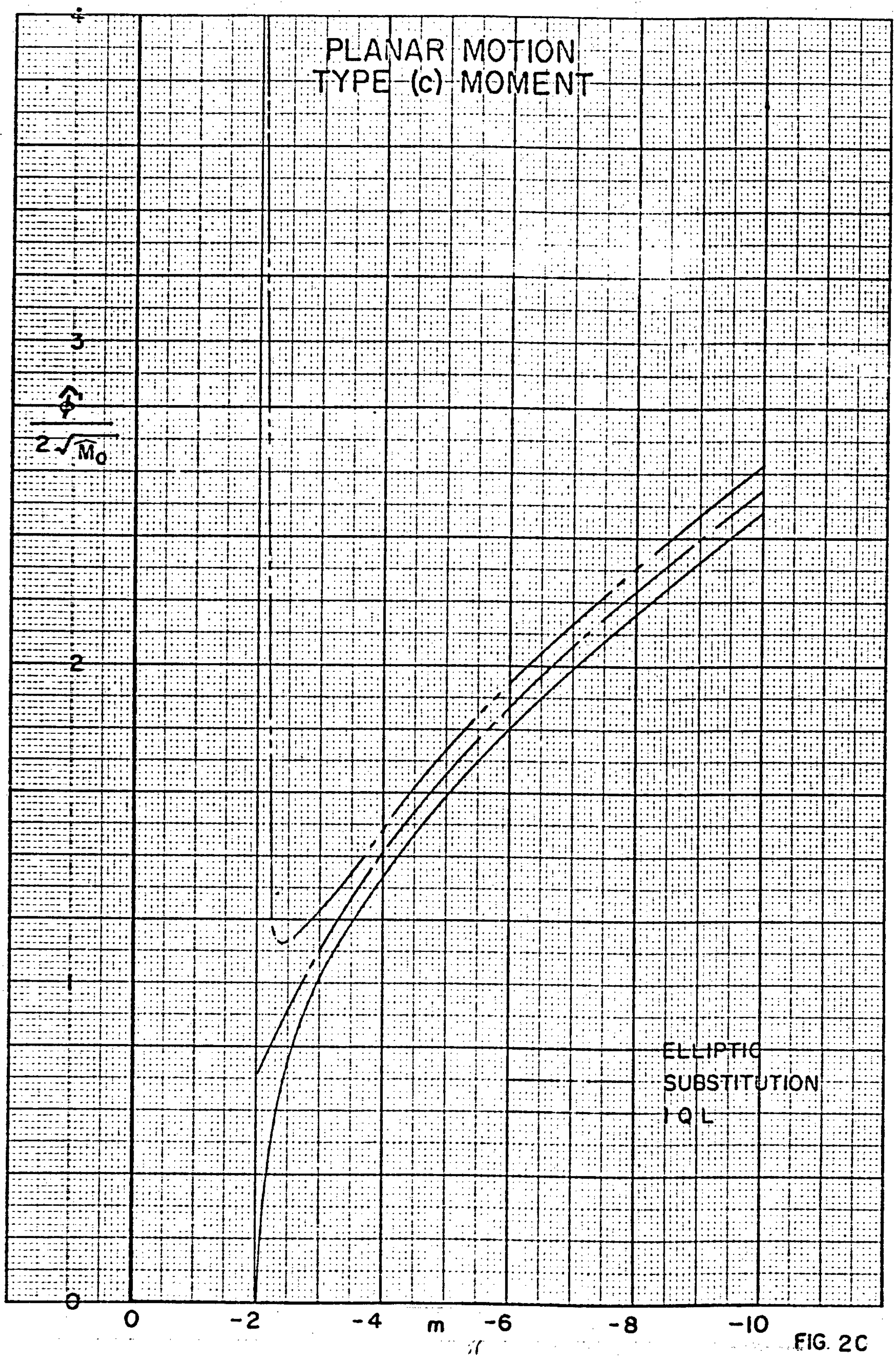

PLANAR MOTION
TYPE (c) MOMENT
$\frac{\hat{\varphi}}{2\sqrt{\hat{M}_0}}$
3
2
1
0
0
-2
-4
m
-6
-8
-10
ELLIPTIC
SUBSTITUTION
I Q L

FIG. 2C

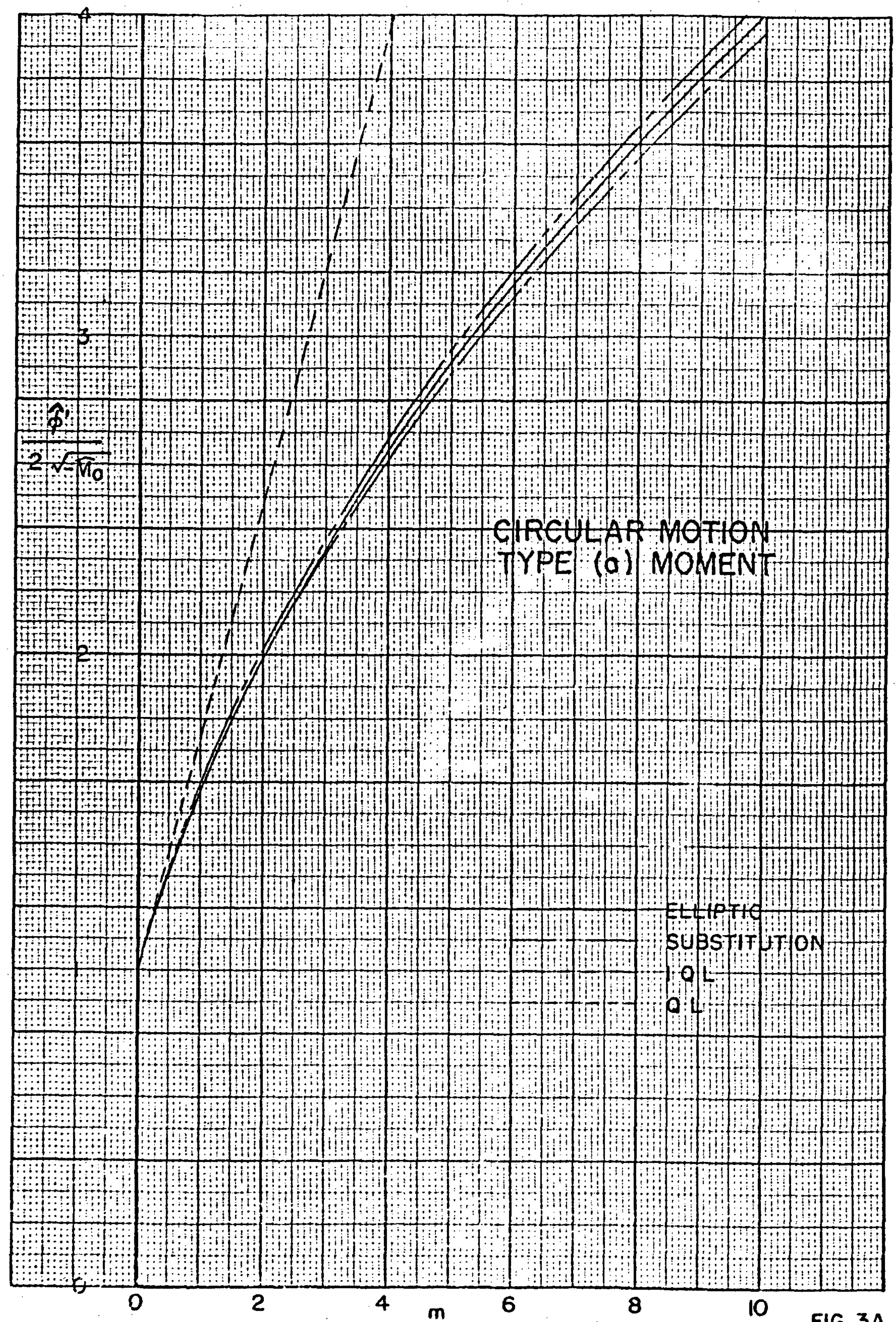
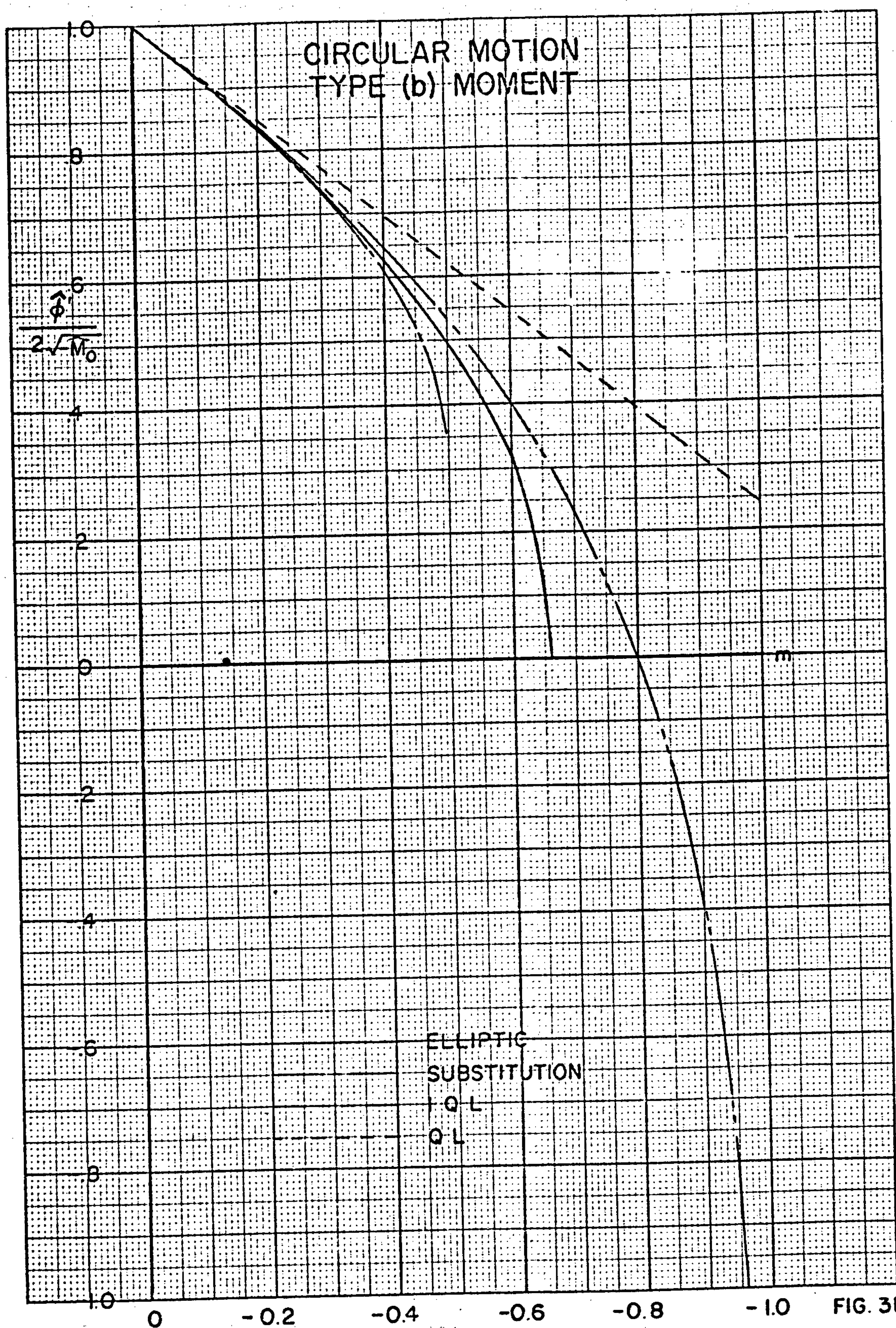

CIRCULAR MOTION
TYPE (a) MOMENT
ELLIPTIC
SUBSTITUTION
I Q L
Q L
4
3
2
0
0
2
4
6
8
10
m

FIG. 3A


CIRCULAR MOTION
TYPE (b) MOMENT
$\frac{\hat{\phi}'}{2\sqrt{M_0}}$
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
m
ELLIPTIC
SUBSTITUTION
I Q L
Q L
0
- 0.2
-0.4
-0.6
-0.8
- 1.0
FIG. 3B

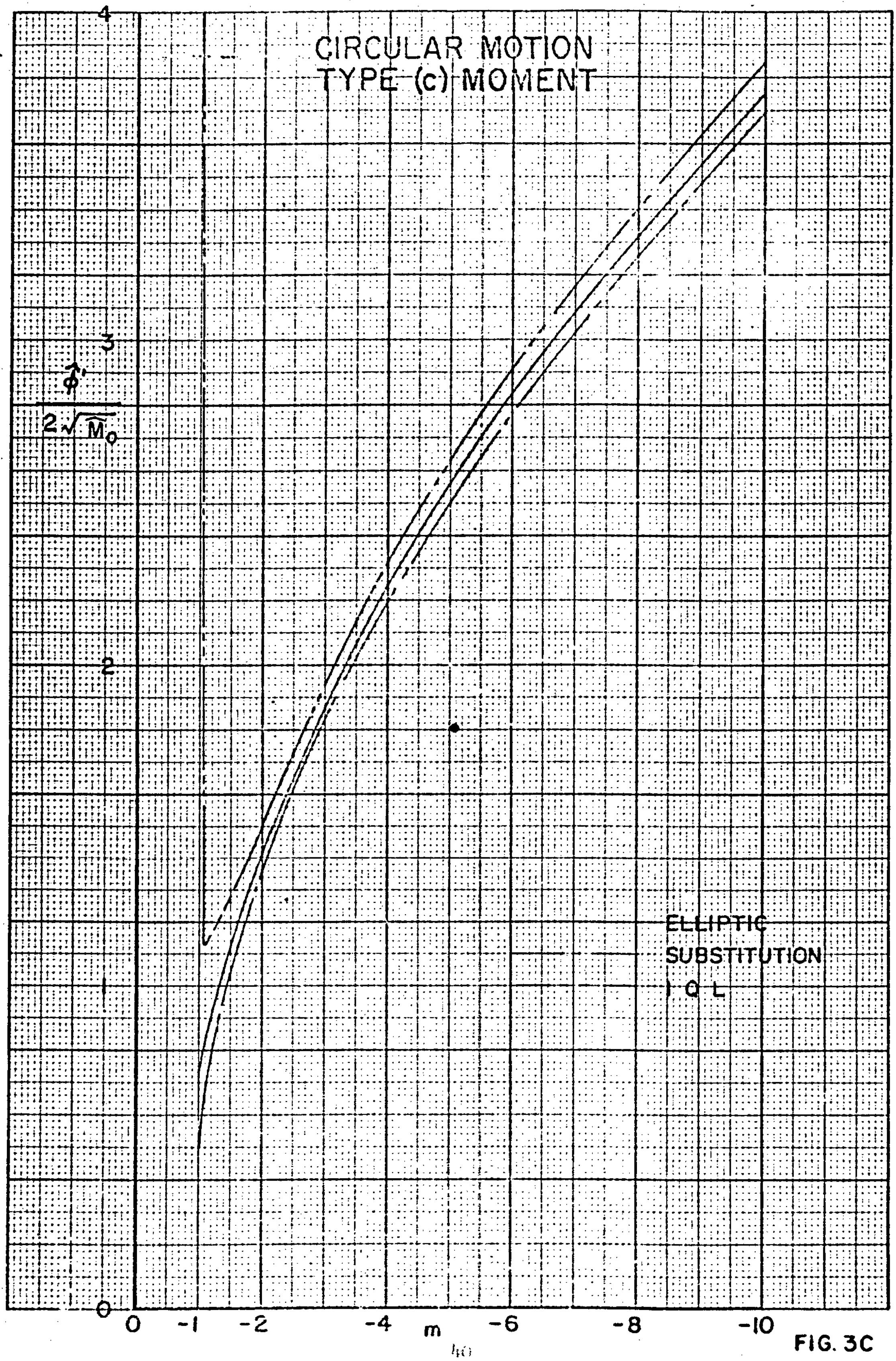

CIRCULAR MOTION
TYPE (c) MOMENT
ELLIPTIC
SUBSTITUTION
I Q L
4
3
2
1
0
0 -1 -2 -4 -6 -8 -10
m

FIG. 3C

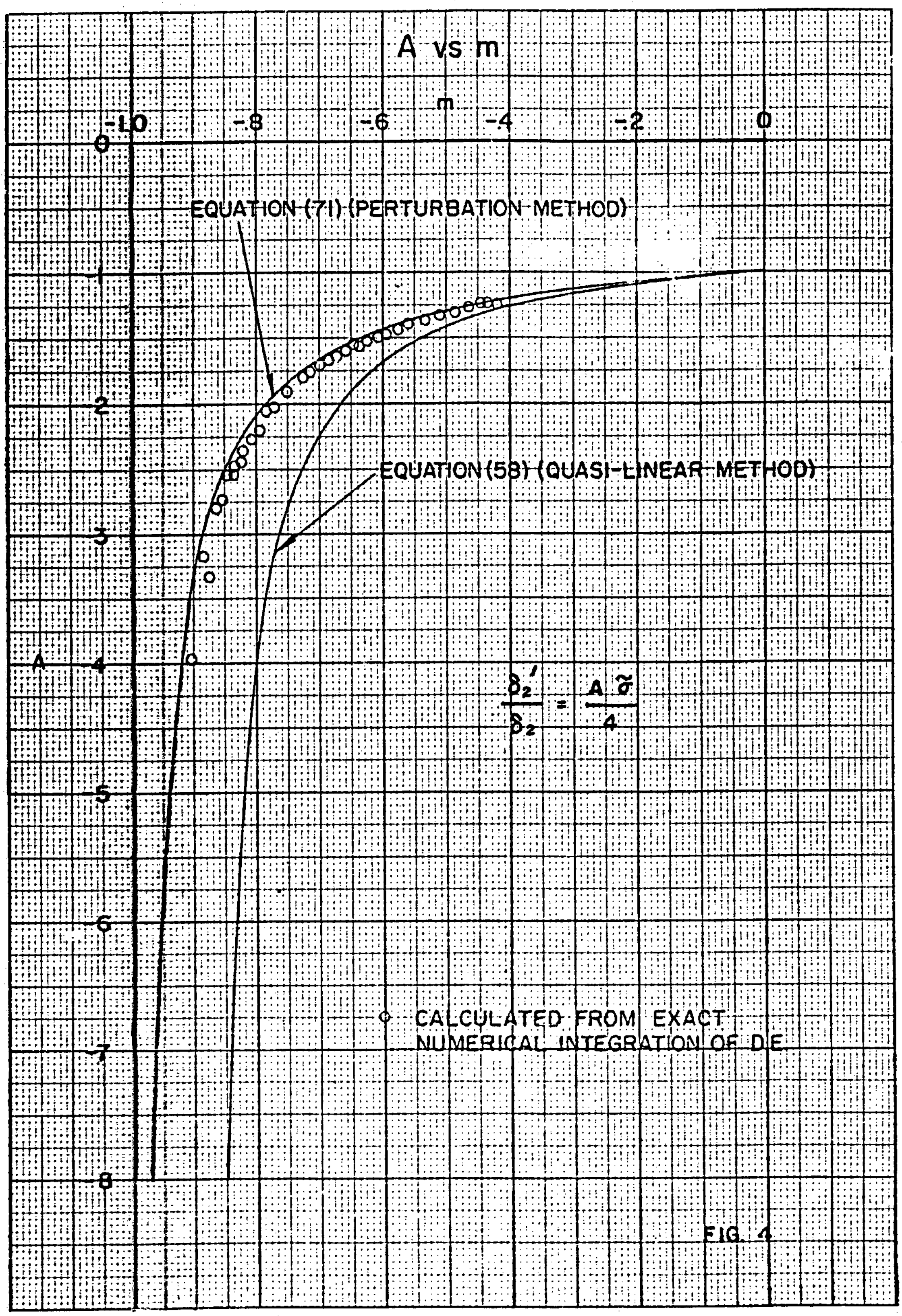

A vs m
m
-1.0
-.8
-.6
-.4
-.2
0
0
.1
.2
.3
.4
.5
.6
.7
.8
A
EQUATION (71) (PERTURBATION METHOD)
EQUATION (58) (QUASI-LINEAR METHOD)
$\frac{\delta_2'}{\delta_2} = \frac{A\,\tilde{\sigma}}{4}$
CALCULATED FROM EXACT NUMERICAL INTEGRATION OF D.E.

FIG. 4

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APPENDIX

A QUADRI-EXPONENTIAL ATMOSPHERE

In Reference 12, the ARDC model atmosphere is approximated by a set of four exponentials of the form

$$\rho = \rho_i e^{-\sigma_i z} \qquad z_{i-1} < z < z_i \qquad \text{(A1)}$$

$$T = \frac{M_{g_o}}{\sigma_i R} + \left[T_i - \frac{M_{g_o}}{\sigma_i R} \right] e^{\sigma_i z} \qquad z_{i-1} < z < z_i \qquad \text{(A2)}$$

where $\frac{M_{g_o}}{R} = 1.041 \times 10^{-2}$ oK/ft

z_o = 0 and the other constants

are given in the Table.

The values of σ given in the Table could be used in the theory of this report at the appropriate altitudes instead of 1/22,000 ft.

TABLE*

z_i ft.	$1/\sigma_i$ ft.	ρ_i slugs/ft.^{3}	T_i ^oK
35,000	30,800	2.377×10^{-3}	288.16
140,000	21,000	4.034×10^{-3}	218.91
240,000	26,900	9.471×10^{-4}	279.68
300,000	18,600	5.099×10^{-2}	193.41

* These values are rounded from those of Reference 12.

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EFFECT OF VARYING AIR DENSITY ON THE NONLINEAR PITCHING AND YAWING MOTION OF A SYMMETRIC MISSILE
Charles H. Murphy

Missiles - Exterior ballistics
Missile flight - Air density

BRL Report No. 1162 February 1962

DA Proj No. 503-03-001, OMSC No. 5010.11.814
UNCLASSIFIED Report

Although the effect of an exponentially varying air density on the angular motion of a re-entering missile has been intensively studied for linear aerodynamic moments, this effect for nonlinear moments has only been investigated by a quasi-linear technique and this work was limited to planar motion. In this report, various modifications of the quasi-linear analysis for a cubic static moment are compared with the exact solutions and the best of these used to determine the general effect of varying density on combined pitching and yawing motion. Next the perturbation method which makes use of the exact solution for a cubic static moment is considered and the necessary modifications introduced by the varying density are made. The predicted density gradient induced damping for planar motion is determined for both theories and compared with the numerical integration of the exact differential equation. The perturbation prediction is found to be superior.

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