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OPTIMUM STRUCTURAL REPRESENTATION IN AEROELASTIC ANALYSES

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AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

Project No. 8219, Task No. 821901

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Engineering Associates, an Affiliate of Susquehanna Sciences, Inc.
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FOREWORD

This report was prepared for the Flight Control Laboratory, Directorate of Aeromechanics, Deputy for Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio, by Computer Engineering Associates, Pasadena, California. The research and development work was accomplished under Air Force Contract No. AF 33(616)-7658, "Electronic Analog Computer Study of Approximate Flexible Airframe-Autopilot Transfer Functions". H. M. Davis of the Flight Control Laboratory was the project engineer on this contract. The senior author, Robert G. Schwendler, is a research scientist at Computer Engineering Associates. The principal investigator, Richard H. MacNeal, Ph.D., is director of engineering analysis at Computer Engineering Associates. The research covered by this report was started on 5 October 1960 and completed on 27 October 1961.

ABSTRACT

This report contains an investigation of optimum methods for the mathematical representation of the structure of an airborne vehicle. Particular emphasis in this study was placed on aeroelastic problems associated with stability and control, but the results and conclusions of the study are applicable to any aeroelastic problem.

A method is presented whereby the elasticity of the structure is completely and rigorously included while the mass of the structure is expressed in terms of a finite number of normal coordinates of the system.

Analog computer investigations of the dynamic response of three typical aircraft configurations were made using the method described above. These analog computer investigations clearly demonstrate the utility of the structural representation presented and also define the approximations which can be made in the representation in terms of the requirements of any particular analysis.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER


Charles B. Westbrook
Chief, Aerospace Mechanics Branch
Flight Control Laboratory

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LIST OF SYMBOLS

A. MATRIX NOTATION

$\{ \}$	Column matrix, vector
$[\]$	Rectangular matrix
$\begin{bmatrix} & \\ & \end{bmatrix}$	Diagonal matrix
$\begin{bmatrix} 1 & \\ & \end{bmatrix}$	Unit diagonal matrix
$\{ \}^T, [\]^T$	Transposed matrices
$[\]^{-1}$	Inverted matrix

B. ROOT SYMBOLS

$\{ F \}$	Forces, either physical or normal coordinates
$[K]$	Stiffness coefficient matrix, either physical or normal coordinates
$[m]$	Mass matrix, either physical or normal coordinates
$[Q]$	Aerodynamic force coefficients (aerodynamic force per unit physical coordinate displacement)
$[X]$	Flexibility matrix for physical coordinates (displacement per unit force)
$\{ y \}$	Displacements in physical coordinates
$[Y]$	Mechanical admittance for normal coordinates (generalized force per unit normal coordinate displacement)
$[Z]$	Flexibility matrix for a supported system measured at zero frequency, i.e. static deflection influence coefficients
$\{ \xi \}$	Normal coordinate displacements
$\{ \Phi \}$	Eigenvector
$[\Phi]$	Modal matrix, array of eigenvectors
ω	Frequency, radians/sec

LIST OF SYMBOLS (Cont'd)

C. SUBSCRIPTS AND SUPERSSCRIPTS		Page
f	Finite frequency modes	14
g	Restrained physical coordinates	22
i	Undifferentiated physical coordinates	3
i	Physical coordinates where masses are located	30
m	Unrestrained physical coordinates	22
o	Static value	10
o	Zero frequency modes	19
r	Undifferentiated modal coordinates	4
s	Sensor coordinates	10
s	State of uniform acceleration	26
δ	Control system deflection	10
∞	Infinite frequency modes	14

D. SPECIAL SYMBOLS AND IMPORTANT SYMBOLS

$[A_{gm}]$	$= [\Phi_{go}^T]^{-1} [\Phi_{mo}^T] = \text{reactions at g coordinates due to static loads at m coordinates}$	24
$[B_{mm}]$	$= [\Phi_{mo}] [\Phi_{mo}]^T [m_{mm}]$	32
D	Aeroelastic index, summation of diagonal terms in $[Q_{mm}] [X_{mm}]$	38
g	Damping factor = 2 x per unit critical damping	100
$[K_r]$	Generalized stiffness matrix	8
$[m_r]$	Generalized mass matrix	8
$[\bar{Q}_{ii}]$	$= ([I] - [Q_{ii}] [X_{ii}])^{-1} [Q_{ii}]$	87

LIST OF SYMBOLS (Cont'd)

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$\begin{bmatrix} Q \\ m \end{bmatrix}^0$	Aerodynamic influence coefficient matrix with terms depending on time derivatives omitted	46
s	Derivative operator	12
$\begin{bmatrix} T \end{bmatrix}$	Transfer function matrix	10
$\begin{bmatrix} X_{ii}^{(f)} \end{bmatrix}$	Flexibility matrix due to finite frequency modes	21
$\begin{bmatrix} X_{ii}^{(\infty)} \end{bmatrix}$	Residual flexibility matrix	17
$\begin{bmatrix} Y_r \end{bmatrix}$	$= \begin{bmatrix} K_r \end{bmatrix} - \omega^2 \begin{bmatrix} m_r \end{bmatrix}$	9

SECTION I

INTRODUCTION

Collar's familiar aeroelastic triangle (Reference 1) provides a convenient reference by which to classify the aeromechanical aspects of flight. According to Collar's triangle aeromechanical problems are concerned with three kinds of forces - aerodynamic forces, inertia forces and elastic forces. For a completely rigorous analysis of any aeromechanical problem all three types of forces must be considered simultaneously. There are, however, many problems where significant results are obtained by ignoring one or two of the basic types of forces or where the total problem can be separated into parts involving only one or two of the three basic types of forces.

Stability and control analysis may be defined as an area of aeromechanical investigation that is concerned with the attitude of the vehicle as a whole, i.e., with the rigid body motions of the vehicle. Historically, elastic deformations were at first ignored in the analysis of stability and control. Later the importance of static elastic deformations, especially for certain specialized problems such as divergence and control effectiveness, was recognized. Today, with the advent of highly elastic aircraft and missiles, stability and control analyses should also include a consideration of the elastic-vibratory motions of the vehicles.

This study relates to the approximate formulation of any aeroelastic problem but with particular emphasis on stability and control problems and is concerned with the adequate representation of:

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1. Rigid body motions
2. Static elastic deformations
3. Elastic-vibratory motions

This study seeks to answer the question, "In an aeroelastic problem what approximations are suitable in the mathematical representation of the structure?". It is not concerned with methods of solution or with the representation of aerodynamic forces although the results are generally applicable to most solution methods and to any aerodynamic force representation.

While a cursory examination was made of other types of structural representation, only the representation of a system in terms of its normal coordinates was selected for closer scrutiny because:

1. The modal approach provides a rational means for reducing the order of the equations on the basis of frequency discrimination. The modal representation of structures has been widely used in the field of flutter analysis for this same reason.
2. The modal approach provides a simple, canonical form of the equations of motion which facilitates generalization to any dynamic system. The ordinary dynamic equations of rigid body aircraft motion are a limiting case of these equations.

A generality of application was preserved in this study so that its results are useful not only in the field of stability and control but also in the fields of flutter analysis, dynamic loads analysis and the analyses of steady maneuver loads.

SECTION 2

NORMAL MODE THEORY AND ITS APPLICATION TO AEROELASTIC PROBLEMS

2.1 INTRODUCTION TO THE THEORY OF NORMAL MODES

For the reasons discussed in Section 1 the normal mode approach has been selected as the optimum method for reducing the order of the mathematical representation of the structure involved in an aeroelastic problem. Although normal mode theory has been widely used by dynamicists and is readily available in literature, its basic concepts will be presented in this section to aid the reader who is unfamiliar with them and to provide a continuity of notation with developments to be subsequently derived. The development of normal mode theory presented here is taken from Reference 2 .

The structures considered in this report are exclusively linear, conservative mechanical systems. The effects of structural damping are assumed to be small in comparison with those of aerodynamic damping in the aeroelastic analysis and will be omitted. This analysis deals specifically with lumped constant systems and lumped constant approximations to distributed systems but the results are expressed in forms that can be easily extended to distributed systems.

The equations describing the forced, steady state vibration of a linear, conservative, lumped constant, mechanical system can be written in the following matrix form

$$\left(\left[K_{ii} \right] - \omega^2 \left[m_{ii} \right] \right) \{ y_i \} = \{ F_i \} \quad (2.1)$$

where:

$\left[K_{ii} \right]$ is the stiffness (inverse influence coefficient)

matrix of the system in terms of the physical coordinate system (i).

$[K_{ii}]$ is in general a symmetric, singular but non-negative matrix.

Non-negative means that for any column matrix $\{\beta\}$

$$\{\beta\}^T [K_{ii}] \{\beta\} \geq 0$$

Stated physically this means that the potential energy of the system can never be negative. The physical significance of a singular stiffness matrix will be discussed later in this section.

$[m_{ii}]$ is a symmetrical square matrix representing the inertia of the system in terms of the physical coordinate system (i). $[m_{ii}]$ is also in general a singular but non-negative matrix.

$\{y_i\}$ is a column matrix containing the displacements of the physical coordinates of the system.

$\{F_i\}$ is a column matrix representing the magnitude of the external forces applied to the system in the physical coordinates of the system. These are sinusoidal with radian frequency ω , and all in phase.

The equation describing the free vibrations of the system is

$$\left([K_{ii}] - \omega^2 [m_{ii}] \right) \{y_i\} = 0 \quad (2.2)$$

The solutions of this equation are the eigenvalues, ω_r^2 and the corresponding eigenvectors $\{\phi_i^{(r)}\}$ such that

$$\left([K_{ii}] - \omega_r^2 [m_{ii}] \right) \{\phi_i^{(r)}\} = 0 \quad (2.3)$$

If $[K_{ii}]$ is singular with defect p (meaning that the determinant of $[K_{ii}]$ and all of its first $p-1$ minors vanish) then there exist

p linearly independent vectors $\{\Phi_i^{(p)}\}$ such that

$$[K_{ii}] \{\Phi_i^{(p)}\} = 0 \quad (2.4)$$

The p vectors $\{\Phi_i^{(p)}\}$ are eigenvectors of equation (2.2) with $\omega^2 = 0$. Physically they correspond to the so-called rigid body motions of the system and therefore the physical significance of a singular stiffness matrix $[K_{ii}]$ is that it represents the spring constants of a system free in p coordinates of space.

Similarly if $[m_{ii}]$ is singular with defect q , then there exist q linearly independent vectors $\{\Phi_i^{(q)}\}$ such that

$$[m_{ii}] \{\Phi_i^{(q)}\} = 0 \quad (2.5)$$

The q vectors $\{\Phi_i^{(q)}\}$ are eigenvectors of the equation

$$\left(\frac{1}{\omega^2} [K_{ii}] - [m_{ii}] \right) \{y_i\} = 0 \quad (2.6)$$

with $1/\omega^2 = 0$. We shall consider them to be eigenvectors of equation (2.2) with $\omega^2 = \infty$. The physical significance of a singular mass matrix $[m_{ii}]$ is that it represents the inertia properties of a system described in the physical coordinates (i), where q coordinates have neither mass nor "mutual mass" associated with them. For physically realizable systems this condition may be considered as the limit of a realistic condition where q coordinates of a structure have significant stiffness coupling to the other i - q coordinates but have insignificant masses associated with them.

These "massless" coordinates may be easily eliminated from the equations in a solution of the normal modes of vibration of a system.

However, since the expressions for the aerodynamic forces on a system may be written in terms of such coordinates, they may not generally be conveniently eliminated from an aeroelastic solution.

We wish next to show that the highest power of ω^2 occurring in the characteristic equation of equation (2.2) is $(\omega^2)^{i-q}$. This characteristic equation is the determinant

$$\left| \begin{bmatrix} K_{ij} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{ij} \end{bmatrix} \right| = 0 \quad (2.7)$$

In the case of a diagonal mass matrix $\begin{bmatrix} m_{ij} \end{bmatrix}$, $i-q$ is obviously the highest power of ω^2 in equation (2.7). In the general case, if equation (2.7) is expanded by minors, $(\omega^2)^i$ will be multiplied by the determinant of $\begin{bmatrix} m_{ij} \end{bmatrix}$. $(\omega^2)^{i-1}$ will be multiplied by a sum of terms each proportional to a first minor of $\begin{bmatrix} m_{ij} \end{bmatrix}$, $(\omega^2)^{i-2}$ will be multiplied by a sum of terms each proportional to a second minor of $\begin{bmatrix} m_{ij} \end{bmatrix}$, etc. Since the determinant and the first $q-1$ minors of $\begin{bmatrix} m_{ij} \end{bmatrix}$ vanish, the remaining highest power of ω^2 is $i-q$.

Furthermore equation (2.7) has p equal roots $\omega^2 = 0$, or in other words, $(\omega^2)^p$ is a factor of the equation. This may be seen by similar argument applied to the terms with the lowest powers of ω^2 which have the determinant and minors of $\begin{bmatrix} K_{ij} \end{bmatrix}$ as factors in their coefficients. Hence there remain $i-p-q$ roots of equation (2.7) (eigenvalues of equation (2.2)) which are different from zero and infinity. It may be shown (Reference 3), using arguments based on the assumed non-negative character of $\begin{bmatrix} K_{ij} \end{bmatrix}$ and $\begin{bmatrix} m_{ij} \end{bmatrix}$, that these roots are real and positive but not necessarily distinct. To each of them, distinct or not, there corresponds a vector $\{\Phi_i^{(r)}\}$ which is linearly

independent from all other such vectors such that

$$\left(\left[K_{ii} \right] - \omega_r^2 \left[m_{ii} \right] \right) \{ \Phi_i^{(r)} \} = 0 \quad (2.8)$$

Hence there are in total i eigenvectors of equation (2.2), p of them corresponding to eigenvalues, $\omega^2 = 0$; q of them corresponding to eigenvalues $\omega^2 = \infty$; and $i-p-q$ corresponding to real, positive eigenvalues.

Although eigenvectors corresponding to the same eigenvalues are not necessarily orthogonal among themselves, it has been shown in References 4 and 2 that there do exist i linearly independent eigenvectors which are orthogonal with reference to $\left[K_{ii} \right]$ and $\left[m_{ii} \right]$ and correspond to the i eigenvalues of equation (2.2). This orthogonality relationship of eigenvectors $\{ \Phi_i^{(1)} \}$ and $\{ \Phi_i^{(2)} \}$ can be expressed as follows

$$\{ \Phi_i^{(1)} \}^T \left[K_{ij} \right] \{ \Phi_i^{(2)} \} = 0 \quad (2.9)$$

$$\{ \Phi_i^{(1)} \}^T \left[m_{ij} \right] \{ \Phi_i^{(2)} \} = 0 \quad (2.10)$$

Any vector $\{ \beta \}$ with n components can be expressed as a linear combination of n other linear independent vectors so that the eigenvectors $\{ \Phi_i^{(r)} \}$ of equation (2.2) form a complete set of vectors suitable for any problem connected with the system. For example the displacement vector $\{ y_i \}$ of equation (2.1) may be represented as follows

$$\{ y_i \} = \left[\Phi_{ir} \right] \{ \xi_r \} \quad (2.11)$$

where $\left[\Phi_{ir} \right]$ is a nonsingular square matrix (called the modal matrix)

whose r^{th} column is the r^{th} eigenvector and $\{\xi_r\}$ is any vector with i components. Equation (2.11) may be regarded as a nonsingular transformation from coordinates $\{y_i\}$ to coordinates $\{\xi_r\}$. The coordinate deflections $\{\xi_r\}$ are called normal coordinate deflections.

In regard to the inhomogeneous steady state vibration problem stated by equation (2.1), a transformation to normal coordinates gives rise to a system of equations which are uncoupled. In equation (2.1), replacing $\{y_i\}$ by $[\Phi_{ir}] \{\xi_r\}$ (equation (2.11)) and premultiplying by $[\Phi_{ir}]^T$, we obtain

$$[\Phi_{ir}]^T ([K_{ii}] - \omega^2 [m_{ii}]) [\Phi_{ir}] \{\xi_r\} = [\Phi_{ir}]^T \{F_i\} \quad (2.12)$$

We now define

$$[m_r] = [\Phi_{ir}]^T [m_{ii}] [\Phi_{ir}] \quad (2.13)$$

$$[K_r] = [\Phi_{ir}]^T [K_{ii}] [\Phi_{ir}] \quad (2.14)$$

$$\{F_r\} = [\Phi_{ir}]^T \{F_i\} \quad (2.15)$$

$[m_r]$, the generalized mass matrix, and $[K_r]$, the generalized stiffness matrix are diagonal matrices because of the orthogonality of the eigenvectors $\{\Phi_i^{(r)}\}$ as shown in equations (2.9) and (2.10).

Equation (2.12) may be written in terms of the normal coordinates

$$([K_r] - \omega^2 [m_r]) \{\xi_r\} = \{F_r\} \quad (2.16)$$

These equations are scalar equations for each of the normal coordinates.

Defining the diagonal matrix of mechanical admittance

$$[Y_r] = [K_r]^{-1} - \omega^2 [m_r]^{-1} \quad (2.17)$$

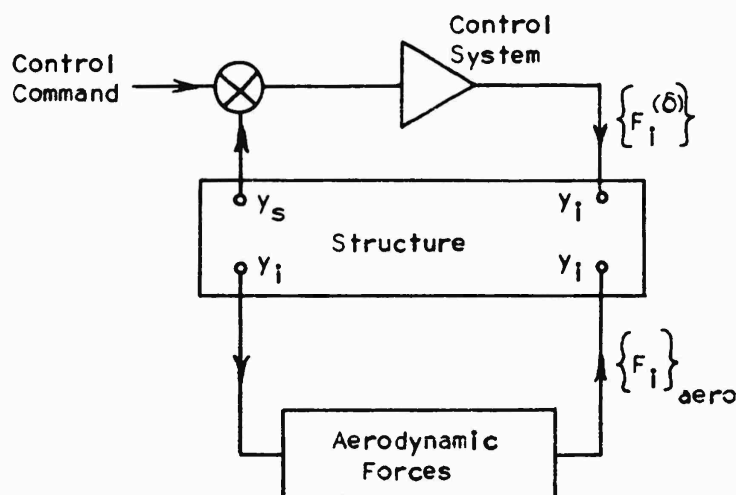
Then equation (2.16) may be rewritten

$$[Y_r] \{\xi_r\} = \{F_r\} \quad (2.18)$$

In general the equations describing the forced, steady state vibration of a linear, conservative system may be written in a simpler, more canonical form in terms of the normal coordinates of the system (equation (2.16)) than when expressed in terms of the physical coordinates (equation (2.1)). It should be noted however that equation (2.16) is of the same order as equation (2.1). A reduction in the order of the system equations can only result from an assumption or in other words a departure from rigor.

2.2 FORMULATION OF THE EQUATIONS FOR A GENERAL AEROELASTIC PROBLEM

A description of the interaction of the chief aeromechanical aspects of flight involved in the solution of the combined auto-pilot-airframe aeroelastic problem, are shown in the following conceptual model.



The concepts of servo-mechanism theory provide a convenient means for separating the analysis of the automatically controlled system pictured above into an analysis of the uncontrolled system, an analysis of the control system and a final interaction analysis. This study concerns itself primarily with the analysis of the uncontrolled system with emphasis on the evaluation of the system parameters required in the final interaction analysis. The parameters of the uncontrolled system directly applicable to the interaction analysis are most conveniently summarized as the transfer function matrix $[T]$ presented to the control system. The matrix $[T]$ is defined by the relationship

$$\{y_s\} = [T] \{F_i^{(\delta)}\} \quad (2.19)$$

The vector $\{y_s\}$ represents the motions of the airframe at control system sensor locations. The vector $\{F_i^{(\delta)}\}$ represents forces on the airframe produced by the control system.

In the evaluation of the transfer function matrix $[T]$, the structure may be represented in its normal coordinates by equation (2.18). The aerodynamic force vector $\{F_i\}_{\text{aero}}$ consists of a part that is linearly proportional to the physical coordinate deflections plus an independent part:

$$\{F_i\}_{\text{aero}} = [Q_{ii}] \{y_i\} + \{F_i^{(0)}\} \quad (2.20)$$

The total force vector applied to the structure $\{F_i\}$ is the sum of the aerodynamic forces and the control system forces, thus

$$\{F_i\} = [Q_{ii}] \{y_i\} + \{F_i^{(0)}\} + \{F_i^{(\delta)}\} \quad (2.21)$$

The generalized force vector $\{F_r\}$ may be obtained by combining equations (2.15), (2.11) and (2.21)

$$\{F_r\} = [\Phi_{ir}]^T [Q_{ii}] [\Phi_{ir}] \{\xi_r\} + [\Phi_{ir}]^T \{F_i^{(0)}\} + [\Phi_{ir}]^T \{F_i^{(\delta)}\} \quad (2.22)$$

Then by substitution of equation (2.22) into equation (2.18), the general equation of motion of the system is

$$[Y_r] \{\xi_r\} = [\Phi_{ir}]^T [Q_{ii}] [\Phi_{ir}] \{\xi_r\} + [\Phi_{ir}]^T (\{F_i^{(0)}\} + \{F_i^{(\delta)}\}) \quad (2.23)$$

or

$$([Y_r] - [\Phi_{ir}]^T [Q_{ii}] [\Phi_{ir}]) \{\xi_r\} = [\Phi_{ir}]^T (\{F_i^{(0)}\} + \{F_i^{(\delta)}\}) \quad (2.24)$$

The control system sensor deflection vector $\{y_s\}$ is related to the normal coordinate deflections by the modal matrix $[\Phi_{sr}]$, equation (2.11):

$$\{y_s\} = [\Phi_{sr}] \{\xi_r\} \quad (2.25)$$

Therefore, by solving equation (2.24) for the normal coordinate deflection vector and substituting into equation (2.25),

$$\{y_s\} = [\Phi_{sr}] ([Y_r] - [\Phi_{ir}]^T [Q_{ii}] [\Phi_{ir}])^{-1} [\Phi_{ir}]^T (\{F_i^{(0)}\} + \{F_i^{(\delta)}\}) \quad (2.26)$$

The transfer function matrix $[T]$ of equation (2.19) may be obtained from equation (2.26) by setting $\{F_i^{(0)}\}$ to zero; then

$$[T] = [\Phi_{sr}] ([Y_r] - [\Phi_{ir}]^T [Q_{ii}] [\Phi_{ir}])^{-1} [\Phi_{ir}]^T \quad (2.27)$$

For the case of a single sensor and a single control system force, $[T]$ is a single scalar function of the derivative operator, s . More generally the number of elements in $[T]$ is equal to the product of the number of sensors and the number of control forces.

The system equations shown in equations (2.23) and (2.27) include a rigorous representation of the structure only when all normal modes of the system are included in the r normal coordinates. The following section of this report will describe a reasonable approximation which can be used to reduce the order of the matrices in the modal representation of a structure.

SECTION 3

THE RESIDUAL FLEXIBILITY APPROXIMATION

3.1 DEFINITION OF APPROXIMATION

The equations describing the forced, steady-state vibration of a linear, conservative system, expressed in terms of the normal coordinates of the system is given by equation (2.18). This equation can also be written

$$\{ \xi_r \} = [Y_r]^{-1} \{ F_r \} \quad (3.1)$$

where $[Y_r]$ is the mechanical admittance as given by equation (2.17) $([K_r] - \omega^2 [m_r])^{-1}$.

By substitution of equations (2.11) and (2.15) in equation (3.1) we obtain

$$\{ y_i \} = [\Phi_{ir}] [Y_r]^{-1} [\Phi_{ir}]^T \{ F_i \} \quad (3.2)$$

an expression for the deflection of the physical coordinates of the system in terms of the forces in the physical coordinates and the mass and stiffness parameters expressed in the normal coordinates of the system.

One of the more important advantages of the use of the normal coordinates in the representation of a dynamic system is that it provides the means to make simplifying assumptions based on frequency discrimination. By representing the system in its normal coordinates we may make simplifying assumptions on only the generalized parameters corresponding to eigenvalues outside some given frequency spectrum. In the formulation of such assumptions it is convenient to write

equation (3.2) in the partitioned form

$$\{y_i\} = \begin{bmatrix} \Phi_{if} & | & \Phi_{i\infty} \end{bmatrix} \begin{bmatrix} Y_f & | & 0 \\ \hline 0 & | & Y_\infty \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{if}^T \\ \hline \Phi_{i\infty}^T \end{bmatrix} \{F_i\} \quad (3.3)$$

where $[Y_f]$ and $[Y_\infty]$ are respectively the generalized mechanical admittance of the modes to be included rigorously in the analysis and the generalized mechanical admittance of all other modes. Repeating equation (2.17), the generalized mechanical admittance matrix of the r^{th} mode is

$$Y_r = K_r - \omega^2 m_r \quad (3.4)$$

where K_r and m_r are the generalized stiffness and mass of the mode (equations (2.13) and (2.14)).

The resonant frequency, ω_r of the r^{th} mode is

$$\omega_r = \sqrt{\frac{K_r}{m_r}} \quad (3.5)$$

therefore

$$m_r = K_r \left(\frac{1}{\omega_r} \right)^2 \quad (3.6)$$

and

$$\omega^2 m_r = K_r \left(\frac{\omega}{\omega_r} \right)^2 \quad (3.7)$$

Then in equation (3.4), if $\omega^2 \ll \omega_r^2$

$$\omega^2 m_r = K_r \left(\frac{\omega}{\omega_r} \right)^2 \ll K_r \quad (3.8)$$

and

$$Y_r \simeq K_r \quad (3.9)$$

Therefore a reasonable approximation to the generalized mechanical admittance $[Y_\infty]$ of modes above the frequency spectrum of the analysis is the generalized stiffness of these higher modes, $[K_\infty]$.

The use of the infinity symbol (∞) as a subscript is explained as follows. A normal mode may be characterized by its eigenvector, by its frequency and either by its generalized mass or by its generalized stiffness. By reducing the generalized mass to zero, the frequency is made to approach infinity. Thus modes where the generalized mass is ignored may be called "infinite frequency modes". The discussion of Section 2 regarding singular mass matrices is relevant.

Using the assumption that

$$[Y_\infty] = [K_\infty] \quad (3.10)$$

equation (3.3) may be written

$$\{y_i\} = \begin{bmatrix} \Phi_{if} & | & \Phi_{i\infty} \end{bmatrix} \begin{bmatrix} Y_f & | & 0 \\ \hline 0 & | & K_\infty \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{if}^T \\ \hline \Phi_{i\infty}^T \end{bmatrix} \{F_i\} \quad (3.11)$$

and since, from equation (3.4)

$$[Y_f] = [K_f] - \omega^2 [m_f] \quad (3.12)$$

equation (3.11) may be rewritten

$$\{y_i\} = \begin{bmatrix} \Phi_{if} \end{bmatrix} \left(\begin{bmatrix} K_f \end{bmatrix} - \omega^2 \begin{bmatrix} m_f \end{bmatrix} \right)^{-1} \begin{bmatrix} \Phi_{if}^T \end{bmatrix} + \begin{bmatrix} \Phi_{i\infty} \end{bmatrix} \begin{bmatrix} K_\infty \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{i\infty}^T \end{bmatrix} \{F_i\} \quad (3.13)$$

From a knowledge of the f mode shapes and frequencies and the mass distribution of the system, all matrices of equation (3.13) can be evaluated except the product $[\Phi_{i\infty}][K_{\infty}]^{-1}[\Phi_{i\infty}]^T$ which we will call the "residual flexibility matrix" of all higher modes. In equation (3.13)

$$[m_f] = [\Phi_{if}]^T [m_{ii}] [\Phi_{if}] \quad (3.14)$$

and from equation (3.5)

$$[K_f] = [\omega_f^2 m_f] \quad (3.15)$$

Since the "residual flexibility matrix" $[\Phi_{i\infty}][K_{\infty}]^{-1}[\Phi_{i\infty}]^T$ is a function of only the stiffness parameters it can most easily be evaluated at zero frequency. At zero frequency equation (3.13) becomes

$$\{y_i\} = \left([\Phi_{if}][K_f]^{-1}[\Phi_{if}]^T + [\Phi_{i\infty}][K_{\infty}]^{-1}[\Phi_{i\infty}]^T \right) \{F_i\} \quad (3.16)$$

3.2 APPLICATION TO SUPPORTED SYSTEMS

For a free system, due to the presence of rigid body or zero frequency modes, the generalized stiffness matrix, $[K_f]$, is singular so that its inverse does not exist. However, for a supported system $[K_f]$ is a nonsingular diagonal matrix. Also for a supported system the influence coefficient matrix, $[Z_{ii}]$, may easily be evaluated by any one of many methods of structural analysis. The influence coefficient matrix is defined by

$$\{y_i\} = [Z_{ii}]\{F_i\} \quad (3.17)$$

From equations (3.16) and (3.17) it is obvious that for a supported system

$$[Z_{ii}] = [\Phi_{if}] [K_f]^{-1} [\Phi_{if}]^T + [\Phi_{i\infty}] [K_{\infty}]^{-1} [\Phi_{i\infty}]^T \quad (3.18)$$

and therefore we may define the "residual flexibility matrix", $[X_{ii}^{(\infty)}]$, in terms of quantities available without knowledge of the higher modes.

$$[X_{ii}^{(\infty)}] = [\Phi_{i\infty}] [K_{\infty}]^{-1} [\Phi_{i\infty}]^T = [Z_{ii}] - [\Phi_{if}] [K_f]^{-1} [\Phi_{if}]^T \quad (3.19)$$

Although the words "residual flexibility" are used to describe this term, it should not be concluded that this term is negligible. The word "residual" refers to the fact that this term accounts for the stiffness properties of the system that are not included in the stiffness matrix, $[K_f]$, of the finite frequency modes explicitly included in the modal representation of the system. If no modes were included in an analysis, then X_{ii} would properly account for all the stiffness of the system. Equation (3.13) may be written for a supported system

$$\{y_i\} = \left([\Phi_{if}] [K_f - \omega^2 m_f]^{-1} [\Phi_{if}]^T + [X_{ii}^{(\infty)}] \right) \{F_i\} \quad (3.20)$$

Equation (3.20) is the equation describing the forced vibration of a linear, conservative, supported system in terms of the lowest f normal coordinates where knowledge of the higher modes is not required. The generalized masses of these higher modes have been shown (equations (3.8) and (3.9)) to probably have an insignificant effect on the response of the system and have been ignored in writing equation (3.20).

Many dynamic analyses of systems have been made in the past where the representation of the systems used was identical to that previously

derived but where the "residual flexibility" of the higher modes was ignored. Reference 2 presents an analysis of a torsion rod which demonstrates the effect of the omission of the "residual flexibility" of the higher modes.

Reference 2 presents the calculations of the response of a cantilever uniform torsion bar excited by a steady state sinusoidal force at frequencies of zero, $\frac{1}{2}$ and $1\frac{1}{2}$ times the first mode frequency of the torsion bar. The results of these calculations are presented in Table 3-1, page 40. It is quite evident from Table 3-1 that the accuracy of representation by a given number of modes is much improved by the inclusion of the "residual flexibility" of the higher modes. These results for a uniform torsion bar can immediately be extended to all conservative systems in which the mode frequencies are well separated because, from the point of view of normal mode theory, there is nothing else that distinguishes a torsion bar from any other conservative system.

In summary, the system parameters which must be known to represent a supported system by equation (3.20) are:

1. $[\Phi_{if}]$, the modal matrix of the modes to be included in the representation.
2. $[K_f]$ and $[m_f]$ the generalized stiffness matrices of included modes.
3. $[X_{ii}^{(\infty)}]$ the "residual flexibility" of the higher modes of the system. $[X_{ii}^{(\infty)}]$ may be evaluated for supported systems with knowledge of the static influence coefficient matrix, $[Z_{ii}]$, of the system, the generalized stiffness matrix of the included modes $[K_f]$ and the modal matrix $[\Phi_{if}]$. From equation (3.19)

$$\begin{bmatrix} x_{ii}^{(\infty)} \end{bmatrix} = \begin{bmatrix} z_{ii} \end{bmatrix} - \begin{bmatrix} \phi_{if} \end{bmatrix} \begin{bmatrix} K_f \end{bmatrix}^{-1} \begin{bmatrix} \phi_{if} \end{bmatrix}^T \quad (3.21)$$

3.3 APPLICATION TO FREE SYSTEMS

The preceding development is basically identical to that presented in Reference 2 .

The direct application of the preceding equations to an unsupported system is impossible due to the singularity of the generalized stiffness matrix and the related nonexistence of the influence coefficient matrix $\begin{bmatrix} z_{ii} \end{bmatrix}$. However, the assumption expressed by equation (3.10) is no less applicable to unsupported systems than it was to supported systems and the "residual flexibility" of the higher modes not explicitly included in a modal representation of a system remains expressible in terms of quantities "measurable" at zero frequency.

Equation (3.3) may be further partitioned to separate the parameters of the zero frequency modes

$$\begin{Bmatrix} y_i \end{Bmatrix} = \begin{bmatrix} \phi_{io} & \phi_{if} & \phi_{i\infty} \end{bmatrix} \begin{bmatrix} Y_o & 0 & 0 \\ 0 & Y_f & 0 \\ 0 & 0 & Y_\infty \end{bmatrix}^{-1} \begin{bmatrix} \phi_{io}^T \\ \phi_{if}^T \\ \phi_{i\infty}^T \end{bmatrix} \begin{Bmatrix} F_i \end{Bmatrix} \quad (3.22)$$

where $\begin{bmatrix} Y_o \end{bmatrix}$ is the "mechanical admittance" matrix of the zero frequency or rigid body modes

$$\begin{bmatrix} Y_o \end{bmatrix} = \omega^2 \begin{bmatrix} m_o \end{bmatrix} \quad (3.23)$$

$\begin{bmatrix} Y_f \end{bmatrix}$ is the "mechanical admittance" matrix of the finite frequency modes to be explicitly included in the modal representation of the

system

$$\begin{bmatrix} Y_f \end{bmatrix} = \begin{bmatrix} K_f \end{bmatrix} + \omega^2 \begin{bmatrix} m_f \end{bmatrix} \quad (3.24)$$

$\begin{bmatrix} Y_\infty \end{bmatrix}$ is the "mechanical admittance" matrix of all higher modes of the system. By the assumption of equation (3.10)

$$\begin{bmatrix} Y_\infty \end{bmatrix} = \begin{bmatrix} K_\infty \end{bmatrix} \quad (3.25)$$

To represent a system by equation (3.22) the following parameters of the system must be known.

1. $\begin{bmatrix} \Phi_{i0} \end{bmatrix}$, the mode shapes of the zero frequency modes.
2. $\begin{bmatrix} m_o \end{bmatrix}$, the generalized masses of the zero frequency modes.
These generalized masses are the inertia properties of the rigid system defined at the center of gravity (e.g. total mass of the system and mass moments of inertia along principal coordinate axes).
3. $\begin{bmatrix} \Phi_{if} \end{bmatrix}$, the mode shapes of the finite frequency modes to be included in the representation of the system.
4. $\begin{bmatrix} m_f \end{bmatrix}$ and $\begin{bmatrix} K_f \end{bmatrix}$, the generalized mass and stiffness in the finite frequency modes to be included. $\begin{bmatrix} K_f \end{bmatrix}$ may be defined in terms of $\begin{bmatrix} m_f \end{bmatrix}$ and the natural frequencies of the modes, ω_f (equation (3.15)).
5. $\begin{bmatrix} \Phi_{i\infty} \end{bmatrix} \begin{bmatrix} K_\infty \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{i\infty} \end{bmatrix}^T$, the residual flexibility of all higher modes of the system, symbolized in equation (3.19) by $\begin{bmatrix} X_{ii}^{(\infty)} \end{bmatrix}$.

Properties of the zero and finite frequency modes of a system have been discussed in Section 2 of the report and may be determined by any of a number of methods. The remaining problem in the use of equation (3.22) is the defining of the "residual flexibility" matrix, $\begin{bmatrix} X_{ii}^{(\infty)} \end{bmatrix}$

in terms of measurable quantities. It is clear that $[X_{ii}^{(\infty)}]$ is a stiffness parameter of the system and, therefore, is independent of frequency. As in the case of the supported system, previously discussed, $[X_{ii}^{(\infty)}]$ will be defined in terms of system properties that are measurable at zero frequency because these are most readily obtainable.

At zero frequency, $[Y_o]$ of equation (3.23) is equal to zero, hence

$$\lim_{\omega \rightarrow 0} [Y_o]^{-1} \rightarrow \infty$$

Therefore in equation (3.22) in order to have a finite deflection vector $\{y_i\}$ at zero frequency we must have

$$[\Phi_{io}]^T \{F_i\} = 0 \quad (3.26)$$

Equation (3.26) is identical to the static equilibrium equations of the system. Also the product

$$[Y_o]^{-1} [\Phi_{io}]^T \{F_i\}$$

is the zero frequency mode deflection vector and will be given the symbol $\{\xi_o\}$. Then from equation (3.22) and (3.19), at all frequencies

$$\{y_i\} = [\Phi_{io}] \{\xi_o\} + [\Phi_{if}] [Y_f]^{-1} [\Phi_{if}]^T \{F_i\} + [X_{ii}^{(\infty)}] \{F_i\} \quad (3.27)$$

We will define $[X_{ii}^f]$ as the generalized flexibility (inverse generalized stiffness) of the f modes of the system, transformed to the i physical coordinates of the system. In matrix form

$$\begin{aligned} [X_{ii}^{(f)}] &= \lim_{\omega \rightarrow 0} [\Phi_{if}] [Y_f]^{-1} [\Phi_{if}]^T \\ &= [\Phi_{if}] [K_f]^{-1} [\Phi_{if}]^T \end{aligned} \quad (3.28)$$

Then at zero frequency, from equations (3.27) and (3.28)

$$\{y_i\} = [\Phi_{io}] \{\xi_o\} + \left([X_{ii}^{(f)}] + [X_{ii}^{(\infty)}] \right) \{F_i\} \quad (3.29)$$

The validity of equation (3.29) is not impaired by any arbitrary positioning of the datum planes for the $\{y_i\}$ deflection vector. Therefore, we may position these datum planes such that a number of y_i coordinate deflections are equal to zero. These coordinates will be given the symbol y_g and will be equal in number to the zero frequency modes of the free system. The remaining y_i coordinates will be given the symbol y_m . Equation (3.29) will be rewritten where the g and m coordinates are separated by matrix partitioning

$$\begin{Bmatrix} y_g \\ y_m \end{Bmatrix} = \begin{bmatrix} \Phi_{go} \\ \Phi_{mo} \end{bmatrix} \{\xi_o\} + \begin{bmatrix} (X_{gg}^{(f)} + X_{gg}^{(\infty)}) & (X_{gm}^{(f)} + X_{gm}^{(\infty)}) \\ (X_{gm}^{(f)} + X_{gm}^{(\infty)})^T & (X_{mm}^{(f)} + X_{mm}^{(\infty)}) \end{bmatrix} \begin{Bmatrix} F_g \\ F_m \end{Bmatrix} \quad (3.30)$$

Equation (3.26) may also be partitioned into g and m coordinates

$$\begin{bmatrix} \Phi_{go}^T & \Phi_{mo}^T \end{bmatrix} \begin{Bmatrix} F_g \\ F_m \end{Bmatrix} = 0 \quad (3.31)$$

The g coordinates must be selected so that, if the free system were restrained at these coordinates, there would be no remaining rigid body degrees of freedom and no "extra" constraints. With this condition the force vector $\{F_g\}$ is statically determinate and the modal matrix $[\Phi_{go}]$ is nonsingular.

From the upper portion of equation (3.30) for the free system, we may write

$$\{\xi_o\} = [\Phi_{go}]^{-1} \{y_g\} - [\Phi_{go}]^{-1} [X_{gg}^{(f)} + X_{gg}^{(\infty)}] \{F_g\} - [\Phi_{go}]^{-1} [X_{gm}^{(f)} + X_{gm}^{(\infty)}] \{F_m\} \quad (3.32)$$

Also, from equation (3.31)

$$\{F_g\} = - [\Phi_{go}^T]^{-1} [\Phi_{mo}^T] \{F_m\} \quad (3.33)$$

Then by substitution of equations (3.32) and (3.33) into the lower portion of equation (3.30) we obtain

$$\{y_m\} = [\Phi_{mo}] [\Phi_{go}]^{-1} \{y_g\} + [Z_{mm}] \{F_m\} \quad (3.34)$$

where

$$\begin{aligned} [Z_{mm}] &= [X_{mm}^{(f)} + X_{mm}^{(\infty)}] - [X_{gm}^{(f)} + X_{gm}^{(\infty)}]^T [\Phi_{go}^T]^{-1} [\Phi_{mo}^T] \\ &- [\Phi_{mo}] [\Phi_{go}]^{-1} [X_{gm}^{(f)} + X_{gm}^{(\infty)}] + [\Phi_{mo}] [\Phi_{go}]^{-1} [X_{gg}^{(f)} + X_{gg}^{(\infty)}] [\Phi_{go}^T]^{-1} [\Phi_{mo}^T] \end{aligned} \quad (3.35)$$

When the datum planes for the $\{y_i\}$ deflection vector are selected such that $\{y_g\}$ equals 0 as previously discussed, equation (3.34) becomes

$$\{y_m\} = [Z_{mm}] \{F_m\} \quad (3.36)$$

From equation (3.36) the matrix $[Z_{mm}]$ is shown to be the influence coefficient matrix of the system when restrained at the g coordinates as defined for the supported system by equation (3.17). Therefore $[Z_{mm}]$ can be determined independently by measurement or calculation at zero frequency.

Equation (3.35) can be rewritten for $[X_{mm}^{(f)} + X_{mm}^{(\infty)}]$

$$\begin{aligned} [X_{mm}^{(f)} + X_{mm}^{(\infty)}] &= [Z_{mm}] + [X_{gm}^{(f)} + X_{gm}^{(\infty)}]^T [\Phi_{go}^T]^{-1} [\Phi_{mo}^T] \\ &+ [\Phi_{mo}] [\Phi_{go}]^{-1} [X_{gm}^{(f)} + X_{gm}^{(\infty)}] - [\Phi_{mo}] [\Phi_{go}]^{-1} [X_{gg}^{(f)} + X_{gg}^{(\infty)}] [\Phi_{go}^T]^{-1} [\Phi_{mo}^T] \end{aligned} \quad (3.37)$$

Equation (3.37) may be simplified by recognizing from equation (3.33) that the matrix product $[\Phi_{go}^T]^{-1} [\Phi_{mo}^T]$ may be obtained by consideration of static equilibrium of the system. For static loads $\{F_m\}$ applied at the m coordinates and reacted at the g coordinates:

$$\{F_g\} = - [A_{gm}] \{F_m\} \quad (3.38)$$

The matrix $[A_{gm}]$ may easily be calculated from geometric parameters of the system because as stated earlier, the force vector $\{F_g\}$ is statically determinate. Therefore

$$[\Phi_{go}^T]^{-1} [\Phi_{mo}^T] = [A_{gm}] \quad (3.39)$$

and by the rules of matrix algebra

$$[\Phi_{mo}] [\Phi_{go}]^{-1} = [A_{gm}^T] \quad (3.40)$$

Also by matrix algebra

$$[X_{gm}^{(f)} + X_{gm}^{(\infty)}]^T [A_{gm}] = \left([A_{gm}^T] [X_{gm}^{(f)} + X_{gm}^{(\infty)}] \right)^T \quad (3.41)$$

Then by substitution of equations (3.39), (3.40) and (3.41) in equation (3.37) we obtain

$$\begin{aligned} [X_{mm}^{(f)} + X_{mm}^{(\infty)}] &= [Z_{mm}] + [A_{gm}] [X_{gm}^{(f)} + X_{gm}^{(\infty)}]^T + \left([A_{gm}] [X_{gm}^{(f)} + X_{gm}^{(\infty)}]^T \right)^T \\ &\quad - [A_{gm}^T] [X_{gg}^{(f)} + X_{gg}^{(\infty)}] [A_{gm}] \end{aligned} \quad (3.42)$$

Equation (3.42) states that the system flexibility matrix for the m coordinates, $[X_{mm}^{(f)} + X_{mm}^{(\infty)}]$, is equal to the influence coefficient matrix of the system when restrained at the g coordinates, $[Z_{mm}]$, plus additional terms that involve coupling between the m and g coordinates. Thus it is apparent that, by measuring the influence coefficient matrix, $[Z_{mm}]$, we determine only that part of the system flexibility involved with the m coordinates. The system flexibility involved with the g coordinates and that coupling the g and the m coordinates are yet to be determined. An approximation that might be made is to assume that the flexibility matrices $[X_{gm}^{(\infty)}]$ and $[X_{gg}^{(\infty)}]$ are zero. In an analysis where only the rigid body modes are explicitly represented, this assumption would be equivalent to assuming that the mass properties of the system are concentrated at the g coordinates. This physical equivalence of the assumption can best be seen by considering equation (3.32). If the $[X_{gg}]$ and $[X_{gm}]$ matrices are assumed to be zero then the deflection vector of the zero frequency modes, $\{\xi_0\}$, is related only to the $\{y_g\}$ deflection vector and this relationship is expressed by the modal matrix, $[\Phi_{g0}]$, of the zero frequency or rigid body modes. Hence the total inertia of the system, which in this case is defined in terms of only the rigid body modes, may be expressed in terms of only the g coordinates. If the g coordinates are selected to be the translations and rotations at the center of gravity of the system, then this assumption is equivalent to concentrating the mass inertia properties at the center of gravity.

It is clear from this discussion that the evaluation of the $[X_{gg}]$ and $[X_{gm}]$ matrices of equation (3.30) and (3.42) is related to the evaluation of "inertia relief" in a more conventional aeroelastic analysis. In other words it is an evaluation of the effect of the mass distribution on the static elastic behavior of the free system.

It will now be demonstrated that the terms $[X_{gm}^{(\infty)}]$ and $[X_{gg}^{(\infty)}]$ can be evaluated by considering the system in states of constant translational or angular acceleration caused by external forces applied at the g coordinates. These accelerations give rise to a set of constant inertia forces that deform the structure. Deformations relative to the g coordinates form the basis for the calculation of $[X_{gm}^{(\infty)}]$ and $[X_{gg}^{(\infty)}]$.

The system acceleration will be defined as s independent combinations of accelerations in the zero frequency normal coordinates and will be symbolized by the matrix $[\ddot{\xi}_{os}]$. Each column of $[\ddot{\xi}_{os}]$ is a vector $\{\ddot{\xi}_o^{(s)}\}$ expressing the steady accelerations of the rigid body modes in one state of acceleration applied to the system. The accelerations at the i coordinates of the system (at which the masses of system are located) due to one s state of system acceleration are

$$\{\ddot{y}_i\} = [\Phi_{io}] \{\ddot{\xi}_o^{(s)}\} \quad (3.43)$$

Then the accelerations at the i coordinates in all states of acceleration defined by the matrix $[\ddot{\xi}_{os}]$ can be expressed by the matrix $[\ddot{y}_{is}]$, where

$$[\ddot{y}_{is}] = [\Phi_{io}] [\ddot{\xi}_{os}] \quad (3.44)$$

The selection of the matrix $\begin{bmatrix} \ddot{\xi}_{os} \end{bmatrix}$ is arbitrary except that it must:

1. Express an independent combination of zero frequency normal coordinates.
2. Include a number of s combinations of normalized accelerations equal in number to rigid body modes (i.e. it must be a square matrix).
3. Be nonsingular (i.e. it must be arranged so that all diagonal elements are finite numbers).

It is many times desirable, but not at all necessary, to choose $\begin{bmatrix} \ddot{\xi}_{os} \end{bmatrix}$ to be a unit matrix (i.e. a diagonal matrix where all diagonal elements are equal to one).

The inertia forces at the i coordinates of the system due to all s states of acceleration are

$$\begin{bmatrix} F_{is} \end{bmatrix} = \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} \ddot{y}_{is} \end{bmatrix}$$

or

$$\begin{bmatrix} F_{is} \end{bmatrix} = \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} \Phi_{io} \end{bmatrix} \begin{bmatrix} \ddot{\xi}_{os} \end{bmatrix} \quad (3.45)$$

The generalized forces in the rigid body normal coordinates of the system due to the s states of acceleration are

$$\begin{bmatrix} F_{os} \end{bmatrix} = \begin{bmatrix} m_o \end{bmatrix} \begin{bmatrix} \ddot{\xi}_{os} \end{bmatrix} \quad (3.46)$$

The only external forces applied to the system are at the g coordinates.

Therefore, for the s states of acceleration

$$\begin{bmatrix} F_{gs} \end{bmatrix} = - \begin{bmatrix} \Phi_{go}^T \end{bmatrix}^{-1} \begin{bmatrix} F_{os} \end{bmatrix}$$

or

$$\begin{bmatrix} F_{gs} \end{bmatrix} = - \begin{bmatrix} \phi_{go}^T \end{bmatrix}^{-1} \begin{bmatrix} m_o \end{bmatrix} \begin{bmatrix} \ddot{\xi}_{os} \end{bmatrix} \quad (3.47)$$

The deflection of the i coordinates relative to the g coordinates, resulting from the inertia loading F_{is} will be symbolized $\begin{bmatrix} y_{is} \end{bmatrix}$ and the deflections at the m coordinates $\begin{bmatrix} y_{ms} \end{bmatrix}$. These deflections may be calculated from the equation

$$\begin{bmatrix} y_{is} \end{bmatrix} = \begin{bmatrix} Z_{ii} \end{bmatrix} \begin{bmatrix} F_{is} \end{bmatrix} \quad (3.48)$$

Since the i coordinates contain the m coordinates as well as others the matrix $\begin{bmatrix} y_{ms} \end{bmatrix}$ is merely the rows of the $\begin{bmatrix} y_{is} \end{bmatrix}$ matrix corresponding to the m coordinates. $\begin{bmatrix} Z_{ii} \end{bmatrix}$ is the influence coefficient matrix of the system when restrained at the g coordinates, previously defined.

$\begin{bmatrix} y_{is} \end{bmatrix}$ may be completely expressed in terms of all the normal coordinates of the system

$$\begin{bmatrix} y_{is} \end{bmatrix} = \begin{bmatrix} \phi_{io} \end{bmatrix} \begin{bmatrix} \xi_{os} \end{bmatrix} + \begin{bmatrix} \phi_{if} \end{bmatrix} \begin{bmatrix} \xi_{fs} \end{bmatrix} + \begin{bmatrix} \phi_{i\infty} \end{bmatrix} \begin{bmatrix} \xi_{\infty s} \end{bmatrix} \quad (3.49)$$

By premultiplication of both sides of equation (3.49) by $\begin{bmatrix} \phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix}$; and, since by orthogonality

$$\begin{bmatrix} \phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} \phi_{if} \end{bmatrix} = \begin{bmatrix} \phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} \phi_{i\infty} \end{bmatrix} = 0$$

and

$$\begin{bmatrix} \phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} \phi_{io} \end{bmatrix} = \begin{bmatrix} m_o \end{bmatrix}$$

then

$$\begin{bmatrix} \phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} y_{is} \end{bmatrix} = \begin{bmatrix} m_o \end{bmatrix} \begin{bmatrix} \xi_{os} \end{bmatrix} \quad (3.50)$$

Therefore

$$\begin{bmatrix} \xi_{os} \end{bmatrix} = \begin{bmatrix} m_o \end{bmatrix}^{-1} \begin{bmatrix} \phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} y_{is} \end{bmatrix} \quad (3.51)$$

Equation (3.51) expresses the deflection of center of gravity of the system relative to the g coordinates, resulting from the s states of acceleration applied. Since the distributed masses of the system shift their position relative to the g coordinates due to elastic deformation of the system, there is a corresponding shift in the center of gravity position relative to the g coordinates of the system.

In applying these acceleration conditions to equation (3.30) we must recall that:

1. Since the physical coordinate deflections are referenced to the g coordinates, $\{y_g\} = 0$.
2. Since external loads are applied to the system at the g coordinates, $\{F_m\} = 0$.

Then, for these conditions of steady accelerations equation (3.30) becomes

$$\begin{bmatrix} 0 \\ \vdots \\ y_{ms} \end{bmatrix} = \begin{bmatrix} \Phi_{go} \\ \vdots \\ \Phi_{mo} \end{bmatrix} \begin{bmatrix} \xi_{os} \end{bmatrix} + \begin{bmatrix} X_{gg}^{(f)} + X_{gg}^{(\infty)} \\ \vdots \\ X_{gm}^{(f)} + X_{gm}^{(\infty)} \end{bmatrix} \begin{bmatrix} F_{gs} \end{bmatrix} \quad (3.52)$$

where

$\begin{bmatrix} y_{ms} \end{bmatrix}$ is defined in equation (3.48)

$\begin{bmatrix} \xi_{os} \end{bmatrix}$ is defined by equation (3.51)

$\begin{bmatrix} F_{gs} \end{bmatrix}$ is defined by equation (3.47)

By decomposing equation (3.52), expressions are obtained for the $\begin{bmatrix} X_{gg} \end{bmatrix}$ and $\begin{bmatrix} X_{gm} \end{bmatrix}$ matrices of equation (3.30) in terms of measurable quantities as follows:

$$\begin{bmatrix} X_{gg}^{(f)} + X_{gg}^{(\infty)} \end{bmatrix} = - \begin{bmatrix} \Phi_{go} \end{bmatrix} \begin{bmatrix} m_{oo} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} y_{is} \end{bmatrix} \begin{bmatrix} F_{gs} \end{bmatrix}^{-1} \quad (3.53)$$

$$\begin{bmatrix} X_{gm}^{(f)} + X_{gm}^{(\omega)} \end{bmatrix}^T = \left(\begin{bmatrix} y_{ms} \end{bmatrix} - \begin{bmatrix} \Phi_{mo} \end{bmatrix} \begin{bmatrix} m_o \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{io} \end{bmatrix}^T \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} y_{is} \end{bmatrix} \right) \begin{bmatrix} F_{gs} \end{bmatrix}^{-1} \quad (3.54)$$

Equations (3.42), (3.53) and (3.54) can be used to evaluate the matrix $\begin{bmatrix} X_{nn}^{(f)} + X_{nn}^{(\omega)} \end{bmatrix}$ from data "measurable" at zero frequency by the following procedure

1. Designate g restrained coordinates and m free coordinates of equation (3.30) at the coordinates desired in the $\begin{bmatrix} X_{nn} \end{bmatrix}$ matrix where the n coordinates include the g and m coordinates but not necessarily all coordinates associated with inertia of the system. The coordinates associated with inertia will be designated by the subscript i . Measure, or calculate, $\begin{bmatrix} z_{mm} \end{bmatrix}$, the deflection influence coefficient matrix at the m coordinates.

2. Obtain the force matrix $\begin{bmatrix} F_{is} \end{bmatrix}$ from equation (3.45) where, assuming unit acceleration in each of the rigid body modes $\left(\begin{bmatrix} \ddot{\xi}_{os} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \right)$

$$\begin{bmatrix} F_{is} \end{bmatrix} = \begin{bmatrix} m_{ii} \end{bmatrix} \begin{bmatrix} \Phi_{io} \end{bmatrix}$$

This $\begin{bmatrix} F_{is} \end{bmatrix}$ matrix must contain all coordinates associated with inertia terms.

3. Obtain the deflection matrices $\begin{bmatrix} y_{is} \end{bmatrix}$ and $\begin{bmatrix} y_{ms} \end{bmatrix}$. These deflection matrices may be calculated from equation (3.48) or measured, as was done in this study, by means of a physical analogy of the system. These data could conceivably be obtained by a physical test of the system itself, but, from the results of the numerical studies discussed in Section 6, this procedure is probably impractical.
4. Determine the matrices $\begin{bmatrix} X_{gg}^{(f)} + X_{gg}^{(\omega)} \end{bmatrix}$ and $\begin{bmatrix} X_{gm}^{(f)} + X_{gm}^{(\omega)} \end{bmatrix}$ from equations (3.53) and (3.54). These matrices will contain only g and m coordinates.

5. Determine $[X_{mm}^{(f)} + X_{mm}^{(\infty)}]$ from equation (3.42).
6. The results of steps 4 and 5 above may be united into the flexibility matrix

$$\begin{bmatrix} (X_{gg}^{(f)} + X_{gg}^{(\infty)}) & | & (X_{gm}^{(f)} + X_{gm}^{(\infty)}) \\ \hline (X_{gm}^{(f)} + X_{gm}^{(\infty)})^T & | & (X_{mm}^{(f)} + X_{mm}^{(\infty)}) \end{bmatrix}$$

Equations (3.37), (3.53) and (3.54) may be somewhat simplified by making the following arbitrary choices available to the analyst:

1. Normalize the zero frequency modes so that the generalized mass is unity ($m_o = 1$).
2. Choose $[\ddot{\xi}_{os}]$ as a unit matrix (i.e. a diagonal matrix with all diagonal elements equal to one).
3. Define the m free coordinates to be identical to the i coordinates. The g restrained coordinates will then have no inertia associated with them. This may be accomplished conveniently in the general case by defining a desired set of g coordinates that are redundant with some i coordinates. In other words two coordinates may be defined at g locations on the system, of which one will be defined as an i coordinate and have the inertia of that point associated with it, and the other coordinate will be defined as a g coordinate.

If these arbitrary choices are made, then by identity

$$\begin{aligned} [y_{ms}] &\equiv [y_{is}] \\ [z_{mm}] &\equiv [z_{ii}] \\ [m_{mm}] &\equiv [m_{ii}] \\ [\phi_{mo}] &\equiv [\phi_{io}] \end{aligned} \tag{3.55}$$

From equation (3.47)

$$[F_{gs}] = -[\Phi_{go}^T]^{-1} \quad (3.56)$$

and from equation (3.45)

$$[F_{is}] = [m_{mm}][\Phi_{mo}] \quad (3.57)$$

Therefore from equations (3.48), (3.55) and (3.57)

$$[y_{is}] = [y_{ms}] = [Z_{mm}][m_{mm}][\Phi_{mo}] \quad (3.58)$$

Equation (3.53) may now be written

$$[X_{gg}^{(f)} + X_{gg}^{(\infty)}] = [\Phi_{go}][\Phi_{mo}^T][m_{mm}][Z_{mm}][m_{mm}][\Phi_{mo}][\Phi_{go}^T] \quad (3.59)$$

and equation (3.54) becomes

$$[X_{gm}^{(f)} + X_{gm}^{(\infty)}]^T = -\left([I] - [\Phi_{mo}][\Phi_{mo}^T][m_{mm}]\right)[Z_{mm}][m_{mm}][\Phi_{mo}][\Phi_{go}^T]^T \quad (3.60)$$

For convenience in writing the matrix equation to follow, the matrix $[B_{mm}]$ will be defined as follows

$$[B_{mm}] = [\Phi_{mo}][\Phi_{mo}^T][m_{mm}] \quad (3.61)$$

Now equations (3.59) and (3.60) will be substituted into equation (3.37)

and equation (3.37) becomes

$$[X_{mm}^{(f)} + X_{mm}^{(\infty)}] = [Z_{mm}] - [Z_{mm}][B_{mm}]^T - [B_{mm}][Z_{mm}] + [B_{mm}][Z_{mm}][B_{mm}]^T$$

or by matrix algebra

$$[X_{mm}^{(f)} + X_{mm}^{(\infty)}] = \left([I] - [B_{mm}]\right)[Z_{mm}]\left([I] - [B_{mm}]\right)^T \quad (3.62)$$

In the use of equation (3.62) advantage can be gained from the fact that since $[Z_{mm}]$ is always a symmetric matrix

$$[Z_{mm}][B_{mm}]^T = ([B_{mm}][Z_{mm}])^T$$

Since the g coordinates are now duplicated by the m coordinates, they are unnecessary in the representation of the free system and are used only in the evaluation of the influence coefficient matrix $[Z_{mm}]$. It is also evident from equation (3.61) and (3.62) that knowledge of the location of the g coordinates, used in the evaluation of $[Z_{mm}]$, is not required in the determination of the "flexibility matrix" of the free system.

The matrix $[X_{mm}^{(f)} + X_{mm}^{(\infty)}]$ is the influence coefficient matrix of the free system when restraints are placed on the zero frequency or rigid body normal coordinate deflections. It may be used quite effectively in steady state aeroelastic analyses because in such an analysis the desired restraints on the system are the zero frequency normal coordinates rather than restraints of some particular physical coordinates.

It is of interest to note that when equation (3.61) is postmultiplied by the modal matrix $[\Phi_{mo}]$

$$[B_{mm}][\Phi_{mo}] = [\Phi_{mo}][\Phi_{mo}]^T [m_{mm}][\Phi_{mo}]$$

and since by previous normalization

$$[\Phi_{mo}]^T [m_{mm}][\Phi_{mo}] = [m_o] = [I]$$

then

$$[B_{mm}][\Phi_{mo}] = [\Phi_{mo}]$$

If $\begin{bmatrix} \Phi_{m0} \end{bmatrix}$ is nonsingular (can be inverted) then

$$\begin{bmatrix} B_{mm} \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}$$

and from equation (3.62)

$$\begin{bmatrix} X_{mm}^{(f)} + X_{mm}^{(\infty)} \end{bmatrix} = 0$$

The modal matrix $\begin{bmatrix} \Phi_{m0} \end{bmatrix}$ is nonsingular only when the number of coordinates of the system is equal to the number of zero frequency modes such that the inertia properties of the system are completely expressable in terms of these coordinates. In this case, from the physical properties of the system, it can be deduced that all degrees of freedom of the system are zero frequency modes. Therefore, since there are neither finite nor infinite frequency modes of the system in terms of these coordinates

$$\begin{bmatrix} X_{mm}^{(f)} + X_{mm}^{(\infty)} \end{bmatrix} = 0$$

In a dynamic analysis the "residual flexibility" $\begin{bmatrix} X_{mm}^{(\infty)} \end{bmatrix}$, of the modes higher than the f modes explicitly represented is desired. This "residual flexibility" matrix may easily be determined by subtraction of $\begin{bmatrix} X_{mm}^{(f)} \end{bmatrix}$ from the matrix $\begin{bmatrix} X_{mm}^{(f)} + X_{mm}^{(\infty)} \end{bmatrix}$ defined in equations (3.42) or (3.62), where from equation (3.28)

$$\begin{bmatrix} X_{mm}^{(f)} \end{bmatrix} = \begin{bmatrix} \Phi_{mf} \end{bmatrix} \begin{bmatrix} K_f \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{mf} \end{bmatrix}^T$$

and

$$\begin{bmatrix} K_f \end{bmatrix} = \begin{bmatrix} \omega_f^2 m_f \end{bmatrix}$$

A simple example illustrating the calculation of the flexibility matrix for a free system is worked out in Appendix I, page 153.

3.4 APPLICATION TO AEROELASTIC ANALYSIS

The "residual flexibility" matrix $\begin{bmatrix} X_{mm}^{(\infty)} \end{bmatrix}$ of the higher modes of a system can most easily be included in a general aeroelastic analysis as follows.

The generalized force vector $\{F_r\}$, of a conservative system is, by definition

$$\{F_r\} = \begin{bmatrix} \Phi_{mr} \end{bmatrix}^T \{F_m\} = \begin{bmatrix} Y_r \end{bmatrix} \{\xi_r\} \quad (3.63)$$

The aerodynamic forces on the system are stated in terms of the physical coordinates of the system as follows

$$\{F_m\}_{\text{aero}} = \begin{bmatrix} Q_{mm} \end{bmatrix} \{y_m\} \quad (3.64)$$

Other external forces on the system are included in the force vector $\{F_m^{(a)}\}$. Then the total applied forces $\{F_m\}$ on the system (excluding inertia forces since they originate within the system and are not considered to be "applied") are

$$\{F_m\} = \begin{bmatrix} Q_{mm} \end{bmatrix} \{y_m\} + \{F_m^{(a)}\} \quad (3.65)$$

The subsequent equations will be written for an analysis including k modes explicitly, where the k modes include the zero frequency as well as the finite frequency modes. The total deflection of the system $\{y_m\}$ may be expressed by the sum of the system deflections in the k normal coordinates $\{\xi_k\}$ and the deflection due to the "residual flexibility" of all higher modes.

$$\{y_m\} = \begin{bmatrix} \Phi_{mk} \end{bmatrix} \{\xi_k\} + \begin{bmatrix} X_{mm}^{(\infty)} \end{bmatrix} \{F_m\} \quad (3.66)$$

The total applied force vector $\{F_m\}$ may be written by combining equations (3.65) and (3.66)

$$\{F_m\} = [Q_{mm}] [\Phi_{mk}] \{\xi_k\} + [Q_{mm}] [X_{mm}^{(\omega)}] \{F_m\} + \{F_m^{(a)}\} \quad (3.67)$$

Solving equation (3.67) for $\{F_m\}$

$$\{F_m\} = ([I] - [Q_{mm}] [X_{mm}^{(\omega)}])^{-1} ([Q_{mm}] [\Phi_{mk}] \{\xi_k\} + \{F_m^{(a)}\}) \quad (3.68)$$

Combining equations (3.63) and (3.68)

$$[Y_k] \{\xi_k\} = [\Phi_{mk}]^T ([I] - [Q_{mm}] [X_{mm}^{(\omega)}])^{-1} ([Q_{mm}] [\Phi_{mk}] \{\xi_k\} + \{F_m^{(a)}\}) \quad (3.69)$$

The general solution of equation (3.69) for the vector $\{\xi_k\}$ is difficult because $[Q_{mm}]$ contains the time derivative operator s and the inversion of the matrix $([I] - [Q_{mm}] [X_{mm}^{(\omega)}])$ is required. However it is probably reasonable in many applications to omit the terms containing s from this matrix. At low frequencies these terms are small. When the s terms have been omitted from the matrix $([I] - [Q_{mm}] [X_{mm}^{(\omega)}])$, it effectively becomes a static aeroelastic correction matrix whose inverse is applied to aerodynamic matrix $[Q_{mm}]$ and the force vector $\{F_m^{(a)}\}$. It is of interest to write equation (3.69) for an aircraft in steady maneuvering flight. In this case the k modes include only the rigid body modes of the system and the applied force vector $\{F_m^{(a)}\}$ is zero.

$$[Y_o] \{s^2 \xi_k\} = [\Phi_{mk}]^T ([I] - [Q_{mm}] [X_{mm}^{(\omega)}])^{-1} [Q_{mm}] [\Phi_{mk}] \{\xi_k\} \quad (3.70)$$

In this case, where no finite frequency modes are included in the

representation, the "residual flexibility" matrix $\begin{bmatrix} X_{nn}^{(\infty)} \end{bmatrix}$ expresses all of the system flexibility and will be defined as $\begin{bmatrix} X_{nn} \end{bmatrix}$, without a superscript. The terms $s^2 \xi_k$ on the left side of the equation are "load factors" for the various rigid body modes (i.e. plunge, pitch, etc.). In this steady state analysis there would be no time derivative terms in either aerodynamic matrix. It is of interest to note in this example that the singularity of the matrix $\left(\begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} Q_{nn} \end{bmatrix} \begin{bmatrix} X_{nn} \end{bmatrix} \right)$ defines a static divergence condition.

It should be noted that the deflection vector $\{\xi_k\}$ of equations (3.69) and (3.70) expresses k deflections of the normal coordinates of the system and does not express the total deflection of the system. The total deflection of the system may be found by combining equations (3.65) and (3.66)

$$\{y_n\} = \left(\begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} X_{nn}^{(\infty)} \end{bmatrix} \begin{bmatrix} Q_{nn} \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \Phi_{nk} \end{bmatrix} \{\xi_k\} + \begin{bmatrix} X_{nn}^{(\infty)} \end{bmatrix} \{F_n^{(a)}\} \right) \quad (3.71)$$

It is obvious from equation (3.71) that in the analysis of the response of a particular physical coordinate of a system which is susceptible to aeroelastic phenomenon due to a given forcing function, the effects of residual flexibility are not limited to normal coordinate response. The effects of residual flexibility must also be considered in the relationship of the physical coordinate response to the response in the normal coordinates.

It was previously noted that a steady state divergence condition is defined by the singularity of the matrix $\left(\begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} Q_{nn} \end{bmatrix} \begin{bmatrix} X_{nn} \end{bmatrix} \right)$ when $\begin{bmatrix} X_{nn} \end{bmatrix}$ includes all the flexibility of the system. The matrix $\begin{bmatrix} Q_{nn} \end{bmatrix} \begin{bmatrix} X_{nn} \end{bmatrix}$ therefore can be used to provide an "aeroelastic index", indicating the

approximate magnitude of steady state aeroelastic effects on the system. The singularity of the matrix $\left(\begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} X_{mm} \end{bmatrix} \right)$ is defined by the vanishing of its determinant. Introducing the variable parameter K, we may write its determinant

$$\left| \begin{bmatrix} 1 \end{bmatrix} - K \begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} X_{mm} \end{bmatrix} \right| = 1 - DK - D_2 K^2 \dots D_n K^n \quad (3.72)$$

A first approximation to the singularity of $\left(\begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} X_{mm} \end{bmatrix} \right)$ can be obtained from $D = 1$ where D is the summation of the diagonal elements of the matrix product $\begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} X_{mm} \end{bmatrix}$.

The summation of the diagonal elements of the matrix $\begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} X_{mm} \end{bmatrix}$ may, therefore, be used as an "aeroelastic index". The following rules will apply to this index D.

1. $(0.1) > D > (-0.1)$ indicates rather small steady state aeroelastic effects on the system.
2. $D < 0$ indicates a loss in aerodynamic effectiveness of the system.
3. $D \geq 1.0$ indicates that the system is near or beyond its steady state divergence speed.
4. $1.0 > D > 0$, indicates that the dynamic pressure, $\left(\frac{1}{2} \rho v^2 \right)$ used in calculating the aerodynamic matrix $\begin{bmatrix} Q_{mm} \end{bmatrix}$, is approximately a factor D times the dynamic pressure at steady state divergence.

The particular value of the analysis of steady state aeroelastic behavior presented above lies in the rigorous treatment of "inertia relief" which is included. The above estimates of aeroelastic effect are made for a free system and do not possess the inherent difficulties of analyses which use approximate corrections to compensate for the effects of restraints on the system.

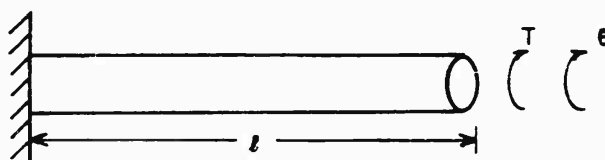
3.5 SUMMARY

The preceding section developed a method of obtaining the "flexibility matrix" of a free system and demonstrated its use in providing the "residual flexibility" approximation to all modes higher than those explicitly included in a modal representation of a system. These parts of the preceding section will be discussed later in this report because they were used in the development of electric analogies to aeroelastic systems studied in this project.

The incorporation of the "residual flexibility" approximation in the equations representing a general aeroelastic system, was also demonstrated in this section. These equations were used by Systems Technology Inc. in a companion study to the one reported here. The results of the study made by Systems Technology Inc. are reported in Reference 5 .

TABLE 3-1

Response of a Cantilever Uniform Torsion Bar
(From Reference 2)



ω_1 = Natural Frequency of 1st Mode

Number of Nodes Included	Mechanical Impedance $\frac{\theta(l)}{T}$			
	$\frac{\omega}{\omega_1} = \frac{1}{2}$		$\frac{\omega}{\omega_1} = 1.5$	
	Without Res. Flex.	With Res. Flex.	Without Res. Flex.	With Res. Flex.
0	0	1.00000	0	1.0000
1	1.08076	1.27019	-.64846	-.45903
2	1.17339	1.27276	-.52538	-.42901
3	1.20614	1.27309	-.49275	-.42580
4	1.22276	1.27317	-.47541	-.42500
5	1.23289	1.27320	-.46502	-.42471
Exact	1.27323		-.42441	

SECTION 4

DISCUSSION OF EXISTING METHODS

This section of the report will be devoted to a discussion of methods, presented in the existing literature, concerning the general aeroelastic response problem. The analysis of this problem is primarily concerned with two fields of engineering knowledge.

1. Stability and control of aircraft.
2. Flutter and response to gusts.

Stability and control analyses of aircraft historically were concerned with the aircraft response in a low frequency spectrum, presumably well below the elastic mode frequencies of aircraft. Aeroelasticity, if included in a stability and control analysis, was assumed to have only second order effects. Flutter on the other hand, is by definition an aeroelastic phenomenon, and historically, flutter analyses were concerned with a frequency spectrum including the major elastic mode frequencies. The same is true of gust analysis.

The unification of these two fields of analysis has been studied for many years (Reference 6 and 7). Due primarily to the significant differences in application of the two fields of analysis and to differences in approach, relatively little has been accomplished toward this unification.

Since methods of analysis commonly used in the fields previously discussed are not directly applicable to the general aeroelastic response problem, but since both have some application to the problem, this discussion of existing methods will briefly summarize analyses commonly used in both the flutter field and in stability and control

problems.

All aeroelastic response analyses of a system must include the mathematical representation of inertia and stiffness properties of the system and an expression of the aerodynamic forces on the system. This discussion will be limited to methods of representing inertia and stiffness properties of the structure involved in an aeroelastic problem. This limitation does not imply that the problem of aerodynamic force representation is simple or not worthy of further investigation. On the contrary the complexity of the problem necessitated its omission from the scope of this study. This section will attempt to classify, compare, and discuss some examples of structural representation included in existing methods of aeroelastic response analysis.

Methods of aeroelastic response analysis may first be classified with regard to coordinate systems used in the analyses. The three coordinate systems used are:

1. Physical coordinates of the system.
2. Normal coordinates of the free-free system.
3. Generalized coordinates other than normal coordinates.

The term physical coordinate here refers to deflection coordinates of particular points on the structure. The expression of the inertia properties of a structure in terms of its physical coordinates is normally accomplished by lumping the mass of the system at discrete points. The stiffness properties of the system may easily be expressed as stiffness coefficients relating the deflections of the physical coordinates to the applied loads at these same coordinates. Physical coordinates are the most convenient means of expressing the structural

properties of a system and are commonly used in aeroelastic analyses as a stepping stone in the representation of the structure in another coordinate system. The transformation is made to other coordinate systems because of one or both of the following complexities of obtaining aeroelastic solutions in terms of the physical coordinates.

1. The accurate representation of a practical structure in terms of its physical coordinates normally results in equations of relatively high order. Also it is normally impossible to reduce the order of the equations while maintaining any assurance of obtaining results to engineering accuracy.
2. The matrix of stiffness coefficients is singular for a free system. This singularity does not render the aeroelastic solution impossible in these coordinates but it does add to the complexity of the solution in many instances.

Examples of aeroelastic response analyses made in terms of the physical coordinates of the systems are the "exact" analyses presented in Appendix A of this report. These analyses were made using the CEA Passive Analog Computer. This computing technique is particularly applicable to the representation of structures with many degrees of freedom in terms of the physical coordinates of the structures. The representation of aerodynamic forces in these coordinates is also convenient.

The difficulties, previously mentioned, experienced in representing the structure involved in an aeroelastic response problem in the physical coordinates of the system do not apply when using the Passive Analog

Computer as the computing technique. However these difficulties do apply when a digital technique is used in obtaining a solution. Therefore when the solution is obtained by digital means the analysis is usually made in some generalized coordinate system.

The most common generalized coordinate system used in aeroelastic response problems, in both the flutter and stability and control fields, is the normal coordinate system of the structure involved in the problem. Normal coordinates of the system are the shapes the structure assumes in its normal modes. This subject is discussed at some length in Section 2 of this report.

The inertia and stiffness properties of a structure are rigorously represented in normal coordinates if all normal coordinates of the structure are included. However in practical methods of analysis, approximations are made by reducing the number of normal coordinates to gain the resulting reduction in the order of the equations involved in the solution. Classification of methods of aeroelastic response analysis using normal coordinates can be made on the basis of differences in these approximations. The three major categories of approximations made are:

1. Representation of the system by its rigid body modes and the stiffness properties of the structure.
2. Representation of the system by some of its normal modes, perhaps including the rigid body modes, but ignoring all other normal modes.
3. Representation of the system by some of its normal modes, including the rigid body modes, and the "residual flexibility"

of all other modes.

The representation of a system by its rigid body modes and the stiffness properties of the structure has been effectively used by stability and control analysts although not conventionally stated in these words. Classical stability equations are written in terms of coordinates whose reference is fixed either to the structure or to the airstream direction. These coordinates may be expressed as combinations of the rigid body modes of the system. Therefore when the total inertia of the system is expressed in terms of these coordinates at the center of gravity of the system it is equivalent to expressing the total inertia as the generalized masses of the rigid body modes of the system.

The effects of the elasticity of the system are accounted for in these stability analyses by the modification of the rigid system aerodynamic stability derivatives used in the analysis. The effects of aeroelasticity on aerodynamic stability derivatives has been treated by many authors, for example References 8 and 10.

Methods of analysis of dynamic response where the system is represented by its rigid body modes and the elasticity of the structure may be summarized by applying these assumptions to equation (3.69) of this report.

$$\begin{bmatrix} m_o & \omega^2 \end{bmatrix} \begin{Bmatrix} \xi_o \end{Bmatrix} = \begin{bmatrix} \Phi_{mo} \end{bmatrix}^T \left(\begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} X_{mm} \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} Q_{mm} \end{bmatrix} \begin{bmatrix} \Phi_{mo} \end{bmatrix} \begin{Bmatrix} \xi_o \end{Bmatrix} + \begin{Bmatrix} F_m^a \end{Bmatrix} \right) \quad (4.1)$$

where:

m_o = generalized masses of the rigid body modes

ξ_o = deflection of the rigid body modes

$\begin{bmatrix} \Phi_{mo} \end{bmatrix}^T$ = modal matrix relating physical coordinates
(m) to normal coordinates (o) of the rigid
body modes

$\begin{bmatrix} Q_{mn}^o \end{bmatrix}$ and $\begin{bmatrix} Q_{mn} \end{bmatrix}$ = aerodynamic influence coefficient matrix of
the rigid system

$\begin{bmatrix} X_{mn} \end{bmatrix}$ = flexibility matrix of the free system

$\{F_m^a\}$ = vector of external forces applied to the
physical coordinates

For convenience in the solution of equation (4.1) the terms of the aerodynamic influence coefficient matrix $\begin{bmatrix} Q_{mn} \end{bmatrix}$ containing the time derivative (d/dt) are omitted from the matrix $\begin{bmatrix} Q^o \end{bmatrix}$. $\begin{bmatrix} Q^o \end{bmatrix}$ is, therefore, an expression of the aerodynamic forces on the rigid system in steady flight. The matrix $\left(\begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} Q_{mn}^o \end{bmatrix} \begin{bmatrix} X_{mn} \end{bmatrix} \right)^{-1}$ may be considered to be an "aeroelastic correction" matrix which modifies the aerodynamic influence coefficient matrix $\begin{bmatrix} Q_{mn} \end{bmatrix}$ which could be equivalent to a modification of the aerodynamic stability derivatives contained in the matrix $\begin{bmatrix} Q_{mn} \end{bmatrix}$.

The $\begin{bmatrix} X_{mn} \end{bmatrix}$ matrix of equation (4.1) is the flexibility matrix of the free system. As shown in Section 3, this matrix may be based on the influence coefficient matrix of the system when restrained at some coordinates provided proper account is taken of mass distribution or "inertia relief". However this consideration given to the mass distribution does not alter the assumption made in writing equation (4.1) that the inertia of the system is represented only in the rigid body modes of the system.

The representation of a system by some of its normal modes, ignoring all others is widely used in the field of flutter analysis. This representation of a system is also used in the analysis of aeroelastic response to gusts and other disturbances (see for example References 8 and 9).

As has been demonstrated in Section 2 of this report, the inertia and elasticity of a system are rigorously represented by the generalized mass and generalized stiffness of all the normal coordinates of a system. When representing a system by some of its normal modes the assumption is made that the generalized mass and stiffness of all other modes have a negligible effect on the response of the system in the frequency spectrum of the analysis. The validity of this assumption is completely dependent on the choice of the number of normal coordinates included in the representation. Although this method of representation has been used successfully in the prediction of flutter speed it will be shown later in this report that equal success should not be expected from its use in general aeroelastic response analyses.

The analysis of dynamic response presented in Reference 11, although not identical to the method previously discussed, may be classified with it. The chief difference is in the choice of generalized coordinates. The analysis shown in Reference 11 represents the system in the normal coordinates of the structure when restrained at some point in addition to the rigid body modes of the free system.

The method of representation of the structure presented in Section 3 of this report uses some of the normal coordinates of the system and includes the generalized stiffness of all other modes. This method of

representation may be considered to be a combination of the previous two modal methods discussed because:

1. Only some of the normal coordinates are included.
2. The aerodynamic influence coefficient matrix is modified to account for the "residual flexibility" of all other modes.

In other words the elasticity of the system is always completely represented.

The representation of a system in generalized coordinates other than the normal coordinates of the system is sometimes used in the general field of aeroelasticity. An example of this approach is the "primitive mode" method developed at Lockheed Aircraft Co. for use in flutter analysis. The "primitive modes" of a wing are a set of pre-selected deflection shapes consisting of curved and straight sections whose mass and stiffness properties are easily computible. These modes are not orthogonal.

SECTION 5

NUMERICAL STUDIES

From preceding sections of this report, it is clear that a very convenient method of representing structure in an analysis of aeroelastic response is in terms of the normal coordinates of the system. One of the advantages of using normal coordinates is the ability it provides the analyst to make judicious approximations in the representation of the system. Numerical studies were made in this project to gain some understanding of the effect of these approximations. Three practical configurations were analyzed.

The approximation evaluation consisted in:

1. A study of the effect of varying the number of normal modes included in the representation of the system
2. A study of the effect of the inclusion of the "residual flexibility" approximation to all higher modes

From these numerical studies understanding was also gained in regard to the practicality of using the "residual flexibility" approximation to higher modes. Comparisons are available which show:

1. The effect of inertia relief
2. Examples of the "aeroelastic index" discussed in Section 3
3. The digital accuracy required in the calculation of "residual flexibility"

The numerical studies also provide some illustrative examples of the effects of elasticity on the dynamic response of three typical aircraft configurations. Although the three configurations were

admittedly chosen to display rather strong aeroelastic effects, they are believed to be reasonable aircraft configurations.

5.1 DESCRIPTION OF CONFIGURATIONS

The three configurations used in the numerical studies were*:

Configuration 2 - missile configuration with "canard"
control surface and aft stabilizing
surface

Configuration 3 - high aspect ratio swept wing airplane

Configuration 4 - delta wing airplane

5.1.1 Configuration 2

Configuration 2 is shown in Figure 5-1. It is a missile with an aerodynamic "canard" control surface at station 250 and stabilizing surface at station 900.

The body of the missile was considered to be flexible in bending with an elastic axis at the centerline of the missile. The flexibility of the body is shown in Figure 5-2 as a plot of the flexibility parameter $1/EI$ versus body station where:

E = Young's Modulus in lb/in.^2

I = Moment of Inertia of the cross-section in in.^4

The mass of the missile body including engines and fuel was represented as lumped masses assigned at the numbered points of Figure 5-3, along the centerline of the missile. The magnitudes and locations of these lumped masses are shown in Table 5-1 for the two mass conditions included in this study.

* The number 1 was assigned to a configuration which was not used in this study.

The aerodynamic surfaces were considered to be rigid but to have an elastic connection to the body of the missile. The flexibility of the aerodynamic surfaces is defined by mass and pitching mass moment of inertia of the surfaces and the natural frequencies of uncoupled pitching and bending degrees of freedom of the surfaces when the missile body is completely restrained. The inertia properties of the surfaces are listed in Table 5-1. The uncoupled natural frequencies of the surfaces when the missile body is restrained are:

Canard Surface (station 250)

Bending degree of freedom - 25 cps

Pitching degree of freedom - 250 cps

Stabilizing Surface (station 900)

Bending degree of freedom - 10 cps

Pitching degree of freedom - 20 cps

The aerodynamic forces on all three configurations of this study were represented in a rudimentary fashion. Since the purpose of these numerical studies was mainly the comparison of predicted response resulting from various assumptions in the structural representation, it was not necessary to include in the analyses the complicating details of a refined representation of the aerodynamic forces.

The aerodynamic forces on the surfaces of configuration 2 were represented as follows:

1. The forces on the body of the missile were assumed to be zero
2. The lift forces on the surfaces, normal to the surfaces and positive upward are given by

$$L = -\frac{1}{2} \rho v^2 S C_{L_\alpha} \left(\theta + \frac{sz}{V} + \frac{cs\theta}{2V} + \delta \right) \quad (5.1)$$

where Z is the normal deflection (positive up) and θ is the elastic pitching slope (positive nose down) of the surface at the aerodynamic center of the surface. δ is the pitching slope increment due to a control command.

3. The moments about the centers of pressure of the surfaces are given by

$$M = -\frac{1}{2} \rho V^2 S \frac{\pi c^2}{8} s \theta \quad (5.2)$$

4. The effects of downwash were neglected.

5.1.2 Configuration 3

Configuration 3 is shown schematically in Figure 5-4. It is a high aspect ratio swept wing airplane having planform geometry and mass distribution similar to the B-47 airplane.

The wing of the airplane is considered to have uncoupled bending and torsional flexibility with reference to the elastic axis shown in Figure 5-4. The elastic axis is a straight line, swept back 35 degrees from the normal to the plane of symmetry, and is located at the 35 per cent chord of the wing planform measured parallel to the plane of symmetry. Engine nacelles are attached to the wing at span stations 230 and 575. These engine nacelles are represented by lumped masses mounted on rigid bars extending forward from the elastic axis.

The fuselage of the airplane was assumed to have an elastic axis in the plane of symmetry. The horizontal stabilizer is assumed to be rigid and attached to the fuselage at fuselage station 1100. Elevator control is provided by movement of the entire horizontal stabilizer.

The bending and torsional flexibility of the wing is shown in Figure 5-5 as plots of the flexibility parameters $1/EI$ and $1/GJ$ versus span station. The bending flexibility of the fuselage is shown in

Figure 5-6 as a plot of $1/EI$ versus fuselage station.

The inertia properties of the airplane are represented by lumped masses located at the numbered points of Figure 5-7. The lumped masses of the wing and fuselage are located on the elastic axes and their magnitudes are listed in Table 5-2 for the three mass conditions considered in this study. The mass points of the wing have, in addition to the lumped masses, mass moments of inertia, related to the lumped masses by the radii of gyration listed in Table 5-2. These radii of gyration are in a direction normal to the elastic axis.

The aerodynamic forces on configuration 3 were represented as follows:

1. For purposes of describing aerodynamic forces, the wing was segmented into three strips. The strips are assumed to be rigid planes whose deflection is described by the elastic deflection of the wing structure at the plunge and streamwise pitch coordinates shown in Figure 5-8.
2. The aerodynamic center of the strips was assumed to be at the $1/4$ chord of the mean span station of the strip.
3. The lift force on each wing strip is given by

$$L = -\frac{1}{2} \rho V^2 S C_{L_\alpha} \left(\theta + \frac{sZ}{V} + \frac{cs\theta}{2V} \right) \quad (5.3)$$

where: Z is the plunge deflection at the aerodynamic center of the strip.

θ is the streamwise pitching slope of the strip.

4. The aerodynamic moment about the center of pressure on each wing strip is given by

$$M = -\frac{1}{2} \rho V^2 S \frac{\pi c^2}{8} \frac{1}{V} s\theta \quad (5.4)$$

5. The aerodynamic center of the horizontal stabilizer is at fuselage station 1100.

6. The lift force on the horizontal stabilizer is given by

$$L = -\frac{1}{2} \rho V^2 S C_{L_\alpha} \left(\theta + \frac{sZ}{V} + \frac{cs\theta}{2V} - .6a_{WI} + \delta \right) \quad (5.5)$$

where: θ is the pitch slope of fuselage station 1100.

Z is the plunge deflection of fuselage station 1100.

a_{WI} is the effective angle of attack of the inboard

strip (strip I) of the wing ($a_{WI} = \theta_{WI} + \frac{sZ_{WI}}{V} + \frac{cs\theta_{WI}}{2V}$).

The term $.6a_{WI}$ in equation (5.5) approximates the effect of wing downwash on the lift of the horizontal stabilizer.

δ is the incremental pitch slope of the horizontal stabilizer resulting from an elevator control command.

7. The moment on the horizontal stabilizer is given by

$$M = -\frac{1}{2} \rho V^2 S \frac{\pi c^2}{8 V} s \theta \quad (5.6)$$

8. Aerodynamic loads on the fuselage were ignored.

5.1.3 Configuration 4

Configuration 4 is a low aspect ratio, delta wing airplane with large stores located in nacelles near the wing tips. The planform geometry of configuration 4 is shown in Figure 5-9. The solid lines of Figure 5-9 shows the spar-rib network which forms the primary structure of the wing. All effective bending material of the wing is lumped in the bending moments of inertia of the spars and ribs shown in Figures 5-10 and 5-11. The slopes of the spars and ribs are elastically connected by "torque boxes" formed by spar and rib webs and the top and bottom cover skin of the wing. These torque boxes, with their associated skin thickness and effective depth, are shown in Figure 5-12.

The fuselage of configuration 4 is represented as an elastic beam extending from fuselage station zero to 800. The bending moment of inertia of the fuselage varies linearly between the values shown in Figure 5-11.

The inertia of configuration 4 is represented by lumped masses at the numbered points of Figure 5-13 and the pitching mass moment of inertia of the large stores mounted near the wing tip. The magnitudes of the lumped masses and mass moment of inertia are shown in Table 5-3 for the three mass conditions considered in this study.

The aerodynamic forces on configuration 4 were represented as follows:

1. For purposes of describing aerodynamic forces, the wing was divided into the three strips shown in Figure 5-14. The strips are assumed to be rigid planes whose deflection is defined by the plunge deflections of the 1/4 and 3/4 chord coordinates shown in Figure 5-14. The aerodynamic lift and moment on each strip are rigidly beamed to these same 1/4 and 3/4 chord coordinates of the elastic structure.
2. The aerodynamic center of each strip is located on the mean chord of the strip, shown in Figure 5-14, and at a distance aft of the effective leading edge of the strip a distance xc (c = mean chord length of strip). x was varied in the study.
3. The aerodynamic lift force on a strip is given by (see Figure 5-15):

$$L = -\frac{1}{2} \rho V^2 s c_{L_\alpha} \left(\theta + \frac{sZ}{V} + \frac{2xc s \theta}{V} + w_1 \delta \right) \quad (5.7)$$

where: θ is the pitching slope of the strip $\left(\theta = \frac{2}{c} \left[z_{3/4} - z_{1/4} \right] \right)$.

Z is the plunge deflection at the aerodynamic center.

$Z_{1/4}$ and $Z_{3/4}$ are the plunge deflections at the $1/4$ and $3/4$ chord points of the elastic structure, then

$$Z = Z_{1/4} \left(\frac{1}{2} + 2x \right) + Z_{3/4} \left(\frac{1}{2} - 2x \right).$$

δ is the control surface deflection shown in Figure 5-15

W_1 is a control surface coefficient defined in

Figure 5-15.

4. The aerodynamic moment about the center of pressure on a strip is given by

$$M = - \frac{1}{2} \rho V^2 S \frac{\pi}{8} \left(\frac{c^2 s \theta}{V} + c W_5 \delta \right) \quad (5.8)$$

where W_5 is a control surface coefficient defined in Figure 5-15

5. The elevator is attached to only strip II and the aerodynamic loads resulting from its deflection appear on only strip II. Since the spanwise width of the elevator is 150 in. and the width of strip II is 100 in., the control surface coefficients W_1 and W_5 given in Figure 5-15 are multiplied by 1.5.
6. The aerodynamic loads on the portion of the fuselage forward of the wing are ignored.

5.2 SCOPE OF NUMERICAL STUDIES

The analyses made of the configurations previously described consisted in

1. Determination of the free-free normal modes and associated natural frequencies of the configurations.
2. Determination of the influence coefficient matrix of selected coordinates of the configurations when the structures were supported at defined coordinates.

3. Determination of the deflection of all coordinates of the restrained structure associated with inertia, due to static loading conditions defined by

$$\{F_i\} = [m_i] \{\ddot{\phi}_i^0\}$$

4. Calculation, for selected mass conditions of the three configurations, of the following matrices defined in Section 3

- a. $[X_{ii}]$ - flexibility matrix of the free-free system
- b. $[X_{ii}^5]$ - flexibility matrix of the lowest 5 finite frequency modes of the system
- c. $[X_{ii}^\infty]$ - "residual flexibility" matrix of all higher modes

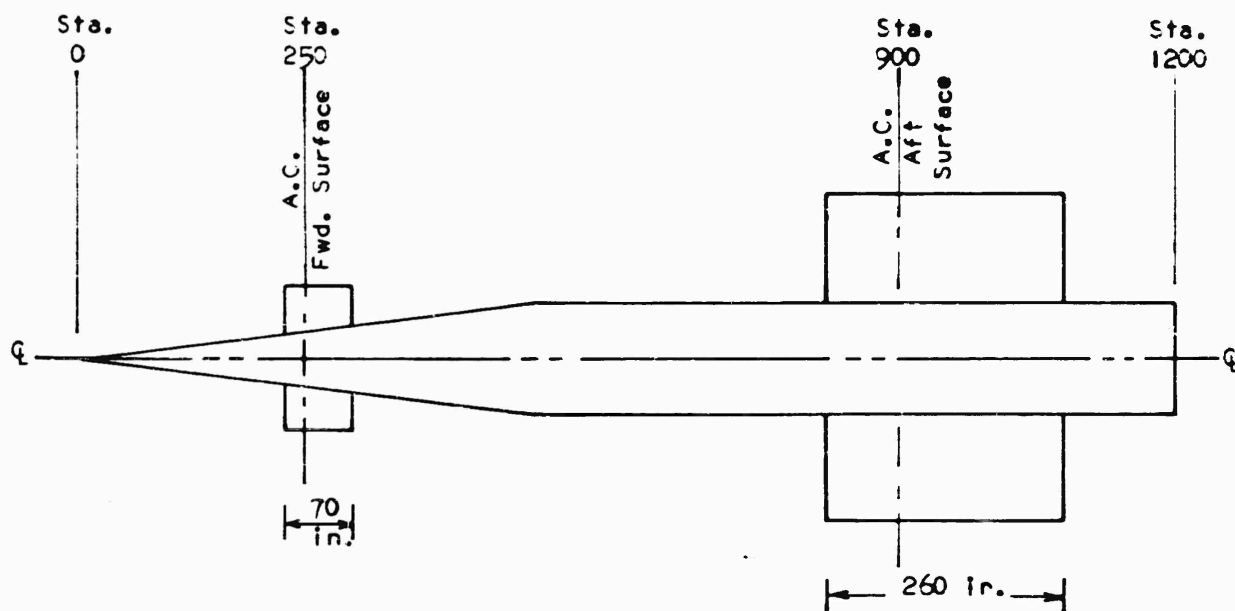
5. Measurement of the response of selected coordinates of the system resulting from sinusoidal control surface motion and vertical gusts. These aeroelastic transfer functions were measured with various assumptions incorporated in the representation of the structure.

The analyses summarized in 1, 2, 3 and 5 above were made using the CEA Direct Analog Computer. The calculations summarized in 4 above were made using desk calculating machines. A discussion of the use of the Direct Analog Computer in these analyses is contained in Appendix A of this report.

The analyses summarized in 1, 2 and 3 above were made using the finite difference representation of the structure. Since the results of these analyses were the basis for all representations of the configurations in their normal modes, these finite difference representations are called "exact" representations in this report. These same "exact"

representations were used in some of the analyses summarized in 5 above.

All analyses made in this study considered only motion symmetric about the plane of symmetry of the configuration. Therefore in all analyses a boundary condition is established along the plane of symmetry whereby roll, yaw and deflection normal to the plane of symmetry is constrained to be zero.



Note: Stations shown in inches

Area of forward surface (total for both sides) = 7000 in.^2

Area of aft surface (total for both sides) = $62,400 \text{ in.}^2$

Figure 5-1. Configuration 2

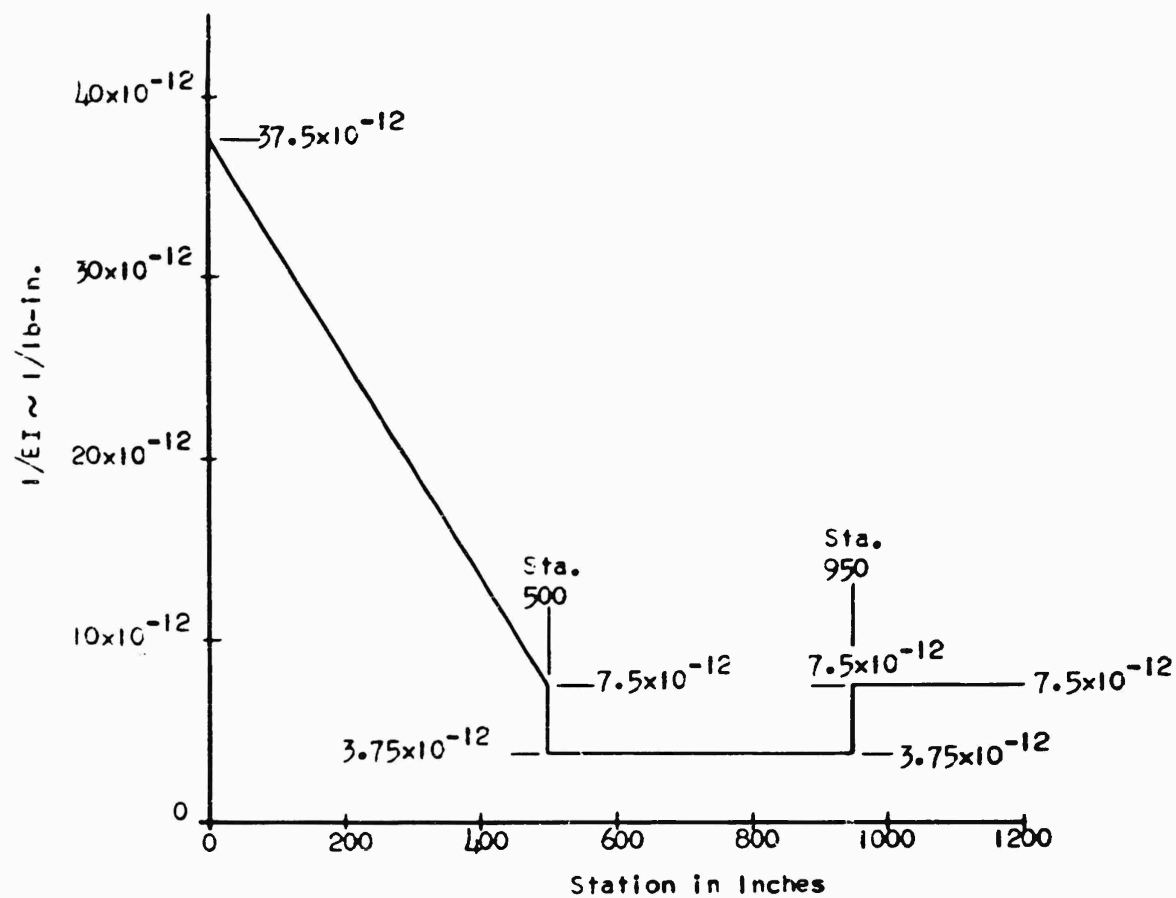


Figure 5-2. Body Flexibility of Configuration 2

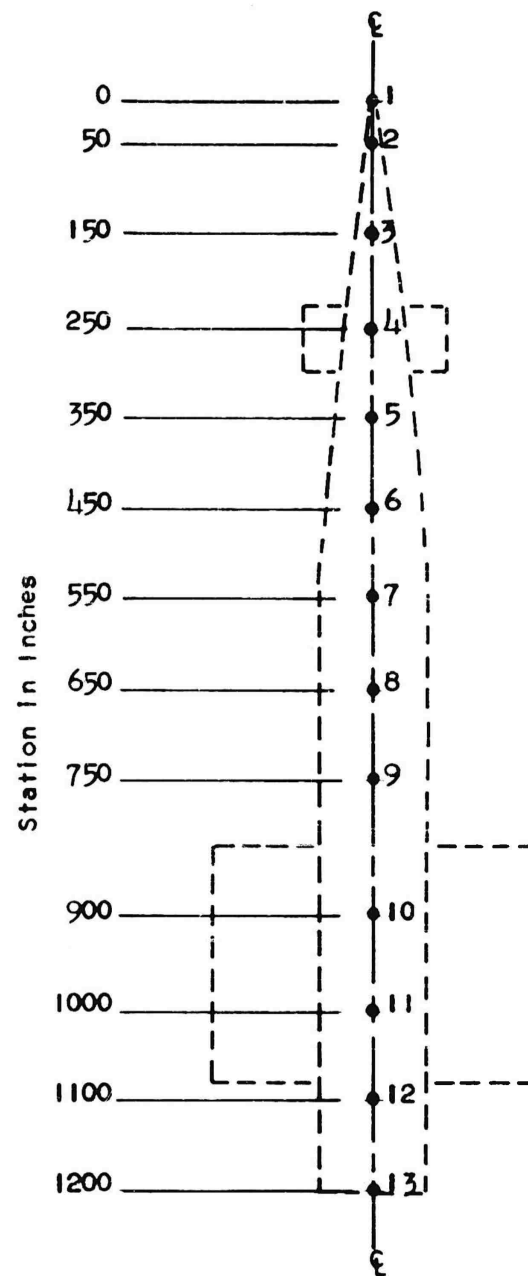


Figure 5-3. Mass Locations of Configuration 2

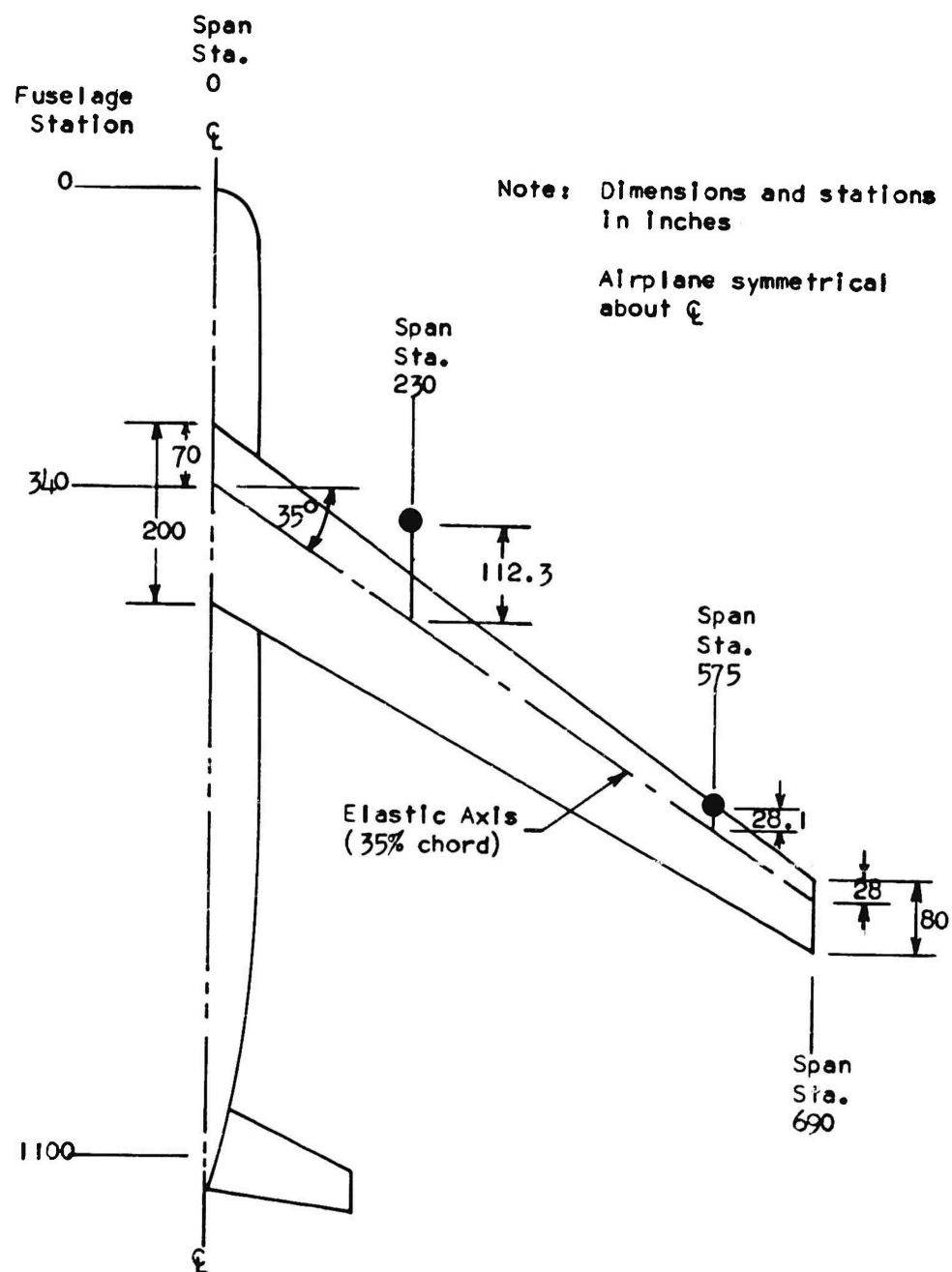


Figure 5-4. Configuration 3

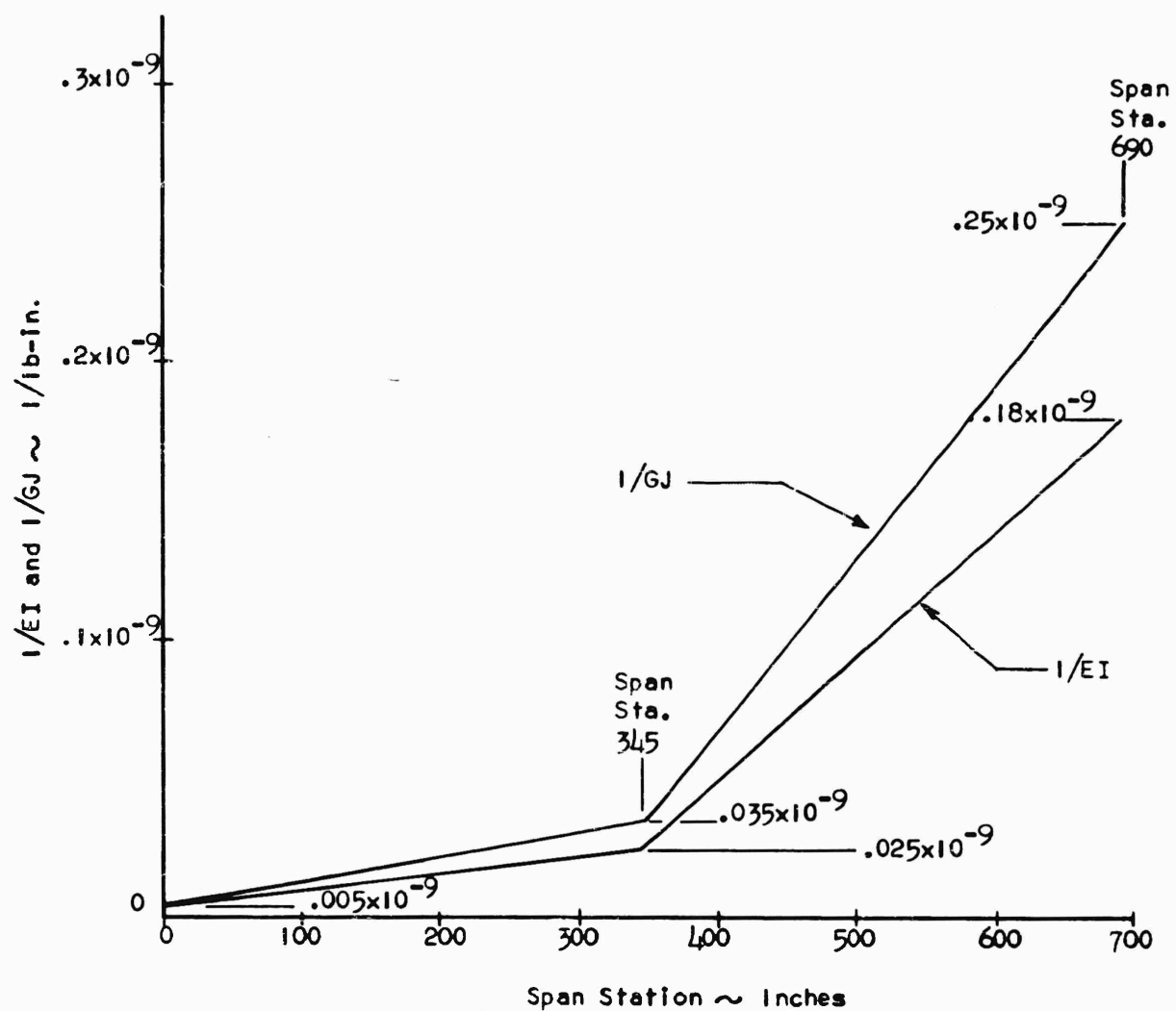


Figure 5-5. Wing Bending and Torsion Flexibility Configuration 3

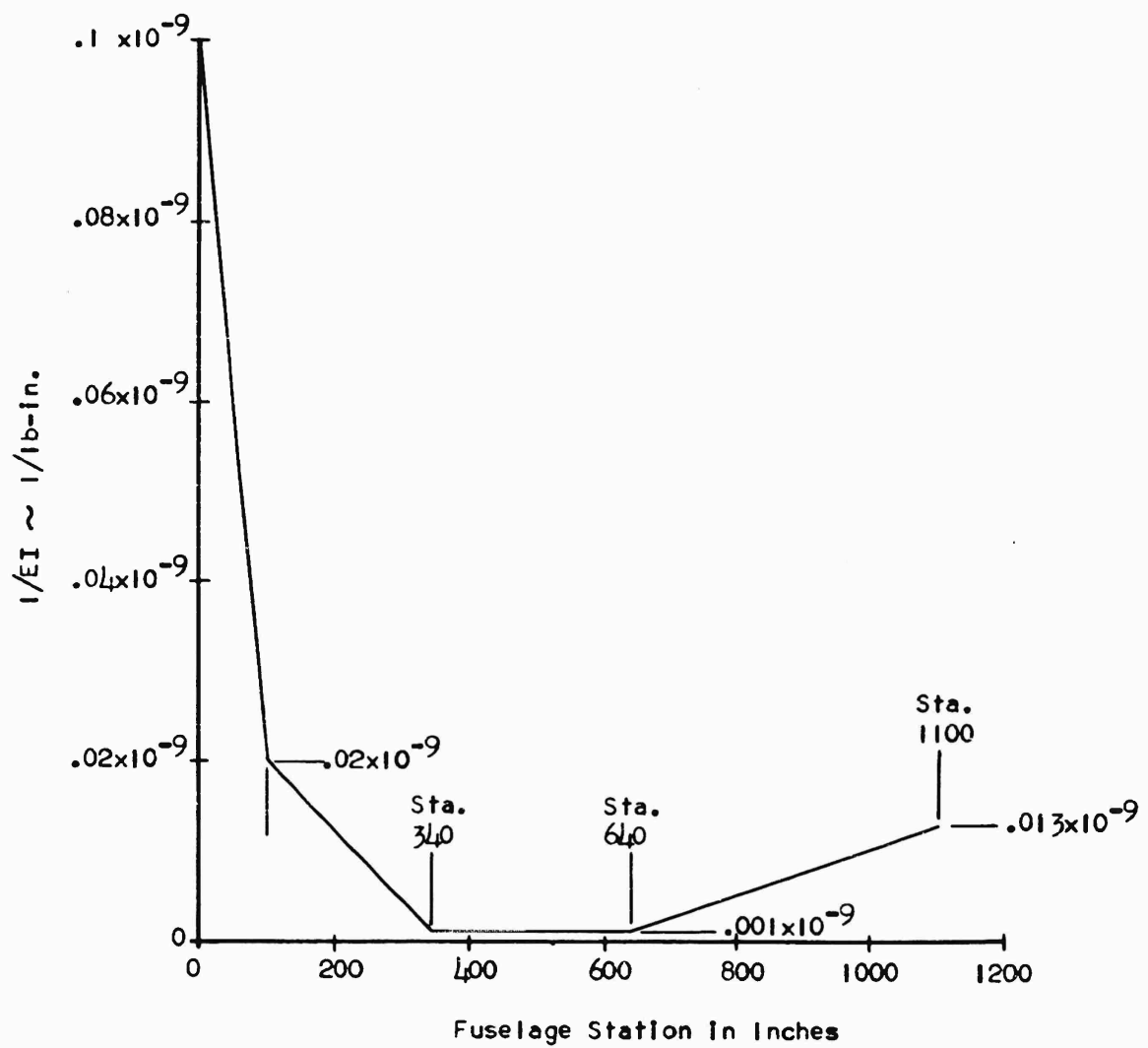


Figure 5-6 Bending Flexibility for Total Fuselage Configuration 3

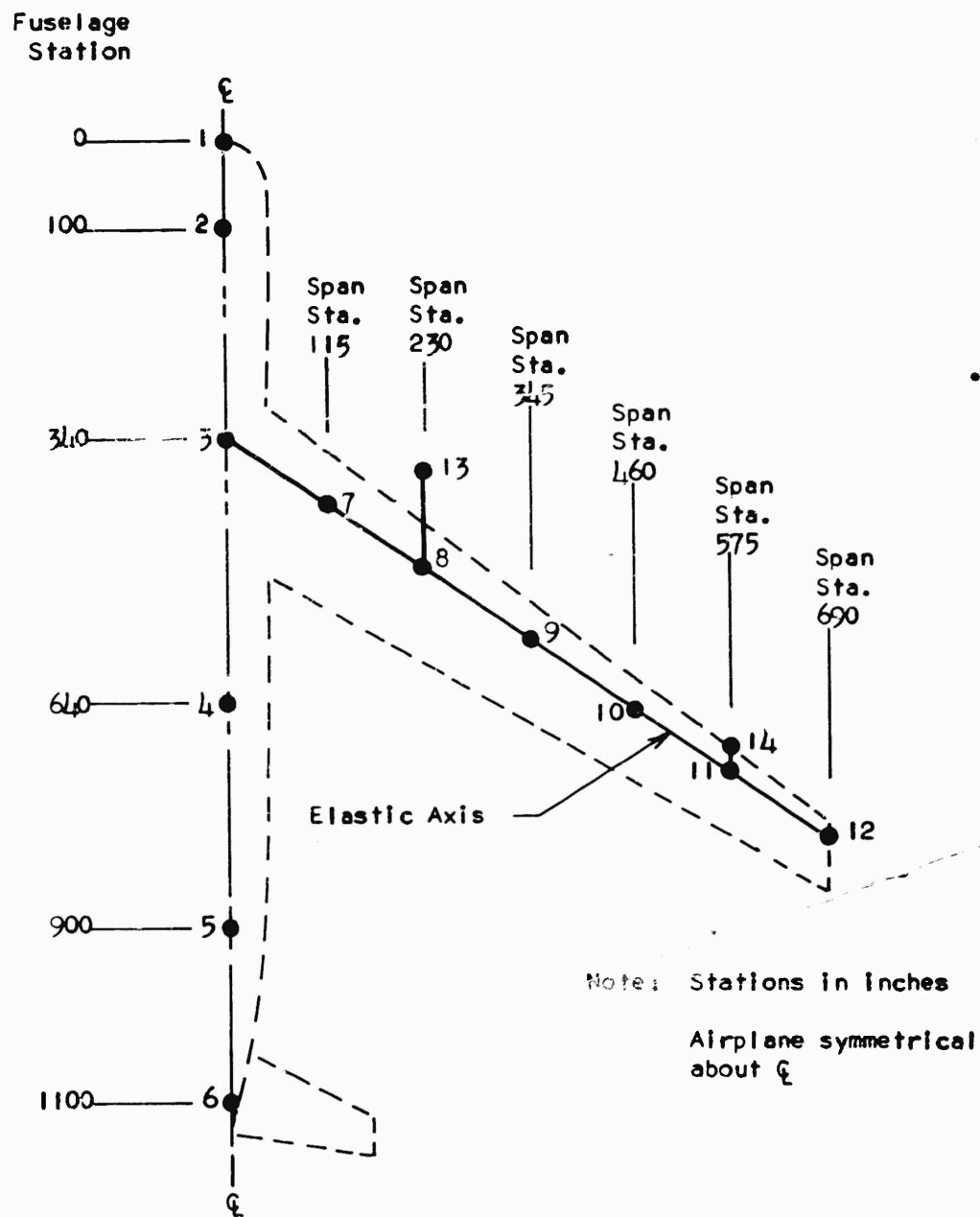


Figure 5-7. Mass Locations of Configuration 3

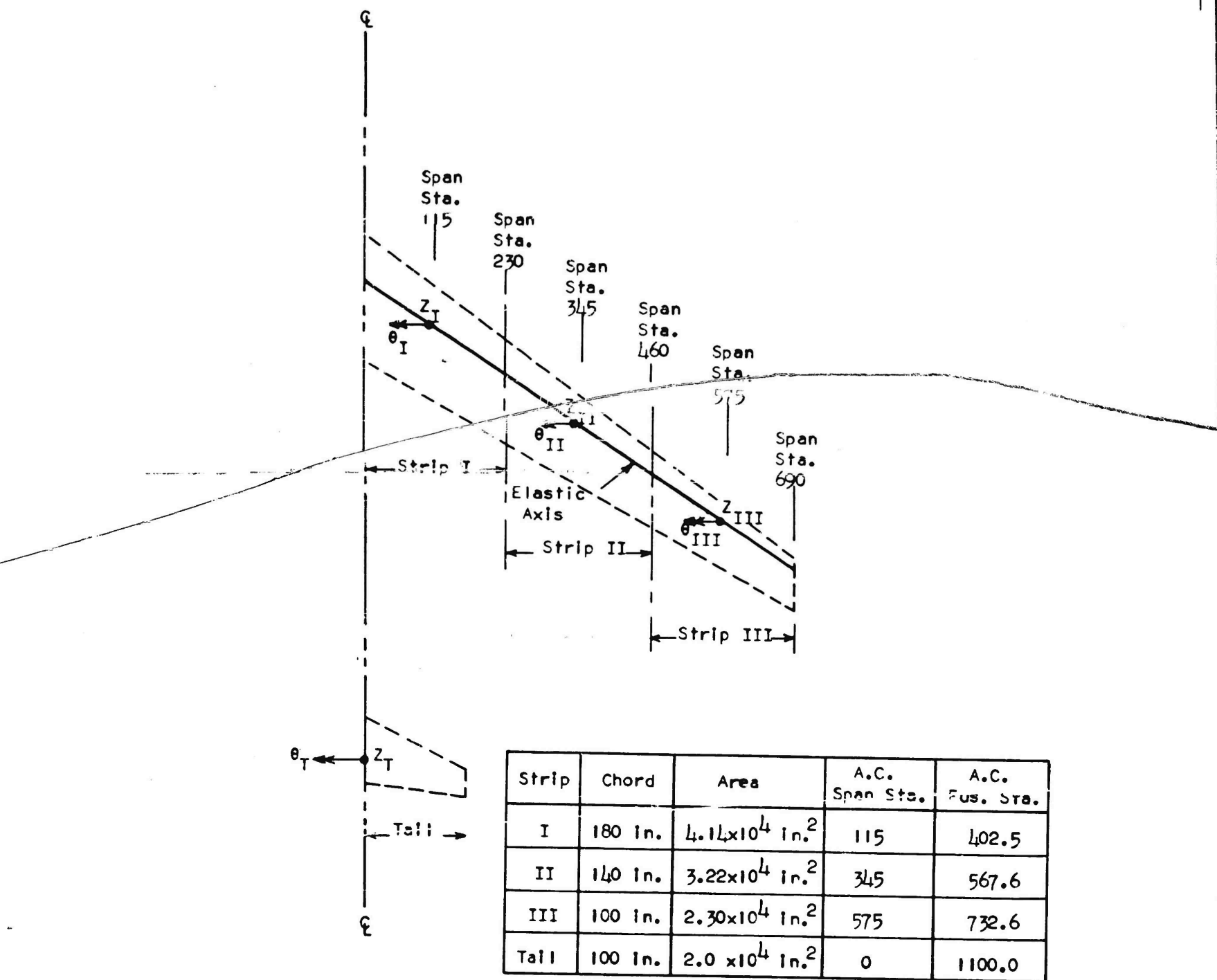


Figure 5-8. Aerodynamic Strips Configuration 3

Fuselage
Station

0

ζ

Note: Stations in inches

Airplane symmetrical about ζ

350

400

450

500

550

600

650

700

750

800

Wing Leading Edge

Wing Mounted Store

Control
Surface

Wing Trailing Edge

Span
Sta.
25

Span
Sta.
100

Span
Sta.
175

Span
Sta.
200

Span.
Sta.
275

Figure 5-9. Configuration 4

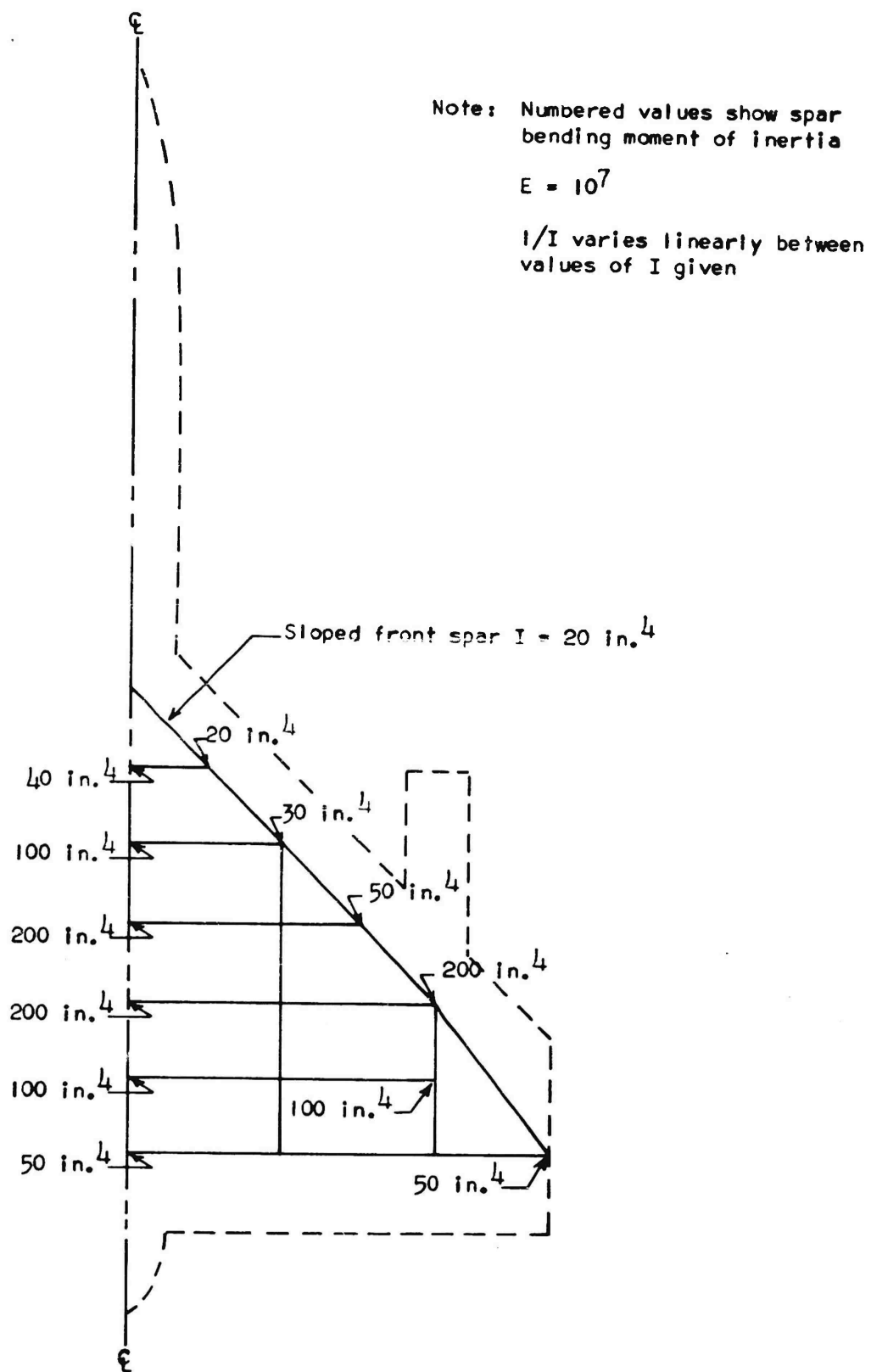


Figure 5-10. Spar Stiffness of Configuration I_4

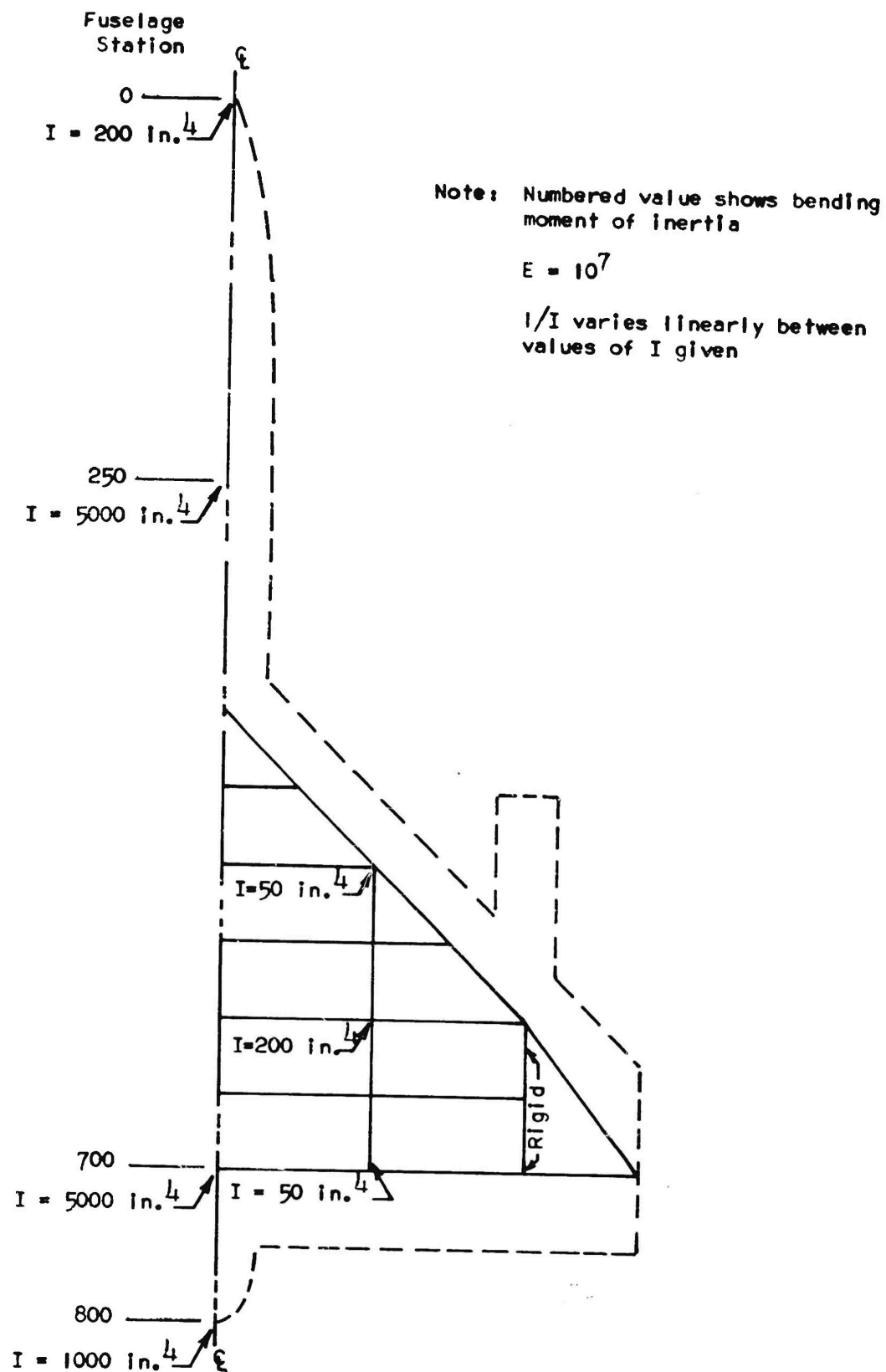
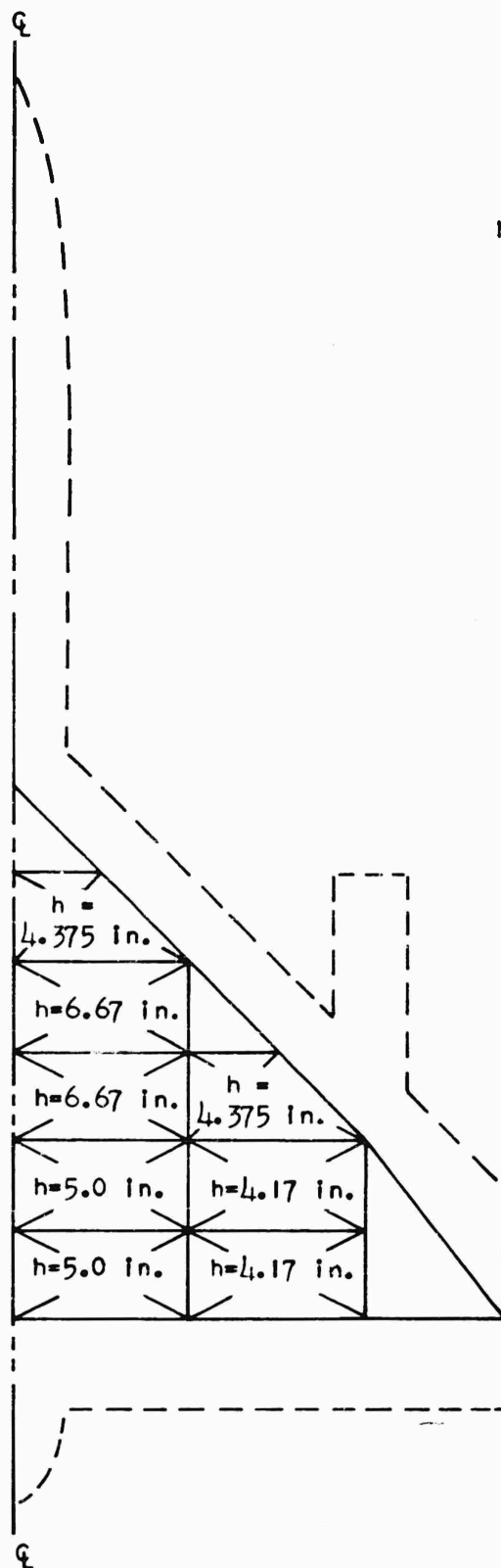


Figure 5-11. Fuselage and Rib Stiffness of Configuration 4



Note:  = torque box

h = average depth
of torque box

Skin thickness top
and bottom = .15 in.

$$G = 4 \times 10^6$$

Figure 5-12. Torque Boxes of Configuration 4

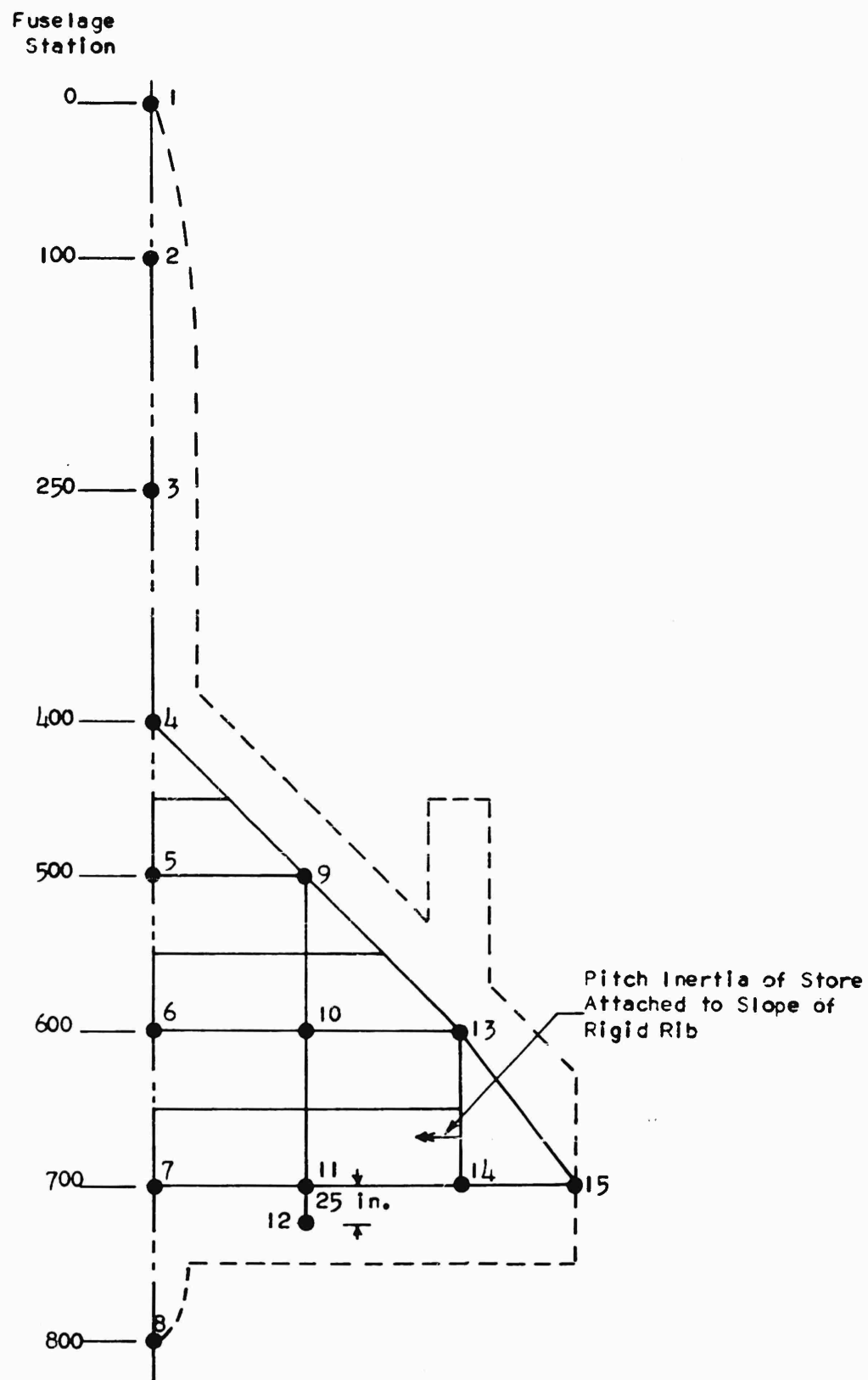


Figure 5-13. Mass Locations of Configuration 4

Strip	Chord	Area	Span Sta. of A.C.
I	100 in.	20,000 in. ²	0
II	300 in.	30,000 in. ²	100 in.
III	200 in.	20,000 in. ²	200 in.

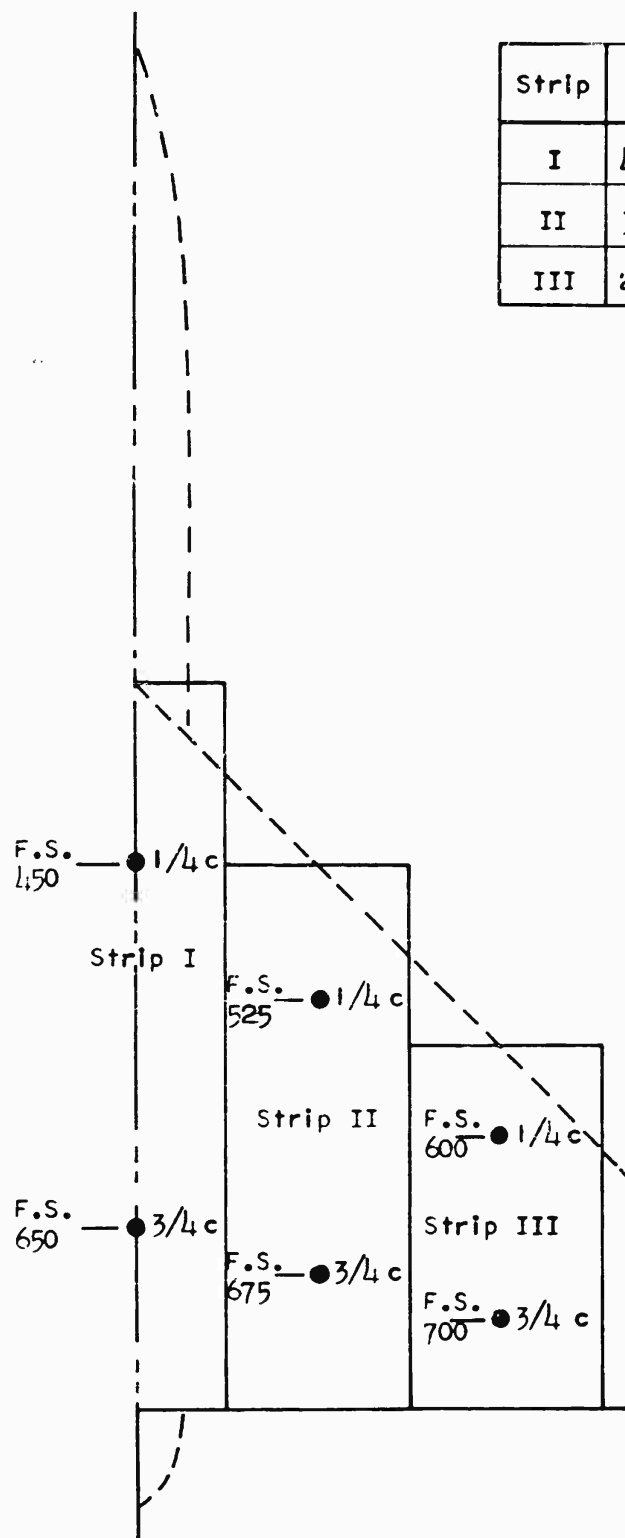
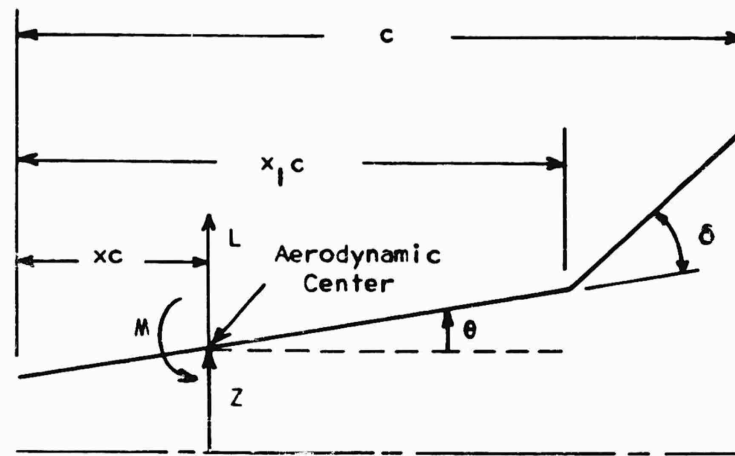


Figure 5-14. Aerodynamic Strips of Configuration 4



$$\Phi = \cos^{-1}(2x_1 - 1)$$

$$\text{Control Lift Coefficient } w_l = \frac{1}{\pi} (\Phi + \sin \Phi)$$

$$\text{Control Moment Coefficient } w_5 = \frac{l_1}{\pi} [\sin \Phi (1 + \cos \Phi)]$$

Figure 5-15. Control Surface Terminology for Configuration 4

TABLE 5-1

MASS DISTRIBUTION FOR CONFIGURATION 2

Mass No. (Figure 5-3)	Station	Condition 30 Mass	Condition 31 Mass
	(In.)	lb-sec ² /in.	lb-sec ² /in.
1	0	4.728	4.728
2	50	18.911	18.911
3	150	18.911	18.911
4	250	31.605	31.605
5	350	56.734	18.911
6	450	82.122	27.374
7	550	94.816	31.605
8	650	94.816	31.605
9	750	51.812	17.270
10	900	51.812	17.270
11	1000	189.632	63.210
12	1100	45.854	15.285
13	1200	22.929	7.642
Forward Surface	250	1.554	1.554
Aft Surface	900	15.544	15.544
Pitching Mass Moment of Inertia			
		lb-sec ² -in.	lb-sec ² -in.
Forward Surface	250	155.4	155.4
Aft Surface	900	2487.	2487.

TABLE 5-2
MASS DISTRIBUTION FOR CONFIGURATION 3

Location	Mass No. (Figure 5-7)	Condition 2 Mass	Condition 5 Mass	Condition 6 Mass	Radius of Gyration Perpendicular to Elastic Axis
		lb-sec ² /in.	lb-sec ² /in.	lb-sec ² /in.	in.
Fuselage  Fuselage	1	1.813	1.813	1.813	- -
	2	11.658	6.477	11.658	- -
	3	43.524	17.617	43.524	- -
	4	32.384	18.912	29.353	- -
	5	17.617	7.254	10.311	- -
	6	6.736	4.145	6.736	- -
Wing  Wing	7	5.414	5.414	5.414	37.5
	8	4.586	4.586	4.586	37.5
	9	3.782	3.782	3.782	36.0
	10	2.927	2.927	2.927	31.5
	11	2.124	2.124	2.124	27.5
	12	.725	.725	10.363	23.0
Nacelles	13	20.726	20.726	20.726	- -
Nacelles	14	8.549	8.549	8.549	- -

TABLE 5-3

MASS DISTRIBUTION FOR CONFIGURATION 4

Mass No. (Figure 5-13)	Condition 10 Mass	Condition 11 Mass	Condition 12 Mass
	lb-sec ² /in.	lb-sec ² /in.	lb-sec ² /in.
1	5.187	5.187	9.875
2	10.365	10.365	20.730
3	20.730	20.730	41.460
4	23.706	23.706	23.706
5	29.302	29.302	29.302
6	21.491	21.491	21.491
7	15.924	15.924	15.924
8	10.365	10.365	10.365
9	.776	.776	.776
10	1.552	1.552	1.552
11	.776	.776	.776
12	.393	.393	.393
13	52.560	52.560	52.560
14	.776	.776	.776
15	.393	.393	.393
Mass Moment of Inertia			
	lb-sec ² -in.	lb-sec ² -in.	lb-sec ² -in.
Store Inertia	3.109×10^5	$.777 \times 10^5$	3.109×10^5

SECTION 6

RESULTS OF NUMERICAL STUDIES

6.1 MODE AND DEFLECTION ANALYSES

Analyses were made of the configurations described in Section 5, using the exact representations discussed in Appendix A, to find:

1. Free-free symmetric normal modes and their associated natural frequencies.
2. The deflection influence coefficient matrices of the structures when restrained by supports at selected coordinates.
3. The deflection matrices $[y_{ms}]$ defined in Section 3. These deflection matrices contain the deflections of the structure, restrained at the same coordinates as in 2 above, resulting from loading conditions defined by the mass distribution and zero frequency modes of the system.

The complete results of the above analyses are contained in Appendix B of this report. This section will contain only a brief discussion and summary of these results.

6.1.1 Configuration 2

Symmetric free-free mode shapes and natural frequencies are presented in Appendix B for both mass condition 30 and condition 31. The lowest 5 elastic mode frequencies are listed in Table 6-1 for mass condition 30 since only this mass condition was used in further numerical studies.

Deflection analyses were made of configuration 2 with the structure

restrained in plunge and pitch at station 900. A deflection influence coefficient matrix was measured for those coordinates of the structure shown in Figure 6.1. This influence coefficient matrix tabulates the deflections, in inches or radians, at the selected coordinates, due to a unit load (one pound or one inch-pound) applied at one of the coordinates. The "inertia relief matrix" $[y_{is}]$, of equation (3.48), was measured on the analog computer by applying the force matrix, $[F_{is}]$ of equation (3.45), to the finite difference representation of the structure. Columns of the "inertia relief matrix", $[y_{is}]$, contain deflections of the structure, relative to the "restrained" coordinates, due to a state of unit acceleration in one of the zero frequency modes (rigid body plunge or pitch). The $[y_{is}]$ matrices for the two mass conditions of configuration 2 are presented in Appendix B.

6.1.2 Configuration 3

Symmetric free-free mode shapes and natural frequencies are presented in Appendix B for the three mass conditions of configuration 3. The lowest 5 elastic mode frequencies are listed in Table 6-1 for mass condition 2.

Deflection analyses were made for configuration 3 with the structure restrained in pitch at the intersection of the wing elastic axis and the plane of symmetry of the airplane. The structure is restrained in plunge at the 1/4 chord of the inboard aerodynamic strip. The deflection influence coefficient matrix was measured for selected coordinates. This matrix is shown in Figure 6-2. The coordinates selected for inclusion in the $[Z_{mm}]$ matrix are, for the most part, those associated with aerodynamic forces since these coordinates are required in the

modal representation of the structure used in an aeroelastic analysis. The number of coordinates included in the matrix is kept to a minimum since the matrix was used in subsequent arithmetic calculations made on desk calculators.

The deflections of the structure due to zero frequency mode accelerations were also measured with the same coordinates restrained as in the influence coefficient matrix. These deflections are presented in Appendix B.

6.1.3 Configuration 4

The symmetric free-free modes of configuration 4 were measured for the 3 mass conditions of Section 5. The lowest 5 elastic mode frequencies of mass condition 10 are listed in Table 6-1. The complete tabulation of normal modes and frequencies of configuration 4 are presented in Appendix B.

The deflection measurements made on the finite difference representation of configuration 4 were made with the structure restrained in plunge at fuselage station 400 and in pitch at fuselage station 325. Both supports of the structure were located in the plane of symmetry of the airplane.

The deflection influence coefficient of selected coordinates of the structure is shown in Figure 6-3. The "inertia relief matrices" for the 3 mass conditions of configuration 4 are presented in Appendix B.

6.2 STIFFNESS CALCULATIONS

The measured data discussed in Section 5.1 was used in the calculation of the following data for the 3 configurations.

1. The "flexibility matrices" of the free systems $[X_{ii}]$.
2. The "flexibility matrices" of the lowest 5 finite frequency modes of the systems $[X_{ii}^5]$.
3. The "residual flexibility matrices" of all higher modes of the systems $[X_{ii}^\infty]$.
4. The product of the "flexibility matrix" of the free system and the steady state aerodynamic influence coefficient matrix, $[Q_{ii}^0][X_{ii}]$. The sum of the diagonal elements of this product matrix is described in Section 3 as the "aeroelastic index" of the system.

6.2.1 "Flexibility Matrices" of the Free Systems

The "flexibility matrices" $[X_{ii}]$ for one mass condition each of configurations 2, 3 and 4 are shown in Figures 6-4, 6-5 and 6-6 respectively. The difference between elements of these matrices and corresponding elements of the deflection influence coefficient matrices of Figures 6-1, 6-2 and 6-3 is due to the phenomenon commonly called inertia relief. Inertia relief is that property of a free system, whereby externally applied forces are "reacted" by the distributed inertia forces of the system rather than by the concentrated reactions of a supported system.

The effect of inertia relief is graphically shown by the comparison of the first row of the matrix of Figure 6-1 with that of Figure 6-4. The first rows of these matrices list the deflections of configuration 2 due to a unit load applied at station 250. Figure 6-1 contains deflections of the structure when supported in pitch and plunge at station 900 and Figure 6-4 contains deflections of the structure when "reaction" to the applied load is provided by the inertia of the system or, more correctly

stated, when the zero frequency modes of the system are constrained to have zero deflection. A comparison of the deflected shape of the missile in these two support conditions is shown in Figure 6-7.

The effects of inertia relief as noted by the comparison of "influence coefficients" of a supported and free structure will, of course, be dependent on the location of restraints in the supported case. In this regard the comparison presented in Figure 6-7 exaggerates this effect to some extent since the restraints in the supported case are somewhat aft of the center of gravity of the system. However, regardless of where the system is supported, when the inertia is well distributed rather than concentrated near the supports, the effect of inertia relief can be expected to be quite large.

The preceding discussion of inertia relief in the analyses of free systems is certainly not meant as the definition of a new problem. This phenomenon has been recognized and considered in aeroelastic analyses for many years. However it is believed that the demonstration of inertia relief obtained in these analyses is of interest to the analyst and serves to demonstrate the strength and generality of application of the method of analysis presented in Section 3.

6.2.2 "Residual Flexibility Matrices"

The term "residual flexibility matrix", as defined in Section 3 of this report, is the approximation of all finite frequency modes of a system, higher than some selected number, by the generalized flexibility of these higher modes. The "flexibility matrix" of a free system, discussed above, is the "residual flexibility matrix"

of all modes of the system higher than zero frequency modes. As more finite frequency modes are explicitly included in the representation of the system the numerical values of the elements of the "residual flexibility matrix" of all higher modes will decrease. If all modes of a system were explicitly included in the representation, the "residual flexibility matrix" would, of course, be a null matrix, or a matrix of zeros.

The intermediate step between the determination of the flexibility matrix of a free system and the calculation of the "residual flexibility matrix" of all modes higher than a selected number is the calculation of the $[X_{ii}^{(f)}]$ matrix. The $[X_{ii}^{(f)}]$ matrices for the lowest five finite frequency modes of one mass condition of configurations 2, 3 and 4 are shown in Figures 6-8, 6-9 and 6-10 respectively. The symbol $[X_{ii}^5]$ is used for these matrices to indicate that 5 finite frequency modes are included. The $[X_{ii}^{(f)}]$ matrix is obtained by use of equation (3.28).

The "residual flexibility matrix", $[X_{ii}^{\infty}]$, is given by the following equation

$$[X_{ii}^{\infty}] = [X_{ii}] - [X_{ii}^f] \quad (6.1)$$

The "residual flexibility matrices" of all modes higher than the fifth finite frequency mode for one mass condition of configurations 2, 3 and 4 are shown in Figures 6-11, 6-12 and 6-13 respectively.

A physical meaning may be obtained for the residual flexibility matrix from a comparison of the matrices $[X_{ii}]$ and $[X_{ii}^{\infty}]$ of configuration 2 of Figures 6-4 and 6-11. It may be noted that the most significant term of the "residual flexibility matrix" of Figure 6-11 is

the diagonal or self term associated with the slope of the aft aerodynamic surface (at station 900). It may also be noted that the terms associated with the plunge coordinate of the same aerodynamic surface (Fin 900) are so small that they have been set equal to zero. These observed relationships of terms in the "residual flexibility matrix" are attributable to the facts that the cantilevered plunge mode frequency of the aft surface is 10 cps, within the frequency spectrum of the first 5 modes, whereas the cantilevered pitching mode is 25 cps, or well above this spectrum. Most of the bending flexibility of the missile body is included in the generalized flexibility of the first 5 finite frequency modes and most of the plunge flexibility of the aft surface is included in the fifth finite frequency mode (10.12 cps). On the other hand most of the pitch flexibility of the aft surface is included in the generalized flexibility of a mode higher than the fifth and is, therefore, included in the "residual flexibility matrix" of all modes higher than the fifth.

6.2.3 Accuracy of "Residual Flexibility Matrices"

The determination of the flexibility matrices of the free systems, as performed in this project, involved the summation of four matrices. The elements of the matrix resulting from this summation are generally smaller in magnitude than the corresponding elements of the matrices which are summed. This reduction in the magnitude of the matrix elements is the result of inertia relief. Therefore, in general there is a loss of digital accuracy in the calculation of the flexibility matrix of the free system from data "measured" for the system when restrained.

The elements of the "residual flexibility matrix" are, in general, smaller in magnitude than those of the flexibility matrix of the free system from which they are determined. Therefore there will be an additional loss of digital accuracy in the calculation of the "residual flexibility matrix".

A judicious choice of restraints to be used in the evaluation of the influence coefficient matrix may minimize this loss of digital accuracy since this choice effects the apparent magnitude of the inertia relief "correction". However, even with an optimum choice of restrained coordinates, the loss of two or more significant digits can be experienced. Such a loss of digital accuracy would not be serious if the complete analyses were made using a digital computer where commonly eight or more significant digits are used. However the loss of two significant digits is serious in the calculation where matrices are used which are based on data obtained from analog analyses or physical test.

The matrices of the free systems, calculated in this study for configurations 2 and 3, were considered to be of satisfactory engineering accuracy. The loss of digital accuracy experienced in the calculation of these matrices for configuration 4 was considered to be excessive. Therefore in the case of configuration 4 the flexibility matrix of the free system is approximated from the generalized flexibility of the lowest 10 elastic modes. This mode summation process does not lead to digital inaccuracy in the calculation of residual flexibility because the terms for higher modes are progressively smaller and because the diagonal terms are all positive. The practical disadvantage of this process is that an excessively large number of modes may be required

for satisfactory accuracy.

6.2.4 Effects on Steady State Aeroelasticity

Some of the effects of steady state aeroelasticity were evaluated for the three configurations at steady state flight conditions typical of those considered in dynamic aeroelastic analyses to be discussed later in this report. The flight conditions considered in the steady state analyses are as follows.

Configuration 2

Dynamic pressure $(1/2 \rho v^2) = 90 \text{ lb/in.}^2$

Lift curve slope $(C_{L_\alpha}) = 1.5$

A.C. location on each strip = at elastic axes

Configuration 3

Dynamic pressure $(1/2 \rho v^2) = 8.31 \text{ lb/in.}^2$

Lift curve slope $(C_{L_\alpha}) = 6.0$

A.C. location on each strip = 25% chord

Configuration 4

Dynamic pressure $(1/2 \rho v^2) = 11.9 \text{ lb/in.}^2$

Lift curve slope $(C_{L_\alpha}) = 5.0$

A.C. location on each strip = 35% chord

The flexibility matrix of the free system, $[X_{ii}]$, was pre-multiplied by the aerodynamic influence coefficient matrix $[Q_{ii}]$ evaluated for the flight conditions shown above. The "aeroelastic index" D , defined in Section 3 as the sum of the leading diagonal elements of the $[Q_{ii}][X_{ii}]$ product matrix, was then evaluated.

The "aeroelastic indices" for the above flight conditions of the configurations are as follows.

Configuration 2 ; $D = .327$

Configuration 3 ; $D = - 3.84$

Configuration 4 ; $D = .718$

From the discussion of the "aeroelastic index" in Section 3 it is evident that for the flight conditions considered, configurations 2 and 4 will show an increase in aerodynamic effectiveness due to aeroelasticity since, in both cases, $D > 0$. Also, assuming the aerodynamic center and lift curve slope constant, the dynamic pressure considered in the analysis of configuration 2 is approximately $1/3$ of q_D , the dynamic pressure at steady state divergence. With the same assumptions the dynamic pressure of configuration 4 is about $3/4$ of q_D .

Configuration 3 has a negative "aeroelastic index" and will therefore show a loss of aerodynamic effectiveness.

It is of interest to note here that the dynamic pressure at steady state divergence of the free system is defined by the singularity of the matrix $\left(\begin{bmatrix} 1 \end{bmatrix} - [Q_{ii}][X_{ii}] \right)$. The dynamic pressure at divergence indicated by an analysis of a supported portion of the structure is given by the singularity of $\left(\begin{bmatrix} 1 \end{bmatrix} - [Q_{ii}][Z_{ii}] \right)$, where $[Z_{ii}]$ is the deflection influence coefficient matrix of that portion of the supported structure. In the numerical studies performed in this project a large difference was observed between $[X_{ii}]$ and $[Z_{ii}]$. Although the supports used in the evaluations of $[Z_{ii}]$ were not chosen as being ideal for prediction of divergence, it may be concluded that

predictions of steady state divergence based on the analyses of supported systems cannot be reliably used as predictions for the systems free of supports.

The "modified Q matrix" has been calculated for configurations 2 and 3. The name "modified Q matrix" refers to the product of the aerodynamic influence coefficient matrix $[Q_{ii}]$, premultiplied by $([I] - [Q_{ii}][X_{ii}])^{-1}$ and will be symbolized $[\bar{Q}_{ii}]$. The matrix $([I] - [Q_{ii}][X_{ii}])^{-1}$ is dimensionless, therefore $[\bar{Q}_{ii}]$ has the same dimensions as $[Q_{ii}]$ and may be considered to be the aerodynamic influence coefficient matrix of the aeroelastic system.

The $[Q_{ii}]$ and $[\bar{Q}_{ii}]$ matrices for the steady state flight conditions previously stated are shown in Figures 6-14 and 6-15 for configurations 2 and 3 respectively. The simplified aerodynamic force representation used in this study results in a steady state aerodynamic influence coefficient matrix which contains non-zero coefficients only at elements which denote aerodynamic loads proportional to pitching slopes of the surface. The matrices shown in Figures 6-14 and 6-15 show only the quadrant of the partitioned $[Q_{ii}]$ and $[\bar{Q}_{ii}]$ matrices which contain these non-zero coefficients. The remaining quadrants of the $[Q_{ii}]$ and $[\bar{Q}_{ii}]$ matrices contain only zero elements.

The comparisons of $[Q_{ii}]$ and $[\bar{Q}_{ii}]$ shown in Figures 6-14 and 6-15 display some interesting effects of steady state aeroelasticity on the configurations of this study. The elements of the $[\bar{Q}_{ii}]$ matrices are, in many cases, quite different in magnitude from the corresponding

elements of the $[Q_{ii}]$ matrices. Of particular interest is the apparent "aerodynamic coupling" shown by finite elements of the $[\bar{Q}_{ii}]$ matrix where no coupling is shown in the $[Q_{ii}]$ matrix by a zero element. The nature of this coupling between coordinates of the system should probably best be called aeroelastic rather than aerodynamic.

The summation of the rows of the $[Q_{ii}]$ and $[\bar{Q}_{ii}]$ matrices results in the distribution of aerodynamic loads on the structures due to a one radian deflection of the rigid body pitching mode. The resulting load distributions of configuration 3 is shown below. The rigid system loads are obtained from the $[Q_{ii}]$ matrix and the elastic system load distribution is obtained from the $[\bar{Q}_{ii}]$ matrix. The pitch angle at each point may be found by application of equation (3.71).

	System Rigid	System Elastic
Force at A.C. of Tail	- .399 x 10 ⁶ lb.	- .4849 x 10 ⁶ lb.
Force at A.C. of Strip I	-2.064 x 10 ⁶ lb.	-1.777 x 10 ⁶ lb.
Force at A.C. of Strip II	-1.605 x 10 ⁶ lb.	- .8075 x 10 ⁶ lb.
Force at A.C. of Strip III	-1.147 x 10 ⁶ lb.	- .4099 x 10 ⁶ lb.
Total Aero Load on Airplane	-5.215 x 10 ⁶ lb.	-3.479 x 10 ⁶ lb.
A.C. of Total Airplane	Station 579.3	Station 576.9

The load distribution is interesting in that it shows that, due to elasticity of the airplane, the "aerodynamic effectiveness" of the wing decreased quite radically while the effectiveness of the tail increased. The total lift on the elastic airplane is shown to be about 2/3 that on the rigid system, but the resultant of these lift forces remains virtually unchanged between the rigid and elastic systems.

The preceding examples of steady state aeroelastic analyses are not presented here as recommended procedures for obtaining an accurate evaluation of steady state aeroelastic loads on an airplane. The aerodynamic force representation used in this study was overly simplified for such an analysis and the formulation of the problem was not sufficiently general to serve a complete steady state aeroelastic analysis. The preceding examples do provide an insight into the susceptibility of the configurations used in the present study to aeroelastic phenomenon. Also the methods used in these examples could be extended to provide a completely general method of steady state aeroelastic analysis.

6.3 AEROELASTIC TRANSFER FUNCTION DATA

The amplitudes of response of selected "sensor" coordinates of the 3 configurations studied in this project were plotted versus the frequency of control surface oscillation or aerodynamic gust excitation. In some cases the phase angle between the response and excitation was plotted in addition to the response amplitude. Table 6-2 presents a summary index of all transfer function data taken in this study. Copies of all transfer function data are contained in Appendix C of this report. Selected examples of these data, for each of the three configurations, will be presented in this section.

The transfer functions of "sensor" response due to control surface or gust excitation were obtained with the structure represented by means of various approximations to provide a comparative evaluation of the effect of these approximations on the predicted response of the systems. These approximations have been discussed at some length in

previous sections of this report. The transfer functions are presented for the following representations of the systems.

1. "Exact" representation - the structure is simulated by the finite difference representation described in Appendix A. Since all modal data were also based on this representation, it can be considered "exact" for purposes of comparison.
2. Modal representation including "residual flexibility" approximation of all modes higher than those explicitly included in the representation. The two zero frequency modes are included in all cases and from zero to five finite frequency modes are explicitly included. The "residual flexibility" matrix changes with varying number of modes explicitly included in the representation.
3. Modal representation where all modes not explicitly included are ignored completely. As in the previous case, the two zero frequency modes and from zero to five finite frequency modes are included. In this representation, when no finite frequency modes are included the system is thereby assumed to be rigid.

6.3.1 Configuration 2

Figures 6-16 through 6-28 present the measured transfer functions of the pitching velocity at station 650 of configuration 2 due to the pitching rotation of the forward aerodynamic surface (at station 250). This transfer function is shown as obtained from the representations of the structure summarized in Table 6-3.

The differences between the measured transfer functions presented are only in the representation of the structure. In all cases the aerodynamic force representations correspond to flight condition C. The aerodynamic parameters of flight condition C are

Velocity = 2250 mph

Altitude = sea level

Dynamic pressure = $1/2 \rho v^2 = 90 \text{ lb/in.}^2$

Lift curve slope = $C_{L_\alpha} = 1.5$

Figure 6-16 presents the transfer function obtained from the "exact" representation of the system. The lowest frequency peak in Figure 6-16 ($f = .73 \text{ cps}$) is the peak in the response of the system due to the "short period" mode. The following four peaks correspond to the lowest four elastic modes of the system. The fifth elastic mode shape contains primarily bending of the aft aerodynamic surface and its amplitude, at the pitching coordinate used as the "sensor" in the transfer function, is very small. The fifth elastic mode, therefore, does not appear in the transfer functions shown.

The term "mode", used with reference to the transfer functions, does not refer to an uncoupled normal mode of the system, but rather it refers to roots of the aeroelastic analysis which include aerodynamic forces as well as stiffness and mass.

The first interesting comparison of transfer functions is between Figures 6-16 and 6-17. The transfer functions plotted in these two figures are identical between .20 cps and 8.0 cps within about $1 \frac{1}{2} \text{ db}$. Some, and possibly most of the differences between Figures 6-16 and 6-17 is

attributable to analog computer inaccuracy since (as discussed in Appendix A) the electric analog circuits used to obtain these transfer functions bear no apparent physical similarity to each other.

Further comparisons of the transfer functions for configuration 2 will be made using Figure 6-17 as the "exact" basis of comparison since it is virtually identical to the "exact" solution and the small differences between the electric analog circuits used in the "exact" and modal representations will thereby be eliminated in the comparison.

The comparison of Figure 6-17 with Figure 6-19 shows the effect of omitting the fourth and fifth elastic modes from the modal representation while retaining the "residual flexibility" of all modes higher than the third finite frequency mode. The highest frequency peak of Figure 6-18 corresponds to the fourth elastic mode. This peak does not appear in Figure 6-19 but the remainder of the transfer function remains unchanged. Progressing to Figures 6-20, 6-21 and 6-22, it is found that as modes are omitted from the representation, the corresponding peaks of the transfer function are also deleted, while the transfer function remains virtually unchanged for the frequency spectrum below that of the mode omitted.

Comparisons may also be made between system representations which include the "residual flexibility" approximation of higher modes and system representations which do not. The most striking comparison is between Figures 6-23 and 6-22. The representation used to obtain the transfer function of Figure 6-23 includes 2 zero frequency modes and 5 elastic modes. The representation used in Figure 6-22 included only the 2 zero frequency modes and the "residual flexibility" approximation to

all elastic modes. The peak corresponding to the short period mode of Figure 6-23 differs from that of Figure 6-17 by about $5\frac{1}{2}$ db while that obtained from the simpler representation of Figure 6-22 differs by only about $1\frac{1}{2}$ db. The reason for the poor accuracy of the transfer function of Figure 6-23 apparently is the exclusion in the representation of the torsional flexibility of the aft aerodynamic surface. This flexibility is primarily expressed in the generalized flexibility of a higher mode.

It is interesting to note that the representation of the system used to obtain the transfer function of Figure 6-23 included 7 modes of a system which, as defined by the exact representation, has a total of 16 degrees of freedom. Apparently, for the prediction of short period mode response of configuration 2, it is much better to represent the system by its rigid body modes and the "residual flexibility" of all the elastic modes than to represent the system in 7 of its 16 degrees of freedom. This statement implies that confidence should not be placed in the representation of a system merely because it includes a seemingly large number of modes.

A comparison of the transfer functions contained in Appendix C, showing vertical acceleration due to rotation of the forward aerodynamic surface, lead to identical conclusions. The comparison of phase angle plots, associated with the transfer function amplitude curves, also lead to the same conclusions although differences between transfer function phase angles are generally less significant than differences in amplitude.

6.3.2 Configuration 3

Figures 6-29 through 6-39 present some of the transfer function amplitude plots measured for flight condition A of configuration 3. The aerodynamic parameters of flight condition A are:

Velocity = 684 mph

Altitude = sea level

Dynamic pressure = $1/2 \rho v^2 = 8.31 \text{ lb/in.}^2$

Lift curve slope = $C_{L_\alpha} = 6.0$

The transfer functions in Figures 6-29 through 6-39 show the pitching velocity amplitude at fuselage station 340 (the wing-fuselage intersection) due to a sinusoidal vertical gust.

The transfer functions of Figures 6-29 through 6-39 were obtained using the representations of the structure of configuration 3 summarized in Table 6-4.

Figure 6-29 presents the transfer function obtained from the "exact" representation of configuration 3. The lowest frequency peak in response ($f = \text{about } .7 \text{ cps}$) shown in Figure 6-29 corresponds to the short period mode of the system. The slight change in slope at a frequency of about 1.9 cps corresponds to the first elastic mode of the system. The uncoupled natural frequency of the first normal mode is 1.20 cps but due to aerodynamic "stiffening" of the wings of the airplane, the corresponding root of the aeroelastic analysis appears to have an associated frequency of about 1.9 cps. The following peaks at frequencies of about 4.7 cps and 5.4 cps correspond to the second and third elastic modes of the system. The fourth elastic mode has a very small amplitude

at the slope coordinate used as a "sensor" in these transfer functions and therefore does not appear as a peak on Figure 6-29. The last peak of Figure 6-29 at about 9.1 cps corresponds to the fifth elastic mode.

A comparison of the transfer functions obtained from the "exact" representation and the best modal representation is shown by the comparison of Figures 6-29 and 6-30. Throughout most of the frequency spectrum the difference between the transfer functions of Figures 6-29 and 6-30 is only about 1 db. However the difference does become large in the frequency range between 2.2 cps and 4.2 cps and again in a narrow frequency band at about 6.9 cps. It is difficult to ascertain the cause of the differences between these transfer functions. However it is evident from comparison of the other records that these regions are sensitive to small changes in representation. Thus most of the difference is probably attributable to small inaccuracies in the electric analog circuits.

By comparison of Figure 6-30 with Figure 6-35, the effect on the transfer function of omitting the "residual flexibility" of elastic modes higher than the fifth can be seen to be quite small. The effect of the omission of the "residual flexibility" when only three elastic modes are explicitly included is seen to be much larger by comparison of Figures 6-31 and 6-36. The transfer function of Figure 6-31, where the representation included the "residual flexibility" approximation to the higher modes, is quite similar to that of Figures 6-29 and 6-30 up to the frequency of the third elastic mode. The transfer function of Figure 6-36, where no approximation was made to higher modes, is quite different throughout most of the frequency spectrum shown. The frequency of the short period mode in Figure 6-36 is about .15 cps higher than that of the "exact" representation

shown in Figure 6-29, and the amplitude of response above 2.0 cps of Figure 6-36 bears little similarity to that of Figure 6-29.

The inclusion of the "residual flexibility" approximation in the modal representation of configuration 3 has been shown to improve the accuracy of the representation. The improvement is not as clearly evident as it was shown to be in the case of configuration 2, but the conclusions previously drawn are certainly sustained by the comparisons for configuration 3.

Many additional transfer function amplitude and phase plots for configuration 3 are presented in Appendix C. A summary index of these data is shown in Figure 6-2. Comparisons of these additional data lead to the conclusions previously stated.

The aerodynamic parameters of flight condition B, considered in the data of Appendix C, are as follows.

Velocity = 684 mph

Altitude = 20,000 ft.

Dynamic pressure = $1/2 \rho v^2 = 4.43 \text{ lb/in.}^2$

Lift curve slope = $C_{L_\alpha} = 6.0$

6.3.3 Configuration 4

The effects of aeroelasticity on the dynamic response of configuration 4 were found to be quite different than those previously discussed. The analysis routine used on configuration 4 was changed from that followed in the analyses of the 2 previous configurations because of the presence of dynamic instabilities in the vicinity of the flight condition F

chosen for the analysis. The aerodynamic parameters of flight condition F are:

Velocity = 1655 mph

Altitude = 40,000 ft.

Dynamic pressure = 11.9 lb/in.²

Lift curve slope = $C_{L_\alpha} = 5.0$

The dynamic analysis of configuration 4 at flight condition F yielded the stability boundaries shown in Figure 6-40. The abscissa of Figure 6-40 is a dimensionless flexibility parameter which expresses the fraction of the "basic" structural flexibilities given in Section 5 and applies to all flexible elements of the structure. The dimensionless flexibility parameter may be also interpreted as a dimensionless dynamic pressure parameter expressing the fraction of the "basic" dynamic pressure of flight condition F with all other parameters of the system constant. The ordinate of Figure 6-40 shows the assumed aerodynamic center for all 3 strips when the control surface deflection (δ) relative to the wing is zero.

The stability boundary labeled "A" on Figure 6-40 defines the parameter combinations which result in a conventional binary flutter instability with a flutter frequency near the natural frequency of the seventh elastic mode (about 11.5 cps at basic flexibility). The stability boundary labeled "B" on Figure 6-40 defines the limits of a conventional binary flutter instability with a flutter frequency near the natural frequency of the fourth elastic mode (about 6.5 cps at basic flexibility). The stability boundary labeled "C" on Figure 6-40 defines the limits of a dynamic instability with an associated frequency below the natural frequency of the first elastic mode. In the language of the flutter

analyst, this instability is the result of the coupling or interaction of the first elastic mode with the zero frequency modes of the system.

From the viewpoint of the stability and control analyst, this latter instability could be recognized as an unstable short period mode resulting from interaction with the first elastic mode. However, regardless of the semantic definition given to this phenomenon, it is certainly an example of the possible effect of elastic mode interaction on the low frequency response of an airborne vehicle. Also, regardless of the area of engineering analysis considered to have the responsibility for the prediction of the instability, this phenomenon in its subcritical case is of great importance in stability and control analyses of the system.

The following data were obtained for configuration 4:

1. The locations of the stability boundaries already discussed.
2. The determination of typical examples of transfer functions for parameter combinations where the system was stable.
3. A limited study of some effects of system representation on the transfer functions.

Figures 6-41 through 6-46 present transfer functions of the pitching velocity at fuselage station 325 due to sinusoidal elevator rotation. All 6 transfer functions presented were obtained from an analysis where the aerodynamic center of each strip was assumed to be at the 35% chord of that strip. They were obtained using the "exact" representation of the system. The differences between the transfer functions of Figures

6-41 through 6-46 are due only to the variation of the flexibility of the system. When the flexibility of the system is a fraction c times the "basic" flexibility, all uncoupled elastic mode frequencies of the system are $\sqrt{1/c}$ times the frequency at "basic" flexibility and the corresponding mode shapes at the two flexibility conditions are identical.

The transfer functions presented correspond to the following flexibility conditions.

Figure 6-41 - $.7 \times$ "basic" flexibility

Figure 6-42 - $.5 \times$ "basic" flexibility

Figure 6-43 - $.3 \times$ "basic" flexibility

Figure 6-44 - $.2 \times$ "basic" flexibility

Figure 6-45 - $.1 \times$ "basic" flexibility

Figure 6-46 - $0 \times$ "basic" flexibility, or a rigid system

The lowest frequency peak in all transfer functions corresponds to the short period mode of the airplane. The short period mode frequency is usually considered to be relatively independent of the flexibility of the system. All higher frequency peaks are related to elastic modes of the system and, therefore, their frequencies are approximately inversely proportional to the square root of the flexibility factor.

Of particular interest in the transfer functions of Figures 6-41 through 6-46 is the rather remarkable variation in the short period mode response with changes in the flexibility of the system. The system parameters used in Figure 6-41 appear in Figure 6-40 as corresponding to a marginally stable case. If, however, the stability margin is expressed in terms of the conventional parameters of flexibility or

dynamic pressure, the margin of safety would appear to be rather large. The critical system parameter for the stability boundary labeled C on Figure 6-40 is the assumed aerodynamic center.

Other evidence demonstrating the sensitivity of the system stability to changes in aerodynamic center location is presented in Figure 6-47. The ordinate of Figure 6-47 shows the damping factor (ζ) of the low frequency root of the system. The abscissa of Figure 6-47 is the assumed location of the aerodynamic centers of the strips in units of percent of the mean chord of the strips. The damping curve of Figure 6-47 is plotted for configuration 4 when all flexibilities of the structure are 1/2 of the "basic" flexibilities presented in Section 5. Of particular interest in Figure 6-47 is the radical change in the stability of the system resulting from a small change in the assumed aerodynamic center location at the 34% chord.

Budgeting limitations of this study prevented a more complete investigation of this phenomenon. It was only possible in this project to identify the "mode interaction" phenomenon as a problem area in the field of stability and control. The problems of attaining an adequate understanding of the phenomenon and the development of satisfactory analysis procedures are left for further investigation.

The study of the effect of the type of structural representation used in an analysis on the predicted response of the system was greatly hampered by the proximity of instabilities of the system. Due to small errors in the determination of the total flexibility of the system, it was found that the system was unstable whenever a modal representation

including the calculated "residual flexibility" was employed. It was possible, however, to use a modal representation of the system including the two zero frequency modes and the lowest 5 finite frequency modes. Also a smaller number of elastic modes was explicitly included in a modal representation including "residual flexibility" of only the higher elastic modes up to the fifth elastic mode. The assumption is made in this representation that the total flexibility of the system is defined by the generalized flexibility of the lowest five elastic modes.

Transfer functions, similar to those presented for configurations 2 and 3, were measured using various representations of the system. A summary index of the transfer functions measured for configuration 4 is shown in Table 6-2 and the transfer functions are included in Appendix C.

The results of the study of the effect of the system representation on the measured transfer functions of configuration 4 is not nearly as conclusive as was the case with configurations 2 and 3. However the results obtained with configuration 4 do limit the general application of conclusions drawn from the analyses of the former configurations.

The analyses of configuration 4 show that in order to obtain an adequate prediction of the response amplitude of the short period mode for this system, some elastic modes must be explicitly included in the modal representation of the system. An adequate prediction of system response for configuration 4 in the frequency spectrum below the first elastic mode cannot be made with system representations which omit the generalized mass of all elastic modes.

An interesting example of the effect of "residual flexibility" was observed with the modal representation of configuration 4, explicitly including two rigid body modes and the first elastic mode. When the generalized flexibility of all higher modes was ignored, the system was unstable but when the generalized flexibility of the second through the fifth mode was included the system became stable.

Figure 6-1

[z_{mm}]

Configuration 2, Condition 30

	Body 4	Body 8	Fin 250	Fin 900	Fin Slope 250	Fin Slope 900	Body Slope 8
Body 4	$.383 \times 10^{-3}$	$.0676 \times 10^{-3}$	$.393 \times 10^{-3}$	0	-1.06×10^{-6}	0	$-.494 \times 10^{-6}$
Body 8	$.0676 \times 10^{-3}$	$.0207 \times 10^{-3}$	$.0676 \times 10^{-3}$	0	$-.118 \times 10^{-6}$	0	$-.118 \times 10^{-6}$
Fin 250	$.383 \times 10^{-3}$	$.0676 \times 10^{-3}$	$.406 \times 10^{-3}$	0	-1.06×10^{-6}	0	$-.494 \times 10^{-6}$
Fin 900	0	0	0	16.4×10^{-6}	0	$.000878 \times 10^{-6}$	0
Fin Slope 250	-1.06×10^{-6}	$-.118 \times 10^{-6}$	-1.06×10^{-6}	0	5.26×10^{-9}	0	$.934 \times 10^{-9}$
Fin Slope 900	0	0	0	$.000878 \times 10^{-6}$	0	2.45×10^{-9}	0
Body Slope 8	$-.494 \times 10^{-6}$	$-.118 \times 10^{-6}$	$-.494 \times 10^{-6}$	0	$.934 \times 10^{-9}$	0	$.941 \times 10^{-9}$

Figure 6-2

[Z_{mm}]

Configuration 3, Condition 2

	Fuselage 6	1/4 Chord II	1/4 Chord III	Fus. Slope 6	Stream Slope I	Stream Slope II	Stream Slope III
Fuselage 6	.498 x 10 ⁻³	-16.6 x 10 ⁻⁶	-.0353 x 10 ⁻³	1.445 x 10 ⁻⁶	-.00864 x 10 ⁻⁶	-.00864 x 10 ⁻⁶	-.00864 x 10 ⁻⁶
1/4 Ch. II	-.0166 x 10 ⁻³	212.	.494 x 10 ⁻³	.00200 x 10 ⁻⁶	.225 x 10 ⁻⁶	.494 x 10 ⁻⁶	.494 x 10 ⁻⁶
1/4 Ch. III	-.0353 x 10 ⁻³	484.	1.62 x 10 ⁻³	.00408 x 10 ⁻⁶	.446 x 10 ⁻⁶	1.58 x 10 ⁻⁶	2.67 x 10 ⁻⁶
Fus. Slope 6	1.445 x 10 ⁻⁶	.00200	.00408 x 10 ⁻⁶	7.07 x 10 ⁻⁹	.013 x 10 ⁻⁹	.013 x 10 ⁻⁹	.013 x 10 ⁻⁹
Stream Slope I	-.00864 x 10 ⁻⁶	.225	.446 x 10 ⁻⁶	.013 x 10 ⁻⁹	1.70 x 10 ⁻⁹	1.70 x 10 ⁻⁹	1.70 x 10 ⁻⁹
Stream Slope II	-.00864 x 10 ⁻⁶	.494	1.58 x 10 ⁻⁶	.013 x 10 ⁻⁹	1.70 x 10 ⁻⁹	8.31 x 10 ⁻⁹	8.31 x 10 ⁻⁹
Stream Slope III	-.00864 x 10 ⁻⁶	.494 x 10 ⁻⁶	2.67 x 10 ⁻⁶	.013 x 10 ⁻⁹	1.70 x 10 ⁻⁹	8.31 x 10 ⁻⁹	36.0 x 10 ⁻⁹

Figure 6-3
 $\left[Z_{nm} \right]$
 Configuration 4, Condition 10

	Wing Pt. II	1/4 Chord Strip I	1/4 Chord Strip II	1/4 Chord Strip III	3/4 Chord Strip I	3/4 Chord Strip II	3/4 Chord Strip III
Wing Pt. II	.536 $\times 10^{-3}$	.030 $\times 10^{-3}$	.104 $\times 10^{-3}$	.3155 $\times 10^{-3}$	.2175 $\times 10^{-3}$	.456 $\times 10^{-3}$	.662 $\times 10^{-3}$
1/4 Chord Strip I	.030	.0054	.0126	.0200	.02515	.02755	.0299
1/4 Chord Strip II	.104	.0126	.137	.178	.069	.106	.150
1/4 Chord Strip III	.3155	.0200	.178	.676	.129	.306	.610
3/4 Chord Strip I	.2175	.02515	.069	.129	.176	.196	.215
3/4 Chord Strip II	.456	.02755	.106	.306	.196	.400	.572
3/4 Chord Strip III	.662 $\times 10^{-3}$	.0299 $\times 10^{-3}$	.150 $\times 10^{-3}$	.610 $\times 10^{-3}$	.215 $\times 10^{-3}$	.572 $\times 10^{-3}$	1.44 $\times 10^{-3}$

Figure 6-4

 $[X_{11}]$

Configuration 2, Condition 30

	Body 4	Body 8	Body 10	Fin 250	Fin 900	Fin Slope 250	Fin Slope 900	Body Slope 8	Body Slope 10
Body 4	$.01597 \times 10^{-3}$	$-.01307 \times 10^{-3}$	-3.08×10^{-6}	$.01646 \times 10^{-3}$	-2.984×10^{-6}	$-.1262 \times 10^{-6}$	$.06439 \times 10^{-6}$	$.007596 \times 10^{-6}$	$.06505 \times 10^{-6}$
Body 8	$-.01306$	$.0159$	3.216	$-.0143$	2.766	$.153$	$-.07940$	$-.004$	$-.07940$
Body 10	$-.00308$	$.003216$	4.388	$-.003324$	3.713	$.001537$	$-.02190$	$.01941$	$-.02191$
Fin 250	$.01646 \times 10^{-3}$	$-.0143 \times 10^{-3}$	-3.324	$.039 \times 10^{-3}$	-2.933	$-.12 \times 10^{-6}$	$.06712$	$.010$	$.06712$
Fin 900	-2.984×10^{-6}	2.77×10^{-6}	3.713	-2.331×10^{-6}	19.55	3.888×10^{-9}	$-.02532 \times 10^{-6}$	$.02139 \times 10^{-6}$	$-.02369 \times 10^{-6}$
Fin Slope 250	$-.1262$	$.153$	$.001537$	$-.12$	$.003893$	3.21	$-.7384 \times 10^{-9}$	$-.4412 \times 10^{-9}$	$-.7384 \times 10^{-9}$
Fin Slope 900	$.06439$	$-.0794$	$-.02190$	$.06712$	$-.02532$	$-.7384$	3.02	$-.03856$	$.5692$
Body Slope 8	$.007594$	$-.004$	$.01941$	$.010$	$.02138$	$-.4412$	$-.07856$	$.295$	$-.03856$
Body Slope 10	$.06502 \times 10^{-6}$	$-.07940 \times 10^{-6}$	$-.02191 \times 10^{-6}$	$.06712 \times 10^{-6}$	$-.02369 \times 10^{-6}$	$-.7384 \times 10^{-9}$	$.5692 \times 10^{-9}$	$-.03856 \times 10^{-9}$	$.5692 \times 10^{-9}$

Figure 6-5

 $[x_{11}]$

Configuration 3, Condition 2

	Fuselage 6	1/4 Chord I	1/4 Chord II	1/4 Chord III	Fus. Slope 3	Fus. Slope 6	Stream Slope I	Stream Slope II	Stream Slope III
Fuselage 6	.2493 $\times 10^{-3}$	.01353 $\times 10^{-3}$	-120.3 $\times 10^{-6}$	-.4000 $\times 10^{-3}$	-.03980 $\times 10^{-6}$	.7566 $\times 10^{-6}$	-.1606 $\times 10^{-6}$	-.4407 $\times 10^{-6}$	-.6614 $\times 10^{-6}$
1/4 Ch. I	.01353	.0126	-10.29	-.06312	.02756	.007080	.0144	-.01688	-.05120
1/4 Ch. II	-.1203	-.01029	173.0	.3142	-.02319	-.1905	.1650	.3440	.2653
1/4 Ch. III	-.4000 $\times 10^{-3}$	-.06312 $\times 10^{-3}$	314.2	1.257 $\times 10^{-3}$	-.2345	-.5571 $\times 10^{-6}$	.1424 $\times 10^{-6}$	1.128 $\times 10^{-6}$	2.095 $\times 10^{-6}$
Fus. Slope 3	-.03980 $\times 10^{-6}$	.02756 $\times 10^{-6}$	-.02319	-.2345 $\times 10^{-6}$	.0001736	-.02892 $\times 10^{-9}$	.0576 $\times 10^{-9}$	-.06752 $\times 10^{-9}$	-.2348 $\times 10^{-6}$
Fus. Slope 6	.7566	.007080	-.1905	-.5571	.00002032	5.683	-.6158	-.9710	-1.240
Stream Slope I	-.1606	.0144	.1650	.1424	.0000576	-.6158	1.570	1.474	1.206
Stream Slope II	-.4394	-.01688	.3440	1.128	-.00006752	-.9710	1.474	7.729	7.160
Stream Slope III	-.6614 $\times 10^{-6}$	-.05120 $\times 10^{-6}$	.2655 $\times 10^{-6}$	2.095 $\times 10^{-6}$	-.0002048 $\times 10^{-6}$	-1.240 $\times 10^{-9}$	1.206 $\times 10^{-9}$	7.160 $\times 10^{-9}$	34.88 $\times 10^{-9}$

Figure 6-6

[X₁₁]Configuration L₄, Condition 10

	Fus. Pt. L ₄	Wing Pt. II	1/4 Chord Strip I	1/4 Chord Strip II	1/4 Chord Strip III	3/4 Chord Strip I	3/4 Chord Strip II	3/4 Chord Strip III	Fus. Slope Sta. 325
Fus. Pt. L ₄	.01724 × 10 ⁻³	.01078 × 10 ⁻³	.01724 × 10 ⁻³	.01716 × 10 ⁻³	-.07646 × 10 ⁻³	.03590 × 10 ⁻³	.008684 × 10 ⁻³	-.03280 × 10 ⁻³	.1694 × 10 ⁻⁶
Wing Pt. II	.01078	.2242	.006682	-.01901	-.06564	.01676	.1699	.1464	-.1166
1/4 Chord Strip I	.01724	.008682	.01925	.02078	-.08106	.04396	.007704	-.05095	.1933
1/4 Chord Strip II	.01716	-.01901	.02078	.1206	-.02280	.01562	-.006186	-.06902	.1290
1/4 Chord Strip III	-.07646	-.06564	-.08106	-.02280	.2343	-.1337	-.05125	.09099	-.2008
3/4 Chord Strip I	.03590	.01676	.04396	.01562	-.1337	.09077	.01478	-.1335	.1410
3/4 Chord Strip II	.008684	.1699	.007704	-.006186	-.05125	.01478	.1325	.1004	-.07719
3/4 Chord Strip III	-.03280 × 10 ⁻³	.1464 × 10 ⁻³	-.05095 × 10 ⁻³	-.06502 × 10 ⁻³	.09099 × 10 ⁻³	-.1335 × 10 ⁻³	.1004 × 10 ⁻³	.7071 × 10 ⁻³	-.5501 × 10 ⁻⁶
Fus. Slope Sta. 325	.1694 × 10 ⁻⁶	-.1166 × 10 ⁻⁶	.1933 × 10 ⁻⁶	.1290 × 10 ⁻⁶	-.2008 × 10 ⁻⁶	.1410 × 10 ⁻⁶	-.07719 × 10 ⁻⁶	-.5501 × 10 ⁻⁶	1.523 × 10 ⁻⁹

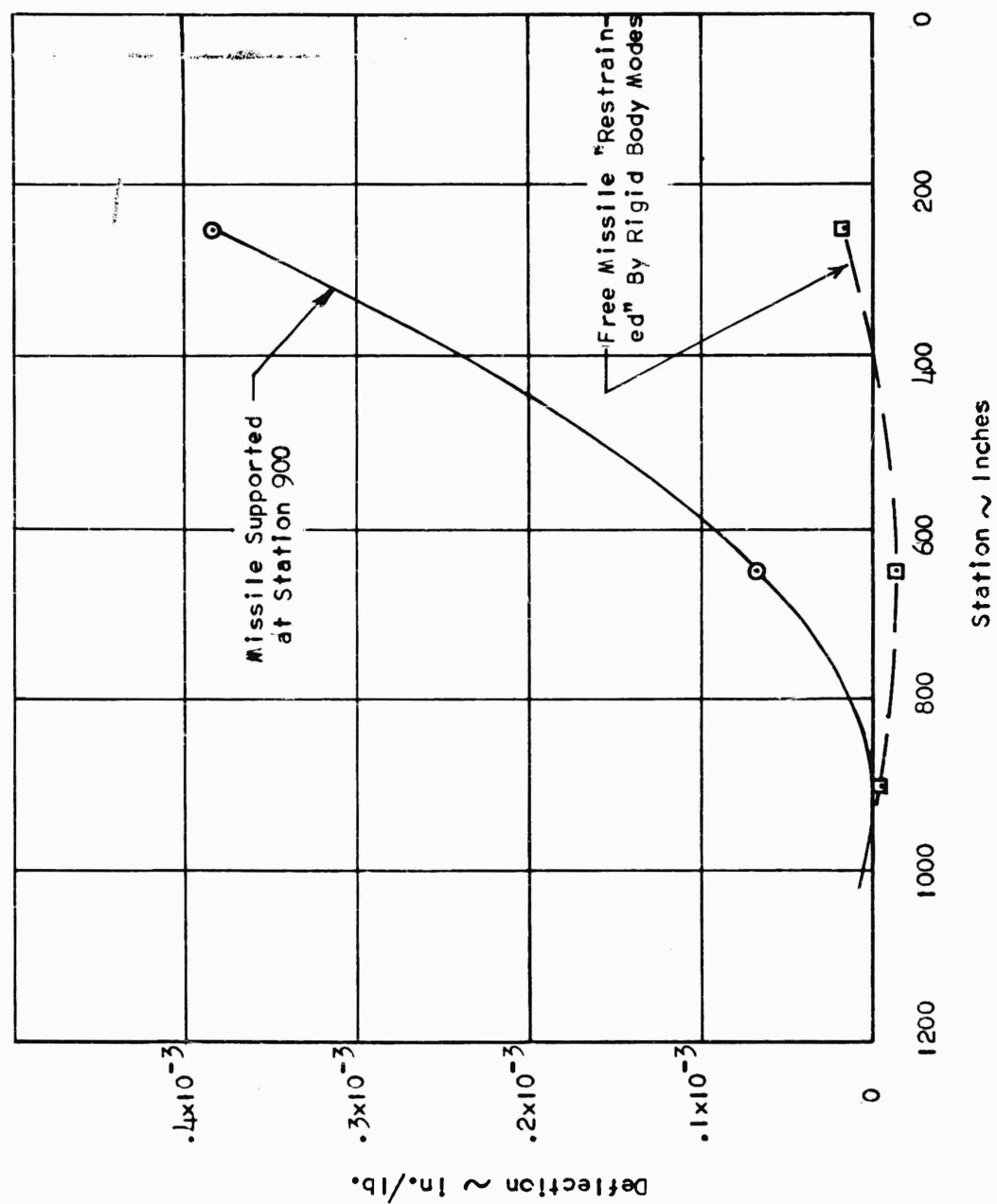


Figure 6-7. Deflection of Configuration 2 Due to Load at Station 250

Figure 6-8

$$\begin{bmatrix} 5 \\ x_{11} \end{bmatrix}$$

Configuration 2, Condition 30

	Body 4	Body 8	Body 10	FIn 250	FIn 900	FIn Slope 250	FIn Slope 900	Body Slope 8	Body Slope 10
Body 4	.01597 $\times 10^{-3}$	-.01307 $\times 10^{-3}$	-3.08 $\times 10^{-6}$	.01646 $\times 10^{-3}$	-2.984 $\times 10^{-6}$	-.1262 $\times 10^{-6}$	.06439 $\times 10^{-6}$	.007596 $\times 10^{-6}$	.06505 $\times 10^{-6}$
Body 8	-.01306	.01532	3.29	-.01311	2.766	.1465	-.08351	-.005390	-.08151
Body 10	-.00308	.003290	3.64	-.003009	3.713	-.00123	-.01895	.01543	-.01372
FIn 250	.01646 $\times 10^{-3}$	-.01312 $\times 10^{-3}$	-3.009	.01699 $\times 10^{-3}$	-2.933	-.1286 $\times 10^{-6}$	.06388	.005739	.06477
FIn 900	-2.984 $\times 10^{-6}$	2.77 $\times 10^{-6}$	3.713	-2.931 $\times 10^{-6}$	19.55	3.888 $\times 10^{-9}$	-.02532 $\times 10^{-6}$	.02139 $\times 10^{-6}$	-.02369 $\times 10^{-6}$
FIn Slope 250	-.1262	.1465	-.001224	-.1286	.003893	2.726	-.6300 $\times 10^{-9}$	-.1891 $\times 10^{-9}$	-.6616 $\times 10^{-9}$
FIn Slope 900	.06439	-.08351	-.01895	.06388	-.02532	-.6237	.4977	-.0216	.4737
Body Slope 8	.007594	-.00540	.01843	.008739	.02138	-.1491	-.02137	.1872	-.009212
Body Slope 10	.06502 $\times 10^{-6}$	-.08151 $\times 10^{-6}$	-.01873 $\times 10^{-6}$	.06477 $\times 10^{-6}$	-.02369 $\times 10^{-6}$	-.6614 $\times 10^{-9}$	.4737 $\times 10^{-9}$	-.00923 $\times 10^{-9}$	.4542 $\times 10^{-9}$

Figure 6-9

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

Configuration 3, Condition 2

	Fuselage 6	1/4 Chord I	1/4 Chord II	1/4 Chord III	Fus. Slope 3	Fus. Slope 6	Stream Slope I	Stream Slope II	Stream Slope III
Fuselage 6	.2092 $\times 10^{-3}$	.01251 $\times 10^{-3}$	-108.2 $\times 10^{-6}$	-.3719 $\times 10^{-3}$	-.05876 $\times 10^{-6}$	.7371 $\times 10^{-6}$	-.1598 $\times 10^{-6}$	-.1179 $\times 10^{-6}$	-.5547 $\times 10^{-6}$
1/4 Ch. I	.01251	.006104	-11.70	-.06206	.00212	.001023	.01180	-.01258	-.03977
1/4 Ch. II	-.1082	-.01172	121.8	.2812	-.01694	-.1331	.1393	.3800	.1383
1/4 Ch. III	-.3719 $\times 10^{-3}$	-.06206 $\times 10^{-3}$	281.2	1.117 $\times 10^{-3}$	-.2034	-.5573 $\times 10^{-6}$	.1231 $\times 10^{-6}$	1.021 $\times 10^{-6}$	1.725 $\times 10^{-6}$
Fus. Slope 3	-.05876 $\times 10^{-6}$	.00212 $\times 10^{-6}$	-.01684	-.2029 $\times 10^{-6}$	.0003749	-.7219 $\times 10^{-9}$	.1662 $\times 10^{-9}$	-.1173 $\times 10^{-9}$	-.5117 $\times 10^{-9}$
Fus. Slope 6	.7371	.001056	-.1831	-.5573	-.0007219	4.154	-.7548	-.8764	-.5294
Stream Slope I	-.1598	.01178	.1393	.1231	.0001656	-.7323	.4458	1.000	.2523
Stream Slope II	-.1179	-.01976	.3800	1.021	-.0001173	-.8790	1.000	4.458	4.332
Stream Slope III	-.5547 $\times 10^{-6}$	-.03977 $\times 10^{-6}$	.1386 $\times 10^{-6}$	1.722 $\times 10^{-6}$	-.0005117 $\times 10^{-6}$	-.5269 $\times 10^{-9}$	.2520 $\times 10^{-9}$	4.382 $\times 10^{-9}$	9.195 $\times 10^{-9}$

Figure 6-10

 $[x_1^5]$

Configuration 4, Condition 10

	Fus. Pt. 4	Wing Pt. II	1/4 Chord Strip I	1/4 Chord Strip II	1/4 Chord Strip III	3/4 Chord Strip I	3/4 Chord Strip II	3/4 Chord Strip III	Fus. Slope Sta. 325
Fus. Pt. 4	.04176 $\times 10^{-3}$	9.728 $\times 10^{-6}$	.04559 $\times 10^{-3}$	.01977 $\times 10^{-3}$	-.07655 $\times 10^{-3}$	.03802 $\times 10^{-3}$	8.463 $\times 10^{-6}$	-.03285 $\times 10^{-3}$	.1447 $\times 10^{-6}$
Wing Pt. II	9.728 $\times 10^{-6}$	.1070 $\times 10^{-3}$	8.554 $\times 10^{-6}$	-.01676	-.06335	.01888	.07361 $\times 10^{-3}$	.1516	-.1154
1/4 Chord Strip I	.04559 $\times 10^{-3}$	8.554 $\times 10^{-6}$	.04810 $\times 10^{-3}$	.02258	-.08412	.04544	8.117 $\times 10^{-6}$	-.05103	.1752
1/4 Chord Strip II	.01977	-.01676 $\times 10^{-3}$	.02258	.01848	-.02232	.01771	-.01070 $\times 10^{-3}$	-.06905	.1472
1/4 Chord Strip III	-.07655	-.06335	-.08412	-.02232	.2343	-.1337	-.04937	.09098	-.2015
3/4 Chord Strip I	.03802 $\times 10^{-3}$	.01888	.04544 $\times 10^{-3}$	.01771	-.1337	.08866	.01638	-.1336	.1643
3/4 Chord Strip II	8.463 $\times 10^{-6}$	.07961	8.117 $\times 10^{-6}$	-.01070	-.04937	.01638	.05952	.1044	-.07010
3/4 Chord Strip III	-.03285 $\times 10^{-3}$	.1516 $\times 10^{-3}$	-.05103 $\times 10^{-3}$	-.06905 $\times 10^{-3}$	.09098 $\times 10^{-3}$	-.1336 $\times 10^{-3}$	.1044 $\times 10^{-3}$	.7072 $\times 10^{-3}$	-.5510 $\times 10^{-6}$
Fus. Slope Sta. 325	.1447 $\times 10^{-6}$	-.1154 $\times 10^{-6}$	.1752 $\times 10^{-6}$	.1472 $\times 10^{-6}$	-.2015 $\times 10^{-6}$	.1643 $\times 10^{-6}$	-.07010 $\times 10^{-6}$	-.5510 $\times 10^{-6}$	1.232 $\times 10^{-9}$

Figure 6-11

$$\begin{bmatrix} x_{11}^8 \\ x_{11}^9 \end{bmatrix}$$

Configuration 2, Condition 30

$$\begin{bmatrix} x_{11}^8 \\ x_{11}^9 \end{bmatrix} = \begin{bmatrix} x_{11}^8 \\ x_{11}^9 \end{bmatrix} - \begin{bmatrix} x_{11}^5 \\ x_{11}^5 \end{bmatrix}$$

	Body 4	Body 8	Body 10	Fin 250	Fin 900	Fin Slope 250	Fin Slope 900	Body Slope 8	Body Slope 10
Body 4	0	0	0	0	0	0	0	0	0
Body 8	0	0	0	0	0	0	0	0	0
Body 10	0	0	.750 x 10 ⁻⁶	-.315 x 10 ⁻⁶	0	.000307 x 10 ⁻⁶	-.00295 x 10 ⁻⁶	.00098 x 10 ⁻⁶	.00319 x 10 ⁻⁶
Fin 250	0	0	.315 x 10 ⁻⁶	.0220 x 10 ⁻³	0	.0100 x 10 ⁻⁶	.0032 x 10 ⁻⁶	.00100 x 10 ⁻⁶	.00235 x 10 ⁻⁶
Fin 900	0	0	0	0	0	0	0	0	0
Fin Slope 250	0	0	.000307 x 10 ⁻⁶	.0100 x 10 ⁻⁶	0	.48 x 10 ⁻⁹	-.1084 x 10 ⁻⁹	.0771 x 10 ⁻⁹	-.0768 x 10 ⁻⁹
Fin Slope 900	0	0	-.00295	.0032	0	-.1084	2.52	-.01696	.0955
Body Slope 8	0	0	.00098	.0010	0	.0771	-.01696	.108	-.02935
Body Slope 10	0	0	.00319 x 10 ⁻⁶	.00235 x 10 ⁻⁶	0	-.0768 x 10 ⁻⁹	.0955 x 10 ⁻⁹	-.02935 x 10 ⁻⁹	.1150 x 10 ⁻⁹

Figure 6-12

$$\begin{bmatrix} x_{11}^6 \\ x_{11}^5 \end{bmatrix}$$

Configuration 3, Condition 2

$$\begin{bmatrix} x_{11}^6 \\ x_{11}^5 \end{bmatrix} = \begin{bmatrix} x_{11}^6 \\ x_{11}^5 \end{bmatrix}$$

	Fuselage 6	1/4 Chord I	1/4 Chord II	1/4 Chord III	Fus. Slope 3	Fus. Slope 6	Stream Slope I	Stream Slope II	Stream Slope III
Fuselage 6	.0401 x 10 ⁻³	.001020 x 10 ⁻³	-12.10 x 10 ⁻⁶	-.0251 x 10 ⁻³	.01396 x 10 ⁻⁶	.01950 x 10 ⁻⁶	-.0008 x 10 ⁻⁶	-.0221 x 10 ⁻⁶	-.0167 x 10 ⁻⁶
1/4 Ch. I	.001020	.006196	1.420	-.00106	.00514	.006040	.0261	.00279	-.01143
1/4 Ch. II	-.0121	.00142	57.20	.03300	-.04655	-.00740	.02570	-.0560	.1270
1/4 Ch. III	-.0251 x 10 ⁻³	-.00106 x 10 ⁻³	33.00	.1400 x 10 ⁻³	-.0313	.000200 x 10 ⁻⁶	.01930	.1070	.3715
Fus. Slope 3	.01396 x 10 ⁻⁶	.00514 x 10 ⁻⁶	-.00665	-.0318 x 10 ⁻⁶	.0000387	.6936 x 10 ⁻⁹	-.0001083 x 10 ⁻⁶	.00004978 x 10 ⁻⁶	.0003069 x 10 ⁻⁶
Fus. Slope 6	.01950	.006040	-.00740	.000200	.0006936	1.529	.1390 x 10 ⁻⁹	-.0933 x 10 ⁻⁹	-.7118 x 10 ⁻⁹
Stream Slope I	-.0008	.00261	.02570	.01930	-.0001083	.1365	1.124	.4740	.9538
Stream Slope II	-.02210	.002790	-.0360	.1070	.00004978	-.0933	.4740	3.271	3.078
Stream Slope III	-.0167 x 10 ⁻⁶	-.01113 x 10 ⁻⁶	.1269 x 10 ⁻⁶	.3715 x 10 ⁻⁶	.0003069 x 10 ⁻⁶	-.7118 x 10 ⁻⁹	.9538 x 10 ⁻⁹	3.078 x 10 ⁻⁹	25.68 x 10 ⁻⁹

Figure 6-13

$$\begin{bmatrix} x_{11}^2 \end{bmatrix}$$

Configuration 14, Condition 10

$$\begin{bmatrix} x_{11}^2 \end{bmatrix} = \begin{bmatrix} x_{11}^4 \end{bmatrix} - \begin{bmatrix} x_{11}^5 \end{bmatrix}$$

	Fus. Pt. 4	Wing Pt. 11	1/4 Chord Strip I	1/4 Chord Strip II	1/4 Chord Strip III	3/4 Chord Strip I	3/4 Chord Strip II	3/4 Chord Strip	Fus. Slope 325
Fus. Pt. 4	.00248 $\times 10^{-3}$	.001952 $\times 10^{-3}$	.00165 $\times 10^{-3}$	-.00261 $\times 10^{-3}$	.090 $\times 10^{-6}$	-.00212 $\times 10^{-3}$	.221 $\times 10^{-6}$	.05 $\times 10^{-6}$	.0247 $\times 10^{-6}$
Wing Pt. 11	.001952	.1172 $\times 10^{-3}$	.128 $\times 10^{-6}$	-.00225	-.00229 $\times 10^{-3}$	-.00212	.07029 $\times 10^{-3}$	-.0052 $\times 10^{-3}$	-.0012
1/4 Chord Strip I	.00165	.128 $\times 10^{-6}$	.00115 $\times 10^{-3}$	-.00130	.06 $\times 10^{-6}$	-.00148	-.113 $\times 10^{-6}$	.08 $\times 10^{-6}$	.0181
1/4 Chord Strip II	-.00261 $\times 10^{-3}$	-.00225 $\times 10^{-3}$	-.00180 $\times 10^{-3}$	.1021 $\times 10^{-3}$	-.18 $\times 10^{-6}$	-.00209 $\times 10^{-3}$	.004514 $\times 10^{-3}$	.03	-.0182 $\times 10^{-6}$
1/4 Chord Strip III	.090 $\times 10^{-6}$	-.00229	.06 $\times 10^{-6}$	-.113 $\times 10^{-6}$	0	0	-.00188	.01	.70 $\times 10^{-9}$
3/4 Chord Strip I	-.00212 $\times 10^{-3}$	-.00212	-.00119 $\times 10^{-3}$	-.00209 $\times 10^{-3}$	0	.00211 $\times 10^{-3}$	-.00150	.10 $\times 10^{-6}$	-.0233 $\times 10^{-6}$
3/4 Chord Strip II	.221 $\times 10^{-6}$	.07029	-.113 $\times 10^{-6}$	.001514 $\times 10^{-3}$	-.00198 $\times 10^{-3}$	-.00160 $\times 10^{-3}$	.07298	-.004 $\times 10^{-3}$	-.00709 $\times 10^{-6}$
3/4 Chord Strip III	.05 $\times 10^{-6}$	-.0052 $\times 10^{-3}$	.08 $\times 10^{-6}$	.03 $\times 10^{-6}$	.01 $\times 10^{-6}$	.10 $\times 10^{-6}$	-.004 $\times 10^{-3}$	-.10 $\times 10^{-6}$	.90 $\times 10^{-9}$
Fus. Slope 325	.0247 $\times 10^{-6}$	-.0012 $\times 10^{-6}$	.0181 $\times 10^{-6}$	-.0152 $\times 10^{-6}$	.70 $\times 10^{-9}$	-.0233 $\times 10^{-6}$	-.00709 $\times 10^{-6}$	.90 $\times 10^{-9}$	.231 $\times 10^{-9}$

Steady State Aerodynamic Influence Coefficient Matrix -

$[Q_{ii}]$ - Configuration 2

	Pitching Slope of Forward Surface	Pitching Slope of Aft Surface
Force at A. C. of Forward Surface	$- .945 \times 10^6$	0
Force at A. C. of Aft Surface	0	-8.424×10^6

Modified Steady State Aerodynamic Influence Coefficient Matrix - $[\bar{Q}_{ii}]$ -
Configuration 2

	Pitching Slope of Forward Surface	Pitching Slope of Aft Surface
Force at A. C. of Forward Surface	-1.069×10^6	$.04481 \times 10^6$
Force at A. C. of Aft Surface	$.7684 \times 10^6$	-10.710×10^6

Note: Dimensions of matrix coefficients are lb/rad.

Figure 6-14

Steady State Aerodynamic Influence Coefficient Matrix -

$[Q_{ii}]$ - Configuration 3

	Pitching Slope of Tail	Pitching Slope of Strip I	Pitching Slope of Strip II	Pitching Slope of Strip III
Force at A.C. of Tail	$-.997 \times 10^6$	$.598 \times 10^6^*$	0	0
Force at A.C. of Strip I	0	-2.064×10^6	0	0
Force at A.C. of Strip II	0	0	-1.605×10^6	0
Force at A.C. of Strip III	0	0	0	-1.147×10^6

* Term due to wing downwash on tail

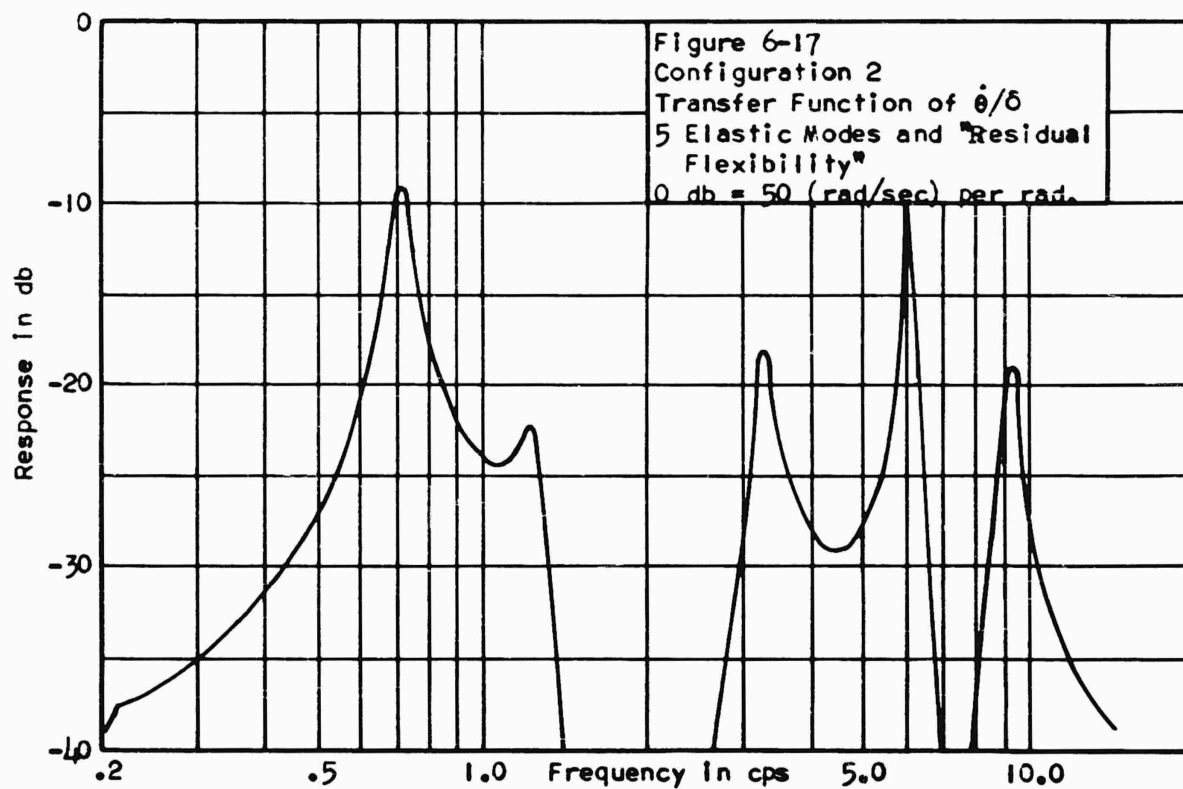
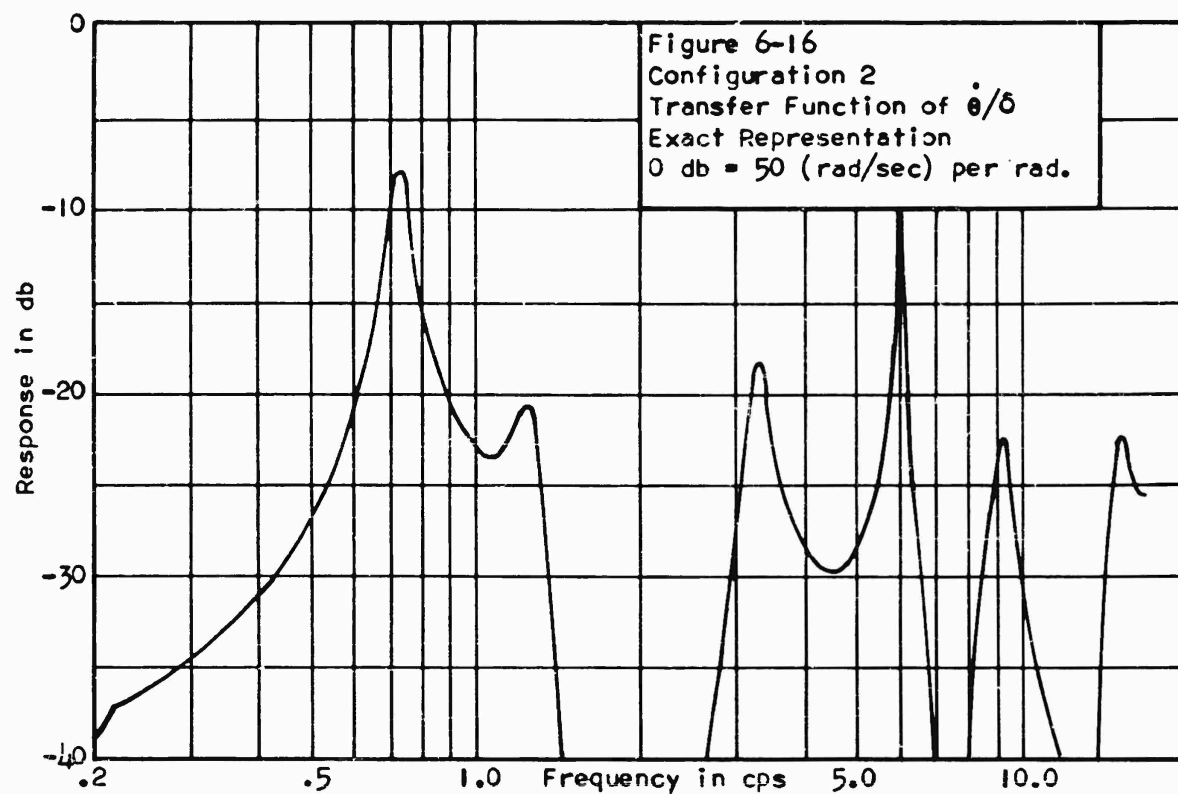
Modified Steady State Aerodynamic Influence Coefficient Matrix -

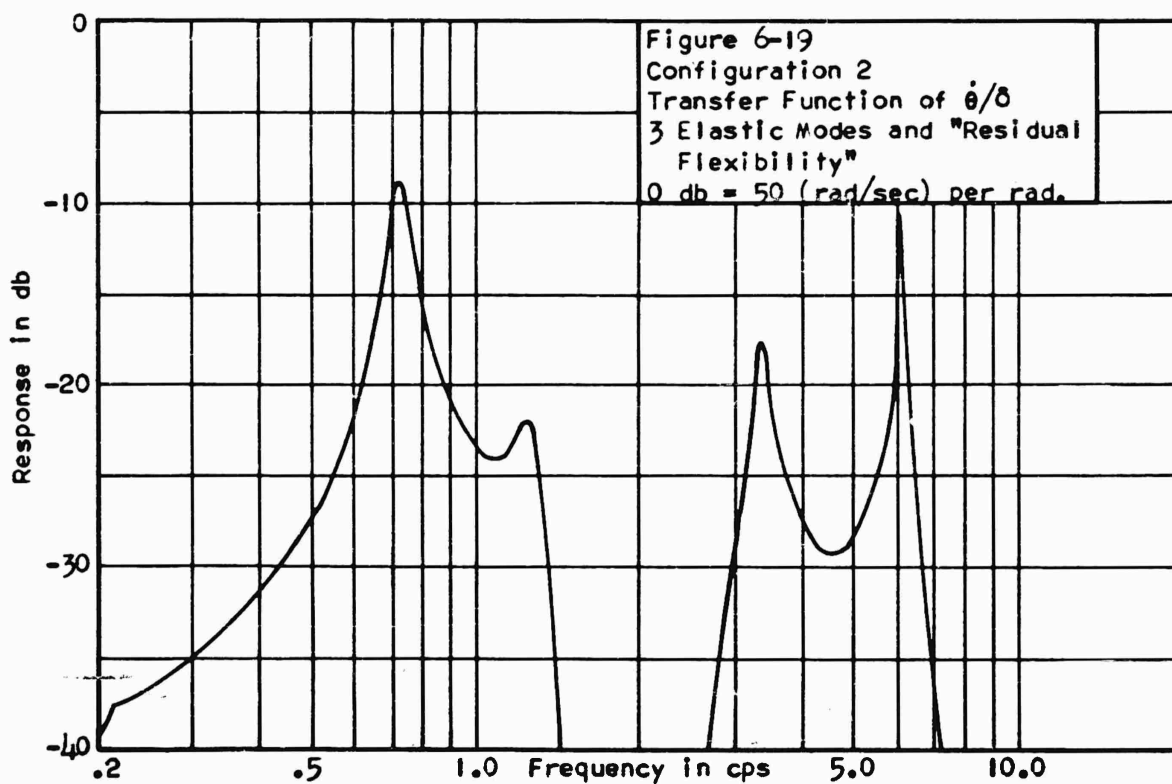
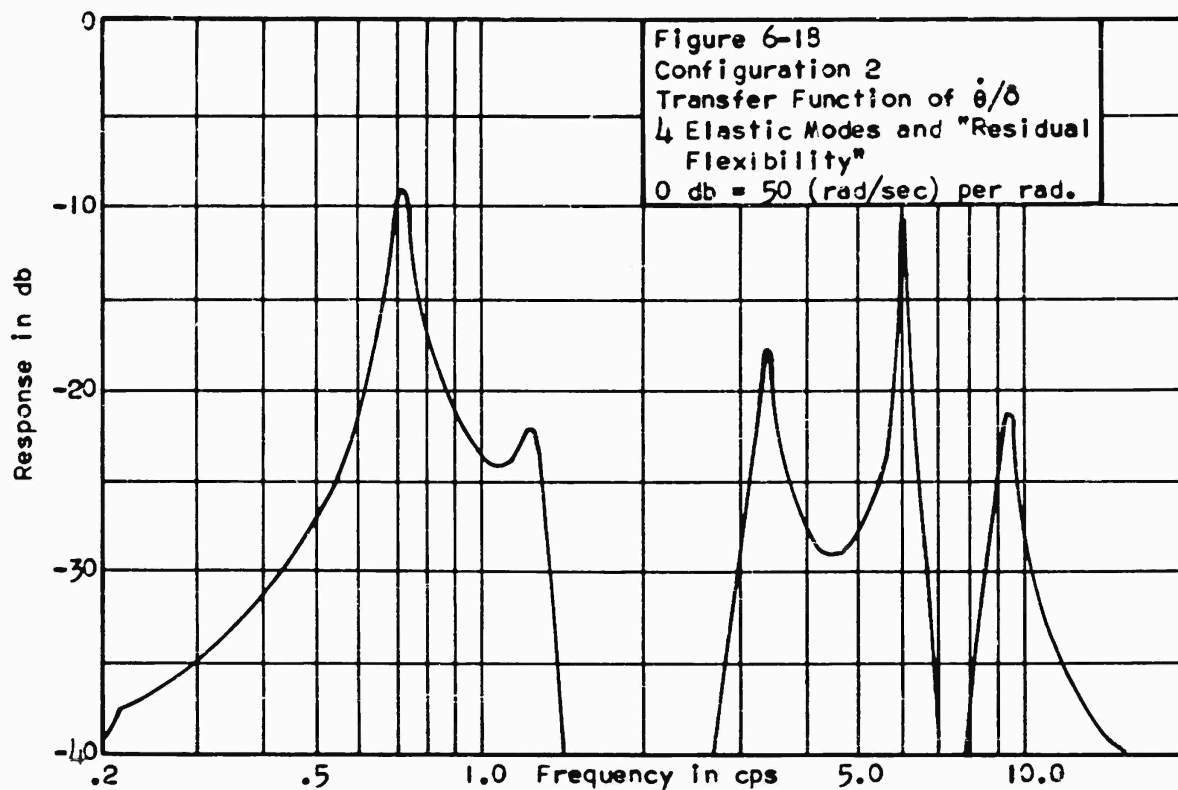
$[\bar{Q}_{ii}]$ - Configuration 3

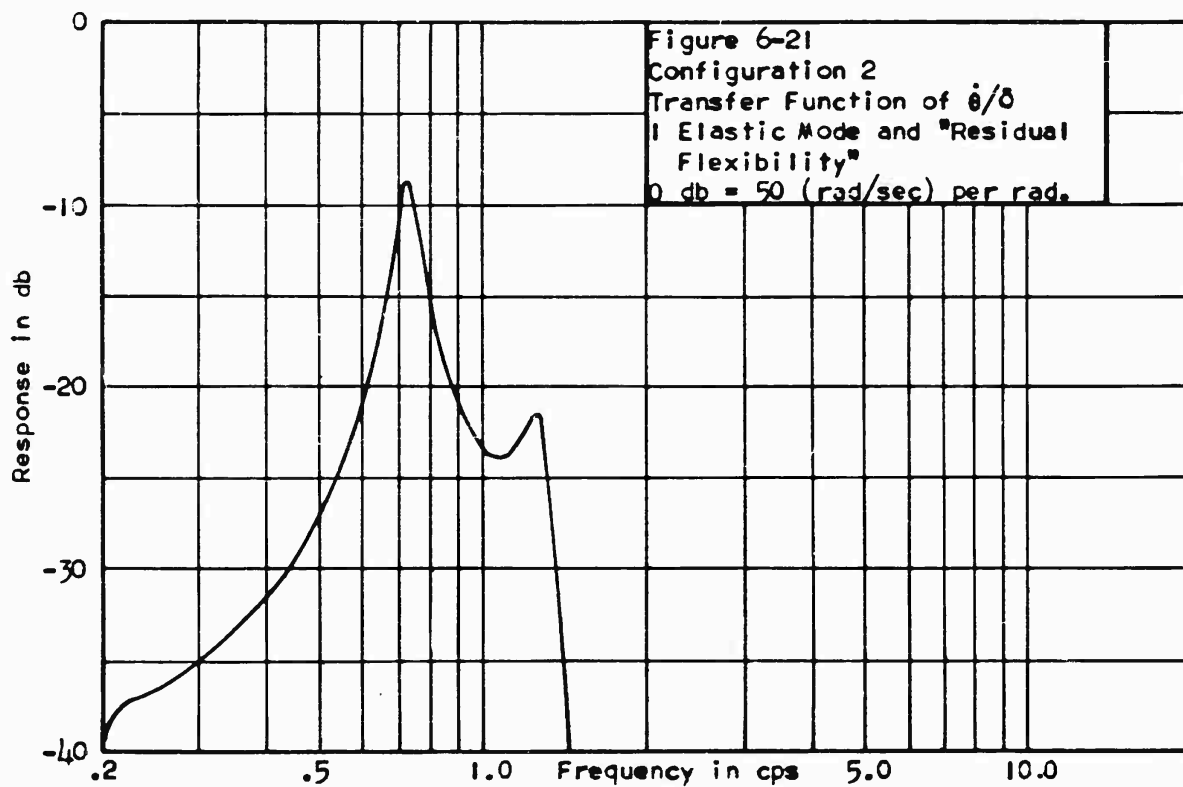
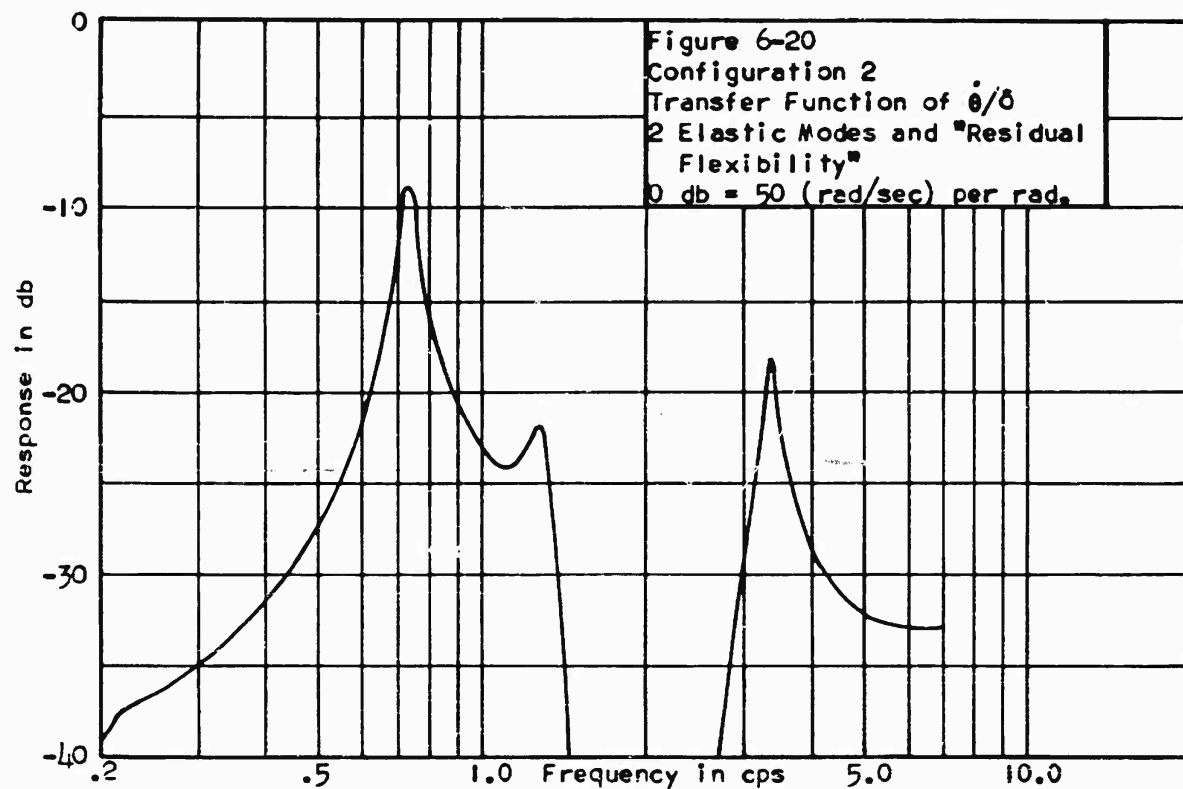
	Pitching Slope of Tail	Pitching Slope of Strip I	Pitching Slope of Strip II	Pitching Slope of Strip III
Force at A.C. of Tail	$-.6056 \times 10^6$	$.3489 \times 10^6$	$-.1589 \times 10^6$	$-.0693 \times 10^6$
Force at A.C. of Strip I	$-.1176 \times 10^6$	-1.929×10^6	$.3206 \times 10^6$	$-.0509 \times 10^6$
Force at A.C. of Strip II	$-.1105 \times 10^6$	$.0691 \times 10^6$	-1.190×10^6	$.4239 \times 10^6$
Force at A.C. of Strip III	$-.1430 \times 10^6$	$.0478 \times 10^6$	$.0766 \times 10^6$	$-.3913 \times 10^6$

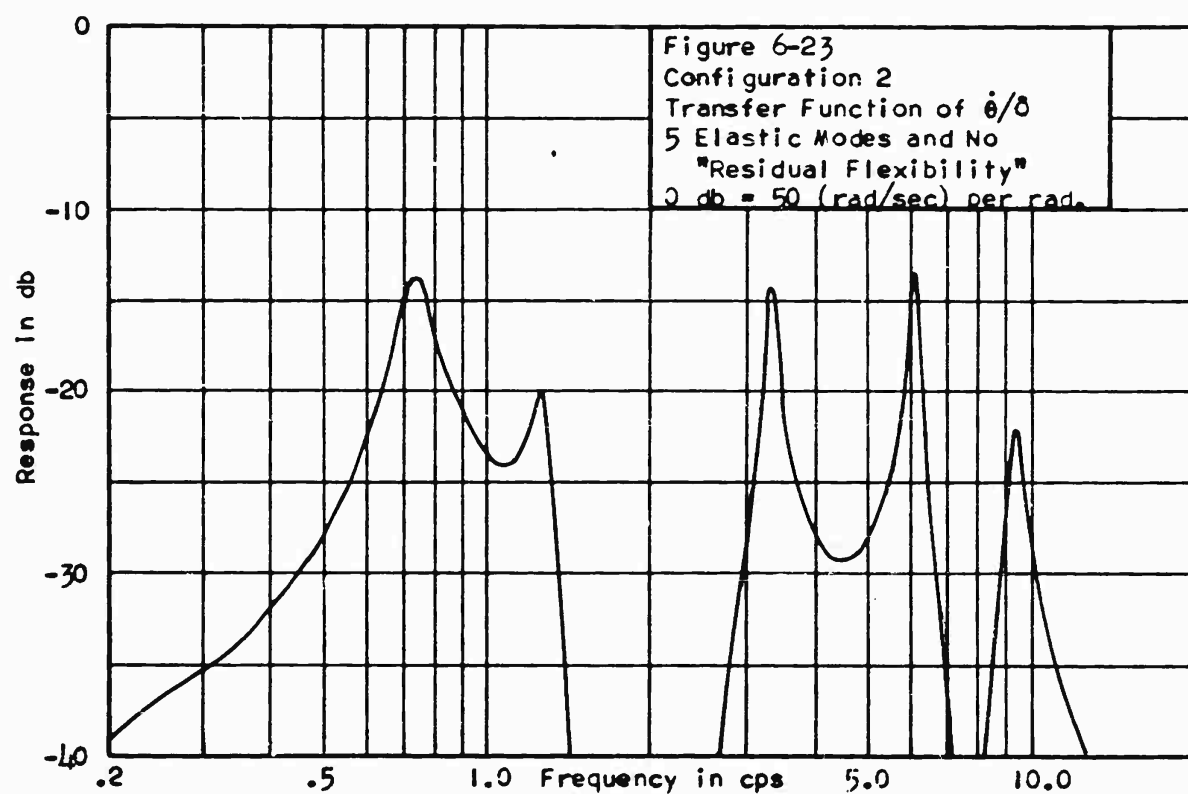
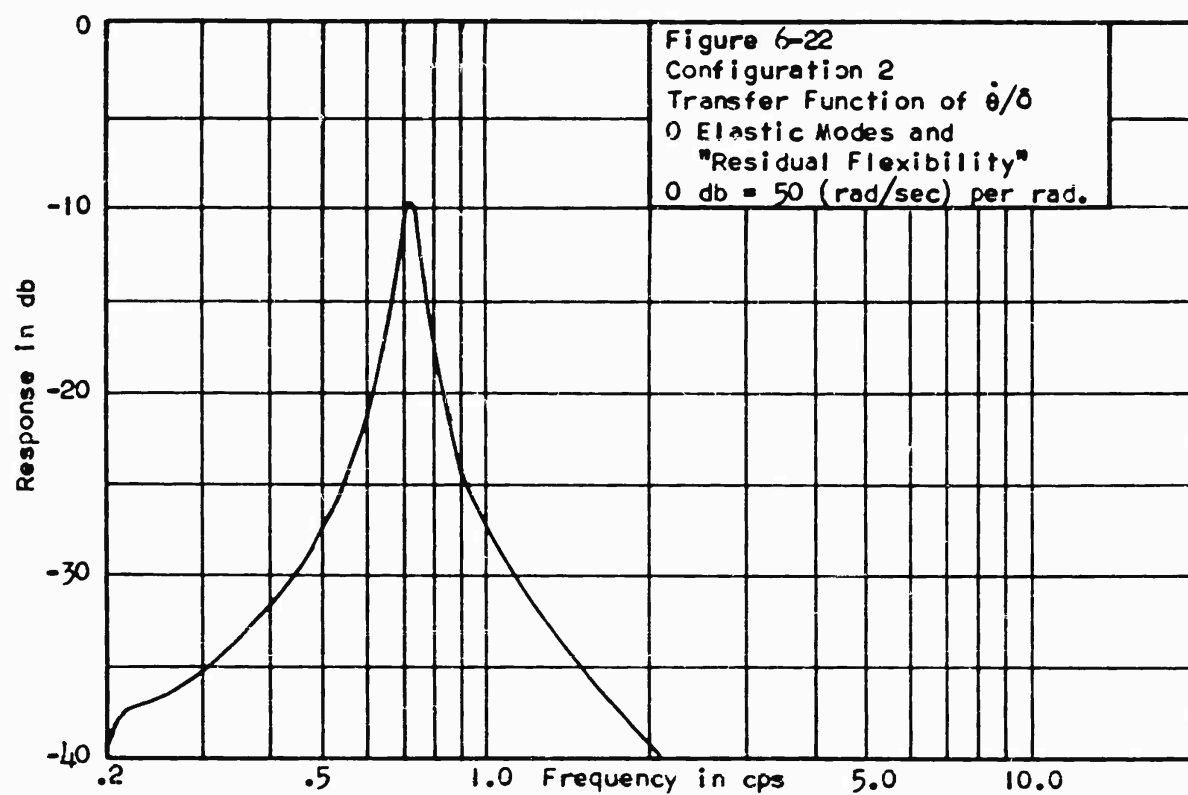
Note: Dimensions of matrix coefficients are lb/rad.

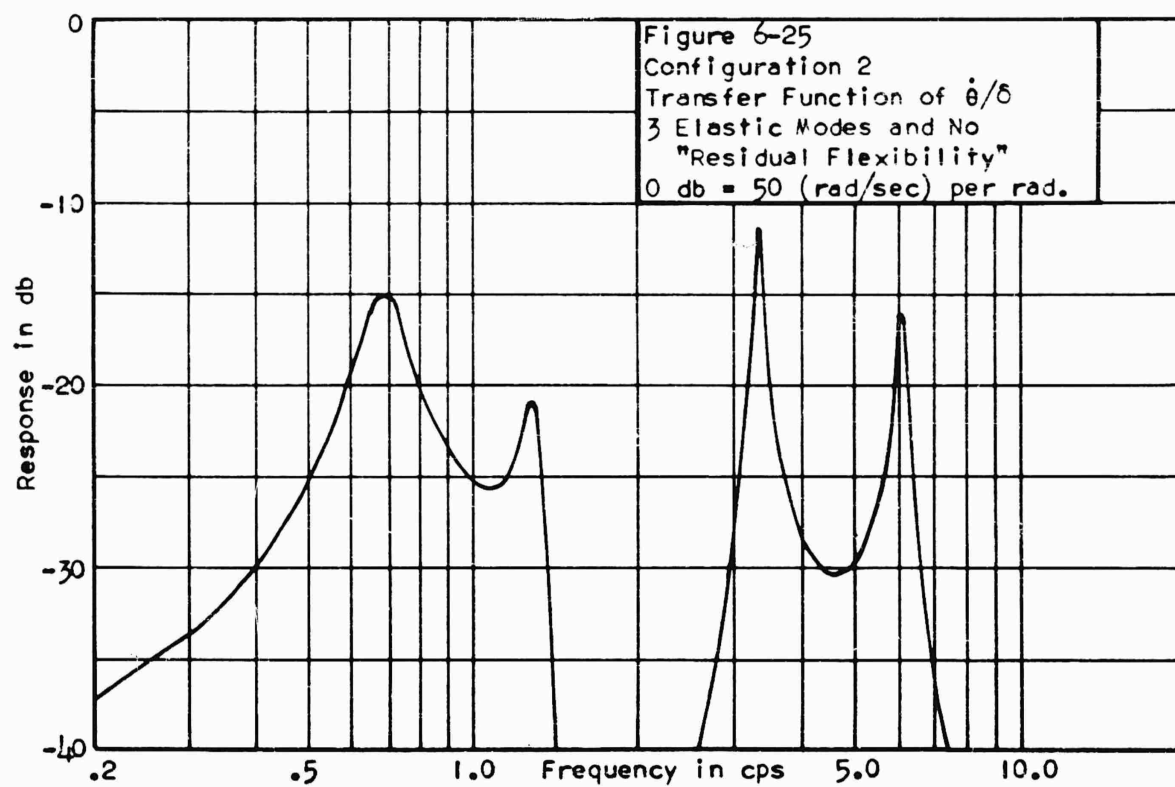
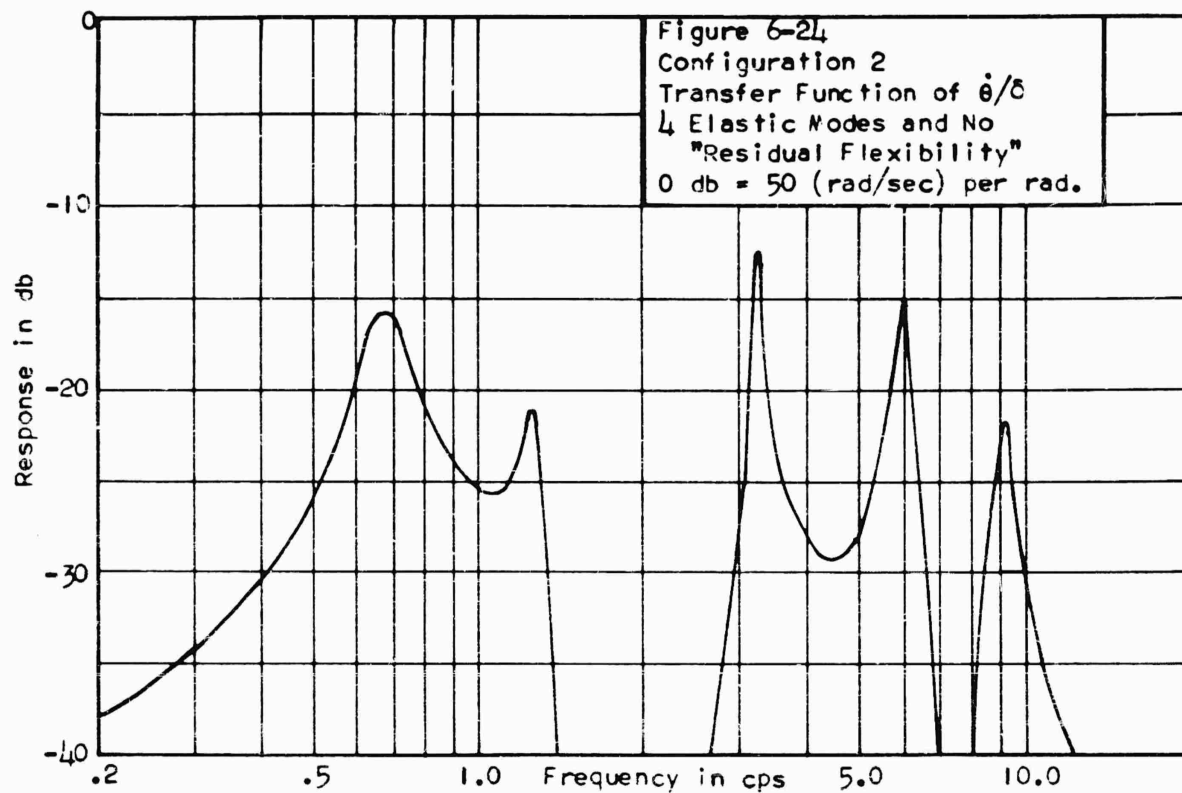
Figure 6-15

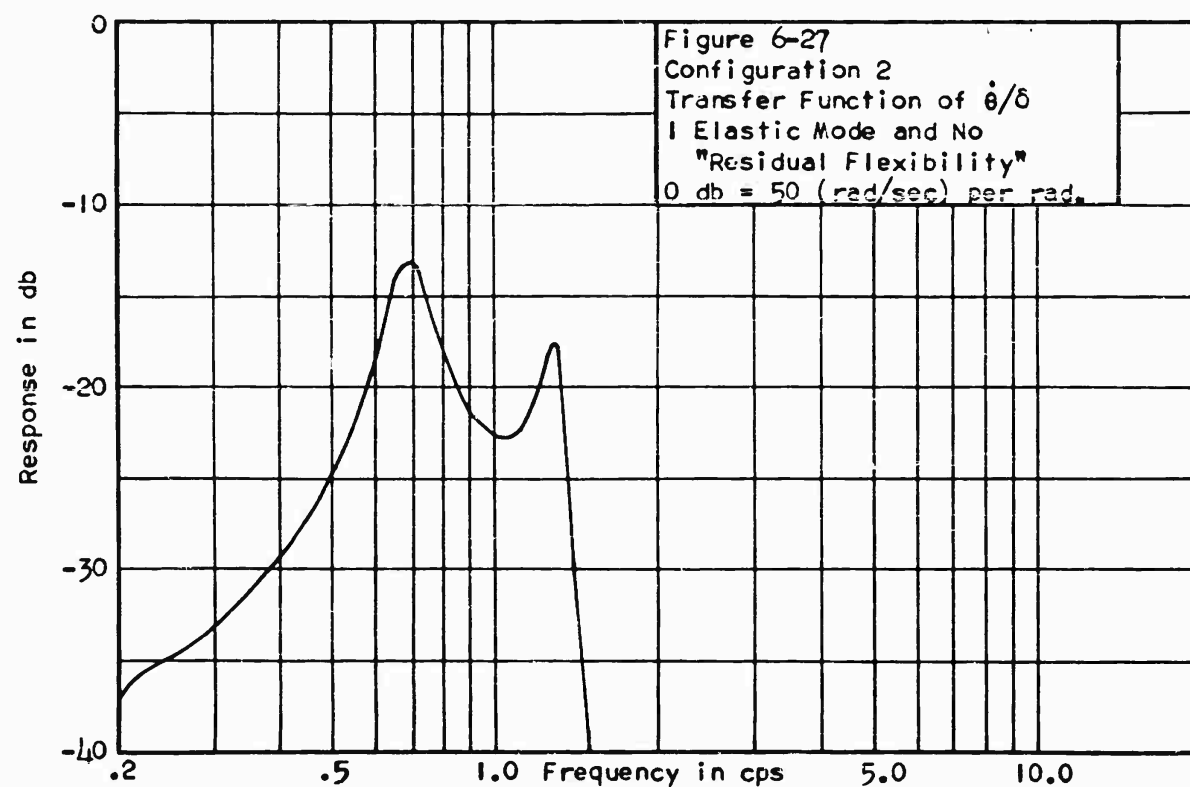
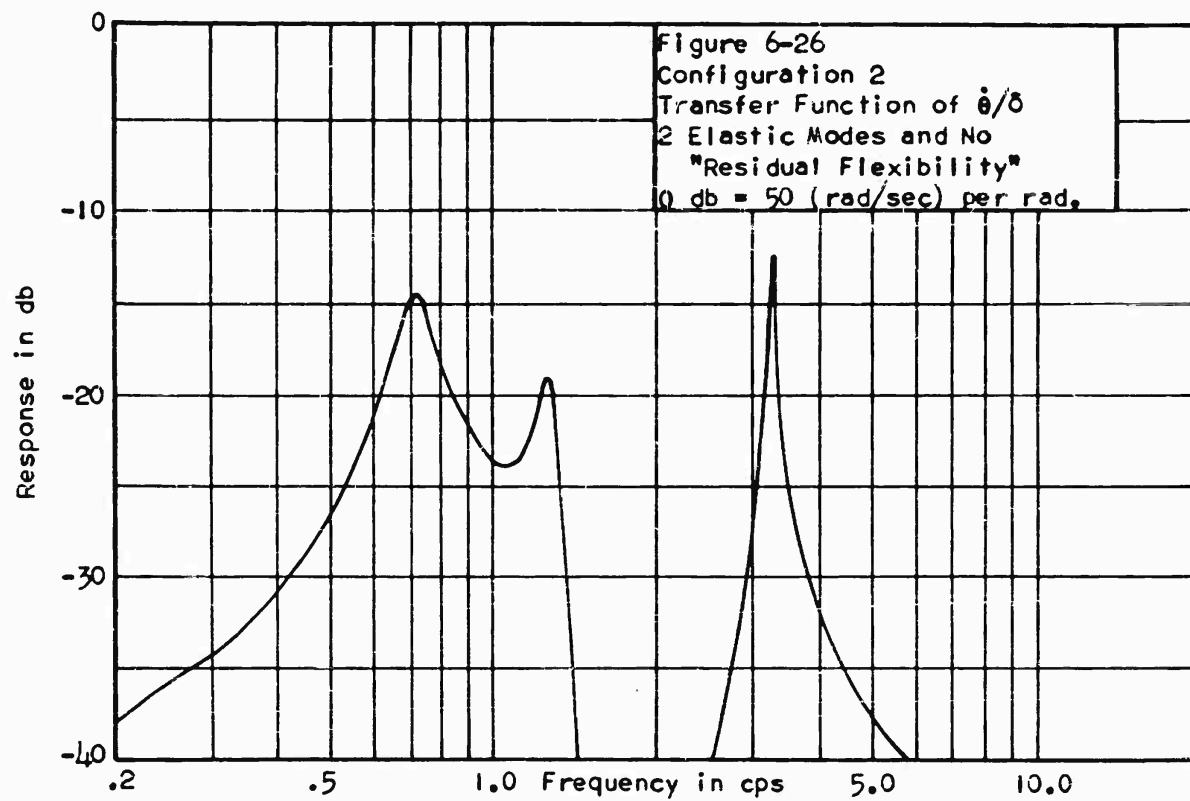


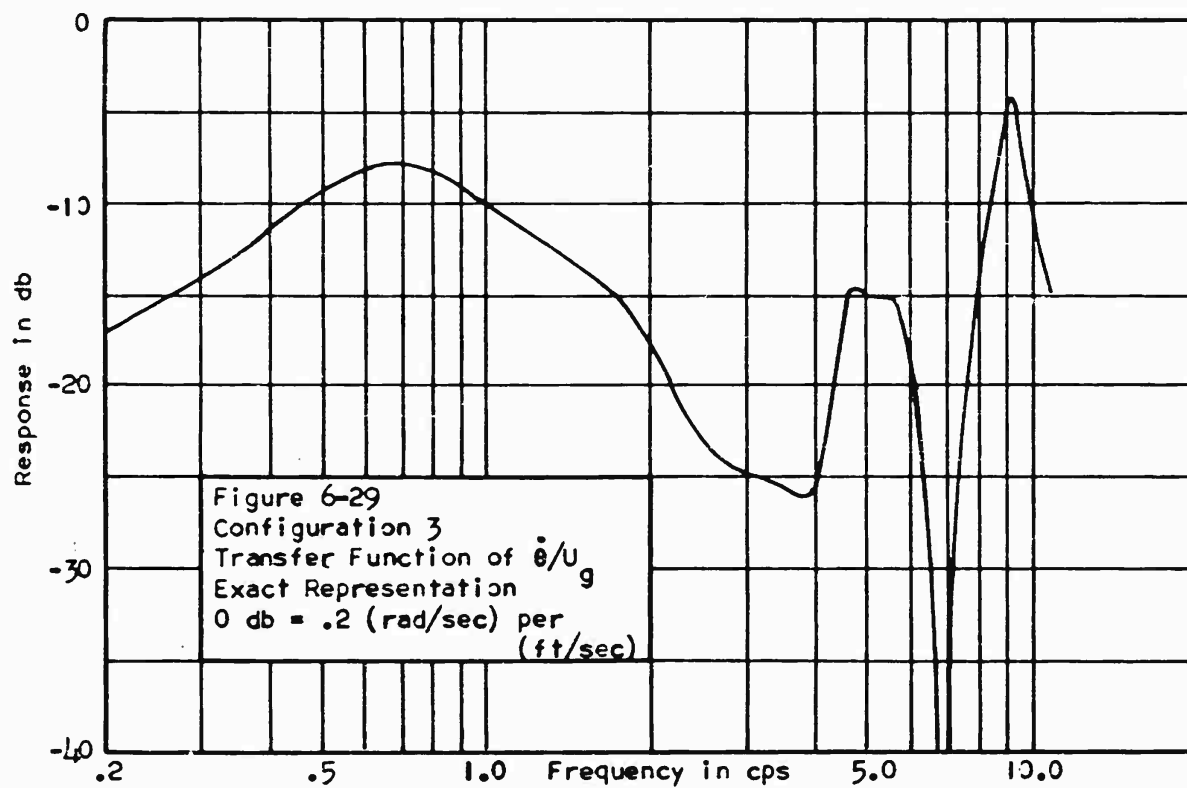
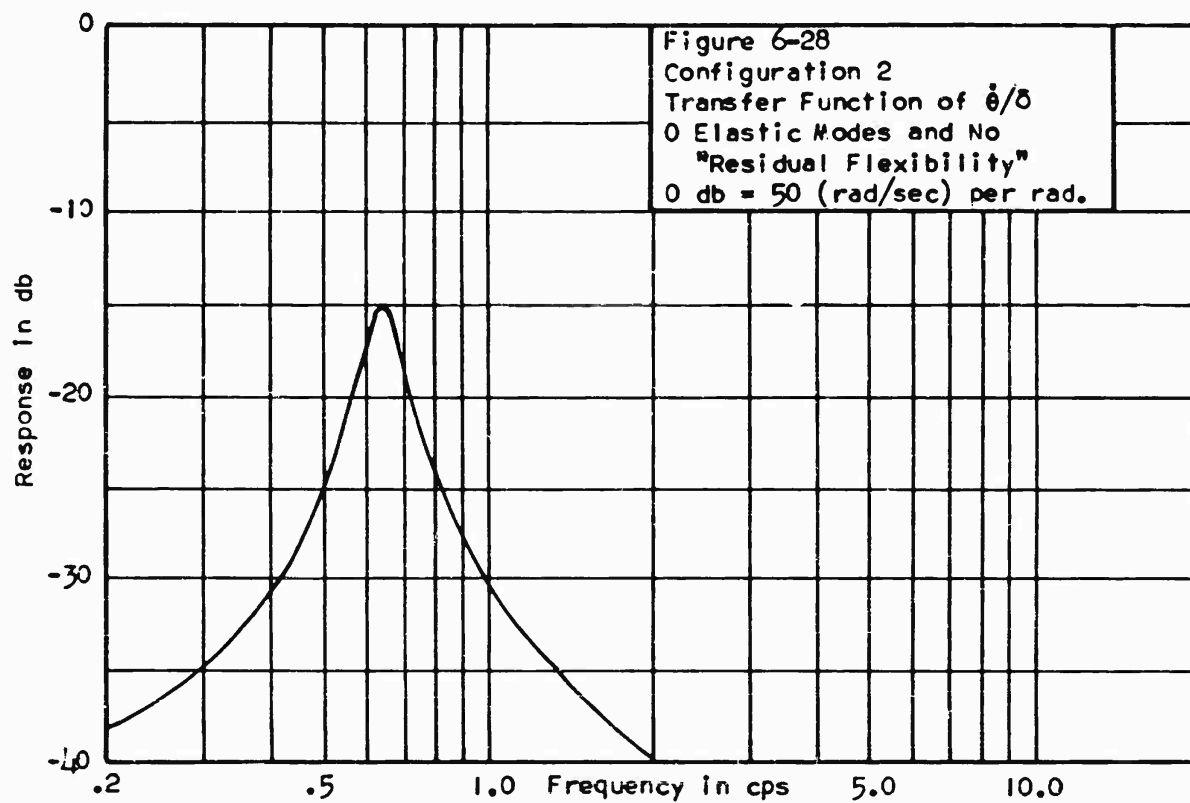


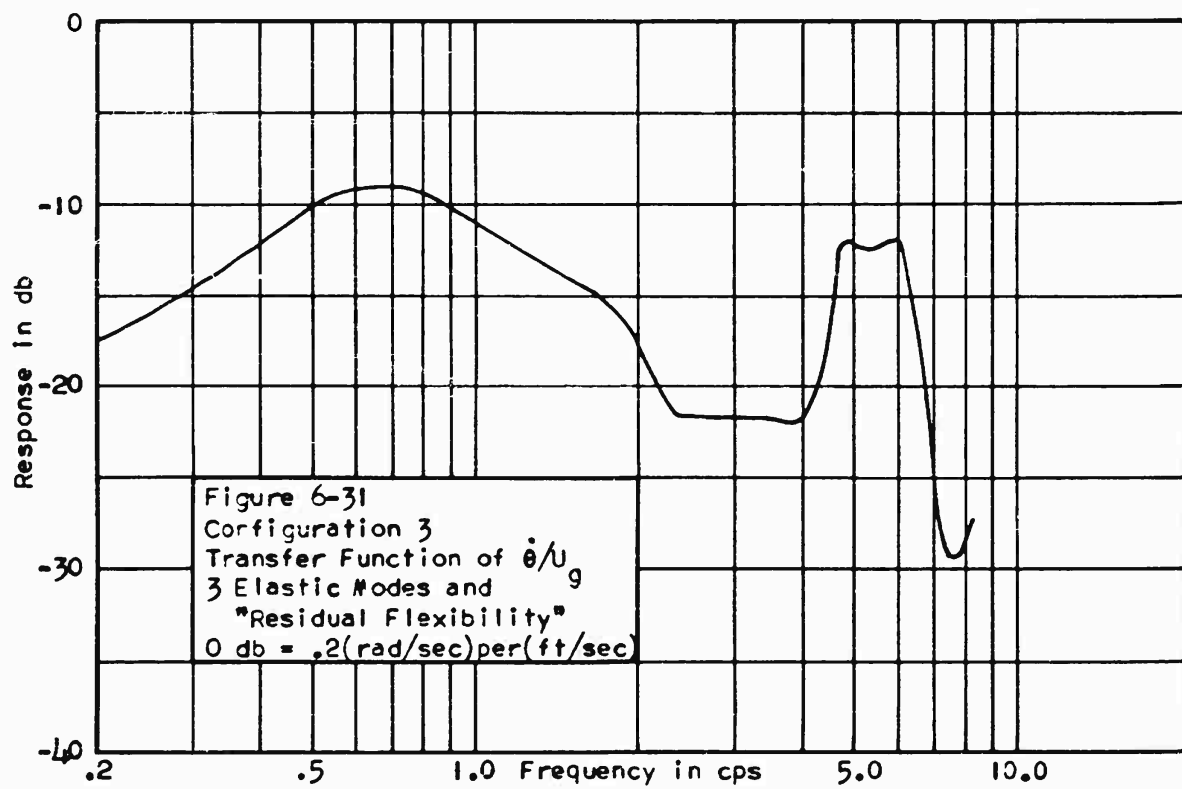
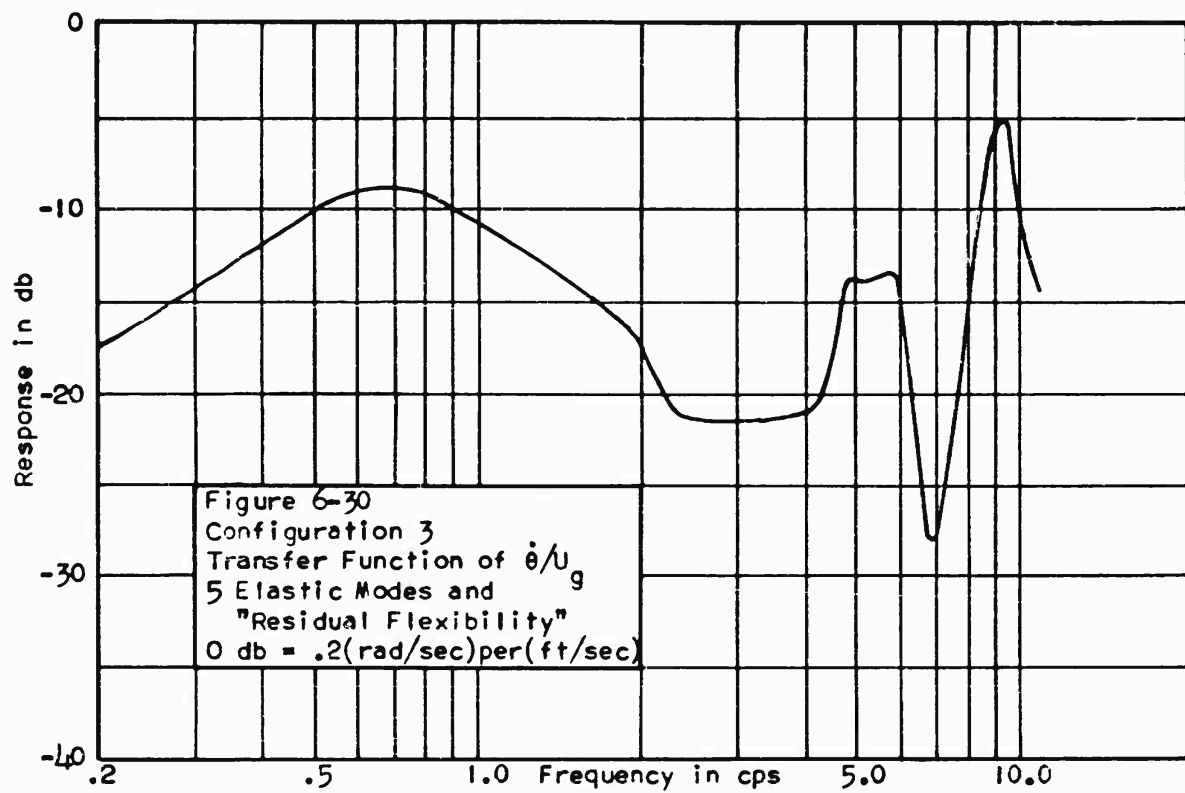


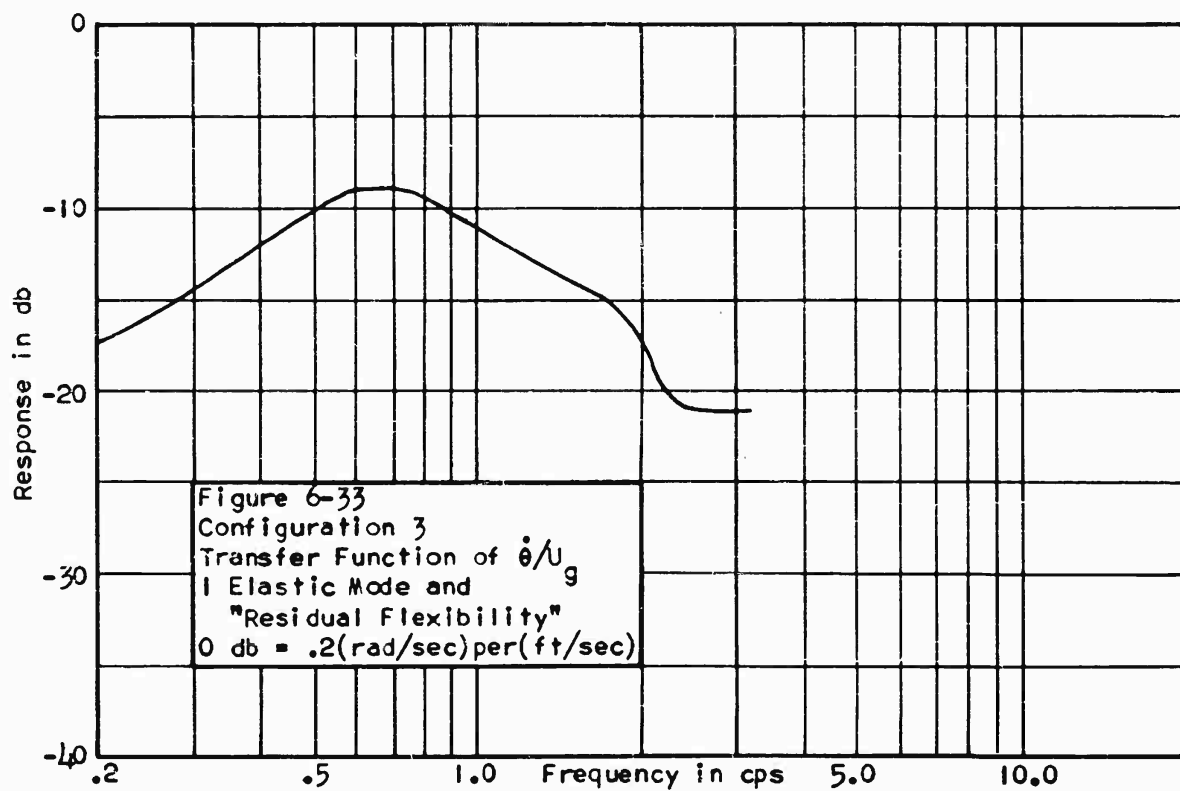
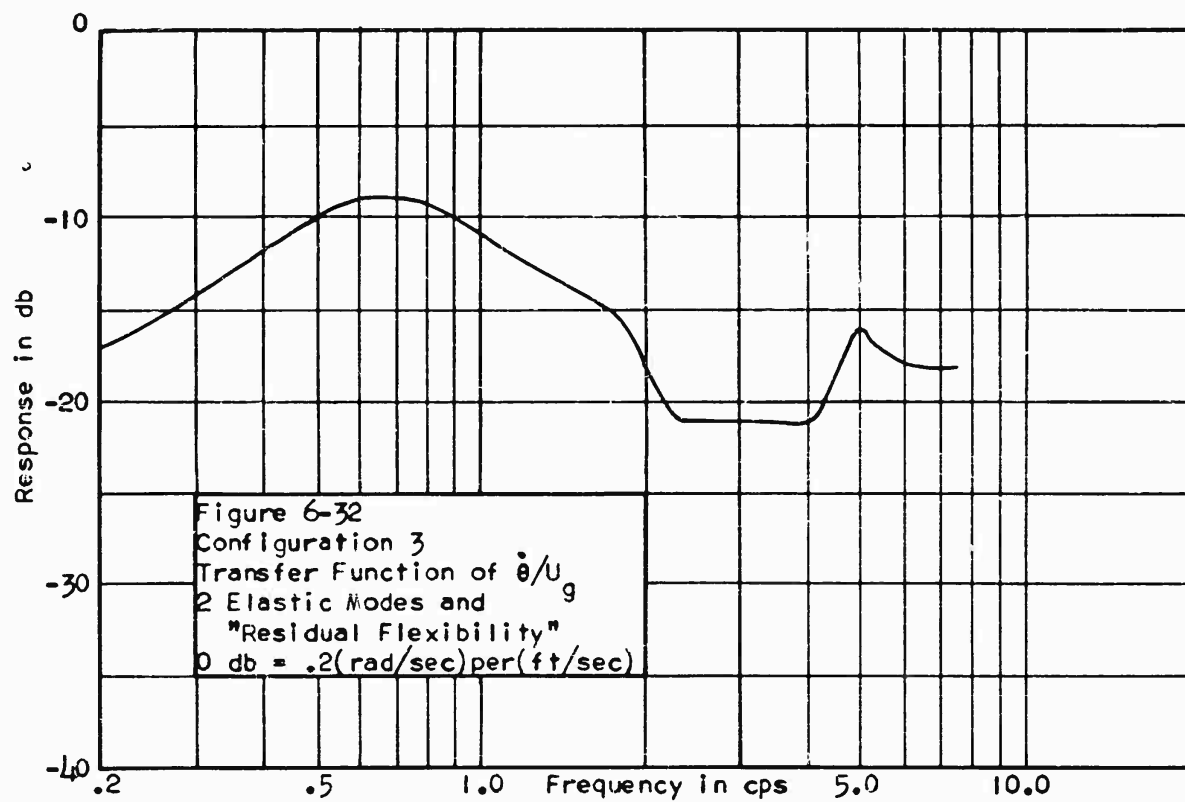


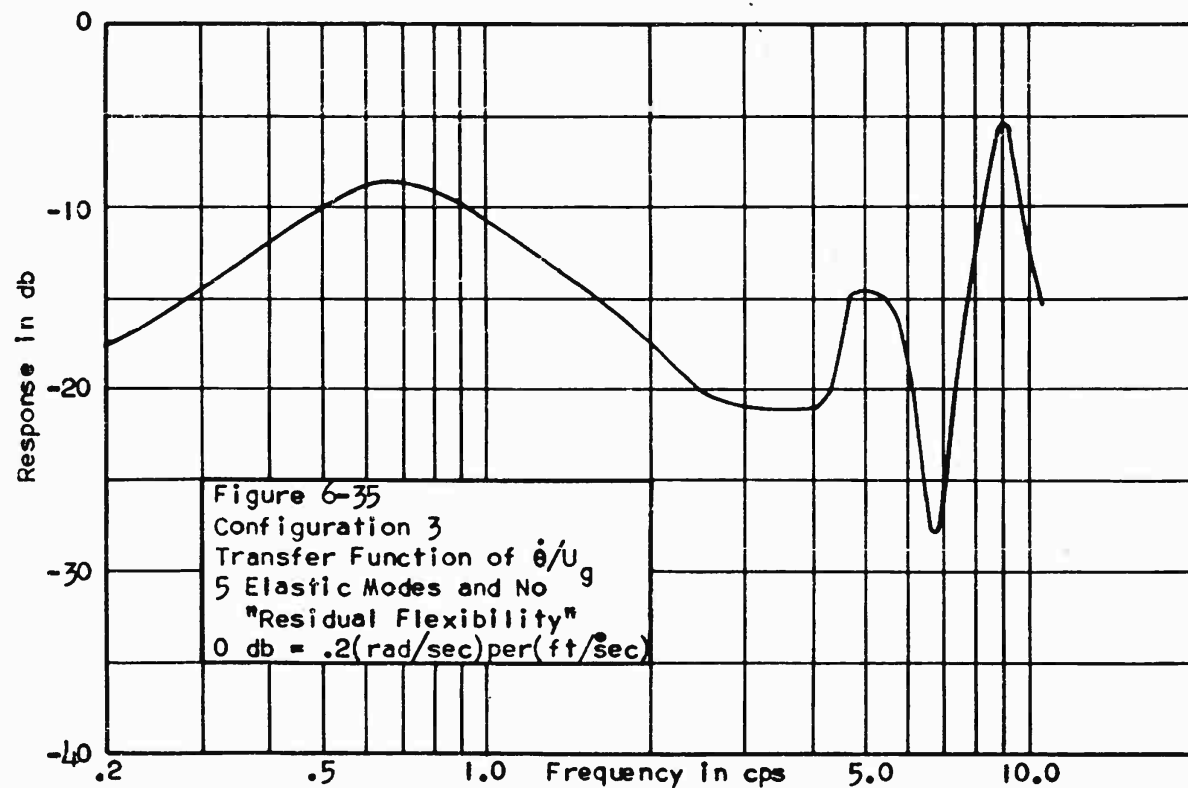
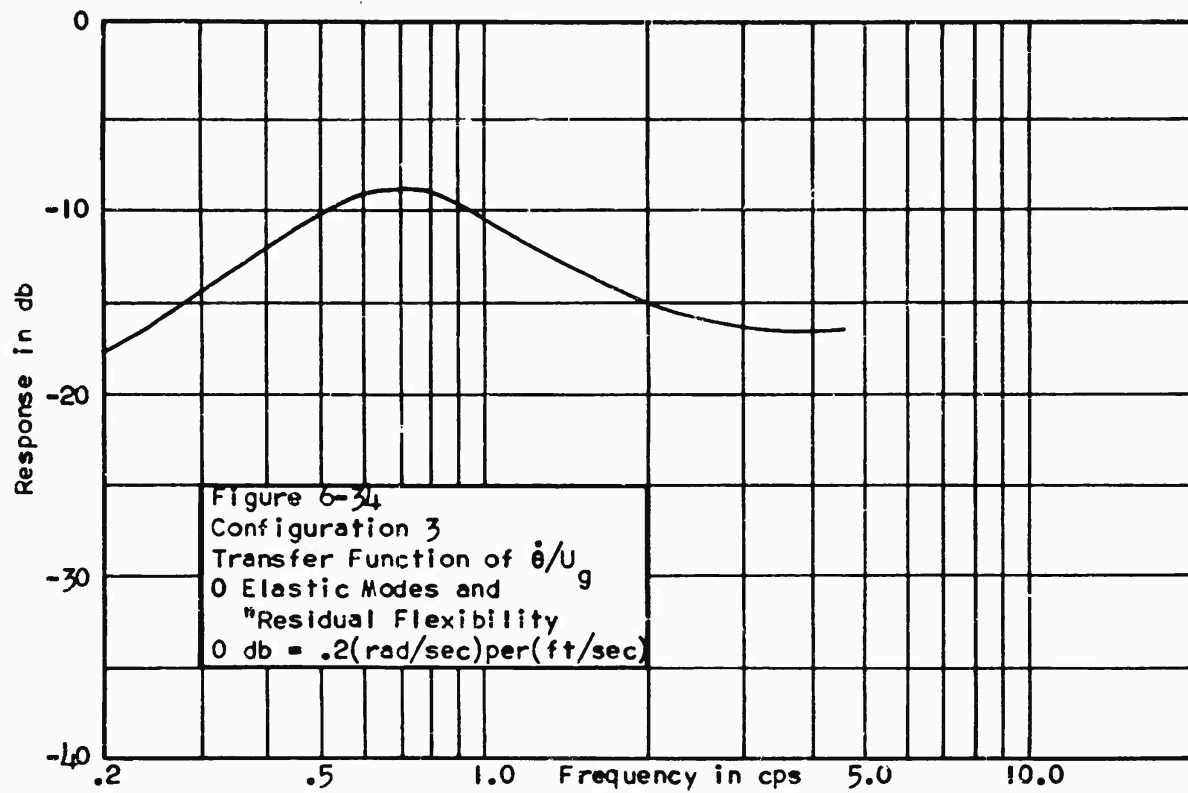


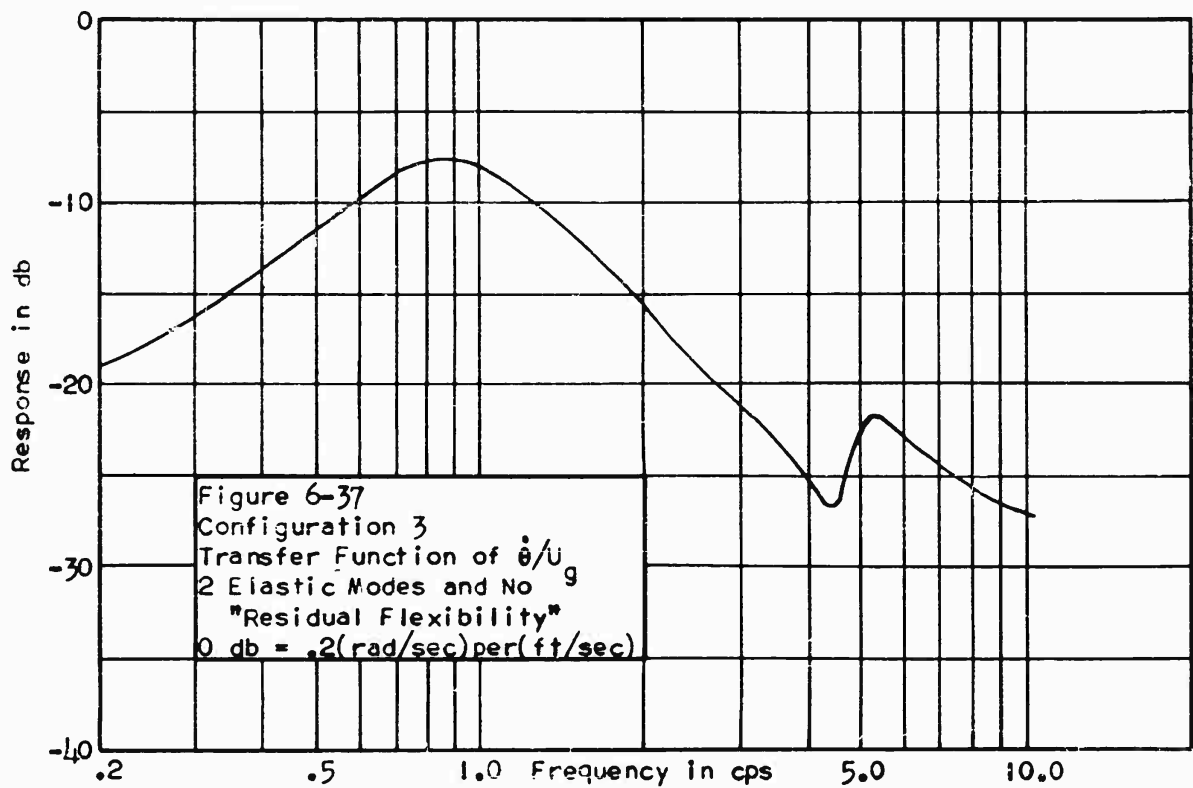
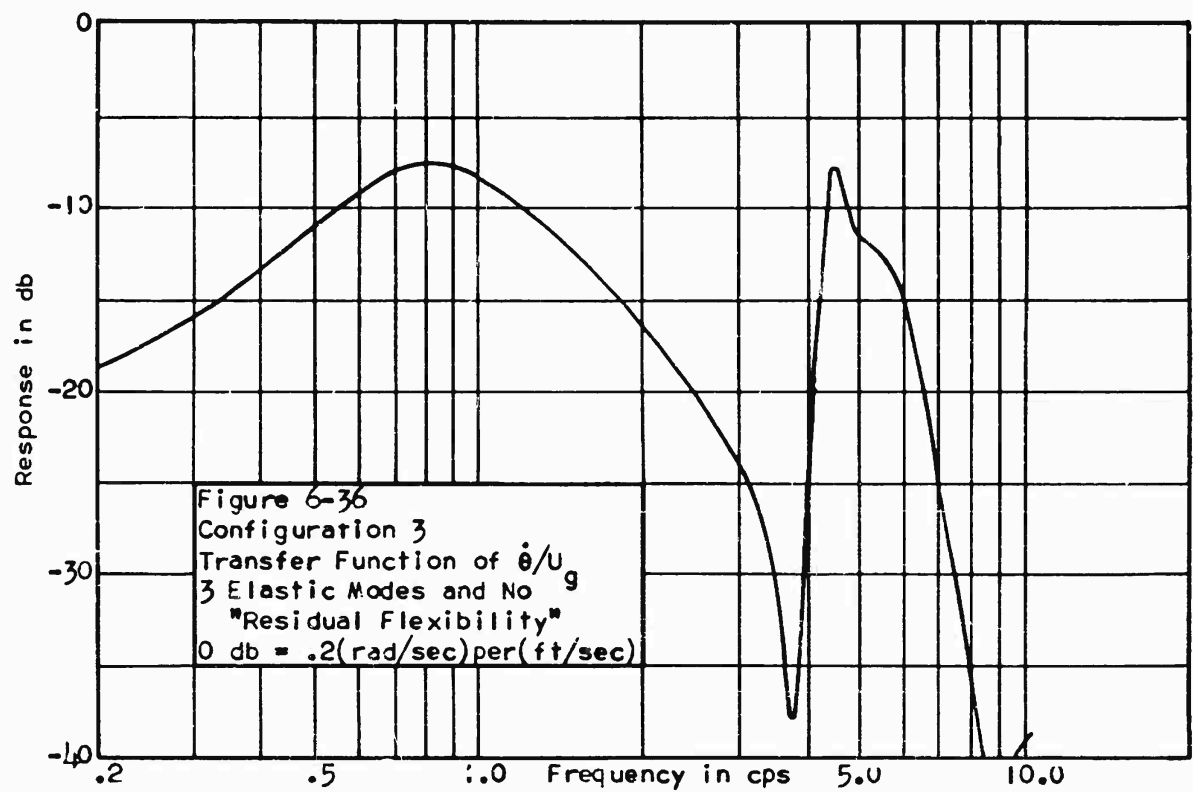


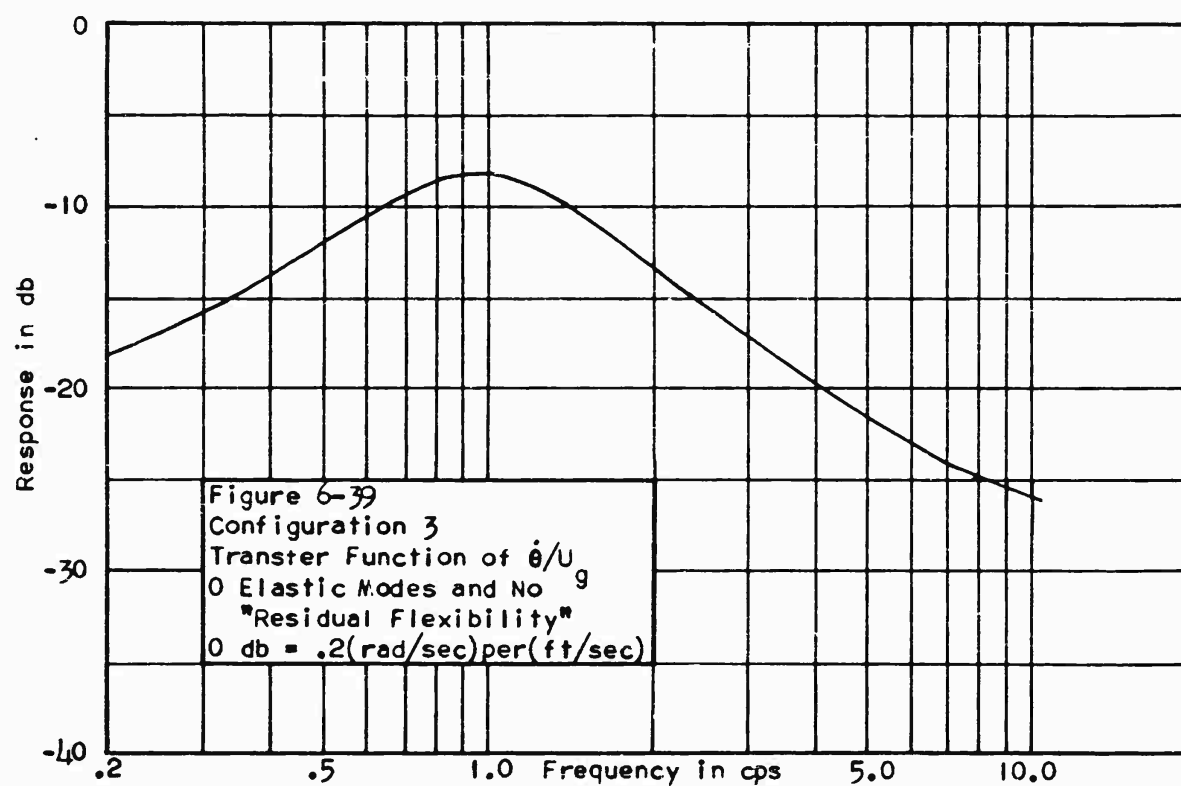
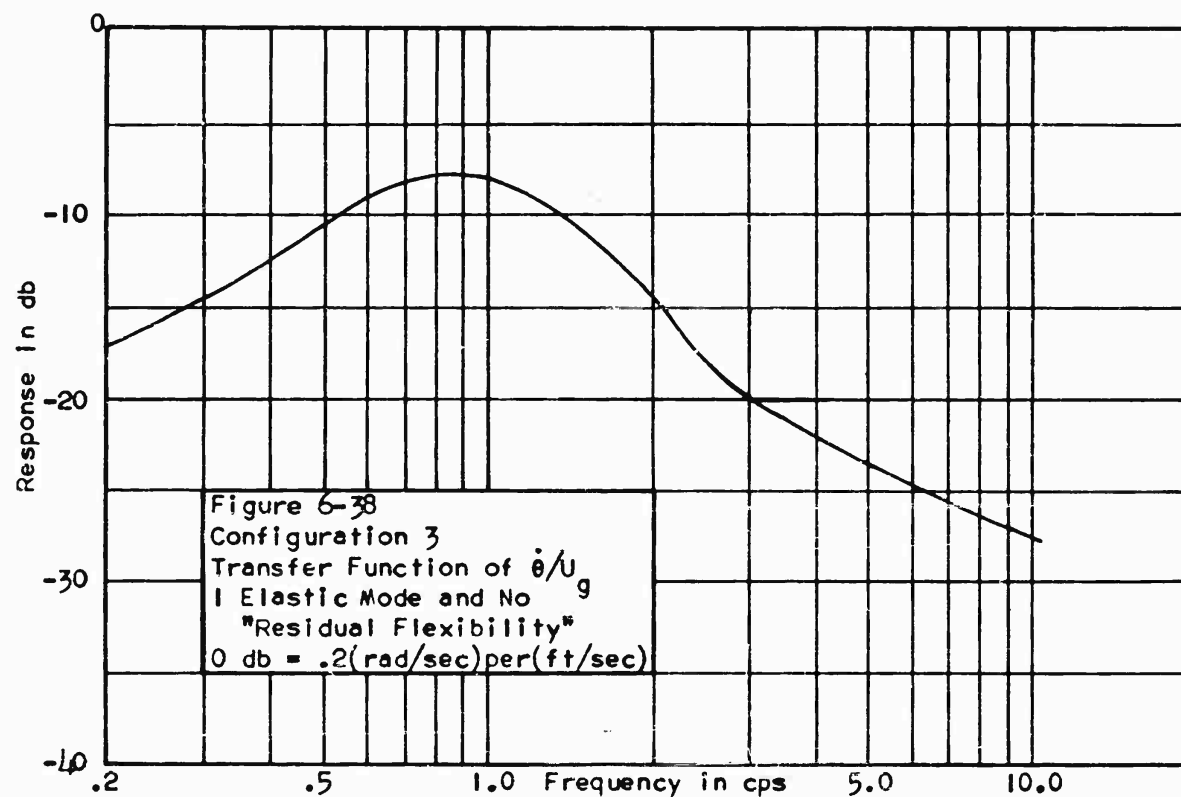












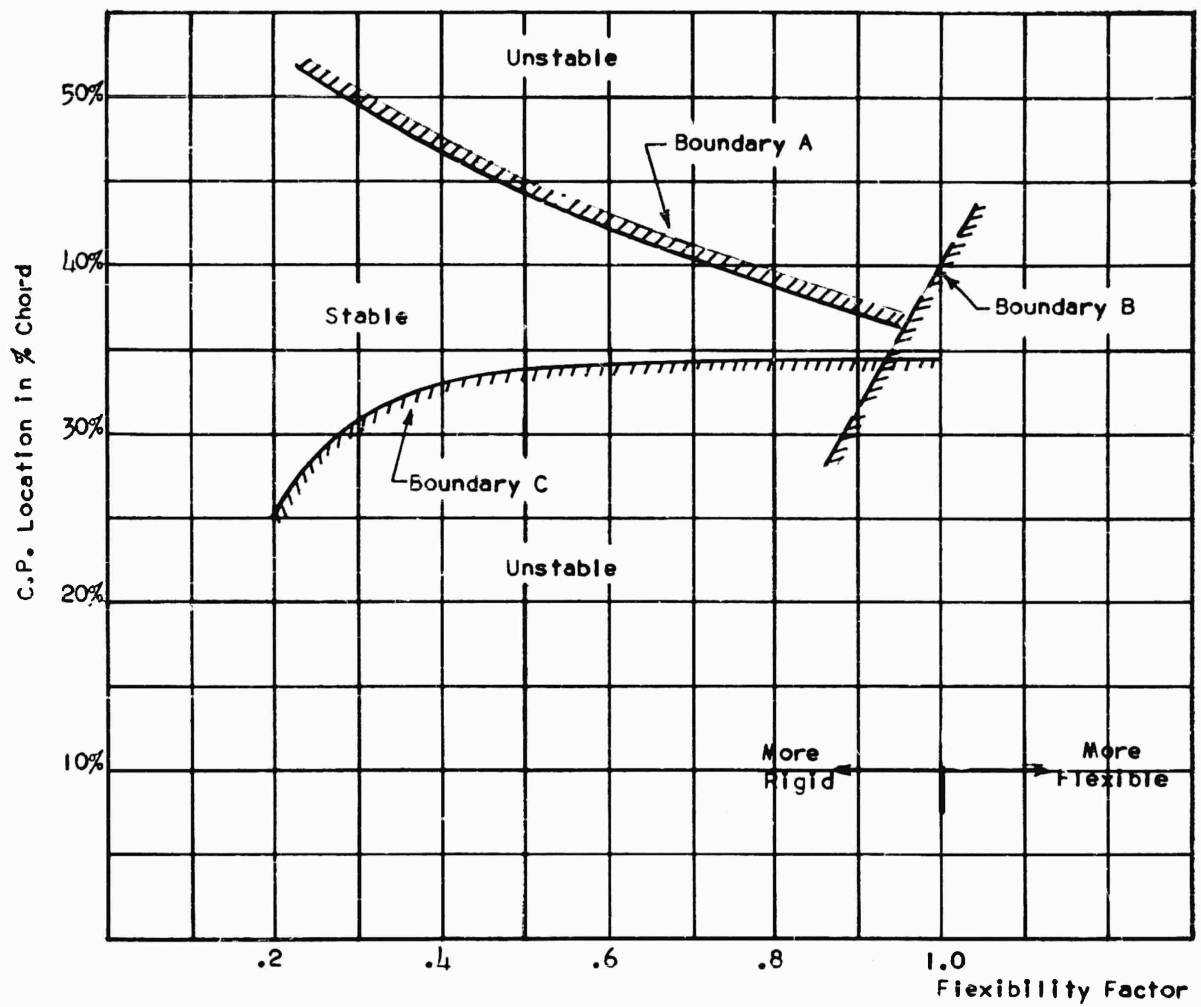
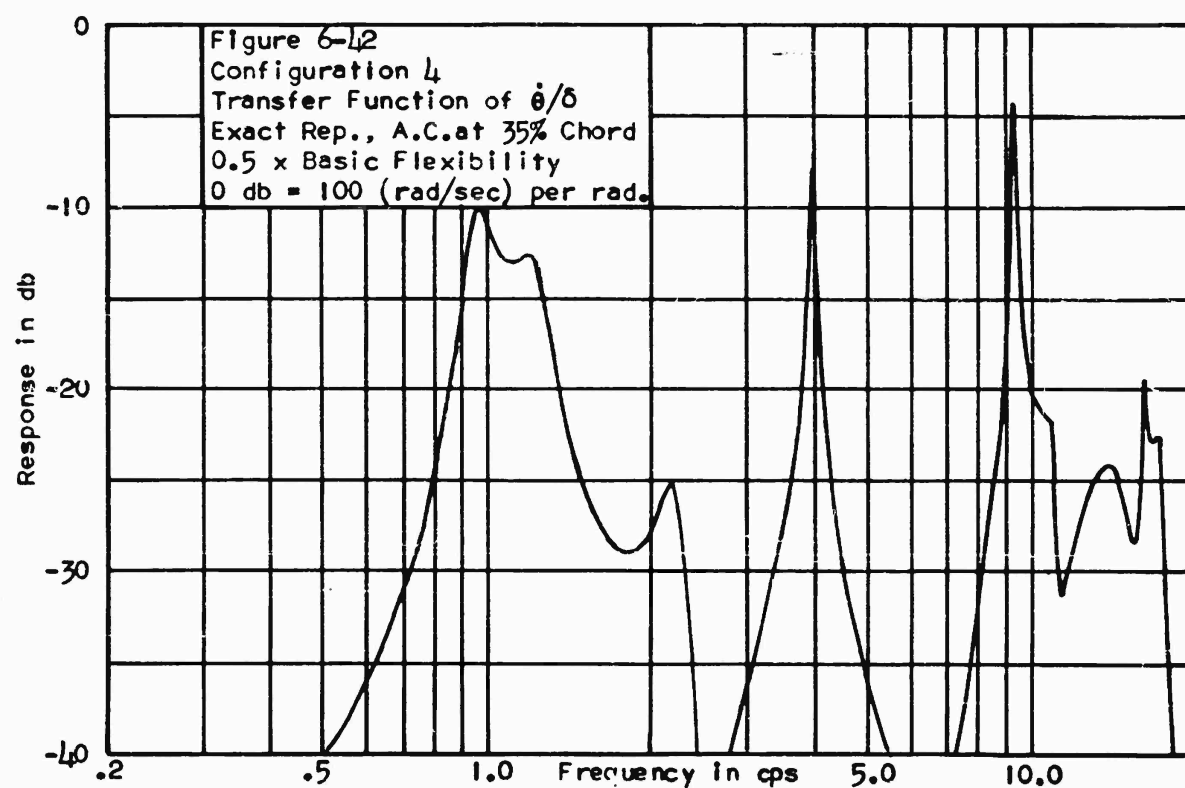
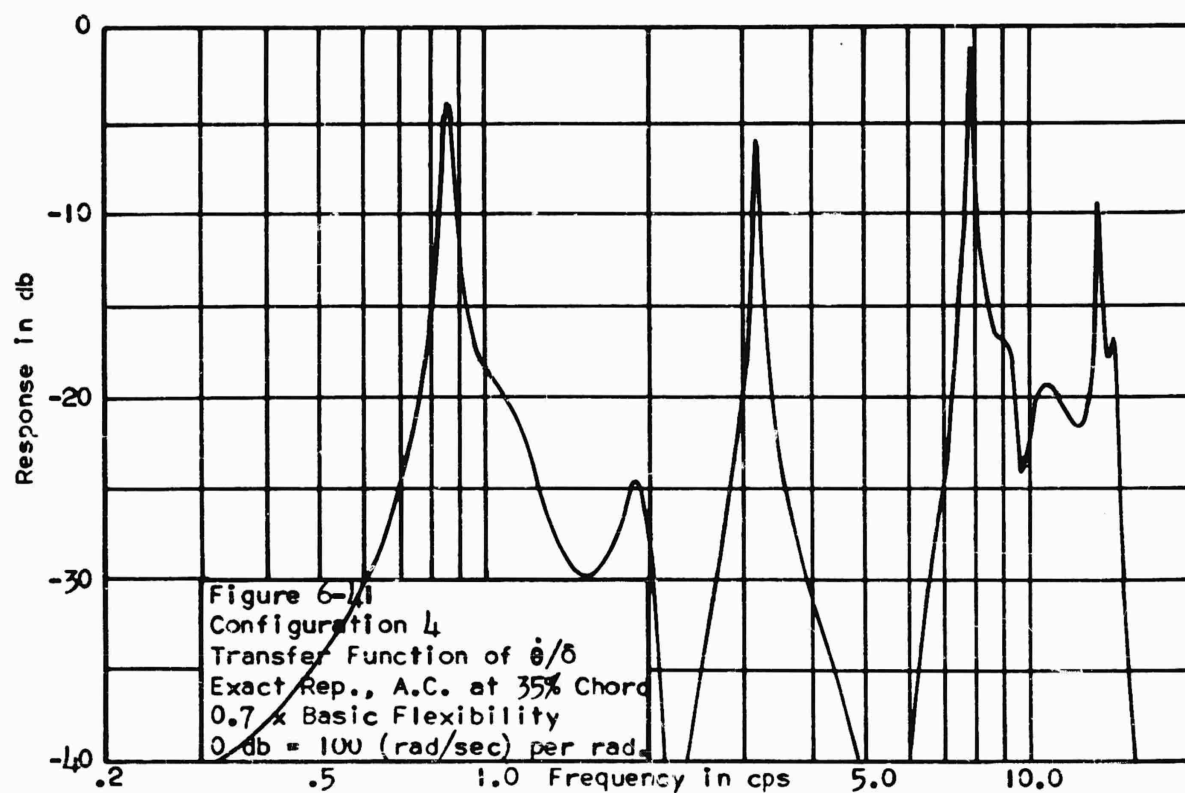
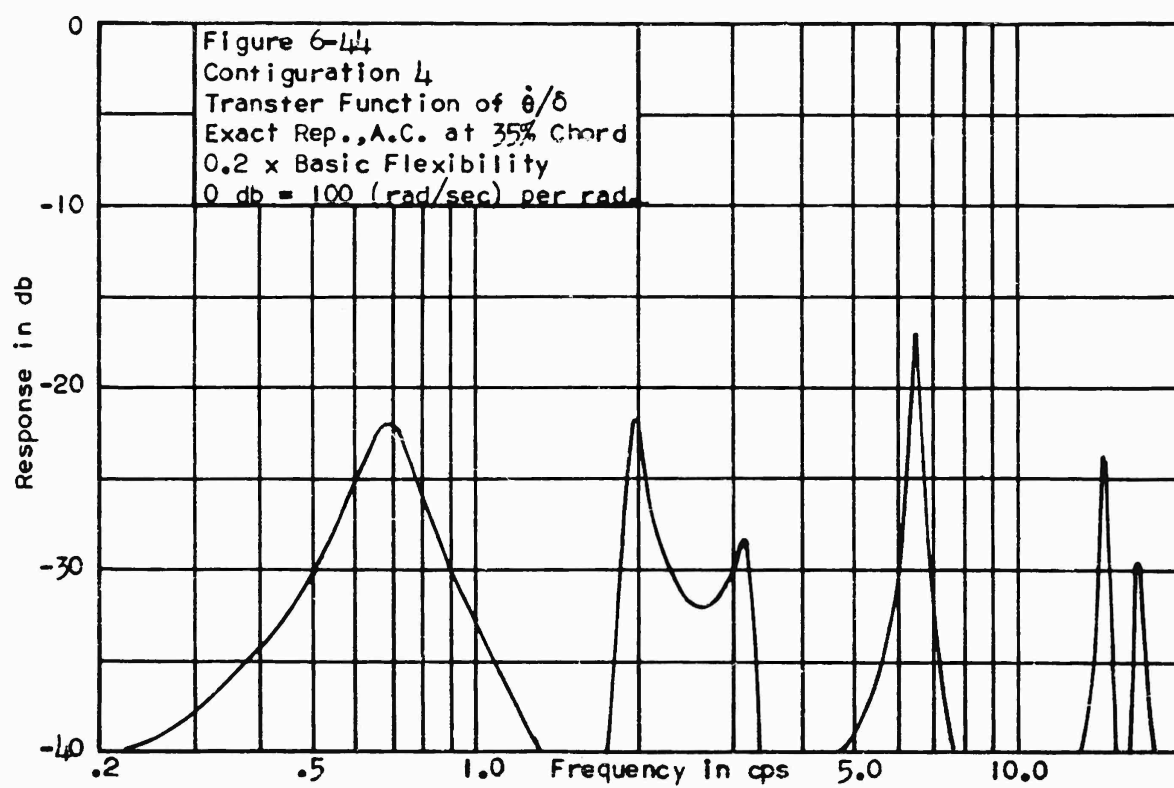
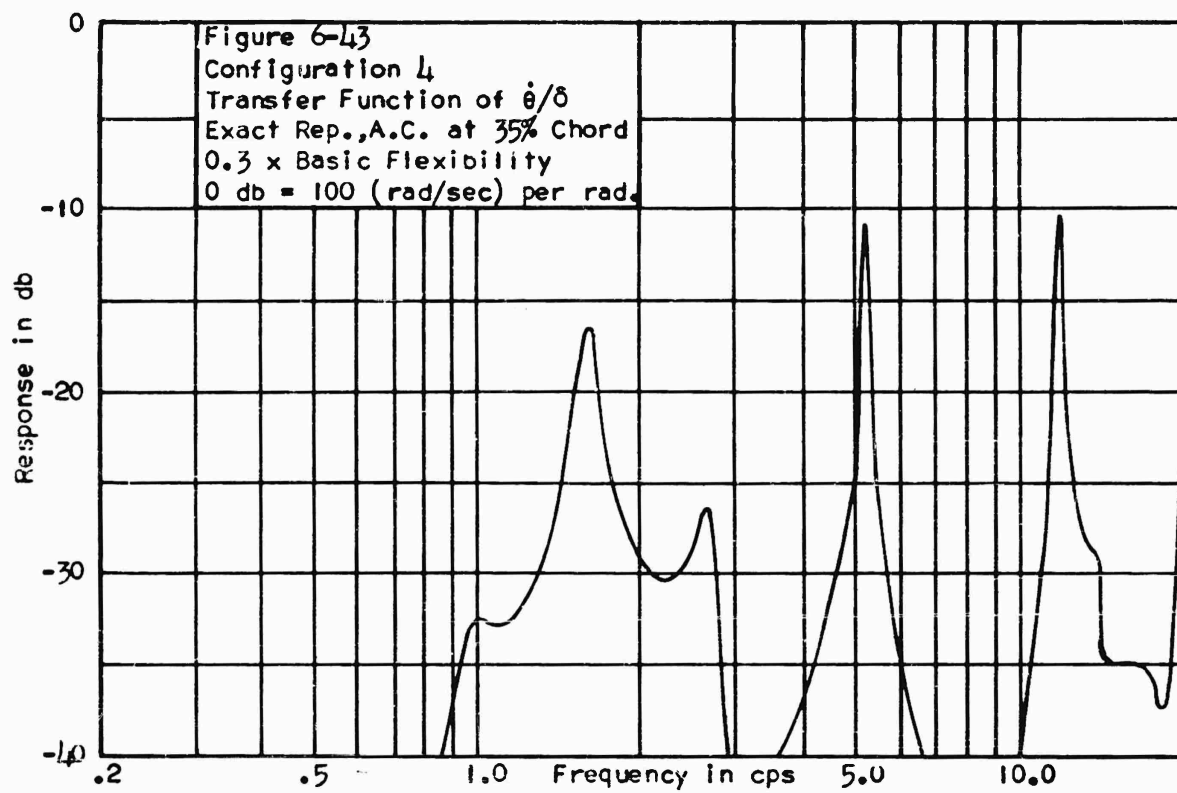
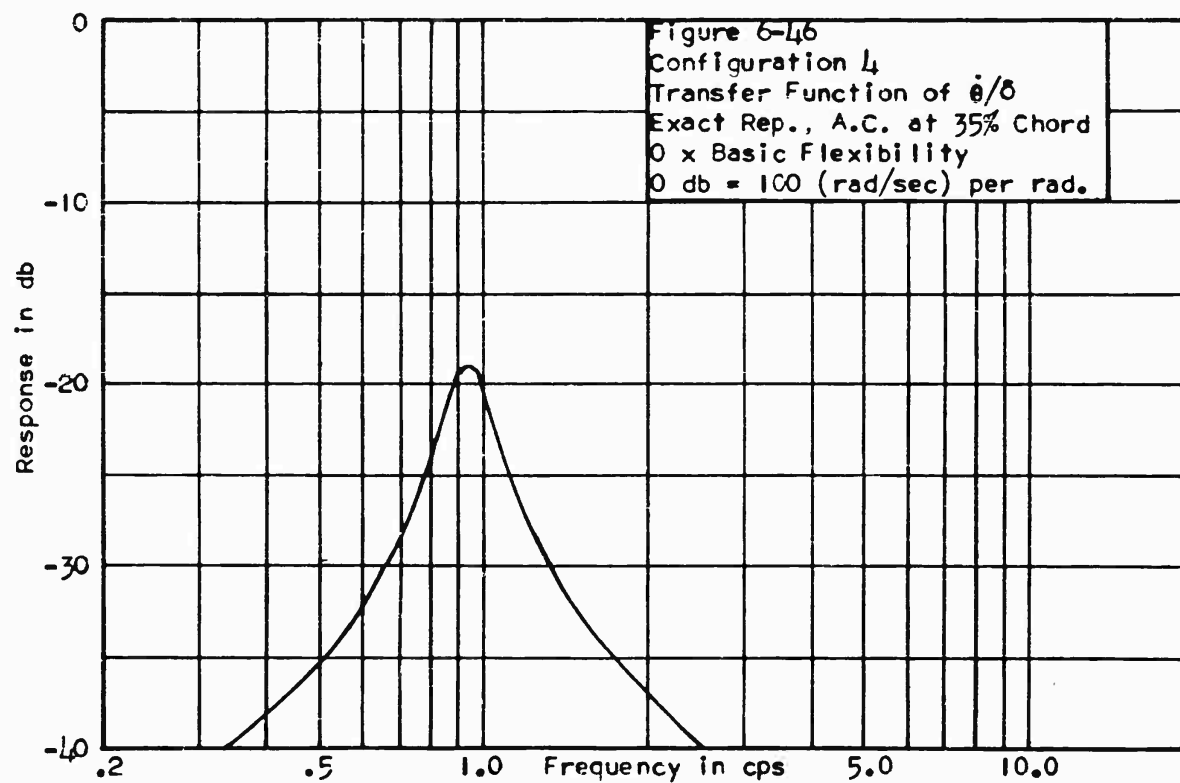
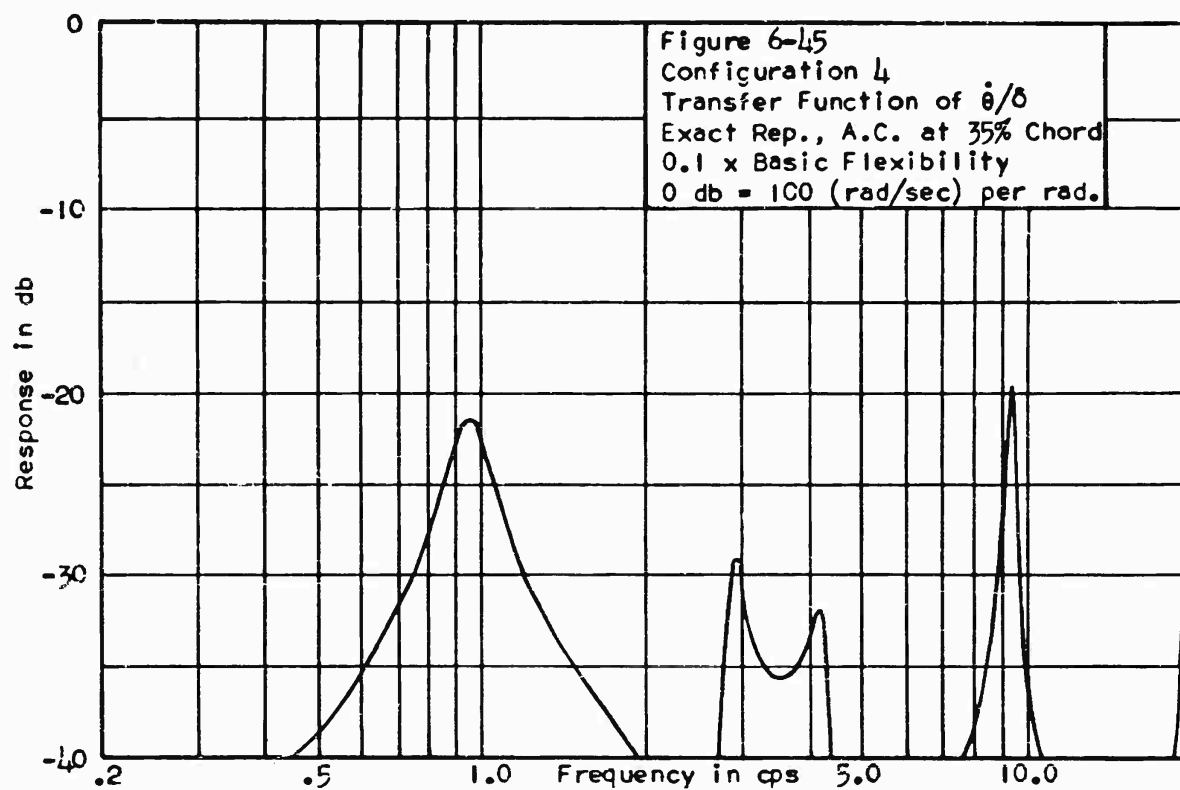


Figure 6-40. Stability Boundaries of Configuration 4







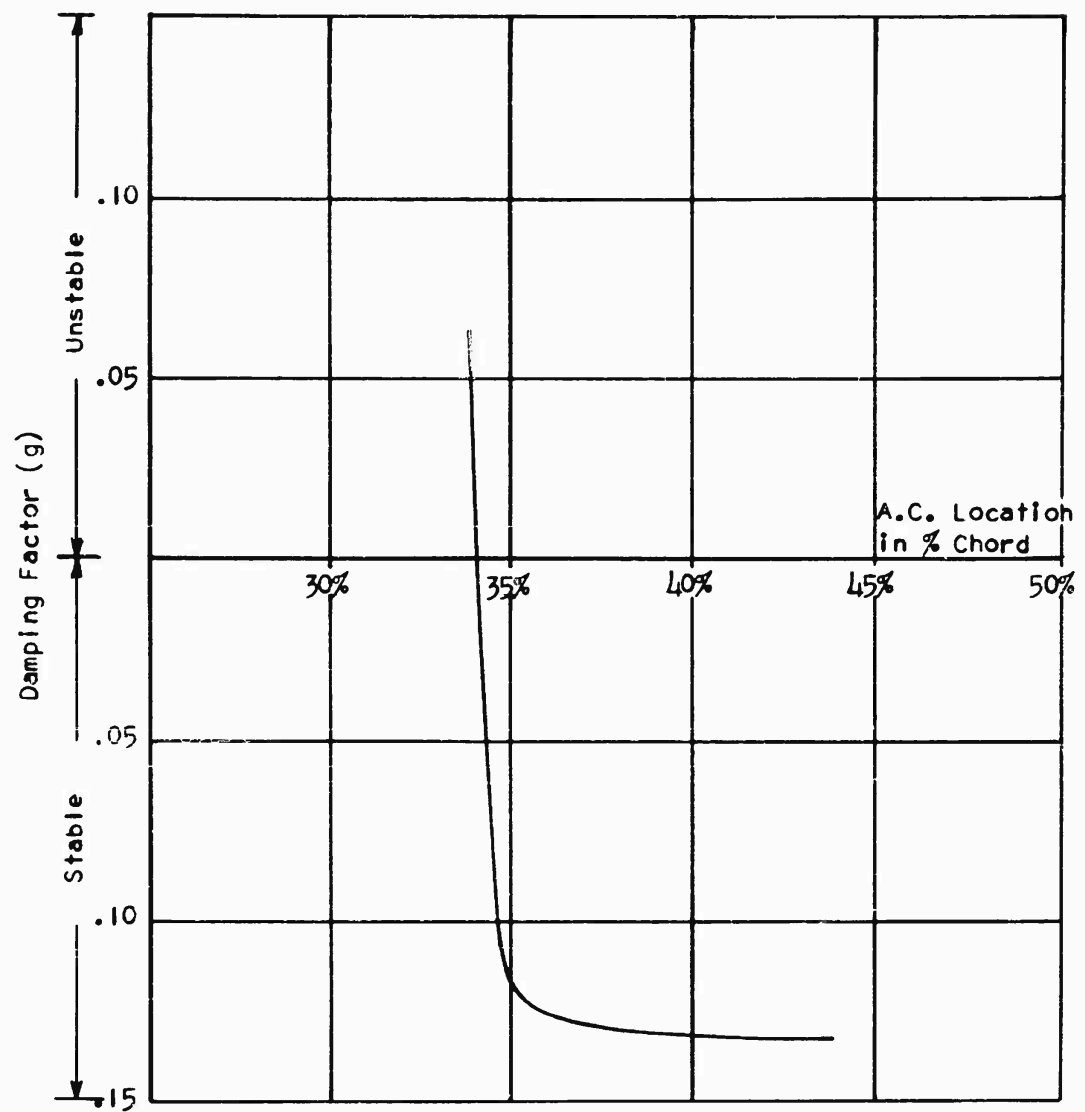


Figure 6-47. Damping of Low Frequency Root of Configuration 4 vs A.C. Location

Flight Condition F

Flexibility = .5 x Basic

$g = 2(\zeta) = 2 \times (\text{Per Unit Critical Damping})$

TABLE 6-1

Lowest 5 Elastic Mode Frequencies

Mode No.	Configuration 2 Condition 30 (cps)	Configuration 3 Condition 2 (cps)	Configuration 4 Condition 10 (cps)
1	1.42	1.20	.948
2	3.42	4.30	1.23
3	5.87	5.19	2.90
4	9.17	6.73	6.40
5	10.12	8.43	7.39

TABLE 6-2

Summary of Transfer Function Charts Presented
Configuration 2, Mass Condition 30, Flight Condition C

$\dot{\theta}/\delta$ = Pitch velocity at station 650 due to cannard rotation

$\ddot{h}/\delta$ = Vertical acceleration at station 650 due to cannard rotation

Chart No.	Type of Representation		Elastic Modes Included in Representation	Residual Flexibility Included in Representation	Type of Measurement
	Modal	Exact			
1		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude ↓ $\dot{\theta}/\delta$ Magnitude
2	X		5	Yes	
3	X		4	↓	
4	X		3	↓	
5	X		2	↓	
6	X		1	↓	
7	X		0	Yes	
8	X		5	No	
9	X		4	↓	
10	X		3	↓	
11	X		2	↓	
12	X		1	↓	
13	X		0	No	
14		X	- - -	- - -	$\ddot{h}/\delta$ Magnitude ↓ $\ddot{h}/\delta$ Magnitude
15	X		5	Yes	
16	X		4	↓	
17	X		3	↓	
18	X		2	↓	
19	X		1	↓	
20	X		0	Yes	
21	X		5	No	
22	X		4	No	
23	X		3	No	

TABLE 6-2. Cont.

Summary of Transfer Function Charts Presented
Configuration 2, Mass Condition 30, Flight Condition C

Chart No.	Type of Representation		Elastic Modes Included in Representation	Residual Flexibility Included in Representation	Type of Measurement
	Modal	Exact			
24	X		2	No	$\ddot{h}/\delta$ Magnitude
25	X		1	No	$\ddot{h}/\delta$ Magnitude
26	X		0	No	$\ddot{h}/\delta$ Magnitude
27		X	- - -	- - -	$\dot{\theta}/\delta$ Phase Angle
28	X		5	Yes	
29	X		2	Yes	
30	X		5	No	
31	X		2	No	
32		X	- - -	- - -	$\ddot{h}/\delta$ Phase Angle
33	X		5	Yes	
34	X		2	Yes	
35	X		5	No	
36	X		2	No	

TABLE 6-2 Cont.

Summary of Transfer Function Charts Presented
Configuration 3, Mass Condition 2

$\dot{\theta}/\delta$ = Pitch velocity at station 340 due to elevator deflection

$\dot{\theta}/u_g$ = Pitch velocity at station 340 due to vertical gust

$\ddot{h}/\delta$ = Vertical acceleration at station 402.5 due to elevator deflection

$\ddot{h}/u_g$ = Vertical acceleration at station 402.5 due to vertical gust

$u_1/\delta, u_2/\delta, u_3/\delta$ = Generalized velocity of the first 3 normal coordinates
due to elevator deflection

Chart No.	Type of Representation		Elastic Modes Included in Representation	Residual Flexibility Included in Representation	Type of Measurement	Flight Condition
	Modal	Exact				
37		X	- - -	- - -	θ/δ Magnitude	A
38	X		5	Yes	↓	↓
39	X		3			
40	X		2			
41	X		1	↓		
42	X		0	Yes		
43	X		5	No		
44	X		4	↓		
45	X		3			
46	X		2			
47	X		1	↓		
48	X		0	No	θ/δ Magnitude	A
49		X	- - -	- - -	θ/u_g Magnitude	A
50	X		5	Yes	↓	↓
51	X		3			
52	X		2			
53	X		1	↓		
54	X		0	Yes		
55	X		5	No		
56	X		3	No	θ/u_g Magnitude	A

TABLE 6-2. Cont.

Summary of Transfer Function Charts Presented, Configuration 3, Mass Condition 2

Chart No.	Type of Representation		Elastic Modes Included in Representation	Residual Flexibility Included in Representation	Type of Measurement	Flight Condition
	Modal	Exact				
57	X		2	No	$\dot{\theta}/u_g$ Magnitude	A
58	X		1	No	$\dot{\theta}/u_g$ Magnitude	A
59	X		0	No	$\dot{\theta}/u_g$ Magnitude	A
60		X	- - -	- - -	$\ddot{h}/\delta$ Magnitude	A
61	X		5	Yes	↓ $\ddot{h}/\delta$ Magnitude	↓ A
62	X		3			
63	X		2			
64	X		1			
65	X		0	Yes		
66	X		5	No		
67	X		4			
68	X		3			
69	X		2			
70	X		1			
71	X		0	No	$\ddot{h}/\delta$ Magnitude	A
72		X	- - -	- - -	$\ddot{h}/u_g$ Magnitude	A
73	X		5	Yes	↓ $\ddot{h}/u_g$ Magnitude	↓ A
74	X		2			
75	X		1			
76	X		0	Yes		
77	X		5	No		
78	X		3			
79	X		1			
80	X		0	No	$\ddot{h}/u_g$ Magnitude	A
81		X	- - -	- - -	$\dot{\theta}/\delta$ Phase Angle	A
82	X		5	Yes	↓ $\dot{\theta}/\delta$ Phase Angle	↓ A
83	X		2	Yes		
84	X		0	Yes		
85	X		2	No		
86	X		0	No	$\dot{\theta}/\delta$ Phase Angle	A

TABLE 6-2. Cont.

Summary of Transfer Function Charts Presented, Configuration 3, Mass Condition 2

Chart No.	Type of Representation		Elastic Modes Included in Representation	Residual Flexibility Included in Representation	Type of Measurement	Flight Condition
	Modal	Exact				
87	X		1	Yes	u_1/δ Magnitude	A
88	X		1	↓ Yes	u_1/δ Phase Angle	↓ A
89	X		1		u_2/δ Magnitude	
90	X		1		u_2/δ Phase Angle	
91	X		1		u_3/δ Magnitude	
92	X		1		u_3/δ Phase Angle	A
93		X	---	---	$\dot{\theta}/\delta$ Magnitude	B
94	X		5	Yes	↓ $\dot{\theta}/\delta$ Magnitude	↓ B
95	X		2	Yes		
96	X		0	Yes		
97	X		2	No		
98	X		0	No		B
99		X	---	---	$\dot{\theta}/u_g$ Magnitude	B
100	X		5	Yes	↓ $\dot{\theta}/u_g$ Magnitude	↓ B
101	X		2	Yes		
102	X		0	Yes		
103	X		2	No		
104	X		0	No		B
105		X	---	---	$\ddot{h}/\delta$ Magnitude	B
106	X		5	Yes	↓ $\ddot{h}/\delta$ Magnitude	↓ B
107	X		2	Yes		
108	X		0	Yes		
109	X		2	No		
110	X		0	No		B
111		X	---	---	$\dot{\theta}/\delta$ Phase Angle	B
112	X		5	Yes	↓ $\dot{\theta}/\delta$ Phase Angle	↓ B
113	X		2	Yes		
114	X		0	Yes		

TABLE 6-2. Cont.

Summary of Transfer Function Charts Presented, Configuration 3, Mass Condition 2

Chart No.	Type of Representation		Elastic Modes Included in Representation	Residual Flexibility Included in Representation	Type of Measurement	Flight Condition
	Modal	Exact				
115	X		2	No	$\dot{\theta}/\delta$ Phase Angle	B
116	X		1	No	$\dot{\theta}/\delta$ Phase Angle	B
117	X		0	No	$\dot{\theta}/\delta$ Phase Angle	B
118	X		1	Yes	u_1/δ Magnitude	B
119	X		1	↓ Yes	u_1/δ Phase Angle	↓ B
120	X		1		u_2/δ Magnitude	
121	X		1		u_2/δ Phase Angle	
122	X		1		u_3/δ Magnitude	
123	X		1		u_3/δ Phase Angle	

TABLE 6-2 Cont.

Summary of Transfer Function Charts Presented
Configuration 4, Mass Condition 10, Flight Condition F

$\dot{\theta}/\delta$ = Pitch velocity at station 325 due to elevator deflection

Chart No.	Type of Representation		Elastic Modes Included in Representation	General Flexibility Through 5th Mode Included	Type of Measurement	Flexibility Factor of Basic Flexibility	A.C. in % Chord
	Modal	Exact					
124		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude	.5	35%
125	X		4	Yes	$\dot{\theta}/\delta$ Magnitude	.5	35%
126	X		3				
127	X		2				
128	X		1				
129	X		0	Yes			
130	X		5	No	$\dot{\theta}/\delta$ Magnitude	.5	35%
131	X		4				
132	X		3				
133	X		2				
134	X		0	No			
135		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude	.7	35%
136		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude	.7	37.5%
137		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude	.1	25%

TABLE 6-2. Cont.

Summary of Transfer Function Charts Presented

Configuration 4, Mass Condition 10, Flight Condition F

Chart No.	Type of Representation		Elastic Modes Included in Representation	General Flexibility Through 5th Mode Included	Type of Measurement	Flexibility Factor of Basic Flexibility	A.C. in % Chord
	Modal	Exact					
138		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude ↓	.1	35%
139		X	- - -	- - -		.1	50%
140		X	- - -	- - -		0	50%
141		X	- - -	- - -		0	35%
142		X	- - -	- - -		0	25%
143		X	- - -	- - -		.3	43.75%
144		X	- - -	- - -		.3	35%
145		X	- - -	- - -		.5	43.75%
146		X	- - -	- - -		.5	37.5%
147		X	- - -	- - -		.5	36.25%
148		X	- - -	- - -	$\dot{\theta}/\delta$ Magnitude	.2	35%

TABLE 6-3

SUMMARY OF TRANSFER FUNCTIONS SHOWN FOR CONFIGURATION 2

Figure No.	Structural Representation	Number of Elastic Modes Explicitly Included in Modal Representation*	"Residual Flexibility" Approximation Included
6-16	Exact	- - -	- - -
6-17	Modal	5	Yes
6-18	↓	4	Yes
6-19		3	Yes
6-20		2	Yes
6-21		1	Yes
6-22		0	Yes
6-23		5	No
6-24		4	No
6-25		3	No
6-26		2	No
6-27		1	No
6-28	Modal	0	No

* 2 zero frequency modes were included in all cases.

TABLE 6-4
SUMMARY OF TRANSFER FUNCTIONS SHOWN FOR CONFIGURATION 3

Figure No.	Structural Representation	Number of Elastic Modes Explicitly Included in Modal Representation*	"Residual Flexibility" Approximation Included
6-29	Exact	- - -	- - -
6-30	Modal	5	Yes
6-31	↓	3	Yes
6-32		2	Yes
6-33		1	Yes
6-34		0	Yes
6-35		5	No
6-36		3	No
6-37		2	No
6-38		1	No
6-39	Modal	0	No

* 2 zero frequency modes were included in all cases.

SECTION 7

CONCLUSIONS

1. An estimate of the effects of aeroelasticity on the stability and control of an airborne vehicle can be obtained from the "aeroelastic index". The aeroelastic index is the summation of the diagonal elements of the matrix product $[Q_{mm}][X_{mm}]$, where $[Q_{mm}]$ is the aerodynamic influence coefficient matrix and $[X_{mm}]$ is the flexibility matrix of the unsupported system. Since the system is unsupported the flexibility matrix must be evaluated with rigid restraints applied to the zero frequency normal coordinates. Elements of $[X_{mm}]$ depend on the mass distribution, in such a way that inertia relief effects are properly accounted for.

The following qualitative estimates can be based on the aeroelastic index, D .

1. $(0.1) > D > (-0.1)$ indicates rather small steady state aeroelastic effects on the system.
2. $D < 0$ indicates a loss in aerodynamic effectiveness of the system.
3. $D \geq 1.0$ indicates the system is near or beyond its steady state divergence speed.
4. $1.0 > D > 0$ indicates that the dynamic pressure, $\left(\frac{1}{2} \rho v^2\right)$ used in calculating the aerodynamic matrix $[Q_{mm}]$, is approximately a factor D times the dynamic pressure at steady state divergence.

2. An accurate approximation for the aeroelastic behavior of a structure may be made in terms of its normal coordinates by including some of its modes explicitly and the "residual flexibility" approximation to all higher modes. The modes explicitly included should be all those in the frequency range from zero to the maximum frequency of interest plus the next higher mode.
3. The inclusion of the "residual flexibility" in the representation suggested in "2" is very important. Although the magnitude of the "residual flexibility" decreases as the number of modes explicitly included in the representation increases, it is inadvisable to attempt to supplant the "residual flexibility" by additional modes. In the examples considered here better accuracy was obtained at frequencies below the first elastic mode by including one mode explicitly plus residual flexibility than was obtained by including five modes explicitly and no residual flexibility.
4. An exception to the rule of "2" occurs in the case of "mode interaction", which may be defined as a condition of potential or incipient aeroelastic instability involving one elastic mode and one rigid body mode. This condition, as observed in the example of configuration 4 of this report, will require further investigation before conclusive statements can be made concerning adequate structural representation.
5. All configurations studied showed that aeroelasticity can have very large effects on the response of a system even at frequencies well below the first elastic mode.

SECTION 8

RECOMMENDATIONS FOR FURTHER STUDY

8.1 MODE INTERACTION

The obvious fundamental problem in the case of "mode interaction" is the understanding of the mechanism of the phenomenon and the definition of the stability margin. Although flutter analysis methods lend some aid in this regard it is believed that further study of the interaction of rigid and elastic modes of a system are required.

Of equal importance to the analysis of the stability of a vehicle with "mode interaction" is the effect of the potential instability beyond the maximum capabilities of the aircraft on the stability and control characteristics within the flight envelope. It has been shown in this study that the presence of this potential instability can have a very large effect on the response of the aircraft at speeds well below the critical speed. Since the scope of this study did not permit more than a cursory examination of this phenomenon, it is suggested that a more complete evaluation of the consequences of "mode interaction" on the stability and control of airborne vehicles is required.

8.2 LATERAL PROBLEM

This study has been limited to the consideration of symmetric aircraft motions. It is suggested that the scope of the results be extended by a similar study considering lateral motions of the system. The possible effects of "mode interaction" should be investigated in the proposed study.

8.3 APPLICATION OF "RESIDUAL FLEXIBILITY" APPROACH TO STEADY STATE PROBLEMS

It was found in this study that the "residual flexibility" approach to the representation of free-free systems could be very useful in the analysis of steady state aeroelastic problems such as divergence, control effectiveness and steady maneuver loads. The advantages of this approach are:

1. Rigorous treatment of "inertia relief".
2. Applicable to symmetric or antisymmetric flight conditions.
3. Applicable to any aerodynamic force representation.

It is suggested that a general procedure be developed for steady state aeroelastic analysis of free-free systems using the "residual flexibility" approach.

8.4 GENERAL DEVELOPMENT OF A UNIFIED METHOD OF SOLUTION OF AEROELASTIC PROBLEMS

A number of fields of analysis of aircraft are concerned with aeroelastic problems. These fields are primarily:

1. Stability and control analysis.
2. Flutter analysis.
3. Analysis of aerodynamic load distribution.

Each of these fields of analysis currently has its own approach to aeroelastic problems which differs sufficiently in procedure and assumptions that none can be applied easily to problems in another field.

An obvious disadvantage of this segregation of problems is the resulting duplication of effort in the development of analysis routines.

An additional disadvantage was noted in this study in the case of "mode interaction". This phenomenon lies between the conventional scope of flutter analysis and stability and control analysis. Many analysis methods used in these fields would not even discover this problem much less completely identify it.

We suggest that there is a great need for a unified method of solution applicable to all aeroelastic problems, from which approximate methods that are useful in special cases may be rigorously and conveniently derived.

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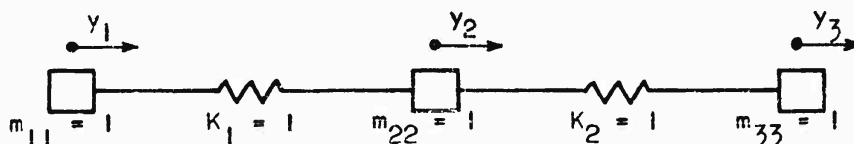
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APPENDIX I

EXAMPLE OF THE CALCULATION OF THE FLEXIBILITY MATRIX FOR A FREE SYSTEM

Consider the simple spring-mass system shown below



The flexibility matrix of this free system, $[X_{ii}] = [X_{ii}^{(f)} + X_{ii}^{(\infty)}]$, will be calculated in two ways; first from consideration of the vibration modes of the system and, second, from the deflection influence coefficient matrix with one point supported, using equation (3.62).

The modal matrix and mode frequencies of the system can easily be computed by elementary analysis. The results are:

$$\begin{aligned} & \{\Phi_i^{(0)}\} \quad \{\Phi_i^{(1)}\} \quad \{\Phi_i^{(2)}\} \\ & [\Phi_i] = \begin{bmatrix} \frac{1}{\sqrt{3}} & 1 & 1 \\ \frac{1}{\sqrt{3}} & 0 & -2 \\ \frac{1}{\sqrt{3}} & -1 & 1 \end{bmatrix} \\ & \omega_r = \quad 0 \quad 1 \quad \sqrt{3} \end{aligned}$$

The generalized masses and stiffness of the modes are

$$[m_r] = \begin{matrix} 1, & 2, & 6 \end{matrix} \quad (\text{from equation 2.13})$$

$$[K_r] = \begin{matrix} 0, & 2, & 18 \end{matrix} \quad (\text{from equation 3.15})$$

Note that the mode shape of the rigid body mode has been selected to give unit generalized mass.

The flexibility matrix of the finite frequency modes can be calculated by equation (3.28). Since there are two finite frequency modes and no infinite frequency modes, equation (3.28) gives the complete flexibility matrix for the system.

$$[x_{ii}^{(f)}] = [x_{ii}] = [\Phi_{if}] [K_f]^{-1} [\Phi_{if}]^T$$

$$[x_{ii}] = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/18 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \quad (\text{from equation 3.28})$$

$$[x_{ii}] = \begin{bmatrix} 5/9 & -1/9 & -4/9 \\ -1/9 & 2/9 & -1/9 \\ -4/9 & -1/9 & 5/9 \end{bmatrix}$$

The first step in the calculation of $[x_{ii}]$ from static influence coefficients is to restrain one coordinate. Let y_3 be that coordinate. The static influence coefficient matrix may then be computed by elementary analysis. The result is

$$[z_{mm}] = [z_{ii}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that, in accordance with the discussion on page 31, the coordinates of all mass points are retained in $[z_{mm}]$. The matrix $[B_{mm}]$ is next computed from equation (3.61):

$$[B_{mm}] = [\Phi_{m0}] [\Phi_{m0}]^T [m_{mm}] \quad (3.61)$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[B_{mm}] = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence by equation (3.62):

$$[x_{ii}] = [x_{mm}] = ([I] - [B_{mm}])[z_{mm}]([I] - [B_{mm}])^T \quad (3.62)$$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 & -1/9 & -4/9 \\ -1/9 & 2/9 & -1/9 \\ -4/9 & -1/9 & 5/9 \end{bmatrix}$$

As a simple example of the application of these results consider that the external forces applied to the system have sufficiently little

high frequency content that the generalized masses of the finite frequency modes may be ignored in calculating the response of the system. In this case the flexibility matrix of the system $[X_{ii}]$ becomes the residual flexibility $[X_{ii}^{(\infty)}]$, and the response as calculated from equation (3.27) is

$$\{y_i\} = [\Phi_{i0}] \{\xi_0\} + [X_{ii}] \{F_i\}$$

The response of the only zero frequency mode, ξ_0 , is obtained from

$$m_0 s^2 \xi_0 = [\Phi_{i0}]^T \{F_i\} \quad \text{(from equations 2.15 and 2.16)}$$

or, since $m_0 = 1$,

$$s^2 \xi_0 = \frac{1}{\sqrt{3}} (F_1 + F_2 + F_3)$$

In this equation s is the derivative operator. The responses of the physical coordinates are

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \left(\frac{F_1 + F_2 + F_3}{\sqrt{3} s^2} \right) + [X_{ii}] \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \left(\frac{F_1 + F_2 + F_3}{3 s^2} \right) \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \begin{bmatrix} 5/9 & -1/9 & -4/9 \\ -1/9 & 2/9 & -1/9 \\ -4/9 & -1/9 & 5/9 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

The first term indicates that the sum of the forces should be integrated twice. This result, which is highly accurate for the conditions postulated, is much simpler than the solution of the full dynamic equations.

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(over)

is presented whereby the elasticity of the structure is completely and rigorously included while the mass of the structure is expressed in terms of a finite number of normal coordinates of the system. Analog computer investigations of the dynamic response of three typical aircraft configurations were made using the method described above. These analog computer investigations clearly demonstrate the utility of the structural representation presented and also define the approximations which can be made in the representation in terms of the requirements of any particular analysis.

1. Aeroelasticity
2. Flutter, mathematical analysis
3. Stability, mathematical analysis
- I. AFSC Project 8219, Task 821901

II. Contract AF33(616)-7658

III. Computer Eng. Associates, Pasadena, Calif.

IV. R. G. Schwendler, R. H. MacNeal

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