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MEMORANDUM  
RM-3199-PR  
JUNE 1962

SIMPLEX METHOD AND THEORY  
Notes on Linear Programming  
and Extensions - Part 62

A. W. Tucker

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MEMORANDUM

RM-3199-PR

JUNE 1962

SIMPLEX METHOD AND THEORY  
Notes on Linear Programming  
and Extensions - Part 62

A. W. Tucker

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PREFACE

Part of the RAND research program consists of basic support studies in mathematics. One aspect of this, of very considerable applicability in large governmental, military, and industrial operations, is concerned with linear programming.

In the present Memorandum the author discusses the basic structure of this theory.

This Memorandum will appear as part of a book, Symposium on Mathematical Optimization Techniques to be published by the University of California Press.

SUMMARY

In this Memorandum, the author discusses the simplex method of linear programming in a format designed to exhibit over-all structure rather than specific operational details. Various terminal possibilities are represented schematically and geometrically, and it is shown that transposition duality theorems can be regarded as corollaries of the duality theorem for a homogeneous linear program.

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## SIMPLEX METHOD AND THEORY

### 1. INTRODUCTION

The simplex method (1947) of George B. Dantzig [1] is much more than the basic computational tool of linear programming. It is a combinatorial algorithm that provides constructive means of establishing fundamental theorems of linear programming [2]—as well as like theorems in cognate areas, such as von Neumann's Minimax Theorem for matrix games [3] and Farkas' Theorem for linear inequalities. Its characteristic pivot transformations are related in an essential way to Gauss-Jordan elimination [4] and to a combinatorial equivalence of matrices [5].

This paper discusses the simplex method in a format designed to exhibit over-all structure rather than specific operational details. The various terminal possibilities are represented schematically and geometrically. Also, it is shown that transposition duality theorems [6], such as the classical ones of Gordan, Farkas, Stiemke and Motzkin, can be regarded as corollaries of the duality theorem for a "homogeneous linear program."

The schemata and block-pivot transformations used in this paper seem to be important methodological devices. They follow closely along lines developed by the author in a previous paper concerned with solutions of matrix games by linear programming [7].

2. DUAL LINEAR SYSTEMS

This section and the next develop underlying concepts and format for use in later sections.

The schema

$$\begin{array}{cccccc}
 & -y_1 & -y_2 & \cdots & -y_n & \\
 \xi_1 & a_{11} & a_{12} & \cdots & a_{1n} & = x_1 \\
 \xi_2 & a_{21} & a_{22} & \cdots & a_{2n} & = x_2 \\
 \vdots & \vdots & \vdots & & \vdots & \vdots \\
 \xi_m & a_{m1} & a_{m2} & \cdots & a_{mn} & = x_m \\
 & = \eta_1 & = \eta_2 & & = \eta_n & 
 \end{array}
 \tag{2.1}$$

is a convenient device for the joint presentation of two systems of linear equations: a column system

$$\begin{array}{cccc}
 \xi_1 a_{11} + \xi_2 a_{21} + & & + \xi_m a_{m1} & = \eta_1 \\
 \xi_1 a_{12} + \xi_2 a_{22} + & & + \xi_m a_{m2} & = \eta_2 \\
 \vdots & \vdots & \vdots & \vdots \\
 \xi_1 a_{1n} + \xi_2 a_{2n} + & & + \xi_m a_{mn} & = \eta_n
 \end{array}
 \left. \vphantom{\begin{array}{cccc} \xi_1 a_{11} + \xi_2 a_{21} + & & + \xi_m a_{m1} & = \eta_1 \\ \xi_1 a_{12} + \xi_2 a_{22} + & & + \xi_m a_{m2} & = \eta_2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1 a_{1n} + \xi_2 a_{2n} + & & + \xi_m a_{mn} & = \eta_n \end{array}} \right\} \Xi A = H
 \tag{2.2}$$

and a row system



### 3. BLOCK-PIVOT TRANSFORMATION

Let  $A_{11}$  be a nonsingular square submatrix of  $A$ , and  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$  the remaining submatrices of  $A$ . Then the schema (2.1) can be rewritten as

$$(3.1) \quad \begin{array}{cc|cc} & -Y_1 & -Y_2 & & \\ \varepsilon_1 & A_{11} & A_{12} & = & X_1 \\ \varepsilon_2 & A_{21} & A_{22} & = & X_2 \\ & = H_1 & = H_2 & & \end{array}$$

Since  $A_{11}^{-1}$  exists, the subsystems

$$\varepsilon_1 A_{11} + \varepsilon_2 A_{21} = H_1 \quad \text{and} \quad -A_{11}Y_1 - A_{12}Y_2 = X_1$$

can be solved for  $\varepsilon_1$  and  $Y_1$  to obtain

$$\varepsilon_1 = H_1 A_{11}^{-1} - \varepsilon_2 A_{21} A_{11}^{-1} \quad \text{and} \quad Y_1 = -A_{11}^{-1} X_1 - A_{11}^{-1} A_{12} Y_2.$$

Substitution for  $\varepsilon_1$  and  $Y_1$  in the subsystems

$$\varepsilon_1 A_{12} + \varepsilon_2 A_{22} = H_2 \quad \text{and} \quad -A_{21}Y_1 - A_{22}Y_2 = X_2$$

yields

$$H_1 A_{11}^{-1} A_{12} + \varepsilon_2 (A_{22} - A_{21} A_{11}^{-1} A_{12}) = H_2$$

and

$$A_{21} A_{11}^{-1} X_1 - (A_{22} - A_{21} A_{11}^{-1} A_{12}) Y_2 = X_2.$$

These results are exhibited by the column and row system of the schema

$$(3.2) \begin{array}{cc} & \begin{array}{c} -X_1 \\ -Y_2 \end{array} \\ \begin{array}{c} H_1 \\ \varepsilon_2 \end{array} & \begin{array}{|c|c|} \hline A_{11}^{-1} & A_{11}^{-1}A_{12} \\ \hline -A_{21}A_{11}^{-1} & A_{22}-A_{21}A_{11}^{-1}A_{12} \\ \hline \end{array} \begin{array}{c} =Y_1 \\ =X_2 \end{array} \\ & \begin{array}{c} = \varepsilon_1 \\ = H_2 \end{array} \end{array}$$

The schema (3.2) is equivalent to the schema (3.1) in the sense that the column equation systems of (3.1) and (3.2) have the same solutions  $\varepsilon, H$  and the row equation systems of (3.1) and (3.2) have the same solutions  $X, Y$ .

Let  $r$  be the order of the nonsingular square submatrix  $A_{11}$ , the choice of which determines uniquely the transformation from the schema (3.1) to the equivalent schema (3.2). Then the transformation from (3.1) to (3.2) is called a block-pivot transformation of order  $r$ , the nonsingular square submatrix  $A_{11}$  of order  $r$  being called the block-pivot. It can readily be verified that the inverse of the block-pivot transformation from (3.1) to (3.2) is a block-pivot transformation from (3.2) to (3.1), the block-pivot being  $A_{11}^{-1}$ .

Any nonzero entry of the matrix  $A$  determines a block-pivot  $A_{11}$  of order one; the corresponding pivot transformation of order one is called an elementary pivot transformation.

Elementary pivoting, utilized so effectively in the simplex method, has its roots in the classical process of Gauss-Jordan (complete) elimination.

Note that the block-pivot transformation of order  $r$  from (3.1) to (3.2) exchanges  $r$  of the individual marginal labels at the left with  $r$  labels at the bottom and  $r$  parallel labels at the right with  $r$  parallel labels at the top, signs being reversed in the latter exchange. Such a block-pivot transformation can always be decomposed into a succession of elementary pivot transformations, exchanging just one label on a margin at a time; conversely, any finite succession of elementary pivot transformations is summarized by a single block-pivot transformation (as explained in [5] and illustrated in [7]).

The  $m$  by  $n$  matrices in (3.1) and (3.2), or any row and/or column permutations thereof, are combinatorially equivalent in a sense discussed by the author in [5].

#### 4. DUAL LINEAR PROGRAMS

Here the format developed in the two previous sections will be used, with some change of symbols, to discuss dual linear programs.

The schema

$$\begin{array}{r}
 (4.1) \\
 \lambda_1 \\
 \lambda_2 \\
 \vdots \\
 \lambda_M \\
 1
 \end{array}
 \begin{array}{cccc|c}
 -x_1 & -x_2 & \dots & -x_N & 1 \\
 a_{11} & a_{12} & \dots & a_{1N} & b_1 \\
 a_{21} & a_{22} & \dots & a_{2N} & b_2 \\
 \vdots & \vdots & & \vdots & \vdots \\
 a_{M1} & a_{M2} & \dots & a_{MN} & b_M \\
 \hline
 c_1 & c_2 & \dots & c_N & d
 \end{array}
 \begin{array}{l}
 = 0 \\
 = 0 \\
 \vdots \\
 = 0 \\
 = w \\
 = \xi_1 \\
 = \xi_2 \\
 \dots \\
 = \xi_N \\
 = \omega
 \end{array}$$

exhibits row and column equation systems

$$\left. \begin{array}{l} -AX + B = 0 \\ -CX + d = w \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \Lambda A + C = \Xi \\ \Lambda B + d = \omega \end{array} \right.$$

which pertain to the following pair of linear programs:

$$(4.2) \quad \text{Primal Program.} \quad \text{To maximize } w = d - CX \\
 \text{constrained by } AX = B, X \geq 0.$$

$$(4.3) \quad \text{Dual Program.} \quad \text{To minimize } \omega = d + \Lambda B \\
 \text{constrained by } \Lambda A + C = \Xi \geq 0.$$

(A vector inequality holds for each component—i.e.,  $X \geq 0$  means  $x_1 \geq 0, x_2 \geq 0, \dots, x_N \geq 0$ .) The "parameters"  $(\lambda_1, \lambda_2, \dots, \lambda_M) = \Lambda$  in the Dual Program are unrestricted in sign.

Let  $A_{11}$  be a nonsingular square submatrix of the matrix  $A$  above. Then the schema (4.1) can be recast as

$$(4.4) \quad \begin{array}{l} \Lambda_1 \\ \Lambda_2 \\ 1 \end{array} \begin{array}{|ccc} -X_1 & -X_2 & 1 \\ A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & d \end{array} \begin{array}{l} = 0 \\ = 0 \\ = w \end{array}$$

$$\begin{array}{ccc} =\varepsilon_1 & =\varepsilon_2 & =\omega \end{array}$$

(Of course, the  $\Lambda_2$ -headed row in (4.4) will be vacuous if the submatrix  $A_{11}$  omits no row of  $A$ , and the  $X_2$ -headed column in (4.4) will be vacuous if  $A_{11}$  omits no column of  $A$ .)

$$\begin{aligned} \bar{A}_{11} &= A_{11}^{-1}, & \bar{A}_{12} &= A_{11}^{-1}A_{12}, & \bar{B}_1 &= A_{11}^{-1}B_1, \\ \bar{A}_{21} &= -A_{21}A_{11}^{-1}, & \bar{A}_{22} &= A_{22} - A_{21}A_{11}^{-1}A_{12}, & \bar{B}_2 &= B_2 - A_{21}A_{11}^{-1}B_1, \\ \bar{C}_1 &= -C_1A_{11}^{-1}, & \bar{C}_2 &= C_2 - C_1A_{11}^{-1}A_{12}, & \bar{d} &= d - C_1A_{11}^{-1}B_1. \end{aligned}$$

Then the schema

$$(4.5) \quad \begin{array}{l} \varepsilon_1 \\ \Lambda_2 \\ 1 \end{array} \begin{array}{|ccc} 0 & -X_2 & 1 \\ \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{C}_2 & \bar{d} \end{array} \begin{array}{l} = X_1 \\ = 0 \\ = w \end{array}$$

$$\begin{array}{ccc} =\Lambda_1 & =\varepsilon_2 & =\omega \end{array}$$

results from the schema (4.4) by the block-pivot transformation having  $A_{11}$  as block-pivot.

The new schema (4.5) is equivalent to the old schema (4.4). That is, the row equation system of one schema is satisfied by any  $X, w$  satisfying the row equation system of the other schema, and the column equation system of one schema is satisfied by any  $\Lambda, \bar{\epsilon}, \omega$  satisfying the column equation system of the other schema. Hence the Primal Program (4.2) calls now for maximizing  $w$  subject to the row equation system of (4.5) and the inequalities  $X_1 \geq 0, X_2 \geq 0$ , and the Dual Program (4.3) calls now for minimizing  $\omega$  subject to the column equation system of (4.5) and the inequalities  $\bar{\epsilon}_1 \geq 0, \bar{\epsilon}_2 \geq 0$ .

If the schema (4.5) is such that

$$(4.6) \quad \bar{A}_{22} = 0, \bar{B}_2 = 0 \text{ (or are vacuous)}$$

and

$$(4.7) \quad \bar{B}_1 \geq 0, \bar{C}_2 \geq 0,$$

then optimal (basic) solutions of the Primal and Dual Programs, (4.2) and (4.3), can be read directly from (4.5) by setting variables at top and left margins equal to zero. These optimal solutions are

$$X_1 = \bar{B}_1 (\geq 0), X_2 = 0; w = \bar{d}$$

and

$$\Lambda_1 = \bar{c}_1, \Lambda_2 = 0; \bar{\varepsilon}_1 = 0, \bar{\varepsilon}_2 = \bar{c}_2 (\geq 0); \omega = \bar{d}.$$

That  $\bar{d}$  is the maximal  $w$  follows from  $\bar{c}_2 \geq 0$ , because

$$w = \bar{d} - \bar{c}_2 x_2 \leq \bar{d} \text{ for all } x_2 \geq 0;$$

and that  $\bar{d}$  is the minimal  $\omega$  follows from  $\bar{B}_1 \geq 0$ , because

$$\omega = \bar{d} + \bar{\varepsilon}_1 \bar{B}_1 \geq \bar{d} \text{ for all } \bar{\varepsilon}_1 \geq 0.$$

The Dantzig simplex method, starting from an initial "presentation" of the pair of linear programs (4.2) and (4.3), employs a finite succession of elementary pivot transformations to achieve, if possible, a terminal "re-presentation" corresponding to a schema (4.5) for which (4.6) and (4.7) hold.

##### 5. CANONICAL REPRESENTATION

A canonical representation ("re-presentation") of the pair of linear programs (4.2) and (4.3) is provided by any schema

$$(5.1) \quad \begin{array}{c} \varepsilon_1 \\ \Lambda_2 \\ 1 \end{array} \begin{array}{|c|c|c|} \hline 0 & -x_2 & 1 \\ \hline \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \hline \bar{A}_{21} & 0 & 0 \\ \hline \bar{C}_1 & \bar{C}_2 & \bar{d} \\ \hline \end{array} \begin{array}{l} = x_1 \\ = 0 \\ = w \end{array}$$

$$\begin{array}{ccc} =\Lambda_1 & =\varepsilon_2 & =\omega \end{array}$$

for which (4.6) holds. To have  $\bar{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12} = 0$ , it is necessary and sufficient that the order of the block-pivot  $A_{11}$  equal the rank  $m$  of the matrix  $A$ , since then

$$[A_{21}, A_{22}] = A_{21}A_{11}^{-1}[A_{11}, A_{12}] .$$

If  $\bar{A}_{22} = 0$ , then  $\bar{B}_2 = 0$  also, unless  $AX = B$  is an inconsistent system of linear equations.

A partly reduced canonical schema

$$(5.2) \quad \begin{array}{c} \varepsilon_1 \\ 1 \end{array} \begin{array}{|c|c|c|} \hline 0 & -x_2 & 1 \\ \hline \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{C}_2 & \bar{d} \\ \hline \end{array} \begin{array}{l} = x_1 \\ = w \end{array}$$

$$\begin{array}{ccc} =\Lambda_1 & =\varepsilon_2 & =\omega \end{array}$$

results from (5.1) through deletion of the  $\Lambda_2$ -headed row in (5.1). The schema (5.2) contains the same information as (5.1) with redundant parameters  $\Lambda_2$  eliminated.

A fully reduced canonical schema

$$(5.3) \quad \begin{array}{cc|cc} & -x_2 & & 1 \\ \varepsilon_1 & \bar{A}_{12} & | & \bar{B}_1 \\ \hline 1 & \bar{C}_2 & | & \bar{d} \\ \hline & =\varepsilon_2 & & =\omega \end{array} = \begin{array}{l} X_1 \\ w \end{array}$$

results through further deletion of the 0-headed column of (5.2). The schema (5.3) contains all the parameter-free information in (5.1) or (5.2).

If  $\bar{B}_1 \geq 0$  in the above schemata, the canonical representation is primal feasible with  $X_1 = \bar{B}_1 (\geq 0)$ ,  $X_2 = 0$  yielding a feasible (basic) solution of the Primal Program (4.2). If  $\bar{C}_2 \geq 0$  in the above schemata, the canonical representation is dual feasible with  $\Lambda_1 = \bar{C}_1$ ,  $\Lambda_2 = 0$  and  $\varepsilon_1 = 0$ ,  $\varepsilon_2 = \bar{C}_2 (\geq 0)$  yielding a feasible (basic) solution of the Dual Program (4.3). If both  $\bar{B}_1 \geq 0$  and  $\bar{C}_2 \geq 0$ , the canonical representation is optimal with the above-stated feasible (basic) solutions as optimal (basic) solutions of the Primal and Dual Programs.

## 6. GEOMETRIC INTERPRETATION

Let the matrix A in schema (4.1) have rank m and let the number of columns of A be  $N = m + n$ . Let [A, B] also have rank m, so that  $AX = B$  is a consistent system of linear equations. Let S be a space of  $N = m + n$  dimensions with a specified coordinate system, so that there is a one-to-one coorespondence between points

(or vectors) of  $S$  and ordered coordinate  $N$ -tuples, written as  $\xi_1, \xi_2, \dots, \xi_N$  for row usage and as  $x_1, x_2, \dots, x_N$  for column usage. Then the solution sets

$$P = \{\xi | \xi = \Lambda A + C, \text{ all } \Lambda\} \quad \text{and} \quad Q = \{X | AX = B\}$$

are linear manifolds of complementary dimensions  $m$  and  $n$  in the space  $S$ . Let  $\xi = \Lambda A + C$  and  $\xi' = \Lambda' A + C$  be any two points of  $P$ , and  $X$  and  $X'$  any two points of  $Q$ . Then the equation

$$\begin{aligned} (\xi' - \xi)(X' - X) &= (\Lambda' A - \Lambda A)(X' - X) = \\ &= (\Lambda' - \Lambda)(AX' - AX) = 0 \end{aligned}$$

shows that  $P$  and  $Q$  are complementary orthogonal linear manifolds in  $S$ .

Let

$$R = \{\xi | \xi \geq 0\} = \{X | X \geq 0\}$$

be the nonnegative orthant in  $S$ . Then the feasible solution sets

$$\{\xi | \xi = \Lambda A + C, \xi \geq 0\} \quad \text{and} \quad \{X | AX = B, X \geq 0\}$$

of the Dual and Primal Programs (4.3) and (4.2) are the polyhedral convex sets  $P \cap R$  and  $Q \cap R$ , respectively.

In a canonical schema (5.3) the complementary orthogonal linear manifolds P and Q are represented by equation systems in the following "slope-intercept" form,

$$(6.1) \quad P: \quad \varepsilon_2 = \varepsilon_1 \bar{A}_{12} + \bar{C}_2$$

and

$$(6.2) \quad Q: \quad \varepsilon_1 = \varepsilon_2 (-\bar{A}_{12}^T) + \bar{B}_1^T,$$

the latter being obtained by transposing  $X_1 = -\bar{A}_{12}X_2 + \bar{B}_1$  and substituting  $\varepsilon_1$  and  $\varepsilon_2$  for  $X_1^T$  and  $X_2^T$ . In (6.1) the  $m$  by  $n$  matrix  $\bar{A}_{12}$  is the " $\varepsilon_2$ :  $\varepsilon_1$ -slope" of P (with  $\varepsilon_2$  as "rise" and  $\varepsilon_1$  as "run") and  $\bar{C}_2$  is the " $\varepsilon_2$ -intercept" of P. In (6.2) the negative-transpose matrix  $-\bar{A}_{12}^T$  is the " $\varepsilon_1$ :  $\varepsilon_2$ -slope" of Q (with  $\varepsilon_1$  as "rise" and  $\varepsilon_2$  as "run") and  $\bar{B}_1^T$  is the " $\varepsilon_1$ -intercept" of Q. This canonical "slope-intercept" representation of P and Q, introduced by the author in [8], is illustrated in Fig. 1.

Let  $\bar{p}$  and  $\bar{q}$  be the intercept points (vectors)

$$\varepsilon_1 = 0, \quad \varepsilon_2 = \bar{C}_2 \quad \text{and} \quad \varepsilon_1 = \bar{B}_1^T, \quad \varepsilon_2 = 0$$

determined by (6.1) and (6.2). Note that the inner (scalar) product

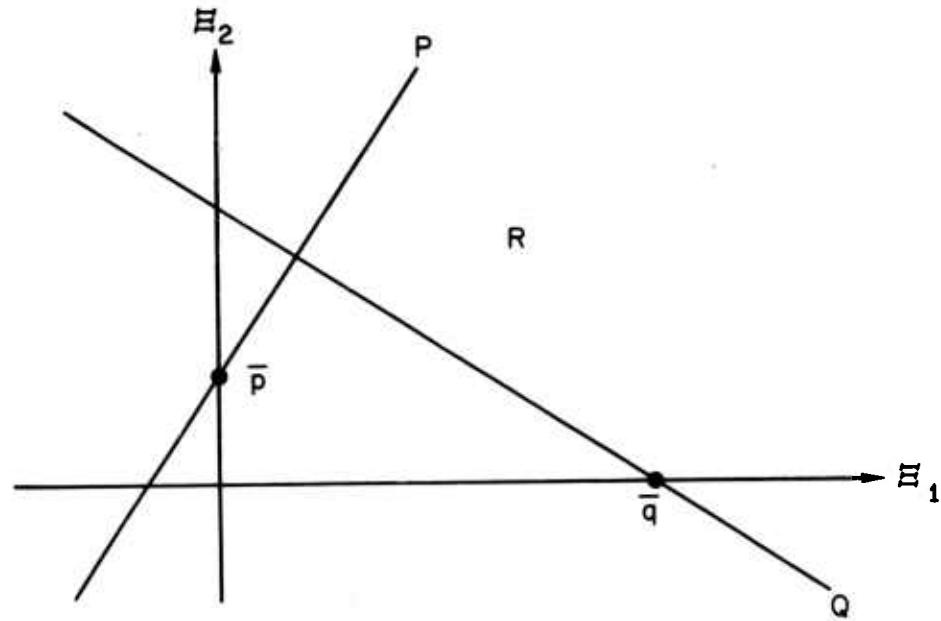


Fig. 1 — Canonical slope representation of complementary orthogonal linear manifolds

$$\bar{p} \cdot \bar{q} = [0, \bar{c}_2] \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = 0.$$

As canonically represented in schema (5.3), the Dual Program is to minimize

$$\omega = \bar{d} + \varepsilon_1 \bar{B}_1 = \bar{d} + [\varepsilon_1, \varepsilon_2] \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = \bar{d} + p \cdot \bar{q}$$

for  $p$  in  $P \cap R$ , and the Primal Program is to maximize

$$w = \bar{d} - \bar{c}_2 x_2 = \bar{d} - [0, \bar{c}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{d} - \bar{p} \cdot q$$

for  $q$  in  $Q \cap R$ . If  $\bar{p}$  belongs to  $P \cap R$  and  $\bar{q}$  belongs to  $Q \cap R$ , then  $p \cdot \bar{q} \geq 0$  for every  $p$  in  $P \cap R$  and  $\bar{p} \cdot q \geq 0$  for every  $q$  in  $Q \cap R$  (since any two vectors in  $R$  have a non-negative inner product). Hence, since  $\bar{p} \cdot \bar{q} = 0$ , it is clear that

$$\omega = \bar{d} + p \cdot \bar{q} \geq \bar{d} + \bar{p} \cdot \bar{q} = \bar{d} \quad \text{for every } p \text{ in } P \cap R$$

and that

$$w = \bar{d} - \bar{p} \cdot q \leq \bar{d} - \bar{p} \cdot \bar{q} = \bar{d} \quad \text{for every } q \text{ in } Q \cap R.$$

That is, the desired minimum and maximum are attained at  $p = \bar{p}$  and  $q = \bar{q}$  if these points both belong to  $R$ . (The intercept points  $\bar{p}$  or  $\bar{q}$  which belong to  $R$  are the extreme points of the polyhedral convex set  $P \cap R$  or  $Q \cap R$ .)

In summary, this geometric interpretation of a pair of linear programs (4.2) and (4.3) involves complementary orthogonal linear manifolds  $P$  and  $Q$  in a space  $S$  with nonnegative orthant  $R$ . If  $P \cap R$  is nonvacuous, the Dual Program is feasible; if  $Q \cap R$  is nonvacuous, the Primal Program is feasible. A canonical representation of these programs involves a joint "slope-intercept" representation of  $P$  and  $Q$ . The resulting intercept points  $\bar{p}$  and  $\bar{q}$  yield optimal solutions if they both belong to  $R$ .

7. SIMPLEX METHOD; TERMINAL POSSIBILITIES

Let  $AX = B$  have a solution  $X \geq 0$ , i.e.,  $Q \cap R$  is nonvacuous and the Primal Program (4.2) is feasible. Then a proof of the simplex method, such as [9], demonstrates the existence of a finite succession of elementary pivot transformations that terminates in a canonical representation for which the matrix

$$(7.1) \quad \begin{array}{|c|c|} \hline A_{12} & B_1 \\ \hline C_2 & d \\ \hline \end{array}$$

of the schema (5.3) has either the schematic form

$$(7.2) \quad \begin{array}{|c|c|} \hline & \oplus \\ & \vdots \\ & \oplus \\ \hline \oplus & \dots & \oplus & * \\ \hline \end{array}$$

or the schematic form

(7.3)

$\ominus$	$\oplus$
$\vdots$	$\vdots$
$\ominus$	$\oplus$
$-$	

where each  $\oplus$  denotes an entry positive or zero, each  $\ominus$  an entry negative or zero, and  $-$  a negative entry. The  $\oplus$ -row and  $\oplus$ -column in (7.2) determine optimal extreme points of  $P \cap R$  and  $Q \cap R$ , the corner entry  $*$  being the common minimum and maximum value. In (7.3) the  $\oplus$ -column determines an extreme point  $\bar{q}$  of  $Q \cap R$  and the  $\ominus$ -column determines the direction of an extreme ray of  $Q \cap R$  issuing from  $\bar{q}$ , along which the objective function  $w \rightarrow +\infty$  because of the corresponding minus entry at the bottom. At the same time the  $(\ominus, -)$ -column in (7.3) shows that  $P \cap R$  is vacuous and the Dual Program is infeasible.

If  $AX = B$  is a consistent system having no solution  $X \geq 0$ , so that  $Q$  exists but  $Q \cap R$  is vacuous and the Primal Program (4.2) is infeasible, then it can be shown that there exists a finite succession of elementary pivot transformations terminating in a canonical representation for which the matrix (7.1) of the schema (5.3) has either the schematic form

(7.4)

$\oplus$	$\dots$	$\oplus$	$-$
$\oplus$	$\dots$	$\oplus$	

or the schematic form

(7.5)

		$\ominus$				
$\oplus$	$\dots$	$\vdots$	$0$	$\dots$	$\oplus$	$-$
		$\vdots$	$\ominus$			
		$-$				

In (7.4) the  $\oplus$ -row at the bottom determines an extreme point  $\bar{p}$  of  $P \cap R$  and the other  $\oplus$ -row determines the direction of an extreme ray of  $P \cap R$  issuing from  $\bar{p}$ , along which the objective function  $\omega \rightarrow -\infty$  because of the corresponding minus entry at the right. In (7.5) the nonpositive column with negative entry at bottom shows that  $P \cap R$  is vacuous and the Dual Program is infeasible. The  $(\oplus, -)$ -row in (7.4) and (7.5) confirms that  $Q \cap R$  is vacuous and the Primal Program is infeasible.

In summary, the terminal possibilities for the simplex method are, in the format of this paper:

Form (7.2) - Primal feasible ( $Q \cap R \neq \phi$ ),

Dual feasible ( $P \cap R \neq \phi$ ).

Form (7.3) - Primal feasible ( $Q \cap R \neq \phi$ ),

Dual infeasible ( $P \cap R = \phi$ ).

Form (7.4) - Primal infeasible ( $Q \cap R = \phi$ ),

Dual feasible ( $P \cap R \neq \phi$ ).

Form (7.5) - Primal infeasible ( $Q \cap R \neq \phi$ ),

Dual feasible ( $P \cap R = \phi$ ).

From any initial presentation (4.1) of a pair of linear programs (4.2) and (4.3), provided  $AX = B$  is a consistent system of linear equations (so that  $Q$  exists), it is possible through a finite succession of elementary pivot transformations to reach a terminal canonical representation for which the matrix (7.1) of the schema (5.3) has one of the above four forms (7.2), (7.3), (7.4), (7.5).

#### 8. HOMOGENEOUS LINEAR PROGRAMS AND TRANSPOSITION— DUALITY THEOREMS

In the pair of linear programs (4.2) and (4.3) take  $B = 0$  and  $d = 0$  to get a homogeneous linear program,

$$(8.1) \quad \text{Minimize } CX \text{ constrained by } AX = 0, X \geq 0,$$

and its dual program,

$$(8.2) \quad \text{Solve } UA + C \geq 0.$$



Take  $C < 0$ . Then  $UA + C \geq 0$  implies  $UA \geq -C > 0$ . Also,  $CX < 0$  for  $X \geq 0$  if, and only if,  $X \neq 0$ . Hence, Theorem 1 yields the following classical theorem of Gordan (and later Stiemke), which seems to have been the earliest known Transposition Duality Theorem (see [6]):

Theorem 3.  $AX = 0$  for some  $X \geq 0$  (i.e.,  $X \geq 0$  and  $\neq 0$ ) if, and only if,  $UA > 0$  for no  $U$ .

Now form the schema

$$(8.4) \quad \begin{array}{c} x_0 \quad X' \quad (\geq 0) \\ U \quad \boxed{\begin{array}{cc} -B & A \\ \hline -1 & 0 \end{array}} \quad = 0 \\ 1 \\ \geq 0 \quad \geq 0 \end{array} = \min$$

where  $A$  is a matrix, and  $-B$  an additional column. Clearly  $-UB - 1 \geq 0$  implies  $UB \leq -1 < 0$ , and  $-Bx_0 + AX' = 0$  for  $x_0 > 0$ ,  $X' \geq 0$  implies  $AX = B$  for  $X = (X'/x_0) \geq 0$ . Hence the alternatives of Theorem 1, applied to (8.4), establish the following classical theorem of Farkas concerning "convex linear dependence":

Theorem 4. If  $UB \geq 0$  for all  $U$  such that  $UA \geq 0$ , then  $B = AX$  for some  $X \geq 0$  (and conversely).

Next form the schema

$$(8.5) \quad \begin{array}{c} U \\ 1 \end{array} \begin{array}{cccc} x_1 & x_2 & x_3^+ & x_3^- \\ \hline A_1 & A_2 & A_3 & -A_3 \\ -1 & 0 & 0 & 0 \\ \hline \geq 0 & \geq 0 & \geq 0 & \geq 0 \end{array} \begin{array}{l} = 0 \\ = \min \end{array}$$

where  $-1$  denotes a row of  $-1$ 's. Observe that  $UA_1 \geq 1$  implies  $UA_1 > 0$  and that  $UA_3 \geq 0$ ,  $-UA_3 \geq 0$  implies  $UA_3 = 0$ . Let  $x_3 = x_3^+ - x_3^-$ . Then Theorem 1, applied to (8.5) establishes the general transposition theorem of T. S. Motzkin:

Theorem 5. Either  $UA_1 > 0$ ,  $UA_2 \geq 0$ ,  $UA_3 = 0$  for some  $U$  or  $A_1x_1 + A_2x_2 + A_3x_3 = 0$  for some  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $x_3$  unrestricted.

### 9. THEOREMS FOR SKEW AND DUAL LINEAR SYSTEMS

Let  $K$  be a skew-symmetric (square) matrix, i.e.,  $K^T = -K$ , and  $I$  the identity matrix of equal order. Form the homogeneous linear program and its dual:

$$(9.1) \quad \begin{array}{c} U \\ 1 \end{array} \begin{array}{ccc} x & y & z \quad (\geq 0) \\ \hline K+I & K & I \\ -1 & 0 & 0 \\ \hline \geq 0 & \geq 0 & \geq 0 \end{array} \begin{array}{l} = 0 \\ = \min \end{array}$$

where  $-1$  denotes a row of  $-1$ 's. Premultiply

$$(K + I)x + Ky + Iz = 0$$

by  $(X + Y)^T$  to get

$$(X + Y)^T K(X + Y) + (X + Y)^T I(X + Z) = 0.$$

Then, since  $(X + Y)^T K(X + Y) = 0$ ,

$$X^T X + X^T Z + Y^T X + Y^T Z = 0.$$

However, this holds for  $X \geq 0$ ,  $Y \geq 0$ ,  $Z \geq 0$  if, and only if, each term is zero; and  $X^T X = 0$  if, and only if,  $X = 0$ . Hence, the homogeneous linear program specified by the rows of (9.1) has a zero minimum and the dual program specified by the columns of (9.1) is feasible. That is, there exists some  $U^*$  satisfying the column inequalities of (9.1):

$$U(K + I) \geq 1 > 0, \quad UK \geq 0, \quad UI \geq 0.$$

This establishes the following "skew-symmetric matrix theorem" (see [6], Theorem 5):

Theorem 6. The system  $UK \geq 0$  of homogeneous linear inequalities, where  $K^T = -K$ , possesses a solution  $U^* \geq 0$  such that  $U^* + U^* K > 0$ .

Apply Theorem 6 to the matrix

$$K = \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}.$$

Then the inequality

$$[\varepsilon, Y^T] \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \geq 0$$

possesses a solution  $\varepsilon^* \geq 0, Y^* \geq 0$  such that

$$[\varepsilon^*, Y^{*T}] + [\varepsilon^*, Y^{*T}] \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} > 0.$$

This establishes the following theorem (see [6], Theorem 3) concerning the dual linear systems of schema (2.1) in Sec. 2:

Theorem 7. The column and row equation systems of the schema

$$\begin{array}{c} -Y \\ \varepsilon \begin{bmatrix} A \end{bmatrix} = X \\ = H \end{array}$$

possess solutions  $\varepsilon^* \geq 0, H^* \geq 0$  and  $X^* \geq 0, Y^* \geq 0$  such that

$$\varepsilon^* + X^{*T} > 0 \text{ and } H^* + Y^{*T} > 0.$$

Apply Theorem 7 to

$$\begin{array}{r}
 \varepsilon_1 \\
 \varepsilon_2^+ \\
 \varepsilon_2^- \\
 = 1 \quad = 2 \quad = 2
 \end{array}
 \begin{array}{|ccc}
 -Y_1 & -Y_2^+ & -Y_2^- \\
 A_{11} & A_{12} & -A_{12} \\
 A_{21} & A_{22} & -A_{22} \\
 -A_{21} & -A_{22} & A_{22}
 \end{array}
 = \begin{array}{l}
 X_1 \\
 X_2^+ \\
 X_2^-
 \end{array}$$

where  $A_{11}$  is an arbitrary submatrix of a matrix  $A$  and  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$  are the remaining submatrices. Then there exist nonnegative solutions (starred) of the column and row equation systems of (9.2) such that

$$[\varepsilon_1^*, \varepsilon_2^{+*}, \varepsilon_2^{-*}] + [X_1^{*T}, X_2^{+*T}, X_2^{-*T}] > 0$$

and

$$[H_1^*, H_2^{+*}, H_2^{-*}] + [Y_1^{*T}, Y_2^{+*T}, Y_2^{-*T}] > 0.$$

Since the sum of the last two columns of (9.2) is zero, and also the sum of the last two rows,

$$H_2^{+*} + H_2^{-*} = 0 \quad \text{and} \quad X_2^{+*} + X_2^{-*} = 0.$$

Hence  $H_2^{+*}$ ,  $H_2^{-*}$  and  $X_2^{+*}$ ,  $X_2^{-*}$ , being nonnegative, are all zero. Now set

$$\begin{array}{l}
 \text{and} \\
 \varepsilon_2 = \varepsilon_2^+ - \varepsilon_2^-, \quad H_2 = H_2^+ - H_2^- \\
 X_2 = X_2^+ - X_2^-, \quad Y_2 = Y_2^+ - Y_2^-
 \end{array}$$

to obtain the following general transposition duality theorem for dual linear systems (see [6], Theorem 6):

Theorem 8. The column and row equation systems of the schema

$$\begin{array}{r} \varepsilon_1 \\ \varepsilon_2 \end{array} \begin{array}{cc} -Y_1 & -Y_2 \\ \boxed{\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}} \\ =H_1 & =H_2 \end{array} = \begin{array}{l} X_1 \\ X_2 \end{array}$$

possess solutions

$$\varepsilon_1^* \geq 0, \varepsilon_2^* \leq 0, H_1^* \geq 0, H_2^* = 0$$

and

$$X_1^* \geq 0, X_2^* = 0, Y_1^* \geq 0, Y_2^* \leq 0$$

such that

$$\varepsilon_1^* + X_1^{*T} > 0 \quad \underline{\text{and}} \quad H_1^* + Y_1^{*T} > 0 .$$

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