Technical Report No. 7

DISPERSION ON A SPHERE: RAYLEIGH'S AND FISHER'S SOLUTIONS AND WATSON'S TEST FOR RANDOMNESS

by

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Research sponsored by the Office of Naval Research under Contract No. Nonr-3423(00) with the Research Triangle Institute

July 2, 1962

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Summary

The distribution of R, the size of the resultant of N unit vectors randomly oriented in three dimensions, was solved by Rayleigh [6] and in a quite different form by Fisher [4]. These forms are proved equivalent. An improved significance table is given for a test for randomness, proposed by Watson [9], and based on this distribution. An approximate test is suggested and compared with the exact test. The power of the exact test against a given alternative distribution, suggested by Fisher, is used to calculate a table of sample sizes.

1. Introduction

1.1 The exact distribution of R, the size of the resultant of N unit vectors randomly oriented in three dimensions, has received much attention. It can be stated in two ways which are not readily seen to be equivalent. Rayleigh [6] gave a solution, in which he used the technique of Kluyver, who had first solved the problem in two dimensions. Rayleigh's density function is

$$f(R) = \frac{2}{\pi}R \int_0^\infty \frac{1}{x^{N-1}} \sin Rx \, (\sin x)^N \, dx \qquad (1)$$

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*This work was done while the author was partially supported by the Office of Naval Research.

and he gave a method for deriving a polynomial expression for f(R). This takes different forms for the successive intervals $N \geq R \geq N-2$, $N-2 \geq R \geq N-4$, and so on until the last interval, which is $1 \geq R \geq 0$ if N is odd and $2 \geq R \geq 0$ if N is even. Rayleigh comments on the tediousness of his procedure.

1.2 Fisher [4] found the distribution from another point of view. Using a geometrical argument, he first found the distribution of X, the component of R on a chosen polar axis. For three dimensions it then happens that the distribution of R can be found very quickly. Fisher's solution for the density function is, using F(R) to distinguish from Rayleigh's form,

$$F(R) = \frac{R}{2^{N-1}(N-2)!} \sum_{s=0}^{N/2} \binom{N}{s} (-1)^s < N - R - 2s >^{N-2} \qquad (2)$$

in which the notation $\langle x \rangle$ means

$$\begin{aligned} \langle x \rangle &= x \quad \text{if} \quad x > 0 \\ \langle x \rangle &= 0 \quad \text{if} \quad x \leq 0. \end{aligned} \qquad (3)$$

Equation (2) is in effect a more compact way of stating the polynomials which arise in Rayleigh's solution: F(R) will be called Fisher's form of the density function. It forms the basis of an exact test of randomness discussed in Section 3. It is easy to use to calculate the significance points and, with an electronic computer, is very fast.

1.3 A proof that Rayleigh's integral could be integrated in the form of (2) was given by Quenouille [5] in 1947; a method of induction is used and in fact the more general problem of steps of different lengths is solved. A proof also appears in van der Pol and Bremmer [8]. This is an interesting example of modern methods, since it is based on the two-sided Laplace integral in 3 dimensions; to a reader unfamiliar with the technique or the notation, however, the proof may be somewhat difficult. It might be interesting to prove the equivalence by following and extending Rayleigh's method; this has been done in Section 2.

1.4 Limiting form as N becomes large.

The asymptotic distribution as N becomes large has many applications. Rayleigh gave this distribution as

$$f(R) \simeq \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{R^2}{N^{3/2}} e^{-3R^2/2N} \qquad (4)$$

using the limiting form of Equation (1). This distribution is re-derived from a statistical point of view in Section 4, where it is used for an approximate test for randomness when N becomes large. Rayleigh's method, and other techniques for random walk problems will be found in a very complete paper by Chandrasekhar [2].

2. Proof of the equivalence of Rayleigh's and Fisher's forms of the distribution function of R.

2.1 It will be proved that $\frac{f(R)}{R} = \frac{F(R)}{R}$ where $f(R)$, $F(R)$ are Rayleigh's and Fisher's forms, respectively defined in equations (1), (2). The proof will be given for N even. It is required to prove:

$$\frac{2}{\pi}\int_0^\infty \frac{(\sin x)^N}{x^{N-1}} \sin Rx \, dx = \frac{1}{2^{N-1}(N-2)!} \sum_{s=0}^{N} \binom{N}{s} (-1)^s \langle N-R-2s \rangle^{N-2}. \qquad (5)$$

The proof will follow the following lines, after several introductory results:

(a) The $(N-3)$-derivative of $\frac{f(R)}{R}$ is shown to be equal to the $(N-3)$-derivative of $\frac{F(R)}{R}$.

(b) Successive integrations are performed, until the two forms are separately built up and shown to be equal.

2.2 Preliminary results.

It is of advantage to state two preliminary results.

(a) It may easily be shown that, for N even,

$$\sin^N x = \frac{(-1)^{N/2}}{2^N} \left[2 \cos Nx - \binom{N}{1} 2 \cos (N-2)x \ldots + \binom{N}{\frac{N}{2}} (-1)^{N/2} \cos(N-N)x\right] \tag{6}$$

$$= \frac{(-1)^{N/2}}{2^{N-1}} \left\{ \sum_{s=0}^{N/2} [\cos(N-2s)x](-1)^s \binom{N}{s} \right\} - \frac{1}{2^N} \binom{N}{\frac{N}{2}} .$$

(b) Expansion of $(1-1)^N$ gives, for N even:

$$\sum_{s=0}^{N/2} \binom{N}{s} (-1)^s = \frac{(-1)^{N/2}}{2} \binom{N}{\frac{N}{2}} \tag{7}$$

2.3 Two Lemmas.

Consider the functions

$$H_1(R) = \int_0^\infty \frac{1}{x^p} \sin Rx \sin^N x \, dx = \int_0^\infty h_1(x, R)dx$$

and

$$H_2(R) = \int_0^\infty \frac{1}{x^p} \cos Rx \sin^N x \, dx = \int_0^\infty h_2(x, r)dx$$

where N is an even integer, p is an integer such that $2 \leq p \leq N$, and the variable R takes values in the interval $0 \leq R \leq N$.

<u>Lemma 1</u>. $H_1(R)$ and $H_2(R)$ may be integrated with respect to R by changing the order of integration.

A set of sufficient conditions for changing the order of integration, and the formal conclusion, are stated below, applied to $H_1(R)$. (See, e.g., Titchmarsh [7] pp. 49-51).

If (a) $h_1(x, R)$ is continuous in the intervals

$$0 \leq x < \infty$$
$$0 \leq R \leq N ,$$

and (b) $\int_0^\infty h_1(x, r)\, dR$ is uniformly convergent to $H_1(R)$ in the interval $0 \leq R \leq N$, then $\int_0^\infty dx \int_0^N h_1(x, R)\, dR = \int_0^N H_1(R)\, dR$.

These conditions will now be examined.

<u>Condition (a)</u>.

If $0 \leq R \leq N$, condition (a) is easily seen to be satisfied for both $h_1(x, R)$ and $h_2(x, R)$, because for $N \geq p$, $\lim_{x \to 0} \frac{\sin^N x}{x^p} \sin Rx$ and $\lim_{x \to 0} \frac{\sin^N x}{x^p} \cos Rx$ both exist.

<u>Condition (b)</u>.

Consider the integral

$$M\,(p, N) = \int_0^\infty \frac{\sin^N x}{x^p}\, dx \quad \text{for } p \text{ an integer such that } 2 \leq p \leq N.$$

This can be written as

$$M\,(p, N) = \int_0^1 \frac{\sin^N x}{x^p}\, dx + \int_0^\infty \frac{\sin^N x}{x^p}\, dx.$$

The first integral converges since $\lim_{x \to 0} \frac{\sin^N x}{x^p}$ exists.

The second integral converges, since it is dominated by $\int_1^\infty \frac{1}{x^p}\,dx$, which converges for $p > 1$. Thus $M(p, N)$ is convergent. The integrand of $H_1(R)$ satisfies the inequality

$$\left| \frac{\sin Rx}{x^p} \sin^N x \right| \leq \frac{\sin^N x}{x^p} \quad \text{for} \quad 0 < x < \infty.$$

Thus by the Weierstrass M-test, $H_1(R)$ is uniformly convergent, since $M(p, N)$ is convergent. A similar proof holds for $H_2(R)$. It follows that $H_1(R)$ and $H_2(R)$ are continuous functions of R.

Lemma 2.

$$H_1(N) = \int_0^\infty \frac{\sin Nx}{x^p} (\sin x)^N dx = 0 \quad \text{if} \quad p \quad \text{is odd.} \tag{8}$$

$$H_2(N) = \int_0^\infty \frac{\cos Nx}{x^p} (\sin x)^N dx = 0 \quad \text{if} \quad p \quad \text{is even.} \tag{9}$$

This may be shown by starting with a result of Störmer quoted in Bromwich [1], p. 518. This is

$$J = \int_0^\infty \frac{\sin Rx}{x^{N+1}} (\sin x)^N dx = \frac{\pi}{2} \quad \text{if } R > N.$$

If the integrand is compared to the function $\frac{(\sin^N x)}{x^{N+1}}$, the integral J may be proved uniformly convergent by the M-test. It is thus continuous, and satisfies the requirements for differentiation under the integral sign (Titchmarsh [7], p. 59). Hence

$$\frac{dJ}{dR} = -\int_0^\infty \frac{\cos Rx}{x^N} (\sin x)^N dx = 0 \quad \text{if } R \geq N.$$

The equality $R = N$ is included because of the continuity of this integral already established. Further differentiation gives the results of equations (8), (9) above.

2.4 <u>To prove</u>:

$$\left(\frac{d}{dR}\right)^{N-3} \frac{f(R)}{R} = \left(\frac{d}{dR}\right)^{N-3} \frac{F(R)}{R}. \qquad (10)$$

Proceeding formally, the left hand side of equation (10) is

$$\left(\frac{d}{dR}\right)^{N-3}\left(\frac{f(R)}{R}\right) = (-1)^{N/2} \frac{2}{\pi} \int_o^\infty \frac{\cos Rx}{x^2} (\sin x)^N dx \qquad (11)$$

already proved uniformly convergent, and the right hand side is

$$\left(\frac{d}{dR}\right)^{N-3}\left(\frac{F(R)}{R}\right) = \frac{-1}{2^{N-1}} \sum_{s=o}^{N/2} (-1)^s \binom{N}{s} \langle N - R - 2s \rangle. \qquad (12)$$

Writing (11) more precisely, and substituting (6) gives

$$\left(\frac{d}{dR}\right)^{N-3}\left(\frac{f(R)}{R}\right)$$

$$= \frac{1}{2^{N-2}\pi} \lim_{a \to o} \int_a^\infty \left\{\frac{\cos Rx}{x^2} \sum_{s=o}^{N/2} \binom{N}{s} (-1)^s \cos(N-2s)x\right\} dx$$

$$- \binom{N}{N/2} \frac{(-1)^{N/2}}{2^{N-1}\pi} \lim_{a \to o} \int_a^\infty \frac{\cos Rx}{x^2} dx. \qquad (13)$$

A typical term in the first integral is, omitting the numerical factor,

$$I = \lim_{a \to o} \int_a^\infty \frac{\cos Rx \cos (N-2s)x}{x^2} dx$$

$$= \lim_{a \to o} \int_a^\infty \frac{1}{x^2}\left\{1 - \sin^2\left(\frac{N+R-2s}{2}\right) - \sin^2\left(\frac{N-R-2s}{2}\right)\right\} dx \qquad (14)$$

The integral $\lim_{a \to o} \int_a^\infty \frac{\sin^2 cx}{x^2} dx = \pi/2|c|$ and is uniformly convergent for all c.

Hence (14) becomes

$$I = \lim_{a \to o} \int_a^\infty \frac{1}{x^2} dx - \pi/2 \left[\frac{N+R-2s}{2} + \frac{N-R-2s}{2}\right] \qquad (15)$$

if $0 < 2s \leq N-R$

and

$$I = \lim_{a \to 0} \int_a^\infty \frac{1}{x^2}\, dx - N/2 \left[\frac{N+R-2s}{2} - \frac{N-R-2s}{2}\right] \qquad (16)$$

if $N - R < 2s \leq N$.

Using (16), equation (13) becomes

$$\left(\frac{d}{dR}\right)^{N-3} \left(\frac{f(R)}{R}\right) = \frac{1}{2^{N-2}\pi} \lim_{a \to 0} \int_a^\infty \left(\sum_{s=0}^{N/2} (-1)^s \binom{N}{s}\right) \frac{dx}{x^2}$$

$$- \frac{1}{2^{N-1}} \sum_{s=0}^{s'} (N-2s)(-1)^s \binom{N}{s}$$

$$- \frac{1}{2^{N-1}} \sum_{s=s'+1}^{N/2} (-1)^2 \binom{N}{s} R$$

$$- \binom{N}{\frac{N}{2}} \frac{(-1)^{N/2}}{2^{N-1}} \lim_{a \to 0} \int_a^\infty \frac{\cos Rx}{x^2}\, dx \qquad (17)$$

where s' is the greatest integer less than or equal to $\frac{N-R}{2}$.

The first and last terms of (17) are, using (7)

$$\frac{(-1)^{N/2}}{2^{N-1}} \binom{N}{\frac{N}{2}} \lim_{a \to 0} \int_a^\infty \frac{1 - \cos Rx}{x^2} = \frac{(-1)^{N/2}}{2^{N-1}\pi} \binom{N}{\frac{N}{2}} \frac{\pi R}{2} . \qquad (18)$$

Using (7) the summation terms in (17) can be condensed to

$$- \frac{1}{2^{N-1}} \sum_{s=0}^{s'} (N-2s)(-1)^s \binom{N}{s} + \frac{1}{2^{N-1}} \sum_{s=0}^{s'} R(-1)^s \binom{N}{s} - \frac{(-1)^{N/2}}{2^N} \binom{N}{\frac{N}{2}} R . \qquad (19)$$

Thus (17) becomes

$$\left(\frac{d}{dR}\right)^{N-3}\left(\frac{f(R)}{R}\right) = \frac{(-1)}{2^{N-1}}\sum_{s=0}^{s'} (N-2s - R)(-1)^s \binom{N}{s}$$

which equals the right hand side of (12).

Thus using the < > notation;

$$(-1)^{N/2}\,\frac{2}{\pi}\int_0^\infty \frac{\cos Rx}{x^2}\,\sin^N x\,dx = \frac{-1}{2^{N-1}}\sum_{s=0}^{N/2} (-1)^s \binom{N}{s}< N-R-2s> \qquad (20)$$

that is,

$$\left(\frac{d}{dR}\right)^{N-3}\frac{f(R)}{R} = \left(\frac{d}{dR}\right)^{N-3}\frac{F(R)}{R}\;.$$

2.5 It is now possible to prove the main result:

$$\frac{f(R)}{R} = \frac{F(R)}{R}\;.$$

Integrate equation (19) with respect to R from R' to N:

$$(-1)^{N/2}\,\frac{2}{\pi}\int_{R'}^{N}\int_0^\infty \frac{\cos Rx}{x^2}\,\sin^N x\,dx\,dR = \frac{-1}{2^{N-1}}\int_{R'}^{N}\sum_{s=0}^{N/2} (-1)^s\binom{N}{s}< N-R-2s>\,dR\;.$$

Applying Lemma 1 to the left-hand side to change the order of integration:

$$(-1)^{N/2}\,\frac{2}{\pi}\int_0^\infty\int_{R'}^{N} \cos Rx\,dR\,\frac{\sin^N x}{x^3}\,dx = \frac{-1}{2^{N-1}}\sum_{s=0}^{N/2} (-1)\binom{N}{s}\,\frac{< N-R'-2s>^2}{2}\;.$$

When the R-integration is performed, and the upper limit inserted, the left-hand side vanishes by Lemma 2. Thus

$$(-1)^{N/2}\,\frac{2}{\pi}\int_0^\infty \sin R's\,\frac{\sin^N x}{x^3}\,dx = \frac{1}{2^{N-1}}\sum_{s=0}^{N/2} (-1)^s\binom{N}{s}\,\frac{< N'R'-2s>^2}{2}\;. \qquad (21)$$

Dropping the dash on R, one may integrate again with respect to R between R' and N; and this process is continued until $N-3$ integrations have been performed, giving:

$$(-1)^{N/2}\frac{2}{\pi}\int_0^\infty \frac{\sin Rx}{x^{N-1}}\sin^N x\,dx = \frac{1}{2^{N-1}}\frac{1}{(N-2)!}\sum_{s=0}^{N/2}(-1)^s\binom{N}{s}\langle N-R-2s\rangle^{N-2} \tag{22}$$

that is:

$$\frac{f(R)}{R} = \frac{F(R)}{R} .$$

Thus the identity of the Rayleigh and Fisher forms is established. For N an odd integer, the proof is modified for the expansion of $\sin^N x$ but follows the same lines as that above.

3. An Exact Test for Randomness.

3.1 Since $f(R)$ is known (Fisher's form will be used), one can calculate the probability that $R > R_o$,

$$\Pr(R > R_o) = \int_{R_o}^{N} f(R)dR = \alpha \tag{23}$$

and use this as a test for randomness.

Watson [9] gave this test, and included significance points R_o for $\alpha = 5\%$, and $\alpha = 1\%$, $N = 5$ to 20. The values of R_o can be calculated by inverse interpolation using the expression, which can be obtained from (23),

$$\Pr(R > R_o) = a_o(R_o) - a_1(R_o) + a_2(R_o) - \dots , \tag{24}$$

where

$$a_s(R_o) = \binom{N}{s}\frac{1}{(N-2)!}\left\langle\frac{N-R_o-2s}{2}\right\rangle^{N-1}\left\{\frac{NR_o+N-R_o-2s}{N(N-1)}\right\}$$

$$= \frac{1}{s!(N-s)!}\left\langle\frac{N-R_o-2s}{2}\right\rangle^{N-1}\left\{NR_o+N-R_o-2s\right\} . \tag{25}$$

The table has been recomputed, using an IBM 650 computer at the University of Toronto Computation Centre, and the values for four values of α are given in Table 1.

3.2 Exact Test for Randomness.

The null hypothesis is that the N given vectors are randomly distributed in 3 dimensions.

(a) Find R, the size of the vector resultant of the N vectors.

(b) Find R_o from Table 1 for appropriate N and significance level α.

(c) If $R > R_o$, reject the hypothesis at significance level α.

Table 1

α%	10	5	2	1
N 3				
4	2.848	3.103	3.351	3.490
5	3.186	3.501	3.827	4.023
6	3.497	3.853	4.239	4.480
7	3.784	4.178	4.611	4.888
8	4.050	4.480	4.955	5.263
9	4.299	4.762	5.276	5.612
10	4.535	5.028	5.579	5.940
11	4.760	5.281	5.867	6.252
12	4.974	5.523	6.140	6.548
13	5.179	5.754	6.402	6.832
14	5.376	5.976	6.653	7.103
15	5.567	6.190	6.896	7.365
16	5.751	6.397	7.130	7.618
17	5.929	6.598	7.356	7.863
18	6.103	6.793	7.576	8.100
19	6.271	6.982	7.790	8.330
20	6.435	7.166	7.998	8.554
21	6.595	7.346	8.200	8.773
22	6.751	7.521	8.398	8.986
23	6.904	7.692	8.591	9.194
24	7.053	7.860	8.780	9.397
25	7.199	8.023	8.965	9.597

4. An Approximate Test for Randomness.

4.1 Watson, in the same paper, also observed that Rayleigh's approximation (Section 1.4) for the density of R when N is large shows that $\frac{3}{N}R^2$ is distributed as χ_3^2 and hence suggested the test in this section. It might be interesting to derive this result from a more statistical viewpoint, which can be extended to the corresponding test in 2 dimensions (see, e.g. Curray [3]) and in fact to a similar problem in higher dimensions.

Let x_i, y_i, z_i be the components of the i^{th} vector, in a rectangular system whose origin is the centre of the sphere. Using polar coordinates, $z = \cos\theta$, $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$. The distribution of z_i has mean 0 and variance

$$E(z_i^2) = \int_0^{\pi}\int_0^{2\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi = \frac{1}{3} .$$

By symmetry the distributions of y and z are the same, though they are not mutually independent.

As N becomes large, an application of the Central Limit Theorem gives the result that $X = \sum_{i=1}^{N} x_i$ will tend to a distribution which is $N(0, \frac{N}{3})$ and similarly for Y, Z, and these distributions will approach independence.

Thus as $N \to \infty$, the distribution of $\frac{3}{N}X^2$, $\frac{3}{N}Y^2$, $\frac{3}{N}Z^2$ are each χ_1^2 and are independent.

Thus $\frac{3}{N}R^2 = \frac{3}{N}(X^2 + Y^2 + Z^2) \quad \chi_3^2$ approximately. (26)

The approximate test is now given.

4.2 Approximate Test for Randomness.

The test is of the same null hypothesis of Section 3.2

(a) Find R as in Section 3.2.

(b) Find R_0 from the equation $R_0^2 = \frac{N}{3}\chi_3^2(\alpha)$ where $\chi_3^2(\alpha)$ is the α-significance level of the χ_3^2 distribution, upper tail.

4.3 Merits of the Approximate Test.

The true significance levels, α', of values of R_o calculated as in Section 4, have been calculated using (24). In Table 2 are given the correct values of R_o, the values of R_o using the approximation (called R_o'), and the true level α', for several values of N.

It can be seen that the approximation gives R_o too large, and thus $\alpha' < \alpha$; as N becomes larger, however, $\alpha' \rightarrow \alpha$.

Table 2

	$\alpha = 10\%$			$\alpha = 5\%$			$\alpha = 2\%$		
N	R_o	R_o'	$\alpha'\%$	R_o	R_o'	$\alpha'\%$	R_o	R_o'	$\alpha'\%$
4	2.848	2.887	9.09	3.103	3.228	3.279	3.351	3.662	0.4
10	4.535	4.564	9.63	5.028	5.104	4.45	5.579	5.726	1.5
25	7.200	7.217	9.859	8.623	8.070	4.79	8.964	9.054	1.82

5. Power of test for randomness

Suppose the test for randomness is used when the vectors in fact come from the distribution suggested by Fisher [4] in his paper, and used in examining paleomagnetic data (see, e.g. Watson and Irving [10].) In this distribution the density function of the spherical polar coordinates θ, ϕ of the vectors is

$$g(\theta, \phi) = \frac{\kappa}{2 \sinh \kappa} \exp(\kappa \cos \theta) \sin \theta .$$

The distribution of R is now

$$f_{\kappa}(R) = \left(\frac{\kappa}{\sinh \kappa}\right)^{N} \frac{\sinh \kappa R}{\kappa R} f(R) . \qquad (27)$$

When $\kappa = 0$, the distribution is random: $f_{\kappa}(R)$, from (27) then becomes $f(R)$.

The hypothesis of randomness will now be rejected with a probability $\lambda(N, \kappa, \alpha)$ given by

$$\lambda(N, \kappa, \alpha) = \int_{R_o}^{N} f_{\kappa}(R)\, dR \qquad (28)$$

where R_o is the appropriate significance point for the test of size α, for the value of N, from Table 1. This gives the power of the test against the above alternative, and can be regarded as "detecting" κ with probability $\lambda(N, \kappa, a)$. A practical question is to ask what N is needed to detect a given κ with a probability λ, when the test used has size a. This has been calculated by finding the value of λ from (28), and is presented in Table 3. The size of the randomness test being applied is given first (a), then the κ it is desired to detect and the Table gives the N required for several values of λ. For large values of N, Rayleigh's approximation (4) has been used in (28), both for R_o and for $f_{\kappa}(R)$: this gives λ slightly too small. Thus in the higher values of the table, it is possible that any N-value is 1 too large, a conservative error.

ACKNOWLEDGMENT: I acknowledge with thanks the assistance of Professor J.H.H. Chalk of this University in preparing the final form of the proof of Section 2, and the general supervision of Professor G.S. Watson.

Number N, in sample, needed to detect a value of k, with probability λ, when applying Watson's test for randomness. The table is presented for 4 values of a, the size of the test of randomness; and for 4 values of λ. Note: The table does not include values of N greater than 70.

Size of test α	k	Detection prob. λ% 90	95	98	99	Size of test α	k	Detection prob. λ% 90	95	98	99
10%	1.0	38	47	58	66	5%		46	56	67	
	1.2	28	34	41	47			33	40	48	54
	1.4	21	26	31	35			25	30	36	41
	1.6	17	20	25	28			20	24	30	33
	1.8	14	17	20	23			17	20	24	27
	2.0	12	14	17	20			14	17	20	22
	2.2	10	12	15	17			12	14	17	19
	2.4	9	11	13	15			11	13	15	17
	2.6	8	10	12	13			10	11	13	15
	2.8	7	9	10	12			9	10	12	13
	3.0	7	8	9	11			8	9	11	12
	3.2	7	7	9	10			8	9	10	11
	3.4	6	7	8	9			7	8	10	10
	3.6	6	7	8	8			7	8	9	10
	3.8	5	6	7	8			6	7	8	9
	4.0	5	6	7	8			6	7	8	9
	4.2	5	6	6	7			6	7	8	8
	4.4	5	5	6	7			6	6	7	8
	4.6	5	5	6	7			5	6	7	8
	4.8	4	5	6	6			5	6	7	7
	5.0	4	5	6				5	6	6	7

Size of test α	k	Detection prob. $\lambda\%$ 90	95	98	99	Size of test α	k	Detection prob. $\lambda\%$ 90	95	98	99
2%	1.0	56	66			1%		63			
	1.2	40	48	56	63			46	53	62	69
	1.4	30	36	43	48			35	41	48	52
	1.6	24	29	36	38			27	32	38	42
	1.8	20	23	29	31			23	26	31	34
	2.0	17	20	23	26			20	22	26	30
	2.2	15	17	20	22			17	19	22	26
	2.4	13	15	18	20			15	17	20	22
	2.6	12	14	16	17			13	15	17	19
	2.8	11	12	14	16			12	14	16	17
	3.0	10	11	13	14			11	13	14	16
	3.2	9	10	12	13			10	12	13	15
	3.4	9	10	11	12			10	11	12	13
	3.6	8	9	10	11			9	10	12	13
	3.8	8	9	10	11			9	10	11	12
	4.0	7	8	9	10			8	9	10	11
	4.2	7	8	9	10			8	9	10	11
	4.4	7	7	8	9			7	8	9	10
	4.6	6	7	8	9			7	8	9	10
	4.8	6	7	8	8			7	8	9	9
	5.0	6	7	8	8			7	7	8	9

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