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ON RECOVERABLE INTERNAL ENERGY IN LINEAR VISCOELASTICITY

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S. Breuer and E. T. Onat

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On Recoverable Internal Energy in Linear Viscoelasticity*

by

S. Breuer** and E. T. Onat***

Summary. A linear viscoelastic solid is subjected to a given deformation history. A portion of the work done by the stresses during this deformation is converted into heat, while the remaining portion increases the internal energy (per unit volume) of the solid. A fraction of the increase in internal energy can be recovered by subjecting the solid to an appropriate future deformation. The paper is concerned with the question of maximizing the recoverable energy by means of an optimum future deformation.

It is shown that the determination of the optimum deformation requires the solution of an integral equation of the Wiener-Hopf type. This equation is solved in the case where the relaxation modulus is given as a sum of exponential functions. The maximum recoverable internal energy is then expressed as a functional of second degree of the given deformation history.

It is observed that the maximum recoverable energy provides a lower bound to the internal energy of the solid. It is hoped that use could be made of the concept of maximum recoverable energy in studies concerned with the thermodynamics of linear viscoelasticity.

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** Research Assistant, Division of Applied Mathematics, Brown University, Providence, R. I.

*** Professor of Engineering, Brown University, Providence, R.I.

1. Introduction. Consider isothermal deformations of a linear viscoelastic solid in simple tension or compression. Let $\sigma(t)$ and $\epsilon(t)$ denote, respectively, the stress and infinitesimal strain components at time t . We adopt the stress-strain relation in the form [1]*

$$\sigma(t) = \int_{-\infty}^t G(t-\tau) \frac{d\epsilon(\tau)}{d\tau} d\tau, \quad (1)$$

where $G(t)$ is the relaxation modulus, which need be defined only for non-negative values of its argument.

Let the material be subjected to a given deformation $\epsilon(t)$ in the past, i.e. in the time interval

$$-\infty < t < 0,$$

where $t=0$ refers to the present instant. The work done by the stress in this interval is given by

$$W = \int_{-\infty}^0 \sigma(\tau) \frac{d\epsilon(\tau)}{d\tau} d\tau. \quad (2)$$

By using the constitutive equation (1) and by extending the range of definition of $G(t)$ with the relation

$$G(-t) = G(t), \quad (3)$$

(2) can be written in the following form,

* Numbers in square brackets refer to the bibliography at the end of the paper.

$$\overset{0}{W} = \frac{1}{2} \int_{-\infty}^0 \int_{-\infty}^0 G(t-\tau) v(t) v(\tau) dt d\tau, \quad (4)$$

where $v(t)$ denotes the rate of strain at time t ,

$$v(t) = \frac{de}{dt}. \quad (5)$$

It was shown in [2] that the second order functional $\overset{0}{W}$ is positive definite if $G(t)$ is a steadily decreasing function of time (for positive times) which is convex from below and tends to a non-negative asymptotic value as $t \rightarrow \infty$.* In the present paper we shall be concerned with the relaxation moduli which possess the above mentioned properties.

Since viscoelastic solids are dissipative, a certain portion of the work $\overset{0}{W}$ will be converted into heat during the course of the deformation, while the remaining portion will increase the internal energy of the solid. We now consider the possibility of converting the internal energy of the solid into useful work by purely mechanical means. For this purpose, we subject the solid to further deformation in the future, i.e. in the time interval

$$0 \leq t < \infty,$$

* Less restrictive conditions on $G(t)$ which still ensure positive definiteness of $\overset{0}{W}$ have been discussed in [3].

and denote by $\int_0^{\infty} W$ the work done by the stress in this interval.

If the future deformation is chosen in such a way that

$$\int_0^{\infty} \sigma(\tau)v(\tau) d\tau < 0, \quad (6)$$

then it is said that useful work is extracted from the material in the interval $0 \leq t < \infty$. In other words, deformations in $0 \leq t < \infty$ satisfying (6) enable one to recover a portion of the internal energy of the solid which has been subjected to a given deformation in $-\infty < t < 0$, the recovered portion of the internal energy, E_r , being given by the obvious relation

$$E_r = - \int_0^{\infty} W. \quad (7)$$

The present paper is concerned with the following problem: Given $v(t)$ in the interval $-\infty < t < 0$, what function $v(t)$, defined in the interval $0 \leq t < \infty$, maximizes the recovered internal energy E_r . The corresponding maximum value of the recoverable internal energy will be denoted by $\int_{-\infty}^0 E_{rm}$.

It will be seen that $\int_{-\infty}^0 E_{rm}$ will depend on the relaxation modulus of the material and on the given strain history in $-\infty < t < 0$. If one assigns the value zero to the internal energy of the unstressed virgin material, then $\int_{-\infty}^0 E_{rm}$ will provide a lower bound to the internal energy of the solid at the termination of the application of the given deformation in $-\infty < t < 0$.

This remark follows from the observation that the optimum extraction process in $0 \leq t < \infty$ will be accompanied, in general, by further dissipation. It is hoped that the availability of a lower bound to the internal energy of solids may be useful in the study of thermodynamics of linear viscoelastic solids.

In section 2 it is shown that the above problem of maximization leads to an integral equation of the Wiener-Hopf type for $v(t)$ in $0 \leq t < \infty$. Section 3 is devoted to the solution of the integral equation in the case where $G(t)$ is given as a sum of exponentials. It will be seen that in this case, the optimum straining which maximizes E_r involves a sudden application of strain at $t=0$, followed by strain rates which decrease in magnitude exponentially with increasing time. Section 4 is concerned with the evaluation of E_{rm}^0 and comparison of it with the free energy of the solid.

It may be appropriate to note here that although the discussion in the present paper is restricted to the case of simple tension or compression, there is no essential difficulty in extending the present concepts and analysis to the general deformations of a viscoelastic solid.

2. Maximum recoverable internal energy. Basic integral equation. Using the definition of work, the recovered internal energy, E_r , can be expressed in the following manner,

$$E_r = - \frac{\int_{-\infty}^{\infty} W}{0} = \frac{0}{-\infty} - \frac{\int_{-\infty}^{\infty} W}{-\infty}, \quad (8)$$

where, according to (4), $\frac{0}{-\infty}$ depends entirely upon the given deformation in $-\infty < t < 0$, but

$$\frac{\int_{-\infty}^{\infty} W}{-\infty} = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t-\tau)v(t)v(\tau)dt d\tau, \quad (9)$$

and hence E_r , depend upon the entire history of strain. In view of (8) the problem of maximizing E_r reduces to the following problem: Given $v(t)$ in the interval $-\infty < t < 0$, to find $v(t)$ in $0 \leq t < \infty$ which minimizes the total work $\frac{\int_{-\infty}^{\infty} W}{-\infty}$, i.e. (9).

Let $v(t)$ be the function minimizing (9). Consider the set of functions given by

$$u(t) = v(t) + \epsilon \omega(t), \quad -\infty < t < \infty, \quad (10)$$

where ϵ is a real parameter and $\omega(t)$ is arbitrary except that

$$\omega(t) \equiv 0, \quad \text{for } -\infty < t < 0. \quad (11)$$

Denoting by $\frac{\int_{-\infty}^{\infty} W[u]}{-\infty}$ the work done by the stress during the time interval $(-\infty, \infty)$ in the course of the deformation characterized by $u(t)$, we obtain from an expression similar to (9),

$$\frac{\int_{-\infty}^{\infty} W[u]}{-\infty} - \frac{\int_{-\infty}^{\infty} W[v]}{-\infty} = \epsilon^2 \frac{\int_{-\infty}^{\infty} W[\omega]}{-\infty} + \epsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t-\tau)v(t)\omega(\tau)dt d\tau. \quad (12)$$

Since $\int_{-\infty}^{\infty} W[\omega]$ is positive definite for the class of relaxation moduli considered in this paper, (12) shows that $\int_{-\infty}^{\infty} W[v]$ will be a minimum provided that

$$\int_{-\infty}^{\infty} \omega(t) \int_{-\infty}^{\infty} G(t-\tau)v(\tau)d\tau dt = 0. \quad (13)$$

On the other hand, since $\omega(t)$ is arbitrary for $t \geq 0$ and zero otherwise, we must have, if (13) is to hold,

$$\int_{-\infty}^{\infty} G(t-\tau)v(\tau)d\tau = 0, \quad 0 \leq t < \infty. \quad (14)$$

Moreover, since $v(\tau)$ is known for $-\infty < \tau < 0$, (14) reduces to

$$f(t) = \int_0^{\infty} G(t-\tau)v(\tau)d\tau, \quad 0 \leq t < \infty, \quad (15)$$

where $f(t)$ is a known function, defined by

$$f(t) = - \int_{-\infty}^0 G(t-\tau)v(\tau)d\tau, \quad 0 \leq t < \infty. \quad (16)$$

Equation (15) is the desired integral equation for the unknown function $v(t)$ in $0 \leq t < \infty$ which maximizes E_p , the recovered internal energy. (15) is of the Wiener-Hopf type and its solution may be arrived at by the Wiener-Hopf technique [4]. In the present study, however, we shall deal with a particular class of relaxation moduli for which the solution of (15) may be obtained by more elementary means.

Before going on to the discussion of this case, we shall establish a result concerning $\int_{-\infty}^0 E_{rm}$. For this purpose,

let us, by using (3), rewrite (8) in the following form,

$$E_r = - \int_0^{\infty} v(t) \int_{-\infty}^0 G(t-\tau)v(\tau)d\tau dt - \frac{1}{2} \int_0^{\infty} \int_0^{\infty} G(t-\tau)v(t)v(\tau)dtd\tau. \quad (17)$$

If we now regard $v(t)$, $0 \leq t < \infty$, as the solution of (15), then the left hand side of (17) becomes $\int_{-\infty}^0 E_{rm}$, while the first term

on the right hand side, in view of (14), takes the form,

$$\int_0^{\infty} v(t) \int_0^{\infty} G(t-\tau)v(\tau)d\tau dt ,$$

so that (17) reduces to

$$\int_{-\infty}^0 E_{rm} = \frac{1}{2} \int_0^{\infty} \int_0^{\infty} G(t-\tau)v(t)v(\tau)dtd\tau . \quad (18)$$

Equation (18) shows that $\int_{-\infty}^0 E_{rm}$ is a positive definite

functional of the optimum extraction process. This implies that for the class of materials considered in this paper it is always possible to extract useful work from a body which is subjected to non-trivial (i.e. non-zero) deformations in the past.

We also note, for future reference, that (18) together with (15) yields

$$2 \int_{-\infty}^0 \dot{E}_{rm} = \int_0^{\infty} f(t)v(t)dt, \quad (19)$$

where $v(t)$, $0 \leq t < \infty$, is the solution of (15).

3. Solution of the integral equation. We shall restrict our attention in this section to the class of relaxation moduli defined by

$$G(t) = \sum_{i=1}^N c_i e^{-a_i |t|}, \quad (20)$$

where c_i and a_i are positive constants and $a_i < a_{i+1}$. For convenience, we define

$$K_i = \int_{-\infty}^0 e^{a_i t} v(t) dt, \quad i=1,2,\dots,N, \quad (21)$$

where $v(t)$ is the given strain rate history defined in $-\infty < t < 0$. By combining (20) and (21) with (16), the basic integral equation (15) can be written in the following form,

$$-\sum_{i=1}^N c_i K_i e^{-a_i t} = \int_0^{\infty} G(t-\tau)v(\tau)d\tau, \quad 0 \leq t < \infty, \quad (22)$$

where $G(t)$ is given by (20).

In seeking a solution of (22) we may adopt the following reasoning. Suppose $N=1$ in (20). Then, as it is well known, the viscoelastic material is a Maxwell body, represented by a combination in series of a spring and a dashpot. The internal energy of the material is stored in the spring, and it can be recovered by a sudden displacement which restores the spring to its

original length. Mathematically, this observation implies that the solution of (22), in the case of $N=1$, is of the form

$$v(t) = A \delta(t), \quad (23)$$

where $\delta(t)$ is the Dirac delta function* and A is a constant related to the past deformation of the body. On the other hand, in the case of $N=2$, (22) becomes

$$\begin{aligned} -c_1 K_1 e^{-a_1 t} - c_2 K_2 e^{-a_2 t} &= \int_0^t [c_1 e^{-a_1(t-\tau)} + c_2 e^{-a_2(t-\tau)}] v(\tau) d\tau \\ &+ \int_t^\infty [c_1 e^{a_1(t-\tau)} + c_2 e^{a_2(t-\tau)}] v(\tau) d\tau. \quad (24) \end{aligned}$$

The integrals on the right hand side of (24) may be eliminated by repeated differentiation of (24) with respect to t . Carrying out this procedure one obtains a differential equation for $v(t)$ which in turn leads to the solution of (24) of the form

$$v(t) = B\delta(t) + Ce^{-\lambda t}, \quad (25)$$

where B , C and λ are suitable constants.

The form of (25) now suggests that in the general case of (22) we may try a solution of the form

$$v(t) = A_1 \delta(t) + \sum_{j=2}^N A_j e^{-b_j t}, \quad (26)$$

*For the present purposes the Dirac delta function is defined in the following manner:

$$\delta(t) = 0, \text{ when } t \neq 0, \text{ and } \int_0^\infty \delta(t) dt = 1.$$

where the constants A_k , b_j are to be determined by further considerations. For this purpose, we introduce (26) into (22) and obtain

$$\begin{aligned}
 - \sum_{i=1}^N c_i K_i e^{-a_1 t} &= A_1 \sum_{i=1}^N c_i e^{-a_1 t} - \sum_{j=2}^N A_j \sum_{i=1}^N \frac{c_i e^{-a_1 t}}{a_1 - b_j} \\
 &+ 2 \sum_{j=2}^N A_j e^{-b_j t} \sum_{i=1}^N \frac{c_i a_1}{a_1^2 - b_j^2}, \quad (27)
 \end{aligned}$$

where b_j are subject to the obvious requirements

$$b_j > 0, \quad b_j \neq a_k. \quad (28)$$

We now demand that

$$\sum_{i=1}^N \frac{c_i a_1}{a_1^2 - b_j^2} = 0, \quad j=2,3,\dots,N, \quad (29)$$

i.e. we desire to choose b_j^2 as the $N-1$ roots of the equation

$$\sum_{i=1}^N \frac{c_i a_1}{a_1^2 - x} = 0. \quad (30)$$

It is clear that b_j^2 determined from (30) will satisfy

$$b_j \neq a_k. \quad (31)$$

- Moreover, it can easily be shown that (30) possesses $N-1$ distinct, positive roots, so that (28) will be fulfilled if b_j are taken as

$$b_j = + \sqrt{x_j}, \quad j=2,3,\dots,N, \quad (32)$$

where x_j are the roots of (30).

Using (29) and observing that $e^{-a_1 t}$ are linearly independent, we obtain from (27) the following system of equations for A_1 ,

$$-K_1 = A_1 - \sum_{j=2}^N \frac{A_j}{a_1 - b_j}, \quad i=1,2,\dots,N. \quad (33)$$

It is shown in the appendix that the solution of (33) is given by

$$A_1 = - \sum_{i=1}^N K_i \frac{\prod_{s=2}^N (a_1 - b_s)}{\prod_{\substack{r=1 \\ r \neq 1}}^N (a_1 - a_r)}, \quad (34)$$

$$A_j = - \frac{1}{\prod_{\substack{q=2 \\ q \neq j}}^N (b_q - b_j)} \sum_{i=1}^N K_i \frac{\prod_{\substack{p=1 \\ p \neq 1}}^N (a_p - b_j) \prod_{s=2}^N (a_1 - b_s)}{\prod_{\substack{r=1 \\ r \neq 1}}^N (a_1 - a_r)}, \quad j=2,3,\dots,N. \quad (35)$$

Equation (26), supplemented by (29), (32), (34) and (35), determines the solution of (22) explicitly.

Equation (26), in conjunction with (5), reveals that the optimum loading program consists of a sudden displacement followed, for $N > 1$, by the straining program characterized by the sum of exponentials in (26).

4. Evaluation of the maximum recoverable internal energy.

Comparison with free energy. We may start the evaluation of E_{rm}^0 by combining (19) and (26) with (16), (20) and (21):

$$2 E_{rm}^0 = - \int_0^{\infty} [A_1 \delta(t) + \sum_{j=2}^N A_j e^{-b_j t}] \sum_{i=1}^N c_i K_i e^{-a_i t} dt. \quad (36)$$

Carrying out the integration in (36) we obtain

$$2 E_{rm}^0 = -A_1 \sum_{i=1}^N c_i K_i - \sum_{i=1}^N c_i K_i \sum_{j=2}^N \frac{A_j}{a_i + b_j}. \quad (37)$$

Now we make use of (34) and (35) to express (37) in terms of K_i .

After some manipulations, given in the appendix, we obtain

$$E_{rm}^0 = \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij} K_i K_j, \quad (38)$$

where

$$\alpha_{ii} = \frac{a_i c_i^2}{2 \sum_{p=1}^N c_p a_p} \frac{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i + a_r)^2}{\prod_{s=2}^N (a_i + b_s)^2} > 0, \quad (39)$$

and

$$\alpha_{ij} = \alpha_{ji} = \frac{c_i c_j a_i a_j (a_i + a_j)}{\sum_{p=1}^N c_p a_p} \frac{\prod_{\substack{r=1 \\ r \neq i, j}}^N (a_i + a_r)(a_j + a_r)}{\prod_{s=2}^N (a_i + b_s)(a_j + b_s)} > 0, \quad i \neq j. \quad (40)$$

In view of (21), we see that $\overset{0}{E}_{-∞}^{rm}$ is the following second order functional of the given strain history in $-∞ < t < 0$,

$$\overset{0}{E}_{-∞}^{rm} = \int_{-∞}^0 \int_{-∞}^0 \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij} e^{a_i t + a_j \tau} v(t)v(\tau) dt d\tau. \quad (41)$$

For the class of materials considered in this section, it is possible to determine the elastic energy stored in the material by regarding the solid as a network of linear elastic and viscous elements (Staverman and Schwarzl [5], Bland [6] and Hunter [7]). These authors deduced, in independent works, the following expression for the elastic energy stored per unit volume of the material subjected to simple tension or compression,

$$\overset{0}{F}_{-∞} = \frac{1}{2} \int_{-∞}^0 \int_{-∞}^0 G(t+\tau)v(t)v(\tau) dt d\tau, \quad (42)$$

where $\overset{0}{F}_{-∞}$ denotes the elastic energy stored at the termination

of the given deformation $v(t)$ applied in $-∞ < t < 0$, and where $G(t)$ is given by (20). Since the deformation has taken place under constant temperature, it is appropriate to interpret $\overset{0}{F}_{-∞}$, from

the point of view of thermodynamics, as the free energy in the sense of Helmholtz.

We note that (42) can be written in the following form by making use of (20) and (21),

$$\frac{0}{F} = \frac{1}{2} \sum_{i=1}^N c_i K_i^2 . \quad (43)$$

Since the extraction of internal energy discussed in the previous sections entails dissipation due to viscous effects we expect the following inequality to hold,

$$\frac{0}{E_{rm}} \leq \frac{0}{F} . \quad (44)$$

Indeed, as shown in the appendix, by using (38), (39), (40) and (43), we obtain

$$\frac{0}{F} - \frac{0}{E_{rm}} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij} (K_i - K_j)^2 . \quad (45)$$

Since $\alpha_{ij} > 0$ [cf. (40)], (45) shows that (44) is fulfilled.

Appendix

Our first task is to solve the system (33), which we repeat here for convenience

$$-K_1 = A_1 - \sum_{j=2}^N \frac{A_j}{a_1 - b_j}, \quad i=1,2,\dots,N. \quad (46)$$

Denote the determinant of the system (46) by Δ . Then

$$\Delta = (-1)^{N-1} \begin{vmatrix} 1 & \frac{1}{a_1 - b_2} & \dots & \frac{1}{a_1 - b_N} \\ 1 & \frac{1}{a_2 - b_2} & \dots & \frac{1}{a_2 - b_N} \\ \dots & \dots & \dots & \dots \\ 1 & \frac{1}{a_N - b_2} & \dots & \frac{1}{a_N - b_N} \end{vmatrix}. \quad (47)$$

Next, subtract the first row of the determinant in (47) from each succeeding row to obtain

$$\Delta = (-1)^{N-1} \frac{\prod_{i=2}^N (a_1 - a_i)}{\prod_{j=2}^N (a_1 - b_j)} \Delta_{2,N}, \quad (48)$$

where

$$\Delta_{k,N} = \begin{vmatrix} \frac{1}{a_k - b_k} & \frac{1}{a_k - b_{k+1}} & \dots & \frac{1}{a_k - b_N} \\ \frac{1}{a_{k+1} - b_k} & \frac{1}{a_{k+1} - b_{k+1}} & \dots & \frac{1}{a_{k+1} - b_N} \\ \dots & \dots & \dots & \dots \\ \frac{1}{a_N - b_k} & \frac{1}{a_N - b_{k+1}} & \dots & \frac{1}{a_N - b_N} \end{vmatrix}, \quad 2 \leq k \leq N, \quad (49)$$

$$\Delta_{N,N} = \frac{1}{a_N - b_N} \quad (49)$$

In (49) we now subtract the first row from each succeeding row to obtain

$$\Delta_{k,N} = \frac{\prod_{r=k+1}^N (a_k - a_r)}{\prod_{p=k}^N (a_k - b_p)} \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{1}{a_{k+1} - b_k} & \frac{1}{a_{k+1} - b_{k+1}} & \dots & \frac{1}{a_{k+1} - b_N} \\ \frac{1}{a_{k+2} - b_k} & \frac{1}{a_{k+2} - b_{k+1}} & \dots & \frac{1}{a_{k+2} - b_N} \\ \dots & \dots & \dots & \dots \\ \frac{1}{a_N - b_k} & \frac{1}{a_N - b_{k+1}} & \dots & \frac{1}{a_N - b_N} \end{vmatrix} \quad (50)$$

Finally, in the determinant in (50) we subtract the first column from each succeeding column to obtain

$$\Delta_{k,N} = \frac{\prod_{r=k+1}^N (a_k - a_r) \prod_{s=k+1}^N (b_s - b_k)}{\prod_{p=k}^N (a_k - b_p) \prod_{q=k+1}^N (a_q - b_k)} \Delta_{k+1,N} \quad (51)$$

Hence, by (48) and (51) we conclude

$$\Delta = (-1)^{N-1} \frac{\prod_{i=1}^{N-1} \prod_{j=i+1}^N (a_i - a_j) \prod_{p=2}^{N-1} \prod_{q=p+1}^N (b_q - b_p)}{\prod_{r=1}^N \prod_{s=2}^N (a_r - b_s)} \neq 0 \quad (52)$$

Next, from (46) we have

$$A_1 = \frac{(-1)^N}{\Delta} \begin{vmatrix} K_1 & \frac{1}{a_1-b_2} & \frac{1}{a_1-b_3} & \dots & \frac{1}{a_1-b_N} \\ K_2 & \frac{1}{a_2-b_2} & \frac{1}{a_2-b_3} & \dots & \frac{1}{a_2-b_N} \\ \dots & \dots & \dots & \dots & \dots \\ K_N & \frac{1}{a_N-b_2} & \frac{1}{a_N-b_3} & \dots & \frac{1}{a_N-b_N} \end{vmatrix} \quad (53)$$

Now in the determinant in (53), the coefficient of K_1 is seen to be $\Delta_{2,N}$ defined in (49), while the coefficient of K_j , for $j > 1$, is equal to the expression obtained from $-\Delta_{2,N}$ upon replacing a_j by a_1 . Using these facts we obtain

$$A_1 = - \sum_{i=1}^N K_i \frac{\prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)}, \quad (54)$$

which establishes (34).

Similarly, (46) gives,

$$A_j = \frac{(-1)^{N-1}}{\Delta} \begin{vmatrix} 1 & \frac{1}{a_1-b_2} & \cdots & \frac{1}{a_1-b_{j-1}} & K_1 & \frac{1}{a_1-b_{j+1}} & \cdots & \frac{1}{a_1-b_N} \\ 1 & \frac{1}{a_2-b_2} & \cdots & \frac{1}{a_2-b_{j-1}} & K_2 & \frac{1}{a_2-b_{j+1}} & \cdots & \frac{1}{a_2-b_N} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{a_N-b_2} & \cdots & \frac{1}{a_N-b_{j-1}} & K_N & \frac{1}{a_N-b_{j+1}} & \cdots & \frac{1}{a_N-b_N} \end{vmatrix},$$

j=2,3,\dots,N.

(55)

In the determinant in (55), the coefficient of K_i , $i=1,2,\dots,N$, is given by $(-1)^{i+j} \Delta^{i,j}$, where $\Delta^{i,j}$ is equal to the determinant obtained from $(-1)^{N-1} \Delta$ upon striking its i th row and j th column, Δ being given by (47). Using this observation we find, after some calculations,

$$A_j = - \frac{1}{\prod_{\substack{q=2 \\ q \neq j}}^N (b_q - b_j)} \sum_{i=1}^N K_i \frac{\prod_{\substack{p=1 \\ p \neq i}}^N (a_p - b_j) \prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)}, \quad j=2,3,\dots,N.$$

(56)

Thus (35) is established.

In order to establish (38), we first show that the numbers b_j^2 defined in (29), necessarily satisfy the relations

$$\frac{c_i a_i}{\sum_{p=1}^N c_p a_p} = \frac{\prod_{s=2}^N (a_i^2 - b_s^2)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i^2 - a_r^2)}, \quad i=1,2,\dots,N. \quad (57)$$

To prove (57), define D_i , $i=1,2,\dots,N$, by means of

$$c_1 a_1 = D_1 \frac{\prod_{j=2}^N (a_1^2 - b_j^2)}{\prod_{\substack{r=1 \\ r \neq 1}}^N (a_1^2 - a_r^2)}, \quad i=1,2,\dots,N. \quad (58)$$

(57) will be proved once we show that (29) implies

$$D_1 = D_2 = \dots = D_N = \sum_{p=1}^N c_p a_p. \quad (59)$$

We expand the coefficient of D_1 in (58) as follows:

$$\frac{\prod_{j=2}^N (a_1^2 - b_j^2)}{\prod_{\substack{r=1 \\ r \neq 1}}^N (a_1^2 - a_r^2)} = 1 - \sum_{\substack{p=1 \\ p \neq 1}}^N \frac{\prod_{j=2}^N (a_p^2 - b_j^2)}{\prod_{\substack{s=1 \\ s \neq p}}^N (a_p^2 - a_s^2)}, \quad (60)$$

so that in view of (58), (60) states that

$$\frac{c_1 a_1}{D_1} = 1 - \sum_{\substack{p=1 \\ p \neq 1}}^N \frac{c_p a_p}{D_p}, \quad (61)$$

i.e.

$$\sum_{i=1}^N \frac{c_i a_i}{D_i} = 1. \quad (62)$$

Equation (62) gives one relation connecting the D_i .

In order to find $N-1$ more relations, we divide the i th equation

of (58) by $D_i(a_i^2 - b_j^2)$, for some j , to obtain

$$\frac{1}{D_i} \frac{c_i a_i}{a_i^2 - b_j^2} = \frac{\prod_{\substack{k=2 \\ k \neq j}}^N (a_i^2 - b_k^2)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i^2 - a_r^2)}, \quad i=1,2,\dots,N; \quad j=2,3,\dots,N. \quad (63)$$

The right hand member of (63) expands as follows,

$$\frac{\prod_{\substack{k=2 \\ k \neq j}}^N (a_i^2 - b_k^2)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i^2 - a_r^2)} = - \sum_{\substack{s=1 \\ s \neq i}}^N \frac{\prod_{\substack{k=2 \\ k \neq j}}^N (a_s^2 - b_k^2)}{\prod_{\substack{r=1 \\ r \neq s}}^N (a_s^2 - a_r^2)} = - \sum_{\substack{s=1 \\ s \neq i}}^N \frac{1}{D_s} \frac{c_s a_s}{a_s^2 - b_j^2}, \quad (64)$$

the last equality following from (63). Finally, (62) and the combination of (63) and (64) yield the system of equations

$$\sum_{i=1}^N \frac{1}{D_i} \frac{c_i a_i}{a_i^2 - b_j^2} = 0, \quad j=2,3,\dots,N, \quad (65)$$

$$\sum_{i=1}^N \frac{c_i a_i}{D_i} = 1,$$

containing N equations for the N unknowns $\frac{1}{D_i}$. Denote the determinant of the system (65) by Δ' . Then

$$\Delta' = \prod_{p=1}^N c_p a_p \begin{vmatrix} \frac{1}{a_1^2 - b_2^2} & \frac{1}{a_2^2 - b_2^2} & \cdots & \frac{1}{a_N^2 - b_2^2} \\ \frac{1}{a_1^2 - b_3^2} & \frac{1}{a_2^2 - b_3^2} & \cdots & \frac{1}{a_N^2 - b_3^2} \\ \dots & \dots & \dots & \dots \\ \frac{1}{a_1^2 - b_N^2} & \frac{1}{a_2^2 - b_N^2} & \cdots & \frac{1}{a_N^2 - b_N^2} \\ 1 & 1 & \dots & 1 \end{vmatrix} \quad (66)$$

We see from (66) that Δ' is proportional to Δ , defined in (47), provided we replace a_i and b_j in (47) by a_i^2 and b_j^2 , respectively. Hence $\Delta' \neq 0$, since $\Delta \neq 0$ by (52), and (65) has a unique solution. Now from (29) we see that the first $N-1$ equations in (65) will be met if we put

$$D_1 = D_2 = \dots = D_N = D, \quad (67)$$

where D is any constant. On the other hand, the last of (65) will be satisfied if we put

$$D = \sum_{p=1}^N c_p a_p. \quad (68)$$

This completes the proof of (57).

We proceed to establish (38). It is obvious from (34), (35) and (37), that $\sum_{-\infty}^0 E_{rm}$ may be written in the form

$$\sum_{r=1}^N \sum_{j=1}^N \alpha_{ij} K_i K_j, \quad (69)$$

where the coefficients α_{ij} must be determined.

Consider the expression $\sum_{j=2}^N \frac{A_j}{a_k + b_j}$, for some k . By

(56) we have

$$\sum_{j=2}^N \frac{A_j}{a_k + b_j} = - \sum_{i=1}^N K_i \frac{\prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)} \sum_{j=2}^N \frac{\prod_{\substack{p=1 \\ p \neq i}}^N (a_p - b_j)}{(a_k + b_j) \prod_{\substack{q=2 \\ q \neq j}}^N (b_q - b_j)}. \quad (70)$$

Equation (70) can be rewritten as

$$\sum_{j=2}^N \frac{A_j}{a_k + b_j} = - \sum_{i=1}^N K_i \frac{\prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)} \left[\frac{\prod_{\substack{p=1 \\ p \neq i}}^N (a_p + a_k)}{\prod_{s=2}^N (a_k + b_s)} - 1 \right], \quad (71)$$

which may be verified by expanding the term contained in square brackets in (71) in terms of $\frac{1}{a_k + b_j}$. From (71) we find that

$$\sum_{i=1}^N c_i K_i \sum_{j=2}^N \frac{A_j}{a_i + b_j} = - \sum_{i=1}^N c_i K_i \frac{\prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)} \left[\frac{\prod_{p=1}^N (a_p + a_i)}{\prod_{s=2}^N (a_i + b_s)} - 1 \right]$$

$$- \sum_{i=1}^N c_i K_i \sum_{\substack{j=1 \\ j \neq i}}^N K_j \frac{\prod_{s=2}^N (a_j - b_s)}{\prod_{\substack{r=1 \\ r \neq j}}^N (a_j - a_r)} \left[\frac{\prod_{p=1}^N (a_p + a_i)}{\prod_{s=2}^N (a_i + b_s)} - 1 \right]. \quad (72)$$

Similarly, we find from (54) that

$$\begin{aligned}
 A_1 \sum_{i=1}^N c_i K_i &= - \sum_{i=1}^N c_i K_i^2 \frac{\prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)} \\
 &\quad - \sum_{i=1}^N c_i K_i \sum_{\substack{j=1 \\ j \neq i}}^N K_j \frac{\prod_{s=2}^N (a_j - b_s)}{\prod_{\substack{r=1 \\ r \neq j}}^N (a_j - a_r)}.
 \end{aligned} \tag{73}$$

Combining (72) and (73) and taking account of (37), we reach,

$$\begin{aligned}
 2 \sum_{r=1}^N E_{rm} &= \sum_{i=1}^N c_i K_i^2 \frac{\prod_{s=2}^N (a_i - b_s)}{\prod_{\substack{r=1 \\ r \neq i}}^N (a_i - a_r)} \frac{\prod_{p=1}^N (a_p + a_i)}{\prod_{p \neq i}^N (a_i + b_p)} \\
 &\quad + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_i K_i K_j \frac{\prod_{s=2}^N (a_j - b_s)}{\prod_{\substack{r=1 \\ r \neq j}}^N (a_j - a_r)} \frac{\prod_{p=1}^N (a_p + a_j)}{\prod_{p \neq j}^N (a_j + b_p)}.
 \end{aligned} \tag{74}$$

At this point we make use of the relations (57) satisfied by a_i , b_j and c_k . If we employ (57), we are able to write (74) in the alternative form

$$\begin{aligned}
 2 E_{-\infty}^0 = & \frac{1}{N \sum_{p=1}^N c_p a_p} \sum_{i=1}^N c_i^2 a_i K_i^2 \frac{\prod_{r=1, r \neq i}^N (a_i + a_r)^2}{\prod_{s=2}^N (a_i + b_s)^2} \\
 & + \frac{2}{N \sum_{p=1}^N c_p a_p} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_i c_j a_i a_j (a_i + a_j) K_i K_j \frac{\prod_{r=1, r \neq i, j}^N (a_i + a_r)(a_j + a_r)}{\prod_{s=2}^N (a_i + b_s)(a_j + b_s)}.
 \end{aligned}
 \tag{75}$$

If we now define α_{11} and α_{1j} as in (39) and (40), respectively, we see that (75) becomes identical with (38). Thus (38) is established.

It remains to establish (45). To this end, we observe from (38) and (43) that

$$2 F_{-\infty}^0 - 2 E_{-\infty}^0 = \sum_{i=1}^N (c_i - 2\alpha_{11}) K_i^2 - 2 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \alpha_{1j} K_i K_j.
 \tag{76}$$

By (39) we have, in view of (57),

$$c_1^{-2\alpha_{11}} = c_1 \frac{\prod_{\substack{p=1 \\ p \neq 1}}^N (a_1 + a_p)}{\prod_{s=2}^N (a_1 + b_s)} \left[\frac{\prod_{s=2}^N (a_1 + b_s)}{\prod_{\substack{p=1 \\ p \neq 1}}^N (a_1 + a_p)} - \frac{\prod_{s=2}^N (a_1 - b_s)}{\prod_{\substack{p=1 \\ p \neq 1}}^N (a_1 - a_p)} \right] \quad (77)$$

Expanding the bracket in (77), we obtain

$$c_1^{-2\alpha_{11}} = c_1 \frac{\prod_{\substack{p=1 \\ p \neq 1}}^N (a_1 + a_p)}{\prod_{s=2}^N (a_1 + b_s)} \left[1 + \sum_{\substack{j=1 \\ j \neq 1}}^N \frac{\prod_{s=2}^N (b_s - a_j)}{(a_1 + a_j) \prod_{\substack{r=1 \\ r \neq 1, j}}^N (a_r - a_j)} - 1 - \sum_{\substack{j=1 \\ j \neq 1}}^N \frac{\prod_{s=2}^N (a_j - b_s)}{(a_1 - a_j) \prod_{\substack{r=1 \\ r \neq 1, j}}^N (a_j - a_r)} \right] \quad (78)$$

Combining the terms in the brackets in (78), we get

$$c_1^{-2\alpha_{11}} = 2c_1 a_1 \frac{\prod_{\substack{p=1 \\ p \neq 1}}^N (a_1 + a_p)}{\prod_{s=2}^N (a_1 + b_s)} \sum_{\substack{j=1 \\ j \neq 1}}^N \frac{1}{a_1 + a_j} \frac{\prod_{s=2}^N (a_j - b_s)}{\prod_{\substack{r=1 \\ r \neq j}}^N (a_j - a_r)} \quad (79)$$

By (57) and (40), (79) yields

$$c_1^{-2} \alpha_{11} = \sum_{\substack{j=1 \\ j \neq 1}}^N 2 \alpha_{1j} . \quad (80)$$

Combining (80) and (76) we finally reach

$$2 \underset{-\infty}{\overset{0}{F}} - 2 \underset{-\infty}{\overset{0}{E}}_{rm} = 2 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq 1}}^N \alpha_{ij} (K_i^2 - K_i K_j) . \quad (81)$$

since $\alpha_{ij} = \alpha_{ji}$ by (40), we see that (81) is identical with (45).

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