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Estimation of Doppler Shift and Spectral Width of a Random Process

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ABSTRACT

The estimation of the Doppler shift, frequency scale factor and amplitude of the spectrum of a stationary Gaussian random process from a record of limited duration is considered. With the approximation that the coefficients of the Fourier series expansion of a realization of long time duration are independent, maximum likelihood estimates are derived and the Cramer-Rao lower bounds to the variances are evaluated.

I. Introduction

The problem of estimating parameters of the spectrum of a stationary random process from a record of limited duration arises in a number of applications. For example, in radar astronomy the signal returned from the moon or a planet is a random process. The radial velocity of such a body can be inferred from the Doppler shift and the rotation rate from the spectral width.[1,2] Considerable attention has been directed toward estimation of over-all spectral shapes.[3-5] However, for situations like the above in which the spectral shape is assumed to be known except for one or more parameters the treatment has been quite limited. The only reference we can cite is that by Swerling[6] who has determined an estimate of the frequency scale factor h which is optimum among a certain class. He first forms an over-all spectral estimate $\Phi^*(f)$ from the data by the methods of Blackman and Tukey[4] and then, assuming a knowledge of the approximate value of h and Gaussian statistics, selects as an estimator that linear functional of $\Phi^*(f)$ subject to certain restrictions which has a minimum mean square error. His expression for the mean square error is found to be approximately equivalent to (21) or (27) below when the spectrum is symmetrical about some center frequency and $\Phi^*(f)$ is estimated at a large number of points, and is generally larger otherwise.

II. The Estimation of Doppler Shift

The approach followed here is to apply the method of maximum likelihood[7,8] with certain approximations. The following assumptions are made:

a) A realization of the random process $x(t)$ is observed for $0 \le t \le T$.

b) The process is stationary, Gaussian, and zero-mean.

c) The single-sided spectrum of the process is

$$\Phi(f) = P(f-f_d)$$

where $P(f-f_d)$ is known except for the translation parameter f_d, the Doppler shift, which is to be estimated. If the process consists of a signal plus an independent noise, $\Phi(f)$ is merely the sum of the two individual spectra.

d) f_d lies within a known finite interval, $f_{dmin} \leq f_d \leq f_{dmax}$.

e) For any value of f_d within this interval only values of $\Phi(f)$ within a finite range $0 < f_1 < f < f_2$ are dependent upon f_d. f_1 and f_2 are independent of f_d and, for convenience, are integral multiples of $1/T$. This assumption eliminates unimportant end effects from the development and is valid if $P(f-f_d)$ has some constant value Φ_o outside of a fixed frequency range or if $x(t)$ is band-limited after the Doppler shift has been imparted. See Figure 1.

f) $P(f-f_d)$ obeys certain general regularity conditions so that it is free of discontinuities, varies slowly over any interval of width $1/T$ and is non-zero for $f_1 \leq f \leq f_2$.

The likelihood function of the realization is now expressed in terms of the Fourier coefficients

$$a_n = \frac{2}{\sqrt{T}} \int_0^T x(t) \cos(2\pi n f_o t)\, dt \tag{1}$$

$$b_n = \frac{2}{\sqrt{T}} \int_0^T x(t) \sin(2\pi n f_o t)\, dt \tag{2}$$

$$c_n^2 = a_n^2 + b_n^2 \tag{3}$$

where

$$f_o = 1/T \tag{4}$$

These coefficients are taken as the "observable coordinates"[8] of the process. This is plausible since under general conditions x(t) can be represented for almost every sample function by the limit in the mean of its Fourier series expansion.[9] The a_n and b_n have a joint Gaussian distribution[9,10] and with the above definitions

$$\lim_{T \to \infty} \operatorname{Var} a_n = \lim_{T \to \infty} \operatorname{Var} b_n = \Phi(nf_o) \tag{5}$$

For large T these coefficients approach independence. See [9], especially Theorem 7, The principal approximation introduced here, upon which all following equations depend, is that they are independent for finite T. This is quite accurate when assumption f) holds. Then with assumption e) from the multivariate Gaussian distribution the logarithm of the likelihood function becomes

$$\mathcal{L}(f_d) = -\frac{1}{2}\left[\sum_{n=N_1}^{N_2} 2 \log 2\pi\, P(nf_o - f_d) + \sum_{n=N_1}^{N_2} \frac{c_n^2}{P(nf_o - f_d)}\right] + C_1 \tag{6}$$

where $N_1 = Tf_1$, $N_2 = Tf_2$, and C with a subscript denotes a quantity independent of the parameter to be estimated. The first summation is independent of f_d to an excellent approximation from assumption f). Hence the maximum likelihood estimate $\hat{f}_d$ can be approximated by minimizing

$$\Lambda(f_d) = \sum_{n=N_1}^{N_2} \frac{c_n^2}{P(nf_o - f_d)} \tag{7}$$

This can be thought of as a cross-correlation between the observed spectrum c_n^2 and the reciprocal of $P(nf_o - f_d)$. Knowledge of the absolute amplitude level of the spectrum is not required.

A necessary condition that a maximum likelihood estimate must satisfy is $\mathcal{L}'(f_d) = 0$ where the prime denotes differentiation with respect to the argument. Applied to (7), this shows that $\hat{f}_d$ must satisfy

$$\sum_{n=N_1}^{N_2} c_n^2 \frac{P'(nf_o - f_d)}{P^2(nf_o - f_d)} = 0 \qquad (8)$$

When the approximate minimum of (7) is found the determination of the point at which this cross-correlation passes through zero locates the minimum exactly.

If

$$P(f - f_d) = \Phi_o + p(f - f_d) \qquad (9)$$

where max $| p(f - f_d) | \ll \Phi_o$ then

$$-\Lambda(f_d) \approx \frac{1}{\Phi_o^2} \sum_n c_n^2 \; p(nf_o - f_d) + C_2 \qquad (10)$$

and $\hat{f}_d$ is given approximately by maximizing this summation.

In calculating $\hat{f}_d$ nearby values of c_n^2 can be averaged providing that $\Phi(f)$ does not vary appreciably over the included interval. Therefore $\Lambda(f_d)$ can be approximated by

$$T \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(f - f_d)} \, df \qquad (11)$$

where $\Phi^*(f)$ is a smoothed spectral estimate obtained by the Blackman-Tukey method or by a spectrum analyzer as long as all of the c_n^2 are effectively included and the detailed structure of the spectrum is not obscured.

The sampling errors of these estimates are now investigated by applying the following general results which hold under suitable regularity conditions. For a parameter

α with true value α_o and any estimate α^*, the Cramer-Rao lower bound[7] for the mean square error is

$$E(\alpha^* - \alpha_o)^2 \geq \frac{[1 + (db/d\alpha_o)]^2}{E[\mathcal{L}'(\alpha_o)]^2} \tag{12}$$

where

$$\mathcal{L}'(\alpha_o) = \left.\frac{\partial \mathcal{L}(\alpha)}{\partial \alpha}\right|_{\alpha = \alpha_o} \tag{13}$$

and the bias of α^* is

$$b = E\alpha^* - \alpha_o \tag{14}$$

If α^* is based on observations of a set of independent random variables the over-all value of $E[\mathcal{L}'(\alpha_o)]^2$ is the sum of the values for the individual random variables since $E\mathcal{L}'(\alpha_o) = 0$. Furthermore, the maximum likelihood estimate $\hat{\alpha}$ is asymptotically efficient and asymptotically Gaussian. This implies that as the number of observations becomes large $\hat{\alpha}$ tends to a Gaussian distribution with mean α_o and variance $1/E[\mathcal{L}'(\alpha_o)]^2$.

In the present case, for a single coefficient c_n^2,

$$\mathcal{L}_n(f_d) = -\frac{1}{2}\left[2 \log 2\pi P(nf_o - f_d) + \frac{c_n^2}{P(nf_o - f_d)}\right] + C_3 \tag{15}$$

$$E[\mathcal{L}_n'(f_d)]^2 = \frac{1}{4} E\left[\frac{2P'(nf_o - f_d)}{P(nf_o - f_d)} - \frac{c_n^2 P'(nf_o - f_d)}{P^2(nf_o - f_d)}\right]^2 \tag{16}$$

$$= \left[\frac{P'(nf_o - f_d)}{P(nf_o - f_d)}\right]^2$$

since

$$E\, c_n^2 = 2P(nf_o - f_d) \tag{17}$$

and

$$E\, c_n^4 = 8P^2(nf_o - f_d) \tag{18}$$

With f_d set equal to its true value

$$E[\mathcal{L}_n'(f_d)]^2 = \left[\frac{\Phi'(n f_o)}{\Phi(n f_o)}\right]^2 \tag{19}$$

and for the entire set of coefficients

$$E[\mathcal{L}'(f_d)]^2 = \sum_{n=N_1}^{N_2} \left[\frac{\Phi'(n f_o)}{\Phi(n f_o)}\right]^2 \tag{20}$$

Therefore, from the asymptotic properties of maximum likelihood estimates, for large $T, \hat{f}_d$ tends to a Gaussian distribution with mean f_d and variance

$$\left\{T \int_{f_1}^{f_2} \left[\frac{\Phi'(f)}{\Phi(f)}\right]^2 df\right\}^{-1} \tag{21}$$

where the summation (20) has been approximated by an integral. The variations of the structure of the spectrum which permit estimation of f_d are thus measured by the integral in (21). The expression (21) is also the Cramer-Rao lower bound to the mean square error of any estimate, providing $db/df_d = 0$, which will be true of any spectrum cross-correlation type of estimate from symmetry considerations.

As a digression, an interesting phenomenon concerning (21) is pointed out. In general, the Cramer-Rao lower bound (12) for the variance of an unbiased estimate can be written as

$$\left\{ E[\mathcal{L}'(\alpha_o)]^2 \right\}^{-1} = \left\{ \int \left[\frac{f'(\underline{x}, \alpha_o)}{f(\underline{x}, \alpha_o)} \right]^2 f(\underline{x}, \alpha_o)\, d\underline{x} \right\}^{-1}$$

where $\underline{x}$ is the vector random variable of the observations and $f(\underline{x})$ is the probability density. This is almost identical in form to (21) although $f(\underline{x})$ and $\Phi(f)$ represent entirely different quantities. This similarity, which has also been found to hold for other functions of $\mathcal{L}$ in this situation, is entirely unexpected and does not occur in other parameter estimation problems.

III. Estimation of Frequency Scale Factor (Spectral Width)

The assumptions of Section II are modified as follows:

c') The single sided spectrum of the process is

$$\Phi(f) = P(h(f - f_c))$$

where f_c is a known center frequency and $P(h(f - f_c))$ is known except for the frequency scale factor h which is to be estimated.

d') h lies within a known finite interval $0 < h_{min} \le h \le h_{max}$

e') and f') are the same as e) and f) except that f_d is replaced by h.

See Figure 2.

Note that if $f_c = 0$ and the non-constant part of $\Phi(f)$ is confined to a narrow high-frequency band then the estimation of h is approximately equivalent to that of f_d.

Proceeding as before, $\hat{h}$ is obtained by maximizing

$$\mathcal{L}(h) = -\frac{1}{2}\left[\sum_{n=N_1}^{N_2} 2 \log 2\pi P(h(nf_o - f_c)) + \sum_{n=N_1}^{N_2} \frac{c_n^2}{P(h(nf_o - f_c))} \right] + C_4$$

$$\approx -\frac{T}{2}\left[\int_{f_1}^{f_2} 2 \log 2\pi P(h(f - f_c))\, df + \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(h(f - f_c))}\, df \right] + C_4$$

$$= -\frac{T}{2}[I_1 + I_2] + C_4 \qquad (22)$$

The evaluation of I_1 can be simplified by writing

$$P(h(f - f_c)) = \Phi_o[1 + p(h(f - f_c))] \tag{23}$$

Then

$$I_1 = 2(f_2 - f_1)\, 2 \log 2\pi\, \Phi_o + \int_{f_1}^{f_2} 2 \log 2\pi \left[1 + p(h(f - f_c))\right] df$$

$$= C_5 + I'_1 \tag{24}$$

On the basis of the previous assumptions the entire non-zero part of the integrand of I'_1 is always included within the limits of integration (see Fig. 3). By a simple transformation of variables

$$I'_1 = \frac{1}{h} \int_{f_1}^{f_2} 2 \log 2\pi \left[1 + p(f - f_c)\right] df$$

$$= \frac{K}{h} \tag{25}$$

Hence

$$\mathcal{L}(h) \approx - \frac{T}{2} \left[\frac{K}{h} + \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(h(f - f_c))} df \right] + C_6 \tag{26}$$

The asymptotic properties of maximum likelihood estimates previously mentioned show that as T becomes large $\hat{h}$ tends to a Gaussian distribution with mean h and variance

$$\left\{ T \int_{f_1}^{f_2} (f - f_c)^2 \left[\Phi'(f)/\Phi(f)\right]^2 df \right\}^{-1} \tag{27}$$

which is calculated in the same way as (21). The effect of spectrum irregularities is thus emphasized by the quadratic factor $(f - f_c)^2$. For estimates of h, db/dh may not be zero so the Cramer-Rao lower bound is not so informative as in the previous case.

IV. Estimation of Spectrum Amplitude†

It is interesting to investigate the estimation of spectrum amplitude by the same method. Assumptions a) and b) of Section II are retained and the following are added:

c'') $\Phi(f) = AP(f) + N_o$

where $P(f)$ and N_o are known and A is to be estimated.

d'') The Fourier coefficients of x(t) can be determined only over the finite frequency range $f_1 \le f \le f_2$.

Proceeding as before, there is obtained

$$\mathcal{L}(A) = -\frac{1}{2} \sum_{n=N_1}^{N_2} \left\{ 2 \log 2\pi \left[AP(nf_o) + N_o \right] + \frac{c_n^2}{AP(nf_o) + N_o} \right\} + C_7 \qquad (28)$$

and

$$\mathcal{L}'(A) = -\frac{1}{2} \sum_{n=N_1}^{N_2} \left[\frac{P(nf_o)}{AP(nf_o) + N_o} \right] \left[1 - \frac{c_n^2}{2\left[AP(nf_o) + N_o \right]} \right] \qquad (29)$$

For any unbiased estimate A*, the Cramer-Rao lower bound is found to be

$$\frac{\mathrm{Var}\, A^*}{A^2} \ge \frac{1}{\sum_{n=N_1}^{N_2} \left[\frac{AP(nf_o)}{AP(nf_o) + N_o} \right]^2} \qquad (30)$$

or approximately

$$\frac{\mathrm{Var}\, A^*}{A^2} \ge \frac{1}{T \int_{f_1}^{f_2} \left[\frac{AP(f)}{AP(f) + N_o} \right]^2 df} \qquad (31)$$

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†This problem was brought to the author's attention by Prof. W. L. Root.

A general explicit solution for the maximum likelihood estimate $\hat{A}$ is not known. However there are two significant special cases. The first is for $N_o = 0$. Then setting $\mathcal{L}'(A) = 0$, there is obtained

$$\hat{A} = \frac{\sum c_n^2/2P(nf_o)}{T(f_2 - f_1)}$$

$$\approx \frac{1}{f_2 - f_1} \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(f)} \, df \qquad (32)$$

This is an unbiased estimate and the variance is found directly to be

$$\frac{\text{Var } \hat{A}}{A^2} = \frac{1}{T(f_2 - f_1)} \qquad (33)$$

Therefore as $(f_2 - f_1)$ approaches infinity the variance approaches zero for any fixed T. This result would be expected from the well known singularity of this situation.

A second special case occurs when $AP(f) \ll N_o$ for all f. $\hat{A}$ can be approximated by setting

$$AP(nf_o) + N_o \approx N_o$$

so (29) becomes

$$\mathcal{L}'(A) \approx \sum \frac{P(nf_o)}{N_o^2} \left[AP(nf_o) + N_o - \frac{c_n^2}{2} \right] = 0 \qquad (34)$$

and

$$\hat{A} \approx \frac{\int_{f_1}^{f_2} \left[\Phi^*(f) - N_o \right] P(f) \, df}{\int_{f_1}^{f_2} P^2(f) \, df} \qquad (35)$$

The estimate is also unbiased and gives

$$\frac{\operatorname{Var} \hat{A}}{A^2} \approx \frac{N_o^2}{T \int_{f_1}^{f_2} [AP(f)]^2 \, df} \tag{36}$$

which agrees with known radiometer formulas. The variances (33) and (36) are found to be the same as the Cramer-Rao lower bounds so these estimates are efficient.

V. Application to Radar Astronomy Measurements

The preceding analysis applies directly to the situation where a target is illuminated by a long pulse and the parameters of the spectrum of the return signal are to be estimated. It also applies to a dual situation in which the range or extent of a target is to be estimated. Consider a target with a time delay τ_o such that when it is illuminated by a unit impulse the reflected signal $r(t - \tau_o)$ is a Gaussian random process with mean zero and

$$E\, r(t - \tau_o)\, r(t_1 - \tau_o) = R(t - \tau_o)\, \delta(t - t_1)$$

so that the received signal is a non-stationary white noise process. This model is consistent with the representation of the target as a collection of a large number of point scatterers randomly and independently distributed with a spatial density corresponding to $\sqrt{R(t - \tau_o)}$.

Suppose that $R(t - \tau_o)$ is known except for the translation parameter τ_o which is to be estimated. The following additional assumptions are made:

a) The target is illuminated by a unit impulse bandlimited to $0 \le f \le W$. This assumption is necessary to represent the practical situation and to avoid a degenerate result. If an RF pulse has been transmitted the received signal is assumed to have been translated to zero frequency.

b) τ_o lies within a known infinite interval.

c) $R(t - \tau_o)$ is of finite duration and lies within the interval $t_1 \leq t \leq t_2$, for any allowed value of τ_o. For convenience take $t_1 = N_1/2W$ and $t_2 = N_2/2W$ where N_1 and N_2 are integers.

d) $R(t - \tau_o)$ is slowly varying over any interval $\Delta t = 1/2W$ and has no discontinuities. (Unfortunately this assumption is not fulfilled for many physical targets.)

e) The return signal is perturbed by additive stationary white Gaussian noise of single-sided spectral density N_o.

f) The return signal plus noise are passed through an ideal bandpass filter $0 \leq f \leq W$ to give a received signal $s(t)$.

On the basis of these assumptions the bandlimited signal $s(t)$ can be represented by $2(t_2 - t_1) W + 1$ sampled values $s(n\Delta t)$. To a good approximation the $s(n\Delta t)$ are independent and

$$\text{Var } s(n\Delta t) = \left[R(n\Delta t - \tau_o) + N_o \right] W \tag{37}$$

The logarithm of the likelihood of the set of samples is, with this approximation,

$$\mathcal{L}(\tau_o) = -\frac{1}{2} \sum_{n=N_1}^{N_2} \frac{s^2(n\Delta t)}{\left[R(n\Delta t) - \tau_o) + N_o \right] W} + C_8 \tag{38}$$

The maximum likelihood estimate $\hat{\tau}_o$ is seen to be analogous to the estimate $\hat{f}_d$ of Section II. Therefore as W becomes large the variance of the asymptotic distribution of $\hat{\tau}_o$ is

$$\left\{ W \int_{t_1}^{t_2} \left[\frac{R'(t - \tau_o)}{R(t - \tau_o) + N_o} \right]^2 dt \right\}^{-1} \tag{39}$$

If the effective duration of the band-limited impulse is taken as $T_e = 1/W$ the variance becomes

$$\frac{T_e}{\int_{t_1}^{t_2} \left[\frac{R'(t - \tau_o)}{R(t - \tau_o) + N_o} \right]^2 dt} \tag{40}$$

Since the exact transmitted pulse shape should not greatly influence the final result, this expression could be taken as an order of magnitude approximation for other pulse shapes.

The depth (spatial extent) of a distributed target is a scale parameter and therefore can be estimated in a manner exactly analogous to that of Section III.

VI. Conclusions

The approximations which have been employed are of increasing accuracy as the observation time becomes large and seem to lead to useful results. However, a further quantitative justification would certainly be desirable. The joint estimation of several parameters is a straightforward extension of the previous analysis. The same methods are applicable to the estimation of other spectral parameters.

The estimates of f_d and h can be realized by determining the coefficients c_n^2 or a spectral estimate $\Phi^*(f)$ by some standard method and then carrying out the search for the maximum likelihood estimate on a digital computer. An explicit interpolation method such as Swerling's appears practical only when the parameters are known to lie within a range which is narrower than the width of the principal fluctuations of the spectrum.

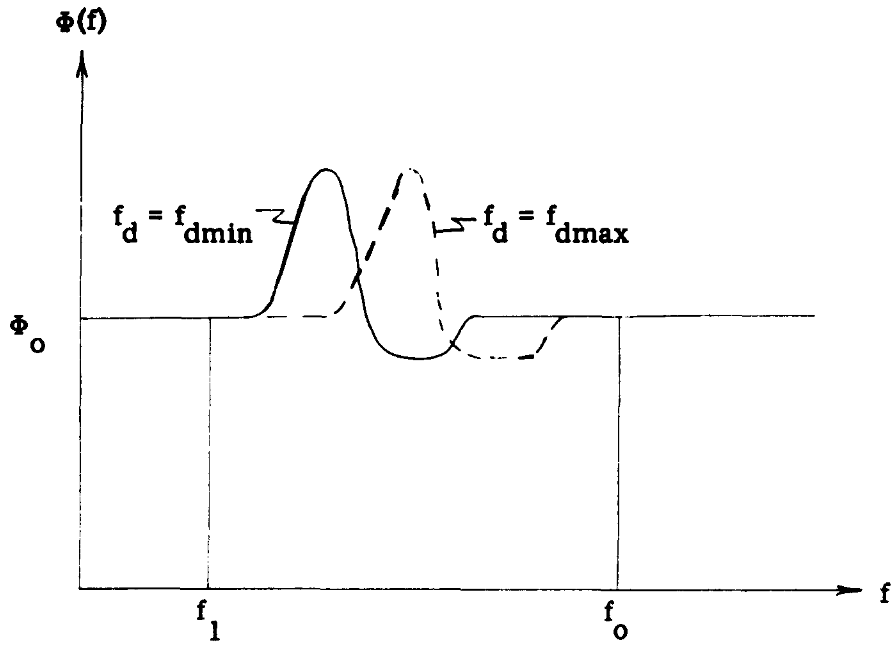

Figure 1 - Assumed Spectral Shape for Estimation of f_d

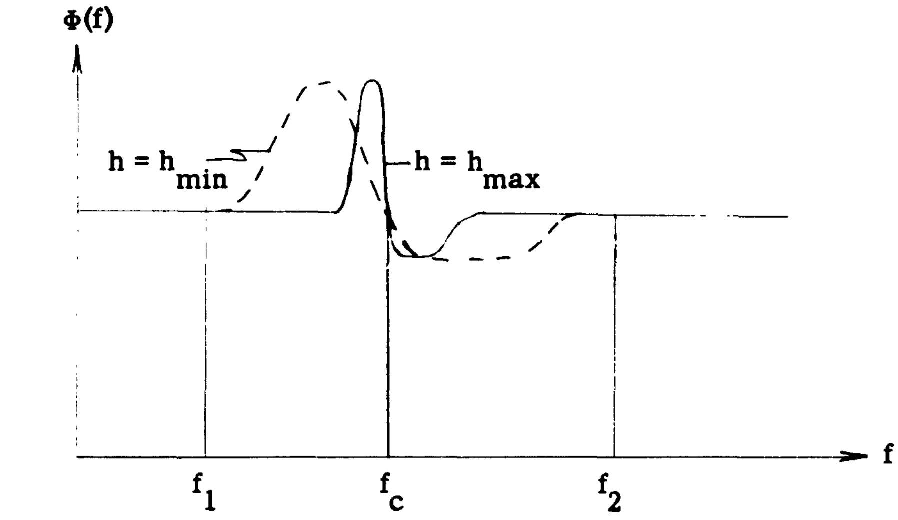

Figure 2 - Assumed Spectral Shape for Estimation of h.

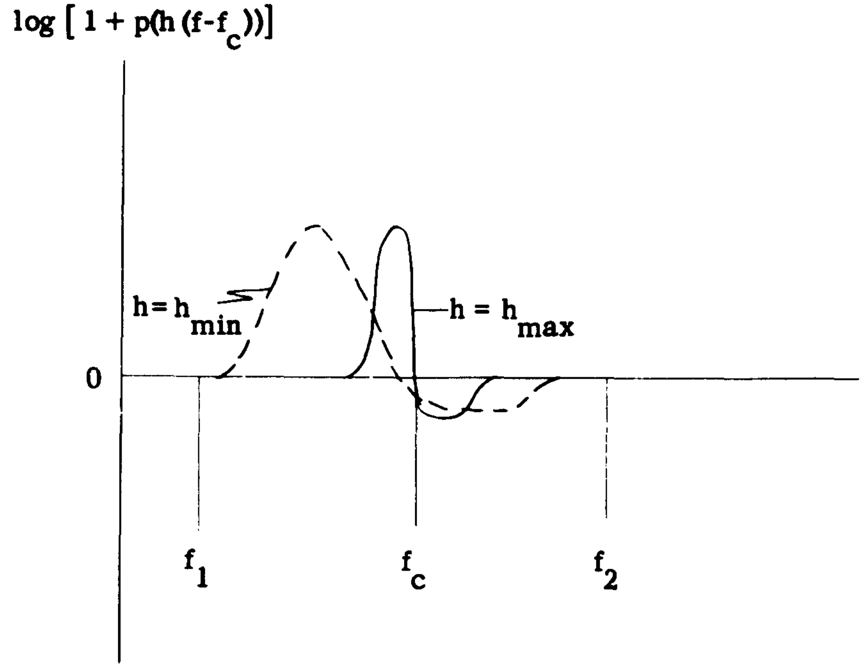

Figure 3 - Pertinent to the Evaluation of $\hat{h}$

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