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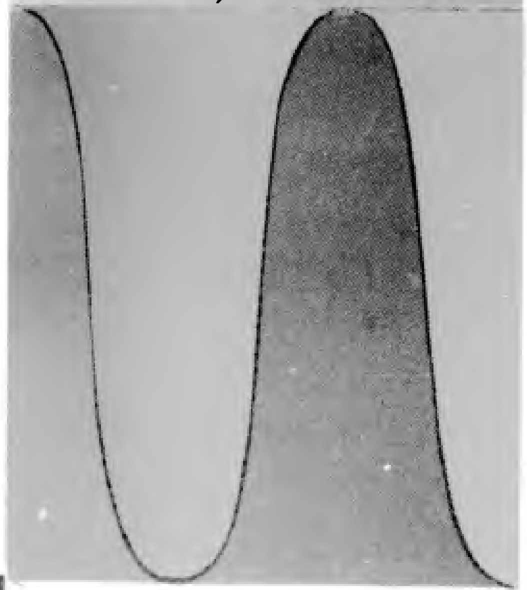

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SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES

III. PROBLEMS OF THE GEOMETRY AND ANALYSIS OF THE HIGHER EUCLIDEAN SPACES

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SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES III.

PROBLEMS OF THE GEOMETRY AND ANALYSIS OF THE HIGHER EUCLIDEAN SPACES

L. C. Young

III 1. Introduction. We shall discuss various aspects of what will turn out to be, basically, a single problem of an extremal character, connected with the notion of convexity. However many further problems will be mentioned, as they play an important part in the background: they range from rigid statics and line geometry to the rewriting of the whole differential calculus with Kneser partial derivatives. Nor is this all, for the next note [13] shows that there is scope for applying methods which come from far outside this background.

In regard to the title of the present note, even elementary students, unlike the general public, are apt to view higher dimensions with scant respect, as if it were merely a question of adding one more variable to the number of coordinates of a point. However this is due to a misconception which is somewhat widespread. Eventually, students will find that, in

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analysis and in geometry, the study of Euclidean and projective spaces is not a mere study of their points; and that we have to treat, besides the points, or vectors, their duals, the hyperplanes, or pseudo-vectors, and the whole series of linear entities of the intermediate dimensions.

Actually, it is in regard to this intermediate series of entities, the so-called multivectors, or k-vectors of n-space, that our basic information, even of the most elementary kind, is so deficient. The deficiency is particularly extensive when $n > 8$ and $2 < k < n-2$. Its persistence was natural in the days when n-space, with $n > 3$, or perhaps $n > 4$, or 5, seemed a remote abstraction. Today the number of variables that affect the flight of a plane, taking account of viscosity, and of the thermo-dynamic and electro-dynamic phenomena, is something like $n=30$, a far cry from the fourth or fifth dimension of Relativity. The most elaborate safety measures do not quite replace a basic understanding of a space of that number of dimensions, and of the phenomena to which it gives rise.

Our information is not quite so deficient when $k=2$, or when $k=n-2$ (the dual case), because the study of bivectors, in the form in which they occur in both line geometry and in three-dimensional rigid statics, is quite old, at any rate when $n=4$. Actually, what information we possess, even for general k and n , is perhaps arrived at most directly, as in W. H. Young's early papers [6, 7, 8, 9], by using the rigid statics approach. For instance, the latter provides quite simply a minimal independent set of the quadratic relations, which are satisfied by, and only by, the system of components of a k-vector in n-space [8, p 64].

As a result of these quadratic relations, multivectors become points of a so-called Grassmann cone in a space of the proper number of dimensions, i.e. they constitute a non-linear space, which has a linear extension. The points of this linear extension are termed composite multivectors, and each of them is representable as a formal sum of the original, or simple, multivectors. The number of simple multivectors needed, in order to represent by their formal sum an arbitrary composite one, is known only in the cases $k=2$ or $n-2$; except that, for $k=3$ and $n\leq 8$, an estimate of this number, due to Weitzenbock [5], gives the right figure. The rigid statics approach, which furnishes a simple proof [6, p 483] for $k=2$ or $n-2$, leads also easily to this same estimate. So far, however, in the questions that we shall discuss, even for $k=2$ and for low values of $n\geq 4$, neither the approach of rigid statics, nor that of line geometry, has been of any material assistance. These questions concern, on the one hand the relations of multivectors to simplicial complexes, and in particular to closed polytopes, so that we are soon faced with topological, as well as algebraic, complications; and on the other hand the notion of function of a multivector, a notion on which we naturally know even less than we know on the subject of the multivectors which occur as variables.

Here, for the first time, a satisfactory derivation theory for such functions is initiated, based on a suggestion made by H. Kneser [2] in a special case, that of a function which satisfies a homogeneity condition:

$$\text{(III 1.1)} \quad F(tj) = tF(j) \text{ for } t \geq 0.$$

The removal of this restriction renders possible the formation of successive derivatives. However theorems such as the inversion of order of mixed partial derivatives are still unknown, and the whole differential calculus will have to be developed ab initio.

The continuous functions $F(j)$, of a k-vector j, which satisfy (III 1.1), are particularly important in analysis; they correspond to the integrands $F(x, j)$ of k-dimensional variational problems in n-space, for fixed positions of the point x of that space. One of our main problems here concerns the notion of convexity for such homogeneous F. It has become traditional for writers on the calculus of variations to make an ad hoc convexity assumption in work on k-dimensional problems, simply because the condition is relevant in the case $k=1$. What is actually relevant, in the problems studied by these writers, is a weak form of convexity which includes the classical midpoint condition, and which may, or may not, be equivalent to it.

These various complications are due to the fact that functions of a k-vector are defined only in a subset of the linear space of composite k-vectors, and that neither the study of their extensions to the whole of this space, nor that of their local restrictions to small neighbourhoods on the Grassmann cone, provides what we really need.

III 2. Clifford numbers. It is convenient here to supplement the account of multivectors, given in [10] and in the literature there cited, by introducing a notation advocated, for instance, by Marcel Riesz [3]. The notation is applicable not only in Euclidean spaces, but also locally in Riemann spaces where the square of the line element is a quadratic form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{III 2.1}$$

It is helpful to bear this in mind in the discussion of maps, for instance in connection with the transformation formulae and invariance theorems of [10, §7], but for the present we set $\{g_{\mu\nu}\} = \{\delta_{\mu\nu}\}$, the diagonal unit matrix.

This being so, we write the line element, which is a vector, in the form

$$e_\mu dx^\mu, \tag{III 2.2}$$

where the coefficients are units, or unit vectors, termed Clifford numbers, which we subject to algebraic laws which ensure that the square of this line element

$$e_\mu e_\nu dx^\mu dx^\nu \tag{III 2.3}$$

can be identified with our previous expression for ds^2; and further that the algebra of polynomials in the e_μ subject to this same condition, be associative and distributive, with the reals (or for that matter, the complex numbers) as coefficients.

Since the product $dx^{\mu}dx^{\nu}$ of two coefficients of the expression (III 2.2) is commutative, this simply means that we have the identities

$$\text{(III 2.4)} \qquad e_{\mu}e_{\nu}+e_{\nu}e_{\mu} = 2g_{\mu\nu},$$

which express the fact that the square of a Clifford number (in the Euclidean metric) is unity, and that different Clifford numbers (in this metric) anti-commute.

Let now α denote an arbitrary increasing sequence of k positive integers which do not exceed n, and write e_{α} for the ordered product of the e_{μ} where $\mu\epsilon\alpha$. A linear form in the e_{α} may then be identified with the notion of a composite k-vector; and we observe that, in virtue of (III 2.4), an arbitrary polynomial in the e_{μ} reduces, in exactly one way, to a sum of composite multivectors of different dimensions.

If a vector v and a composite k-vector j are expressed as linear forms in the e_{μ} and e_{α} their product vj in the Clifford algebra is a polynomial in the e_{μ} which is expressible as the sum of a composite $(k+1)$-vector and a composite $(k-1)$-vector. These are precisely the composite multivectors which we denoted in [10] by the exterior and co-products $v\times j$ and $j\otimes v$.

This remark leads, not only to alternative definitions of exterior product and co-product, but also, very easily, to the fundamental identity

$$\text{(III 2.5)} \qquad v\times(j\otimes V)+(v\times j)\otimes V = (vV)j,$$

in which v and V are vectors, and (vV) is their scalar product.

III 3. The algebraic and the geometric resultant. In the sequel, the terms multivector and k-vector will mean simple multivector and simple k-vector; and the term system will be used for a finite class in which repetitions are permitted. More precisely a system s of k-vectors j is defined by a non-negative integer valued function s(j), termed multiplicity of j in the system s , such that s(j) vanishes except for a finite number of values of j, termed members of the system s. Similarly we define a system of systems, etc. We term combination of two systems s_1 and s_2 the system defined by the sum of their multiplicities. More generally, we define similarly the combination of a system of systems.

A system of k-vectors will be termed trivial if it consists of the exterior products of a corresponding system of vectors with a fixed (k-1)-vector. A system of k-vectors will be termed singular, if it is made up of pairs of the form j and -j, i.e. if the multiplicities s(j) and s(-j) are the same for each j. Given $\epsilon > 0$, we term singular ϵ-system, or simply ϵ-system, a singular system of k-vectors such that ϵ exceeds the sum of their magnitudes, i.e. the sum $\sum' s(j)|j|$. We term ϵ-modification of a system of k-vectors, a system obtained by combining it with an ϵ-system.

A k-vector will be said to be parallel to another, if it is a positive real multiple of it, anti-parallel if it is a negative real multiple. We term subdivision the operation of replacing a k-vector j by parallel k-vectors of the form $a_\nu j$ such that $\sum a_\nu = 1$; we term amalgamation the inverse operation. In a system s, subdivision of the k-vector j, of multiplicity ≥ 1, means decreasing by unity its multiplicity s(j) and increasing correspondingly the multiplicities of the $a_\nu j$ by one or more units, according to whether they have distinct coefficients a_ν or not. Two systems of k-vectors, which differ only as a result of such replacements, will be termed equivalent by subdivision. We shall use also an operation, akin to subdivision, which we term duplication of a system. It consists in replacing the latter by the combination of a suitable number of replicas on a different scale, or on several scales, in such a manner that the total sum of magnitudes is not affected. By a replica we mean here a system, derived from a system s, by assigning to it a multiplicity s(aj) where $0 < a \leq 1$. The notions of subdivision and duplication will also be applied, with obvious meanings, to polytopes.

An oriented k-dimensional simplex, whose vertices in the positive order, prescribed up to an even permutation, we denote by $x_0, x_1, \ldots, x_k$, will be associated with the k-vector j, if the latter is the multiple $1/(k!)$ of the exterior product of the k differences $x_1 - x_0, \ldots, x_k - x_0$; this product is of course not altered by an even permutation of vertices; more generally, a simplex will be associated with a system of multivectors, if the latter are all parallel, and the simplex is associated with their amalgamation.

We shall understand by an oriented polytope, or simply a polytope, an oriented simplicial complex of a given dimension. A polytope Π will be associated with a system s of multivectors, if s is equivalent by subdivision to the combination of the systems associated with the individual simplices of Π . A system s of k-vectors will be said to be in strict geometric equilibrium, if there exist a closed polytope Π associated with s ; in particular, if in addition Π can be chosen of the type of the k-sphere, s is said to be in simple strict geometric equilibrium. A system s of k-vectors will be said to be in geometric (or loose geometric) equilibrium, if it possesses for each $\epsilon > 0$, an ϵ-modification in strict geometric equilibrium; and in simple geometric equilibrium, if it possesses for each $\epsilon > 0$, an ϵ-modification in simple strict geometric equilibrium.

A system of k-vectors will be said to possess a strict geometric resultant, consisting of the k-vector j , if there exist a positive integer m and a simplex Δ , such that $m\Delta$ is associated with j and has the same boundary as a polytope Π associated with the given system. In particular, if we can here choose $m=1$ and Π of the type of the k-cell, we say that the system has the simple strict geometric resultant j. In the case $j=0$, these definitions are to be interpreted as meaning strict geometric equilibrium, and simple strict geometric equilibrium, as the case may be. Finally, we omit the word strict, if the property in question holds, not necessarily for the given system, but for an ϵ-modification, where $\epsilon > 0$ is arbitrarily small.

A system of k-vectors will be said to be in algebraic equilibrium if the sum of the corresponding linear forms in the e_α, introduced in the preceding section, vanishes. It will be said to have an algebraic resultant, consisting of a k-vector j, if the combination of the given system and that consisting of the single term $-j$ is in algebraic equilibrium. We shall allow ourselves to drop the term algebraic when no confusion is thereby caused. Moreover we write $\sum j_\nu = 0$, or $\sum j_\nu = j$, in the case of k-vectors j_ν which are in equilibrium, or which possess the resultant j, in the algebraic sense. This agrees with the notation of [10].

(III 3.1) Theorem. A system of k-vectors has an algebraic resultant j, if and only if it has the geometric resultant j.

The proof of this theorem will occupy the next note [13]. So far we have not succeeded in finding an elementary proof. It would also be desirable to find out whether the existence of an algebraic resultant implies that of a simple geometric one, and if not, how the existence of the latter can be characterized geometrically. Similar questions suggest themselves in regard to the special case of (III 3.1) in which it is assumed that $j = 0$, i.e. in the case of equilibrium.

At the present time, we can supply but little information on these questions, even if we specialize k and n (the dimension of the underlying space), unless the system is trivial, or otherwise restricted. However the remarks and methods of the next section may prove helpful.

III 4. Superposition theorems and supplementary information of a special nature. Given a system S whose terms are systems s of k-vectors, we shall understand by the superposition of S, or of the systems s, the system obtained by combining all the systems s. Further, if the systems s all possess resultants in one of the senses of the preceding section, we term resultant system of S, or of the systems s, the system of the resultants in this sense of the systems s. It is convenient to think of superposition in this case, as the operation of replacing, in the resultant system, each resultant by the appropriate s. In this form superposition can also be performed cumulatively: from an m-th system of k-vectors, we derive an (m+1)-st, by replacing one or more terms by systems of which they are the resultants in the relevant sense.

In this connection, we make the following two remarks:

(III 4.1) Theorem. The superposition of any systems of k-vectors in algebraic, geometric, or simple geometric, equilibrium, is itself in equilibrium in the same sense.

(III 4.2) Theorem. Let S be a system of systems s of k-vectors, and suppose that S has a resultant system in the algebraic, geometric, or simple geometric, sense. Then, if this resultant system has a resultant of its own, or is in equilibrium, in the same sense, the same is true of the superposition of S.

In these statements, the algebraic case is immediate, and so is the remainder of the first statement, if we bear in mind that a sum of closed polytopes, or of k-spheres , is itself of the same type; moreover in the first statement we could substitute strict geometric for geometric, without affecting its validity. In the case of the second statement, this substitution is not possible and the proof is less straight forward. We shall limit ourselves to the geometric case, since it is the more complicated, and the proof in this case can serve as a model for that of the simple geometric one. This proof will depend on the following consequence of Vitali's well-known measure-theoretic lemma:

(III 4.3) Lemma. Let s be a system of k-vectors with the strict geometric resultant j, and let m, Δ, Π be chosen as in the definition of this resultant; further, let Δ' denote any simplex, or other flat k-dimensional figure, associated with j/m. Then given $\epsilon > 0$, there exists an ϵ-modification s' of s, such that $m\Delta'$ has the same boundary as some polytope Π' associated with s'.

By a flat figure, we mean here a polytope situated in a k-dimensional hyperplane of n-space, such that its faces have disjoint interiors in this hyperplane. To prove (III 4.3), we denote by Δ'' a figure containing Δ', such that their difference has k-dimensional measure $\epsilon/(2m)$, and we cover all but measure precisely $\epsilon/(2m)$ of Δ'' by a finite number of disjoint simplices, similar to Δ and parallel to it, which we denote by Δ_ν $(\nu=1, 2, \ldots, N)$. This is possible, since we can so cover all but measure $\leq \epsilon/(2m)$ by Vitali's lemma, and we can then reduce the size of one or more Δ_ν, if necessary. We now express the sets $\Delta'' - \sum \Delta_\nu$ and $\Delta'' - \Delta'$ as sums of simplices, and we orient the former to make them parallel to j , and the latter anti-parallel. We denote by Π_0 a polytope consisting of m times these two sets of simplices; evidently Π_0 is associated with a singular ϵ-system. For each Δ_ν ($\nu = 1, 2, \ldots, N$), we now construct a polytope Π_ν , which is a replica of Π on a suitable scale and has the same boundary as $m\Delta_\nu$. We denote by Π' the sum of the polytopes Π_ν for $\nu = 0, 1, \ldots, N$. By construction, Π' then has the same boundary as $m\Delta'$, and is associated with the ϵ-modification of s , obtained by combining s with a singular ϵ-system associated with Π_0 .

In addition to the lemma just proved, we shall make use of the following remarks. We observe that by duplication, we can, in the definition of a strict geometric resultant, modify the quantities m, Δ, Π , so that Π becomes the sum of a preassigned number r of identical polytopes, and therefore so that m becomes divisible by a preassigned r . We note further that duplication of an ϵ-modification leads to an ϵ-modification of the duplicated system, with the same ϵ ; and that the combination of ϵ_ν-modifications of systems s_ν ($\nu = 1, \ldots, N$) is, for $\epsilon = \sum \epsilon_\nu$, an ϵ-modification of the combination of the s_ν.

We pass on to the proof of (III 4.2) in the case of geometric resultants. We may remove from S any terms which are systems in geometric equilibrium, and therefore suppose that its resultant system s_0 contains no vanishing term. We may further effect any desired preliminary ϵ-modification of s_0 and of the superposition system s of S , by augmenting S by corresponding new systems, consisting of single k-vectors which constitute together the appropriate ϵ-system; we may therefore suppose that s_0 has a strict geometric resultant, and we denote by m_0, Δ_0, Π_0 the quantities by which the latter is defined. We may also suppose, by preliminary duplications of systems of S, that the faces δ_ν of Π_0 are associated one-to-one with the terms of s_0, and so with the geometric resultants j_ν of the systems s_ν of S ($\nu = 1, \ldots, N$).

Next, by an ϵ-modification of s, which results, for $\epsilon = \sum \epsilon_\nu$, from ϵ_ν-modifications of the s_ν ($\nu = 1, \ldots, N$), we may suppose that each s_ν has a strict geometric resultant, and we denote by $m_\nu, \Delta_\nu, \Pi_\nu$ the quantities by which the latter is defined. Here we may further suppose, by duplication of Π_ν, that each m_ν is replaced by the least common multiple, and therefore that all the m_ν ($\nu = 1, 2, \ldots, N$) have the same value r. Finally, by duplication, we replace the polytope Π_0 by one of the form $r\Pi^*$, where the faces Δ_ν^* of Π^* are now associated one to one with the same k-vectors j_ν/r ($\nu = 1, 2, \ldots, N$) as the simplices Δ_ν.

We now effect a further ϵ-modification of s, which again results, for $\epsilon = \sum \epsilon_\nu$, from ϵ_ν-modifications of the s_ν, so as to arrange, in accordance with lemma (III 4. 3), that the simplices Δ_ν now become identical with the corresponding faces Δ_ν^* of Π^*. The new polytopes Π_ν then have the same boundaries as the corresponding parts $r\Delta_\nu^*$ of $r\Pi^*$; by replacing in $r\Pi^*$ each of these parts by the corresponding Π_ν, we obtain a polytope Π, associated with s, which has the same boundary as $r\Pi^*$. Since $r\Pi^*$, by construction, was associated with s_0 and had the same boundary as some multiple of a simplex, we see that s and s_0 have a same strict geometric resultant after our various reductions; and this completes the proof.

The superposition theorems (III 4.1) and (III 4.2) effect a considerable reduction of some of the questions, raised in the preceding section in connection with theorem (III 3.1), since for instance a system of bivectors in equilibrium in 4-space is the superposition of such systems with not more than seven terms. However even for systems in equilibrium with this small number of terms, a direct verification of (III 3.1) is no easy task. In fact, already systems in equilibrium with only four terms may need to be subdivided indefinitely, and modified, in order to be associated with the relevent closed polytopes. It will be found that systems of four k-vectors in algebraic equilibrium are actually in simple geometric equilibrium: this will result from their being expressible, when suitably ordered, as differences J_1-J_0, J_2-J_1, J_3-J_2, J_0-J_3, where J_0, J_1, J_2, J_3 are themselves k-vectors.

We shall establish two theorems, which concern special cases in which there is a simple geometric resultant, or simple geometric equilibrium. A system s of k-vectors will be said to be in cyclic equilibrium, if its terms, when suitably ordered, are differences $J_\nu - J_{\nu-1}$ $(\nu = 1, 2, \ldots, N)$ where the J_ν are k-vectors and $J_0 = J_N$. A system of k-vectors will be said to possess an inductive resultant, if it can be so ordered as a sequence $j_1, j_2, \ldots, j_n$, that for each $\nu = 1, 2, \ldots, N$ the system $s_\nu = \{j_1, j_2, \ldots, j_\nu\}$ has an algebraic resultant. Finally, a system of k-vectors will be termed trivially reducible, if it is the result of a finite number of superpositions of trivial systems with trivial resultant systems.

(III 4.4) Theorem. Every trivially reducible system has a simple geometric resultant; and every system which possesses an inductive resultant is trivially reducible.

(III 4.5) Theorem. Every system in cyclic equilibrium is also in simple geometric equilibrium.

Proofs. We first dispose of (III 4.5) by reducing it to (III 4.4) . To this effect we denote by s a system in cyclic equilibrium, and we show that s is equivalent by subdivision to a system s* in cyclic equilibrium, such that s* has an ϵ-modification with an inductive resultant. If s consists of the k-vectors $J_\nu - J_{\nu-1}$ $(\nu = 1, 2, \ldots, N)$, where the J_ν are k-vectors subject to $J_0 = J_N$, then by writing $J^*_\nu = J_\nu / r$, where r is a positive integer, and extending the definition for $\nu > N$ by the periodicity relation $J^*_{\nu+N} = J^*_\nu$, we determine a cyclic system s* , consisting of the k-vectors $J^*_\nu - J^*_{\nu-1}$ $(\nu = 1, 2, \ldots, rN)$. If we choose r large enough, s* will have an ϵ-modification s** , obtained by combining it with the pair of k-vectors $\pm J^*_0 = \pm J_0 / r$, each of which has magnitude $< \epsilon/2$ by choice of r . We verify at once that s** has an inductive resultant, and this completes the proof, since s* arises from s by duplication and is therefore equivalent to s by subdivision.

It remains to establish (III 4.4). It will be sufficient, by induction and by (III 4.2), to treat systems with only two terms. Our assertion is then that a pair of k-vectors, with an algebraic resultant, consists of the exterior products of two vectors with a same (k-1)-vector, and has a simple geometric resultant. We may suppose (by orthogonalization of their factors) that the pair of k-vectors consists of

$$v_1 \times j_1 \text{ and } v_2 \times j_2$$

where j_1, j_2 are (k-1)-vectors orthogonal to both v_1 and v_2. Hence, if v denotes a linear combination of v_1 and v_2 for which both the scalar products vv_1 and vv_2 differ from 0, and if J denotes the resultant of $v_1 \times j_1$ and $v_2 \times j_2$, the co-product $J \otimes v$ will be the resultant of two non-vanishing multiples of j_1 and j_2. It follows by an easy induction, firstly that j_1 and j_2 have a common factor, and secondly, by removing this factor from the original pair, that the latter are indeed the exterior products of a pair of vectors with a same (k-1)-vector j.

Our pair thus constitutes a system s of the form

$$\{v_1 \times j,\ v_2 \times j\},$$

where v_1 and v_2 may be taken orthogonal to j, and here we may clearly arrange, that, given $\epsilon^* > 0$, the magnitudes of v_1 and v_2 be $O(\epsilon^*)$, provided that of j is $O(1/\epsilon^*)$. We now denote by t a triangle, associated with $v_1 \times v_2$, whose sides are vectors proportional to v_1, v_2 and $-v_1-v_2$, the proportionality factor being then $\sqrt{2}$; and we write $\sum$ for the Cartesian product of t with a $(k-1)$-dimensional cube Q , associated with j .

With suitable orientation conventions for $\sum$ and so for its boundary, we see that the boundary of $\sum$ then includes faces associated with the k-vectors

$$v_1 \times j,\ v_2 \times j, \text{ and } (-v_1-v_2) \times j \ ;$$

we denote these faces by Δ_1, Δ_2, Δ . The remainder of the boundary of $\sum$ consists of the Cartesian product of t with the boundary of Q , and so of pairs of anti-parallel faces of measure $O(\epsilon^*)$, which by choice of ϵ^* together form an ϵ-system. Hence, if we denote by Π the polytope, derived from the boundary of $\sum'$ by removing the face Δ , we obtain a polytope of the type of the k-cell , associated with an ϵ-modification of s . The boundary of this polytope Π is that of a flat k-cell , anti-parallel to Δ . Hence, by a simplified version of lemma (III 4.3) in which now $m = 1$, we derive from Π a polytope, associated with an ϵ-modification of s and of the type of the k-cell , such that its boundary coincides with that of a simplex. Hence s has a simple geometric resultant, and this completes the proof.

III 5. The Kneser gradient. Functions of k-vectors are defined on the Grassmann cone, a subset of the linear space whose points are the composite k-vectors. However, they are also functions of the exterior product of k independent vectors , $v_1, v_2, \ldots, v_k$. This means that they can be treated as functions of nk real variables, defined for all values of these variables, but subject to the invariance condition which states that they do not alter when the vectors v_i $(i = 1, 2, \ldots, k)$ are replaced by vectors u_i $(i = 1, 2, \ldots, k)$ with the same exterior product, or what amounts to the same, by vectors u_i whose matrix is derived from that of the v_i by multiplying by a $(k \times k)$-matrix of determinant one.

In passing from the Grassmann cone to the space of the k vectors v_i, the following lemma is convenient.

(III 5.1) Lemma. Let j', j'' be a pair of unit k-vectors subject to $|j'-j''| < \epsilon$, where $0<\epsilon<1$. Then there exist representations $j' = v_1' \times v_2' \times \ldots \times v_k'$, $j'' = v_1'' \times v_2'' \times \ldots \times v_k''$, such that $|v_i'-v_i''|<\epsilon$ for $i = 1, 2, \ldots, k$.

To see this we first rewrite the inequality assumed in the form $j' j'' > 1-\frac{1}{2}\epsilon^2$, and we suppose j', j'' expressed as exterior products of two sets of unit vectors v_i' and v_i'', so that $j' j''$ becomes the determinant Δ of the matrix of the scalar products $v_r' v_s''$ $(r, s = 1, 2, \ldots, k)$. We denote by A the inverse of the unit matrix obtained by dividing each of these elements by $\Delta^{1/k}$. Then j'' is the exterior product of the unit vectors derived from the v_i'' by taking the rows of the product matrix of A with the matrix of the v_i''. If we rename these rows v_i'' $(i = 1, 2, \ldots, k)$, the scalar products $v_r' v_s''$ will constitute a diagonal matrix, of determinant $>1-\frac{1}{2}\epsilon^2$, whose diagonal elements are ≤ 1. It follows that each diagonal element exceeds $1-\frac{1}{2}\epsilon^2$, i.e. that $v_i' v_i'' > 1-\frac{1}{2}\epsilon^2$, which means, since

v_i', v_i'' are unit vectors, that $|v_i'-v_i''| < \epsilon$. This proves the lemma.

As a corollary, we remark that if dj denotes a differential of a k-vector , i.e. a differential in the space of composite k-vectors which is tangent to the Grassmann cone at a point j of that cone, and we suppose further that $j \neq 0$, then dj can be expressed in the usual manner, as the differential of the exterior product of any k vectors v_i $(i = 1, 2, \ldots, k)$ for which this product is j ; i.e. we may write $dj = \sum d_i j$ where $d_i j$ is the k-vector derived from $j = v_1 \times v_2 \times \ldots \times v_k$ by replacing the vector v_i by its differential dv_i .

From this it follows further that a function $f(j)$, defined on the Grassmann cone, is differentiable at a point $j \neq 0$, if and only if the function $F(v_1, v_2, \ldots, v_k) = f(v_1 \times v_2 \times \ldots \times v_k)$ is differentiable at a corresponding point. We suppose f kept fixed and we denote by V_i the vector $(\partial/\partial v_i)F$. Further we denote by ρ , and term contraction coefficient of f , the quantity

$$\frac{\partial}{\partial t} f(tj) \Big]_{t=1}$$

We shall suppose f differentiable at j , so that this quantity exists, and in addition we shall suppose that $\rho \neq 0$.

(III 5.2) Lemma. Let w_i $(i = 1, 2, \ldots, k)$ be vectors whose matrix is derived from that of the vectors v_i by multiplication by a unit $(k \times k)$-matrix and let W_i be the vector $(\partial/\partial w_i)\, F\, (w_1, w_2, \ldots, w_k)$. Then

$$V_1 \times V_2 \times \dots \times V_k = W_1 \times W_2 \times \dots \times W_k \ .$$

In fact, by partial derivation in v_i, we deduce from the identity $F(v_1, v_2, \dots, v_k) = \overset{\circ}{F}(w_1, w_2, \dots, w_k)$ that

$$V_i = \sum_s W_s \frac{\partial w_s}{\partial v_i} \ ,$$

and the identity to be established follows from the fact that the matrix $\partial w_s / \partial v_i$ is unitary.

(III 5.3) Lemma. We have, for $i, s = 1, 2, \dots, k$, the relations $v_i V_s = \rho \delta_{is}$, where (δ_{is}) is the diagonal unit matrix.

To see this it is sufficient to form the derivative in t, at $t = 0$, of the function obtained by substituting for v_i the vector $v_i + tv_s$, in $F(v_1, v_2, \dots, v_k)$.

(III 5.4) Definition. The k-vector

$$\rho^{-(k-1)} V_1 \times V_2 \times \dots \times V_k$$

is termed the Kneser gradient of f at $j = v_1 \times v_2 \times \dots \times v_k$, and denoted by f_j.

We note that, by lemma (III 5.2), the k-vector, defining this gradient, does in fact depend only on j.

(III 5.5) Lemma. Let $d_i j$ denote the k-vector derived from $j = v_1 \times v_2 \times \dots \times v_k$ by replacing the vector v_i by its differential dv_i. Then $f_j \, d_i j = V_i \, dv_i$.

To establish this relation, in which the right hand side is a single term, without summation convention, we observe that the product of the matrix of the V_i with the matrix used to define the k-vector $d_i j$, is, by (III 5.3), a $(k \times k)$-matrix M, which differs from a diagonal matrix, with diagonal elements ρ, only in the i-th column, where the diagonal element is $V_i dv_i$. Hence the determinant of M has the value $V_i dv_i \rho^{k-1}$. But this determinant is also, by Lagrange's rule, the scalar product of the k-vectors, defined by $d_i j$ and by the exterior product of the V_i, and its value by the definition (III 5.4) is thus $(d_i j)(f_j \rho^{k-1})$. The desired relation follows by dividing by ρ^{k-1}, and this completes the proof.

We observe that (III 5.5) implies $jf_j = v_i V_i = \rho$ and also $f_j dj = \sum_i V_i dv_i = dF = df$. The following statement includes these facts together with others that are easily verified.

(III 5.6) Theorem. Let f be a differentiable function of a k-vector j. Then its Kneser gradient exists at any such $j \neq 0$, at which the contraction coefficient ρ does not vanish, and constitutes a (k-vector)-valued function f_j of j. It is subject to the identities $jf_j = \rho$ and $f_j dj = df$ for any differential dj tangent to the Grassmann cone. Moreover, if $h(z)$ is a differentiable function of a real variable z with non-vanishing derivative h', then the composite function $h\{f(j)\}$ has the Kneser gradient $h'(f).f_j$. Finally, for $k = 1$, and, provided $\rho \neq 0$, for $k = n-1$, n, or 0, the Kneser gradient reduces to the gradient, or to the derivative, in the usual sense.

We shall establish also a less immediate property:

(III 5.7) Theorem. Let $\bar{j}$ denote the normal to j, and let $\bar{f}$ denote the function of an (n-k)-vector which is defined by writing $\bar{f}(\bar{j}) = f(j)$. Then the Kneser gradient of $\bar{f}$ at $\bar{j}$ is the normal to that of f at j.

In proving this, we may suppose, by replacing f by a function of the form $af(bj)$, where a, b are reals, that j is a unit k vector and that $\rho = 1$. We can then express j and $\bar{j}$ as exterior products $v_1 \times v_2 \times \ldots \times v_k$ and $\bar{v}_1 \times \bar{v}_2 \times \ldots \times \bar{v}_{n-k}$ of unit vectors v_i $(i = 1, 2, \ldots, k)$ and $\bar{v}_s$ $(s = 1, 2, \ldots, n-k)$ which together constitute an orthonormal system in n-space. We write $\bar{V}_s$ for the vectors defined, similarly to the V_i, for the function $\bar{f}$. Further, by forming for $t = 0$ the derivative of the functions of t obtained by replacing in f and $\bar{f}$ the vectors v_i and $\bar{v}_s$ (for a fixed pair of i and s) by the expressions

$$\frac{v_i + t\bar{v}_s}{1 + t^2} \quad \text{and} \quad \frac{\bar{v}_s - tv_i}{1 + t^2} ,$$

we find that

$$V_i \bar{v}_s = -\bar{V}_s v_i \quad (i = 1, 2, \ldots, k;\ s = 1, 2, \ldots, n-k) ,$$

since the relevant functions of t are equal, by definition of $\bar{f}$.

By expressing V_i and $\bar{V}_s$ as linear combinations of the vectors of the orthonormal system, we deduce from the relations just proved, together with those of lemma (III 5.3) and the analogous ones derived from the function $\bar{f}$, that

$$V_i \overline{V}_s = 0 \ (i = 1, 2, \ldots, k; \ s = 1, 2, \ldots, n-k) \ .$$

If we bear in mind that $\rho = \overline{\rho} = 1$, it follows that f_j , the exterior product of the V_i $(i = 1, 2, \ldots, k)$, is orthogonal to $\overline{f}_{\overline{j}}$, which is that of the $\overline{V}_s$ $(s = 1, 2, \ldots, n-k)$. The former is thus a real multiple c of the normal to the other, and the same constant c is then also the ratio of their scalar products with the unit k-vector j and its normal $\overline{j}$. Therefore $c = 1$, since both these scalar products have the value $\rho = \overline{\rho} = 1$, by theorem (III 5.6) . This completes the proof.

Many questions remain, concerning Kneser gradients and their components, the Kneser partial derivatives: to the latter, doubtless some of the differential calculus will apply unaltered, and some must be modified modulo the defining relations of the Grassmann cone. There is also the question of weakening the restriction $\rho \neq 0$: thus Barthel [1] proposed a definition of second derivatives for an $F(j)$ subject to (III 1.1) ; it may be equivalent to a device such as defining $f_{yz} = f^{-1} . \{(\partial/\partial y)(f f_z) - f_y f_z\}$. That author also considered third derivatives, but they are now covered by the case $\rho \neq 0$. Further questions concern the extension of a function, and of its Kneser gradient, to the linear space of composite vectors, and whether, for instance, this can preserve convexity. Thus, cf [1] , the gradient of certain quadratics is their Kneser gradient.

III 6. Convexity of integrands. Let $F(j)$ denote, as in the introduction, a k-integrand, or more precisely the restriction to constant x of a k-integrand $F(x, j)$, i.e. a continuous function of the k-vector j, subject to the homogeneity condition (III 1.1). On any k-dimensional polytope Π associated with a system of k-vectors j_ν ($\nu = 1, 2, \ldots, N$), the integral of $F(j)$ is then the sum $\sum_\nu' F(j_\nu)$. We shall be concerned more particularly with polytopes Π whose boundary is the same as that of a k-dimensional simplex Δ, or else a multiple of that of Δ, and we consider the following extremal problem: Under what conditions on F, is the integral of F over Δ the least value of the integral over a polytope Π of a certain type with the same boundary, or more generally, the least value of the expression

$$\frac{1}{m}\int_\Pi F ,$$

for polytopes Π whose boundary is m times that of Δ? Clearly this question is related to those considered in sections III3 and III4. We shall therefore be able to answer it only in part.

The k-integrand F will be termed convex at j, if it is subject to the inequality

$$\text{(III 6.1)} \qquad \sum_\nu F(j_\nu) \geq F(j) ,$$

for all systems of k-vectors j_ν ($\nu = 1, 2, \ldots, N$) which satisfy the relation

$$\text{(III 6.2)} \qquad \sum_\nu j_\nu = j ,$$

i.e. for all systems of k-vectors j_ν which possess j as their algebraic resultant. We term convex, an integrand which is convex at every j .

Our main theorem is as follows:

(III 6.3) Theorem. In order that the k-integrand F be convex at j , it is necessary and sufficient that, for each k-simplex Δ associated with j , the integral of F over Δ be the minimum value of the expression

$$\frac{1}{m}\int_\Pi F ,$$

for polytopes Π whose boundary is m times that of Δ .

We shall deduce this from theorem (III 3.1) . To this effect, we observe that the condition stated expresses the validity of (III 6.1) for all systems of k-vectors j_ν with the strict geometric resultant j . By continuity of F , this is equivalent to the validity of the same inequality for all systems of k-vectors j_ν with the geometric resultant j , or what comes to the same in view of (III 3.1) , with the same algebraic resultant j , i.e. for all such systems subject to (III 6.2) . This completes the proof.

The result proved holds in particular when $j = 0$. We shall say, as in [10] , that the integrand F is potentially non-negative, if it is convex at 0 . Our assertion is then:

(III 6.4) Corollary. In order that the integrand F be potentially non-negative, it is necessary and sufficient that $\int_\Pi F \geq 0$ for every closed polytope Π .

The preceding theorem and its corollary provide for the first time some basis for k-dimensional convexity assumptions which are frequently made in the calculus of variations. In the 1920s and 1930s, it was emphasized many times by Carathéodory, in conversations, that these assumptions were improperly motivated, by a false analogy with the case $k = 1$, and by the temptations of a facile generalization. To quote Hilbert: "A tree does not grow only in height, but by its roots".

However the precise extent to which these assumptions are now motivated, depends on the admissibility of two operations. One operation consists in replacing a piece of a minimizing variety by the fraction $1/m$ of a variety with m times the same boundary. The other is that of attaching by a thread, to a point of a minimizing variety, a closed polytope Π of arbitrarily high connectivity. Very few writers have been concerned with problems in which one or both these operations are admissible.

Convexity assumptions are sometimes stated in an equivalent form, which is dual to the one used above, in virtue of the following statement:

(III 6.5) Alternative definitions. In order that F be potentially non-negative, it is necessary and sufficient that its convex minorant be continuous, or again, that there exist a linear integrand aj such that $F(j)-aj \geq 0$ for all k-vectors j. In order that a differentiable F be convex at j, it is necessary and sufficient that, for all k-vectors J, the "excess function" $F(J)-F(j)-(J-j)F_j$ be non-negative. Here F_j denotes the Kneser gradient of F.

In the above statement, the convex minorant of F denotes a function G defined for composite k-vectors j by writing

$$G(j) = \mathrm{Inf} \sum_{\nu} F(j_{\nu})$$

where the infimum is for all systems of k-vectors j_ν whose algebraic sum is the composite k-vector j. The proof of (III 6.5) depends on arguments which are sufficiently known.

Variational problems in which the two operations described earlier are not admitted, justify only apparently much weaker forms of convexity. We shall say that the integrand F satisfies the mid-point convexity condition, if $F(j_1) + F(j_2) \geq F(j)$ whenever j_1, j_2 and j are k-vectors subject to $j_1 + j_2 = j$. (This condition is the analogue, for homogeneous functions, of the usual midpoint condition.)

(III 6.6) Theorem. In order that a k-integrand F satisfy the midpoint condition, it is necessary and sufficient that it be subject to (III 6.1) for all trivially reducible systems which satisfy (III 6.2).

The proof of this theorem is the same as that of the equivalence of the midpoint condition with convexity, in the case of a homogeneous function defined in the whole of a linear space. We observe that by combining theorems (III 6.6) and (III 4.4) with the remarks below, it will be found that the midpoint convexity assumption could indeed be variationally justified.

We shall term simple geometric system, a system of k-vectors with a simple geometric resultant. An integrand, which satisfies (III 6.1) for all simple geometric systems subject to (III 6.2), will be termed restrictedly convex at j.

(III 6.7) Theorem. In order that the k-integrand F be restrictedly convex at j, it is necessary and sufficient that, for each k-simplex Δ associated with j , the integral of F over Δ be the minimum value of the integral

$$\int_{\Pi} F ,$$

for polytopic k-cells Π with the same boundary as Δ .

The proof is similar to that of (III 6.3) and may be omitted. The theorem indicates that an assumption of restricted convexity might be justified in problems for simply-connected or fixed types, in the calculus of variations. However we should point out that, at the present time, we do not possess a single example of an integrand which satisfies either the midpoint condition, or that of restricted convexity, without being convex.

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