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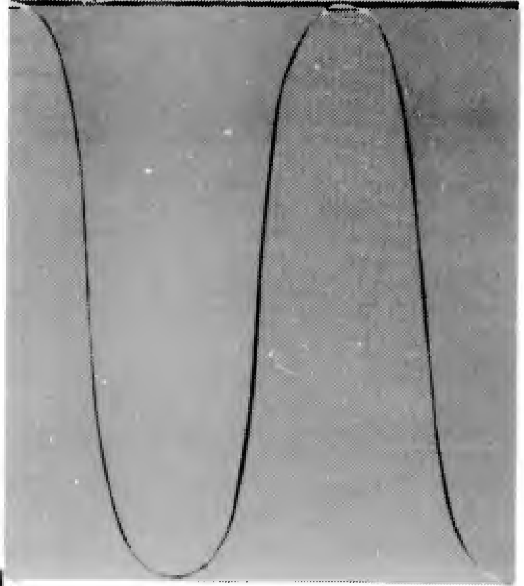

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SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES

IV. THE ALGEBRAIC AND THE GEOMETRIC RESULTANT, AN APPLICATION OF VARIATIONAL METHODS

W. H. Fleming and L. C. Young

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SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES IV

THE ALGEBRAIC AND THE GEOMETRIC RESULTANT, AN APPLICATION OF VARIATIONAL METHODS

W. H. Fleming and L. C. Young *)

IV 1. Introduction. The object of this note is to prove the theorem announced in the previous one, (III 3.1) of [13] , i.e. to establish for a system of multivectors, or in particular for a system of forces acting on a 3-dimensional rigid body, the identity of the algebraic and the geometric resultant, whenever one of them exists. The extremal nature of this theorem, already brought out by its rôle in the convexity theorem (III 6.3) of the same note, will be shown again by its proof, which uses the techniques of modern variational theory.

We recall that this theory, initiated in [7], is to a large extent independent of convexity assumptions: this is because it seeks, not only classical solutions, but also "hidden" ones, which belong, in the first and older form of the theory, to the closure of the class of classical admissible varieties with given boundaries, or in the later form, to its convex hull.

The first form concerned those "restricted" problems, in which varieties were limited to particular topological types [8, 9, 4]; in common with the

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traditional work of about that time and subsequently, it was limited by Dirichlet techniques, to 2-dimensional varieties. In this case, the structure of the admissible generalized varieties was completely characterized, so that generalized solutions automatically reduced to parametric ones, in any restricted problem whose integrand was subject to restricted convexity**) . We have already remarked, in the preceding note [13] , that there is then a proper motivation for this restricted convexity, but none for the convexity assumption so far always present in traditional work, at any rate when $n > 3$.

The second form of the theory concerns problems in which no topological restriction is made, and which the classical approach is therefore not fit to tackle. In this case a convexity assumption on the integrand is not very helpful, since it merely ensures that each generalized solution whose track happens to be parametric, should then coincide with the latter. In general, however, as shown by an example due to Fleming [3] and by that provided by the next note [14] , already in the Plateau problem the tracks need not be parametric, nor even expressible as integral currents [2] . Nevertheless it is this unrestricted form of the theory which is relevant here.

It is remarkable that to study the notion of resultant, and therefore the traditional condition of "regularity" of integrands, we have to appeal to a form of variational problem outside the scope of the classical theory.

**) A statement of C. B. Morrey [6, §1] , which appears to overlook this, was perhaps meant to refer to the very different state of affairs which exists in the unrestricted form of the theory. The paper will be found a most useful reference regarding the traditional approach.

IV 2. Basic variational lemmas. The variational character of our proof necessitates our distinguishing clearly between an integrand $F(x, j)$, and the function of j only, which is its restriction for constant x. The terms: polytopes, weighted polytopes, generalized varieties, tracks, boundaries, etc. will be understood henceforth as in [10] ; they are thus linear functionals of certain kinds, on appropriate spaces of integrands. We speak of micro-polytopes, micro-varieties, etc. when referring to appropriate functionals, considered on the spaces of restrictions of integrands to constant x, and we say that they are concentrated at the point x.

(IV 2.1) Lemma. Let G denote an open bounded convex domain of n-space, and P a closed weighted $(k-1)$-polytope in G, where $0<k<n$; further let $K(P)$ denote the class of weighted k-polytopes in G, possessing the perimeter P, and suppose given a k-integrand F_0, subject to

$$\int_\Pi F_0 \geq 0$$

for every closed k-polytope Π in G. Then, given positive reals ϵ_ν $(\nu = 1, 2, \ldots)$ tending to 0, there exist exact k-integrands $\varphi_\nu(x, j)$ $(\nu = 1, 2, \ldots)$ and a measurable function $\varphi(x, j)$, defined for the points x and the k-vectors j of n-space, such that:

(i) $\varphi_\nu(x,j) \le F_0(x,j) + \epsilon_\nu |j|$ for all ν, x, j ;

(ii) $\varphi(x,j) = \lim_\nu \varphi_\nu(x,j)$ for all j and almost all x ;

(iii) $\int_\Pi \varphi = \lim_\nu \int_\Pi \varphi_\nu$ for every weighted k-polytope Π ;

(iv) $\int_{\Pi_0} \varphi = \mathrm{Inf}_{\Pi \epsilon K(P)} \int_\Pi F_0$ for each $\Pi_0 \epsilon K(P)$.

(IV 2.2) Lemma. Let G, P, $K(P)$ be as in the previous lemma, and let L_0 be any generalized k-variety, situated in G , which possesses the perimeter P . Then there exist weighted k-polytopes Π_ν in $K(P)$ $(\nu = 1, 2, \ldots)$, such that $L_0 = \lim_\nu \Pi_\nu$, i.e. so that, for each k-integrand F, $L_0(F) = \lim_\nu \Pi_\nu(F)$, where

$$\Pi_\nu(F) = \int_{\Pi_\nu} (F) \ .$$

The proof of this pair of lemmas is the same as that of (5,1) and (7.1) of [5] , where it was supposed that $k=2$. The former depends on the Hahn-Banach theorem [1, p 55] and on well-known results of the theory of the integral. The latter, on a separation theorem in the conjugate of a Banach space and on the former of the two lemmas. Both proofs remain valid with the obvious changes, when k is unrestricted.

(IV 2.3) Lemma. Let L_0 be a generalized k-variety with the same track and support as the oriented k-simplex Δ_0 , or else with track 0 and support consisting of a single point, in which case Δ_0 will be regarded as reducing to this point and as having a vanishing track and boundary. Then there exist weighted k-polytopes Π^*_ν $(\nu = 1, 2, \ldots)$, with rational weights, such that

(i) $L_0 = \lim_\nu \Pi_\nu^*$;

(ii) each Π_ν^* has the same boundary as Δ_0 ;

(iii) each Π_ν^* has its support in a given neighbourhood of that of Δ_0 .

This lemma follows from the preceding one and from an elementary lemma, analogous, in the dimension k, to (3.3) of [5, p. 461].

IV 3. The weak closure theorem. It is convenient, momentarily, to extend slightly the notion of integrand, by allowing it to be vector-valued. If L is a generalized k-variety and $f(x, j)$ a vector-valued function, defined by a finite number N of components f_ν, each of which is a real-valued k-integrand, we shall write $L(f)$ for the vector with the N components $L(f_\nu)$. In the case in which $f(x, j) = j$, where j is a k-vector, to be regarded as a point of the enlarged vector-space, consisting of composite k-vectors, the value of $L(f)$ will be termed the flux of the generalized k-variety L ; in the case $k=2$, it is sometimes termed the oriented area, or the algebraic area, of L . In general this flux need not be a k-vector, but only a composite one; however, if it does reduce to a k-vector, we say that L has a resultant flux. Further, given any generalized k-variety L, and a point, or vector,

x_0 , the generalized variety L_0 defined by writing, for each k-integrand f ,

$$L_0(f) = L(f_0) \text{ where } f_0(x, j) = f(x-x_0, j) ,$$

will be termed translation of L by the vector x_0 , and the micro-variety, defined by the same equation when $f_0(x, j) = f(x_0, j)$, will be termed pattern of L concentrated at the point x_0 , or simply, pattern of L , when x_0 is kept fixed.

(IV 3.1) Theorem. Let M be a k-dimensional micro-variety which possesses a resultant flux j_0 . Then, for a suitable subsequence $\nu = \nu_1, \nu_2, \ldots$ of the positive integers, there exists a polytope Π_ν whose boundary coincides with that of ν repetitions of an oriented k-simplex Δ_ν with the flux j_0/ν , such that if M_ν is the pattern of Π_ν concentrated at the support of M , we have $M = \lim_\nu M_\nu$.

It is understood that, when $j_0 = 0$, we make the same conventions regarding Δ_ν , as we made for the simplex Δ_0 of lemma (IV 2.3) . In proving our theorem, we shall exclude this case, which merely requires a minor simplification of our argument. The proof will consist in applying that lemma, after exhibiting M as the pattern of a generalized k-variety L_0 , which is subject to the conditions there described.

To this effect, we denote by Δ_0 a k-simples , parallel to j_0 and of unit k-dimensional measure, and we designate by u the points of Δ_0 , and by du the k-dimensional measure of its subsets. We denote further by

M_u the translation of M by the vector u , and we choose for L_0 the mixture $\int M_u du$, i.e. we define L_0 by writing, for each k-integrand f

$$L_0(f) = \int M_u(f)du .$$

We may suppose the support of M at the origin, and we observe that M_u has the pattern M concentrated there, and that, for every linear k-integrand f , $M_u(f) = f(u, j_0)$. Consequently, the pattern of L_0 concentrated at the origin is $\int M du = M \int du = M$, and moreover, for every linear k-integrand f , we find that $L_0(f) = \int f(u, j_0)du$. This last relation means that L_0 has the same track as Δ_0 .

To the generalized variety L_0 we now apply our lemma (IV 2.3) , and we denote by N_ν a positive integer, which tends to infinity with ν and contains as factors the denominators of the rationals, which occur as weights in the weighted polytope Π_ν^* of that lemma. By replacing the sequence $\nu = 1, 2, \ldots$ by the subsequence defined by N_ν and re-indexing the weighted polytopes, we arrange that $N_\nu = \nu$, where now $\nu = \nu_1, \nu_2, \ldots$, and that the multiple ν of Π_ν^* has weights which are integers , so that it constitutes a k-polytope. From the latter, by a contraction $1/\nu$, we derive a polytope Π_ν with the same boundary as ν repetitions of a k-simplex Δ_ν , derived by this contraction from Δ_0 .

Since Π_ν and Π_ν^* clearly have a same pattern M_ν , consisting of a micro-polytope concentrated at the origin, we now derive the relation $M = \lim_\nu M_\nu$

by applying the relation (i) of (IV 2.3), in the form $L_0(f_0) = \lim_\nu \Pi_\nu^*(f_0)$, to integrands f_0 of the type $f_0(x, j) = f(0, j)$.

IV 4. The strong closure theorem. By the Riesz representation, a micro-variety concentrated at x_0 is specified by a finite measure $d\mu$ on the unit sphere $|j| = 1$ of the k-vectors j , i.e. by the functional $M(f) = \int f(x_0, j)d\mu$ defined in the space of k-integrands f . In the case of a micro-polytope, $d\mu$ vanishes in the complement of a finite set of unit k-vectors . We shall write $|M|$ for the total measure $\mu = \int d\mu$ of the unit sphere $|j| = 1$, and more generally $|M-M'|$ for the total variation of the difference $d\mu - d\mu'$ of the two measures which specify the two micro-varieties M and M' . We say that a micro-variety M is the strong limit of micro-varieties M_ν if $|M-M_\nu| \to 0$.

(IV 4.1) Theorem. Let M be a k-dimensional micro-polytope which possesses a resultant flux j_0 . Then for a suitable subsequence $\nu = \nu_1, \nu_2, \ldots$ of the positive integers, there exists a polytope Π_ν , whose boundary coincides with that of ν repetitions of an oriented k-simplex Δ_ν with the flux j_0/ν , such that if M_ν is the pattern of Π_ν concentrated at the support of M , we have $M = \text{strong} \lim_\nu M_\nu$.

We shall derive this from the weak closure theorem, with the help of the following elementary construction:

(IV 4.2) Lemma. Let j', j'' be a pair of unit k-vectors , and let $\epsilon > k|j'-j''|$. Then, given any k-simplex Δ'' with flux j'' , we can construct a polytope Π' with the same boundary as Δ'' , such that Π' consists of parallel faces of total flux j' together with further faces of total k-dimensional measure $<\epsilon$.

Proof of (IV 4.2) . According to a lemma (III5.1) established in the previous note [13] , we can express j' , j'' as exterior products of unit vectors

$$j' = v_1' \times v_2 \times \ldots \times v_k' \ , \ j'' = v_1'' \times v_2'' \times \ldots \times v_k''$$

where $|v_i' - v_i''| < \epsilon/k$ for each $i = 1, 2, \ldots, k$. This implies that there is an inductive system [13, III] with the resultant j'' , whose terms have the form $j_0, j_1, \ldots, j_k$, where $j_0 = j'$ and $|j_i| < \epsilon/k$ for $i = 1, 2, \ldots, k$. To see this, it is sufficient to set $j' = j_0 = J_0$, and $j_i = J_i - J_{i-1}$ for $i = 1, 2, \ldots, k$, where J_i is the exterior product of the first i factors in our expression for j'' with the last $k-i$ factors of that for j' . The assertion of our lemma therefore follows from theorem (III4.4) of the previous note [13] , in view of the fact that $|j_1| + |j_2| + \ldots + |j_k| < \epsilon$. In fact we can choose Π' to be of the type of the k-cell and to be such that the faces not associated with our inductive system have k-dimensional total measure $<\epsilon - |j_1| - |j_2| - \ldots - |j_k|$. This completes the proof.

Proof of (IV 4.1) . Our hypothesis includes that of (IV 3.1) and we shall now denote by $\widetilde{\Pi}_\nu$ $(\nu = \nu_1, \nu_2, \ldots)$ the polytopes there obtained, and by $\widetilde{M}_\nu$ their patterns. We shall denote further, by E the finite set of unit k-vectors which is the support of the measure $d\mu$, associated with the micro-polytope M . We term E the j-support of M .

We denote provisionally by a_r $(r = 1, 2, \ldots)$ positive numbers tending to 0, which do not exceed half the smallest distance, in the k-vector space, of two distinct members of E; and by $B_{\nu r}$ the set of those faces, if any, of $\widetilde{\Pi}_\nu$, for each of which the unit k-vector j, proportional to the flux of the face, satisfies for some $j' \epsilon E$ the inequality

$$a_r > k|j'-j| \, .$$

The k-dimensional measure of the set of points of $\widetilde{\Pi}_\nu$ not in $B_{\nu r}$ will, moreover, be written provisionally $a_{\nu r}$.

Since, by (IV 3.1), the pattern $\widetilde{M}_\nu$ of $\widetilde{\Pi}_\nu$ has the (weak) limit M, it is clear that, for fixed r, $\lim_\nu a_{\nu r} = 0$. This implies the existence of a function $\nu(r)$, which increases to infinity with r, such that

$$a_{\nu r} < a_r \quad \text{for} \quad \nu \geq \nu(r), \quad \nu = \nu_1, \nu_2, \ldots \, .$$

It follows further, that, for a suitable subsequence of ν, which we may suppose renamed $\nu = \nu_1, \nu_2, \ldots$, there exists a function $r(\nu)$ increasing to infinity with ν, such that

$$a_{\nu r} < a_r \quad \text{for} \quad r = r(\nu), \quad \nu = \nu_1, \nu_2, \ldots \, .$$

We shall write, for brevity, ϵ_ν and B_ν, for the values of a_r and $B_{\nu r}$ when $r = r(\nu)$.

Thus the ϵ_ν are positive numbers tending to 0 as ν describes the sequence $\nu_1, \nu_2, \ldots$; and B_ν is the set of those faces of $\widetilde{\Pi}_\nu$ for each of which the unit k-vector j, proportional to the flux of the face, satisfies for some $j' \in E$ the inequality

$$\epsilon_\nu > k|j'-j| \ ;$$

moreover, the k-dimensional measure of the set of points of $\tilde{\Pi}_\nu$ not in B_ν, is $<\epsilon_\nu$.

This being so, we associate to each simplex $\Delta \in B_\nu$ a polytope Π with the same boundary, derived by similarity from the polytope Π' of lemma (IV 4.2), where we choose for Δ'' a simplex whose flux is the unit k-vector j proportional to the flux of Δ, and where $j' \in E$ and $\epsilon = \epsilon_\nu$. Then Π will consist of faces parallel to $j' \in E$, together with further faces of total k-dimensional measure $<\epsilon_\nu |\Delta|$, where $|\Delta|$ is the k-dimensional measure of Δ.

By replacing each simplex $\Delta \in B_\nu$ by the corresponding polytope Π, we derive from $\tilde{\Pi}_\nu$ a new polytope Π_ν, with the same boundary. We denote by M_ν the pattern of Π_ν, concentrated at the support of M, and it only remains to be shown that $M = \text{strong} \lim_\nu M_\nu$.

To this effect, we write $\Pi_\nu = \Pi_\nu' + \Pi_\nu''$ and $M_\nu = M_\nu' + M_\nu''$, where Π_ν' consists of the faces of Π_ν parallel to k-vectors of E, and Π_ν'' of the remaining faces, and where M_ν' and M_ν'' are the patterns of Π_ν' and Π_ν''. We denote further by A the supremum of the values of $\tilde{M}_\nu(f)$ for the integrand $f(x,j) = |j|$; clearly A is finite since $\tilde{M}_\nu \to M$; moreover A is also the supremum of the k-dimensional measures of the polytopes

$\tilde{\Pi}_\nu$ $(\nu = \nu_1, \nu_2, \ldots)$. The k-dimensional measure of the faces of Π_ν which are not in $\tilde{\Pi}_\nu$ and are not parallel to k-vectors in E is at most

$$\epsilon_\nu \sum_{\Delta \epsilon B_\nu} |\Delta| \leq \epsilon_\nu A ;$$

since the k-dimensional measure of $\tilde{\Pi}_\nu - B_\nu$ is $< \epsilon_\nu$, it follows that that of Π_ν'' is $< \epsilon_\nu (A+1)$, and therefore that $|M_\nu''| < \epsilon_\nu (A+1)$. Thus strong $\lim_\nu M_\nu'' = 0$.

A similar calculation shows that, for each k-integrand f of the form f(j) , independent of x , the difference $\tilde{\Pi}_\nu(f) - \Pi_\nu(f)$ has, for $\nu = \nu_1, \nu_2, \ldots$, the limit 0 . In fact, if ω_ν denotes the oscillation of f for an alteration ϵ_ν of its argument on the unit sphere $|j| = 1$, and Ω is the supremum of $|f|$ on this sphere, we find that the difference in question is majorized in absolute value by

$$2 A \Omega \epsilon_\nu + A \omega_\nu .$$

Hence, by applying this to integrands of the form $f(x_0, j)$, we see that $\lim_\nu (\tilde{M}_\nu - M_\nu) = 0$, and therefore that

$$\lim_\nu M_\nu = M ,$$

and hence that $\lim_\nu M_\nu' = M$, i.e. that $\lim_\nu M_\nu'(f) = M(f)$, for each k-integrand f .

This being so, let $j' \epsilon E$ and let (j') be the set consisting of the single element j'. We denote by α and α_ν the values at (j') of the measures, associated by the Riesz representation with the micro-polytopes M and M_ν'. Since E is a finite set, we can select an integrand f of the form $f(j)$, independent of x, so that $f(j') = 1$ and $f(j) = 0$ when $j \neq j'$, $j \epsilon E$. For this f we find that $M(f) = \alpha$, $M_\nu(f) = \alpha_\nu$, and hence $\alpha = \lim_\nu \alpha_\nu$, i.e. $\lim_\nu |\alpha - \alpha_\nu| = 0$. Since E is a finite set, it follows that $\lim_\nu |M - M_\nu'| = 0$, i.e. that strong $\lim_\nu M_\nu' = M$.

By adding to this the relation strong $\lim_\nu M_\nu'' = 0$, obtained earlier, we find that strong $\lim_\nu M_\nu = M$, as asserted; and this completes the proof.

In regard to the theorem just proved, it is convenient to remark that we can arrange further that the patterns of the polytopes Π_ν shall have the form

$$M_\nu = M + P_\nu ,$$

where P_ν is a closed micro-polytope with limit 0. To see this, it is sufficient to modify M_ν by the addition of a finite number of terms, consisting of the patterns of pairs of k-simplices, corresponding to the elements $j' \epsilon E$, and for which the flux is $\pm(\alpha - \alpha_\nu)j'$. This entails a similar modification of the polytopes Π_ν, which does not affect our other assertions.

IV 5. Final form of the closure theorem and proof of the equivalence of algebraic and geometric resultants. By the adjoint of a generalized variety L, and in particular of a micro-polytope, we mean a generalized variety L^* such that $L^*(f) = L(f^*)$ where $f^*(x, j) = f(x, -j)$.

(IV 5.1) Theorem. Let M be a k-dimensional micro-polytope which possesses a resultant flux j_0. Then for a suitable subsequence $v = \nu_1, \nu_2, \ldots$ of the positive integers, there exists a polytope Π_ν, whose boundary coincides with that of ν repetitions of an oriented k-simplex Δ_ν with the flux j_0/ν, such that the pattern of Π_ν, concentrated at the support of M, has the form

$$M + \widetilde{P}_\nu + \widetilde{P}_\nu^*$$

where $\widetilde{P}_\nu$, $\widetilde{P}_\nu^*$ are micro-polytopes, which are the adjoints of one another, and where $\lim_\nu \widetilde{P}_\nu = \lim_\nu \widetilde{P}_\nu^* = 0$.

We remark that the closure theorem in this form implies the theorem announced in the previous note (III 3.1) of [13], namely:

(IV 5.2) Corollary. Every system of multivectors, which possesses the algebraic resultant j_0, possesses also the geometric resultant j_0, and vice-versa.

In fact, we may ignore the converse part, which is trivial, and suppose given a system with an algebraic resultant j_0. Clearly the system determines uniquely a corresponding micro-polytope M, which is the pattern of every polytope associated with the given system; moreover M possesses the resultant flux j_0. Given $\epsilon > 0$, we can determine, according to theorem (IV 5.1), the micro-polytopes P_ν, P_ν^* so that

$$|P_\nu| + |P_\nu{}^*| < \epsilon \ ,$$

remembering that, for instance, $|P_\nu| = P_\nu(f)$ for $f(x,j) = |j|$. Hence by combining the given system with an ϵ-system which corresponds to $P_\nu + P_\nu{}^*$, we obtain a system of multivectors associated with the polytope Π_ν , and therefore possessing the strict geometric resultant j_0 . This proves the corollary if theorem (IV 5.1) holds.

Before proving the latter, it is convenient to establish the following elementary results:

(IV 5.3) Lemma. Let P^* denote a k-dimensional closed micro-polytope. Then there exist closed micro-polytopes Q, Q^* adjoint to one another, such that $P^* + Q + Q^*$ is the pattern of a closed polytope Π and $|Q| = |Q^*| < n!\ |P^*|$.

(IV 5.4) Lemma. Let s_0 be the system of the components of a k-vector j_0 , and let $\epsilon > 0$. Then there exists an ϵ-system s , such that $s+s_0$ has the strict simple geometric resultant j_0 , i.e. there exists a polytope Π_0 of the type of the k-cell, which is associated with the system $s+s_0$ and has the same boundary as a simplex Δ_0 whose flux is j_0 .

Proof of (IV 5.4) . We prove this first, and in so doing we use induction with respect to the sum $k+n$ of the dimensions of our k-vectors and our underlying Euclidean n-space . We denote by v a unit vector along a co-ordinate axis and by s_1 , s_2 the systems of components of the k-vectors

j_1, j_2 where $j_1 = vx(j\otimes v)$, $j_2 = (vxj)\otimes v$ are the vertical and horizontal projections of j , the vector v being regarded as vertical. By the inductive hypothesis, the statements in our lemma are applicable to the system s_1 of the components of j_1 and to the system s_2 of those of j_2 , since the lemma is evidently valid when $k = 1$, and also when $n = 2$. Hence by the superposition theorem of the previous note (III4.2) of [13] , these statements apply to the combination s_0 of s_1 and s_2 , and this is our assertion.

Proof of (IV 5.3) . Let E be the finite set of the k-vectors on the unit sphere $|j| = 1$ which constitute the j-support of P^* , and for each $j' \epsilon E$, let $\alpha(j')$ be the value at (j') of the measure, associated by the Riesz representation with P^* . We denote by E_0 the set of the k-vectors of the form $j_0 = j'\alpha(j')$. To each such j_0 , we attach, for a sufficiently small $\epsilon>0$, the corresponding polytope Π_0 and simplex Δ_0 of (IV 5.4) , and we observe that the k-dimensional measures $|\Pi_0|$ and $|\Delta_0|$ then satisfy the inequality $|\Pi_0| < \epsilon + N|\Delta_0|$, where $N<n!$ is the number of components of a k-vector in n-space. Thus $|\Pi_0| < n!\,|\Delta_0|$.

We now denote by M_0 the pattern of Π_0 and by Q the sum of the micro-polytopes M_0 , which correspond to the various $j_0 \epsilon E_0$ and have the same support as P^* . Clearly

$$|Q| < n! \sum |j_0| < n!\,|P^*| .$$

We denote further by Q^* the adjoint of Q , and it only remains to be shown that $P^* + Q + Q^*$ is the pattern of a closed polytope.

To this effect, we write $Q = Q' + Q''$, where Q'' , where Q'' is the pattern of the relevant sum of ϵ-systems . Evidently Q'' is the pattern of a sum of closed polytopes, consisting of pairs of adjoint simplices. Further $P^* + Q^*$ is the pattern of a sum of closed polytopes of the form $\Delta_0 + \Pi_0^*$, where Π_0^* is the adjoint of Π_0 . Finally Q' , by construction, is closed, and its j-support on the unit sphere $|j| = 1$ consists of k-vectors parallel, or antiparallel, to coordinate subspaces of dimension k . This is only possible if Q' is the pattern of a sum of pairs of adjoint simplices, parallel to these coordinate subspaces, since each component of the flux of Q' must vanish. Hence by addition, the micro-polytope

$$P^* + Q + Q^* = Q'' + (P^* + Q^*) + Q'$$

is the pattern of a sum of closed polytopes, and therefore that of a closed polytope. This completes the proof.

Proof of (IV 5.1). We define P_ν as in the remark at the end of the preceding section, just after the proof of (IV 4.1) , and we write $P_\nu{}^*$ for the adjoint of P_ν . We then define as in lemma (IV 5.3) micro-polytopes $Q = Q_\nu$, $Q^* = Q_\nu{}^*$ corresponding to $P^* = P_\nu{}^*$. Our assertions are now easily verified, when we set $\widetilde{P}_\nu = P_\nu + Q_\nu$, $\widetilde{P}_\nu{}^* + Q_\nu{}^*$.

REFERENCES

[1] Banach S., Théorie des opérations linéaires, Warsaw 1932.

[2] Federer H. and Fleming W. H., Normal and integral currents, Annals of Math. 72 (1960) 45[-520.

[3] Fleming W. H., An example in the problem of least area, Proc. Amer. Math. Soc. 7 (1956) 1063-1074.

[4] ______________________ Irreducible generalized surfaces, Riv. Mat. Univ. Parma 8 (1957) 251-281.

[5] Fleming W. H. and Young L. C., A generalized notion of boundary, Trans. Amer. Math. Soc. 76 (1954) 457-484.

[6] Morrey C. B. Jr., The parametric variational problem for double integrals, Comm. P. & A. Math. 14 (1961) 569-575.

[7] Young L. C., Generalized curves ..., Comptes Rendus Soc. Sciences et Lettres, Varsovie, Ce III 30 (1937) 325-341.

[8] ______________________ Surfaces parametriques généralisées, Bull. Soc. Math. France 79 (1951) 59-85.

[9] ______________________ Generalized surfaces of finite topological type, Memoirs Amer. Math. Soc. No. 17 (1955).

[10] ______________________ Contours on generalized and extremal varieties, Journ. Math. & Mech.

[11] ______________________ Some extremal questions for simplicial complexes I. Polyhedral geodesic strips. MRC report 304.

[12] ______________________ Some extremal questions for simplicial complexes II. On the 'radius times periphery' problem for area. MRC report 307.

[13] ______________________ Some extremal questions for simplicial complexes III. Problems of the geometry and analysis of the higher euclidean spaces. MRC report 320.

[14] ______________________ Some extremal questions for simplicial complexes V. The relative area of a Klein bottle. MRC report

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