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Some Relationships between the Tchebycheff Approximations on an Interval and on a Discrete Subset of That Interval

E. W. Cheney

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SOME RELATIONSHIPS BETWEEN THE TCHEBYCHEFF APPROXIMATION ON AN INTERVAL AND ON A DISCRETE SUBSET OF THAT INTERVAL

by

Professor E. W. Cheney
University of California, Los Angeles
(Visiting Staff Member)
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1. Introduction

Let there be prescribed an integer n and a continuous function f defined on the interval $[-1,1]$. Among all the polynomials of degree not exceeding n there exists one, P, for which the deviation

$$\max_{-1 \leq x \leq 1} | f(x) - P(x) |$$

is an absolute minimum. In other words, under the "uniform" metric the minimum distance between the function f and the set of all polynomials having degree $\leq n$ is achieved by an appropriate polynomial, P; (as a matter of fact, P is unique.) We may denote this minimum distance by $E(n,f)$.

Now a great deal is known about P and about $E(n,f)$. For example, a theorem of Tchebycheff characterizes P by a certain property of equi-oscillation about f. By the Weierstrass theorem, we know that $E(n,f) \downarrow 0$ as $n \uparrow \infty$. Theorems by Jackson [4] tell us how the smoothness of f influences the rapidity with which $E(n,f) \downarrow 0$. On the other hand, in any but the most elementary cases, an exact determination of the polynomial P of best approximation involves the solution of systems of non-linear algebraic equations which must be attacked using iterative techniques. For practical reasons, refuge is therefore taken in the following device. A finite subset Y of $[-1,1]$ is selected. Assume that Y contains at least $n+1$ points. Then again there will exist exactly one polynomial Q of degree $\leq n$ for which the expression

$$\max_{y \in Y} | f(y) - Q(y) |$$

is a minimum. If the set Y in some sense "fills out" the interval $[-1,1]$ then with some confidence Q is taken as a substitute for P. Some grounds for this confidence will be developed below.

Most of the results that we have in mind remain true in a more general setting. Whenever possible, then, we shall state the most general case, followed by one or more similar theorems worked out in greater detail for the case of algebraic polynomials. The generalized setting will be as follows. Instead of algebraic polynomials of degree n, which are formed as linear combinations of the functions

$$1,\ x,\ x^2, \ldots,\ x^n\ ,$$

we shall consider "generalized polynomials", which are formed as linear combinations of some other prescribed set of functions

$$g_0,\ g_1, \ldots,\ g_n$$

which are assumed to be continuous on the interval $[-1,1]$, and which in addition have the property that for any selection of $n + 1$ points

$$-1 \leq x_0 < x_1 < \ldots < x_n \leq 1$$

the determinant $\operatorname{Det} g_i(x_j)$ is not zero. A set of functions having these two properties is said to be a <u>Tchebycheff system</u>. The non-vanishing of Vandermonde's determinant

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{vmatrix}$$

implies that $\{1, x, x^2, \ldots, x^n\}$ is a Tchebycheff system. Our restriction to the interval $[-1,1]$ is a matter of convenience rather than necessity.

Now corresponding to a Tchebycheff system $\{g_0, \ldots, g_n\}$ we define a function which we call the <u>joint modulus of continuity</u>:

$$\Omega(\delta) = \max_{0 \le i \le n} \max_{|x-y| \le \delta} |g_i(x) - g_i(y)|$$

It is clearly the maximum of the moduli of continuity of the functions g_i.

It will be convenient to introduce some other notations. We define

$$\|f\| = \max_{-1 \le x \le 1} |f(x)|$$

$$\|f\|_Y = \max_{y \in Y} |f(y)|$$

$$|Y| = \max_{-1 \le x \le 1} \min_{y \in Y} |x - y|$$

$$|Y|_\pi = \max_{-1 \le x \le 1} \min_{y \in Y} |\cos^{-1}x - \cos^{-1}y| .$$

The latter two expressions give us measures for how nearly a set Y "fills out" the interval $[-1,1]$.

For a Tchebycheff system $\{g_0, \ldots, g_n\}$ the generalized polynomial of best approximation to a prescribed continuous function is necessarily unique. This is also proved here in Lemma 5 in stronger form.

2. Some Lemmas

We collect here a number of preliminary results which are used later.

Lemma 1 Let $\{g_0, \ldots, g_n\}$ be a Tchebycheff system for $[-1,1]$ and Y a finite subset of $[-1,1]$ containing at least $n+1$ points. Then there is a constant α such that

$$\|P\| \leq \alpha \|P\|_Y$$

for all generalized polynomials P. Furthermore $\alpha \to 1$ as $|Y| \to 0$.

Proof The constant $1/\alpha$ may be taken to be the minimum of the positive continuous functional $\|P\|_Y$ over the compact set defined by $\|P\| = 1$. To complete the proof, let β denote the minimum of the positive continuous functional $\|P\|_Y$ on the compact set defined by $\Sigma |\lambda_i| = 1$, where $P = \Sigma \lambda_i g_i$. Now let P be fixed. Let x be a point where $|P(x)|$ is a maximum. Select $y \in Y$ as close as possible to x. Then $\|P\| = |P(x)| \leq |P(x) - P(y)| + |P(y)| \leq \Sigma |\lambda_i| \, |g_i(x) - g_i(y)| + |P(y)| \leq \Sigma |\lambda_i| \, \Omega(|Y|) + |P(y)| \leq \frac{1}{\beta} \Omega(|Y|) \|P\|_Y + \|P\|_Y$. Since $\Omega(\delta) \to 0$ as $\delta \to 0$, this completes the proof.

Lemma 2 For algebraic polynomials of degree n, we have

$$\|P\| \leq \frac{1}{1 - n^2 |Y|} \|P\|_Y \qquad \text{when } |Y| < n^{-2}$$

$$\|P\| \leq \frac{1}{1 - n |Y|_\pi} \|P\|_Y \qquad \text{when } |Y|_\pi < n^{-1}$$

$$\|P\| \leq \frac{1}{1 - \frac{1}{2} n^2 |Y|_\pi^2} \|P\|_Y \qquad \text{when } |Y|_\pi < \sqrt{2}\, n^{-1}.$$

<u>Proof</u> Markov's inequality states that $\|P'\| \leq n^2 \|P\|$. Taking x and y as before, we have $\|P\| = |P(x)| \leq |P(x) - P(y)| + |P(y)| = |P'(\xi)|.|x - y| + |P(y)| \leq n^2 \|P\|.|Y| + \|P\|_Y$. Solving for $\|P\|$ gives us $\|P\| \leq \dfrac{1}{1 - n^2 |Y|} \|P\|_Y$.

Bernstein's inequality [1,3] states that $|T'(\Theta)| \leq n . \max\limits_{0 \leq \Theta \leq 2\pi} |T(\Theta)|$, when T is any trigonometric polynomial of degree $\leq n$. Taking x and y as before, put $\Phi = \cos^{-1} x$ and $\psi = \cos^{-1} y$. Then $\|P\| = |P(x)| \leq$ $\leq |P(\cos\Phi) - P(\cos\psi)| + |P(y)| = |\frac{d}{d\Theta} P(\cos\Theta)| \, |\Phi - \psi| + |P(y)| \leq$ $\leq n \|P\|.|Y|_\pi + \|P\|_Y$. Solving for $\|P\|$ we get $\|P\| \leq \dfrac{1}{1 - n|Y|_\pi} \|P\|_Y$. We include this argument because of its simplicity; it can be improved by the following trick of Bernstein. [2].

Define $T(\Theta) = P(\cos\Theta)$. Define Φ and ψ as before. Then $T'(\Phi) = 0$. Using Bernstein's inequality, we have $|T'(\Theta)| = |T'(\Theta) - T'(\Phi)| =$ $= |T''(\Theta_0)| \, |\Theta - \Phi| \leq n^2 \|T\| \, |\Theta - \Phi|$. Thus $\|P\| = |T(\Phi)| = |T(\psi) + \int_\psi^\Phi T'(\Theta)\, d\Theta| \leq$ $\leq |P(y)| + n^2 \|P\| \int_\psi^\Phi |\Theta - \Phi)\, d\Theta \leq |P(y)| + n^2 \|P\| \frac{1}{2}(\Phi - \psi)^2 \leq \|P\|_Y +$ $+ \frac{1}{2} n^2 \|P\|.|Y|_\pi^2$. Solving again, we have $\|P\| \leq \dfrac{1}{1 - \frac{1}{2} n^2 |Y|_\pi^2} \|P\|_Y$.

<u>Remark</u> The degree to which Y fills out the interval $[-1,1]$ can be measured by means of any monotone function Φ by defining

$$|Y|_\Phi = \max_{-1 \leq x \leq 1} \min_{y \in Y} |\Phi(x) - \Phi(y)| .$$

To make comparisons meaningful we may require $|\Phi(1) - \Phi(-1)| = 2$. For example, we have already used $\Phi(x) = x$ and $\Phi(x) = \frac{2}{\pi} \cos^{-1} x$. It would be nice to know whether any such function Φ exists which permits a

significant improvement in the preceding lemma.

<u>Lemma 3</u> Let $\{g_0, \ldots, g_n\}$ be a Tchebycheff system for $[-1,1]$. Then there is a constant β such that for any continuous f with modulus of continuity ω, and for any generalized polynomial P,

$$\| f - P \| \leq \| f - P \|_Y + \omega(|Y|) + \beta \|P\| \Omega(|Y|) .$$

<u>Proof</u> Let x be a point where $|f - P|$ is a maximum. Let y be a point of Y as close as possible to x. Let $1/\beta$ be the minimum of the positive continuous functional $\|P\|$ on the compact set defined by $\Sigma |\lambda_i| = 1$, where $P = \Sigma \lambda_i g_i$. Then $\| f - P \| = |f(x) - P(x)| \leq$
$\leq |f(x) - f(y)| + |f(y) - P(y)| + |P(y) - P(x)| \leq$
$\leq \omega(|Y|) + \| f - P \|_Y + \Sigma|\lambda_i| \Omega(|Y|)$. The proof is completed by noting that $\Sigma |\lambda_i| \leq \beta \|P\|$.

<u>Lemma 4</u> In the case of algebraic polynomials, the two last terms in the inequality of Lemma 3 may be replaced by

$$\omega(|Y|) + n^2 \| P \| . |Y| \quad \text{or} \quad \omega(|Y|) + n \| P \| . |Y|_\pi .$$

<u>Lemma 5</u> Let $\{g_0, \ldots, g_n\}$ be a Tchebycheff system for $[-1,1]$ and let f be a continuous function on $[-1,1]$. Then there is a constant $\gamma > 0$ such that

$$\| f - Q \| \geq \| f - P \| + \gamma \|P - Q\|$$

where P and Q are generalized polynomials, Q being arbitrary and P being the best approximation to f.

<u>Proof</u> By the Tchebycheff-Bernstein theorem there exist points $x_i \in [-1,1]$

$(i = 0,\ldots, n+1)$ and signs σ_i such that $P(x_i) - f(x_i) = \sigma_i \| P - f \|$. For each $x \in [-1,1]$ define an $(n+1)$-tuple $\hat{x} = (g_0(x), g_1(x),\ldots, g_n(x))$. By a theorem of [6], the origin in $(n+1)$-space lies in the convex hull of the points $\sigma_i \hat{x}_i$. Thus an equation $0 = \Sigma \lambda_i \sigma_i \hat{x}_i$ holds true, with $\lambda_i \geq 0$ and $\Sigma \lambda_i = 1$. From the definition of a Tchebycheff system, $\lambda_i > 0$. Let $R = \Sigma c_j g_j$ be any generalized polynomial of norm 1. Then $0 = \sum_{ij} c_j \lambda_i \sigma_i g_j(x_i) = \sum_i \lambda_i \sigma_i R(x_i)$. Not all the numbers $\sigma_i R(x_i)$ can be zero since $\|R\| = 1$, and since $\lambda_i > 0$, we conclude that at least one of them is positive and one negative. Thus the number $\gamma = \min_{\|R\|=1} \max_i \sigma_i R(x_i)$ is positive, being the minimum of a positive continuous functional on a compact set. Now let Q be an arbitrary generalized polynomial. If $Q = P$ the inequality to be proved is trivial. Otherwise put $R = (P-Q)/\|P-Q\|$. Then select i so that $\sigma_i R(x_i) \geq \gamma$. We have $\| f - Q \| \geq \sigma_i \{f(x_i) - P(x_i)\} + \sigma_i \{ P(x_i) - Q(x_i)\} = \| f - P \| + \|P - Q\| \sigma_i R_i \geq \| f - P \| + \gamma \| P - Q \|$. This lemma tells us that the function $\Delta(c_0,\ldots, c_n) = \| f - \Sigma c_i g_i \|$ has a graph which comes to a sharp point at its minimum. This is not true for the least-squares norm.

<u>Lemma 6</u> $\quad |Y|_\pi \leq \pi \sqrt{\tfrac{1}{2} |Y|}$.

<u>Proof</u> It will suffice to prove $|\cos^{-1} x - \cos^{-1} y| \leq \pi \sqrt{\tfrac{1}{2} |x - y|}$. There is no loss of generality in supposing that $\cos^{-1} x \geq \cos^{-1} y$. Put $\cos^{-1} x = \alpha + \beta$ and $\cos^{-1} y = \alpha - \beta$, where $0 \leq \alpha \pm \beta \leq \pi$. Thus $0 \leq \beta \leq \tfrac{1}{2}\pi$ and $|x - y| = 2 \sin\alpha \sin\beta$. The inequality to be proved now reads $4\beta^2 \leq \pi^2 |\sin\alpha \sin\beta|$. If β is fixed, then $\beta \leq \alpha \leq \pi - \beta$ and consequently the minimum value of $\sin\alpha$ is $\sin\beta$. It therefore will suffice to prove $4\beta^2 \leq \pi^2 \sin^2\beta$, or $2\beta \leq \pi \sin\beta$; but this last inequality

follows immediately from the fact that $\sin\beta$ is a concave function on the interval $[0, \frac{1}{2}\pi]$.

3. The Two Approximations Related

The first type of result to establish is that the error $\| f - Q \|$ is approximately the same for a discrete approximation Q as for the best approximation on the interval. Thus the discrete approximation is a safe substitute for the more-difficult-to-obtain approximation on the interval.

Theorem 1 Let $\{g_0, \ldots, g_n\}$ be a Tchebycheff system for $[-1,1]$ and P the generalized polynomial of best approximation to a continuous function f having modulus of continuity ω . Let Q be the generalized polynomial of best approximation to f on a discrete set Y . Then

$$\| f - Q \| \leq \| f - P \| + \omega(|Y|) + 2\alpha\beta \|f\| \Omega(|Y|) \quad .$$

Proof It is clear that $\| f - Q \|_Y \leq \|f - P\|_Y \leq \| f - P \| \leq \| f - Q \|$. By Lemma 3, the first and last terms of this inequality will approach each other as $|Y| \to 0$. Thus $\| f - Q \| - \| f - P \| \leq \| f - Q \| - \| f - Q \|_Y \leq$ $\leq \omega(|Y|) + \beta \|Q\| \Omega(|Y|)$. To complete the proof, use Lemma 1 : $\|Q\|_Y \leq \|Q - f\|_Y + \|f\|_Y \leq 2\|f\|$, and therefore $\|Q\| \leq 2\alpha \|f\|$.

Theorem 2 For algebraic polynomials the inequality of Theorem 1 may be written, as long as $|Y|_\pi < n^{-1}$,

$$\| f - Q \| \leq \| f - P \| + \omega(|Y|) + 4n \| f \| . |Y|_\pi .$$

The next theorem shows that, as $|Y| \to 0$, the discrete approximation converges uniformly to the best approximation on the interval.

<u>Theorem 3</u> Let $\{g_0, \ldots, g_n\}$ be a Tchebycheff system on $[-1,1]$ and f a continuous function with modulus of continuity ω. As $|Y| \to 0$, the generalized polynomial Q of best approximation to f on Y converges uniformly to the generalized polynomial P of best approximation to f on $[-1,1]$. Specifically

$$\| Q - P \| \leq \gamma^{-1} [\omega(|Y|) + 2\alpha\beta \| f \| \Omega(|Y|)]$$

where α, β, and γ are positive constants defined earlier.

<u>Proof</u> From Lemma 5, $\gamma \| Q - P \| \leq \| f - Q \| - \| f - P \|$. The proof is completed by applying Theorem 1 to this inequality.

<u>Theorem 4</u> For algebraic polynomials, we have the estimate

$$\| Q - P \| \leq \gamma^{-1} [\omega(|Y|) + 4n \| f \| \, |Y|_\pi]$$

as long as $|Y|_\pi < 1/n$.

We turn finally to the situation which is presented when the infinite sequence of monomials

$$1, x, x^2, x^3, \ldots \tag{1}$$

is replaced by another infinite sequence of continuous functions

$$g_0, g_1, g_2, \ldots \quad . \tag{2}$$

Such a system is termed a <u>Markov System</u> if every initial segment

$$g_0, g_1, \ldots, g_n$$

is a Tchebycheff system. We assume further that system (2) is <u>fundamental</u> in $C[-1,1]$. This means that the linear combinations of the functions g_i form a dense set in $C[-1,1]$. Thus, given $f \in C[-1,1]$ and $\epsilon > 0$, there must exist an index n and coefficients $\lambda_0, \ldots, \lambda_n$ such that

$$\| f - \sum_{i=0}^{n} \lambda_i g_i \| < \epsilon .$$

The Weierstrass theorem guarantees, of course, that system (1) is fundamental in $C[-1,1]$. The following theorem results from Theorem 1 .

<u>Theorem 5</u> Let $\{g_0, g_1, \ldots\}$ be a fundamental Markov system on $[-1,1]$. Then it is possible to prescribe a system of finite point sets $Y_0, Y_1, \ldots$ in such a way that the generalized polynomials

$$\sum_{i=0}^{n} \lambda_i g_i$$

of best approximation to a continuous function f on Y_n converge uniformly to f as $n \to \infty$.

A result like Theorem 5 was first given by Curtis (for the case of algebraic polynomials) . [5] . A simple consequence of his result is the following.

<u>Theorem 6</u> Let $Y_0, Y_1, \ldots$ be a sequence of point sets in $[-1,1]$ such that $|Y_n|_\pi < 1/n$. Let Q_n denote the best polynomial approximation of degree $\leq n$ to a continuous function f on the set Y_n . Then $\| Q_n - f \| \to 0$.

<u>Proof</u> From Lemma 2, if $|Y|_\pi < 1/n$ then $\| P \| \leq 2 \| P \|_Y$. Now let

P_n denote the best approximation of degree n to f on $[-1,1]$. Then

$$\begin{aligned}
\|Q_n - f\| &\le \|Q_n - P_n\| + \|P_n - f\| \\
&\le 2\,\|Q_n - P_n\|_{Y_n} + \|P_n - f\| \\
&\le 2\,\|Q_n - f\|_{Y_n} + 2\|f - P_n\|_{Y_n} + \|P_n - f\| \\
&\le 5\,\|P_n - f\| \to 0 ,
\end{aligned}$$

by Weierstrass theorem.

The theorem concerning convergence of approximating polynomials on finite point sets was first given by Motzkin and Walsh in a paper entitled "The Least p-th Power Polynomials on a Finite Point Set", Trans. Am. Math. Soc. 83 (1956) 371-396 (Theorem 7.1) . The theorem was extended to general n-parameter families by Curtis in a paper entitled "N-Parameter Families and Best Approximation", Pac. J. Math. 9 (1959) 1013-1027. Their proofs are different from that given here.

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