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THE ANALYTIC FOUNDATIONS OF LINEAR, TIME-INVARIANT N-PORTS

by

D. C. Youla, L. Kaplan, and D. Stock

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PART I

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D. C. Youla, L. Kaplan, and D. Stock

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ABSTRACT

At a single frequency, a linear, source-free, time invariant n-port N is completely characterized by the n x n normalized scattering matrix S. The work of Mason[1], Deschamps[2], and Schaug-Petterson and Tonning[3], has revealed the very striking connection between the basic performance properties of 2-ports and their invariants under lossless reciprocal embeddings leaving the number of ports unchanged. In general, the embedding of an n-port N in a 2n-port M results in a new n-port $\hat{N}$ whose scattering matrix W is a matrix bilinear transform of S. One of this paper's main objectives is to show that lossless, reciprocal M induces transformations that are strict analogues of those Möbius mappings of the ordinary complex plane which leave the unit circle invariant. The observation serves as the starting point for almost all the investigations reported herein.

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THE ANALYTIC FOUNDATIONS OF THE "SPOT" FREQUENCY THEORY OF LINEAR, TIME INVARIANT N-PORTS

I. INTRODUCTION

At a single frequency, a linear, source-free, time invariant n-port N is completely characterized by the n x n normalized scattering matrix S. The work of Mason[1], Deschamps[2], and Schaug-Petterson and Tonning[3], has revealed the very striking connection between the basic performance properties of 2-ports and their invariants under lossless reciprocal embeddings leaving the number of ports unchanged. In general, the embedding of an n-port N in a 2n-port M results in a new n-port $\hat{N}$ whose scattering matrix W is a matrix bilinear transform of S. One of this paper's main objectives is to show that lossless, reciprocal M induces transformations that are strict analogues of those Möbius mappings of the ordinary complex plane which leave the unit circle invariant. The observation serves as the starting point for almost all the investigations reported herein.

Embedding can also be done so that the number of ports is not an invariant. When the number of ports at the input is less than the number of ports at the output, a rectangular transfer matrix[4] may be formulated. The consideration of this type of embedding leads to results quite similar to 2n-port embedding in many aspects.

II. NOTATION AND PRELIMINARY RESULTS

Let A be an arbitrary matrix. Then A', $\bar{A}$, A^*, A^{-1} and $d(A)$ stand for the transpose, the complex conjugate, the complex conjugate transpose, the inverse of A (whenever it exists) and the determinant of A, respectively. A diagonal n x n matrix A with diagonal elements $u_1, u_2, \ldots, u_n$ is written $A = \text{diag}[u_1, u_2, \ldots, u_n]$. For a hermitian matrix $A = A^*$, $A \geq 0$ means that A is the matrix of a semi-positive definite quadratic form; $A > 0$ means that the form is positive definite; $\underline{a}$, $\underline{b}$, $\underline{v}$, $\underline{I}$, etc., represent column vectors; 0_n and 1_n are the n x n zero and identity matrices. If $A_1, A_2, \ldots, A_k$ are k square matrices, the square matrix

$$A = \begin{bmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & A_k \end{bmatrix}$$

is denoted by $A = A_1 \dotplus A_2 \dotplus \ldots, \dotplus A_k$, a "direct sum" of matrices.

When the dimension of a rectangular matrix is to be displayed explicitly, it will be done by $A_{m,n}$.

Figure 1a is the schematic of a 2n-port M, and figure 1b is that of a N-port M. From this point on it must be clearly understood that T is normalized to some set of 2n or N non-zero real part port numbers. Let:

$$\underline{b}_1 = (b_1, b_2, \ldots, b_n)'$$

$$\underline{b}_2 = (b_{n+1}, b_{n+2}, \ldots, b_{n+k})'$$

and:

$$\underline{a}_1 = (a_1, a_2, \ldots, a_n)'$$

$$\underline{a}_2 = (a_{n+1}, a_{n+2}, \ldots, a_{n+k})'$$

denote the normalized reflected and incident wave amplitudes on the input and output sides of M respectively. In the 2n-port $n + k = 2n$. By definition:

$$\begin{aligned}
\underline{a}_1 + \underline{b}_1 &= \underline{v}_1 = (v_1, v_2, \ldots, v_n)' \\
\underline{a}_1 - \underline{b}_1 &= \underline{I}_1 = (I_1, I_2, \ldots, I_n)' \\
\underline{a}_2 + \underline{b}_2 &= \underline{v}_2 = (v_{n+1}, v_{n+2}, \ldots, v_{n+k})' \\
\underline{a}_2 - \underline{b}_2 &= \underline{I}_2 = (I_{n+1}, I_{n+2}, \ldots, I_{n+k})'
\end{aligned} \tag{1}$$

Of course, the vector port voltages $\underline{v}_1$, $\underline{v}_2$ and currents $\underline{I}_1$, $\underline{I}_2$ are also normalized. The transfer scattering matrix T relates reflected and incident amplitudes at the input to incident and reflected amplitudes at the output. Thus:

$$\left.\begin{matrix} \underline{b}_1 \\ \hline \underline{a}_1 \end{matrix}\right] = T \left.\begin{matrix} \underline{a}_2 \\ \hline \underline{b}_2 \end{matrix}\right] \quad , \tag{2}$$

where the half brackets indicate column matrices (vectors).

Setting:

$$T = \begin{array}{cc} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] & \begin{array}{c} n \\ n \end{array} \\ \begin{array}{cc} k & \;\; k \end{array} & \end{array} \qquad , \qquad (3)$$

and substituting in (3) gives:

$$\underline{b}_1 = A\,\underline{a}_2 + B\,\underline{b}_2$$

$$\underline{a}_1 = C\,\underline{a}_2 + D\,\underline{b}_2 \qquad . \qquad (4)$$

For the 2n-port embedding the output n-ports of M are terminated in an n-port N possessing the normalized scattering matrix S. Then:

$$\underline{a}_2 = S\,\underline{b}_2 \qquad .$$

From (4):

$$\underline{b}_1 = (AS + B)\,\underline{b}_2$$

$$\underline{a}_1 = (CS + D)\,\underline{b}_2$$

$$\underline{b}_1 = (AS + B)\,(CS + D)^{-1}\,\underline{a}_1 \qquad .$$

In other words, the scattering matrix W of the resultant n-port $\hat{N}$ (Fig. 2) is:

$$W = (AS + B)\,(CS + D)^{-1} = T(S) \qquad , \qquad (5)$$

a matrix bilinear transform of S.

For the rectangular transfer matrix the following holds:

$$\begin{bmatrix} \underline{b}_1 \\ \underline{a}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \underline{a}_2 \\ \underline{b}_2 \end{bmatrix}$$

$$\underline{b}_1 = W \underline{a}_1$$

$$\underline{a}_2 = S \underline{b}_2$$

$$W \underline{a}_1 = AS \underline{b}_2 + B \underline{b}_2$$

$$W(CS \underline{b}_2 + D \underline{b}_2) = AS \underline{b}_2 + B \underline{b}_2$$

$$W(CS + D) = (AS + B)$$

Define;

$$M_{n,n} = (C_{n,k} S_{k,k} + D_{n,k}) (CS + D)'_{k,n}$$

Thus;

$$W = (AS + B) (CS + D)' M^{-1} \qquad (5')$$

In equation (5') the inverse of a rectangular matrix (CS + D) is calculated explicitly and shown to be

$$(CS + D)' [(CS + D) (CS + D)']^{-1}$$

This may be indicated formally by

$$(CS + D)^{-1} ,$$

where it is understood that a rectangular matrix has only a left or a right inverse. It is assumed that the rank of the rectangular matrix is its maximum value, else no inverse exists. This statement follows from the result that the rank of the product of matrices cannot exceed the rank of any of the factors. In order to emphasize the distinction between inverses of square and rectangular matrices in this paper, the superscript notation "-1" will not be used for rectangular matrices. In cases where the explicit left or right inverse is not shown, the inverse property will be stated explicitly, as in

$$\hat{Q}Q = 1_{n,n}.$$

If the matrix is square, the left inverse equals the right inverse and is simply called the inverse, if it exists. In such a case equation (5') will reduce to equation 5.

To translate the conditions of losslessness and reciprocity onto T in a compact manner, the two matrices:

$$\sigma_2 = \left[\begin{array}{c|c} 1_n & 0_n \\ \hline 0_n & -1_n \end{array}\right] \quad , \qquad (6)$$

and:

$$\sigma_1 = \left[\begin{array}{c|c} 0_n & 1_n \\ \hline -1_n & 0_n \end{array}\right] \quad , \tag{7}$$

are indispensable. They are very closely related to the generalized spin operators of quantum mechanics. The next two criteria are derived and fully discussed in reference 5:

a) The 2n-port, M, is lossless if and only if for all real frequencies:

$$T \sigma_2 T^x = \sigma_2 \quad . \tag{8}$$

Indulging in an abuse of language, T is said to be "lossless".

b) M is reciprocal if and only if for all real frequencies:

$$T \sigma_1 T' = \sigma_1 \quad . \tag{9}$$

Such a T is said to be "reciprocal". When T is a real meromorphic matrix function of the complex frequency variable p, (8) and (9) can be written in the alternative forms:

$$a_1)\ T(p)\,\sigma_2\,T^*(-p) = \sigma_2 \tag{10}$$

$$b_1)\ T(p)\,\sigma_1\,T'(p) = \sigma_1 \quad . \tag{11}$$

in which guise they are valid in the entire p-plane. Note that:

$$\sigma_2^2 = -\sigma_1^2 = 1_{2n} \quad , \tag{12}$$

i.e.:

$$\sigma_1^{-1} = -\sigma_1 \tag{13}$$

$$\sigma_2^{-1} = \sigma_2 \quad . \tag{14}$$

It is easy to show that matrices satisfying either (8), (9), or both, form a group under matrix multiplication. Since the multiplication of transfer scattering matrices corresponds to the cascading of their representative networks, the above statement is also obvious on physical grounds. Furthermore, if T is reciprocal, (lossless), T' is reciprocal (lossless). A proof is most simple. From $T \sigma_1 T' = \sigma_1$, it follows by multiplication on the right with $\sigma_1 T$ that:

$$T \sigma_1 T' \sigma_1 T = -T$$

$$T' \sigma_1 T = \sigma_1 \quad .$$

Q.E.D. The lossless case is treated similarly.

For the rectangular transfer matrix the conditions for reciprocity and lossless are respectively;

$$T_{2n,2k} \, \sigma_{1\ 2k,2k} \, T' = \sigma_{1\ 2n,2n}$$

$$T_{2n,2k} \, \sigma_{2\ 2k,2k} \, T^* = \sigma_{2\ 2n,2n}$$

The rectangular transfer matrix cannot be formulated if the number of ports at the input is larger than the numbers at the output[4] and thus T' does not represent a network. In addition, the form $T' \sigma T$ will have rank if at most n which is less than k; hence, $T' \sigma T$ cannot equal σ .

A 2n-port M with transfer scattering matrix:

$$T = \begin{array}{cc} n & n & \\ \left[\begin{array}{cc} U & 0_n \\ 0_n & V \end{array}\right] & & \begin{array}{c} n \\ n \end{array} \end{array} , \qquad (14a)$$

is called a lossless all-pass when U and V are both unitary. If $V \neq \bar{U}$, it is a non-reciprocal all-pass. Using (5), it is easily seen that:

$$W = T(S) = USV^* \quad . \qquad (14b)$$

When $V = \bar{U}$,

$$W = USU' \quad . \qquad (14c)$$

The reciprocal all-pass network must be a 2n-port.

III. PROPERTIES OF TRANSFER SCATTERING MATRICES

<u>Theorem 1</u>: Let

$$T = \begin{array}{cc} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] & \begin{array}{c} n \\ n \end{array} \\ \begin{array}{cc} n & n \end{array} & \end{array} ,$$

and suppose that T represents a reciprocal 2n-port M, i.e.:

$$T \sigma_1 T' = \sigma_1 \quad . \tag{15}$$

Then:

$$d(T) = +1 \quad . \tag{16}$$

Proof: See reference 11, p. 248, Theorem 28.1.

Theorem 2: Let:

$$T \sigma_1 T' = \sigma_1 \quad .$$

Then, the eigenvalues of T occur in reciprocal pairs, i.e., λ and λ^{-1} are both eigenvalues of T if either one is an eigenvalue.

Proof: By hypothesis:

$$T \sigma_1 T' = \sigma_1$$

$$\sigma_1 T' \sigma_1 - 1 = T^{-1} \quad .$$

Thus T and T^{-1} possess the same eigenvalues. But those of T^{-1} are the reciprocals of those of T, Q.E.D.

Corollary: If T is reciprocal, the eigenvalues +1 and -1 are of even multiplicity.

Proof: From Theorem 1, it is known that $d(T) = +1$. Now $d(T)$ is also the product of its eigenvalues. Invoking Theorem 2:

$$d(T) = (-1)^{\nu} = +1 \qquad ,$$

ν being the multiplicity of the eigenvalues -1. Hence, ν = an even integer. All the eigenvalues λ for which $|\lambda| \neq 1$, occur together with a distinct reciprocal λ^{-1}. The total number, β, of eigenvalues included in these pairs must, of course, be even. Since the multiplicity, μ, of the eigenvalue +1 is $2n - \nu - \beta$, μ is also even, Q.E.D.

Theorem 3: Let the $2n \times 2n$ matrix T represent a lossless 2n-port M. Then, its eigenvalues occur in complex conjugate reciprocal pairs, i.e., λ and $1/\overline{\lambda}$ are both eigenvalues if either one is an eigenvalue.

Proof: By hypothesis, $T \sigma_2 T^* = \sigma_2$.

Hence:

$$\sigma_2 T^* \sigma_2^{-1} = T^{-1} \qquad ,$$

which means T^* and T^{-1} possess the same eigenvalues. But those of T^* and T^{-1} are the complex conjugates and reciprocals of those belonging to T respectively, Q.E.D.

Corollary: Suppose T is both lossless and reciprocal and that λ is an eigenvalue not lying on the unit circle ($|\lambda| \neq 1$). Then T possesses the four distinct eigenvalues:

$$\lambda, \quad \lambda^{-1}, \quad \bar{\lambda}, \quad \bar{\lambda}^{-1} \quad .$$

Proof: Theorems 1, 2, 3.

For the rectangular matrix the concept of eigenvalues is not pertinent. Hence, the first three theorems do not have analogues for the rectangular transfer matrix. From the conditions involving losslessness and reciprocity it is seen that the rank of a transfer matrix representing a lossless or a reciprocal network is n.

Theorem 4: The 2n x 2n matrix T is the transfer scattering matrix of a lossless reciprocal 2n-port M if and only if at real frequencies it possesses the parametric representation:

$$T = \begin{bmatrix} U & 0_n \\ 0_n & \bar{U} \end{bmatrix} \begin{bmatrix} \cosh & \sinh \\ \sinh & \cosh \end{bmatrix} \begin{bmatrix} V & 0_n \\ 0_n & \bar{V} \end{bmatrix} \quad , \qquad (17)$$

where,

1. U and V are two arbitrary unitary n x n matrices
2. λ is an n x n diagonal matrix whose diagonal elements $\mu_1, \mu_2, \ldots, \mu_n$ are real,

i.e.,

$$\lambda = \mathrm{diag}[\mu_1, \mu_2, \ldots, \mu_n] \quad ,$$

$$\mu_r \text{ real}, \ (r = 1, 2, \ldots, n) \quad . \qquad (18)$$

The matrices $\cosh\lambda$ and $\sinh\lambda$ are defined by:

$$\cosh\lambda = \mathrm{diag}[\cosh \mu_1, \ldots, \cosh \mu_n]$$

$$\sinh\lambda = \mathrm{diag}[\sinh \mu_1, \ldots, \sinh \mu_n] \quad ,$$

and are also diagonal.

Proof: See reference 6.

Corollary: The $2n \times 2k$ matrix T is the transfer matrix of a lossless reciprocal m-port if and only if at real frequencies it possesses the parametric representation:

$$T = \begin{bmatrix} U & 0 \\ 0 & \bar{U} \end{bmatrix} \hat{T} \begin{bmatrix} V & 0 \\ 0 & \bar{V} \end{bmatrix} \tag{17'}$$

U is the $n \times n$ unitary matrix and V is a $k \times k$ unitary matrix. $\hat{T}$ is a $2n \times 2k$ matrix such that $\hat{T}\,\hat{T}^*$ is the $2n \times 2n$ matrix:

$$\begin{bmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{bmatrix}$$

Proof: See reference 4.

Physically speaking, (17) is equivalent to saying that at real frequencies any lossless, reciprocal 2n-port M can be considered to be the cascade of three networks ($T = T_c\ T_b\ T_a$), the end ones being lossless, reciprocal all-passes and the middle one a transformer bank of n uncoupled ideal two-port transformers (Fig. 3a).

Relation (17') is different only in the fact that transformer coupling is not accomplished in the manner of n separate two-ports, but as shown in Fig. 3b.

Another useful interpretation is that any lossless reciprocal embedding of an n-port N with scattering matrix S can be realized by performing three special such embeddings in sequence:

a) $S_a = T_a(S) = VSV'$, V unitary; (19)

b) $S = T_b(S_a) = [(\cosh\lambda)\, S_a + \sinh\lambda\,][(\sinh\lambda)\, S_a + \cosh\lambda\,]^{-1}$,

$\lambda = \mathrm{diag}[\mu_1, \mu_2, \ldots, \mu_n]$,

μ_r real, $(r = 1, 2, \ldots, n)$; (20)

It is noted that in the rectangular case T_b is a rectangular transfer matrix and (20) is modified to the form of (5').

c) $W = T_c(S_b) = US_bU'$, U unitary. (21)

These characteristics will prove extremely useful in the later discussion on canonic forms.

IV. HOMOGENOUS COÖRDINATES

As it stands, the bilinear matrix mapping T(S), as expressed in (5) (non-homogenous coördinates), is too cumbersome to work with.

It can be made more tractable by the introduction of homogenous coördinates.

The two n x n matrices X, Y, are said to be the homogenous coördinates of the n x n matrix S if

$$S = -X^{-1} Y \quad . \tag{22}$$

This correspondence is indicated symbolically by:

$$S \doteq (X, Y) \quad . \tag{23}$$

When X is singular, S is the "point at infinity". Obviously:

$$S \doteq (X, Y) \quad ,$$

implies that:

$$S \doteq (QX, QY) \quad ,$$

for any non-singular n x n matrix Q.

Theorem 5: Let Q be any k x n matrix of rank n and suppose that:

$$S \doteq (X, Y) \quad , \quad d(X) \neq 0 \quad ,$$
$$W \doteq (U, V) \quad , \quad d(U) \neq 0 \quad .$$

Then, (5) is equivalent to:

$$[X \mid Y] = Q[U \mid V]\, T \quad , \tag{24}$$

where:

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad ,$$

Proof: Expanding (24):

$$X = Q(UA + VC) \quad ,$$

$$Y = Q(UB + VD) \quad , \qquad (24a)$$

Whence:

$$X^{-1} Y = -S = [Q(UA + VC)]^{-1} [Q(UB + VD)]$$

In a 2n-port, Q is square.

$$-S = (A - WC)^{-1} (B - WD) \quad ,$$

$$(A - WC) S = WD - B$$

$$W(CS + D) = AS + B \quad , \qquad (25)$$

and:

$$W = (AS + B) (CS + D)^{-1} \qquad (26)$$

If Q is not square the proof proceeds as follows:

$$-S = [QU(A - WC)]^{-1} [QU(B - WD)]$$

$$QU(WD - B) = QU(A - WC) S$$

$$QU[WD - B - (A - WC)S] = 0$$

$$WD - B = AS + WCS$$

$$W = (AS + B) (CS + D)' M^{-1} \qquad \text{Q.E.D.} \qquad (26')$$

Conversely, (26) implies (25) and setting $X \doteq Q(UA + VC)$, Q non-singular and arbitrary, (24) is recovered. Similarly (26') implies (24).

The transformations (26) and (26') are uniquely determined by T and is independent of Q as long as the rank of Q is n. Actually (24) is slightly more general than (26) because it continues to make sense even when (CS + D) or M is singular. The "point at infinity" is, in homogenous coördinates, the collection of all pairs (X, Y) such that $d(X) = 0$. Thus (26) is defined only for those n x n matrices S for which $d(CS + D)$ or $d(M) \neq 0$, i.e., T maps the space of all n x n matrices (punctured at the "point" S defined by $d(CS + D) = 0$) onto itself in a one-one manner. However, when viewed as a transformation on the space of all pairs of n x n matrices, there is no exceptional "point", the puncture having been eliminated by the introduction of two "coördinates". Hence, if:

$$[X \mid Y] = Q[U \mid V]\, T \quad ,$$

For the 2n-port:

$$[U \mid V] = Q^{-1}\, [X \mid Y]\, T^{-1} \quad .$$

For the rectangular transfer matrix

$$[U \mid V] = \hat{Q}[X \mid Y]\, \hat{T}$$

Where

$$\hat{Q}\, Q = 1_{n,n}$$

and

$$T \hat{T} = 1_{n,n} \quad .$$

Lemma 1: Let

$$S \doteq (X, Y) \quad ,$$

and:

$$S' \doteq (X_1, Y_1) \quad .$$

Then:

$$[X \mid Y] \, \sigma_1 \, [X_1 \mid Y_1]' = 0_n \quad . \tag{27}$$

In particular, if S is symmetric,

$$S = S' \quad ,$$

and:

$$[X \mid Y] \, \sigma_1 \, [X \mid Y]' = 0_n \quad . \tag{28}$$

Explicitly:

$$X \, Y'_1 - Y \, X'_1 = 0_n \quad , \tag{27a}$$

$$X \, Y' - Y \, X' = 0_n \quad . \tag{28a}$$

Proof: Observe that:

$$S = -X^{-1} \, Y \quad ,$$

$$S' = -X_1^{-1} \, Y_1 \quad .$$

$$Y'_1 (X'_1)^{-1} = X^{-1} \, Y \quad ,$$

$$X \, Y'_1 - Y \, X'_1 = 0_n \quad ,$$

Q.E.D.

Lemma 2: Let:

$$S \doteq (X, Y)$$

and:

$$(S^*)^{-1} \doteq (X_1, Y_1)$$

Then:

$$[X \mid Y] \, \sigma_2 \, [X_1 \mid Y_1]^* = 0_n \quad . \tag{29}$$

In particular, if S is unitary, i.e., $S^* S = 1_n$,

$$[X \mid Y] \, \sigma_2 \, [X \mid Y]^* = 0_n \quad . \tag{30}$$

Explicitly:

$$X X_1^* - Y Y_1^* = 0_n \tag{29a}$$

$$X X^* - Y Y^* = 0_n \quad . \tag{30a}$$

Proof: From:

$$S = -X^{-1} Y \quad ,$$

and:

$$(S^*)^{-1} = -X_1^{-1} Y_1 \quad ,$$

it follows that:

$$XS = -Y, \; X_1 (S^*)^{-1} = -Y_1 \quad .$$

$$X_1 X^* - Y_1 Y^* = 0 \quad ,$$

Q.E.D.

V. PROPERTIES OF MATRIX BILINEAR TRANSFORMATIONS

The bilinear transformation (26) or its homogenous equivalent (24) induces a matrix Möbius mapping of the space of n x n matrices S into itself. The transformation (26') maps the space of k x k matrices onto the space of n x n matrices. The purpose of this section is to study its structure.

Lemma 3: Suppose that the embedding network is reciprocal, i.e., $T \sigma_1 T' = \sigma_1$. Then if $T(S) = W$, $T(S') = W'$.

Proof: Let:

$$S \doteq (X, Y)$$

$$S' \doteq (X_1, Y_1)$$

$$W \doteq (U, V)$$

$$T(S') = (U_1, V_1)$$

From Lemma 1, Eq. (27):

$$[X \mid Y] \sigma_1 [X_1 \mid Y_1]' = 0_n \quad , \tag{31}$$

and from Eq. (24):

$$[X \mid Y] = Q_1 [U \mid V] T \tag{32}$$

$$[X_1 \mid Y_1] = Q_2 [U_1 \mid V_1] T \quad . \tag{33}$$

Thus,

$$0_n = [X \mid Y] \sigma_1 [X_1 \mid Y_1]' = Q_1 [U \mid V] T \sigma_1 T' [U_1 \mid V_1]' Q_2'$$

$$= Q_1 [U \mid V] \sigma_1 [U_1 \mid V_1]' Q_2' \quad .$$

Since Q_1 and Q_2 have rank n:

$$[U \mid V] \sigma_1 [U_1 \mid V_1]' = 0_n \quad .$$

That is to say:

$$T(S') = W'$$

(See Lemma 1), Q.E.D.

<u>Lemma 4</u>: Suppose that the embedding network is lossless, i.e., $T \sigma_2 T^* = \sigma_2$. Then if $T(S) = W$, and $d(S) \neq 0$, $T(S^{*-1}) = (W^*)^{-1}$.

<u>Proof</u>: Let

$$S \doteq (X, Y) \quad ,$$

$$(S^*)^{-1} \doteq (X_1, Y_1) \quad ,$$

$$W \doteq (U, V) \quad ,$$

$$T(S^{*-1}) \doteq (U_1, V_1) \quad .$$

From Lemma 2, Eq. (29):

$$[X \mid Y] \sigma_2 [X_1 \mid Y_1]^* = 0_n \quad ,$$

and from Eqs. (32) and (33):

$$0_n = Q_1 [U \mid V] T \sigma_2 T^* [U_1 \mid V_1]^* Q_2^*$$

$$= Q_1 [U \mid V] \sigma_2 [U_1 \mid V_1] Q_2^* \quad .$$

As before:

$$0_n = [U \mid V] \sigma_2 [U_1 \mid V_1]^* \quad ,$$

that is to say:

$$T(S^{*-1}) = (W^*)^{-1}$$

(see Lemma 2), Q.E.D.

A corollary of the above lemma is that unitary S go to unitary W. Phrased differently, lossless n-ports remain lossless under lossless embeddings, which of course is as it should be.

A less trivial translation of Lemma 4 is available, however. In the space of all k x k matrices S, the "boundary" of the "unit" circle, H, is defined to be the collection of all k x k unitary matrices. Two "points" S_1 and S_2 are said to be inverse with respect to H if $S_1^* S_2 = 1_k$. The unit circle, $\hat{H}$ is defined similarly for the space of all n x n matrices. Thus Lemma 4 states, in effect, that lossless embeddings leave the boundary of the unit circle invariant and send inverse points (with respect to H) into inverse points (with respect to $\hat{H}$). The interior I(H), and exterior, E(H) are, respectively, the set of matrices satisfying:

$$I(H):\quad 1_n - S\,S^* > 0$$

$$E(H):\quad 1_n - S\,S^* < 0 \quad .$$

Similar definitions hold for $I(\hat{H})$ and $E(\hat{H})$.

Does T map I(H) into $I(\hat{H})$? This is the subject of Lemma 5. For 2n-port embedding $\hat{H} = H$.

<u>Lemma 5</u>: Let T be lossless, i.e., $T\,\sigma_2\,T^* = \sigma_2$. Then, I(H) maps into $I(\hat{H})$.

<u>Proof</u>: Expansion of $T\,\sigma_2\,T^* = \sigma_2$ yields:

$$A A^* - B B^* = 1_n \tag{34a}$$

$$A C^* - B D^* = 0_n \tag{34b}$$

$$C C^* - D D^* = -1_n \tag{34c}$$

Assume that if a 2n-port $d(CS + D) = 0$, or, in the arbitrary case, the rank of $(CS + D) < n$. Then, there exists a non-trivial n-vector $\underline{b}$ such that:

$$\underline{b}^*(CS + D) = \underline{0}_n$$

$$\underline{b}^* CS = \underline{-b}^* D$$

$$S^* C^* \underline{b} = -D^* \underline{b}$$

$$\underline{b}^* CSS^* C^* \underline{b} = \underline{b}^* DD^* \underline{b} \quad .$$

Using (34c):

$$0 \leq \underline{b}^* C(1_k - SS^*) C^* \underline{b} = \underline{b}^*(CC^* - DD^*) \underline{b} = -\underline{b}^* \underline{b} < 0 \quad .$$

A contradiction. Thus $d(CS + D) \neq 0$ or $d[(CS + D)(CS + D)'] \neq 0$ for all S belonging to the interior of H. Consequently, if $T(S) = W = (U, V)$, $d(U) \neq 0$. The proof can now be brought to a quick conclusion. Let:

$$S \doteq (X, Y) \quad ,$$

$$d(X) \neq 0 \quad ,$$

$$W \doteq (U, V) \quad .$$

According to what has transpired above, $d(U) \neq 0$. Clearly:

$$X(1_n - SS^*)X^* = XX^* - YY^* = [X \mid Y]\, \sigma_2\, [X \mid Y]^*$$
$$= Q[U \mid V]T \sigma_2 T^*[U \mid V]^*Q^*$$
$$= Q(UU^* - VV^*)Q^* \quad .$$

$$UU^* - VV^* = \hat{Q}X(1_k - SS^*)X^*\hat{Q}^*$$

where $\hat{Q}Q = 1_n$ and $Q^*\hat{Q}^* = 1_n$

Since $d(U) \neq 0$,

$$1_n - VV^* = U^{-1}(UU^* - VV^*)U^{*-1} > 0 \quad , \qquad \textbf{Q.E.D.}$$

The reader is no doubt wondering why so much energy has been expended in proving the apparently obvious fact that lossless embedding of a passive network N yields a passive n-port $\hat{N}$. The reason is that this "obviousness" is illusory because it is based on the implicit and undemonstrated assumption that $T \sigma_2 T^* = \sigma_2$ implies the existence of a physical 2n-port M, composed of lossless elements alone, which possesses the transfer scattering matrix T. Nevertheless, such a synthesis procedure is feasible by first noting, as above, that $T \sigma_2 T^* = \sigma_2$ implies $d(D) \neq 0$, and then finding the scattering matrix S_T corresponding to T. Thus, after some elementary manipulations:

$$S_T = \left[\begin{array}{c|c} 1_n & -B \\ \hline 0_n & -D \end{array}\right]^{-1} \left[\begin{array}{c|c} 0_n & A \\ \hline -1_n & C \end{array}\right] \quad , \qquad (35)$$

where $1_{2n} - S_T S_T^* = 0_{2n}$, i.e., S_T is unitary.

For the rectangular transfer matrix the $(n + k) \times (n + k)$ unitary scattering matrix S_T is found by equation (35) if the inverse is interpreted as that of a rectangular matrix.

Finally, the technique described in reference 7 realizes M as a lossless 2n-port composed of ideal capacitors, inductors, gyrators and transformers.

VI. MÖBIUS MAPPINGS OF THE COMPLEX s-PLANE

At this point it is perhaps advisable in order to motivate the definition of "cross-ratio" matrix, to give a brief review of the properties of the group of Möbius mappings of the complex s-plane. Generically,

$$w(s) = \frac{as + b}{cs + d} \quad , \tag{36}$$

where:

$$1 = ad - bc \quad . \tag{36a}$$

The quantities w, s, a, b, c and d are complex scalars. If $w(s) \neq$ constant, (36a) entails no loss of generality. Let:

$$T = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] ,$$

a 2 x 2 matrix. Then it is immediate that $T \sigma_1 T' = \sigma_1$, in which:

$$\sigma_1 = \left[\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right] ,$$

is also a 2 x 2 matrix $(n = 1)$. Thus any lossless mapping of the z-plane is reciprocal.

Again, suppose that under (36),

$$s_1 \longrightarrow w_1$$
$$s_2 \longrightarrow w_2$$
$$s_3 \longrightarrow w_3$$
$$s_4 \longrightarrow w_4 \quad .$$

From (36) and (36a),

$$w_r - w_k = \frac{s_r - s_k}{(cs_r + d)(cs_k + d)} ; \; (r, k=1, 2, 3, 4) \quad . \tag{37}$$

Introducing the abbreviations:

$$\eta = \frac{1}{(cs_1 + d)(cs_2 + d)(cs_3 + d)(cs_4 + d)} \quad , \tag{38}$$

(37) and (38) yield:

$$(w_1 - w_2)(w_3 - w_4) = \eta\ (s_1 - s_2)(s_3 - s_4) \quad , \qquad (39)$$

$$(w_1 - w_3)(w_2 - w_4) = \eta\ (s_1 - s_3)(s_2 - s_4) \quad . \qquad (40)$$

Dividing (39) by (40) gives:

$$(w_1 - w_2)(w_1 - w_3)^{-1}\ (w_3 - w_4)(w_2 - w_4)^{-1}$$

$$= (s_1 - s_2)(s_1 - s_3)^{-1}\ (s_3 - s_4)(s_2 - s_4)^{-1} \quad . \qquad (41)$$

The right-hand side of (39) is called the cross-ratio of the four points s_1, s_2, s_3, s_4 and is denoted by $R(s_1, s_2, s_3, s_4)$. Equation (39) may be rewritten as:

$$R(w_1, w_2, w_3, w_4) = R(s_1, s_2, s_3, s_4) \quad ,$$

and its content expressed by saying that the cross-ratio of any four points is an invariant under every Möbius transformation.

The most general bilinear transformation mapping the unit circle and its interior $|s| \leq 1$ onto itself ($|w| \leq 1$) is given by[8]:

$$w = e^{i\lambda} \ \frac{s - \alpha}{\bar{\alpha}_s - 1} \quad ,$$

$$i = \sqrt{-1} \qquad\qquad (43)$$

$$\lambda \text{ real}, \ |\alpha| < 1 \quad .$$

After normalization to unity determinant (43) appears as:

$$w = \frac{as + b}{cs + d} \qquad ,$$

in which:

$$a = -\frac{i}{k}\, e^{i\lambda/2} \quad , \qquad b = \frac{\alpha i}{k}\, e^{i\lambda/2} \quad ,$$

$$c = -\frac{i\bar{\alpha}}{k}\, e^{-i\lambda/2} \quad , \qquad d = \frac{i}{k}\, e^{-i\lambda/2} \quad ,$$

$$k = (1 - |\alpha|^2)^{1/2} > 0 \quad .$$

The reader is invited to show for himself that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad ,$$

i.e.:

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ,$$

is lossless. In short the Möbius mappings described in Equation (43) are the two-dimensional analogues of those 2n-port embeddings T which are both lossless and reciprocal. It is not difficult to find the necessary and sufficient conditions for the existence of a mapping T of the type (43) which transforms two given points s_1, s_2, into two given points w_1, w_2, i.e., $w_1 = T(s_1)$, $w_2 = T(s_2)$. First, unless $|s_n| \overset{\geq}{<} 1$ implies $|w_n| \overset{\geq}{<} 1$, ($n = 1, 2$), no solution is possible. Second, the points inverse to s_1, s_2, w_1, w_2, with respect to the unit circle are in the same order:

$$\bar{s}_1^{-1},\ \bar{s}_2^{-1},\ \bar{w}_1^{-1},\ \bar{w}_2^{-1} \quad . \qquad \text{(Fig.4)}$$

Because inverse points map into inverse points[9], the desired transformation T, if it exists, must set up the correspondence:

$$w_1 = T(s_1)$$

$$w_2 = T(s_2)$$

$$\bar{w}_2^{-1} = T(\bar{s}_2^{-1})$$

$$\bar{w}_1^{-1} = T(\bar{s}_1^{-1}) \quad .$$

Invoking the invariance of the cross-ratio (eq. 41):

$$\frac{w_1 - w_2}{w_1 - \bar{w}_2^{-1}} \; \frac{\bar{w}_2^{-1} - \bar{w}_1^{-1}}{w_2 - \bar{w}_1^{-1}} = \frac{s_1 - s_2}{s_1 - \bar{s}_2^{-1}} \; \frac{\bar{s}_2^{-1} - \bar{s}_1^{-1}}{s_2 - \bar{s}_1^{-1}} \quad ,$$

or, after clearing fractions:

$$(w_1 - w_2)(1 - w_1\bar{w}_2)^{-1} \; \overline{(w_1 - w_2)}(1 - \bar{w}_1 w_2)^{-1} \qquad (44)$$

$$= (s_1 - s_2)(1 - s_1\bar{s}_2)^{-1} \; \overline{(s_1 - s_2)}(1 - \bar{s}_1 s_2)^{-1} \quad .$$

More simply:

$$\left|\frac{w_1 - w_2}{1 - w_1\bar{w}_2}\right| = \left|\frac{s_1 - s_2}{1 - s_1\bar{s}_2}\right| \quad , \qquad (45)$$

Equation (45) is a necessary condition for the existence of a Möbius mapping of type (43) which images s_1, s_2, into w_1, w_2. It is also sufficient because (45) implies that:

$$\frac{w_1 - w_2}{1 - w_1\bar{w}_2} = e^{i\theta} \frac{s_1 - s_2}{1 - s_1\bar{s}_2} \quad , \theta \text{ real} \quad , \tag{46}$$

a form which strongly suggests that the required mapping is:

$$\frac{w - w_2}{1 - w\,\bar{w}_2} = e^{i\theta} \frac{s - s_2}{1 - s\,\bar{s}_2} \quad . \tag{47}$$

That (47) meets all requirements is almost obvious by inspection.

One unfortunate circumstance prevents the immediate generalization of the above results to arbitrary n. Namely, a scalar s is necessarily a symmetric 1 x 1 matrix whereas the n x n matrices S of importance in the theory of invariants under lossless, reciprocal embeddings are non-symmetric! As a matter of fact, a symmetric S does not possess invariants of a non-arithmetic nature. To appreciate this remark in the scalar case ($n = 1$), it is sufficient to note that any s_o in the unit circle can be mapped into $s = 0$ by the lossless, reciprocal transformation:

$$w = e^{i\lambda} \frac{s - s_o}{\bar{s}_o\, s - 1} \tag{48}$$

Similarly, any s_o lying outside the unit circle can be mapped into $s = \infty$ by the lossless, reciprocal transformation:

$$w = e^{i\lambda} \frac{\bar{s}_0 s - 1}{s - s_0} \quad . \qquad (49)$$

Theorem 5 extends the result to arbitrary n x n symmetric matrices.

Theorem 5: Let S be an arbitrary symmetric n x n matrix and suppose that $1_n - SS^*$ possesses r_1 positive eigenvalues, r_2 zero eigenvalues, and $n - (r_1 + r_2)$ negative eigenvalues. Then, by lossless reciprocal 2n-port embedding, S can be transformed into

$$W = T(S) = \mathrm{diag}[\overbrace{0, 0, \ldots, 0,}^{r_1} \; \overbrace{1, 1, \ldots, 1,}^{r_2} \; \overbrace{\infty, \infty, \ldots, \infty}^{n-(r_1 + r_2)}] \quad . \qquad (50)$$

Thus, the resultant n-port $\hat{N}$ is nothing more than n uncoupled 2-terminal impedances, in which the first r_1 are equal to the corresponding port normalization numbers, (a match!), the next r_2 are open-circuits and the last $n - (r_1 + r_2)$ are the negative of the associated port numbers (see fig. 5b). For the embedding in the lossless reciprocal network with a rectangular transfer matrix, $T_{m,n}$ $(m < n)$, the resultant m-port $\hat{M}$ is of the same form as $\hat{N}$. However, no statements can be made about invariants under this transformation.●

Proof: There exists[10] an n x n unitary matrix V such that:

$$W_a = VSV' = \mathrm{diag}[\epsilon_1, \epsilon_2, \ldots, \epsilon_n] \quad . \qquad (51)$$

the ϵ's being the non-negative square roots of the eigenvalues of SS^*. By hypothesis:

$$\epsilon_r < 1, \ (r = 1, 2, \ldots, r_1) \quad ,$$

$$= 1, \ (r = r_1 + 1, \ldots, r_1 + r_2)$$

$$\epsilon_r > 1, \ (r \neq 1, 2, \ldots, r_1 + r_2) \quad .$$

The n-port $\hat{N}_a$ corresponding to W_a is obtained by embedding N (with scattering matrix S) in the lossless, reciprocal 2n-port M_a having the transfer scattering matrix:

$$T_a = \begin{bmatrix} V & 0_n \\ 0_n & \ddot{V} \end{bmatrix} \quad .$$

M_a is a reciprocal all-pass. The impedance, z_{re}, seen looking into port r of $\hat{N}_a$ is:

$$z_{ra} = \frac{1 + \epsilon_r}{1 - \epsilon_r} > 1, \ (r = 1, 2, \ldots, r_1) \quad ,$$

$$= \infty, \ (r = r_1 + 1, \ldots, r_1 + r_2) \quad ,$$

$$= \frac{1 + \epsilon_r}{1 - \epsilon_r} < -1, \ (r \neq 1, 2, \ldots, r_1 + r_2) \quad .$$

For the 2n-port case place in the rth port of $\hat{N}_a$ an ideal 2-port transformer with transformer ratio ϵ^{μ} : 1 (see fig. 5a) where:

$$\mu_r = -1/2 \ln z_{ra}, \ (r = 1, 2, \ldots, r_2) \quad ,$$

$$\mu_r = 0, \ (r = r_1 + 1, \ldots, r_1 + r_2) \quad ,$$

$$= -1/2 \ln |z_{ra}|, \ (r \neq 1, 2, \ldots, r_1 + r_2) \quad .$$

This new n-port $\hat{N}$ has all the desired properties;

$$z_r = +1, \ (r = 1, 2, \ldots, r_1) \quad ,$$

$$= \infty, \ (r = r_1 + 1, \ldots, r_1 + r_2) \quad ,$$

$$= -1, \ (r \neq 1, 2, \ldots, r_1 + r_2) \quad .$$

The embedding 2n-port M has the transfer scattering matrix:

$$T = \left[\begin{array}{c|c} (\cosh \lambda)\ V & (\sinh \lambda)\ \bar{V} \\ \hline (\sinh \lambda)\ V & (\cosh \lambda)\ \bar{V} \end{array}\right]$$

where:

$$\lambda = \mathrm{diag}[\mu_1, \mu_2, \ldots, \mu_n] \quad .$$

For the rectangular transfer matrix, W is the same as for the 2n-port. There are many possible couplings since there is no longer a one-to-one correspondence between input and output in the transformer bank. Considering two impedances at the output which map into a single input impedance, there are six arrangements of positive and negative resistances and open circuits at the output. The input is dependent on the load impedances but in some cases the sign or the finiteness of the input impedance is known, and it is clear that all the positive numbers can be made +1 and the negative numbers can be made -1. This means the equivalent circuit of fig. (5b) is valid.

Definition 1: Two n-ports N and $\hat{N}$ are said to be LRE equivalent if one can be derived from the other by lossless reciprocal 2n-port embedding.

The next two corollaries are immediate deductions from Theorem 5:

Corollary 1: Two reciprocal n-ports N and $\hat{N}$ with symmetric matrices S and W, respectively, are LRE equivalent if and only if the two associated "energy" matrices $1_n - SS^*$ and $1_n - WW^*$ have the same ranks and signatures. Recall that the signature of a square matrix is the difference between its positive and negative eigenvalues.

Corollary 2: The only LRE invariants of a reciprocal n-port N possessing the symmetric scattering matrix S are the rank and signature of $1_n - SS^*$ and these are both arithmetic in nature.

According to Corollary 2, the search for algebraic invariants must be restricted to non-reciprocal n-ports.

For embedding networks with rectangular transfer matrices the concept of LRE invariants is not applicable. In addition to the formal difficulty, there are no invariants since the number of eigenvalues at the input is different from the number at the output; hence, there is no way of forming a unique one-to-one relation. It is easily seen that an n-port can be mapped into two different m-ports ($m < n$) M_1 and M_2 which are not LRE. Also conversely M_1 can arise from two n-ports N_1 and N_2 which are not LRE.

VII. THE CROSS-RATIO MATRIX

Equation (41) inspires the following definition for the cross-ratio matrix $R(S_1, S_2, S_3, S_4)$, of four arbitrary n x n matrices S_1, S_2, S_3, S_4:

$$R(S_1, S_2, S_3, S_4) = (S_1-S_3')(S_2-S_3')^{-1}(S_2-S_4')(S_1-S_4')^{-1} \quad . \tag{52}$$

In general, R exists only if the matrices $S_2 - S_3'$ and $S_1 - S_4'$ are non-singular.

Theorem 6: The cross-ratio $R(S_1, S_2, S_3, S_4)$ matrix undergoes a similarity transformation whenever S_1, S_2, S_3, S_4, are subjected to the same reciprocal mapping by a 2n-port.

Proof: What has to be shown is that if (see Eq. 5):

$$
\begin{aligned}
W_1 &= T(S_1) \\
W_2 &= T(S_2) \\
W_3 &= T(S_3) \\
W_4 &= T(S_4) \qquad ,
\end{aligned}
$$

and:

$$T \, \sigma_1 \, T' = \sigma_1 \qquad ,$$

then:

$$R(S_1, S_2, S_3, S_4) = L^{-1} R(W_1, W_2, W_3, W_4)L \qquad (53)$$

for some non-singular matrix L.

The most expeditious way to proceed is to express R in homogenous coördinates. Let:

$$
\begin{aligned}
S_1 &\doteq (X_1, Y_1) \\
S_2 &\doteq (X_2, Y_2) \\
S_3 &\doteq (X_3, Y_3) \\
S_4 &\doteq (X_4, Y_4) \qquad ,
\end{aligned}
$$

and note that:

$$
\begin{aligned}
S_1 - S_3' &= [1_n \mid -S_1] \, \sigma_1 \, [1_n \mid -S_3]' \\
&= [1_n \mid X_1^{-1} Y_1] \, \sigma_1 \, [1_n \mid X_3^{-1} Y_3]' \\
&= X_1^{-1} [X_1 \mid Y_1] \, \sigma_1 \, [X_3 \mid Y_3]' \, (X_3')^{-1} \\
&= X_1^{-1} < S_1, S_3 > (X_3')^{-1} \qquad , \qquad (54)
\end{aligned}
$$

where:

$$< S_1, S_3 > = [X_1 \mid Y_1] \sigma_1 [X_3 \mid Y_3]' \quad . \tag{55}$$

Thus:

$$R(S_1,S_2,S_3,S_4) = X_1^{-1} <S_1,S_3> <S_2,S_3>^{-1} <S_2,S_4> <S_1,S_4>^{-1} X_1 \quad , \tag{56}$$

and therefore, in homogenous coördinates:

$$X_1 R(S_1,S_2,S_3,S_4) X_1^{-1} = <S_1,S_3> <S_2,S_3>^{-1} <S_2,S_4> <S_1,S_4>^{-1} \quad . \tag{57}$$

Let:

$$W_1 = T(S_1) \doteq (U_1,V_1)$$

$$W_2 = T(S_2) \doteq (U_2,V_2)$$

$$W_3 = T(S_3) \doteq (U_3,V_3)$$

$$W_4 = T(S_4) \doteq (U_4,V_4) \quad .$$

Using Eq. (24):

$$[X_1 \mid Y_1] = Q_1[U_1 \mid V_1]T$$

$$[X_2 \mid Y_2] = Q_2[U_2 \mid V_2]T$$

$$[X_3 \mid Y_3] = Q_3[U_3 \mid V_3]T$$

$$[X_4 \mid Y_4] = Q_4[U_4 \mid V_4]T \quad .$$

Moreover, by (55):

$$<S_1,S_3> = Q_1[U_1 \mid V_1]T \sigma_1 T' [U_3 \mid V_3]'Q_3'$$

$$= Q_1<W_1,W_3>Q_3' \quad .$$

Since $T\sigma_1 T' = \sigma_1$ by hypothesis, it is immediate that:

$$X_1 R(S_1,S_2,S_3,S_4)X_1^{-1} = Q_1 \langle W_1,W_3\rangle \langle W_2,W_3\rangle^{-1} \langle W_2,W_4\rangle \langle W_1,W_4\rangle^{-1} Q_1^{-1} \quad ;$$

i.e.:

$$R(S_1,S_2,S_3,S_4) = L^{-1} R(W_1,W_2,W_3,W_4)L \quad , \tag{58}$$

where:

$$L = U_1^{-1} Q_1^{-1} X_1 \quad , \tag{59}$$

Q.E.D.

In non-homogenous coördinates:

$$L = A - W_1 C \quad , \tag{60}$$

as is easily verified by referring to (24a). Theorem 6 is the key result of the entire theory and goes to the very heart of the matter.

In the case when the embedding is accomplished by a network with a rectangular transfer matrix, the cross-ratio matrix may be defined for both the input and the output termination. However, there is no result that is similar to Theorem 6.

Theorem 7: Let N be an n-port possessing the scattering matrix S. Then, the "characteristic" cross-ratio matrix:

$$R(S) = (S-S')(1 -S^*S)\overline{(S-S')}(1_n-S'S)^{-1} \quad , \tag{61}$$

undergoes a similarity transformation whenever N is subjected to any 2n-port embedding of the lossless, reciprocal type.

Proof: Let $W = T(S)$, in which T is LRE. Invoking Lemmas 3 and 4, and Theorem 6

$$W' = T(S')$$
$$(W^*)^{-1} = T(S^{*-1}) \qquad ,$$

and

$$R(S',\bar{S}^{-1},S',\bar{S}^{-1}) = L^{-1}\, R(W',\bar{W}^{-1},W',\bar{W}^{-1})L \qquad .$$

Thus:

$$\begin{aligned} R(S) &= (S'-S)(\bar{S}^{-1}-S)^{-1}(\bar{S}^{-1}-S^{*-1})(S'-S^{*-1})^{-1} \\ &= (S-S')(1_n-S^*S)^{-1}\overline{(S-S')}(1_n-S'\bar{S})^{-1} \qquad , \end{aligned} \tag{62}$$

is similar to $R(W)$, Q.E.D.

The validity of the above theorem hinges solely on the nonsingularity of (1_n-S^*S).

Again it is noted for the non 2n-port embedding $R(S)$ can be defined at both input and output, but the interrelation is not enlightening.

Corollary 1: The eigenvalues and elementary divisors of $R(S)$ are preserved under matrix bilinear mappings of the LRE type.

Proof: $R(S)$ and $R(W)$ are similar, Q.E.D.

Corollary 2: The quantity:

$$|d(R)| = \frac{|d(S-S')|}{|d(1-S^*S)|} \qquad , \tag{63}$$

is invariant under LRE transformations.

Proof: d(R) is the product of the eigenvalues of R(S), Q.E.D.

For n = 2, (63) reduces to Mason's invariant$^{(1)}$. The invariant (63) has also been obtained by Dasher and Meadows[12] in an entirely different manner.

The two matrices:

$$\Sigma = S - S' = - \Sigma' \qquad ,$$

and:

$$E = (1_n - S^* S)^{-1} = E^*$$

are skew-symmetric and hermitian respectively. From (62):

$$R(S) = \Sigma \bar{E} \bar{\Sigma} \bar{E} = \Sigma E \bar{\Sigma} E' \qquad . \tag{64}$$

The eigenvalues of R(S) are determined by either of the two equivalent determinant equations:

$$d(\Sigma E \bar{\Sigma} E' - \lambda 1_n) = 0 \tag{65}$$

$$d[\Sigma E \bar{\Sigma} - \lambda (E')^{-1}] = 0 \tag{66}$$

Theorem 8: The eigenvalues of R(S) have the following properties:

a) λ and $\bar{\lambda}$ are both eigenvalues of R(S) if either one is an eigenvalue;

b) For n even, all the eigenvalues of R(S) are of even multiplicity. For n odd, all the non-zero eigenvalues are of even multiplicity;

Theorem 8: Continued

c) If E is definite (either positive or negative), the eigenvalues of R(S) are all non-positive and real.

Proof: For any square matrix F, $d(F) = 0$ implies $d(F^*) = 0$. Hence, (66), $\Sigma = -\Sigma'$ and $E = E^*$ yield:

$$d[\ \Sigma\ E\bar{\Sigma}\ -\ \bar{\lambda}\,(E')^{-1}\ ] = 0 \quad ,$$

i.e., $\bar{\lambda}$ is also an eigenvalue of R(S) and part a) is proved. As regards part c), note first that $\Sigma\ E\ \Sigma$ is hermitian. Let λ be any eigenvalue of R(S) and $\underline{b}$ a corresponding eigenvector of the matrix $\Sigma\ E\ \bar{\Sigma}\ -\ \lambda(E')^{-1}$. By (66) and definition:

$$\underline{b}^*(\ \Sigma\ E\ \bar{\Sigma}\ )\underline{b} = \lambda\ \ \underline{b}^*(E')^{-1}\underline{b} \qquad .$$

Since E is definite, $(E')^{-1}$ is definite and $\underline{b}^*(E')^{-1}\underline{b} \neq 0$. Therefore:

$$-\ \frac{\underline{b}^*(\ \Sigma\ E\ \bar{\Sigma}\ )\underline{b}}{\underline{b}^*(E')^{-1}\underline{b}} = -\ \frac{\underline{x}^*\ E\ \underline{x}}{\underline{b}^*(E')^{-1}\underline{b}} \quad , \qquad (67)$$

where $\underline{x} = \bar{\Sigma}\ \underline{b}$.

But the numerator and denominator of (67) are real and have the same signs. Thus, $\lambda \leq 0$, Q.E.D. All that remains is part b). Suppose first that n is even and $d(\Sigma\ ) \neq 0$. Then (65) can be rewritten as:

$$d(E \bar{\bar{\Sigma}} E' - \lambda \Sigma^{-1}) = 0 \quad . \tag{68}$$

Now, the matrix $E \bar{\bar{\Sigma}} E' - \lambda \Sigma^{-1}$ is skew-symmetric for all scalars λ , and by a well-known result, its determinant is a perfect square, i.e., each eigenvalue is of even multiplicity. If $d(\Sigma) = 0$, it is possible to find a skew-symmetric matrix $\Omega = (\omega_{rk})$ such that $d(\Sigma_{\epsilon}) = d(\Sigma + \Omega) \neq 0$, and max $|\omega_{rk}| < \epsilon$ where ϵ is arbitrarily small. Thus:

$$d(\Sigma_{\epsilon} E \bar{\bar{\Sigma}} E' - \lambda 1_n) = g^2(\lambda ; \epsilon) \quad ,$$

$g(\lambda ; \epsilon)$ being a polynomial in λ . In the limit:

$$f(\lambda) = d(\Sigma E \bar{\bar{\Sigma}} E' - \lambda 1_n) = \lim_{\epsilon \to 0} g^2(\lambda ; \epsilon)$$

$$= g^2(\lambda ; 0) \quad ,$$

where:

$$g(\lambda ; 0) = \lim_{\epsilon \to 0} g(\lambda ; \epsilon) \quad ,$$

and exists for reasons of continuity. Thus the method used when $d(\Sigma) \neq 0$ will work and there will be a zero eigenvalue of even multiplicity. If n is odd, any skew-symmetric matrix will have a zero determinant. Hence the methods for even n must be modified and one considers a (n + 1)-port where the "new" port is uncoupled from the other ports and has no internal reflection ($S_{n+1, n+1} = 0$). This (n + 1)-port is handled by the previous method; all its eigenvalues are of even multiplicity. However, all the eigenvalues except one that is zero are also associated with the original n-port. Hence, all non-zero eigenvalues are of even multiplicity, while the zero eigenvalue is of odd multiplicity. Q.E.D.

VIII. GENERAL ALL-PASS EMBEDDING

For the two-port the reciprocity condition is ad - bc = 1. If this is not the case, one divides a, b, c, and d, by $\sqrt{ad - bc}$ $(\neq 0)$ and these new $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$, represent a reciprocal network that has the same bilinear transformation as the original network in one direction. For a lossy two-port the unit circle is not invariant, but two points which are inverse with respect to the unit circle will be transformed into two points inverse with respect to the image of the unit circle. For the 2n-port, the losslessness and reciprocity conditions are complicated and it is difficult to obtain results for a general embedding. The special case where the embedding network is an all-pass is amenable to analysis.

The transfer matrix of the all-pass is,

$$T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

Hence:

$$T \sigma_1 T' = \sigma_1 \begin{bmatrix} DA' & 0 \\ 0 & AD' \end{bmatrix} = \begin{bmatrix} AD' & 0 \\ 0 & DA' \end{bmatrix} \sigma_1 .$$

Similarly:

$$T\sigma_2 T^* = \sigma_2 \begin{bmatrix} AA^* & 0 \\ 0 & DD^* \end{bmatrix} = \begin{bmatrix} AA^* & 0 \\ 0 & DD^* \end{bmatrix} \sigma_2 \quad .$$

Of course the additional matrices will be the identity matrix, if the all-pass is lossless and reciprocal.

Theorem 9: If the embedding 2n-port M is an all-pass, and $T(S) = W$, then

$$T(S') = \begin{bmatrix} DA' & 0 \\ 0 & AD' \end{bmatrix}^{-1} W' \quad .$$

Proof: Let

$$S \doteq (X, Y)$$
$$S' \doteq (X_1, Y_1)$$
$$W \doteq (U, V)$$
$$T(S') \doteq (U_1, V_1)$$

From Lemma 1, Eq. (27):

$$[X, Y]\, \sigma_1\, [X_1', Y_1'] = 0_n$$

and from Eq. (24):

$$[X \mid Y] = Q_1[U \mid V]T$$
$$[X_1 \mid Y_1] = Q_2[U_1 \mid V_1]T$$

Thus:

$$O_n = [X \mid Y]\, \sigma_1\, [X_1 \mid Y_1]' = Q_1[U \mid V]T\sigma_1\, T'\, [U_1 \mid V_1]'Q_2'$$

$$= Q_1[U \mid V]\, \sigma_1 \begin{bmatrix} DA' & 0 \\ 0 & AD' \end{bmatrix} [U_1 \mid V_1]'Q_2'$$

Since Q_1 and Q_2 are non-singular

$$[U \mid V]\, \sigma_1 \begin{bmatrix} DA' & 0 \\ 0 & AD' \end{bmatrix} [U_1 \mid V_1]' = O_n$$

That is to say:

$$W' = \begin{bmatrix} DA' & 0 \\ 0 & AD' \end{bmatrix} T(S')$$

$$T(S') = \begin{bmatrix} DA' & 0 \\ 0 & AD' \end{bmatrix}^{-1} W' \qquad \text{Q.E.D.}$$

Theorem 10: If the 2n-port embedding is all-pass, and $T(S) = W$, and $d(S) \neq 0$, then,

$$T(S^{*-1}) = \begin{bmatrix} AA^* & 0 \\ 0 & DD^* \end{bmatrix}^{-1} W^{*-1}$$

Proof: This parallels Lemma 4 exactly as Theorem 9 paralleled Lemma 3.

CONCLUSIONS

The principal object of this research is to obtain canonical forms for the transformation of scattering matrices through lossless reciprocal networks. If the number of ports at the input of the device is the same as at the output of the device, the device is called a 2n-port, represented by a square transfer scattering matrix. Otherwise the device is represented by a rectangular transfer scattering matrix. Interesting results accrue from the search for canonical forms. It is shown that lossless and reciprocal transformations will transform S into W, S' into W', and S^{*-1} into W^{*-1}. Quite similar results hold for a non-lossless or non-reciprocal transformation if the transforming network is an all-pass. The lossless mappings transform the unit circle (i.e., unitary matrices) into itself, and passive networks ($1-SS^*$ positive definite) into other passive networks.

If the embedded network is reciprocal, then the canonical form consists of n uncoupled one-ports which are matched loads, negatives of matched loads and open circuits. These correspond to the dissipative, active, and lossless eigenvalues in the embedded network. In the case for the rectangular transfer matrix, the coallescing of the number of eigenvalues leaves no invariant, but the circuit form is the same as that for the square transfer matrix.

A cross-ratio matrix can be formulated and it is shown that it undergoes a similarity transformation in 2n-port embedding. The special choice of matrices in the cross ratio leads to the "characteristic" cross ratio matrix R(S). This supplies the generalization of the Mason invariant for arbitrary n. Corresponding properties of the eigenvalues of R(S) are obtained.

It is hoped that in succeeding research canonical forms will be obtained for the embedded non-reciprocal network. Some partial results have been obtained, but are stated elsewhere.

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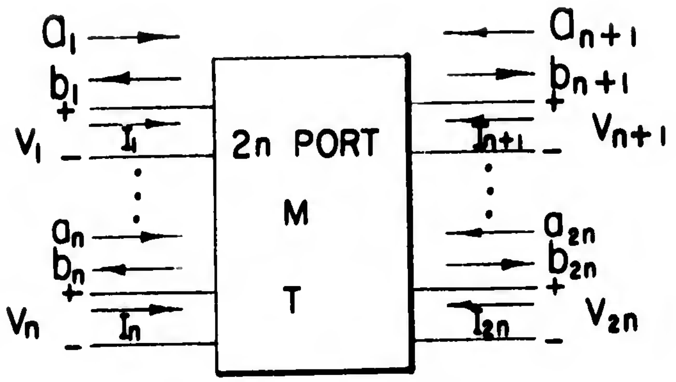
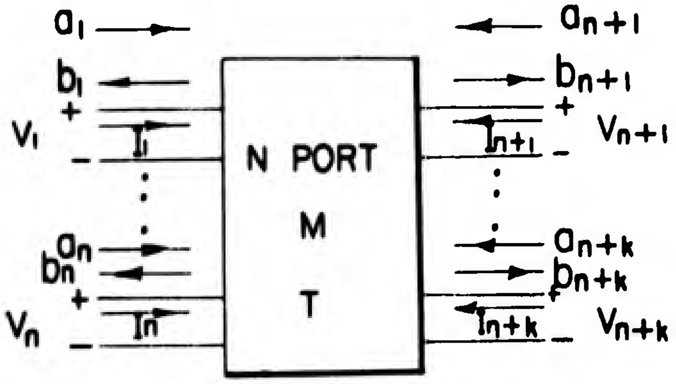
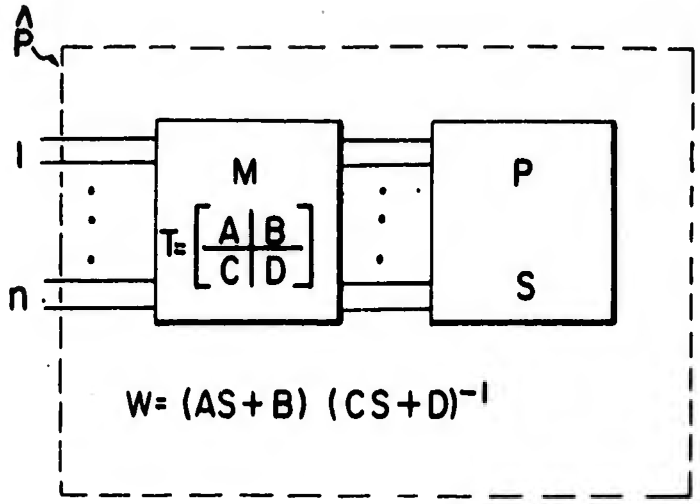


(a) Schematic of a N-port M With a Rectangular Transfer Matrix T
$k > n$, $k + n = N$

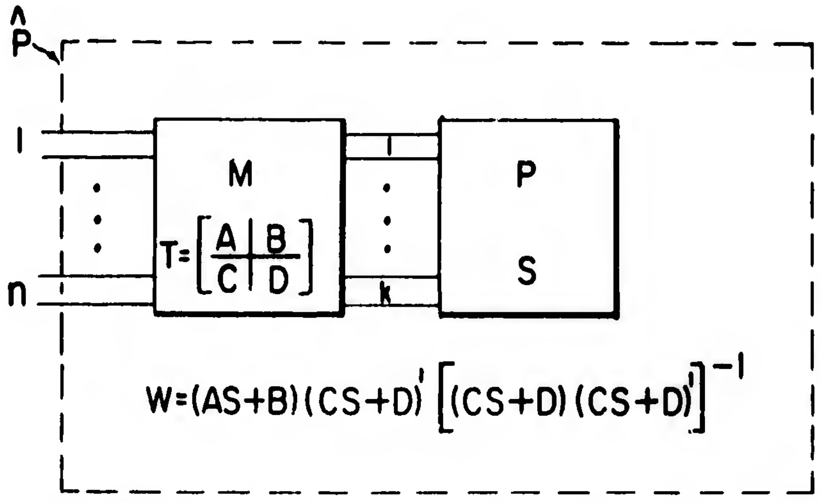


(b) Schematic of a 2n-port M With Transfer Scattering Matrix T

Fig. 1

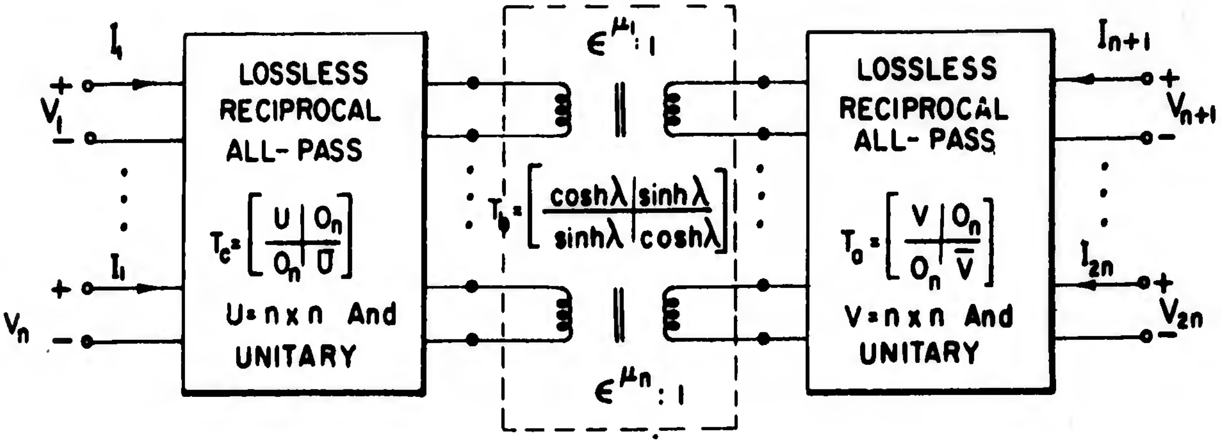


(a) The Embedding of an n-port P, With Scattering Matrix S, in a 2n-port M, With Transfer Scattering Matrix T, to Give a New n-port $\hat{P}$ With Scattering Matrix W

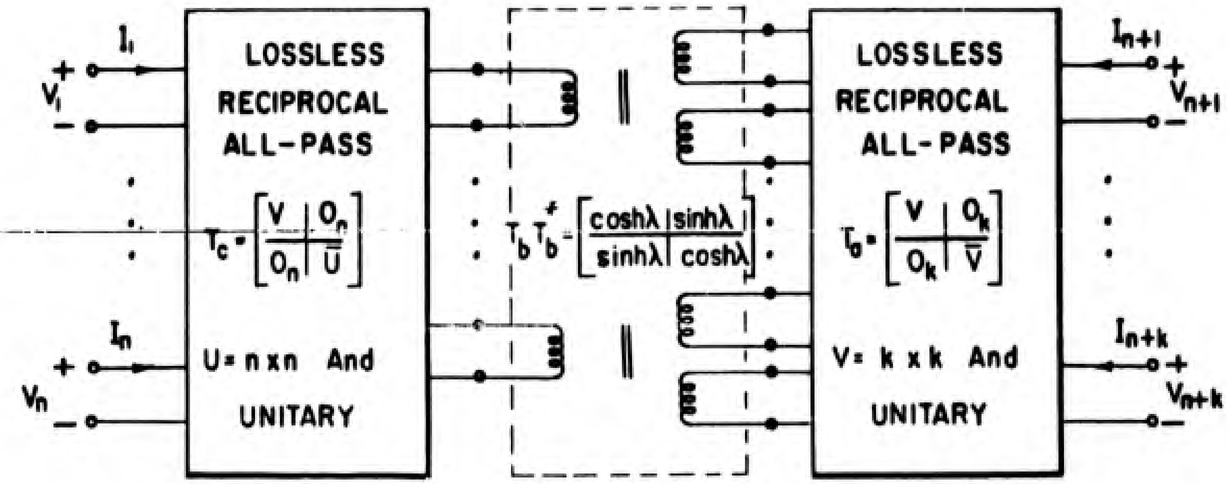


(b) The Embedding of a k-port P, With Scattering Matrix S, in an N-port With Transfer Scattering Matrix T, to Give a New n-port $\hat{P}$ With Scattering Matrix W
$k > n$, $k + n = N$

Fig. 2

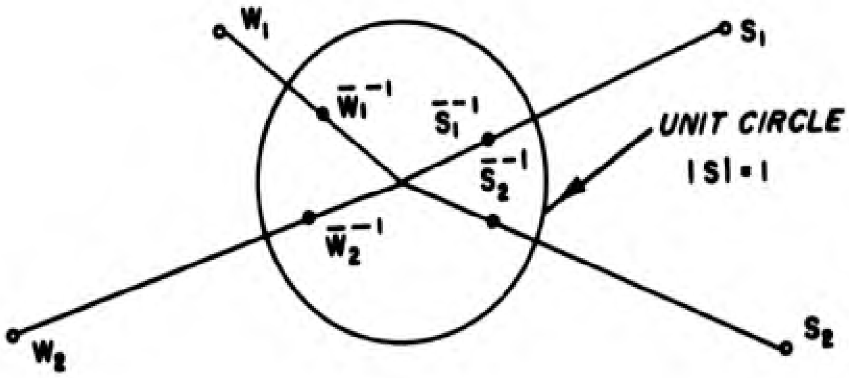


(a) Real Frequency Equivalent Circuit For a Lossless Reciprocal 2n-port M



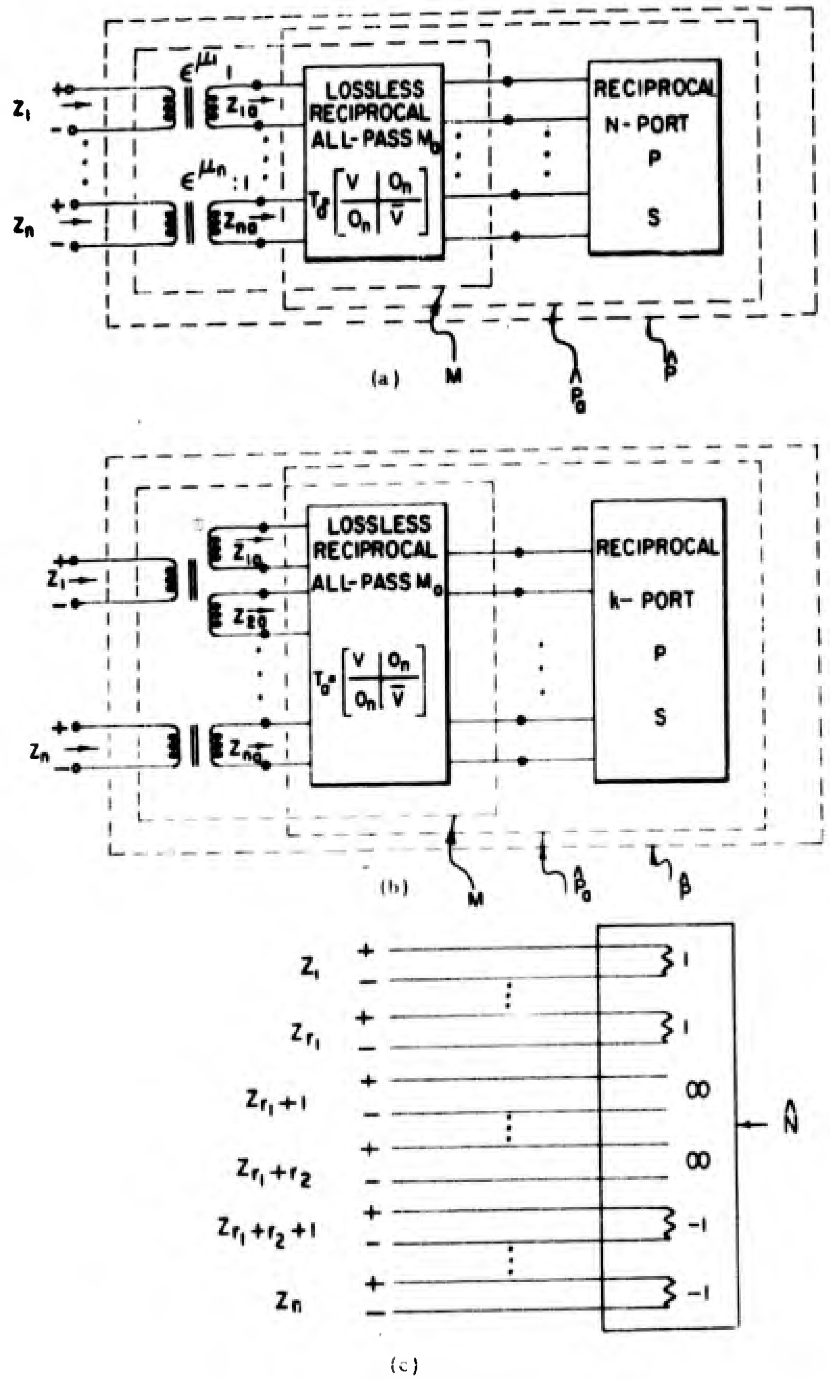
(b) Real Frequency Equivalent Circuit for a Lossless N-port M
k > n, k + n = N

Fig. 3



Four Points in the Complex s-plane
and Their Inverses With Respect to the Unit Circle

Fig. 4



In Fig. (5a) is Shown the Lossless Reciprocal 2n-port M Which Reduces the Reciprocal n-port P to the Canonic Form of Fig. (5c)

In Fig. (5b) is Shown the Lossless Reciprocal N-port (k > n, k + n = N) Which Reduces the Reciprocal k-port P to the Canonical Form of Fig. (5c).

Fig. 5

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