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A BUILDING BLOCK TECHNIQUE FOR THE STATEMENT AND SOLUTION OF PROBLEMS INVOLVING THE BIVARIATE NORMAL DISTRIBUTION

THESIS

Graduate Astronautics

GA/EE/62-1
August 1962

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Capt USAF

Presented to the Faculty of the School of Engineering of

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in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

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Preface

The purpose of this report is to present a simple, general technique for solving problems involving a bivariate normal distribution. The Building Block Technique is particularly designed to be used by the engineer who does not have an extensive background in statistics, but still encounters the necessity to solve problems which involve two normal statistical variables.

The underlying philosophy of this report is to present a technique, and data of sufficient accuracy, so that this type of problem can be solved rapidly at one's desk. Although a few statistical definitions are presented, primary emphasis is placed on the basic concepts of analytical geometry and the geometrical analogs of double integrals. If three decimal places are adequate with some uncertainty in the third place, all of the numerical data required to solve many problems involving the bivariate normal distribution are contained in this report.

I wish to express my deepest appreciation to Dr. H. Leon Harter, Aeronautical Research Laboratories, my Thesis Sponsor, for his thoughtful suggestions and guidance; to Professor A. H. Moore, whose comments helped to crystallize the problems to be solved; and to Mr. Edwin Godfrey, Computer Division, Aeronautical Systems Division, who programed the IBM 7090. My special thanks are due to Professor T. L. Regulinski, my Faculty Thesis Advisor, for his understanding, patience, and guidance over the long interval during which this thesis evolved. I particularly wish to express my indebtedness to my wife and three children, who have sacrificed more than anyone else during the preparation of this material.

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Abstract

The Building Block Technique is a simplified method for solving problems involving two normal (Gaussian) statistical variables. This technique is intended for engineers who are not versed in statistics. Graphs and tables of precomputed statistical functions, and a catalog of these functions in the form of geometrical analogs (building blocks) are presented. The paper is self-contained: if the problem is stated using the building blocks, the answer is the algebraic sum of their values. A new function, $A(\beta,\gamma)$, the probability included in an offset angle of the unit circular normal distribution, is introduced and tabulated to five decimal places.

A BUILDING BLOCK TECHNIQUE FOR THE STATEMENT AND SOLUTION OF PROBLEMS INVOLVING THE BIVARIATE NORMAL DISTRIBUTION

I. Introduction

In real life no operation is exactly duplicated, nor are any two parts, or actions, ever exactly identical. There is always an error of some magnitude. Inspection of machined parts, the time to failure of equipment, random electrical noise, height of workers, performance, and risk calculations all reflect this variability. Although the study of such problems is in the domain of statistics and mathematical probability, it is frequently the responsibility of the engineer to determine the best combination of variables when setting tolerances or establishing performance specifications.

Frequently such problems can be expressed in terms of two normal (Gaussian) statistical variables (for example, the range error and the direction error of a ballistic missile.) These variables may be independant, so that they cannot affect each other; or, they may be correlated, so that a change in one variable produces a proportional change in the expected value of the other variable.

Purpose

The purpose of this thesis is to present a technique which can be used to set up and to solve problems involving two normal statistical variables without using statistics, probability, or an electronic computer.

This paper is intended to be complete in itself. Any problem which can be stated using the building block technique can be solved without using any other reference or technique.

Method

An analog technique using "building blocks" is presented. Each building block is the geometrical area representing the integral of a statistical function whose value is known (Appendix A.) Geometrical manipulation of the building blocks to define a problem is equivalent to the mathematical manipulation of double integrals of intricate exponential functions which do not normally have closed solutions.

Once the appropriate building blocks have been selected, the numerical solution to the problem is the algebraic sum of their values from Appendix A.

Advantages

The building block technique eliminates the need for using higher mathematics. In essence, the problem is reduced to adding and subtracting areas, using building blocks. The building blocks can be combined to arrive at an answer without knowing the underlying statistics, without stating the problem in terms of the double integrals, and without performing numerical integration.

Using the building block technique permits persons who are not versed in statistics or probability to set up and solve problems in normal bivariate distributions, at their desk, using a single reference. No operation is required which cannot be performed with a pencil and paper and, occasionally, a slide rule.

Accuracy

The building block technique does not intrinsically introduce any error into the solution. The numerical accuracy of the answer is limited only by the accuracy with which the value of each of the building blocks is known.

Using the graphs in Appendix A, the numerical solution is accurate to two decimal places, with an error in the third decimal place which is equal to, or less than, the number of building blocks used in the solution. If more significant figures are required, the value of each building block can generally be found to one or two more places in the references.

Order of Presentation

The remainder of this paper is devoted to developing and presenting the Building Block Technique. Chapter II defines the bivariate normal distribution in both mathematical and geometrical terms. Chapter III presents the transformation which is used to map the general bivariate normal distribution in the circular normal distribution for which the values of the building blocks are known. Chapter IV is a discussion of the concept of a building block. Chapter V is a catalog of the building blocks. Chapter VI contains the general rules for using the building blocks and for estimating the accuracy of the solution. Chapter VII contains some examples of problems which have been solved using the building blocks. The summary and conclusions are contained in Chapter VIII. Chapter IX contains some recommendations for further work in this area.

Where extensive analytical proofs or statistical justifications are required, they have been placed in an appendix.

II. The Bivariate Normal Distribution

Mathematical Description

The probability, P(x), that a normally distributed variable, x, will occur between two bounding functions, say $f_1(x)$ and $f_2(x)$, is given by

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{f_1(x)}^{f_2(x)} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \tag{1}$$

where
- μ_x is the mean value of the distribution of x
- σ_x is the standard deviation of x about μ_x
- $f_1(x)$ is the function of x which is the lower bound of the region of interest
- $f_2(x)$ is the function of x which is the upper bound of the region of interest

Similarly, the probability, P(y), that another normally distributed variable, y, will occur between two bounding functions, say $g_1(y)$ and $g_2(y)$, is given by

$$P(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \int_{g_1(y)}^{g_2(y)} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} dy \tag{2}$$

By definition, P(x,y) is the joint probability that a point (X,Y) will fall in the region R, where R is the region bounded by $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(y)$. The equation which gives P(x,y) for the bivariate normal distribution is

$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \iint_R e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} dy\,dx \tag{3}$$

where x and y are the normally distributed variables of equations (1) and (2)

ρ is the correlation coefficient between x and y $(-1 \leq \rho \leq 1)$

In equation (3), the correlation coefficient is most usefully interpreted as that proportion of the standard deviation of x, or of y, which is due to changes in the other variable. The correlation coefficient is a function of the changes in, say, E(y) induced by changes in x alone. ρ is positive if E(y) increases due to a positive change in x; ρ is negative if E(y) decreases due to a positive change in x. If E(y) is unaffected by any change in x, then $\rho = 0$, and the distribution is uncorrelated (Ref 2:359).

For the uncorrelated bivariate normal distribution, $\rho = 0$, and equation (3) reduces to

$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \iint_R e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} dy\,dx \qquad (4)$$

A special case of the uncorrelated distribution occurs when $\sigma_x = \sigma_y = \sigma$. This distribution is the circular normal distribution (Ref 7:xxxi). For the circular normal distribution, equation (4) reduces to

$$P(x,y) = \frac{1}{2\pi\sigma^2} \iint_R e^{-\frac{1}{2\sigma^2}\left[(x-\mu_x)^2 + (y-\mu_y)^2\right]} dy\,dx \qquad (5)$$

For the purposes of this thesis, a normalized distribution will be used. The normalized distribution is defined as a unit circular normal distribution about the origin $(\sigma_u = \sigma_v = 1;\ \mu_u = \mu_v = 0)$.

The equation of the normalized distribution is

$$P(u,v) = \frac{1}{2\pi}\iint_R e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du \tag{6}$$

It should be noted that, in this distribution, rotation of the coordinate axes (or, equally, rotation of R about the origin) does not change the value of integrand (Appendix C). This property will be used extensively.

Geometrical Description

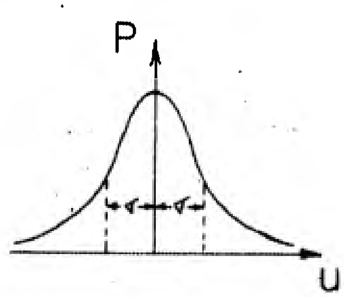


Fig. 1

Bell-shaped Curve
(Univariate)

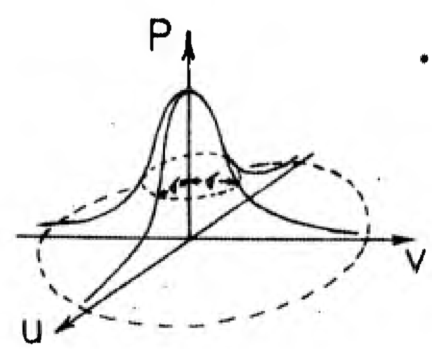


Fig. 2

Bell-shaped Surface
(Bivariate)

Fig. 1 shows a typical "bell-shaped" curve for a single variable. This curve is a plot of the normal probability of a variable whose mean is zero, and whose standard deviation is a constant equal to σ. If this bell-shaped curve is rotated about its mean value, it forms a "bell-shaped" surface, whose cross sections are all circles (Fig. 2). This bell-shaped surface is a circular normal distribution. The mean of the distribution in Fig. 2 is at the origin; if it is assumed that $\sigma = 1$, then Fig. 2 is also a sketch of the normalized distribution.

However, it is difficult to work with three dimensional figures. Therefore, the convention of viewing distributions like Fig. 2 from the top will be adopted. In this form only the trace of the standard deviation is required to show the shape of the distribution. Fig. 3 is the equivalent of Fig. 2.

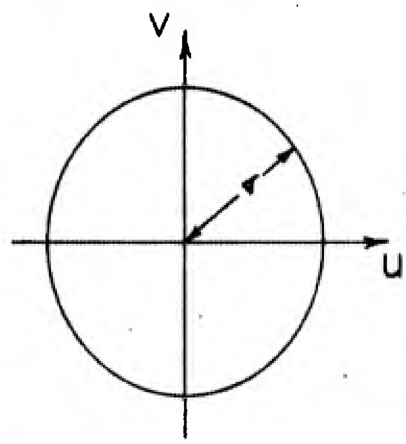


Fig. 3

Equivalent Presentation
Circular Normal Distribution
(Top View of Fig. 2)

In the uncorrelated bivariate normal distribution, $\rho = 0$. Geometrically, this means that the horizontal cross sections of the distribution are ellipses whose principal axes are parallel to the coordinate axes (Ref 2:359). Fig. 4 shows two uncorrelated bivariate normal distributions.

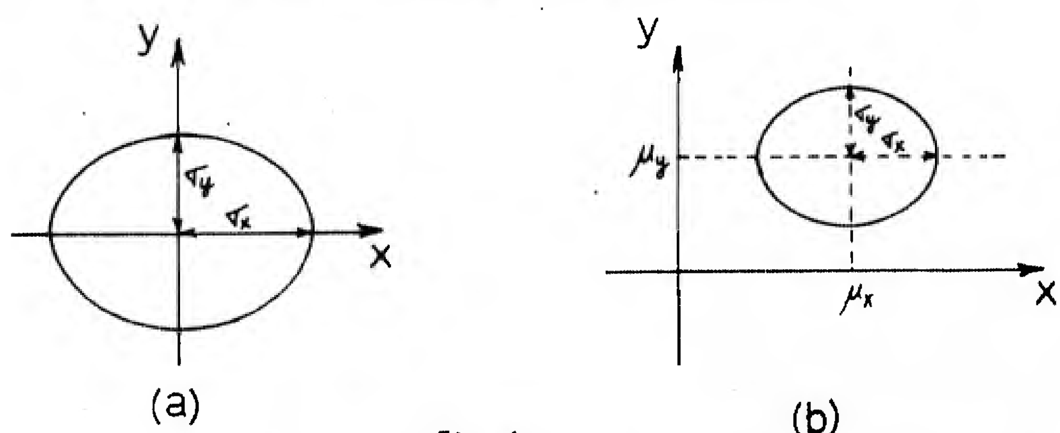


Fig. 4

Uncorrelated Bivariate Normal Distribution ($\rho = 0$), with Principal Axes Through the Origin

Offset, Uncorrelated Bivariate Normal Distribution ($\rho = 0$, but $\mu_x \neq 0$, $\mu_y \neq 0$)

The general bivariate normal distribution is unrestricted in orientation. It is usually offset from the origin, has elliptical cross sections, and does not have its principal axes parallel to the coordinate axes. Fig. 5 shows a typical general bivariate normal distribution.

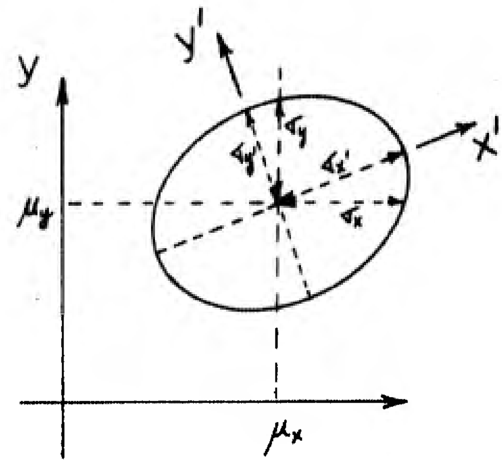


Fig. 5

Typical General Bivariate Normal Distribution

It should be noted that the unit circular normal distribution in Figs. 2 and 3 is a special case of the general bivariate normal distribution. In this special case, which has been selected as the normalized form for purposes of computation, the mean of the distribution is at the origin, the horizontal cross sections are circles, and the distance from the origin to the trace of the standard deviation is the unit of distance.

III. The Normalizing Transformations

The transformations given in this chapter map probabilities in the general bivariate normal distribution into equivalent probabilities in the normalized distribution. The most useful characteristic of these transformations is that they map lines into lines, ellipses into ellipses, convex polygons into convex polygons, concave polygons into concave polygons, hyperbolas into hyperbolas, and so forth (Ref 10:21). Analytical proof of these transformations is given in Appendix B.

Coordinate Transformation

$$u = \frac{x-\mu_x}{\sigma_x} \tag{7}$$

$$v = \frac{v' - \rho u}{\sqrt{1-\rho^2}} \qquad (\rho \neq \pm 1) \tag{8}$$

where

$$v' = \frac{y-\mu_y}{\sigma_y} \tag{9}$$

Transformation of the Slope of a Line

$$m = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\sigma_x}{\sigma_y} m_{xy} - \rho\right) \qquad (\rho \neq \pm 1) \tag{10}$$

where

$$m_{xy} = \frac{y_2 - y_1}{x_2 - x_1} \equiv \text{Slope of the line in the general distribution} \tag{11}$$

IV. The Building Block

A "building block" is defined as the area on the u,v plane of the normalized distribution which corresponds to the integral of a statistical function. The characteristics of a building block are:

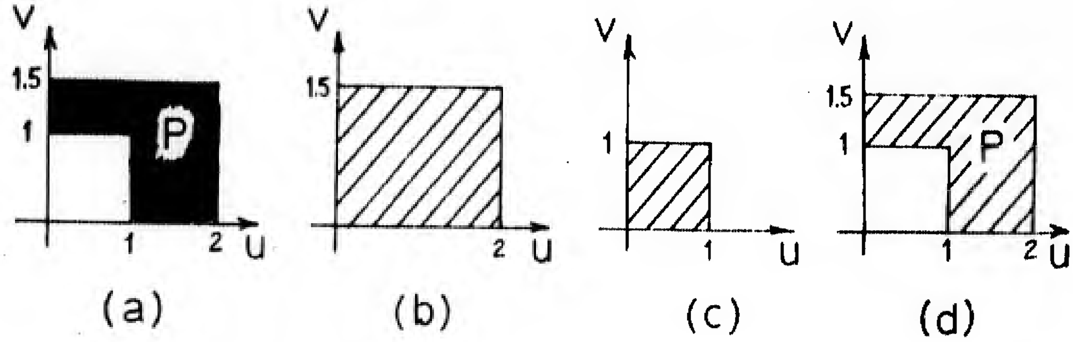
1. A simple geometry in the coordinate system of the normalized distribution.
2. One to four geometrical parameters which uniquely define the function.
3. A known numerical solution in terms of these parameters.

Since the numerical value of each building block is known (Appendix A), it is only necessary to describe the area of interest on the u,v plane using the characteristic parameters of the building blocks. Once the building blocks required have been identified, the numerical solution is reduced to looking up their values and taking the algebraic sum.

To assist in the identification and visualization of the building blocks, a catalog of the functions which can be used for building blocks, their characteristic parameters, and their shape in the u,v plane is presented in Chapter V.

Use of the building blocks can be compared to designing a wall with a window in it using various prefabricated wall sections. The problem is stated by deciding where the window must be, how big it must be, and the shape that it must have. Solving the problem can be compared to the job of the designer who must consider the prefabricated sections which are available and then select the necessary shapes and sizes which will produce the specified wall and window.

An example of building block manipulation is given in Fig. 6. In this example it is desired to find the probability (P) that $0 \le y \le 1.5$ when $1 \le x \le 2$, and $1 \le y \le 1.5$ when $0 \le x \le 1$ in the normalized distribution.



$P = \frac{1}{2\pi}\iint_A e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$ Block 1 Block 2 P = Block 1 - Block 2

Numerical Solution:

Block 1 =	$\frac{\alpha(2)\,\alpha(1.5)}{4}$	=	.207	
- Block 2 = -	$\frac{\alpha(1)\,\alpha(1)}{4}$	=	- .117	
P =	Sum	=	.090	Answer

Fig. 6

Example of Manipulation and Numerical Solution Using Building Blocks

The analytical statement of this problem is given by

$$P = \frac{1}{2\pi}\int_1^2\int_0^{1.5} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du + \frac{1}{2\pi}\int_0^1\int_1^{1.5} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du \qquad (12)$$

$$= \frac{1}{2\pi}\iint_A e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du \qquad (13)$$

The shaded area of Fig. 6a represents the "wall" or desired probability; the unshaded area represents the "window" or area which is not of interest. Figs. 6b and 6c show the two rectangular building blocks (defined in Section 22, Chapter V) which are combined in Fig. 6d to produce the desired result. The lower part of Fig. 6 shows the general format for finding the numerical solution to a problem. In this example, the answer is the numerical difference between the two building blocks. Detailed rules for manipulating the building blocks are given in Chapter VI.

The analog nature of the building blocks is, also, best shown by means of an example. Fig. 7a is a sketch of the analytical arguments of the problem of the offset circle. The probability (P) is defined as the probability that an event in the normalized distribution will occur within a radius (r) of some point (A), where (A) is at a distance (d) from the origin.

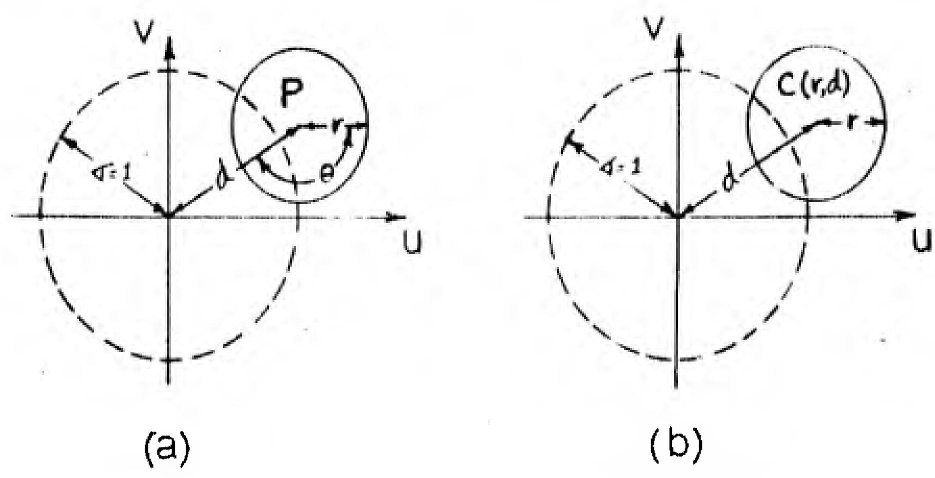


Fig. 7

Sketch of Analytical Problem (For Offset Circle)

Sketch of Building Block C(r,d)

The analytical solution of this problem is the evaluation of the equation

$$P = \frac{e^{-\frac{d^2}{2\sigma^2}}}{2\pi\sigma^2} \int_0^{2\pi}\int_0^{r} e^{-\frac{1}{2\sigma^2}(r^2-2rd\cos\theta)}\, r\,dr\,d\theta \qquad (14)$$

which does not have a closed solution when $d \neq 0$ (Ref 1:1). Direct evaluation of this integral requires numerical integration techniques and an electronic computer.

However, in the normalized distribution the value of equation (14) as a function of r and d is known (Ref 1:5-303; Ref 3:102-105; Ref 11:3-18; and Fig. A-6, Appendix A), and the geometry of the integral in the u,v plane is simple (Fig. 7b). Therefore, the function meets the requirements for a building block. In Chapter V, the Catalog of Building Blocks, this function is identified as the offset circle building block C(r,d). C(r,d) is a circular disc with its center at (A), whose numerical value is equal to the value of the statistical function in equation (14) when $\sigma = 1$. Therefore, it is no longer necessary to integrate equation (14). When r and d are known, the numerical value of C(r,d) can be read directly from Fig. A-6 in Appendix A.

It should be noted that finding the value of a building block does not require any knowledge of statistics or probability. Measuring the geometrical parameters of the building block is sufficient. The power of the Building Block Technique lies in this simplification, and in the addition and subtraction characteristics indicated in Fig. 6. Together they permit the evaluation of the probability integral over complicated regions without any direct use of statistics or higher mathematics.

V. Catalog of the Building Blocks

This chapter presents a catalog of the building blocks. It includes all but one of the statistical functions of the normalized distribution for which numerical solutions exist in the literature. The function T(h,a) was omitted because it is simply the difference between two more versatile building blocks (Ref 10:7). One new function, A(θ,γ), has been included. This function was defined by the author based on a suggestion in one of the references (Ref 7:xxxii). Five place tables of A(θ,γ) are presented in Appendix F.

In this chapter, each building block is identified geometrically in terms of its characteristic geometrical parameters. In addition, a mathematical statement of the statistical function which each building block represents is given, and a reference is made to the appropriate graphical or tabular solution in Appendix A.

The building blocks are grouped into three types:

I. Infinite sectors of the u,v plane

II. Bounded areas of the u,v plane

III. Pieces of other building blocks

Type I building blocks are not bounded in at least one dimension, for at least one variable. Type II building blocks are bounded in both variables. Type III building blocks are derived by "cutting up" Type I or Type II building blocks into smaller pieces in accordance with Rule 6 of Chapter VI. Since an infinite number of Type III building blocks is possible, only a representative sample is listed in this catalog.

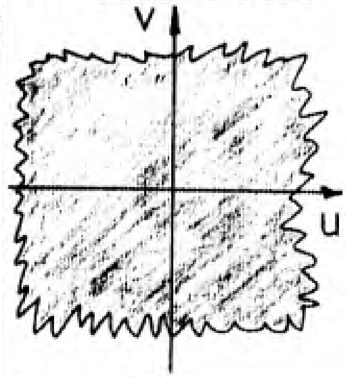
The numerical values of the Type I and Type II building blocks are given in Appendix A. The numerical values of the Type III building blocks will always be some known fraction of a Type I, or Type II, building block. Therefore, no additional graphs or tables are required for the Type III building blocks.

Except for $A(\beta,\gamma)$, the numerical values which were used to construct the graphs and tables in Appendix A have all been taken from the references. The function $A(\beta,\gamma)$ was computed for this thesis by numerical integration of its defining integral using an IBM 7090 electronic computer (Appendix F).

The Catalog of Building Blocks is presented on the next seven pages of this chapter.

Type I Building Blocks: Infinite Sectors

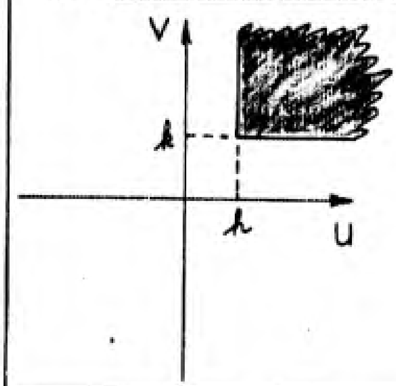
(1) The Whole Plane



$$P(u,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

(Numerical Value: 1.000...)

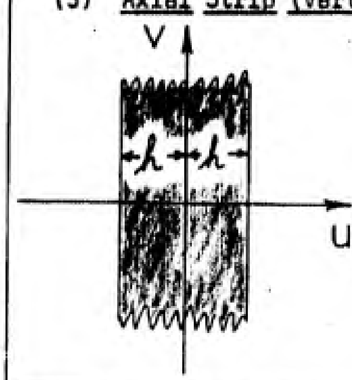
(2) Right Angle Sector, L(h,k)



$$L(h,k) = \frac{1}{2\pi}\int_{h}^{\infty}\int_{h}^{\infty} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

(Numerical Value: Fig. A-1)

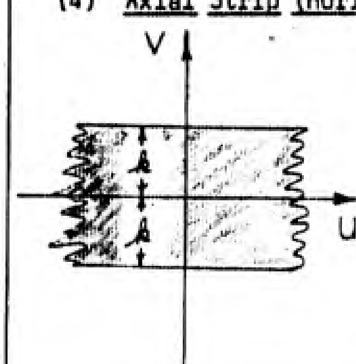
(3) Axial Strip (Vertical), $\alpha(h)$



$$\alpha(h) = \frac{1}{\sqrt{2\pi}}\int_{-h}^{h} e^{-\frac{u^2}{2}}\,du$$

(Numerical Value: Fig. A-2)

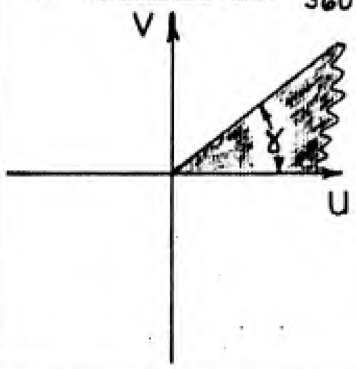
(4) Axial Strip (Horizontal), $\alpha(k)$



$$\alpha(k) = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} e^{-\frac{v^2}{2}}\,dv$$

(Numerical Value: Fig. A-2)

(5) <u>Centered Arc</u>, $\frac{\gamma}{360}$

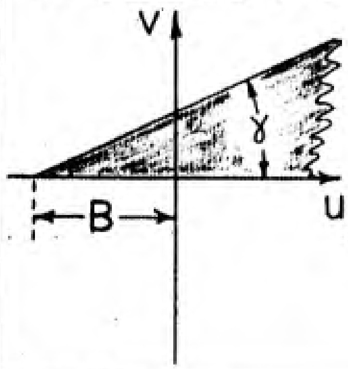


$$\frac{\gamma}{360} = \frac{1}{2\pi}\int_0^{\infty}\int_0^{u\tan\gamma} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

$$= \frac{\gamma\langle radians\rangle}{2\pi} = \frac{\gamma\langle degrees\rangle}{360^\circ}$$

(Numerical Value: Table A-I)

(6) <u>Offset Arc</u>, <u>A(B,γ)</u>

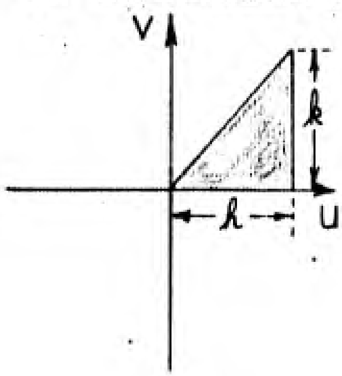


$$A(B,\gamma) = \frac{1}{2\pi}\int_{-B}^{\infty}\int_0^{(u+B)\tan\gamma} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

(Numerical Value: Fig. A-3)

Type II Building Blocks: Bounded Areas

(7) <u>Right Triangle</u>, <u>V(h,k)</u>



$$V(h,k) = \frac{1}{2\pi}\int_0^{h}\int_0^{\frac{\lambda u}{h}} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

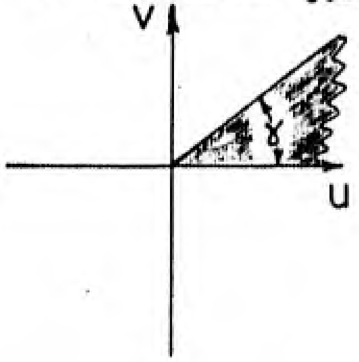
$$= V(h,\lambda) \quad \text{if } h \geq k$$

$$= V(\lambda,k) \quad \text{if } h \leq k$$

where $\lambda = \frac{k}{h}$ or $\frac{h}{k}$, whichever ≤ 1

Numerical Value:
- $V(h,\lambda)$: Fig. A-4
- $V(\lambda,k)$: Fig. A-5

(5) <u>Centered Arc</u>, $\frac{\gamma}{360}$

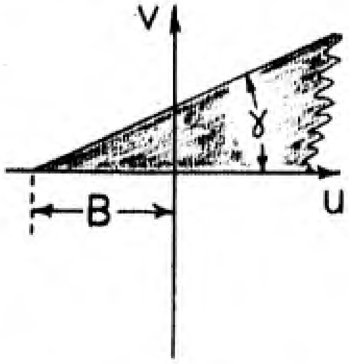


$$\frac{\gamma}{360} = \frac{1}{2\pi}\int_0^{\infty}\int_0^{u\tan\gamma} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

$$= \frac{\gamma\langle radians\rangle}{2\pi} = \frac{\gamma\langle degrees\rangle}{360^\circ}$$

(Numerical Value: Table A-I)

(6) <u>Offset Arc</u>, <u>A(B,γ)</u>

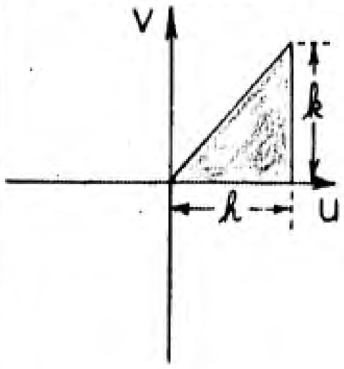


$$A(B,\gamma) = \frac{1}{2\pi}\int_{-B}^{\infty}\int_0^{(u+B)\tan\gamma} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

(Numerical Value: Fig. A-3)

Type II Building Blocks: Bounded Areas

(7) <u>Right Triangle</u>, <u>V(h,k)</u>



$$V(h,k) = \frac{1}{2\pi}\int_0^{h}\int_0^{\frac{\lambda u}{h}} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

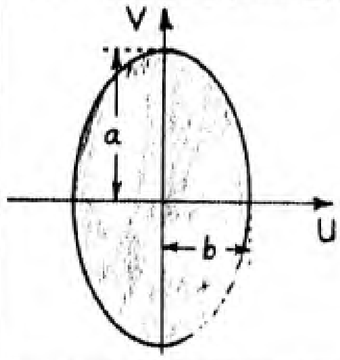
$$= V(h,\lambda) \quad \text{if } h \geq k$$

$$= V(\lambda,k) \quad \text{if } h \leq k$$

where $\lambda = \frac{k}{h}$ or $\frac{h}{k}$, whichever ≤ 1

Numerical Value:
V(h,λ): Fig. A-4
V(λ,k): Fig. A-5

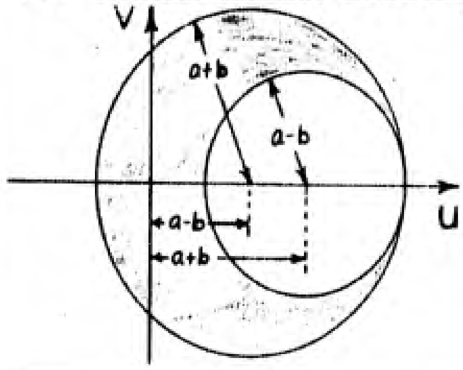
(11) Centered Ellipse, E(a,b,0,0)



$$E(a,b,0,0) = \frac{1}{2\pi}\int_{-b}^{b}\int_{-\frac{a}{b}\sqrt{b^2-u^2}}^{+\frac{a}{b}\sqrt{b^2-u^2}} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

(Numerical Value: Table A-II)

(12) Centered Ellipse, Alternate Method



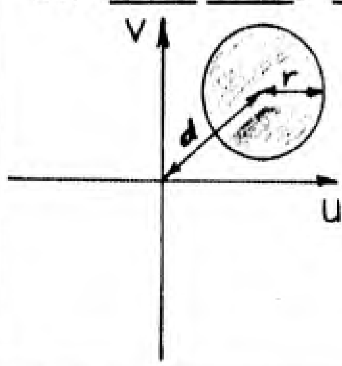
$$E(a,b,0,0) = C[(a-b),(a+b)] - C[(a+b),(a-b)]$$

a = Length of Semi-major Axis

b = Length of Semi-minor Axis

(Numerical Value: Fig. A-6)

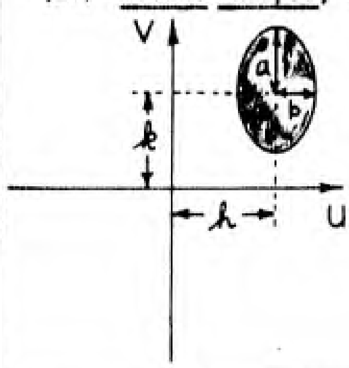
(13) Offset Circle, C(r,d)



$$C(r,d) = \frac{e^{-\frac{d^2}{2}}}{2\pi}\int_0^{2\pi}\int_0^{r} e^{-\frac{1}{2}(r^2-2rd\cos\theta)}\,r\,dr\,d\theta$$

(Numerical Value: Fig. A-6)

(14) Offset Ellipse, E(a,b,h,k)

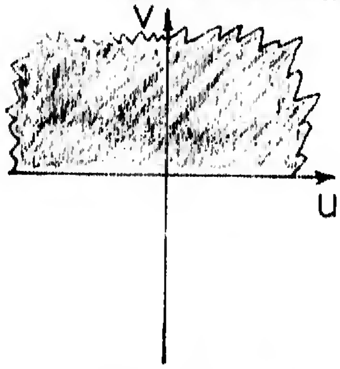


$$E(a,b,h,k) = \frac{1}{2\pi}\int_{h-b}^{h+b}\int_{k-\frac{a}{b}\sqrt{b^2-(u-h)^2}}^{k+\frac{a}{b}\sqrt{b^2-(u-h)^2}} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

(Numerical Value: Table A-II)

Type III Building Blocks: Pieces of Types I & II

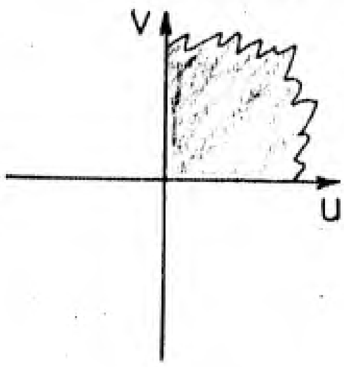
(15) Half of the Plane



$$P_{\frac{1}{2}\text{ plane}} = \frac{1}{2}(P_{\text{whole plane}})$$

(Numerical Value: 0.500...)

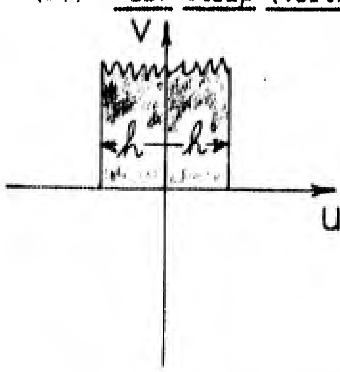
(16) Quarter of the Plane



$$P_{\frac{1}{4}\text{ plane}} = \frac{1}{4}(P_{\text{whole plane}})$$

(Numerical Value: 0.250...)

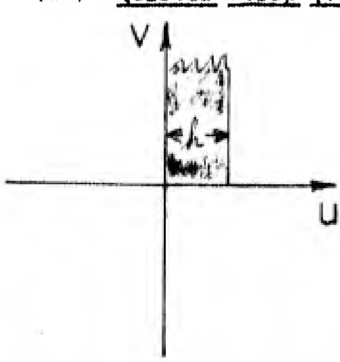
(17) Half Strip (Vertical), $\frac{\alpha(h)}{2}$



$$\frac{\alpha(h)}{2} = \frac{1}{2}\left[\alpha(h)\right]$$

[Numerical Value: $\frac{1}{2}$(Values Fig. A-2)]

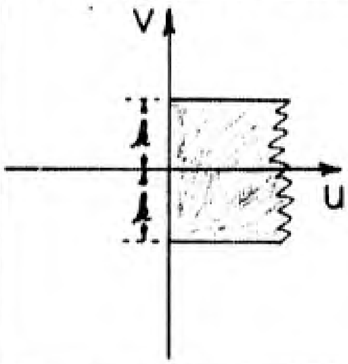
(18) Quarter Strip (Vertical), $\frac{\alpha(h)}{4}$



$$\frac{\alpha(h)}{4} = \frac{1}{4}\left[\alpha(h)\right]$$

[Numerical Value: $\frac{1}{4}$(Values Fig. A-2)]

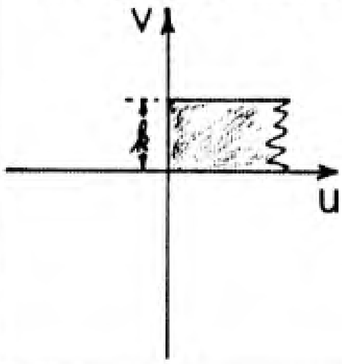
(19) Half Strip (Horizontal), $\frac{\alpha(k)}{2}$



$$\frac{\alpha(k)}{2} = \frac{1}{2}\left[\alpha(k)\right]$$

[Numerical Value: $\frac{1}{2}$(Values Fig. A-2)]

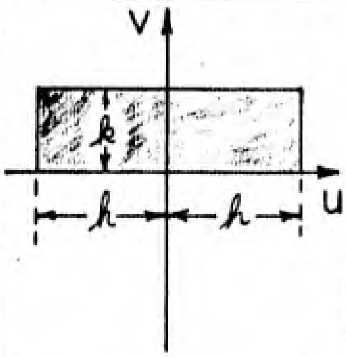
(20) Quarter Strip (Horizontal), $\frac{\alpha(k)}{4}$



$$\frac{\alpha(k)}{4} = \frac{1}{4}\left[\alpha(k)\right]$$

[Numerical Value: $\frac{1}{4}$(Values Fig. A-2)]

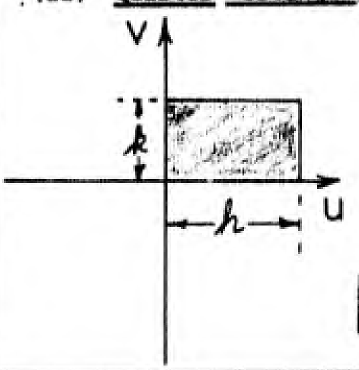
(21) Half Rectangle, $\frac{\alpha(h)\alpha(k)}{2}$



$$\frac{\alpha(h)\alpha(k)}{2} = \frac{1}{2}\left[\alpha(h)\right]\left[\alpha(k)\right]$$

[Numerical Value: $\frac{1}{2}$ Product (Values: Fig. A-2)]

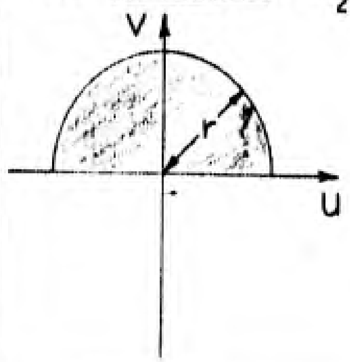
(22) Quarter Rectangle, $\frac{\alpha(h)\alpha(k)}{4}$



$$\frac{\alpha(h)\alpha(k)}{4} = \frac{1}{4}\left[\alpha(h)\right]\left[\alpha(k)\right]$$

[Numerical Value: $\frac{1}{4}$ Product (Values: Fig. A-2)]

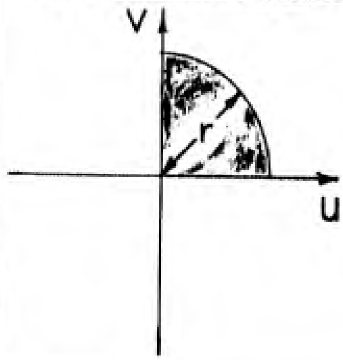
(23) Semi-Circle, $\frac{C(r,0)}{2}$



$$\frac{C(r,0)}{2} = \frac{1}{2}\left[C(r,0)\right]$$

[Numerical Value: ½(Values Fig. A-6a)]

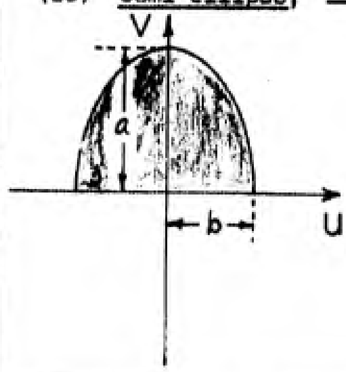
(24) Quadrant of Circle, $\frac{C(r,0)}{4}$



$$\frac{C(r,0)}{4} = \frac{1}{4}\left[C(r,0)\right]$$

[Numerical Value: ¼(Values Fig. A-6a)]

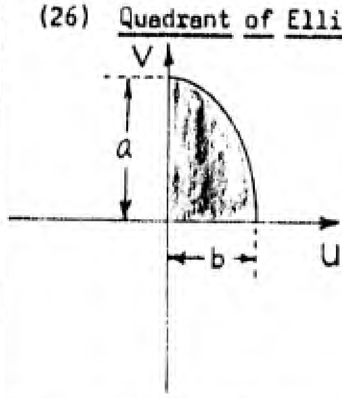
(25) Semi-Ellipse, $\frac{E(a,b,0,0)}{2}$



$$\frac{E(a,b,0,0)}{2} = \frac{1}{2}\left[E(a,b,0,0)\right]$$

[Numerical Value: ½(Values Table A-II)]

(26) Quadrant of Ellipse, $\frac{E(a,b,0,0)}{4}$



$$\frac{E(a,b,0,0)}{4} = \frac{1}{4}\left[E(a,b,0,0)\right]$$

[Numerical Value: ¼(Values Table A-II)]

VI. Use of the Building Blocks

Basic Procedure

The basic procedure of the Building Block Technique was demonstrated in the example in Fig. 6:

1. Define the desired integral geometrically.
2. Apply the normalizing transformations if the original distribution is not the normalized distribution.
3. Construct the "area" of the desired integral using the building blocks.
4. Find the numerical value of each building block using Appendix A.
5. Add, algebraically, the values of the building blocks (+ for added blocks, - for subtracted blocks).

This sum is the desired answer.

Rules for the Manipulation of Building Blocks

1. If a building block is added, add its value to the sum.
2. If a building block is subtracted, subtract its value from the sum.
3. Any building block can be rotated about the origin of the normalized distribution without changing its value (Appendix C).
4. Each building block must be added, or subtracted, in its entirety.
5. Multiplication of building blocks is not permitted, except to form the $\alpha(h)\alpha(k)$ building block (Appendix E).
6. If a building block is, or can be rotated about the origin to be, symmetrical with respect to one, or both, of the coordinate axes, the building block can be divided into smaller related building blocks:

    a. Any building block which is symmetrical with respect to one coordinate axis is divided by that axis into two equal parts (Appendix C).

b. Any building block which is symmetrical with respect to both coordinate axes is divided by the axes into four equal parts, or quarters (Appendix C).

c. Any building block which is symmetrical with respect to the origin can be divided into equal probability areas by equal angles whose vertices are at the origin (Appendix D).

d. If a building block cannot be rotated to be symmetrical with respect to a coordinate axis, it can not be subdivided into symmetrical parts whose values are equal.

e. Each part of a subdivided building block becomes a new building block which conforms to all of the manipulation rules.

7. Maximum accuracy in the solution is obtained by using the minimum number of building blocks.

8. If the semi-major and semi-minor axes of a general bivariate normal distribution are known, it is unnecessary to know the correlation coefficient. Make the x and y axes coincident with the axes of the distribution. By definition, the distribution will then be uncorrelated in x and y.

9. When the proper combination of building blocks has been selected, the answer is the algebraic sum of their values.

10. Sketch the coordinate axes, the problem area, and the building blocks (freehand) to keep track of the positive and negative areas which occur when adding and subtracting building blocks.

Accuracy of the Result

The graphs in Appendix A have been plotted to ±.001 (or 0.1%). Since this accuracy does not depend on the size of the building block, only the number of building blocks used will effect the accuracy of the solution.

When using Appendix A, the following building blocks are assumed to be known without error since they are either exact values or have their error beyond the third decimal place:

1. Whole plane = 1.000...
2. Half plane = .500...

3. Quarter plane = .250...

4. $\frac{\gamma}{360}$

5. Permissible halves of any building block (such as $\frac{C(r,d)}{2}$).

6. Permissible quarters of any building block (such as $\frac{\alpha(h)}{4}$).

Therefore, a reasonable estimate of the maximum error using Appendix A is:

$$\text{Maximum Error} = \pm .001\ (1+n)$$

where n = the number of building blocks used which are not of the types listed in the preceding paragraph.

The actual error will usually be smaller than this maximum, since some of the errors in reading the values of the building blocks will tend to cancel. It should be noted that this error is a function only of the accuracy of the numerical values used for the building blocks. If the numerical value of each building block were known exactly, there would be no error in the final answer.

VIII. Examples

Seven sample problems are presented in this chapter to illustrate some of the properties of the building blocks, and to provide familiarity with the Building Block Technique. The examples illustrate the following items:

Example 1: Building block addition; use of the V(h,k) and V(k,h) building blocks; computation of the quadrant building block $\frac{\alpha(h)\alpha(k)}{4}$; and the relationship between these building blocks.

Example 2: Formation of areas from building blocks of several shapes; rotation of the building blocks; and the relationship between $A(\theta,\gamma)$ and some of the other building blocks.

Example 3: Use of the normalizing transformations; some of the geometrical effects of normalizing; use of the $A(\theta,\gamma)$ building block; and use of the Whole Plane building block.

Example 4: Use of the E(a,b,0,0) building block; the geometrical effects of normalizing a circular area; and subtraction of building blocks.

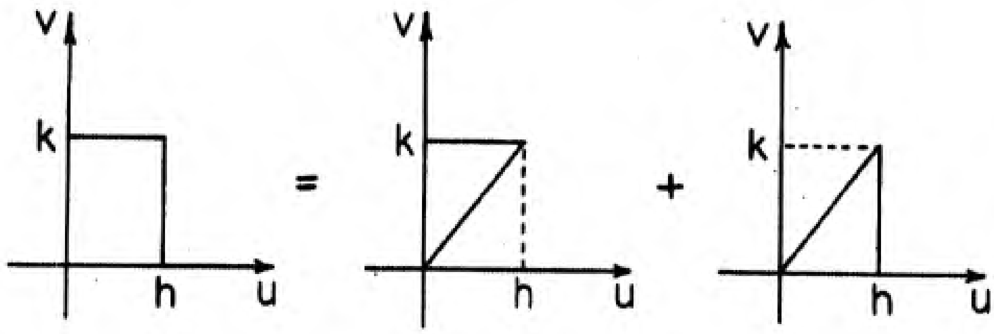
Example 5: Use of the $\frac{\alpha(h)\alpha(k)}{4}$ building block to define offset rectangular areas; and removal of the negative area that can result when overlapping building blocks are subtracted.

Example 6: Use of the V(h,k) building block to solve problems which involve odd shaped polygons, or offset line segments. (The equations for computing h and k for an offset line segment are given in this example.)

Example 7: Use of the alternate method for finding the numerical value of the Centered Ellipse; use of the C(r,d) building block; and the use of orientation information to eliminate the correlation coefficient and thus simplify the transformation equations.

Example 1.

Verify the value of $\frac{\alpha(h)\alpha(k)}{4}$ for h = 0.8 and k = 1.0 by using other building blocks.



$$\frac{\alpha(h)\alpha(k)}{4} = V(k,h) + V(h,k)$$

Fig. 8

Building Block Solution to Example 1

Numerical solution:

$h = 0.8$
$k = 1.0$
$\lambda = \frac{h}{k}$ (since $h < k$) $= 0.8$

$V(k,h) \rightarrow V(k,\lambda) = V(1.0,\ 0.8) = .048$ (Fig. A-4)

$V(h,k) \rightarrow V(\lambda,k) = V(0.8,\ 1.0) = .051$ (Fig. A-5)

$V(k,h) + V(h,k) = .099 = 9.9\%$ <u>Answer</u>.

Check on solution:

$\alpha(h) = \alpha(0.8) = .575$

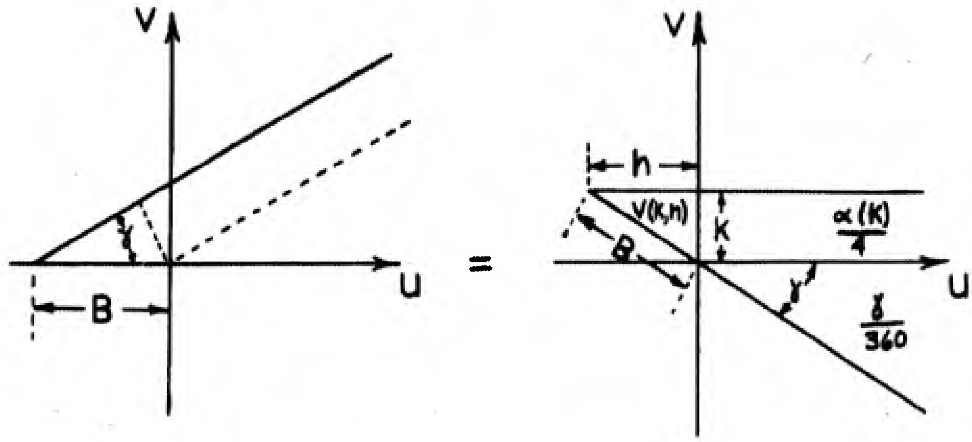
$\alpha(k) = \alpha(1.0) = .683$

$$\frac{\alpha(h)\alpha(k)}{4} = \frac{(.575)(.683)}{4} = .098$$

(To 5 decimal places, the answer should be .09836)

Example 2.

Verify the value of $A(B,\gamma)$ for $B = 1.000$ and $\gamma = 30^{\circ}$ using other building blocks.



$$A(B,\gamma) = V(k,h) + \frac{\alpha(k)}{4} + \frac{\gamma}{360}$$

Fig. 9

Building Block Solution to Example 2

Numerical solution:

$h = (1.000)\cos 30^{\circ} = 0.866$

$k = (1.000)\sin 30^{\circ} = 0.500$

$\lambda = \frac{k}{h}$ (since $h > k$) $= 0.577$

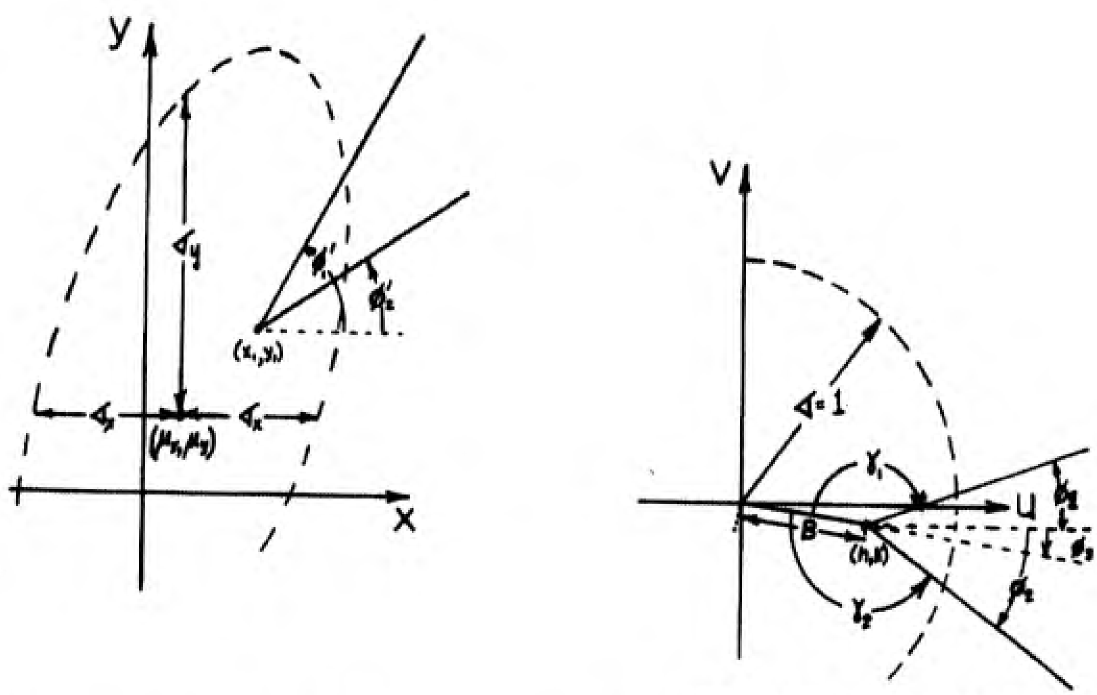
$V(k,h) \Rightarrow V(\lambda,h) = V(0.577, 0.866)$	=	.031	(Fig. A-5)
$\frac{\alpha(k)}{4} = \frac{\alpha(0.500)}{4} = \frac{.382}{4}$	=	.096	(Fig. A-2)
$\frac{\gamma}{360} = \frac{30}{360}$	=	.083	(Table A-I)
$V(k,h) + \frac{\alpha(k)}{4} + \frac{\gamma}{360}$	=	.210	= 21.0% **Answer.**

Check on solution: $A(1.000, 30^{\circ}) = .20961$ (Appendix F)

Example 3.

Find the probability that x and y will occur in the angle between the lines making 30° and 60° angles with the positive direction of the x axis and whose vertex is at the point $x = 1.5$, $y = 2.0$, if

$$\sigma_x = 2; \quad \mu_x = 0.5; \quad \rho = 0.6$$
$$\sigma_y = 4; \quad \mu_y = 1.0$$



Sketch of Original Problem

Sketch of Normalized Problem

Fig. 10

Effect of Normalizing Example 3

Transformations:

$$h = \frac{x_1 - \mu_x}{\sigma_x} = \frac{1.5 - 0.5}{2} = 0.500$$

$$k' = \frac{y_1 - \mu_y}{\sigma_y} = \frac{2.0 - 1.0}{4} = 0.250$$

$$k = \frac{k' - \rho h}{\sqrt{1 - \rho^2}} = \frac{(.250) - (.6)(.5)}{\sqrt{1 - .36}} = -\frac{.05}{.8} = -0.0625$$

$$m_{\phi_1'} = \tan 60^\circ = 1.732$$

$$m_{\phi_1} = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\sigma_x}{\sigma_y} m_{\phi_1'} - \rho\right) = \frac{1}{0.8}\left[\frac{(2)}{(4)}(1.732) - (.6)\right] = 0.332$$

$$\phi_1 = \arctan (0.332) = 18.4^\circ$$

$$m_{\phi_2} = \frac{1}{0.8}\left[\frac{(2)}{(4)}(.577) - (.6)\right] = -0.390$$

$$\phi_2 = \arctan (-0.390) = -21.3^\circ$$

Numerical solution:

The radius B also forms an angle, ϕ_3, with the u axis:

$$\phi_3 = \arctan\left(\frac{k}{h}\right) = \arctan\left(-\frac{.0625}{.500}\right) = -7.1^\circ$$

γ is measured from the radius B. Therefore, from the normalized sketch in Fig. 10

$$\gamma_1 = 180^\circ - \left[|\phi_1| + |\phi_3|\right] = 180^\circ - 25.5^\circ = 154.5^\circ$$

$$\gamma_2 = 180^\circ - \left[|\phi_2| - |\phi_3|\right] = 180^\circ - 14.2^\circ = 165.8^\circ$$

$$B = \sqrt{h^2 + k^2} = \sqrt{.250 + .039} = 0.538$$

The building block solution of the normalized sketch is

$$P = \text{Whole Plane} - \left[A(B,\gamma_1) + A(B,\gamma_2)\right]$$

$A(B,\gamma_1) = A(.538, 154.5^\circ) = .466$ (Fig. A-3)

$A(B,\gamma_2) = A(.538, 165.8^\circ) = .481$ (Fig. A-3)

$A(B,\gamma_1) + A(B,\gamma_2) = .947$

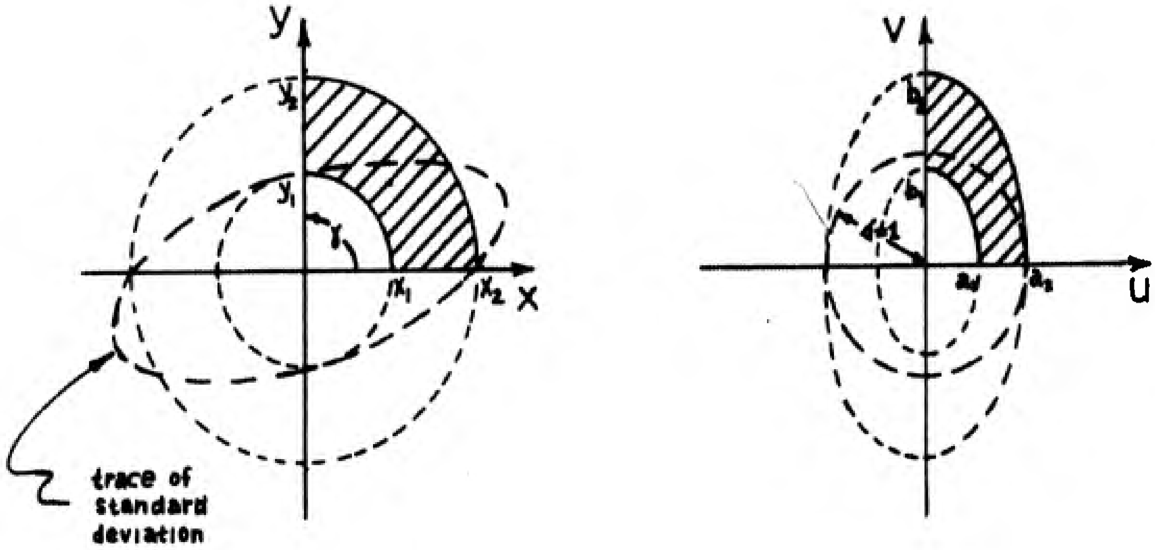
$P = 1 - .947 = .053 = 5.3\%$ Answer.

Example 4.

Find the probability that x and y will occur at a distance, d, from the origin, such that $2 \le d \le 4$, and that x and y will be in the arc, γ, where $0^\circ \le \gamma \le 90^\circ$ measured from the positive direction of the x axis, if

$$\sigma_x = 4; \quad \mu_x = 0; \quad \rho = 0.6$$

$$\sigma_y = 2; \quad \mu_y = 0$$



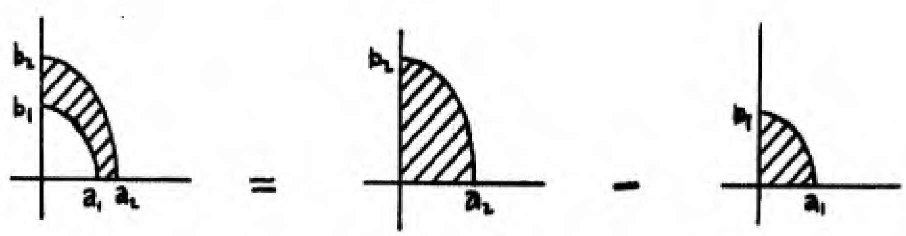
Sketch of Original Problem

Sketch of Normalized Problem

Fig. 11

Effect of Normalizing Example 4

Building block solution:



$$P = \frac{E(a_2,\, b_2,\, 0,\, 0)}{4} - \frac{E(a_1,\, b_1,\, 0,\, 0)}{4}$$

Fig. 12

Building Block Solution of Example 4

Transformations:

Normalizing the semi-axes of the circles,

$$a_1 = \frac{x_1-\mu_x}{\sigma_x} = \frac{2-0}{4} = 0.500$$

$$a_2 = \frac{x_2-\mu_x}{\sigma_x} = \frac{4-0}{4} = 1.000$$

$$b_1' = \frac{y_1-\mu_y}{\sigma_y} = \frac{2-0}{2} = 1.000$$

$$b_1 = \frac{b_1'-\rho a_1}{\sqrt{1-\rho^2}} = \frac{1-(.6)(.5)}{\sqrt{1-.36}} = \frac{.7}{.8} = 0.875$$

$$b_2' = \frac{y_2-\mu_y}{\sigma_y} = \frac{4-0}{2} = 2.000$$

$$b_2 = \frac{b_2'-\rho a_2}{\sqrt{1-\rho^2}} = \frac{2-(.6)(1)}{.8} = \frac{1.4}{.8} = 1.750$$

Numerical solution:

$$\frac{E(a_2, b_2, 0, 0)}{4} = \frac{E(1, 1.750, 0, 0)}{4} = \frac{.554}{4} = .138$$

$$-\frac{E(a_1, b_1, 0, 0)}{4} = -\frac{E(.5, .875, 0, 0)}{4} = -\frac{.191}{4} = -.048$$

$$.090$$

P = sum = .090 = 9.0 % <u>Answer</u>.

<u>Example 5</u>.

After normalizing, the region of the desired probability is the rectangular area shown in Fig. 13. Find this probability.

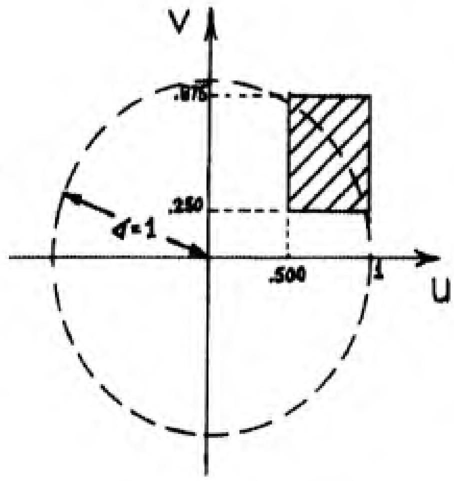
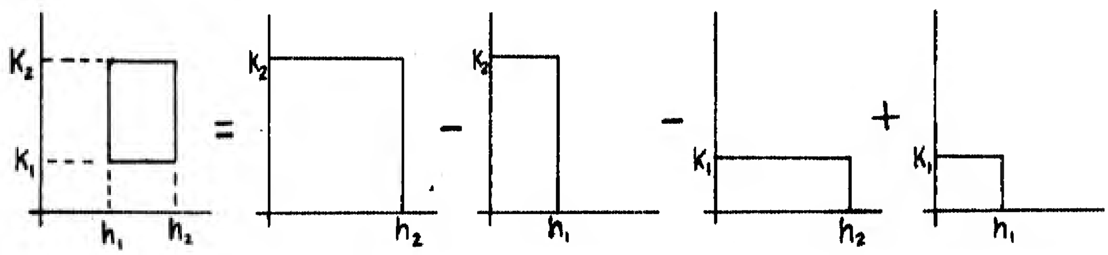


Fig. 13

Sketch of Example 5

Building block solution:



$$P = \frac{\alpha(h_2)\alpha(k_2)}{4} + \frac{\alpha(h_1)\alpha(k_2)}{4} + \frac{\alpha(h_2)\alpha(k_1)}{4} + \frac{\alpha(h_1)\alpha(k_1)}{4}$$

Fig. 14

Building Block Solution of Example 5

Numerical solution:

From Fig. A-2

$$\alpha(h_1) = \alpha(0.500) = .382$$

$$\alpha(h_2) = \alpha(1.000) = .683$$

$$\alpha(k_1) = \alpha(0.250) = .197$$

$$\alpha(k_2) = \alpha(0.875) = .618$$

Therefore

$$\frac{\alpha(h_2)\alpha(k_2)}{4} = \frac{(.683)(.618)}{4} = .106$$

$$-\frac{\alpha(h_1)\alpha(k_2)}{4} = -\frac{(.382)(.618)}{4} = -.059$$

$$-\frac{\alpha(h_2)\alpha(k_1)}{4} = -\frac{(.683)(.197)}{4} = -.034$$

$$\frac{\alpha(h_1)\alpha(k_1)}{4} = \frac{(.382)(.197)}{4} = .019$$

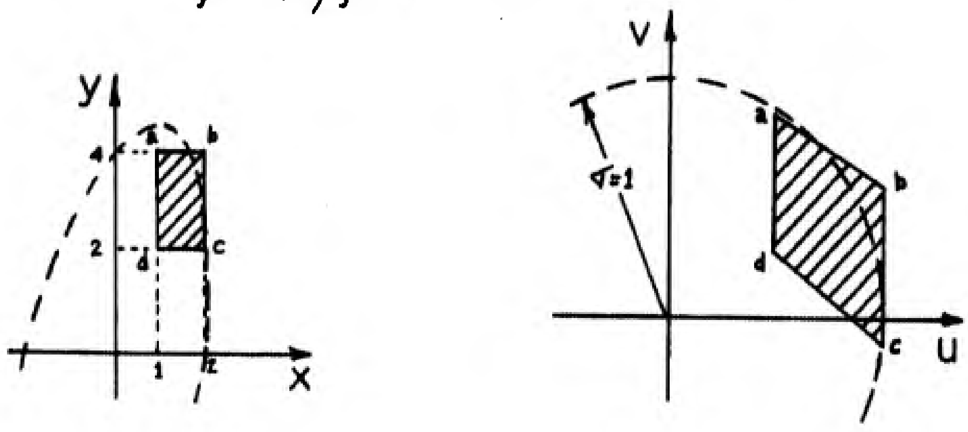
$$P = \text{Sum} = .032 = 3.2\ \%$$ Answer.

<u>Example 6.</u>

Find the probability that $1 \leq x \leq 2$ and $2 \leq y \leq 4$, if

$$\sigma_x = 2; \quad \mu_x = 0; \quad \rho = 0.6$$

$$\sigma_y = 4; \quad \mu_y = 0$$



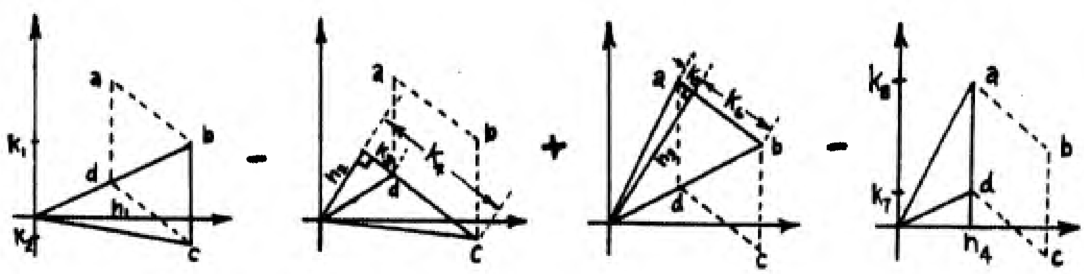
Sketch of Original Problem

Sketch of Normalized Problem

Fig. 15

Effect of Normalizing Example 6

Building block solution:



$$P = \left[V(h_1,k_1) + V(h_1,k_2)\right] - \left[V(h_2,k_4) - V(h_2,k_3)\right] + \left[V(h_3,k_5) + V(h_3,k_6)\right] - \left[V(h_4,k_8) - V(h_4,k_7)\right]$$

Fig. 16

Building Block Solution of Example 6

Transformations:

$$u_a = \frac{x_a - \mu_x}{\sigma_x} = \frac{1-0}{2} = .500$$

$$v_a' = \frac{y_a - \mu_y}{\sigma_y} = \frac{4-0}{4} = 1.000$$

$$v_a = \frac{v_a' - \rho u_a}{\sqrt{1-\rho^2}} = \frac{1-(.6)(.5)}{\sqrt{1-.36}} = \frac{.7}{.8} = .875$$

Similiarly

$$u_b = 1.000; \quad u_c = 1.000; \quad u_d = .500$$

$$v_b = .500; \quad v_c = -.125; \quad v_d = .250$$

Numerical solution:

The following equations are useful for finding the arguments of the V(h,k) building blocks which are associated with the line segment between the points (u_1,v_1) and (u_2,v_2) (Ref 7:xxx):

$$h = \frac{|u_2 v_1 - u_1 v_2|}{\sqrt{(u_2-u_1)^2 + (v_2-v_1)^2}} \tag{15}$$

$$k_1 = \frac{|u_1(u_2-u_1) + v_1(v_2-v_1)|}{\sqrt{(u_2-u_1)^2 + (v_2-v_1)^2}} \tag{16}$$

$$k_2 = \frac{|u_2(u_2-u_1) + v_2(v_2-v_1)|}{\sqrt{(u_2-u_1)^2 + (v_2-v_1)^2}} \tag{17}$$

where, h is the distance along the perpendicular from the line to the origin, and k_1 and k_2 are the distances from the foot of the perpendicular to the end points of the line segement. The two center sketches in Fig. 16 are examples of the two possible cases that can occur.

The only values of h and k in Fig. 16 which are not known from the transformations are h_2, h_3, k_3, k_4, k_5, k_6. These values must be computed using equations (15), (16), and (17).

$$h_2 = \frac{|u_c v_d - u_d v_c|}{\sqrt{(u_c-u_d)^2+(v_c-v_d)^2}} = \frac{|(1)(.250)-(.5)(-.125)|}{\sqrt{(1-.5)^2+(-.125-.250)^2}} = \frac{.312}{.625} = .499$$

$$K_3 = \frac{|u_d(u_c-u_d)+v_d(v_c-v_d)|}{\sqrt{(u_c-u_d)^2+(v_c-v_d)^2}} = \frac{|(.5)(.5)+(.250)(-.375)|}{.625} = \frac{.166}{.625} = .266$$

$$K_4 = \frac{|u_c(u_c-u_d)+v_c(v_c-v_d)|}{\sqrt{(u_c-u_d)^2+(v_c-v_d)^2}} = \frac{|(1)(.5)+(-.125)(-.375)|}{.625} = \frac{.294}{.625} = .470$$

Similarly

$h_3 = 1.000$

$k_5 = .012$

$k_6 = .499$

Evaluating the building blocks:

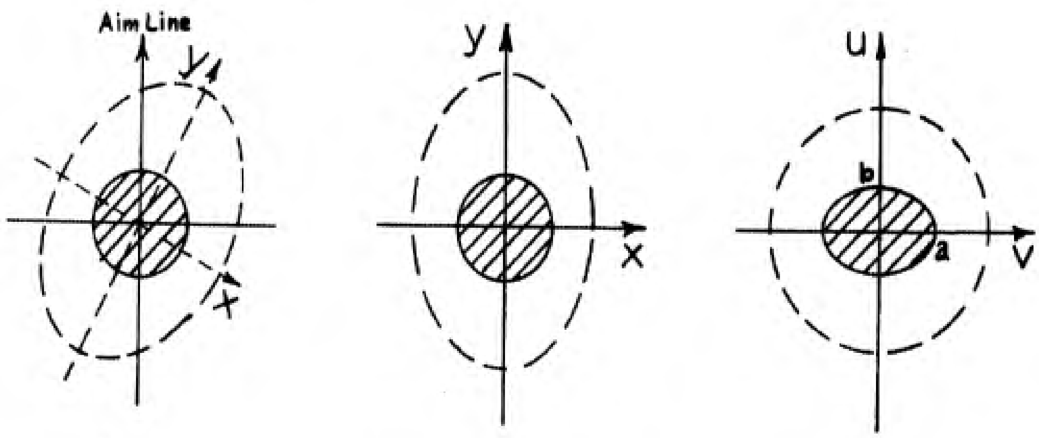
$V(h_1,k_1) = V(u_b,v_b)$	$\Rightarrow V(u_b,\lambda)$	= $V(1, .5)$	=	.031
$V(h_1,k_2) = V(u_c,v_c)$	$\Rightarrow V(u_c,\lambda)$	= $V(1, .125)$	=	.008
$-V(h_2,k_4)$	$\Rightarrow -V(h_2,\lambda)$	= $-V(.499, .942)$	=	$-$ .017
$V(h_2,k_3)$	$\Rightarrow V(h_2,\lambda)$	= $V(.499, .533)$	=	.010
$V(h_3,k_5)$	$\Rightarrow V(h_3,\lambda)$	= $V(1, .012)$	=	.001
$V(h_3,k_6)$	$\Rightarrow V(h_3,\lambda)$	= $V(1, .499)$	=	.031
$-V(h_4,k_8) = -V(u_a,v_a)$	$\Rightarrow -V(\lambda,v_a)$	= $-V(.588, .875)$	=	$-$ .032
$V(h_4,k_7) = V(u_d,v_d)$	$\Rightarrow V(u_d,\lambda)$	= $V(.5, .5)$	=	.009
		Sum	=	.041

P = Sum = .041 Answer

Example 7.

A ballistic missile is to be fired at a target 5000 miles away. From previous tests, this missile is known to have random errors in guidance which cause it to impact in an elliptical pattern about the target. At this range, the major axis of the pattern is oriented 17.5° clockwise from the aiming line; the standard deviation along the major axis is 1.0 miles; and the standard deviation along the minor axis is 0.5 miles.

Find the probability that the missile will impact within a radius of 1/4 mile from the target.



Sketch of Original Problem

Taking x and y Along Axes Makes $\rho = 0$

Sketch of Normalized Problem

Fig. 17

Effect of Normalizing Example 7

Transformations:

$$a = \frac{x_1 - \mu_x}{\sigma_x} = \frac{(.25) - (0)}{(.5)} = .5$$

$$b = \frac{y_1 - \mu_y}{\sigma_y} = \frac{(.25) - (0)}{(1.0)} = .25$$

Building block solution:

$$P = E(.5, .25, 0, 0)$$

But, E(.5, .25, 0, 0) is beyond the range of Table II. However, for the centered ellipse there is an alternate set of building blocks which can be used.

$$E(a, b, 0, 0) = C(a+b, a-b) - C(a-b, a+b)$$

Therefore

$$C(r_1, d_1) = C(a+b, a-b) = C(.75, .25) = .240$$

$$-C(r_2, d_2) = -C(a-b, a+b) = -C(.25, .75) = -.023$$

$$\text{Sum} = .217$$

<u>Answer</u>:

Probability of impacting within 1/4 mile of the target is 21.7 %.

VIII. Summary and Conclusions

Summary

Normalizing transformations are used to express problems of bivariate normal statistics as equivalent problems in the unit circular normal distribution. The equivalent problem is then solved. Numerical solution is accomplished by: using the Catalog of Building Blocks to identify the area of interest in terms of the building blocks; looking up the numerical value of these building blocks in the Appendix; and, adding these numerical values algebraically. This sum is the desired answer.

The Catalog of Building Blocks is a compilation of the functions of the unit circular normal distribution for which tables of values exist in the statistical literature. Graphs and tables are presented which permit the numerical evaluation of each of the building blocks to three decimal places. One new function, $A(B,\gamma)$, the probability included in an offset arc, is introduced. This function is tabulated to five decimal places in the Appendix.

Conclusions

The Building Block Technique is a simplified approach to the solution of problems which involve two normal (Gaussian) variables. Integration of the bivariate normal integral is not required because precalculated statistical functions are used as building blocks to evaluate the problem. Statistical skills are not required to use this technique because each of the building blocks is measured in terms of analytical geometry, and the numerical values of the building blocks are combined by addition and subtraction.

The Building Block Technique makes it possible for one who is not versed in statistics to solve problems in bivariate normal distributions, at his desk, using a single reference.

IX. Recommendations

It is recommended that the complete 750,000 entry table of the numerical value of the offset ellipse building block (Ref 5:8) be computed and published. These tables would reduce the error which is introduced when it is necessary to interpolate in all four of the geometrical parameters of this building block.

It is recommended that all of the tables of the numerical values of the building block functions be compiled and published as a set of volumes. Since three decimal place accuracy is already available in Appendix A, it is recommended that the compiled tables be accurate to not less than five decimal places. Conservatively, it is estimated that these tables would require 1500 to 3000 printed pages. However, their computation and compilation is recommended in order to increase the accuracy of the numerical solution to problems.

To broaden the scope and complexity of problems which can be solved by the Building Block Technique, it is further recommended that similar sets of building blocks be developed for the multivariate normal distributions in three or more variables, and for other continuous distributions in two or more variables. The tables of values of these new building blocks should be added to the set of tables which was recommended in the preceding paragraph.

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Appendix A

Numerical Value of the Building Blocks

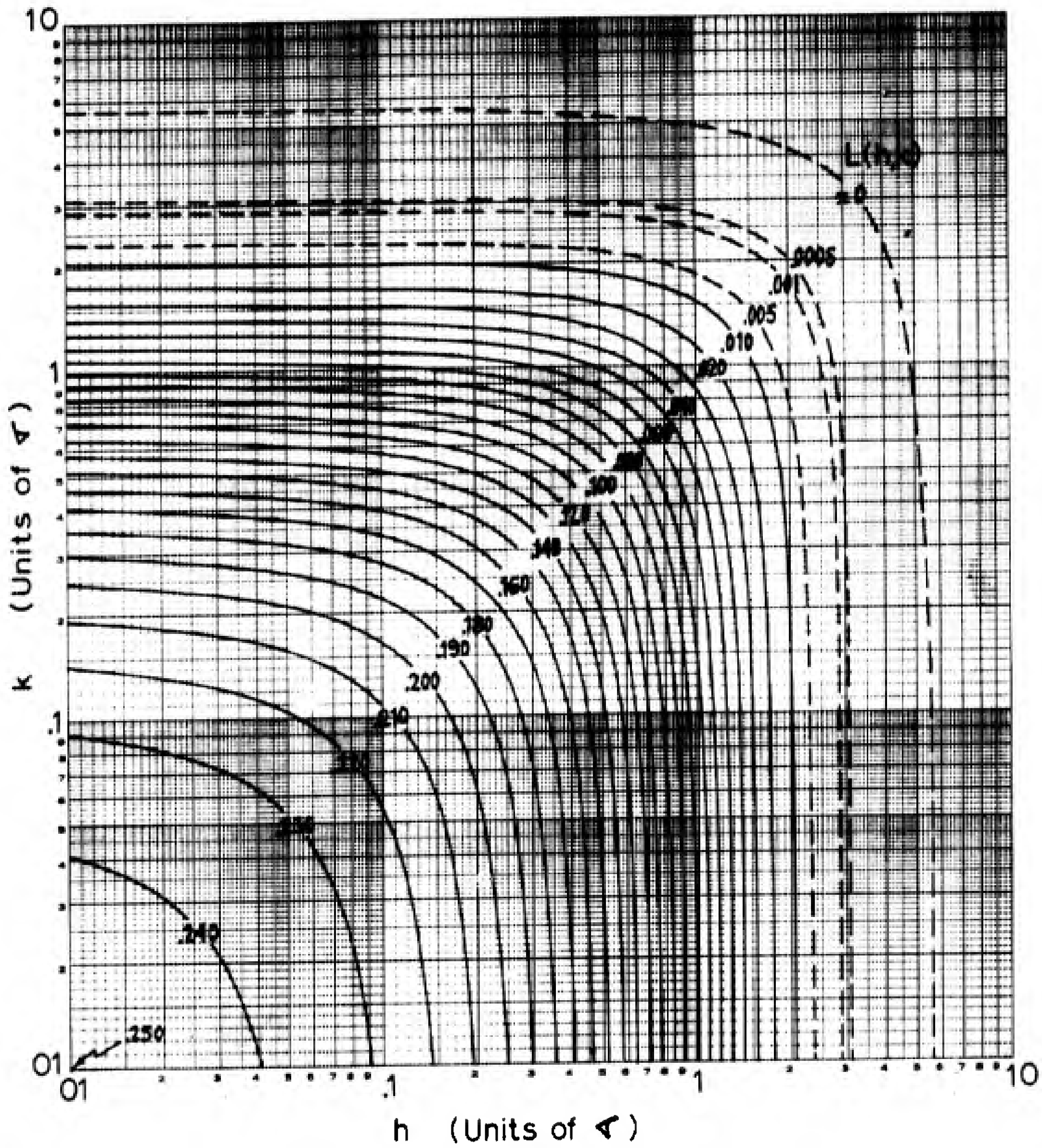


Fig. A-1

Graph of the Numerical Value of the L(h,k) Building Block

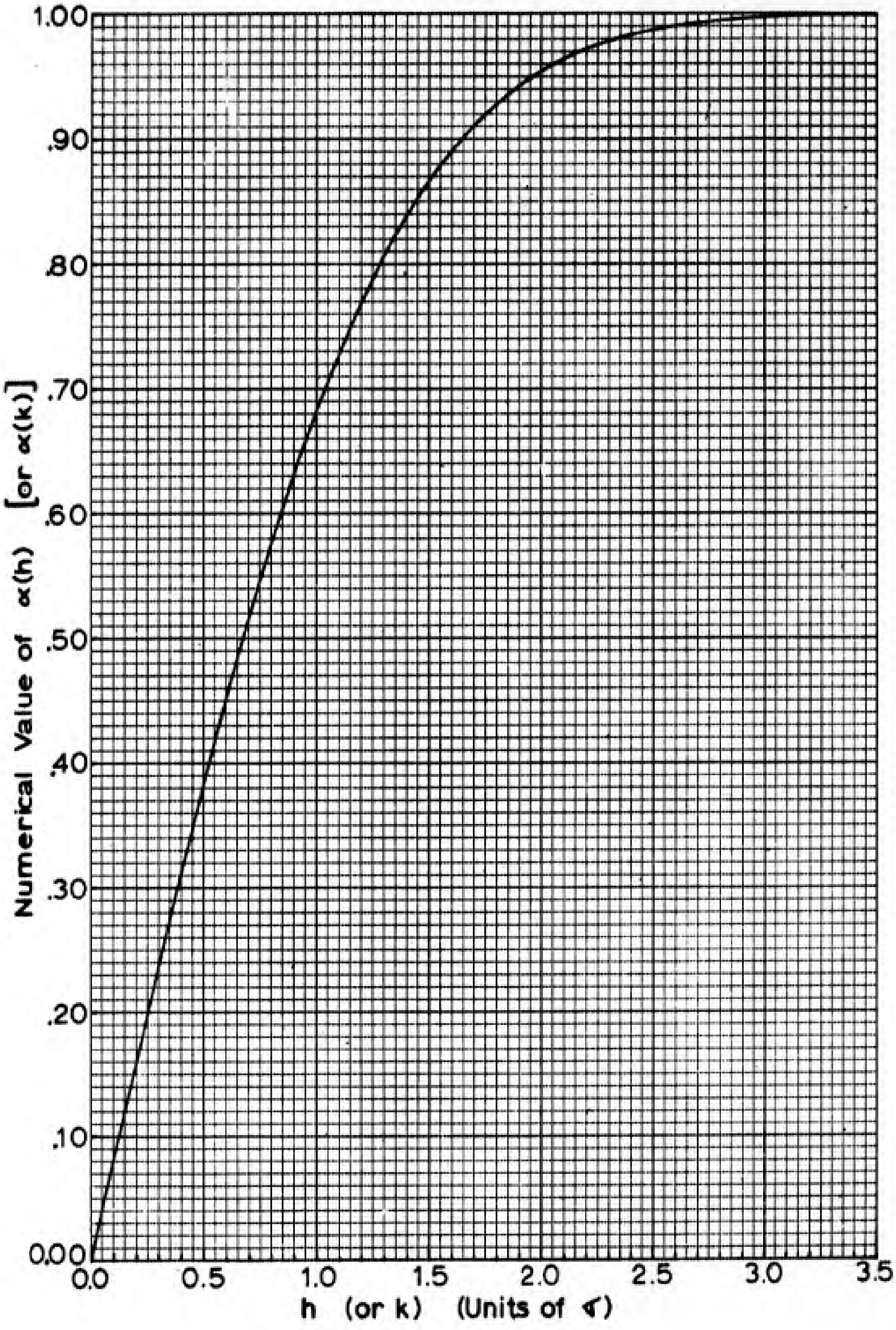


Fig. A-2

Graph of the Numerical Value of the α(h) or α(k) Building Block

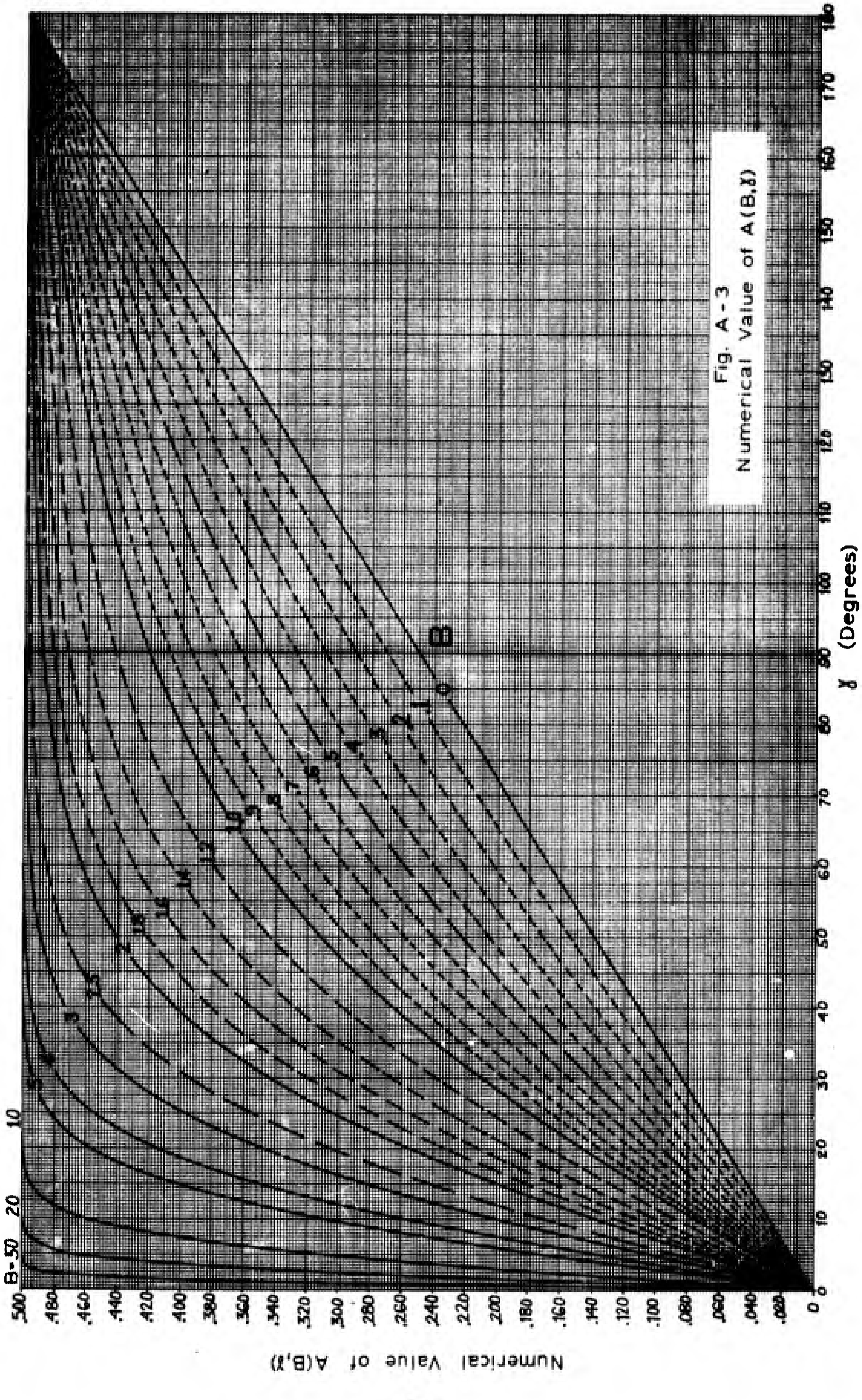


Fig. A-3
Numerical Value of A(B,γ)

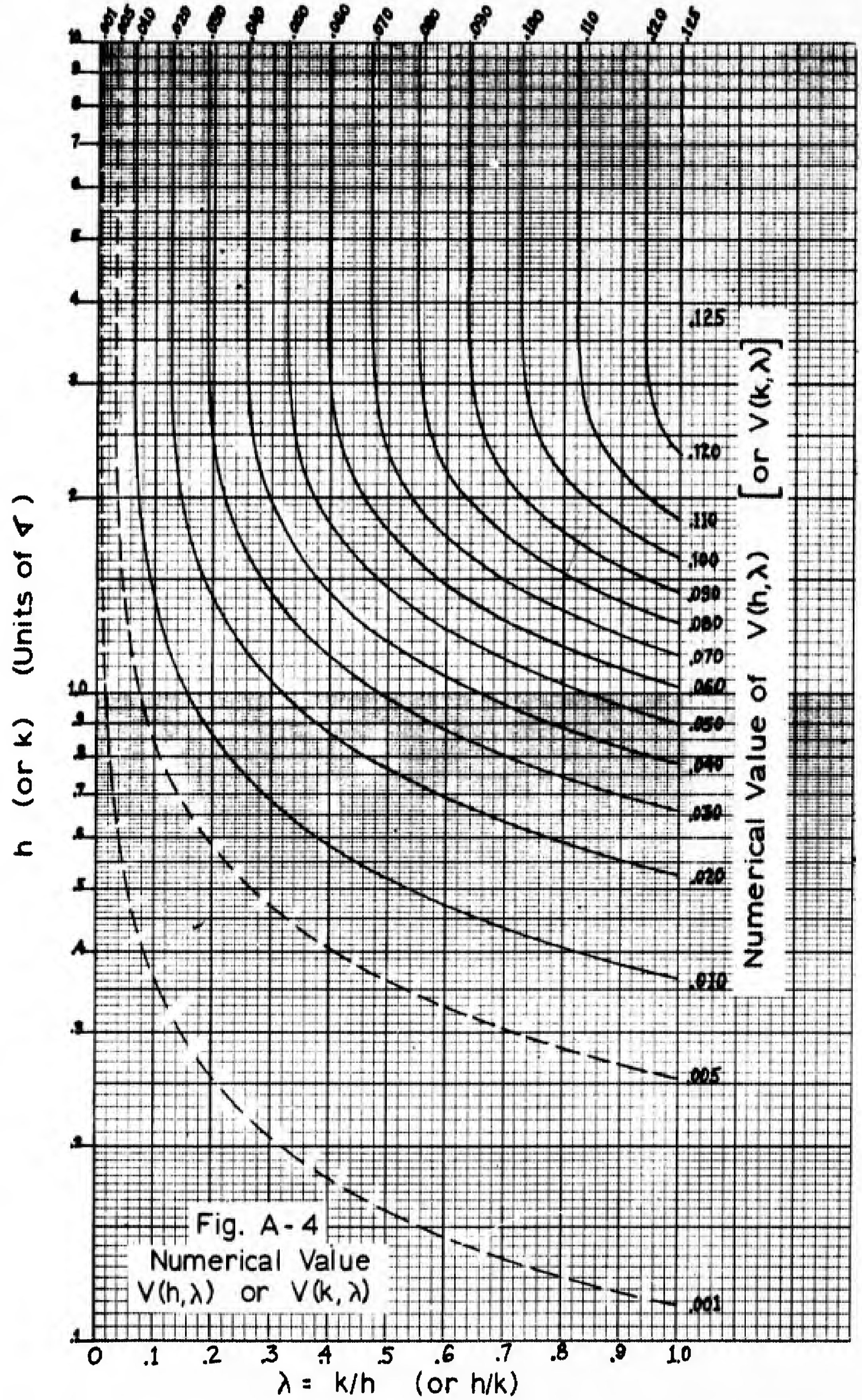


Fig. A-4
Numerical Value
V(h,λ) or V(k,λ)

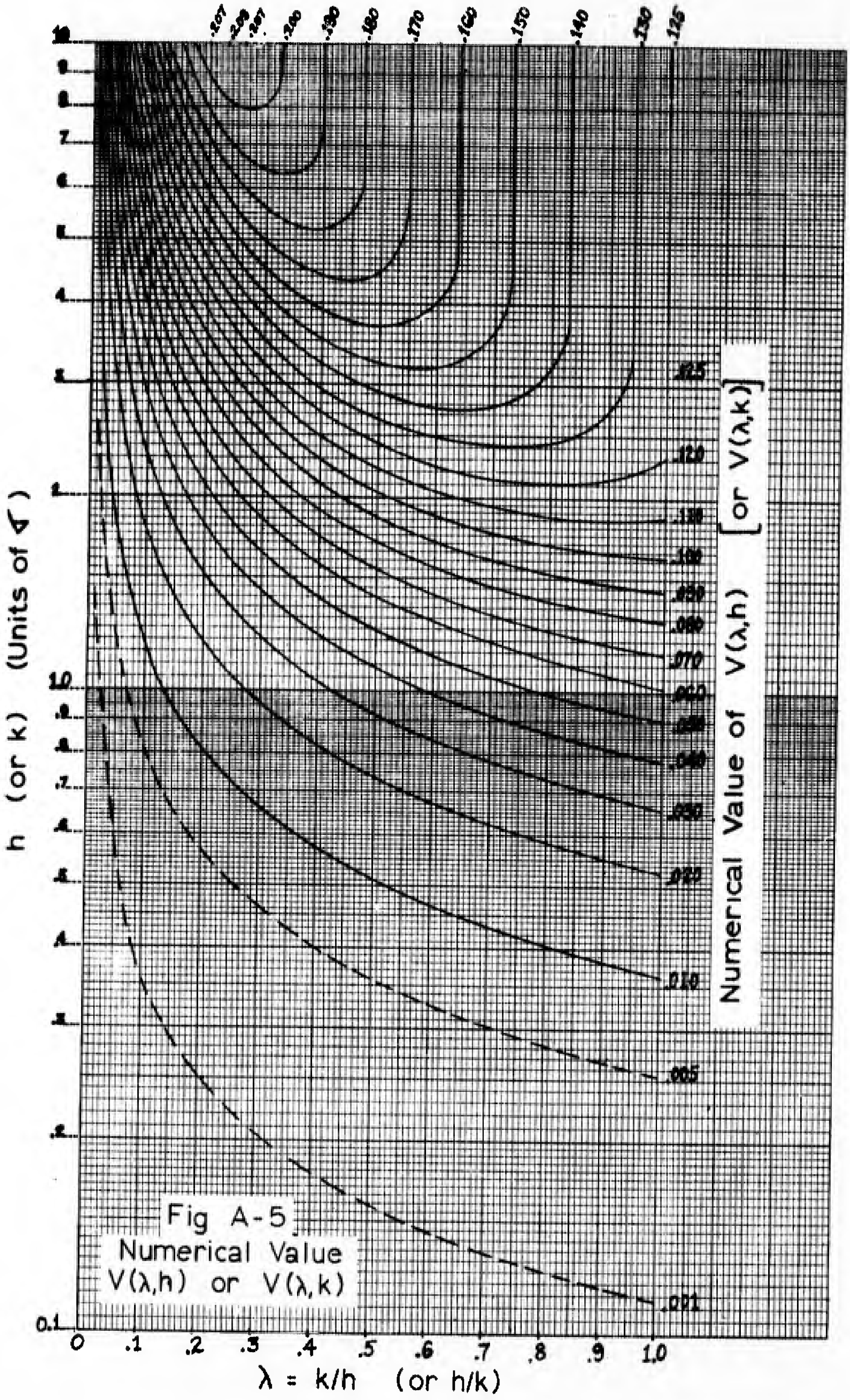


Fig A-5
Numerical Value
V(λ,h) or V(λ,k)

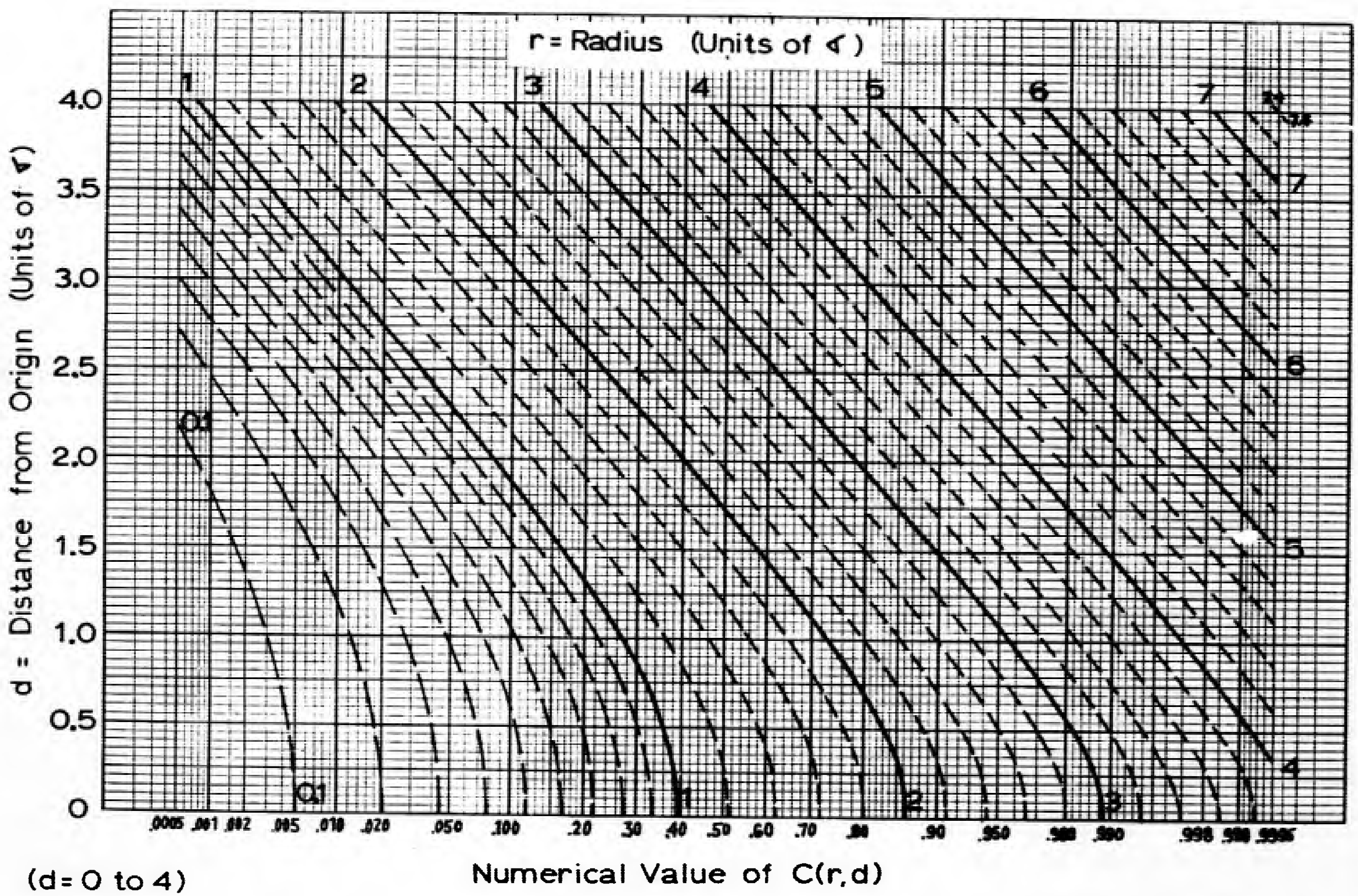
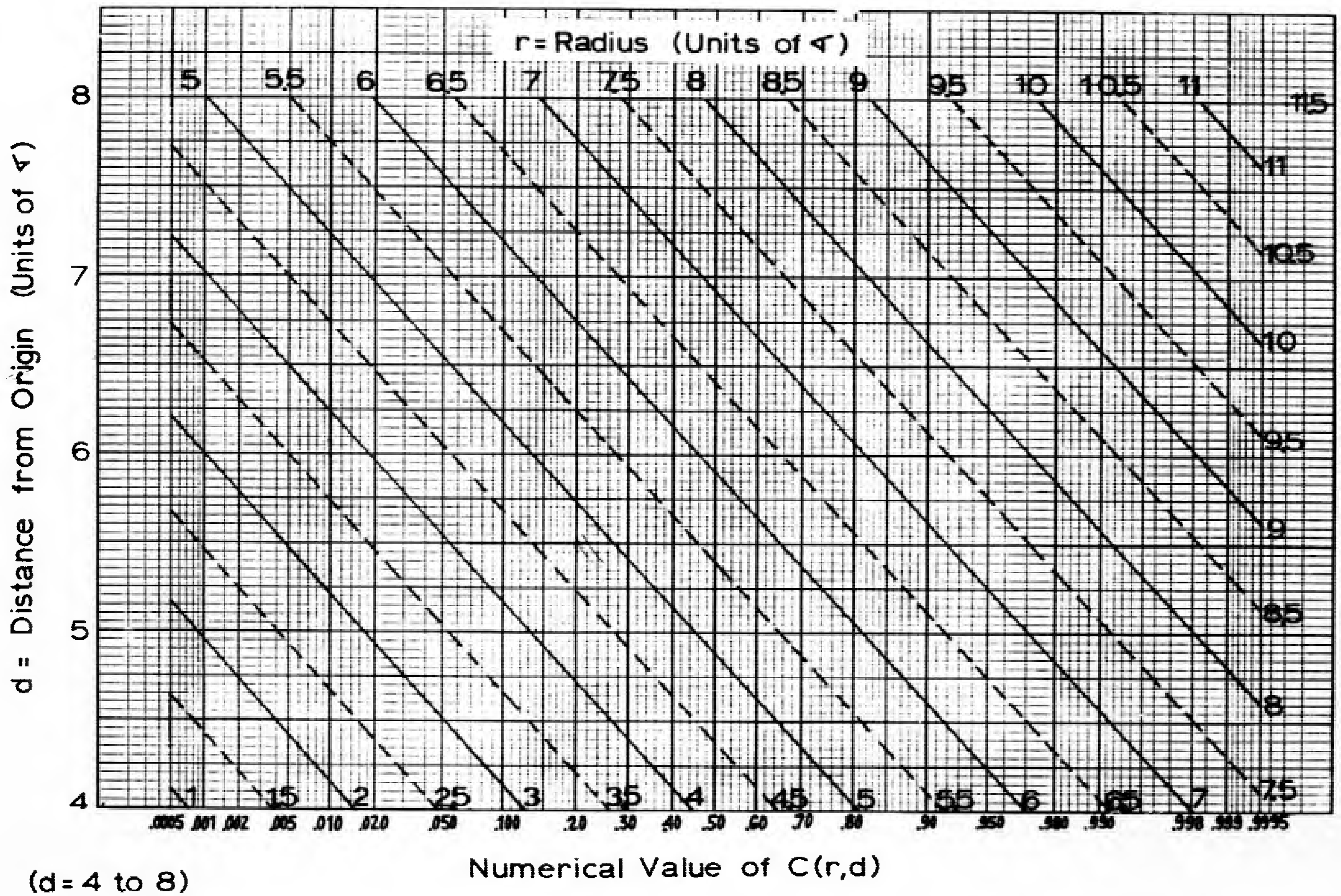

r = Radius (Units of σ)
d = Distance from Origin (Units of σ)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
(d= 0 to 4)
Numerical Value of C(r,d)

Fig A-6a



(d = 4 to 8)

Fig A-6b

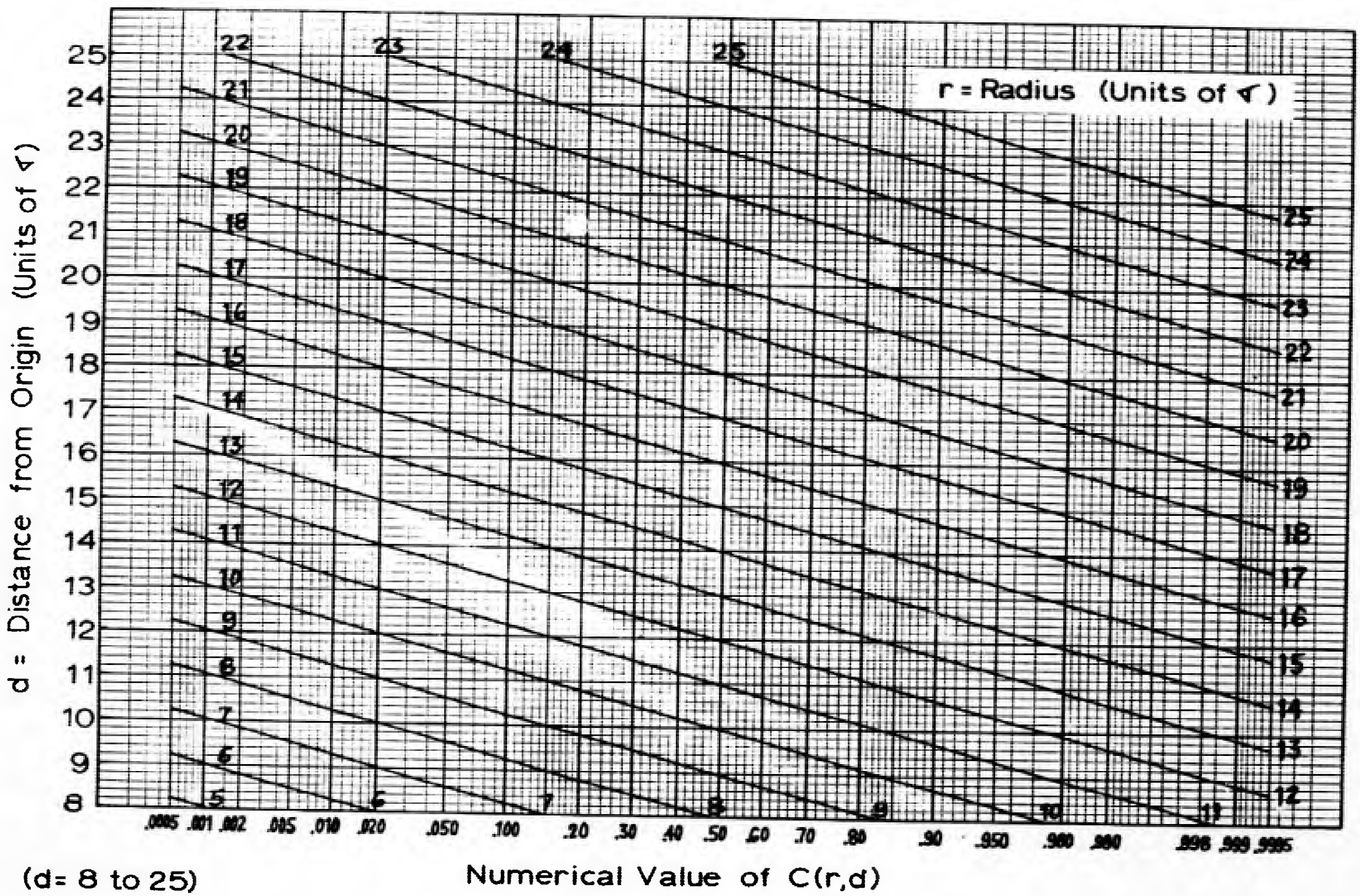

d = Distance from Origin (Units of σ)
25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8
r = Radius (Units of σ)
.0005 .001 .002 .005 .010 .020 .050 .100 .20 .30 .40 .50 .60 .70 .80 .90 .950 .980 .990 .998 .999 .9995
(d= 8 to 25)
Numerical Value of C(r,d)

Fig. A-6c

Table A-I

Numerical Value of the $\frac{\gamma}{360}$ Building Block

(γ in Degrees of Arc)

γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$
1	.00278	37	.10278	73	.20278	109	.30278	145	.40278
2	.00556	38	.10556	74	.20556	110	.30556	146	.40556
3	.00833	39	.10833	75	.20833	111	.30833	147	.40833
4	.01111	40	.11111	76	.21111	112	.31111	148	.41111
5	.01389	41	.11389	77	.21389	113	.31389	149	.41389
6	.01667	42	.11667	78	.21667	114	.31667	150	.41667
7	.01944	43	.11944	79	.21944	115	.31944	151	.41944
8	.02222	44	.12222	80	.22222	116	.32222	152	.42222
9	.02500	45	.12500	81	.22500	117	.32500	153	.42500
10	.02778	46	.12778	82	.22778	118	.32778	154	.42778
11	.03056	47	.13056	83	.23056	119	.33056	155	.43056
12	.03333	48	.13333	84	.23333	120	.33333	156	.43333
13	.03611	49	.13611	85	.23611	121	.33611	157	.43611
14	.03889	50	.13889	86	.23889	122	.33889	158	.43889
15	.04167	51	.14167	87	.24167	123	.34167	159	.44167
16	.04444	52	.14444	88	.24444	124	.34444	160	.44444
17	.04722	53	.14722	89	.24722	125	.34722	161	.44722
18	.05000	54	.15000	90	.25000	126	.35000	162	.45000
19	.05278	55	.15278	91	.25278	127	.35278	163	.45278
20	.05556	56	.15556	92	.25556	128	.35556	164	.45556
21	.05833	57	.15833	93	.25833	129	.35833	165	.45833
22	.06111	58	.16111	94	.26111	130	.36111	166	.46111
23	.06389	59	.16389	95	.26389	131	.36389	167	.46389
24	.06667	60	.16667	96	.26667	132	.36667	168	.46667
25	.06944	61	.16944	97	.26944	133	.36944	169	.46944
26	.07222	62	.17222	98	.27222	134	.37222	170	.47222
27	.07500	63	.17500	99	.27500	135	.37500	171	.47500
28	.07778	64	.17778	100	.27778	136	.37778	172	.47778
29	.08056	65	.18056	101	.28056	137	.38056	173	.48056
30	.08333	66	.18333	102	.28333	138	.38333	174	.48333
31	.08611	67	.18611	103	.28611	139	.38611	175	.48611
32	.08889	68	.18889	104	.28889	140	.38889	176	.48889
33	.09167	69	.19167	105	.29167	141	.39167	177	.49167
34	.09444	70	.19444	106	.29444	142	.39444	178	.49444
35	.09722	71	.19722	107	.29722	143	.39722	179	.49722
36	.10000	72	.20000	108	.30000	144	.40000	180	.50000

Table A-I

Numerical Value of the $\frac{\gamma}{360}$ Building Block

(γ in Degrees of Arc)

γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$	γ	$\frac{\gamma}{360}$
181	.50278	217	.60278	253	.70278	289	.80278	325	.90278
182	.50556	218	.60556	254	.70556	290	.80556	326	.90556
183	.50833	219	.60833	255	.70833	291	.80833	327	.90833
184	.51111	220	.61111	256	.71111	292	.81111	328	.91111
185	.51389	221	.61389	257	.71389	293	.81389	329	.91389
186	.51667	222	.61667	258	.71667	294	.81667	330	.91667
187	.51944	223	.61944	259	.71944	295	.81944	331	.91944
188	.52222	224	.62222	260	.72222	296	.82222	332	.92222
189	.52500	225	.62500	261	.72500	297	.82500	333	.92500
190	.52778	226	.62778	262	.72778	298	.82778	334	.92778
191	.53056	227	.63056	263	.73056	299	.83056	335	.93056
192	.53333	228	.63333	264	.73333	300	.83333	336	.93333
193	.53611	229	.63611	265	.73611	301	.83611	337	.93611
194	.53889	230	.63889	266	.73889	302	.83889	338	.93889
195	.54167	231	.64167	267	.74167	303	.84167	339	.94167
196	.54444	232	.64444	268	.74444	304	.84444	340	.94444
197	.54722	233	.64722	269	.74722	305	.84722	341	.94722
198	.55000	234	.65000	270	.75000	306	.85000	342	.95000
199	.55278	235	.65278	271	.75278	307	.85278	343	.95278
200	.55556	236	.65556	272	.75556	308	.85556	344	.95556
201	.55833	237	.65833	273	.75833	309	.85833	345	.95833
202	.56111	238	.66111	274	.76111	310	.86111	346	.96111
203	.56389	239	.66389	275	.76389	311	.86389	347	.96389
204	.56667	240	.66667	276	.76667	312	.86667	348	.96667
205	.56944	241	.66944	277	.76944	313	.86944	349	.96944
206	.57222	242	.67222	278	.77222	314	.87222	350	.97222
207	.57500	243	.67500	279	.77500	315	.87500	351	.97500
208	.57778	244	.67778	280	.77778	316	.87778	352	.97778
209	.58056	245	.68056	281	.78056	317	.88056	353	.98056
210	.58333	246	.68333	282	.78333	318	.88333	354	.98333
211	.58611	247	.68611	283	.78611	319	.88611	355	.98611
212	.58889	248	.68889	284	.78889	320	.88889	356	.98889
213	.59167	249	.69167	285	.79167	321	.89167	357	.99167
214	.59444	250	.69444	286	.79444	322	.89444	358	.99444
215	.59722	251	.69722	287	.79722	323	.89722	359	.99722
216	.60000	252	.70000	288	.80000	324	.90000	360	1.00000

Table A-II

Numerical Value of the $E(a,b,h,k)$ Building Block

$h = 0.0$, $k =$

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.1175	.1045	.0735	.0408	.0179	.0062	.0017
	1.0	.2153	.1953	.1457	.0890	.0443	.0179	.0058
	1.5	.2838	.2642	.2123	.1457	.0843	.0405	.0159
	2.0	.3254	.3102	.2672	.2046	.1363	.0774	.0367
	2.5	.3482	.3379	.3070	.2567	.1929	.1269	.0714
	3.0	.3603	.3536	.3328	.2963	.2444	.1819	.1189
1.0	1.0	.3935	.3573	.2671	.1638	.0819	.0332	.0108
	1.5	.5168	.4818	.3887	.2683	.1562	.0755	.0299
	2.0	.5901	.5637	.4881	.3765	.2529	.1447	.0691
	2.5	.6291	.6118	.5591	.4714	.3575	.2374	.1347
	3.0	.6491	.6383	.6040	.5423	.4516	.3394	.2238
1.5	1.5	.6753	.6309	.5120	.3563	.2092	.1021	.0408
	2.0	.7666	.7343	.6408	.4995	.3394	.1963	.0947
	2.5	.8129	.7930	.7307	.6232	.4788	.3220	.1848
	3.0	.8351	.8235	.7855	.7134	.6022	.4588	.3065
2.0	2.0	.8647	.8309	.7310	.5763	.3965	.2321	.1133
	2.5	.9116	.8921	.8293	.7163	.5580	.3805	.2211
	3.0	.9321	.9218	.8867	.8153	.6983	.5401	.3659
2.5	2.5	.9561	.9383	.8791	.7679	.6059	.4184	.2461
	3.0	.9737	.9654	.9351	.8693	.7543	.5915	.4061
3.0	3.0	.9889	.9822	.9563	.8962	.7856	.6230	.4325

(From Ref 5:10)

Table A-II

Numerical Value of the E(a,b,h,k) Building Block

h = 0.5, k =

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.1045	.0929	.0653	.0363	.0159	.0055	.0015
	1.0	.1915	.1737	.1296	.0792	.0394	.0159	.0052
	1.5	.2525	.2350	.1888	.1295	.0749	.0359	.0141
	2.0	.2896	.2760	.2377	.1819	.1211	.0687	.0326
	2.5	.3100	.3008	.2731	.2282	.1714	.1127	.0634
	3.0	.3209	.3149	.2962	.2635	.2171	.1615	.1055
1.0	1.0	.3573	.3244	.2423	.1484	.0741	.0300	.0098
	1.5	.4698	.4378	.3528	.2431	.1412	.0681	.0269
	2.0	.5371	.5128	.4433	.3412	.2286	.1304	.0622
	2.5	.5733	.5572	.5083	.4274	.3232	.2140	.1211
	3.0	.5920	.5818	.5497	.4922	.4087	.3062	.2014
1.5	1.5	.6309	.5889	.4768	.3307	.1935	.0940	.0374
	2.0	.7178	.6869	.5975	.4638	.3137	.1806	.0868
	2.5	.7628	.7432	.6826	.5795	.4429	.2964	.1693
	3.0	.7849	.7732	.7353	.6648	.5581	.4229	.2811
2.0	2.0	.8309	.7972	.6986	.5478	.3746	.2180	.1058
	2.5	.8783	.8583	.7945	.6822	.5279	.3576	.2065
	3.0	.9000	.8889	.8517	.7787	.6623	.5086	.3422
2.5	2.5	.9383	.9195	.8579	.7447	.5834	.4000	.2337
	3.0	.9576	.9482	.9151	.8456	.7285	.5668	.3862
3.0	3.0	.9822	.9745	.9459	.8820	.7682	.6048	.4168

(From Ref **5**:10)

Table A-II

Numerical Value of the $E(a,b,h,k)$ Building Block

$h = 1.0, \quad k =$

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.0735	.0653	.0459	.0255	.0112	.0039	.0011
	1.0	.1347	.1222	.0911	.0556	.0277	.0111	.0036
	1.5	.1778	.1655	.1328	.0910	.0525	.0251	.0099
	2.0	.2041	.1945	.1672	.1277	.0849	.0480	.0227
	2.5	.2187	.2121	.1923	.1604	.1201	.0788	.0442
	3.0	.2266	.2223	.2088	.1853	.1523	.1130	.0736
1.0	1.0	.2671	.2423	.1807	.1103	.0549	.0221	.0072
	1.5	.3524	.3280	.2633	.1805	.1042	.0500	.0197
	2.0	.4043	.3853	.3316	.2535	.1686	.0955	.0452
	2.5	.4329	.4200	.3812	.3182	.2387	.1568	.0881
	3.0	.4483	.4398	.4135	.3676	.3026	.2247	.1466
1.5	1.5	.5120	.4768	.3836	.2637	.1527	.0734	.0290
	2.0	.5861	.5592	.4825	.3702	.2472	.1405	.0668
	2.5	.6265	.6085	.5539	.4643	.3497	.2307	.1300
	3.0	.6477	.6361	.5998	.5355	.4429	.3304	.2164
2.0	2.0	.7310	.6986	.6057	.4679	.3146	.1801	.0861
	2.5	.7785	.7576	.6934	.5857	.4443	.2957	.1678
	3.0	.8024	.7896	.7486	.6735	.5619	.4228	.2790
2.5	2.5	.8791	.8579	.7909	.6748	.5182	.3481	.1994
	3.0	.9025	.8904	.8502	.7728	.6523	.4966	.3311
3.0	3.0	.9563	.9459	.9097	.8359	.7145	.5508	.3716

(From Ref 5:11)

Table A-II

Numerical Value of the $E(a,b,h,k)$ Building Block

$h = 1.5$, $k =$

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.0408	.0363	.0255	.0142	.0062	.0022	.0006
	1.0	.0750	.0680	.0506	.0309	.0153	.0062	.0020
	1.5	.0991	.0921	.0738	.0504	.0290	.0139	.0054
	2.0	.1139	.1085	.0931	.0709	.0469	.0265	.0125
	2.5	.1223	.1185	.1072	.0890	.0664	.0434	.0243
	3.0	.1268	.1243	.1165	.1030	.0844	.0623	.0404
1.0	1.0	.1638	.1484	.1103	.0671	.0332	.0133	.0043
	1.5	.2171	.2017	.1611	.1096	.0627	.0298	.0116
	2.0	.2504	.2380	.2034	.1540	.1013	.0567	.0266
	2.5	.2695	.2607	.2348	.1938	.1436	.0931	.0517
	3.0	.2802	.2742	.2558	.2249	.1827	.1339	.0862
1.5	1.5	.3563	.3307	.2637	.1789	.1021	.0483	.0188
	2.0	.4116	.3910	.3334	.2515	.1648	.0919	.0429
	2.5	.4435	.4288	.3853	.3171	.2338	.1509	.0834
	3.0	.4616	.4514	.4205	.3686	.2982	.2174	.1392
2.0	2.0	.5763	.5478	.4679	.3537	.2321	.1296	.0605
	2.5	.6201	.6001	.5404	.4459	.3296	.2131	.1177
	3.0	.6445	.6309	.5892	.5181	.4203	.3070	.1967
2.5	2.5	.7679	.7447	.6748	.5613	.4184	.2725	.1515
	3.0	.7952	.7802	.7332	.6504	.5327	.3924	.2532
3.0	3.0	.8962	.8820	.8359	.7505	.6230	.4650	.3035

(From Ref 5:11)

Table A-II

Numerical Value of the E(a,b,h,k) Building Block

h = 2.0, k =

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.0179	.0159	.0119	.0062	.0027	.0009	.0003
	1.0	.0330	.0299	.0222	.0135	.0067	.0027	.0009
	1.5	.0437	.0406	.0324	.0221	.0127	.0060	.0024
	2.0	.0503	.0478	.0409	.0310	.0205	.0115	.0054
	2.5	.0541	.0524	.0472	.0390	.0290	.0188	.0105
	3.0	.0562	.0550	.0514	.0453	.0369	.0271	.0175
1.0	1.0	.0819	.0741	.0549	.0332	.0163	.0065	.0021
	1.5	.1092	.1012	.0803	.0541	.0306	.0144	.0055
	2.0	.1268	.1201	.1018	.0761	.0493	.0273	.0126
	2.5	.1372	.1323	.1181	.0961	.0701	.0447	.0245
	3.0	.1434	.1399	.1293	.1122	.0896	.0645	.0409
1.5	1.5	.2092	.1935	.1527	.1021	.0572	.0266	.0101
	2.0	.2441	.2308	.1942	.1437	.0920	.0502	.0229
	2.5	.2655	.2554	.2262	.1822	.1310	.0824	.0444
	3.0	.2784	.2710	.2489	.2136	.1683	.1193	.0743
2.0	2.0	.3965	.3746	.3146	.2321	.1481	.0803	.0364
	2.5	.4315	.4149	.3670	.2948	.2111	.1320	.0706
	3.0	.4526	.4405	.4044	.3463	.2720	.1917	.1185
2.5	2.5	.6059	.5834	.5182	.4184	.3011	.1889	.1012
	3.0	.6338	.6178	.5699	.4911	.3879	.2747	.1703
3.0	3.0	.7856	.7682	.7145	.6230	.4986	.3572	.2237

(From Ref 5:12)

Table A-II

Numerical Value of the E(a,b,h,k) Building Block

h = 2.5, k =

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.0062	.0055	.0039	.0022	.0009	.0003	.0001
	1.0	.0114	.0104	.0077	.0047	.0023	.0009	.0003
	1.5	.0152	.0141	.0112	.0076	.0044	.0021	.0008
	2.0	.0176	.0167	.0142	.0107	.0070	.0039	.0018
	2.5	.0189	.0183	.0164	.0135	.0100	.0064	.0036
	3.0	.0197	.0193	.0179	.0157	.0127	.0093	.0059
1.0	1.0	.0332	.0300	.0221	.0133	.0065	.0026	.0008
	1.5	.0445	.0412	.0324	.0216	.0121	.0056	.0021
	2.0	.0521	.0492	.0413	.0304	.0194	.0106	.0048
	2.5	.0567	.0545	.0481	.0386	.0276	.0173	.0093
	3.0	.0596	.0579	.0530	.0453	.0355	.0251	.0156
1.5	1.5	.1021	.0940	.0734	.0483	.0266	.0121	.0045
	2.0	.1203	.1132	.0939	.0681	.0426	.0226	.0101
	2.5	.1320	.1263	.1102	.0868	.0608	.0371	.0194
	3.0	.1395	.1351	.1223	.1027	.0787	.0541	.0327
2.0	2.0	.2321	.2180	.1801	.1296	.0803	.0421	.0185
	2.5	.2555	.2441	.2120	.1658	.1148	.0693	.0358
	3.0	.2704	.2616	.2360	.1967	.1493	.1013	.0603
2.5	2.5	.4184	.4000	.3481	.2725	.1889	.1138	.0586
	3.0	.4423	.4282	.3873	.3237	.2461	.1669	.0990
3.0	3.0	.6230	.6048	.5508	.4650	.3572	.2446	.1462

(From Ref 5:12)

Table A-II

Numerical Value of the E(a,b,h,k) Building Block

h = 3.0, k =

a	b	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.5	.0017	.0015	.0011	.0006	.0003	.0001	.0000
	1.0	.0031	.0028	.0021	.0013	.0006	.0002	.0001
	1.5	.0042	.0039	.0031	.0021	.0012	.0006	.0002
	2.0	.0048	.0046	.0039	.0029	.0019	.0011	.0005
	2.5	.0052	.0050	.0045	.0037	.0027	.0017	.0010
	3.0	.0055	.0053	.0049	.0043	.0034	.0025	.0016
1.0	1.0	.0108	.0098	.0072	.0043	.0021	.0008	.0003
	1.5	.0146	.0135	.0105	.0070	.0038	.0018	.0007
	2.0	.0172	.0162	.0135	.0098	.0061	.0033	.0015
	2.5	.0189	.0181	.0158	.0125	.0088	.0054	.0028
	3.0	.0199	.0193	.0175	.0147	.0113	.0078	.0048
1.5	1.5	.0408	.0374	.0290	.0188	.0101	.0045	.0017
	2.0	.0485	.0454	.0372	.0265	.0162	.0084	.0036
	2.5	.0537	.0512	.0440	.0339	.0231	.0137	.0070
	3.0	.0572	.0551	.0492	.0404	.0301	.0201	.0118
2.0	2.0	.1133	.1058	.0861	.0605	.0364	.0185	.0079
	2.5	.1260	.1196	.1022	.0778	.0522	.0304	.0151
	3.0	.1345	.1294	.1148	.0932	.0684	.0447	.0256
2.5	2.5	.2461	.2337	.1994	.1515	.1012	.0586	.0289
	3.0	.2627	.2527	.2242	.1818	.1331	.0865	.0491
3.0	3.0	.4325	.4168	.3716	.3035	.2237	.1462	.0832

(From Ref 5:13)

Appendix B

Proof of the Normalization Transformations

Proof that the coordinate transformations given in Section III produce an equivalent unit circular normal distribution

By definition, the equation of the general bivariate normal distribution is

$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \iint_R e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}} dy\,dx \qquad (B\text{-}1)$$

Let

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}} \qquad (B\text{-}2)$$

Further, let

$$u' = f_1(x,y) = \frac{x-\mu_x}{\sigma_x} \qquad (B\text{-}3)$$

$$v' = f_2(x,y) = \frac{y-\mu_y}{\sigma_y} \qquad (B\text{-}4)$$

With the inverse functions

$$x = g_1(u',v') = u'\sigma_x + \mu_x \qquad (B\text{-}5)$$

$$y = g_2(u',v') = v'\sigma_y + \mu_y \qquad (B\text{-}6)$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial u'} & \frac{\partial g_2}{\partial u'} \\ \frac{\partial g_1}{\partial v'} & \frac{\partial g_2}{\partial v'} \end{vmatrix} \qquad (B\text{-}7)$$

SUBSTITUTING THESE PARTIAL DIFFERENTIALS (FROM B-5 AND B-6)

$$J = \begin{vmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{vmatrix} = \sigma_x \sigma_y \qquad \text{(B-8)}$$

SUBSTITUTING B-3 AND B-4 INTO B-2, AND MULTIPLYING BY THE JACOBIAN

$$p(u',v') = \sigma_x \sigma_y \left\{ \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\left[-\frac{1}{2(1-\rho^2)}\left(u'^2 - 2\rho u'v' + v'^2\right)\right]} \right\} \qquad \text{(B-9)}$$

WHICH REDUCES TO

$$p(u',v') = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\left[-\frac{1}{2(1-\rho^2)}\left(u'^2 - 2\rho u'v' + v'^2\right)\right]} \qquad \text{(B-10)}$$

NOW, LET

$$u = f_1(u',v') = u' \qquad \text{(B-11)}$$

$$v = f_2(u',v') = \frac{v' - \rho u'}{\sqrt{1-\rho^2}} \qquad \text{(B-12)}$$

WITH THE INVERSE FUNCTIONS

$$u' = g_1(u,v) = u \qquad \text{(B-13)}$$

$$v' = g_2(u,v) = \rho u + \sqrt{1-\rho^2}\, v \qquad \text{(B-14)}$$

THE JACOBIAN OF THIS TRANSFORMATION IS

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial u'} & \frac{\partial g_2}{\partial u'} \\ \frac{\partial g_1}{\partial v'} & \frac{\partial g_2}{\partial v'} \end{vmatrix} = \begin{vmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{vmatrix} = \sqrt{1-\rho^2} \qquad \text{(B-15)}$$

ALSO, FROM B-11 AND B-12

$$u'^2 - 2\rho u'v' + v'^2 = u^2 - 2\rho u\left(\rho u + \sqrt{1-\rho^2}\,v\right) + \left(\rho u + \sqrt{1-\rho^2}\,v\right)^2$$

$$= u^2 - 2\rho^2 u^2 - 2\rho u v\sqrt{1-\rho^2} + \rho^2 u^2 + 2\rho u v\sqrt{1-\rho^2} + (1-\rho^2)v^2$$

$$= (1-\rho^2)(u^2+v^2) \qquad \text{(B-16)}$$

THEREFORE, SUBSTITUTING B-11 AND B-12 INTO B-10, AND MULTIPLYING BY THE JACOBIAN

$$p(u,v) = \sqrt{1-\rho^2}\left\{\frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{\left[-\frac{1}{2(1-\rho^2)}(1-\rho^2)(u^2+v^2)\right]}\right\} \qquad \text{(B-17)}$$

WHICH REDUCES TO

$$p(u,v) = \frac{1}{2\pi}\, e^{-\frac{1}{2}(u^2+v^2)} \qquad \text{(B-18)}$$

IN INTEGRAL FORM

$$P(u,v) = \frac{1}{2\pi}\iint_R e^{-\frac{1}{2}(u^2+v^2)}\,du\,dv \qquad \text{(B-19)}$$

WHICH IS, BY DEFINITION, THE UNIT CIRCULAR NORMAL DISTRIBUTION. Q.E.D.

PROOF OF THE SLOPE OF A LINE TRANSFORMATION

BY DEFINITION, THE SLOPE OF A STRAIGHT LINE BETWEEN TWO POINTS (x_1, y_1) AND (x_2, y_2) IS

$$m_{xy} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(B-20)}$$

LET (x_1, y_1) AND (x_2, y_2) BE ANY TWO GIVEN POINTS ON A LINE ORIENTED WITH RESPECT TO THE x, y COORDINATE SYSTEM ASSOCIATED WITH A GENERAL BIVARIATE NORMAL DISTRIBUTION. LET

$$u' = \frac{x - \mu_x}{\sigma_x} \qquad (B\text{-}3)$$

$$v' = \frac{y - \mu_y}{\sigma_y} \qquad (B\text{-}4)$$

THEN

$$m_{u'v'} = \frac{v_2' - v_1'}{u_2' - u_1'} = \frac{\frac{y_2 - y_1}{\sigma_y}}{\frac{x_2 - x_1}{\sigma_x}} = \frac{\sigma_x}{\sigma_y} \frac{y_2 - y_1}{x_2 - x_1} \qquad (B\text{-}21)$$

$$m_{u'v'} = \frac{\sigma_x}{\sigma_y} m_{xy} \qquad (B\text{-}22)$$

ALSO, LET

$$u = u' \qquad (B\text{-}11)$$

$$v = \frac{v' - \rho u'}{\sqrt{1 - \rho^2}} \qquad (\rho \neq \pm 1) \qquad (B\text{-}12)$$

THEN

$$m_{uv} = \frac{v_2 - v_1}{u_2 - u_1} = \frac{\frac{(v_2' - \rho u_2') - (v_1' - \rho u_1')}{\sqrt{1 - \rho^2}}}{u_2' - u_1'}$$

$$= \frac{1}{\sqrt{1 - \rho^2}} \; \frac{(v_2' - v_1') - \rho (u_2' - u_1')}{u_2' - u_1'} \qquad (B\text{-}23)$$

$$m_{uv} = \frac{1}{\sqrt{1 - \rho^2}} \left(m_{u'v'} - \rho \right) \qquad (B\text{-}24)$$

OR, SUBSTITUTING B-22 INTO B-24

$$m_{uv} = \frac{1}{\sqrt{1 - \rho^2}} \left(\frac{\sigma_x}{\sigma_y} m_{xy} - \rho \right) \qquad \text{Q.E.D.} \qquad (B\text{-}24)$$

Degenerate Case, $\rho = \pm 1$

When ρ is equal to one of its extremes (± 1), the entire mass of the distribution falls along a single straight line. In this case, x and y are completely dependent variables. For each value of x there is one, and only one, possible value of y; and, conversely, for each value of y there is one, and only one, possible value of x. Each variable is a linear function of the other, and they vary in the same or opposite sense depending on whether ρ is equal to +1 or equal to -1, respectively (Ref 2:278).

Thus, if $\rho = \pm 1$, the distribution is no longer bivariate. It can be expressed as a function of a single normally distributed variable. It is a univariate normal distribution in either x or y. Tables to be used for evaluating the univariate normal distribution are available in most mathematical handbooks, or they can be found in any elementary statistical text or statistical handbook (Ref 3:81-89).

Since this degenerate case is not a bivariate normal distribution, and since real distributions in which ρ is exactly equal to $\pm$ 1 are extremely unlikely, this case will not be discussed further in this report.

Appendix C

Proofs of Rotation of Building Blocks and Subdivision by the Axes

Proof that rotation of a building block about the origin of the normalized distribution does not affect its numerical value.

The equation of the normalized distribution is

$$P(u,v) = \frac{1}{2\pi} \iint_R e^{-\frac{1}{2}(u^2+v^2)} du\,dv \qquad (6)$$

Let U,V be an orthogonal coordinate system whose axes make an angle θ with the u and v axes of the normalized distribution. Further, let the origin of the U,V system be coincident with the origin of the u,v system. These coordinate systems are shown in Figure C-1.

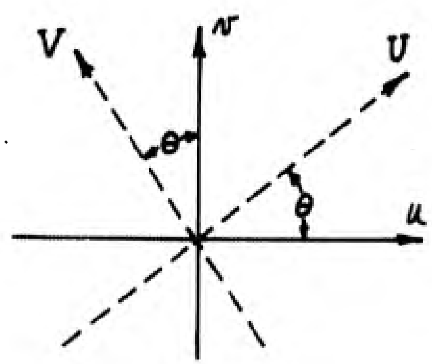


Fig. C-1

Rotation of the Coordinate System

From Fig. C-1, the transformation to the U,V coordinate system is

$$U = u\cos\theta + v\sin\theta \qquad (C\text{-}1)$$

$$V = -u\sin\theta + v\cos\theta \qquad (C\text{-}2)$$

WITH THE INVERSE FUNCTIONS

$$u = g_1(U,V) = U\cos\theta - V\sin\theta \qquad (C\text{-}3)$$

$$v = g_2(U,V) = U\sin\theta + V\cos\theta \qquad (C\text{-}4)$$

THE JACOBIAN FOR THIS TRANSFORMATION IS

$$J = \begin{vmatrix} \frac{\partial u}{\partial U} & \frac{\partial v}{\partial U} \\ \frac{\partial u}{\partial V} & \frac{\partial v}{\partial V} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1 \qquad (C\text{-}5)$$

SUBSTITUTING C-3 AND C-4 INTO (4), AND MULTIPLYING BY THE JACOBIAN

$$P(U,V) = \frac{1}{2\pi} \iint_R e^{-\frac{1}{2}\left[U^2\cos^2\theta - 2UV\sin\theta\cos\theta + V^2\sin^2\theta + U^2\sin^2\theta + 2UV\sin\theta\cos\theta + V^2\cos^2\theta\right]} dV\,dU$$

$$= \frac{1}{2\pi} \iint_R e^{-\frac{1}{2}\left[(U^2+V^2)(\cos^2\theta+\sin^2\theta)\right]} dV\,dU$$

$$P(U,V) = \frac{1}{2\pi} \iint_R e^{-\frac{1}{2}(U^2+V^2)} dV\,dU \qquad (C\text{-}6)$$

BUT

$$U^2+V^2 = u^2\cos^2\theta - 2uv\sin\theta\cos\theta + v^2\sin^2\theta + u^2\sin^2\theta + 2uv\sin\theta\cos\theta + v^2\cos^2\theta$$

$$= (u^2+v^2)(\cos^2\theta + \sin^2\theta)$$

$$= u^2+v^2 \qquad (C\text{-}7)$$

Therefore, the value of the integral is unchanged by rotation about the origin of the normalized distribution.

<u>Proof that any building block which is symmetrical with respect to a coordinate axis is divided, by that axis, into two equal parts.</u>

It follows directly from the preceding proof and the given symmetry of the building block that a rotation of 180 degrees about the origin will not change the value of the whole building block. Similarly, the rotation can not change the integral of either of the symmetrical halves of the building block. Therefore, since the original and final integral are identical, the integrals of the two halves are equal.

Symmetry with Respect to Both Coordinate Axes

It follows directly from the preceding proof that the axes divide the building block into four equal quadrants.

Appendix D

Proof of Subdivision of Building Blocks Which Are Symmetrical About the Origin

Only C(r,0) and the Whole Plane are symmetrical about the origin. But, the Whole Plane can also be expressed as the limit of C(r,0) as r approaches infinity. Therefore, it is desired to prove that equal angles, whose vertices are at the origin, intercept equal probability sectors of C(r,0).

Let γ be the central angle of a sector of C(r,0). Let the probability of being in this sector be

$$P(u,v) = C(r,0)\Big]_0^{\gamma} = \int_0^{\gamma}\int_0^{r} e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta \qquad \text{(D-1)}$$

Further, let V(h,k) be defined by

$$h^2 + k^2 = r^2$$

$$\lambda = \tan\theta$$

$$\theta = \frac{\gamma}{n}$$

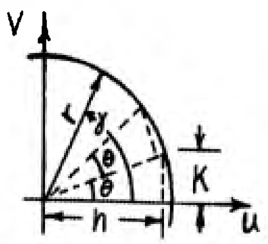


Fig. D-1

Approximation of $C(r,0)\Big]_0^{\gamma}$

Then,

$$\lim_{\substack{K\to 0\\ n\to\infty}} \Big[nV(h, \tan\theta) \Big] = nV(r,\theta) = C(r,0)\Big]_0^{\gamma} \qquad \text{(D-2)}$$

Therefore,

$$P(u,v) = nV(r,\theta) = nV(r, \tfrac{\gamma}{n}) \qquad \text{(D-3)}$$

But,

n = a very large constant which gives the desired accuracy of approximation for $\gamma \leq 360^\circ$

r = a constant for a given C(r,0)

Therefore, for any given C(r,0), P(u,v) is a function of γ alone. Q.E.D.

Appendix E

Multiplication of the Building Blocks

In general, the building blocks can not be multiplied together because they are not independent functions of the variables u and v.

The only exception is the product of $\alpha(h)$ and $\alpha(k)$. This product is permitted because the α function is a function of only one variable:

$$\alpha(q) = \frac{1}{2\pi} \int_{-q}^{q} e^{-\frac{1}{2}t^2} dt \qquad (E\text{-}1)$$

where q can be either $h = f(u)$, or $k = f(v)$, and t is a dummy variable for integration.

The product of $\alpha(h)$ and $\alpha(k)$ is defined in Chapter V, The Catalog of Building Blocks, as the building block $\alpha(h)\alpha(k)$. This is the only building block product which can be formed, since the other building blocks are already functions of both of the variables in the distribution.

Appendix F

Table of A(B,γ)

$$A(B,\gamma) = \frac{1}{2\pi}\int_{-B}^{\infty}\int_{0}^{(u+B)\tan\gamma} e^{-\frac{1}{2}(u^2+v^2)}\,dv\,du$$

B = 0(.01).2(.02).5(.05)2(.1)5(.2)6(.5)8(1)14(2)20(5)50

γ = 0°(.1)1(.2)5(.5)10(1)30(2)60(5)180°

A(B, γ)

B \ γ(°)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	00000	00028	00056	00083	00111	00139	00167	00194	00222	00250
0.01	00000	00028	00056	00084	00113	00141	00169	00197	00225	00253
0.02	00000	00028	00057	00085	00114	00142	00171	00199	00228	00256
0.03	00000	00029	00058	00087	00115	00144	00173	00202	00231	00260
0.04	00000	00029	00058	00088	00117	00146	00175	00204	00234	00263
0.05	00000	00030	00059	00089	00118	00148	00177	00207	00236	00266
0.06	00000	00030	00060	00090	00120	00150	00179	00209	00239	00269
0.07	00000	00030	00061	00091	00121	00151	00182	00212	00242	00273
0.08	00000	00031	00061	00092	00123	00153	00184	00215	00245	00276
0.09	00000	00031	00062	00093	00124	00155	00186	00217	00248	00279
0.10	00000	00031	00063	00094	00126	00157	00188	00220	00251	00283
0.11	00000	00032	00064	00095	00127	00159	00191	00222	00254	00286
0.12	00000	00032	00064	00096	00129	00161	00193	00225	00257	00289
0.13	00000	00033	00065	00098	00130	00163	00195	00228	00260	00293
0.14	00000	00033	00066	00099	00132	00165	00198	00230	00263	00296
0.15	00000	00033	00067	00100	00133	00167	00200	00233	00266	00300
0.16	00000	00034	00067	00101	00135	00169	00202	00236	00270	00303
0.17	00000	00034	00068	00102	00136	00170	00205	00239	00273	00307
0.18	00000	00034	00069	00103	00138	00172	00207	00241	00276	00310
0.19	00000	00035	00070	00105	00140	00174	00209	00244	00279	00314
0.20	00000	00035	00071	00106	00141	00176	00212	00247	00282	00318
0.22	00000	00036	00072	00108	00144	00181	00217	00253	00289	00325
0.24	00000	00037	00074	00111	00148	00185	00222	00259	00295	00332
0.26	00000	00038	00076	00113	00151	00189	00227	00264	00302	00340
0.28	00000	00039	00077	00116	00154	00193	00232	00270	00309	00347
0.30	00000	00039	00079	00118	00158	00197	00237	00276	00316	00355
0.32	00000	00040	00081	00121	00161	00202	00242	00282	00323	00363
0.34	00000	00041	00082	00124	00165	00206	00247	00288	00330	00371
0.36	00000	00042	00084	00126	00168	00210	00253	00295	00337	00379
0.38	00000	00043	00086	00129	00172	00215	00258	00301	00344	00387
0.40	00000	00044	00088	00132	00176	00219	00263	00307	00351	00395
0.42	00000	00045	00090	00134	00179	00224	00269	00314	00359	00403
0.44	00000	00046	00091	00137	00183	00229	00274	00320	00366	00412
0.46	00000	00047	00093	00140	00187	00233	00280	00327	00373	00420
0.48	00000	00048	00095	00143	00191	00238	00286	00333	00381	00429
0.50	00000	00049	00097	00146	00194	00243	00292	00340	00389	00437
0.55	00000	00051	00102	00153	00204	00255	00306	00357	00408	00459
0.60	00000	00054	00107	00161	00214	00268	00321	00375	00428	00482
0.65	00000	00056	00112	00168	00224	00280	00336	00393	00449	00505
0.70	00000	00059	00117	00176	00235	00293	00352	00411	00469	00528

A(B, γ)

B \ γ(°)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.75	00000	00061	00123	00184	00245	00307	00368	00429	00491	00552
0.80	00000	00064	00128	00192	00256	00320	00384	00448	00513	00577
0.85	00000	00067	00134	00201	00267	00334	00401	00468	00535	00602
0.90	00000	00070	00139	00209	00279	00348	00418	00488	00557	00627
0.95	00000	00073	00145	00218	00290	00363	00435	00508	00580	00653
1.00	00000	00075	00151	00226	00302	00377	00453	00528	00603	00679
1.05	00000	00078	00157	00235	00314	00392	00470	00549	00627	00705
1.10	00000	00081	00163	00244	00325	00407	00488	00570	00651	00732
1.15	00000	00084	00169	00253	00338	00422	00506	00591	00675	00760
1.20	00000	00087	00175	00262	00350	00437	00525	00612	00700	00787
1.25	00000	00091	00181	00272	00362	00453	00543	00634	00724	00815
1.30	00000	00094	00187	00281	00375	00468	00562	00656	00749	00843
1.35	00000	00097	00194	00291	00387	00484	00581	00678	00775	00872
1.40	00000	00100	00200	00300	00400	00500	00600	00700	00800	00900
1.45	00000	00103	00206	00310	00413	00516	00619	00723	00826	00929
1.50	00000	00106	00213	00319	00426	00532	00639	00745	00852	00958
1.55	00000	00110	00219	00329	00439	00549	00658	00768	00878	00988
1.60	00000	00113	00226	00339	00452	00565	00678	00791	00904	01017
1.65	00000	00116	00233	00349	00465	00582	00698	00814	00930	01047
1.70	00000	00120	00239	00359	00479	00598	00718	00837	00957	01077
1.75	00000	00123	00246	00369	00492	00615	00738	00861	00984	01107
1.80	00000	00126	00253	00379	00505	00632	00758	00884	01010	01137
1.85	00000	00130	00259	00389	00519	00648	00778	00908	01037	01167
1.90	00000	00133	00266	00399	00532	00665	00798	00931	01064	01197
1.95	00000	00136	00273	00409	00546	00682	00819	00955	01091	01228
2.00	00000	00140	00280	00420	00559	00699	00839	00979	01119	01258
2.10	00000	00147	00293	00440	00587	00733	00880	01027	01173	01320
2.20	00000	00154	00307	00461	00614	00768	00921	01075	01228	01381
2.30	00000	00160	00321	00481	00642	00802	00962	01123	01283	01443
2.40	00000	00167	00335	00502	00669	00836	01004	01171	01338	01505
2.50	00000	00174	00348	00523	00697	00871	01045	01219	01393	01567
2.60	00000	00181	00362	00543	00724	00906	01087	01268	01449	01630
2.70	00000	00188	00376	00564	00752	00940	01128	01316	01504	01692
2.80	00000	00195	00390	00585	00780	00975	01170	01365	01560	01754
2.90	00000	00202	00404	00606	00808	01010	01212	01413	01615	01817
3.00	00000	00209	00418	00627	00836	01044	01253	01462	01671	01879
3.10	00000	00216	00432	00648	00863	01079	01295	01511	01726	01942
3.20	00000	00223	00446	00668	00891	01114	01337	01559	01782	02004
3.30	00000	00230	00460	00689	00919	01149	01378	01608	01838	02067
3.40	00000	00237	00473	00710	00947	01184	01420	01657	01893	02130

A(B, γ)

B \ γ (°)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3.50	00000	00244	00487	00731	00975	01218	01462	01705	01949	02192
3.60	00000	00251	00501	00752	01003	01253	01504	01754	02004	02255
3.70	00000	00258	00515	00773	01030	01288	01545	01803	02060	02317
3.80	00000	00265	00529	00794	01058	01323	01587	01851	02116	02380
3.90	00000	00272	00543	00815	01086	01357	01629	01900	02171	02442
4.00	00000	00279	00557	00835	01114	01392	01671	01949	02227	02505
4.10	00000	00285	00571	00856	01142	01427	01712	01997	02282	02567
4.20	00000	00292	00585	00877	01170	01462	01754	02046	02338	02630
4.30	00000	00299	00599	00898	01197	01497	01796	02095	02394	02692
4.40	00000	00306	00613	00919	01225	01531	01838	02143	02449	02755
4.50	00000	00313	00627	00940	01253	01566	01879	02192	02505	02817
4.60	00000	00320	00641	00961	01281	01601	01921	02241	02560	02880
4.70	00000	00327	00654	00982	01309	01636	01963	02289	02616	02942
4.80	00000	00334	00668	01003	01337	01671	02004	02338	02672	03005
4.90	00000	00341	00682	01023	01364	01705	02046	02387	02727	03067
5.00	00000	00348	00696	01044	01392	01740	02088	02435	02783	03130
5.20	00000	00362	00724	01086	01448	01810	02171	02533	02894	03255
5.40	00000	00376	00752	01128	01504	01879	02255	02630	03005	03380
5.60	00000	00390	00780	01170	01559	01949	02338	02727	03116	03505
5.80	00000	00404	00808	01211	01615	02018	02422	02824	03227	03629
6.00	00000	00418	00835	01253	01671	02088	02505	02922	03338	03754
6.50	00000	00453	00905	01357	01810	02262	02713	03165	03616	04066
7.00	00000	00487	00975	01462	01949	02435	02922	03408	03893	04378
7.50	00000	00522	01044	01566	02088	02609	03130	03650	04170	04689
8.00	00000	00557	01114	01671	02227	02783	03338	03893	04447	05000
9.00	00000	00627	01253	01879	02505	03130	03754	04378	05000	05621
10.00	00000	00696	01392	02088	02783	03477	04170	04862	05552	06241
11.00	00000	00766	01531	02296	03061	03824	04585	05345	06103	06859
12.00	00000	00835	01671	02505	03338	04170	05000	05828	06653	07475
13.00	00000	00905	01810	02713	03616	04516	05414	06310	07202	08090
14.00	00000	00975	01949	02922	03893	04862	05828	06790	07749	08703
16.00	00000	01114	02227	03338	04447	05552	06653	07749	08839	09922
18.00	00000	01253	02505	03754	05000	06241	07475	08703	09922	11131
20.00	00000	01392	02783	04170	05552	06928	08295	09652	10997	12330
25.00	00000	01740	03477	05207	06928	08635	10326	11998	13648	15272
30.00	00000	02088	04170	06241	08295	10326	12330	14301	16234	18126
35.00	00000	02435	04862	07270	09652	11998	14301	16553	18746	20876
40.00	00000	02783	05552	08295	10997	13648	16234	18747	21174	23509
45.00	00000	03130	06241	09314	12330	15273	18126	20876	23510	26016
50.00	00000	03477	06928	10326	13648	16870	19972	22935	25745	28388

A(B, γ)

B \ γ (°)	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8
0.00	00278	00333	00389	00444	00500	00556	00611	00667	00722	00778
0.01	00281	00338	00394	00450	00506	00563	00619	00675	00731	00788
0.02	00285	00342	00399	00456	00513	00570	00627	00684	00740	00797
0.03	00288	00346	00404	00461	00519	00577	00634	00692	00750	00807
0.04	00292	00350	00409	00467	00525	00584	00642	00701	00759	00817
0.05	00296	00355	00414	00473	00532	00591	00650	00709	00768	00827
0.06	00299	00359	00419	00479	00538	00598	00658	00718	00778	00838
0.07	00303	00363	00424	00485	00545	00606	00666	00727	00787	00848
0.08	00307	00368	00429	00490	00552	00613	00674	00736	00797	00858
0.09	00310	00372	00434	00496	00558	00620	00682	00745	00807	00869
0.10	00314	00377	00440	00502	00565	00628	00691	00754	00816	00879
0.11	00318	00381	00445	00508	00572	00635	00699	00763	00826	00890
0.12	00322	00386	00450	00514	00579	00643	00707	00772	00836	00900
0.13	00325	00390	00456	00521	00586	00651	00716	00781	00846	00911
0.14	00329	00395	00461	00527	00593	00658	00724	00790	00856	00922
0.15	00333	00400	00466	00533	00600	00666	00733	00799	00866	00933
0.16	00337	00404	00472	00539	00607	00674	00741	00809	00876	00944
0.17	00341	00409	00477	00546	00614	00682	00750	00818	00886	00955
0.18	00345	00414	00483	00552	00621	00690	00759	00828	00897	00966
0.19	00349	00419	00488	00558	00628	00698	00768	00837	00907	00977
0.20	00353	00424	00494	00565	00635	00706	00776	00847	00918	00988
0.22	00361	00433	00505	00578	00650	00722	00794	00866	00939	01011
0.24	00369	00443	00517	00591	00665	00739	00812	00886	00960	01034
0.26	00378	00453	00529	00604	00680	00755	00831	00906	00982	01057
0.28	00386	00463	00540	00618	00695	00772	00849	00926	01004	01081
0.30	00395	00474	00552	00631	00710	00789	00868	00947	01026	01105
0.32	00403	00484	00565	00645	00726	00806	00887	00968	01048	01129
0.34	00412	00494	00577	00659	00742	00824	00906	00989	01071	01154
0.36	00421	00505	00589	00673	00758	00842	00926	01010	01094	01178
0.38	00430	00516	00602	00688	00774	00860	00946	01032	01118	01203
0.40	00439	00527	00615	00702	00790	00878	00966	01053	01141	01229
0.42	00448	00538	00627	00717	00807	00896	00986	01075	01165	01254
0.44	00457	00549	00640	00732	00823	00915	01006	01098	01189	01280
0.46	00467	00560	00653	00747	00840	00933	01027	01120	01213	01307
0.48	00476	00572	00667	00762	00857	00952	01048	01143	01238	01333
0.50	00486	00583	00680	00777	00874	00972	01069	01166	01263	01360
0.55	00510	00612	00714	00816	00918	01020	01122	01224	01326	01428
0.60	00535	00642	00749	00856	00963	01070	01177	01284	01391	01498
0.65	00561	00673	00785	00897	01009	01121	01233	01345	01457	01570
0.70	00587	00704	00822	00939	01056	01174	01291	01408	01525	01643

A(B, γ)

B \ γ (°)	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8
0.75	00614	00736	00859	00982	01104	01227	01349	01472	01595	01717
0.80	00641	00769	00897	01025	01153	01281	01409	01537	01665	01793
0.85	00668	00802	00936	01069	01203	01336	01470	01604	01737	01870
0.90	00697	00836	00975	01114	01254	01393	01532	01671	01810	01949
0.95	00725	00870	01015	01160	01305	01450	01595	01740	01885	02029
1.00	00754	00905	01056	01207	01357	01508	01659	01809	01960	02111
1.05	00784	00940	01097	01254	01410	01567	01724	01880	02037	02193
1.10	00814	00976	01139	01302	01464	01627	01789	01952	02114	02277
1.15	00844	01013	01181	01350	01519	01687	01856	02024	02193	02361
1.20	00875	01049	01224	01399	01574	01748	01923	02098	02272	02447
1.25	00905	01087	01268	01448	01629	01810	01991	02172	02353	02533
1.30	00937	01124	01311	01498	01686	01873	02060	02247	02434	02621
1.35	00968	01162	01355	01549	01742	01936	02129	02322	02516	02709
1.40	01000	01200	01400	01600	01800	02000	02199	02399	02598	02798
1.45	01032	01239	01445	01651	01858	02064	02270	02476	02682	02887
1.50	01065	01278	01490	01703	01916	02128	02341	02553	02766	02978
1.55	01097	01317	01536	01755	01974	02193	02412	02631	02850	03069
1.60	01130	01356	01582	01808	02033	02259	02484	02710	02935	03160
1.65	01163	01396	01628	01860	02093	02325	02557	02789	03021	03252
1.70	01196	01435	01674	01913	02152	02391	02630	02868	03107	03345
1.75	01230	01475	01721	01967	02212	02458	02703	02948	03193	03438
1.80	01263	01515	01768	02020	02272	02524	02776	03028	03280	03531
1.85	01297	01556	01815	02074	02333	02591	02850	03109	03367	03625
1.90	01330	01596	01862	02128	02393	02659	02924	03189	03454	03719
1.95	01364	01637	01909	02182	02454	02726	02998	03270	03542	03813
2.00	01398	01678	01957	02236	02515	02794	03073	03351	03630	03908
2.10	01466	01759	02052	02345	02638	02930	03222	03515	03806	04098
2.20	01535	01841	02148	02455	02761	03067	03373	03678	03984	04289
2.30	01603	01924	02244	02564	02884	03204	03523	03843	04161	04480
2.40	01672	02007	02341	02674	03008	03341	03674	04007	04340	04672
2.50	01741	02089	02437	02785	03132	03479	03826	04172	04518	04864
2.60	01811	02172	02534	02895	03256	03617	03977	04337	04697	05056
2.70	01880	02255	02631	03006	03381	03755	04129	04503	04876	05249
2.80	01949	02339	02728	03117	03505	03893	04281	04668	05055	05441
2.90	02019	02422	02825	03227	03630	04032	04433	04834	05234	05634
3.00	02088	02505	02922	03338	03754	04170	04585	04999	05413	05826
3.10	02158	02588	03019	03449	03879	04308	04737	05165	05592	06019
3.20	02227	02672	03116	03560	04003	04446	04889	05330	05771	06211
3.30	02296	02755	03213	03671	04128	04585	05040	05496	05950	06404
3.40	02366	02838	03310	03782	04253	04723	05192	05661	06129	06596

A(B, γ)

B \ γ(°)	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8
3.50	02435	02922	03407	03893	04377	04861	05344	05826	06308	06788
3.60	02505	03005	03504	04003	04502	04999	05496	05992	06486	06980
3.70	02574	03088	03602	04114	04626	05137	05647	06157	06665	07172
3.80	02644	03171	03699	04225	04751	05275	05799	06322	06843	07363
3.90	02713	03255	03796	04336	04875	05413	05950	06486	07021	07555
4.00	02783	03338	03893	04446	04999	05551	06102	06651	07199	07746
4.10	02852	03421	03990	04557	05124	05689	06253	06816	07377	07937
4.20	02922	03505	04087	04668	05248	05827	06404	06981	07555	08128
4.30	02991	03588	04184	04778	05372	05964	06555	07145	07733	08319
4.40	03060	03671	04280	04889	05496	06102	06706	07309	07910	08509
4.50	03130	03754	04377	04999	05620	06240	06857	07473	08088	08700
4.60	03199	03837	04474	05110	05744	06377	07008	07637	08265	08890
4.70	03269	03920	04571	05220	05868	06515	07159	07801	08442	09080
4.80	03338	04004	04668	05331	05992	06652	07310	07965	08619	09269
4.90	03407	04087	04765	05441	06116	06789	07460	08129	08795	09459
5.00	03477	04170	04861	05552	06240	06926	07611	08292	08972	09648
5.20	03616	04336	05055	05772	06487	07200	07911	08619	09324	10026
5.40	03754	04502	05248	05992	06734	07474	08211	08945	09676	10403
5.60	03893	04668	05441	06213	06981	07747	08511	09270	10026	10779
5.80	04031	04834	05634	06433	07228	08020	08810	09595	10377	11154
6.00	04170	05000	05827	06652	07474	08293	09108	09919	10726	11528
6.50	04516	05414	06309	07201	08089	08973	09852	10726	11595	12458
7.00	04862	05827	06790	07748	08702	09650	10593	11529	12458	13381
7.50	05207	06240	07270	08294	09312	10324	11329	12327	13316	14296
8.00	05552	06653	07748	08838	09920	10995	12062	13119	14166	15203
9.00	06241	07475	08702	09921	11130	12328	13514	14687	15846	16990
10.00	06927	08294	09651	10996	12328	13645	14947	16230	17495	18740
11.00	07612	09110	10594	12063	13515	14947	16358	17747	19111	20449
12.00	08294	09921	11531	13121	14689	16232	17748	19234	20690	22113
13.00	08974	10729	12461	14169	15849	17498	19112	20691	22231	23730
14.00	09651	11531	13384	15207	16994	18744	20451	22115	23731	25298
16.00	10997	13122	15207	17247	19237	21171	23046	24857	26602	28277
18.00	12329	14690	16995	19237	21410	23506	25521	27450	29290	31038
20.00	13647	16234	18745	21173	23507	25741	27868	29885	31787	33571
25.00	16869	19971	22934	25742	28385	30853	33140	35243	37162	38900
30.00	19971	23509	26821	29889	32699	35245	37526	39549	41323	42861
35.00	22935	26822	30376	33578	36420	38905	41046	42863	44382	45634
40.00	25744	29890	33579	36797	39552	41864	43767	45304	46520	47465
45.00	28388	32701	36421	39553	42124	44185	45796	47024	47939	48603
50.00	30856	35248	38907	41866	44185	45951	47253	48186	48834	49271

A(B, γ)

B \ γ(°)	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8
0.00	00833	00889	00944	01000	01056	01111	01167	01222	01278	01333
0.01	00844	00900	00956	01013	01069	01125	01181	01238	01294	01350
0.02	00854	00911	00968	01025	01082	01139	01196	01253	01310	01367
0.03	00865	00923	00980	01038	01096	01153	01211	01269	01326	01384
0.04	00876	00934	00993	01051	01109	01168	01226	01284	01343	01401
0.05	00887	00946	01005	01064	01123	01182	01241	01300	01359	01418
0.06	00897	00957	01017	01077	01137	01197	01256	01316	01376	01436
0.07	00908	00969	01030	01090	01151	01211	01272	01332	01393	01453
0.08	00920	00981	01042	01103	01165	01226	01287	01349	01410	01471
0.09	00931	00993	01055	01117	01179	01241	01303	01365	01427	01489
0.10	00942	01005	01067	01130	01193	01256	01319	01381	01444	01507
0.11	00953	01017	01080	01144	01207	01271	01334	01398	01461	01525
0.12	00965	01029	01093	01157	01222	01286	01350	01415	01479	01543
0.13	00976	01041	01106	01171	01236	01301	01366	01431	01496	01562
0.14	00988	01053	01119	01185	01251	01317	01383	01448	01514	01580
0.15	00999	01066	01132	01199	01266	01332	01399	01465	01532	01599
0.16	01011	01078	01146	01213	01281	01348	01415	01483	01550	01617
0.17	01023	01091	01159	01227	01295	01364	01432	01500	01568	01636
0.18	01035	01104	01173	01242	01311	01379	01448	01517	01586	01655
0.19	01047	01116	01186	01256	01326	01395	01465	01535	01605	01674
0.20	01059	01129	01200	01270	01341	01411	01482	01553	01623	01694
0.22	01083	01155	01227	01300	01372	01444	01516	01588	01660	01733
0.24	01108	01182	01255	01329	01403	01477	01551	01624	01698	01772
0.26	01133	01208	01284	01359	01435	01510	01586	01661	01736	01812
0.28	01158	01235	01312	01390	01467	01544	01621	01698	01775	01852
0.30	01184	01263	01341	01420	01499	01578	01657	01736	01814	01893
0.32	01210	01290	01371	01451	01532	01613	01693	01774	01854	01935
0.34	01236	01318	01401	01483	01565	01648	01730	01812	01894	01977
0.36	01262	01347	01431	01515	01599	01683	01767	01851	01935	02019
0.38	01289	01375	01461	01547	01633	01719	01805	01891	01976	02062
0.40	01317	01404	01492	01580	01667	01755	01843	01930	02018	02106
0.42	01344	01434	01523	01613	01702	01792	01881	01971	02060	02150
0.44	01372	01463	01555	01646	01737	01829	01920	02011	02103	02194
0.46	01400	01493	01587	01680	01773	01866	01959	02053	02146	02239
0.48	01428	01524	01619	01714	01809	01904	01999	02094	02189	02284
0.50	01457	01554	01651	01748	01845	01942	02039	02136	02233	02330
0.55	01530	01632	01734	01836	01938	02040	02141	02243	02345	02447
0.60	01605	01712	01819	01926	02032	02139	02246	02353	02460	02566
0.65	01682	01794	01905	02017	02129	02241	02353	02465	02577	02688
0.70	01760	01877	01994	02111	02228	02345	02462	02579	02696	02813

A(B, γ)

B \ γ (°)	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8
0.75	01840	01962	02085	02207	02329	02452	02574	02696	02819	02941
0.80	01921	02049	02177	02305	02432	02560	02688	02816	02943	03071
0.85	02004	02137	02271	02404	02537	02671	02804	02937	03070	03203
0.90	02088	02227	02366	02505	02644	02783	02922	03060	03199	03338
0.95	02174	02319	02463	02608	02752	02897	03041	03186	03330	03474
1.00	02261	02412	02562	02712	02863	03013	03163	03313	03463	03613
1.05	02349	02506	02662	02818	02974	03130	03286	03442	03598	03754
1.10	02439	02601	02763	02925	03087	03249	03411	03573	03735	03897
1.15	02529	02698	02866	03034	03202	03370	03538	03706	03873	04041
1.20	02621	02795	02970	03144	03318	03492	03666	03840	04013	04187
1.25	02714	02894	03075	03255	03435	03615	03795	03975	04155	04335
1.30	02807	02994	03181	03367	03553	03740	03926	04112	04298	04484
1.35	02902	03095	03288	03480	03673	03865	04058	04250	04442	04634
1.40	02997	03196	03395	03594	03793	03992	04191	04389	04587	04786
1.45	03093	03299	03504	03710	03915	04120	04325	04529	04734	04938
1.50	03190	03402	03614	03825	04037	04248	04460	04671	04882	05092
1.55	03287	03506	03724	03942	04160	04378	04596	04813	05030	05247
1.60	03385	03610	03835	04060	04284	04508	04732	04956	05180	05403
1.65	03484	03715	03947	04178	04408	04639	04870	05100	05330	05560
1.70	03583	03821	04059	04296	04534	04771	05008	05244	05481	05717
1.75	03683	03927	04171	04415	04659	04903	05147	05390	05633	05875
1.80	03783	04034	04285	04535	04786	05036	05286	05535	05785	06034
1.85	03883	04141	04398	04655	04912	05169	05426	05682	05938	06193
1.90	03984	04248	04512	04776	05040	05303	05566	05829	06091	06353
1.95	04085	04356	04626	04897	05167	05437	05707	05976	06245	06513
2.00	04186	04464	04741	05018	05295	05571	05848	06123	06399	06674
2.10	04389	04680	04971	05261	05552	05841	06131	06420	06708	06996
2.20	04593	04898	05202	05506	05809	06112	06415	06717	07018	07319
2.30	04798	05116	05434	05751	06067	06383	06699	07014	07329	07643
2.40	05003	05335	05666	05996	06326	06655	06984	07313	07640	07967
2.50	05209	05554	05898	06242	06585	06928	07270	07611	07952	08292
2.60	05415	05773	06131	06488	06844	07200	07555	07910	08263	08616
2.70	05621	05992	06363	06734	07104	07473	07841	08208	08575	08941
2.80	05827	06212	06596	06980	07363	07745	08126	08507	08886	09265
2.90	06033	06431	06829	07226	07622	08017	08411	08805	09197	09589
3.00	06239	06651	07062	07472	07881	08289	08696	09103	09508	09912
3.10	06445	06870	07294	07717	08140	08561	08981	09400	09818	10235
3.20	06651	07089	07526	07963	08398	08832	09265	09697	10128	10557
3.30	06856	07308	07758	08208	08656	09103	09549	09994	10437	10878
3.40	07062	07527	07990	08453	08914	09374	09833	10290	10745	11199

A(B, γ)

B \ γ (°)	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8
3.50	07267	07745	08222	08698	09172	09644	10116	10585	11053	11519
3.60	07472	07963	08453	08942	09429	09914	10398	10880	11360	11839
3.70	07677	08182	08684	09186	09686	10184	10680	11174	11667	12157
3.80	07882	08399	08915	09429	09942	10452	10961	11468	11972	12475
3.90	08087	08617	09146	09673	10198	10721	11242	11761	12277	12792
4.00	08291	08834	09376	09916	10453	10989	11522	12053	12582	13108
4.10	08495	09051	09606	10158	10708	11256	11802	12345	12885	13423
4.20	08699	09268	09835	10400	10963	11523	12081	12636	13188	13737
4.30	08903	09485	10065	10642	11217	11789	12359	12926	13490	14051
4.40	09106	09701	10293	10883	11471	12055	12637	13215	13791	14363
4.50	09309	09917	10522	11124	11724	12320	12914	13504	14091	14675
4.60	09512	10133	10750	11365	11976	12585	13190	13792	14390	14985
4.70	09715	10348	10978	11605	12228	12849	13466	14079	14689	15295
4.80	09918	10563	11205	11844	12480	13112	13741	14366	14986	15603
4.90	10120	10778	11432	12083	12731	13375	14015	14651	15283	15910
5.00	10322	10992	11659	12322	12982	13637	14289	14936	15579	16217
5.20	10725	11420	12111	12798	13481	14160	14834	15503	16167	16826
5.40	11126	11846	12561	13272	13978	14680	15376	16067	16752	17432
5.60	11527	12271	13010	13744	14473	15197	15915	16627	17333	18032
5.80	11926	12694	13457	14214	14965	15711	16450	17183	17909	18628
6.00	12325	13116	13902	14682	15455	16222	16982	17735	18481	19219
6.50	13314	14164	15006	15841	16669	17488	18298	19100	19892	20675
7.00	14295	15201	16098	16986	17865	18733	19591	20438	21274	22098
7.50	15266	16227	17177	18115	19042	19957	20860	21749	22624	23486
8.00	16228	17241	18241	19228	20201	21160	22103	23031	23943	24839
9.00	18119	19230	20325	21400	22457	23494	24510	25505	26479	27431
10.00	19964	21165	22343	23497	24625	25728	26803	27852	28872	29864
11.00	21759	23040	24292	25512	26700	27855	28977	30064	31116	32133
12.00	23501	24853	26167	27442	28678	29873	31026	32138	33207	34234
13.00	25186	26598	27964	29283	30554	31775	32948	34070	35143	36166
14.00	26813	28275	29681	31032	32325	33561	34740	35860	36924	37930
16.00	29881	31411	32866	34247	35551	36781	37936	39018	40029	40969
18.00	32692	34250	35713	37081	38355	39537	40630	41635	42557	43399
20.00	35239	36788	38221	39541	40749	41851	42851	43753	44564	45289
25.00	40463	41857	43092	44176	45122	45941	46645	47244	47752	48178
30.00	44180	45300	46240	47020	47661	48181	48599	48932	49194	49397
35.00	46651	47463	48104	48601	48982	49269	49482	49638	49750	49830
40.00	48184	48722	49116	49399	49599	49737	49830	49893	49933	49959
45.00	49074	49400	49619	49764	49857	49915	49951	49972	49985	49992
50.00	49556	49737	49849	49915	49954	49976	49987	49994	49997	49999

A(B, γ)

B \ γ (°)	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
0.00	01389	01528	01667	01806	01944	02083	02222	02361	02500	02639
0.01	01406	01547	01688	01844	01969	02109	02250	02391	02531	02672
0.02	01424	01566	01709	01851	01993	02136	02278	02421	02563	02705
0.03	01442	01586	01730	01874	02018	02162	02306	02451	02595	02739
0.04	01460	01605	01751	01897	02043	02189	02335	02481	02627	02773
0.05	01478	01625	01773	01921	02068	02216	02364	02511	02659	02807
0.06	01496	01645	01795	01944	02094	02243	02393	02542	02692	02841
0.07	01514	01665	01817	01968	02119	02271	02422	02573	02724	02876
0.08	01532	01686	01839	01992	02145	02298	02451	02604	02757	02911
0.09	01551	01706	01861	02016	02171	02326	02481	02636	02791	02946
0.10	01570	01727	01883	02040	02197	02354	02511	02668	02824	02981
0.11	01588	01747	01906	02065	02223	02382	02541	02699	02858	03017
0.12	01607	01768	01929	02089	02250	02411	02571	02732	02892	03053
0.13	01627	01789	01952	02114	02277	02439	02602	02764	02926	03089
0.14	01646	01810	01975	02139	02304	02468	02632	02797	02961	03125
0.15	01665	01832	01998	02164	02331	02497	02663	02829	02996	03162
0.16	01685	01853	02021	02190	02358	02526	02694	02863	03031	03199
0.17	01704	01875	02045	02215	02385	02556	02726	02896	03066	03236
0.18	01724	01896	02069	02241	02413	02585	02757	02929	03101	03273
0.19	01744	01918	02093	02267	02441	02615	02789	02963	03137	03311
0.20	01764	01940	02117	02293	02469	02645	02821	02997	03173	03349
0.22	01805	01985	02165	02345	02526	02706	02886	03066	03246	03425
0.24	01846	02030	02214	02399	02583	02767	02951	03135	03319	03503
0.26	01887	02076	02264	02453	02641	02829	03018	03206	03394	03582
0.28	01929	02122	02315	02507	02700	02892	03085	03277	03469	03661
0.30	01972	02169	02366	02563	02760	02956	03153	03349	03546	03742
0.32	02015	02217	02418	02619	02820	03021	03222	03422	03623	03823
0.34	02059	02265	02470	02676	02881	03086	03291	03496	03701	03906
0.36	02103	02313	02523	02733	02943	03152	03362	03571	03780	03989
0.38	02148	02362	02577	02791	03005	03219	03433	03647	03860	04074
0.40	02193	02412	02631	02850	03068	03287	03505	03723	03941	04159
0.42	02239	02462	02686	02909	03132	03355	03578	03801	04023	04246
0.44	02285	02513	02741	02969	03197	03424	03652	03879	04106	04333
0.46	02332	02565	02797	03030	03262	03494	03726	03958	04190	04421
0.48	02379	02617	02854	03091	03328	03565	03802	04038	04274	04510
0.50	02427	02669	02911	03153	03395	03636	03878	04119	04359	04600
0.55	02549	02803	03057	03311	03564	03818	04071	04324	04577	04829
0.60	02673	02940	03206	03472	03738	04004	04269	04534	04799	05063
0.65	02800	03079	03358	03637	03915	04194	04471	04749	05026	05303
0.70	02930	03222	03514	03806	04097	04388	04678	04968	05258	05547

A(B, γ)

B \ γ(°)	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
0.75	03063	03368	03673	03978	04282	04586	04889	05192	05494	05796
0.80	03198	03517	03835	04153	04470	04787	05104	05420	05735	06050
0.85	03336	03668	04000	04331	04662	04993	05323	05652	05981	06309
0.90	03476	03822	04168	04513	04857	05201	05545	05888	06230	06571
0.95	03618	03979	04338	04697	05056	05414	05771	06127	06483	06838
1.00	03763	04137	04511	04884	05257	05629	06000	06370	06740	07108
1.05	03910	04298	04687	05074	05461	05847	06232	06616	07000	07382
1.10	04058	04461	04864	05266	05667	06068	06467	06866	07263	07659
1.15	04208	04627	05044	05461	05876	06291	06705	07118	07529	07940
1.20	04360	04794	05226	05657	06088	06517	06945	07372	07798	08223
1.25	04514	04962	05410	05856	06301	06745	07188	07630	08070	08509
1.30	04669	05133	05595	06056	06517	06976	07433	07889	08344	08797
1.35	04826	05305	05782	06259	06734	07208	07680	08151	08620	09088
1.40	04984	05478	05971	06463	06953	07442	07929	08415	08899	09381
1.45	05143	05652	06161	06668	07174	07678	08180	08680	09179	09675
1.50	05303	05828	06352	06875	07396	07915	08432	08947	09461	09972
1.55	05464	06005	06545	07083	07619	08153	08686	09216	09744	10269
1.60	05626	06183	06738	07292	07844	08393	08941	09486	10028	10569
1.65	05789	06362	06933	07502	08069	08634	09197	09757	10314	10869
1.70	05953	06542	07129	07713	08296	08876	09454	10029	10601	11170
1.75	06118	06722	07325	07925	08523	09119	09712	10302	10888	11472
1.80	06283	06903	07522	08138	08752	09362	09970	10575	11177	11775
1.85	06449	07085	07719	08351	08980	09607	10230	10849	11466	12078
1.90	06615	07267	07918	08565	09210	09851	10489	11124	11755	12382
1.95	06781	07450	08116	08780	09440	10096	10750	11399	12045	12686
2.00	06949	07633	08315	08994	09670	10342	11010	11674	12334	12990
2.10	07284	08001	08714	09425	10131	10834	11532	12226	12915	13598
2.20	07620	08369	09114	09856	10593	11326	12054	12777	13494	14206
2.30	07957	08738	09515	10288	11056	11818	12576	13328	14073	14813
2.40	08294	09107	09916	10719	11518	12310	13097	13877	14651	15417
2.50	08631	09476	10316	11151	11979	12801	13617	14425	15226	16019
2.60	08969	09845	10716	11581	12440	13291	14135	14971	15799	16618
2.70	09306	10214	11116	12011	12899	13779	14651	15515	16369	17214
2.80	09642	10582	11515	12440	13357	14266	15166	16056	16936	17806
2.90	09979	10950	11913	12868	13814	14751	15678	16594	17500	18394
3.00	10315	11316	12310	13294	14269	15234	16187	17129	18060	18978
3.10	10650	11682	12706	13719	14722	15714	16694	17661	18616	19557
3.20	10984	12047	13100	14143	15173	16192	17198	18190	19168	20132
3.30	11318	12411	13494	14564	15623	16668	17699	18715	19716	20701
3.40	11651	12774	13886	14984	16070	17141	18197	19237	20260	21266

A(B, γ)

B \ γ (°)	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
3.50	11984	13136	14276	15403	16515	17611	18691	19754	20799	21826
3.60	12315	13497	14666	15819	16957	18079	19182	20268	21334	22380
3.70	12645	13857	15053	16234	17398	18543	19670	20778	21864	22929
3.80	12975	14215	15439	16647	17836	19005	20155	21283	22390	23473
3.90	13304	14572	15824	17057	18271	19464	20636	21785	22910	24011
4.00	13631	14928	16207	17466	18704	19920	21113	22282	23426	24543
4.10	13958	15283	16588	17872	19134	20373	21587	22775	23936	25070
4.20	14284	15636	16968	18277	19562	20823	22057	23263	24442	25591
4.30	14608	15988	17345	18679	19987	21269	22523	23748	24942	26106
4.40	14932	16339	17722	19079	20410	21712	22985	24227	25437	26614
4.50	15255	16688	18096	19477	20830	22152	23443	24702	25927	27117
4.60	15576	17035	18468	19872	21246	22589	23898	25172	26411	27614
4.70	15896	17382	18839	20266	21661	23022	24348	25638	26890	28104
4.80	16215	17726	19207	20656	22072	23451	24794	26099	27364	28589
4.90	16533	18070	19574	21045	22480	23878	25236	26555	27832	29067
5.00	16850	18411	19939	21431	22885	24300	25674	27006	28294	29538
5.20	17480	19090	20662	22195	23687	25135	26538	27894	29202	30462
5.40	18105	19762	21378	22950	24476	25955	27383	28761	30087	31360
5.60	18725	20428	22085	23694	25253	26759	28212	29609	30949	32233
5.80	19340	21086	22783	24427	26017	27549	29023	30436	31788	33079
6.00	19949	21738	23473	25150	26768	28323	29815	31242	32603	33898
6.50	21448	23336	25157	26908	28586	30190	31717	33166	34538	35832
7.00	22910	24886	26782	28594	30319	31956	33502	34959	36325	37602
7.50	24334	26388	28347	30207	31965	33620	35171	36619	37965	39212
8.00	25718	27839	29849	31743	33521	35181	36723	38149	39462	40665
9.00	28360	30582	32659	34586	36364	37995	39482	40829	42042	43128
10.00	30828	33108	35205	37119	38852	40410	41800	43031	44113	45058
11.00	33115	35413	37489	39348	40997	42447	43710	44801	45735	46528
12.00	35219	37496	39514	41284	42819	44136	45255	46194	46976	47618
13.00	37140	39362	41291	42944	44344	45514	46479	47267	47901	48405
14.00	38880	41018	42832	44350	45601	46618	47432	48074	48574	48957
16.00	41842	43743	45278	46495	47441	48162	48702	49098	49384	49586
18.00	44165	45776	47005	47921	48587	49060	49388	49610	49757	49852
20.00	45934	47238	48172	48821	49260	49548	49731	49844	49912	49952
25.00	48533	49172	49552	49767	49884	49945	49975	49989	49995	49998
30.00	49553	49798	49914	49966	49987	49995	49998	50000	50000	50000
35.00	49886	49960	49987	49996	49999	50000	50000	50000	50000	50000
40.00	49975	49994	49999	50000	50000	50000	50000	50000	50000	50000
45.00	49996	49999	50000	50000	50000	50000	50000	50000	50000	50000
50.00	49999	50000	50000	50000	50000	50000	50000	50000	50000	50000

A(B, γ)

B \ γ (°)	10	11	12	13	14	15	16	17	18	19
0.00	02778	03056	03333	03611	03889	04167	04444	04722	05000	05278
0.01	02813	03094	03375	03656	03937	04218	04500	04781	05062	05343
0.02	02848	03160	03417	03702	03986	04271	04555	04840	05124	05409
0.03	02883	03171	03459	03747	04072	04323	04611	04899	05187	05475
0.04	02919	03210	03502	03793	04085	04376	04668	04959	05250	05541
0.05	02954	03250	03545	03840	04135	04430	04725	05019	05314	05609
0.06	02991	03289	03588	03887	04185	04484	04782	05080	05378	05676
0.07	03027	03329	03632	03934	04236	04538	04840	05141	05443	05744
0.08	03064	03370	03675	03981	04287	04592	04898	05203	05508	05813
0.09	03101	03410	03720	04029	04338	04647	04956	05265	05574	05882
0.10	03138	03451	03764	04077	04390	04703	05015	05328	05640	05952
0.11	03175	03492	03809	04126	04442	04759	05075	05391	05706	06022
0.12	03213	03534	03854	04175	04495	04815	05134	05454	05773	06092
0.13	03251	03575	03900	04224	04548	04871	05195	05518	05841	06163
0.14	03289	03618	03946	04273	04601	04928	05255	05582	05908	06235
0.15	03328	03660	03992	04323	04655	04986	05316	05647	05977	06307
0.16	03367	03703	04038	04373	04709	05043	05378	05712	06046	06379
0.17	03406	03745	04085	04424	04763	05101	05440	05777	06115	06452
0.18	03445	03789	04132	04475	04818	05160	05502	05843	06184	06525
0.19	03485	03832	04179	04526	04873	05219	05564	05910	06255	06599
0.20	03525	03876	04227	04578	04928	05278	05628	05977	06325	06673
0.22	03605	03964	04323	04682	05040	05398	05755	06112	06468	06823
0.24	03687	04054	04421	04787	05153	05519	05884	06248	06612	06975
0.26	03770	04145	04520	04894	05268	05642	06014	06387	06758	07129
0.28	03853	04237	04620	05002	05384	05766	06146	06526	06906	07284
0.30	03938	04330	04721	05112	05502	05891	06280	06668	07055	07441
0.32	04024	04424	04824	05223	05621	06019	06415	06811	07206	07600
0.34	04111	04519	04927	05335	05741	06147	06552	06956	07359	07761
0.36	04198	04616	05032	05448	05863	06277	06690	07103	07514	07924
0.38	04287	04713	05138	05563	05986	06409	06830	07251	07670	08088
0.40	04377	04812	05246	05679	06111	06542	06972	07400	07828	08254
0.42	04468	04911	05354	05796	06236	06676	07114	07551	07987	08421
0.44	04559	05012	05464	05914	06363	06812	07258	07704	08148	08591
0.46	04652	05114	05574	06034	06492	06949	07404	07858	08311	08761
0.48	04746	05216	05686	06154	06621	07087	07551	08014	08475	08934
0.50	04840	05320	05799	06276	06752	07227	07700	08171	08640	09108
0.55	05081	05584	06086	06586	07085	07582	08077	08570	09061	09550
0.60	05327	05854	06380	06903	07425	07945	08462	08977	09490	10001
0.65	05579	06130	06679	07227	07772	08315	08855	09393	09928	10460
0.70	05836	06412	06986	07557	08126	08693	09256	09817	10374	10928

A(B, γ)

B \ γ (°)	10	11	12	13	14	15	16	17	18	19
0.75	06098	06699	07298	07894	08487	09077	09664	10248	10828	11404
0.80	06365	06991	07615	08236	08854	09468	10079	10686	11289	11887
0.85	06636	07289	07938	08585	09227	09866	10500	11131	11756	12377
0.90	06912	07591	08266	08938	09605	10269	10927	11581	12230	12874
0.95	07192	07897	08599	09296	09989	10677	11360	12038	12710	13376
1.00	07476	08208	08936	09659	10377	11090	11798	12499	13195	13883
1.05	07763	08523	09277	10027	10771	11509	12240	12966	13684	14395
1.10	08054	08841	09622	10398	11168	11931	12687	13436	14178	14912
1.15	08349	09163	09971	10773	11569	12357	13138	13911	14676	15432
1.20	08646	09488	10323	11152	11973	12787	13592	14389	15177	15955
1.25	08946	09816	10678	11534	12381	13220	14050	14870	15680	16480
1.30	09249	10146	11036	11918	12791	13655	14510	15353	16186	17008
1.35	09554	10479	11397	12305	13204	14093	14972	15839	16694	17538
1.40	09861	10814	11759	12694	13619	14533	15436	16326	17204	18068
1.45	10170	11152	12124	13086	14036	14975	15901	16815	17714	18600
1.50	10480	11490	12490	13478	14454	15418	16368	17304	18225	19131
1.55	10792	11831	12858	13872	14874	15862	16835	17794	18736	19662
1.60	11106	12172	13226	14267	15294	16306	17303	18284	19247	20193
1.65	11421	12515	13596	14663	15715	16752	17771	18773	19757	20722
1.70	11736	12858	13967	15060	16137	17197	18239	19262	20266	21250
1.75	12053	13203	14338	15457	16558	17642	18706	19750	20774	21777
1.80	12370	13547	14709	15854	16980	18086	19172	20237	21280	22300
1.85	12687	13893	15081	16250	17400	18530	19637	20723	21784	22822
1.90	13005	14238	15452	16647	17821	18973	20101	21206	22286	23340
1.95	13323	14583	15824	17043	18240	19414	20564	21687	22785	23855
2.00	13641	14928	16195	17439	18659	19855	21024	22167	23281	24367
2.10	14277	15618	16935	18227	19493	20730	21938	23117	24264	25379
2.20	14912	16305	17672	19011	20320	21598	22843	24055	25232	26375
2.30	15545	16990	18405	19789	21140	22456	23736	24980	26185	27351
2.40	16176	17671	19134	20561	21952	23304	24617	25889	27120	28308
2.50	16804	18348	19856	21325	22754	24141	25484	26783	28036	29243
2.60	17428	19020	20572	22082	23547	24966	26337	27660	28933	30156
2.70	18049	19687	21281	22829	24329	25778	27175	28519	29809	31045
2.80	18665	20348	21983	23568	25099	26576	27997	29360	30665	31911
2.90	19276	21003	22677	24296	25858	27360	28802	30181	31498	32753
3.00	19882	21651	23363	25015	26605	28130	29590	30983	32310	33569
3.10	20484	22293	24041	25723	27339	28885	30361	31766	33099	34361
3.20	21080	22928	24709	26421	28060	29625	31114	32528	33866	35128
3.30	21670	23555	25369	27107	28768	30349	31850	33270	34609	35869
3.40	22255	24176	26019	27782	29462	31058	32567	33991	35330	36585

A(B, γ)

B \ γ (°)	10	11	12	13	14	15	16	17	18	19
3.50	22833	24788	26660	28446	30143	31750	33267	34692	36028	37276
3.60	23406	25393	27292	29098	30810	32427	33948	35373	36703	37942
3.70	23973	25991	27914	29739	31464	33088	34611	36033	37356	38583
3.80	24533	26580	28526	30367	32103	33732	35255	36672	37986	39199
3.90	25087	27161	29128	30984	32729	34361	35881	37291	38593	39791
4.00	25635	27734	29720	31589	33340	34973	36489	37890	39178	40359
4.10	26175	28299	30301	32181	33937	35569	37079	38468	39742	40903
4.20	26710	28855	30873	32762	34520	36149	37650	39027	40283	41425
4.30	27237	29403	31434	33330	35089	36713	38204	39566	40804	41924
4.40	27758	29942	31985	33886	35644	37261	38740	40085	41303	42400
4.50	28272	30473	32526	34430	36185	37793	39258	40586	41782	42855
4.60	28779	30995	33056	34961	36711	38309	39759	41068	42241	43288
4.70	29279	31509	33576	35481	37224	38809	40243	41530	42680	43701
4.80	29772	32014	34085	35988	37722	39294	40709	41975	43100	44094
4.90	30258	32510	34584	36483	38207	39764	41159	42402	43501	44468
5.00	30737	32997	35073	36965	38679	40218	41593	42811	43884	44822
5.20	31673	33945	36018	37895	39580	41083	42412	43579	44596	45477
5.40	32580	34858	36922	38777	40429	41889	43168	44281	45241	46063
5.60	33458	35736	37785	39612	41225	42639	43865	44921	45823	46586
5.80	34307	36579	38607	40401	41971	43334	44506	45503	46346	47051
6.00	35127	37387	39389	41144	42668	43978	45092	46030	46814	47461
6.50	37049	39256	41172	42815	44208	45375	46340	47131	47771	48284
7.00	38792	40917	42722	44233	45482	46499	47316	47965	48473	48867
7.50	40360	42379	44054	45421	46519	47388	48065	48584	48977	49269
8.00	41761	43655	45187	46404	47353	48080	48628	49033	49328	49540
9.00	44095	45704	46934	47854	48527	49008	49344	49575	49729	49831
10.00	45876	47181	48120	48776	49222	49518	49708	49827	49900	49943
11.00	47194	48209	48890	49333	49611	49779	49879	49935	49966	49983
12.00	48141	48898	49370	49653	49815	49905	49953	49977	49990	49995
13.00	48801	49344	49656	49827	49917	49962	49983	49993	49997	49999
14.00	49247	49622	49820	49918	49965	49985	49994	49998	49999	50000
16.00	49727	49887	49956	49984	49995	49998	49999	50000	50000	50000
18.00	49911	49970	49991	49997	49999	50000	50000	50000	50000	50000
20.00	49974	49993	49998	50000	50000	50000	50000	50000	50000	50000
25.00	49999	50000	50000	50000	50000	50000	50000	50000	50000	50000
30.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
35.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
40.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
45.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
50.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000

A(B, γ)

B \ γ (°)	20	21	22	23	24	25	26	27	28	29
0.00	05556	05833	06111	06389	06667	06944	07222	07500	07778	08056
0.01	05624	05905	06186	06467	06748	07029	07310	07591	07872	08153
0.02	05693	05977	06262	06546	06830	07114	07398	07682	07966	08250
0.03	05763	06050	06338	06625	06913	07200	07487	07775	08062	08349
0.04	05833	06124	06414	06705	06996	07287	07577	07867	08158	08448
0.05	05903	06197	06492	06786	07080	07374	07667	07961	08254	08547
0.06	05974	06272	06569	06867	07164	07461	07758	08055	08351	08648
0.07	06046	06347	06648	06948	07249	07549	07850	08150	08449	08749
0.08	06118	06422	06726	07031	07335	07638	07942	08245	08548	08851
0.09	06190	06498	06806	07113	07421	07728	08034	08341	08647	08953
0.10	06263	06575	06886	07197	07507	07818	08128	08437	08747	09056
0.11	06337	06652	06966	07281	07595	07908	08222	08535	08847	09160
0.12	06411	06729	07047	07365	07682	07999	08316	08632	08948	09264
0.13	06485	06807	07129	07450	07771	08091	08411	08731	09050	09369
0.14	06560	06886	07211	07535	07860	08184	08507	08830	09152	09474
0.15	06636	06965	07293	07622	07949	08277	08603	08930	09255	09581
0.16	06712	07044	07377	07708	08039	08370	08700	09030	09359	09687
0.17	06788	07125	07460	07795	08130	08464	08798	09131	09463	09795
0.18	06865	07205	07544	07883	08221	08559	08896	09232	09568	09903
0.19	06943	07286	07629	07971	08313	08654	08994	09334	09673	10012
0.20	07021	07368	07714	08060	08405	08750	09094	09437	09779	10121
0.22	07178	07533	07886	08239	08592	08943	09294	09644	09993	10341
0.24	07338	07699	08060	08421	08780	09139	09496	09853	10209	10564
0.26	07499	07868	08236	08604	08971	09336	09701	10065	10428	10789
0.28	07662	08039	08414	08789	09163	09536	09908	10279	10649	11017
0.30	07827	08211	08594	08977	09358	09738	10117	10495	10872	11247
0.32	07994	08386	08777	09166	09555	09942	10329	10713	11097	11479
0.34	08162	08562	08960	09358	09754	10149	10542	10934	11324	11713
0.36	08333	08740	09146	09551	09955	10357	10757	11157	11554	11950
0.38	08505	08920	09334	09747	10158	10567	10975	11381	11786	12188
0.40	08679	09102	09524	09944	10363	10779	11195	11608	12020	12429
0.42	08854	09285	09715	10143	10569	10994	11416	11837	12255	12672
0.44	09032	09471	09908	10344	10778	11210	11640	12067	12493	12917
0.46	09211	09658	10103	10547	10988	11428	11865	12300	12733	13163
0.48	09391	09847	10300	10751	11201	11648	12092	12535	12974	13412
0.50	09574	10037	10499	10958	11415	11869	12321	12771	13218	13662
0.55	10036	10520	11002	11481	11957	12431	12902	13370	13834	14296
0.60	10508	11013	11515	12014	12510	13003	13492	13978	14461	14940
0.65	10989	11515	12038	12557	13073	13585	14093	14597	15097	15594
0.70	11479	12026	12569	13109	13644	14175	14702	15225	15743	16256

A(B, γ)

B \ γ (°)	20	21	22	23	24	25	26	27	28	29
0.75	11976	12545	13109	13668	14223	14774	15319	15860	16396	16926
0.80	12482	13071	13656	14236	14811	15380	15944	16503	17056	17603
0.85	12993	13604	14210	14810	15404	15993	16575	17152	17722	18285
0.90	13512	14144	14770	15391	16005	16612	17213	17806	18393	18973
0.95	14036	14689	15336	15977	16610	17236	17854	18466	19069	19664
1.00	14565	15240	15908	16568	17220	17864	18501	19129	19748	20359
1.05	15099	15795	16483	17163	17834	18496	19150	19794	20430	21056
1.10	15637	16354	17062	17761	18451	19131	19802	20462	21113	21753
1.15	16179	16916	17644	18362	19070	19768	20455	21131	21797	22451
1.20	16723	17481	18229	18966	19692	20406	21110	21801	22481	23148
1.25	17270	18048	18815	19571	20314	21045	21764	22470	23163	23844
1.30	17819	18617	19403	20176	20937	21684	22418	23138	23844	24537
1.35	18369	19187	19991	20782	21559	22322	23070	23804	24523	25227
1.40	18919	19757	20579	21387	22180	22958	23720	24467	25198	25912
1.45	19471	20326	21167	21992	22800	23592	24368	25126	25868	26593
1.50	20022	20896	21753	22594	23417	24223	25012	25782	26534	27268
1.55	20572	21464	22338	23194	24032	24851	25651	26432	27194	27937
1.60	21121	22030	22921	23792	24643	25475	26286	27077	27848	28598
1.65	21668	22595	23501	24386	25250	26094	26916	27716	28495	29252
1.70	22214	23156	24077	24976	25853	26708	27539	28348	29135	29898
1.75	22757	23715	24650	25562	26451	27316	28157	28973	29766	30535
1.80	23297	24271	25220	26144	27043	27918	28767	29591	30389	31163
1.85	23835	24822	25784	26720	27630	28513	29370	30200	31003	31780
1.90	24368	25370	26344	27291	28210	29101	29965	30800	31608	32388
1.95	24898	25913	26899	27856	28784	29682	30552	31392	32203	32984
2.00	25424	26451	27448	28415	29350	30256	31130	31974	32787	33570
2.10	26462	27512	28529	29512	30461	31377	32259	33108	33924	34707
2.20	27481	28551	29584	30581	31541	32464	33351	34202	35017	35797
2.30	28478	29565	30613	31620	32587	33515	34403	35253	36064	36838
2.40	29452	30554	31612	32627	33598	34527	35414	36259	37064	37828
2.50	30403	31515	32581	33601	34573	35501	36383	37221	38015	38768
2.60	31328	32449	33520	34541	35512	36434	37309	38136	38919	39657
2.70	32227	33354	34426	35446	36412	37327	38192	39007	39774	40496
2.80	33099	34229	35301	36316	37275	38180	39032	39832	40582	41285
2.90	33944	35074	36143	37151	38100	38992	39829	40612	41343	42025
3.00	34762	35889	36952	37950	38887	39764	40584	41347	42057	42717
3.10	35553	36674	37728	38715	39637	40497	41297	42039	42727	43363
3.20	36315	37429	38471	39444	40350	41191	41969	42689	43353	43964
3.30	37050	38154	39183	40139	41026	41846	42602	43298	43937	44521
3.40	37757	38849	39862	40801	41667	42464	43196	43867	44480	45038

A(B, γ)

B \ γ (°)	20	21	22	23	24	25	26	27	28	29
3.50	38437	39514	40510	41428	42273	43046	43753	44398	44984	45515
3.60	39090	40150	41127	42024	42844	43593	44274	44892	45450	45954
3.70	39715	40758	41714	42587	43383	44106	44760	45350	45882	46358
3.80	40315	41337	42271	43120	43890	44586	45213	45775	46279	46729
3.90	40888	41889	42799	43623	44366	45035	45634	46169	46645	47067
4.00	41436	42414	43299	44097	44813	45453	46024	46531	46980	47377
4.10	41959	42913	43772	44542	45231	45843	46386	46865	47288	47658
4.20	42457	43386	44218	44961	45621	46205	46720	47172	47568	47914
4.30	42931	43834	44639	45354	45985	46541	47029	47454	47824	48145
4.40	43382	44258	45035	45721	46324	46852	47312	47712	48057	48354
4.50	43811	44659	45408	46065	46640	47140	47573	47947	48268	48543
4.60	44217	45037	45757	46386	46933	47405	47813	48162	48460	48713
4.70	44603	45394	46085	46685	47204	47650	48032	48357	48633	48865
4.80	44967	45730	46392	46964	47455	47875	48232	48534	48788	49002
4.90	45312	46046	46679	47223	47687	48081	48414	48694	48929	49124
5.00	45638	46342	46947	47463	47901	48270	48580	48839	49055	49233
5.20	46234	46880	47429	47891	48279	48601	48868	49088	49268	49415
5.40	46762	47352	47846	48257	48597	48876	49104	49289	49438	49558
5.60	47227	47762	48204	48567	48863	49103	49295	49449	49572	49669
5.80	47636	48117	48510	48828	49084	49288	49450	49577	49676	49754
6.00	47992	48423	48770	49047	49266	49439	49573	49677	49757	49819
6.50	48690	49008	49255	49445	49590	49699	49781	49842	49886	49919
7.00	49167	49394	49563	49688	49779	49845	49892	49926	49949	49966
7.50	49484	49640	49752	49831	49886	49924	49949	49967	49978	49986
8.00	49689	49793	49864	49911	49943	49964	49977	49986	49991	49995
9.00	49896	49937	49963	49978	49987	49993	49996	49998	49999	49999
10.00	49969	49983	49991	49995	49998	49999	49999	50000	50000	50000
11.00	49992	49996	49998	49999	50000	50000	50000	50000	50000	50000
12.00	49998	49999	50000	50000	50000	50000	50000	50000	50000	50000
13.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
14.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
16.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
18.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
20.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
25.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
30.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
35.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
40.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
45.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
50.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000

A(B, γ)

B \ γ (°)	30	32	34	36	38	40	42	44	46	48
0.00	08333	08889	09444	10000	10556	11111	11667	12222	12778	13333
0.01	08433	08995	09556	10118	10679	11240	11801	12361	12922	13482
0.02	08534	09102	09669	10236	10803	11369	11935	12501	13066	13631
0.03	08636	09209	09782	10355	10927	11499	12071	12641	13212	13782
0.04	08738	09317	09896	10475	11053	11630	12207	12783	13358	13933
0.05	08841	09426	10011	10596	11179	11762	12344	12925	13505	14084
0.06	08944	09536	10127	10717	11306	11894	12482	13068	13653	14237
0.07	09048	09646	10243	10839	11434	12028	12620	13211	13801	14390
0.08	09153	09757	10360	10962	11562	12161	12759	13356	13950	14544
0.09	09259	09869	10478	11085	11692	12296	12899	13501	14100	14699
0.10	09365	09981	10596	11210	11821	12432	13040	13646	14251	14854
0.11	09472	10094	10715	11335	11952	12568	13181	13793	14403	15010
0.12	09579	10208	10835	11460	12083	12705	13323	13940	14555	15167
0.13	09687	10322	10955	11587	12215	12842	13466	14088	14707	15324
0.14	09796	10437	11077	11714	12348	12980	13610	14237	14861	15482
0.15	09905	10553	11198	11841	12482	13119	13754	14386	15015	15641
0.16	10015	10669	11321	11970	12616	13259	13899	14536	15170	15800
0.17	10126	10786	11444	12099	12751	13399	14045	14687	15325	15960
0.18	10237	10904	11568	12229	12886	13540	14191	14838	15482	16121
0.19	10349	11022	11692	12359	13022	13682	14338	14990	15638	16282
0.20	10462	11141	11817	12490	13159	13824	14485	15143	15796	16444
0.22	10689	11381	12070	12754	13435	14111	14783	15450	16112	16770
0.24	10918	11623	12324	13021	13713	14400	15082	15759	16431	17098
0.26	11150	11868	12582	13290	13993	14692	15384	16071	16753	17428
0.28	11384	12115	12841	13562	14276	14986	15689	16386	17076	17760
0.30	11621	12365	13103	13836	14562	15282	15995	16702	17402	18094
0.32	11860	12617	13367	14112	14850	15580	16304	17021	17729	18431
0.34	12101	12871	13634	14390	15140	15881	16615	17341	18059	18769
0.36	12344	13127	13903	14671	15432	16184	16928	17664	18390	19108
0.38	12589	13385	14174	14954	15726	16489	17243	17988	18724	19450
0.40	12837	13646	14447	15239	16022	16796	17560	18314	19058	19792
0.42	13086	13909	14722	15526	16320	17105	17879	18642	19395	20137
0.44	13338	14173	14999	15815	16620	17415	18199	18971	19733	20482
0.46	13591	14440	15278	16106	16922	17727	18521	19302	20072	20829
0.48	13847	14708	15559	16398	17226	18041	18844	19635	20412	21177
0.50	14104	14979	15842	16693	17531	18357	19169	19968	20754	21525
0.55	14755	15662	16556	17435	18301	19151	19987	20807	21612	22401
0.60	15416	16355	17279	18187	19079	19954	20812	21652	22475	23280
0.65	16086	17057	18011	18947	19865	20763	21642	22501	23341	24161
0.70	16765	17768	18751	19714	20656	21577	22476	23353	24209	25042

A(B, γ)

B \ γ (°)	30	32	34	36	38	40	42	44	46	48
0.75	17451	18485	19497	20486	21452	22394	23312	24206	25076	25922
0.80	18144	19208	20248	21263	22251	23214	24150	25059	25942	26798
0.85	18843	19937	21003	22042	23052	24034	24986	25910	26804	27670
0.90	19546	20668	21761	22823	23854	24853	25821	26757	27661	28535
0.95	20252	21403	22521	23605	24655	25671	26652	27599	28512	29391
1.00	20961	22139	23281	24386	25454	26484	27478	28435	29355	30238
1.05	21672	22876	24040	25164	26249	27293	28297	29262	30187	31074
1.10	22383	23612	24797	25940	27039	28096	29109	30080	31009	31897
1.15	23094	24346	25552	26711	27824	28891	29912	30887	31819	32706
1.20	23804	25078	26302	27477	28602	29677	30704	31683	32614	33500
1.25	24511	25806	27048	28236	29372	30454	31485	32465	33395	34278
1.30	25215	26530	27787	28988	30132	31220	32253	33233	34161	35038
1.35	25916	27248	28520	29731	30882	31974	33008	33986	34909	35779
1.40	26611	27960	29244	30464	31620	32715	33748	34722	35639	36501
1.45	27301	28664	29959	31187	32347	33442	34472	35441	36351	37203
1.50	27984	29361	30665	31898	33060	34154	35180	36143	37043	37885
1.55	28660	30049	31361	32597	33760	34851	35872	36826	37716	38545
1.60	29328	30727	32045	33284	34445	35531	36545	37489	38367	39183
1.65	29988	31395	32718	33957	35115	36195	37200	38133	38998	39799
1.70	30639	32053	33378	34615	35769	36841	37836	38757	39608	40392
1.75	31280	32699	34025	35260	36407	37470	38453	39360	40195	40963
1.80	31911	33333	34658	35889	37028	38081	39051	39943	40761	41511
1.85	32531	33955	35277	36502	37632	38673	39629	40504	41305	42037
1.90	33140	34564	35883	37099	38219	39246	40187	41045	41828	42539
1.95	33738	35160	36473	37681	38788	39801	40724	41565	42328	43019
2.00	34324	35743	37048	38245	39339	40336	41242	42063	42806	43477
2.10	35458	36866	38153	39325	40388	41350	42218	42998	43699	44327
2.20	36543	37933	39196	40337	41365	42288	43114	43852	44509	45093
2.30	37575	38942	40175	41281	42270	43151	43933	44626	45237	45777
2.40	38554	39894	41092	42159	43105	43941	44677	45324	45890	46385
2.50	39480	40787	41946	42970	43871	44661	45350	45950	46470	46922
2.60	40353	41622	42739	43718	44572	45313	45954	46508	46984	47393
2.70	41173	42401	43473	44404	45209	45901	46495	47003	47435	47803
2.80	41941	43125	44149	45031	45786	46430	46976	47439	47829	48158
2.90	42659	43796	44770	45601	46306	46902	47402	47822	48172	48464
3.00	43328	44415	45338	46119	46774	47322	47778	48156	48468	48726
3.10	43949	44985	45857	46586	47192	47693	48106	48445	48723	48949
3.20	44524	45508	46327	47006	47564	48021	48393	48696	48940	49137
3.30	45056	45986	46754	47383	47894	48309	48643	48910	49125	49296
3.40	45545	46423	47139	47719	48186	48560	48858	49094	49281	49428

A(B, γ)

B \ γ (°)	30	32	34	36	38	40	42	44	46	48
3.50	45995	46820	47485	48018	48443	48779	49043	49250	49412	49538
3.60	46408	47179	47796	48284	48668	48968	49201	49382	49521	49628
3.70	46785	47505	48073	48518	48864	49131	49336	49493	49612	49703
3.80	47129	47798	48321	48725	49035	49271	49451	49586	49687	49764
3.90	47441	48062	48541	48906	49183	49391	49547	49663	49749	49813
4.00	47725	48298	48735	49064	49311	49493	49628	49727	49800	49853
4.10	47982	48510	48907	49202	49420	49580	49696	49780	49841	49885
4.20	48214	48698	49058	49322	49514	49653	49753	49824	49874	49910
4.30	48422	48866	49190	49426	49594	49715	49799	49859	49901	49930
4.40	48610	49014	49306	49515	49662	49766	49838	49888	49923	49946
4.50	48778	49145	49407	49592	49720	49809	49870	49911	49940	49959
4.60	48928	49261	49495	49657	49769	49845	49896	49930	49953	49969
4.70	49061	49362	49571	49713	49810	49874	49917	49945	49964	49976
4.80	49180	49451	49636	49761	49844	49898	49934	49957	49972	49982
4.90	49286	49529	49693	49801	49872	49918	49948	49967	49979	49986
5.00	49379	49597	49741	49835	49896	49935	49959	49974	49984	49990
5.20	49534	49707	49818	49888	49932	49958	49975	49985	49991	49994
5.40	49653	49789	49873	49925	49956	49974	49985	49991	49995	49997
5.60	49744	49850	49913	49950	49972	49984	49991	49995	49997	49998
5.80	49813	49894	49941	49967	49982	49990	49995	49997	49998	49999
6.00	49865	49926	49960	49979	49989	49994	49997	49998	49999	50000
6.50	49942	49971	49986	49993	49997	49999	49999	50000	50000	50000
7.00	49977	49990	49995	49998	49999	50000	50000	50000	50000	50000
7.50	49991	49996	49999	49999	50000	50000	50000	50000	50000	50000
8.00	49997	49999	50000	50000	50000	50000	50000	50000	50000	50000
9.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
10.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
11.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
12.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
13.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
14.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
16.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
18.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
20.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
25.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
30.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
35.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
40.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
45.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
50.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000

A(B, γ)

B \ γ (°)	50	52	54	56	58	60	65	70	75	80
0.00	13889	14444	15000	15556	16111	16667	18056	19444	20833	22222
0.01	14042	14602	15162	15721	16281	16840	18237	19632	21026	22419
0.02	14196	14760	15324	15888	16451	17014	18418	19820	21219	22616
0.03	14351	14919	15488	16055	16622	17188	18601	20009	21413	22813
0.04	14506	15079	15651	16223	16793	17363	18784	20198	21607	23010
0.05	14663	15240	15816	16391	16966	17539	18967	20388	21801	23208
0.06	14820	15401	15982	16561	17139	17715	19151	20578	21996	23405
0.07	14977	15563	16148	16731	17312	17892	19335	20768	22191	23603
0.08	15136	15726	16314	16901	17486	18070	19520	20959	22386	23801
0.09	15295	15889	16482	17072	17661	18248	19706	21150	22581	23999
0.10	15455	16053	16650	17244	17836	18426	19891	21342	22777	24197
0.11	15615	16218	16819	17417	18012	18606	20078	21534	22973	24395
0.12	15776	16383	16988	17590	18189	18785	20265	21726	23169	24594
0.13	15938	16549	17158	17763	18366	18966	20452	21918	23365	24792
0.14	16101	16716	17328	17938	18544	19147	20639	22111	23561	24990
0.15	16264	16883	17500	18112	18722	19328	20827	22304	23758	25189
0.16	16428	17051	17671	18288	18901	19510	21016	22497	23955	25387
0.17	16592	17220	17844	18464	19080	19692	21204	22691	24151	25585
0.18	16757	17389	18016	18640	19259	19875	21393	22885	24348	25784
0.19	16923	17558	18190	18817	19440	20058	21583	23079	24545	25982
0.20	17089	17729	18364	18994	19620	20241	21772	23273	24742	26180
0.22	17423	18071	18713	19351	19983	20609	22153	23661	25136	26576
0.24	17759	18415	19064	19709	20347	20979	22534	24051	25529	26971
0.26	18097	18761	19418	20068	20712	21350	22916	24440	25923	27365
0.28	18438	19109	19772	20429	21080	21722	23299	24830	26316	27758
0.30	18780	19458	20129	20792	21448	22096	23682	25220	26709	28151
0.32	19124	19809	20487	21156	21817	22470	24066	25609	27101	28542
0.34	19470	20162	20846	21521	22188	22845	24450	25999	27492	28932
0.36	19817	20517	21207	21888	22559	23221	24834	26388	27882	29320
0.38	20166	20872	21569	22255	22931	23597	25218	26776	28271	29706
0.40	20516	21229	21931	22623	23304	23974	25603	27164	28659	30091
0.42	20867	21587	22295	22992	23677	24351	25986	27551	29046	30473
0.44	21220	21946	22659	23361	24051	24729	26370	27937	29430	30854
0.46	21573	22305	23024	23731	24425	25106	26753	28321	29814	31232
0.48	21928	22666	23390	24101	24799	25483	27135	28705	30195	31608
0.50	22283	23027	23756	24471	25173	25860	27516	29087	30574	31982
0.55	23174	23931	24672	25397	26106	26800	28465	30035	31513	32903
0.60	24067	24836	25588	26321	27037	27735	29406	30971	32436	33807
0.65	24961	25741	26501	27242	27963	28664	30336	31893	33342	34689
0.70	25853	26643	27411	28157	28881	29585	31254	32800	34229	35550

A(B, γ)

B \ γ (°)	50	52	54	56	58	60	65	70	75	80
0.75	26743	27541	28314	29064	29791	30495	32157	33688	35095	36388
0.80	27628	28432	29210	29962	30689	31392	33045	34557	35938	37200
0.85	28507	29315	30096	30849	31575	32275	33915	35405	36758	37987
0.90	29377	30189	30971	31723	32447	33143	34765	36231	37553	38746
0.95	30237	31051	31832	32583	33302	33993	35595	37033	38322	39478
1.00	31087	31900	32680	33426	34140	34823	36402	37809	39063	40180
1.05	31923	32735	33511	34252	34959	35634	37186	38560	39777	40854
1.10	32745	33554	34325	35059	35758	36423	37945	39284	40462	41498
1.15	33552	34356	35120	35847	36536	37189	38679	39981	41118	42112
1.20	34341	35139	35896	36612	37291	37932	39387	40649	41744	42696
1.25	35113	35903	36651	37356	38022	38650	40068	41289	42342	43250
1.30	35866	36647	37383	38077	38730	39343	40721	41901	42910	43775
1.35	36599	37369	38094	38774	39412	40011	41348	42483	43449	44270
1.40	37311	38070	38781	39447	40069	40652	41946	43037	43958	44737
1.45	38001	38747	39444	40095	40701	41266	42516	43563	44439	45176
1.50	38670	39402	40083	40717	41307	41854	43059	44060	44893	45587
1.55	39316	40033	40698	41314	41886	42416	43574	44529	45318	45971
1.60	39939	40639	41287	41886	42439	42950	44062	44971	45717	46330
1.65	40539	41222	41852	42432	42966	43458	44522	45386	46089	46664
1.70	41115	41780	42391	42953	43468	43940	44957	45776	46437	46973
1.75	41668	42314	42906	43448	43943	44396	45365	46140	46760	47260
1.80	42197	42824	43396	43918	44394	44827	45748	46479	47060	47525
1.85	42703	43310	43862	44364	44819	45233	46107	46849	47338	47769
1.90	43186	43772	44304	44785	45220	45615	46443	47089	47595	47993
1.95	43645	44211	44722	45183	45598	45973	46756	47361	47831	48199
2.00	44082	44627	45117	45557	45953	46308	47046	47612	48048	48386
2.10	44889	45391	45840	46240	46596	46914	47566	48057	48429	48714
2.20	45611	46071	46478	46838	47156	47438	48009	48432	48747	48984
2.30	46252	46670	47036	47358	47641	47889	48385	48745	49009	49204
2.40	46817	47194	47522	47808	48056	48272	48699	49003	49222	49382
2.50	47312	47650	47941	48192	48409	48596	48961	49215	49395	49525
2.60	47743	48043	48300	48519	48707	48868	49176	49387	49534	49637
2.70	48115	48380	48605	48795	48956	49093	49352	49525	49644	49726
2.80	48435	48667	48862	49026	49164	49279	49494	49635	49730	49795
2.90	48708	48910	49078	49218	49334	49431	49608	49722	49797	49848
3.00	48939	49114	49258	49376	49474	49554	49699	49790	49849	49888
3.10	49134	49284	49406	49506	49587	49653	49771	49843	49889	49920
3.20	49296	49425	49528	49611	49678	49732	49827	49883	49919	49943
3.30	49432	49540	49627	49696	49750	49794	49870	49914	49941	49959
3.40	49544	49635	49707	49763	49808	49843	49903	49937	49958	49971

A(B, γ)

B \ γ (°)	50	52	54	56	58	60	65	70	75	80
3.50	49636	49712	49771	49817	49853	49882	49928	49955	49970	49980
3.60	49711	49774	49823	49860	49889	49911	49948	49967	49979	49986
3.70	49772	49824	49863	49893	49916	49934	49962	49977	49985	49990
3.80	49821	49863	49895	49919	49937	49951	49973	49984	49990	49993
3.90	49860	49895	49920	49939	49954	49964	49980	49989	49993	49995
4.00	49891	49919	49940	49955	49966	49974	49986	49992	49995	49997
4.10	49916	49938	49955	49966	49975	49981	49990	49995	49997	49998
4.20	49935	49953	49966	49975	49982	49986	49993	49996	49998	49999
4.30	49951	49965	49975	49982	49987	49990	49995	49997	49999	49999
4.40	49963	49974	49981	49987	49991	49993	49997	49998	49999	49999
4.50	49972	49980	49986	49990	49993	49995	49998	49999	49999	50000
4.60	49979	49986	49990	49993	49995	49997	49998	49999	50000	50000
4.70	49984	49989	49993	49995	49997	49998	49999	50000	50000	50000
4.80	49988	49992	49995	49997	49998	49998	49999	50000	50000	50000
4.90	49991	49994	49996	49998	49998	49999	50000	50000	50000	50000
5.00	49994	49996	49997	49998	49999	49999	50000	50000	50000	50000
5.20	49997	49998	49999	49999	49999	50000	50000	50000	50000	50000
5.40	49998	49999	49999	50000	50000	50000	50000	50000	50000	50000
5.60	49999	49999	50000	50000	50000	50000	50000	50000	50000	50000
5.80	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
6.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
6.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
7.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
7.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
8.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
9.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
10.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
11.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
12.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
13.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
14.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
16.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
18.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
20.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
25.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
30.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
35.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
40.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
45.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
50.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000

A(B, γ)

B \ γ (°)	85	90	95	100	105	110	115	120	125	130
0.00	23611	25000	26389	27778	29167	30556	31944	33333	34722	36111
0.01	23810	25200	26587	27974	29359	30743	32125	33506	34885	36263
0.02	24009	25399	26786	28170	29551	30929	32305	33677	35047	36415
0.03	24208	25599	26984	28366	29743	31115	32484	33848	35209	36566
0.04	24407	25798	27182	28561	29934	31301	32662	34019	35370	36716
0.05	24606	25997	27380	28756	30125	31486	32840	34188	35530	36865
0.06	24805	26196	27578	28951	30315	31670	33018	34357	35689	37013
0.07	25004	26396	27775	29145	30505	31854	33194	34525	35847	37161
0.08	25204	26594	27972	29339	30694	32037	33370	34692	36005	37308
0.09	25403	26793	28169	29532	30882	32220	33545	34859	36161	37454
0.10	25602	26992	28366	29725	31071	32402	33719	35024	36317	37599
0.11	25801	27190	28562	29918	31258	32583	33893	35189	36472	37743
0.12	26000	27388	28758	30110	31445	32763	34066	35353	36626	37886
0.13	26199	27586	28953	30302	31631	32943	34238	35516	36779	38028
0.14	26398	27784	29148	30493	31817	33122	34409	35679	36932	38170
0.15	26596	27981	29343	30683	32002	33301	34580	35840	37083	38310
0.16	26795	28178	29537	30873	32187	33478	34749	36001	37234	38450
0.17	26993	28375	29731	31063	32370	33655	34918	36160	37384	38589
0.18	27191	28571	29924	31251	32553	33831	35086	36319	37532	38727
0.19	27389	28767	30117	31440	32736	34007	35253	36477	37680	38864
0.20	27587	28963	30310	31627	32917	34181	35419	36634	37827	39000
0.22	27981	29353	30693	32001	33279	34528	35749	36946	38118	39269
0.24	28374	29742	31074	32371	33637	34871	36076	37253	38406	39534
0.26	28766	30128	31452	32739	33992	35211	36399	37557	38689	39796
0.28	29157	30513	31828	33104	34343	35547	36718	37857	38969	40054
0.30	29546	30895	32202	33466	34692	35880	37033	38154	39245	40308
0.32	29933	31275	32573	33825	35037	36209	37344	38446	39516	40558
0.34	30318	31653	32940	34181	35378	36534	37652	38734	39784	40805
0.36	30701	32028	33305	34533	35716	36855	37955	39018	40048	41047
0.38	31082	32401	33667	34882	36050	37173	38254	39298	40307	41285
0.40	31460	32771	34026	35227	36380	37486	38549	39574	40563	41520
0.42	31836	33137	34381	35569	36706	37795	38840	39846	40815	41750
0.44	32210	33501	34733	35907	37028	38100	39127	40113	41062	41977
0.46	32581	33862	35081	36241	37346	38401	39410	40376	41305	42199
0.48	32948	34219	35425	36571	37660	38698	39688	40635	41544	42418
0.50	33313	34573	35766	36897	37970	38990	39962	40890	41779	42632
0.55	34211	35442	36601	37694	38725	39701	40627	41508	42347	43150
0.60	35088	36287	37410	38464	39453	40385	41265	42098	42889	43644
0.65	35942	37108	38185	39206	40153	41040	41874	42661	43405	44113
0.70	36772	37902	38941	39920	40824	41667	42456	43197	43895	44557

A(B, γ)

B \ γ (°)	85	90	95	100	105	110	115	120	125	130
0.75	37576	38669	39668	40605	41466	42264	43009	43705	44359	44977
0.80	38353	39408	40373	41260	42078	42833	43534	44187	44798	45373
0.85	39102	40117	41041	41886	42660	43373	44031	44642	45212	45746
0.90	39823	40797	41680	42482	43214	43884	44501	45071	45601	46096
0.95	40515	41448	42288	43048	43738	44367	44943	45474	45966	46423
1.00	41177	42068	42866	43584	44233	44822	45360	45853	46308	46731
1.05	41808	42657	43414	44091	44700	45250	45750	46207	46628	47017
1.10	42410	43217	43932	44568	45138	45651	46115	46538	46926	47284
1.15	42982	43746	44421	45018	45550	46027	46456	46846	47203	47531
1.20	43524	44246	44881	45439	45935	46377	46774	47133	47460	47761
1.25	44036	44717	45312	45834	46294	46703	47069	47399	47698	47973
1.30	44518	45160	45717	46202	46629	47006	47343	47645	47918	48168
1.35	44972	45574	46094	46545	46940	47287	47596	47872	48121	48348
1.40	45398	45962	46446	46864	47228	47547	47829	48081	48308	48514
1.45	45797	46323	46773	47159	47494	47786	48044	48273	48479	48665
1.50	46169	46659	47076	47432	47739	48006	48241	48449	48635	48804
1.55	46516	46971	47356	47684	47965	48209	48422	48610	48778	48930
1.60	46837	47260	47615	47915	48172	48394	48587	48757	48909	49045
1.65	47136	47526	47853	48128	48361	48563	48737	48891	49027	49149
1.70	47411	47771	48071	48322	48534	48716	48874	49012	49134	49244
1.75	47665	47997	48271	48499	48692	48856	48998	49122	49231	49329
1.80	47900	48203	48453	48661	48835	48983	49111	49221	49319	49406
1.85	48114	48392	48620	48808	48965	49098	49212	49311	49398	49475
1.90	48310	48564	48771	48941	49082	49201	49303	49392	49469	49537
1.95	48489	48721	48908	49061	49188	49295	49385	49464	49532	49593
2.00	48652	48863	49032	49169	49283	49378	49459	49529	49589	49643
2.10	48934	49107	49244	49354	49445	49520	49584	49638	49685	49726
2.20	49165	49305	49415	49503	49574	49633	49682	49724	49761	49792
2.30	49352	49464	49551	49620	49676	49722	49760	49792	49820	49844
2.40	49501	49590	49659	49713	49756	49791	49820	49844	49865	49883
2.50	49619	49690	49743	49785	49818	49844	49866	49884	49900	49914
2.60	49712	49767	49808	49840	49865	49885	49901	49915	49927	49937
2.70	49784	49827	49858	49882	49901	49916	49928	49938	49947	49954
2.80	49840	49872	49896	49914	49928	49939	49948	49955	49962	49967
2.90	49884	49907	49925	49938	49948	49956	49963	49968	49972	49976
3.00	49915	49932	49946	49956	49963	49969	49973	49977	49981	49983
3.10	49939	49951	49961	49968	49974	49978	49981	49984	49986	49988
3.20	49956	49965	49973	49978	49982	49985	49987	49989	49990	49992
3.30	49969	49975	49981	49985	49987	49989	49991	49992	49993	49994
3.40	49978	49983	49987	49989	49991	49993	49994	49995	49996	49996

A(B, γ)

B \ γ (°)	85	90	95	100	105	110	115	120	125	130
3.50	49985	49988	49991	49993	49994	49995	49996	49996	49997	49997
3.60	49990	49992	49994	49995	49996	49997	49997	49998	49998	49998
3.70	49993	49994	49996	49997	49997	49998	49998	49998	49999	49999
3.80	49995	49996	49997	49998	49998	49999	49999	49999	49999	49999
3.90	49997	49997	49998	49999	49999	49999	49999	49999	49999	50000
4.00	49998	49998	49999	49999	49999	49999	49999	50000	50000	50000
4.10	49998	49999	49999	49999	50000	50000	50000	50000	50000	50000
4.20	49999	49999	49999	50000	50000	50000	50000	50000	50000	50000
4.30	49999	49999	50000	50000	50000	50000	50000	50000	50000	50000
4.40	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.60	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.70	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.80	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.90	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.20	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.40	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.60	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.80	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
6.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
6.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
7.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
7.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
8.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
9.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
10.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
11.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
12.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
13.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
14.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
16.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
18.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
20.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
25.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
30.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
35.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
40.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
45.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
50.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000

A(B, γ)

B \ γ (°)	135	140	145	150	155	160	165	170	175	180
0.00	37500	38889	40278	41667	43056	44444	45833	47222	48611	50000
0.01	37641	39017	40392	41766	43140	44512	45885	47257	48628	50000
0.02	37780	39144	40505	41865	43223	44580	45936	47291	48646	50000
0.03	37920	39270	40618	41963	43306	44647	45986	47325	48663	50000
0.04	38058	39395	40729	42060	43388	44713	46037	47359	48680	50000
0.05	38195	39520	40840	42157	43469	44779	46086	47392	48696	50000
0.06	38332	39644	40951	42253	43550	44845	46136	47425	48713	50000
0.07	38467	39767	41060	42348	43631	44909	46185	47458	48729	50000
0.08	38602	39889	41169	42442	43710	44974	46234	47491	48746	50000
0.09	38736	40010	41277	42536	43789	45038	46282	47523	48762	50000
0.10	38870	40131	41384	42629	43868	45101	46330	47555	48778	50000
0.11	39002	40251	41490	42722	43946	45164	46377	47587	48794	50000
0.12	39133	40370	41596	42813	44023	45226	46424	47618	48810	50000
0.13	39264	40488	41701	42904	44099	45288	46471	47650	48825	50000
0.14	39394	40605	41805	42994	44175	45349	46517	47680	48841	50000
0.15	39523	40721	41908	43084	44251	45410	46563	47711	48856	50000
0.16	39651	40837	42010	43173	44326	45470	46608	47742	48872	50000
0.17	39778	40951	42112	43261	44400	45530	46653	47772	48887	50000
0.18	39904	41065	42213	43348	44473	45589	46698	47802	48902	50000
0.19	40029	41178	42313	43435	44546	45648	46742	47831	48917	50000
0.20	40153	41290	42412	43521	44618	45706	46786	47861	48931	50000
0.22	40400	41512	42608	43691	44761	45821	46873	47918	48960	50000
0.24	40642	41730	42801	43858	44901	45934	46958	47975	48989	50000
0.26	40881	41945	42991	44022	45038	46044	47041	48031	49017	50000
0.28	41116	42156	43178	44183	45174	46153	47123	48086	49044	50000
0.30	41347	42364	43361	44341	45306	46259	47203	48139	49071	50000
0.32	41575	42568	43541	44496	45436	46364	47282	48192	49097	50000
0.34	41799	42769	43718	44649	45564	46467	47359	48243	49123	50000
0.36	42019	42966	43892	44798	45689	46567	47434	48294	49148	50000
0.38	42235	43160	44062	44945	45812	46666	47509	48343	49173	50000
0.40	42448	43350	44229	45089	45933	46762	47581	48392	49197	50000
0.42	42656	43536	44393	45230	46051	46857	47652	48439	49221	50000
0.44	42861	43719	44554	45369	46166	46949	47722	48486	49244	50000
0.46	43063	43899	44712	45504	46279	47040	47790	48531	49267	50000
0.48	43260	44075	44866	45637	46390	47129	47856	48576	49289	50000
0.50	43454	44248	45018	45767	46498	47216	47922	48619	49311	50000
0.55	43921	44664	45382	46080	46759	47424	48078	48723	49363	50000
0.60	44366	45059	45728	46376	47006	47621	48226	48822	49412	50000
0.65	44787	45433	46054	46655	47238	47807	48365	48915	49459	50000
0.70	45185	45786	46362	46918	47457	47982	48496	49002	49502	50000

A(B, γ)

B \ γ (°)	135	140	145	150	155	160	165	170	175	180
0.75	45562	46119	46652	47166	47663	48146	48619	49083	49543	50000
0.80	45916	46432	46925	47398	47855	48300	48733	49160	49581	50000
0.85	46249	46726	47180	47616	48036	48443	48841	49231	49617	50000
0.90	46561	47001	47419	47819	48204	48577	48941	49298	49650	50000
0.95	46853	47258	47642	48009	48361	48702	49034	49360	49681	50000
1.00	47126	47498	47850	48185	48507	48818	49121	49417	49710	50000
1.05	47380	47721	48043	48349	48642	48926	49201	49471	49736	50000
1.10	47616	47928	48222	48501	48768	49025	49275	49520	49761	50000
1.15	47836	48120	48388	48641	48884	49117	49344	49565	49784	50000
1.20	48038	48297	48541	48771	48991	49202	49407	49607	49804	50000
1.25	48226	48461	48682	48890	49089	49280	49465	49646	49824	50000
1.30	48398	48612	48811	49000	49180	49352	49519	49681	49841	50000
1.35	48557	48750	48931	49101	49262	49417	49567	49714	49857	50000
1.40	48702	48877	49040	49193	49338	49477	49612	49743	49872	50000
1.45	48836	48993	49139	49277	49407	49532	49653	49770	49886	50000
1.50	48957	49098	49230	49353	49470	49582	49690	49795	49898	50000
1.55	49068	49195	49313	49423	49527	49627	49723	49817	49909	50000
1.60	49169	49282	49388	49486	49579	49668	49754	49837	49919	50000
1.65	49260	49361	49455	49543	49626	49705	49781	49855	49928	50000
1.70	49343	49433	49517	49595	49668	49739	49806	49872	49936	50000
1.75	49417	49498	49572	49641	49707	49769	49829	49887	49944	50000
1.80	49484	49556	49622	49683	49741	49796	49849	49900	49950	50000
1.85	49545	49608	49666	49721	49772	49820	49867	49912	49956	50000
1.90	49599	49655	49706	49754	49799	49842	49883	49923	49961	50000
1.95	49647	49697	49742	49784	49824	49861	49897	49932	49966	50000
2.00	49691	49734	49774	49811	49846	49878	49910	49940	49970	50000
2.10	49763	49797	49827	49856	49882	49907	49931	49955	49977	50000
2.20	49821	49846	49869	49891	49911	49930	49948	49966	49983	50000
2.30	49865	49884	49902	49918	49933	49947	49961	49974	49987	50000
2.40	49899	49914	49927	49939	49950	49961	49971	49981	49991	50000
2.50	49926	49936	49946	49955	49963	49971	49979	49986	49993	50000
2.60	49946	49953	49961	49967	49973	49979	49984	49990	49995	50000
2.70	49960	49966	49971	49976	49981	49985	49989	49993	49996	50000
2.80	49972	49976	49979	49983	49986	49989	49992	49995	49997	50000
2.90	49980	49983	49985	49988	49990	49992	49994	49996	49998	50000
3.00	49986	49988	49990	49991	49993	49994	49996	49997	49999	50000
3.10	49990	49991	49993	49994	49995	49996	49997	49998	49999	50000
3.20	49993	49994	49995	49996	49997	49997	49998	49999	49999	50000
3.30	49995	49996	49997	49997	49998	49998	49999	49999	50000	50000
3.40	49997	49997	49998	49998	49998	49999	49999	49999	50000	50000

A(B, γ)

B \ γ (°)	135	140	145	150	155	160	165	170	175	180
3.50	49998	49998	49998	49999	49999	49999	49999	50000	50000	50000
3.60	49999	49999	49999	49999	49999	49999	50000	50000	50000	50000
3.70	49999	49999	49999	49999	50000	50000	50000	50000	50000	50000
3.80	49999	49999	50000	50000	50000	50000	50000	50000	50000	50000
3.90	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.10	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.20	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.30	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.40	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.60	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.70	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.80	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
4.90	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.20	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.40	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.60	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
5.80	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
6.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
6.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
7.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
7.50	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
8.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
9.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
10.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
11.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
12.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
13.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
14.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
16.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
18.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
20.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
25.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
30.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
35.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
40.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
45.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000
50.00	50000	50000	50000	50000	50000	50000	50000	50000	50000	50000

Vita

Arthur Glen Builta [redacted] graduated from the East Detroit High School, East Detroit, Michigan, in 1945, he attended the University of Michigan for one year. In 1946, he entered the United States Naval Academy from which he was graduated in 1950 with the degree of Bachelor of Science. Upon graduation, he accepted his commission in the United States Air Force. In 1951 he attended the University of Chicago for a special one year graduate course in meteorology. His other Air Force assignments prior to entering the Air Force Institute of Technology were as a weather officer with the 15th Weather Squadron on Okinawa during the Korean War; as a Climatic Analyst, Headquarters Air Weather Service; and, as the Pentagon Liaison Officer for the Climatic Center, USAF, Washington, D.C.

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