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SPACE SYSTEM SPECIFIC IMPULSE

by

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JOHN JAY HOPKINS LABORATORY
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ABSTRACT

The high specific impulse (I_{sp}) potentialities of nuclear propulsive devices, which appear so attractive for space vehicle applications, are known to be partly offset by the relatively massive equipment required. A single measure for rating any propulsive system based on its dynamic effectiveness and which includes this mass effect, is derived as a system specific impulse, I_{ss} .

I_{ss} is determined as the equivalent I_{sp} of an idealized system having the same net effectiveness as the actual system but with zero m_s (mass of all structural equipment except payload, m_ℓ). By expressing I_{ss} in terms of actual I_{sp} , r_s (the ratio m_s/m_o , where m_o is the initial gross mass), and the equivalent velocity v_r , a number of important results follow:

1. For chemical systems with $r_s = 0.05$, the ratio I_{ss}/I_{sp} is 0.9 for a satellite mission. For nuclear systems with r_s ranging up to 0.5 , I_{ss}/I_{sp} reduces to as low as 0.3 .

2. For increasingly higher I_{sp}, with r_s and v_r constant, I_{ss} increases rapidly at first, then tends toward a limit $v_r / [g \log (1 - r_s)]$ for $I_{sp} \rightarrow \infty$; which shows that improvements in I_{sp} are highly fruitful up to a point and that little is gained by further increase of I_{sp} unless r_s can be reduced.

3. By representing v_r and $\log G$ as coordinates where $G = m_o/m_\ell$, the design point for any system has vector properties with I_{ss} proportional to the vector argument. The I_{ss} of a multistage vehicle is proportional to the argument of the vector sum of the separate stages.

4. For proportionate multistaging, I_{ss} for the vehicle is identical with that for each stage, showing the vehicle propulsive effectiveness is not improved by staging.

5. The magnitude of G_s (per stage) is determined mainly by the logarithmic form of the basic equation rather than by I_{sp}, so the favorable payload-carrying capability of any stage, whether chemically or nuclearly propelled, is approximately the same (G_{s1} = 3 to 6.)

6. Finally, the principal improvement of the higher I_{sp} of nuclear systems lies in the considerably higher maximum v_r per stage by which more extensive missions (v_r up to 120,000 ft/sec) can be accomplished with one or two stages rather than eight or ten, thus reducing the vehicle mass ratio G from G_{s1}^8 or G_{s1}^{10} to G_{s1} or G_{s1}^2, where G_{s1} is approximately constant for all types.

INTRODUCTION

The specific impulse, I_{sp}, potentialities of nuclear propulsive devices, which range from 800 sec to 15,000 sec and over[1,2], make such devices highly attractive for space application. It is well known, however, that they are massive and consequently not as effective as indicated by their attractive I_{sp} values. Yet in many analyses, as Sutton[1] or Ehricke[2], systems are rated on the basis of I_{sp} without reference to these large mass differences.

I_{sp}, being the total impulse per unit mass of ejected propellant, is an important measure of the capability of an engine-propellant combination to produce "static" thrust. In any application, however, the problem is dynamic, the essential goal being to accelerate a payload to a desired velocity, the thrust being only one of several factors determining this acceleration[3]. A single measure for this dynamic condition is needed. I_{sp} has been used as a convenient approximation but is a usable approximation only as long as the mass of propulsive equipment and structure such as rocket engine, pumps, or tanks (needed to utilize the propellant) is sufficiently small and similar, as for example when comparing two systems using nearly equivalent liquid chemical propellants. When evaluating types with widely differing relative masses, however, I_{sp} alone can be quite misleading. Examples include:

1. In comparing a liquid hydrogen with a kerosene system, the liquid hydrogen has about one-twelfth the density of kerosene and hence requires extra mass for the much larger tank volume as well as special equipment to handle the low temperatures and evaporation. In any application these added masses are necessary and must also be accelerated before the 36% higher I_{sp} can be utilized.

2. In evaluating the comparative merits of a nuclear propulsive system, as mentioned above, the equipment needed for the nuclear system is relatively so much more massive than that for the chemical system that the propellant I_{sp} alone loses all meaning as a measure of the system capability.

PART (1) SINGLE STAGE VEHICLES

(1.1) A Measure for Comparison of Propulsive Systems

This paper derives a single measure for comparing such widely differing schemes of propulsion by defining a vehicle or system specific impulse (represented by I_{ss}) based on the ability of the system to accomplish a reference mission. I_{ss} has the advantage of being a single measure and is derived in a form to be used as an equivalent I_{sp}.

It is observed first that I_{sp}, the engine-propellant specific impulse, is a measure of the system input or applied impulse, and that I_{ss}, the system specific impulse, is a corresponding measure of the input of an ideal system which has the maximum capability of accomplishing the reference mission.

The reference mission is defined in terms of the fundamental equation of motion, from which the equivalent ideal propulsive system is derived. The system specific impulse is determined in terms of this.

(1.2) Useful (Net) Impulse

The end purpose of any reaction propulsive system is to impart a total equivalent useful output impulse to a useful mass or payload, m_ℓ . This useful impulse is

$$I_u = m_\ell v_r \tag{1}$$

where v_r is the equivalent or reference velocity defined later.

(1.3) Applied (Input) Impulse

To obtain this I_u ,

(1) impulse is applied to the system by an "engine" which provides thrust obtained by casting off, stopping, or deflecting a "propellant." The applied impulse is then

$$I_a = \int \dot{m}_p v_e dt = m_p v_e \tag{2}$$

where $\dot{m}_p$ is the time rate of mass ejection; v_e is the "effective exhaust" velocity (of ejection).

(2) the "engine", propellant, and structure necessary to operate the engine and "carry" the useful load all have mass. A portion of the applied momentum, I_a , must be used in accelerating this "non-useful" mass while imparting I_u to m_ℓ.

The specific impulse is defined[4] as

$$I_{sp} = v_e/g \tag{3}$$

By Eq. (2) then

$$I_a = (gI_{sp})m_p \tag{4}$$

We note that I_{sp} is a direct measure of the capability of the propulsive system to produce applied impulse, I_a , rather than useful impulse, I_u . Our purpose here is to determine a measure for the propulsive system, including everything required to impart I_u to m_ℓ . This will then be a measure of the capability of the propulsive system (including necessary structure, etc.) for accomplishing the mission.

(1.4) Equivalent or Reference Velocity

The equation of motion of a reaction propelled vehicle utilizing propellant mass castoff at a rate of $\dot{m}_p$ may be written

$$(m_o - \dot{m}_p t)\left(\frac{dv_a}{dt} + a_i\right) = \dot{m}_p v_e \tag{5}$$

where

v_a = the actual velocity at any time

a_i = the resultant acceleration induced by gravity, drag, etc. expressed as a function of time

m_o = total launch mass

= $m_\ell + m_p + m_s$

m_s = mass of all structure and equipment except m_ℓ and m_p

We define the equivalent or reference velocity as

$$v_r = v_a + \int_0^T a_i dt = \int_0^T \frac{\dot{m}_p v_e dt}{m_o - \dot{m}_p t} = v_e \log \frac{m_o}{m_\ell + m_s} \tag{6}$$

The velocity loss term, $\int_0^T a_i dt$, has been evaluated[3] by Donovan, for example, for typical applications.

(1.5) Relation of I_u to I_a

From Eqs. (1) and (2)

$$I_u = I_a \frac{m_\ell}{m_p} \log \frac{m_o}{m_\ell + m_s} \tag{7}$$

Eq. (7) brings out the distinction between I_a and I_u and shows the relationship between the factors m_o , m_ℓ and m_s and I_u.

From Eq. (1) we have

$$v_r = I_u/m_\ell \tag{8}$$

which shows v_r to be the useful impulse per unit mass of useful load.

Since both I_u and the reference mission are determined by v_r , we concentrate our analysis on it. From Eqs. (3), (4), (7) and (8), v_r may be written in the following convenient forms:

$$v_r = v_e \log \frac{m_o}{m_\ell + m_s} \tag{9}$$

$$= - v_e \log (r_\ell + r_s) \tag{10}$$

$$= - v_e \log \left[\delta + r_\ell (1 - \delta)\right] \tag{11}$$

where

$$r_i = m_i/m_o$$

$$\delta = m_s/(m_s + m_p) = r_s/(1 - r_\ell)$$

and

$$1 = r_\ell + r_p + r_s \tag{12}$$

from

$$m_o = m_\ell + m_p + m_s$$

as defined before.

(1.6) Equivalent Ideal Propulsive System

From the fundamental nature of a reaction propulsive system, there would be no thrust or I_a as shown by Eq. (2) unless a mass of propellant m_p is carried aboard and cast off at a rate $\dot{m}_p$. The most ideal condition possible then is when only m_ℓ, the useful load, and m_p, are present, and m_s, the mass of all the remaining equipment and structure, is zero. For this case, then, m_s, r_s and δ are all zero and Eq. (11) becomes

$$v_{ri} = - v_{ei} \log r_{\ell i} \tag{13}$$

$$= v_{ei} \log G_i \tag{14}$$

where

$$G = 1/r_\ell \tag{15}$$

Eqs. (14) and (15) describe the fundamental relation between the basic performance parameters of the ideal system.

(1.7) System Specific Impulse, I_{ss}

If we compare any actual system described by Eq. (11) with an ideal system which has the same output or useful impulse, then

$$(I_u)_{act.} = (I_u)_i$$

and by Eq. (1) we may also take

$$(v_r)_{act.} = v_{ri} \qquad \text{and} \qquad (r_\ell)_{act.} = r_{\ell i}$$

Thus, by Eqs. (11) and (13)

$$v_{ei} = \frac{v_e \log \left[\delta + r_\ell \; (1 - \delta)\right]}{\log r_\ell} \tag{16}$$

v_{ei} determines the magnitude of an equivalent propellant exhaust velocity which will impart the same useful impulse ($I_u = m_\ell v_r$) to the equivalent ideal system (with $\delta = 0$) as is imparted to the actual system by v_e, the actual effective propellant exhaust velocity.

Since by Eq. (3) $I_{sp} = v_e/g$, then we may define the system specific impulse as

$$I_{ss} = v_{ei}/g$$

Eq. (14) then gives

$$I_{ss} = I_{sp} \frac{\log \left[\delta + r_\ell (1 - \delta)\right]}{\log r_\ell} \tag{17}$$

A graph of I_{ss}/I_{sp} as a function of G is given in Fig. 1 for representative values of δ.

Using Eqs. (3), (11), (12) and (15) we have also

$$I_{ss} = - v_r / \left[g \log (1 - r_p - r_s)\right] \tag{18}$$

$$= - v_r/(g \log r_\ell) \tag{19}$$

$$= v_r/(g \log G) \tag{20}$$

From Eqs. (1) and (12) we have

$$v_r = - v_e \log (1 - r_p) \tag{21}$$

If we define the mission constant as

$$c = \frac{v_r}{v_e} = \frac{v_r}{g\, I_{sp}} \tag{22}$$

then Eq. (21) gives

$$1 - r_p = e^{-c} \tag{23}$$

and Eq. (18) becomes

$$I_{ss} = - v_r / \left[g \log \left(e^{-c} - r_s \right) \right] \tag{24}$$

By Eqs. (11) and (22)

$$c = - \log \left[\delta + r_\ell (1 - \delta) \right] \tag{25}$$

so that Eqs. (15) and (17) give

$$I_{ss} = I_{sp} (c) / \log G \tag{26}$$

(1.8) Limiting Characteristics of I_{ss}

We examine two special cases of Eq. (24).

(1) $I_{sp} \to \infty$

From Eqs. (22) and (23)

$$e^{-c} \to 1$$

and I_{ss} asymptotically approaches a maximum

$$(I_{ss})_{max} = - v_r \left[g \log (1 - r_s) \right] \tag{27}$$

$$= - v_r / \left\{ g \log \left[1 - \delta (1 - r_\ell) \right] \right\} \tag{28}$$

since from Eq. (11) $r_s = \delta (1 - r_\ell)$.

This shows that for any propulsive system I_{ss}, the system capability, is limited to a maximum by r_s, the structural ratio, no matter how high the I_{sp} is.

(2) When the term $(e^{-c} - r_s)$ of Eq. (24) approaches zero, $I_{ss} \rightarrow 0$. For this condition then we have a minimum cutoff

$$(I_{sp})_c = -v_r/(g \log r_s) \tag{29}$$

I_{ss} has only practical value then when

$$I_{sp} > (I_{sp})_c$$

The several curves given in Figs. 2(a), 2(b) and 2(c), based on Eq. (24), illustrate these characteristics, namely, (1) the reduction of I_{sp} to I_{ss} as introduced by the mass ratio r_s; (2) the asymptotic approach of I_{ss} to $(I_{ss})_{max}$; and (3) the minimum usable $(I_{sp})_c$.

(1.9) Physical and Geometric Interpretation of I_{ss}

In addition to the physical meaning as the net propulsive effectiveness of the system, I_{ss} has a fundamental geometrical interpretation.

Since the system net output (i.e., I_u) is determined by v_r (see Eq. (8)) and since the ratio of gross mass to payload is one of the most important design parameters, the functional relation of $v_r = f(\log G)$ defined by the fundamental Eq. (6) or (14) is basic. This relation, stated in terms of I_{ss}, is simply (from Eq. (20)),

$$v_r = gI_{ss} \log G \tag{30}$$

Represented as a two-coordinate system with v_r and $\log G$ as coordinates or vector components, any design point then may be considered as a vector. gI_{ss} is the slope of a line through the design point and the origin and hence is the argument of the vector. This fundamental characteristic of I_{ss}, which is illustrated in Figs. 3 and 4, will be treated more fully in Part (2).

PART (2) - MULTISTAGE VEHICLES

(2.1) Limitations of Single Stage Vehicles

From Eq. (11), if v_e ($= gI_{sp}$) is held constant, then v_r increases as r_ℓ decreases, reaching an asymptotic limit of

$$(v_r)_{max} = gI_{sp} \log \frac{1}{\delta} \qquad (31)$$

This limit differs from that given as Eq. (9) by Malina and Summerfield[5]. Their v_{max} is the velocity at the end of powered flight which is the same as v_a given by Eq. (6) above (or, if $g_o = 0$, by Eq. (21) above.) v_{max} of Ref. 5 is then an operational limit, whereas $(v_r)_{max}$ as given by Eq. (31) is a design limit and consequently of different form.

For a typical chemical system with $I_{sp} = 300$ sec, $\delta = 0.1$ this limit, $(v_r)_{max}$, equals 22,000 ft/sec, as shown in Fig. 3. Even a low altitude satellite requires more than this (30,000 to 35,000).[1] More extensive missions require approximately 35,000 for a 300 mile orbit, 45,000 for a 25,000 mile orbit, and up to 120,000 for a round trip to Mars. This limit is overcome by staging.

From Eq. (17) we observe that as $r_\ell \to 0$ and $v_r \to (v_r)_{max}$, $I_{ss} \to 0$, as shown in Fig. 1. Consequently, if the propulsive system is to have a reasonable magnitude for its effective propulsive capability, namely I_{ss}, it must be operated well below $(v_r)_{max}$. This requires the number of stages to be somewhat greater than $v_r/(v_r)_{max}$.

(2.2) Analysis of I_{ss} for Staged Vehicles

We define the mission or reference velocity per stage as Δv_r. With n proportionately equal stages of the Malina-Summerfield type[5], Eq. (11) gives

$$v_r = n \Delta v_r = - ngI_{sp} \log \left[\delta + r_{\ell s} (1 - \delta)\right] \qquad (32)$$

where $r_{\ell s}$ is now the ratio of stage payload, i.e., $m_{\ell i}$, the mass of all stages it "carries", to m_{oi} , its total mass at launch or stage separation including $m_{\ell s}$.

In Eq. (32) it is assumed that the same values of Δv_r , δ , and $r_{\ell s}$ are common to all n stages. If so, then the ratio of actual payload mass to initial launch mass (in contrast to stage "payload") is

$$r_{\ell o} = \frac{m_\ell}{m_o} = \frac{m_{\ell 1}}{m_{o1}} \cdot \frac{m_{\ell 2}}{m_{o2}} \cdots \frac{m_{\ell n}}{m_{on}} = \prod_{i=1}^{n} \left(\frac{m_{\ell i}}{m_{oi}}\right) = (r_{\ell s})^n \qquad (33)$$

where $m_{\ell 2} = m_{o1}$, etc. by definition of proportionate staging. For a vehicle of n stages, then Eq. (19) gives

$$I_{ss} = - v_r/(g \log r_{\ell o}) \qquad (34)$$

which with Eqs. (13) and (33) becomes respectively

$$I_{ss} = - v_r/\left[g \log (r_{\ell s})^n\right] = - \Delta v_r/(g \log r_{\ell s}) \qquad (35)$$

Combining with Eq. (32) gives

$$I_{ss} = \frac{nI_{sp} \log \left[\delta + r_{\ell s} (1 - \delta)\right]}{\log (r_{\ell s})^n} = I_{sp} \frac{\log \left[\delta + r_{\ell s} (1 - \delta)\right]}{\log r_{\ell s}} \qquad (36)$$

Eqs. (35) and (36) are independent of n , the number of stages, and Eq. (35) is identical with Eq. (17). These show the system specific impulse of the staged vehicle is the same as that for each stage considered separately. This derivation assumes each stage has a common value of $r_{\ell s}$, δ , and I_{sp}. This requirement is shown in the Appendix to be unnecessary.

Restated, Eqs. (35) and (36) show that although higher v_r can be obtained by staging, still the specific propulsive effectiveness of the system, i.e., I_{ss} , is not improved by staging. Actually, in practice I_{ss} of each stage is decreased because δ must be greater because of the added separation equipment. As shown later, small increases in I_{ss} can be obtained indirectly by staging.

From Eq. (25) we have for a single stage vehicle

$$r_{\ell s} = \frac{e^{-\Delta c} - \delta}{1 - \delta} \tag{37}$$

where $\Delta c = \Delta v_r / g I_{sp}$. From Eq. (32) for n stages

$$r_{\ell o} = \left[\frac{e^{-c/n} - \delta}{1 - \delta}\right]^n \tag{38}$$

and

$$G = \frac{1}{r_{\ell o}} = \left[\frac{1 - \delta}{e^{-c/n} - \delta}\right]^n \tag{39}$$

Thus giving for system specific impulse, using Eq. (34),

$$I_{ss} = v_r / (g \log G) \tag{40}$$

$$= v_r \Big/ \left\{ g \log \left[\frac{1 - \delta}{e^{-c/n} - \delta}\right]^n \right\} \tag{41}$$

or

$$I_{ss} = I_{sp} \, (c/n) \Big/ \left\{ \log \left[\frac{1 - \delta}{e^{-c/n} - \delta}\right] \right\} \tag{42}$$

In deriving the I_{ss} of each stage, c would be $n \Delta c$ so that Eq. (42) is also independent of n in agreement with Eq. (36).

(2.3) <u>Geometric Interpretation of I_{ss} for Staging</u>

From Eqs. (32), (35) and (40) we have

$$\Delta v_r = g I_{ss} \log G_s \tag{43}$$

and

$$v_r = n\Delta v_r = gI_{ss}(n \log G_s) \qquad (44)$$

where $G_s = \frac{1}{r_{\ell s}}$ and refers to a single stage.

From Eq. (44) and its graph given in Fig. 4, we observe that staging, represented by integral values of $n \geqq 2$, constitutes a "stretching" of both abscissa and ordinate by the factor n. Although higher magnitudes of v_r are attainable in this way by staging, this is accomplished by increasing the magnitude of log G and hence by downgrading the payload ratio (= 1/G). At the same time, the slope of the line through all points A, B, C and D (Fig. 4) is still the same. Since this equals gI_{ss}, this is in agreement with the result of Eqs. (35) and (36), namely, that staging does not improve the propulsive effectiveness, I_{ss}.

(2.4) Indirect Influence of Staging on I_{ss}

The foregoing analysis resulting in Eqs. (35) and (36) shows that I_{ss} is not influenced by the number of stages, n, as long as v_r, δ, $r_{\ell s}$ and I_{sp} are held constant as n is changed. On the contrary, it is seen from Eq. (41) and from the family of curves for $n = 1, 2, 3$, etc. given in Fig. 4 that design points can be selected for which the slope of the line through the origin and hence I_{ss} is greater (see Point A'.)

(2.5) Maximum I_{ss} Possible with Staging

In particular, it is seen that if Δv_r (per stage) is chosen smaller and n larger, that I_{ss} can be increased to a limit, the tangent to the curve at the origin. This occurs when $\Delta v_r \rightarrow 0$ as $n \longrightarrow \infty$ so that

$$\int n \, d\, v_r = v_r$$

This limit is determined as follows. If v_r and I_{sp} are constant, c $(= v_r/g\, I_{sp})$ will also be constant and Eq. (42) is then a function of n. If n is increased without bound, Eq. (42) gives 0/log 1 which is indeterminant.

The

$$\lim_{n\to\infty} \frac{c/n}{\log\left(\frac{1-\delta}{e^{-c/n}-\delta}\right)} = \lim_{n\to\infty} \frac{\frac{d}{dn}(c/n)}{\frac{d}{dn}\log\left(\frac{1-\delta}{e^{-c/n}-\delta}\right)} = 1-\delta \qquad (45)$$

Hence

$$I_{ss} \to (1-\delta)\, I_{sp} \qquad (46)$$

(2.6) Minimum Number of Stages

An additional point: If the denominator of the term under the log of Eq. (42) is zero or negative, then I_{ss} is respectively zero or negative. Hence, only values for which

$$e^{-c/n} > \delta \qquad (47)$$

are of practical interest. Condition (47) may be written

$$c/n = v_r/n\, gI_{sp} \leqq \log\frac{1}{\delta} \qquad (48)$$

which is Eq. (31) expressed for n stages with $v_r = n\,\Delta v_r$.

From Eq. (48) we can determine the minimum number of stages required for any mission constant c . Table I lists such results for various values of c . Table II lists combinations of I_{sp} and v_r which give the mission constants of Table I.

(2.7) Preferred Number of Stages

From Eqs. (4) and (46) then, it is seen that I_{ss} can be increased by staging within the conditions cited, namely that v_r (= v_r/n) be chosen small and n increased. The ratio, I_{ss}/I_{sp}, given by Eq. (42) is plotted in Fig. 5 for some typical values of c and δ . It is observed that I_{ss}/I_{sp} approaches near its limit for $n = 1$ for $c = 0.2$, $\delta = 0.4$ and for $n = 2$ for $c = 2$, $\delta = 0.1$.

This is generally the case, as c gets larger the minimum number of stages (which must be used as given in Table I) increases and for any number, n , above that minimum, the ratio I_{ss}/I_{sp} is closely equal to the limiting value. Since it is in the interest of low cost, less complication, and higher reliability to keep n down, then n must be confined within these rather narrow limits, between (1) the minimum usable (calculated from Eq. (48), and (2) just enough to realize the principal part of the ratio I_{ss}/I_{sp} .

For the three examples plotted on Fig. 5, the preferred number of stages accordingly is

For $c = 0.2$,	$\delta = 0.4$	$n = 1$	
$c = 2$	$\delta = 0.1$	$n = 2$	
$c = 4$	$\delta = 0.1$	$n = 4$	preferably
		(three could be used)	

Since $c = v_r/g\,I_{sp}$, when considering the advisability of one design over another, it is seen that high I_{sp} which gives low c (such as 0.2), even with the penalty of high structural factor ($\delta = 0.4$), requires only one stage. On the other hand, the low structural factor of $\delta = 0.1$ requires 2 to 4 stages when c is 2 to 4, respectively. This would be the case if the I_{sp} were respectively 1/10 and 1/20 that for $c = 0.2$. Thus increased I_{sp} removes the necessity of staging in many cases, even with high δ .

(2.8) Minimum Gross Mass Ratio

From Eq. (39) we have, after combining with Eqs. (40) and (46) for the limiting case where $n \rightarrow \infty$,

$$G_{\infty} = e^{v_r/gI_{ss}} = e^{c/(1-\delta)} \tag{49}$$

We define $$f_G = G_n/G_{\infty} = \left[\frac{1-\delta}{e^{-c/n} - \delta}\right]^n \Big/ e^{c/(1-\delta)} \tag{50}$$

Thus, Eq. (39) becomes

$$G_n = f_G G_\infty = f_G e^{c/(1 - \delta)} \tag{51}$$

Eq. (49) gives the lowest possible value of G for a given mission constant, c , and structural factor, δ , when the number of stages is infinite. Since an infinite number of stages is practically impossible, then in any actual application G will be larger as given by G_n in Eqs. (39) and (51).

Eq. (50) shows how much greater G_n is than the lowest possible minimum, namely G_∞ . f_G is plotted in Fig. 6 for two typical cases. The dashed line, A , shows a cross plot of f_G vs n for optimistically low values of f_G selected for $n = 4$ and $n = 10$, respectively. $n = 3$ and $n = 8$ could have been chosen, giving curve B , thus nearly doubling f_G but reducing the number of stages by one. It is assumed that the former case would normally be selected in a design, thereby adding one extra stage to reduce the launch mass by nearly 50%. For the analysis, the important points are (1) that f_G is a function of n (whether curve A or B is selected), which can be determined in terms of v_r , δ and I_{sp}; (2) that f_G is at least 1.5 for all missions and designs of interest; and (3) that the longer missions requiring $v_r \to$ 130,000 ft/sec with $I_{sp} = 400$ sec and $\delta = 0.1$, f_G can be as large as 2.8 to 4.6.

For this range of v_r , we conclude the following approximate relation, namely

$$1.5 < f_G < 4.6 \tag{52}$$

Condition (52), with Eq. (51), provides a means of determining the lowest practical magnitude of G_n to be expected in terms of c , δ and n .

(2.9) Apparent Effect of Staging

We can take the relation given by Eq. (10) and apply it in the following way to a vehicle of n stages:

$$v_r = n\Delta v_r = -gI_{spo} \log (1 - r_{po}) \qquad (53)$$

where I_{spo} would be the specific impulse with r_{po} necessary to produce v_r ; and r_{po} is the ratio of the total mass of propellant from all stages to the total launch mass.

If typical values are substituted in Eq. (53), the result gives an I_{spo} which could be interpreted as a system specific impulse, and which would be approximately equal to nI_{sp} , the number of stages times the propellant I_{sp} . From this interpretation it could be concluded that staging increases I_{ss} by a factor n . This result is at variance with that concluded from Eqs. (35) and (36). The distinction is that I_{spo} includes the downgrading of the payload factor (increase in G) mentioned earlier, which is confused with the propulsive system effectiveness.

To show this, Eq. (53) may be written

$$v_r = -gI_{spo} \log (r_{\ell o} + r_{so}) \qquad (54)$$

where r_{so} is the ratio of total structural mass to gross launch mass. But by Eq. (33) this is

$$v_r = -gI_{spo} \log \left[(r_{\ell s})^n + r_{so}\right] \qquad (55)$$

In actual applications r_{so} , the over-all structural mass ratio , is not appreciably different from r_{ss} , that for each separate stage. Likewise, $r_{\ell s}$ and r_{ss} are usually of about the same magnitude. Accordingly, assuming

$$r_{\ell s} = r_{ss} = r_{so} \qquad (56)$$

and Eq. (55) becomes

$$v_r = -gI_{spo} \log \left[(r_{ss})^n + r_{ss}\right] = -gI_{spo} \log \left[r_{ss}(1 + r_{ss}{}^{n-1})\right] \qquad (57)$$

In Eq. (57), for those applications (which include most) where $r_{\ell s}$ is of the order of 0.25 or less, if $n = 3$ we have $(r_{\ell s})^{n-1} << 1$. $(r_{\ell s})^{n-1}$ can be ignored, leaving

$$v_r = - gI_{spo} \log r_{so} = - gI_{spo} \log r_{ss} \tag{58}$$

But since $r_{\ell s} = r_{ss}$ and $r_{\ell s} = (r_{\ell o})^{1/n}$ then approximately

$$I_{spo} = \frac{v_r}{g \log (\frac{1}{r_{\ell o}})^{1/n}} = \frac{nv_r}{g \log \frac{1}{r_{\ell o}}} \tag{59}$$

If I_{spo} is interpreted as I_{ss} , then the result from Eq. (59) (that staging increases the I_{ss} linearly with n , the number of stages) is at variance with that from Eq. (35) (that staging has no direct effect on I_{ss}.) But Eq. (59) requires

(a) $r_{so} = r_{ss}$, the structural factor for each stage to be equal to that for the over-all vehicle; and

(b) $(r_{\ell s})^{n-1} << 1$.

Then, essentially, the payload ratio is negligible and is reduced to $(r_{\ell s})^n$, a much smaller fraction of the launch mass. Because of this, (b) as well as (a) can be utilized to transform Eq. (53) into Eq. (59). The principal point is that $r_{\ell s}$, which is of the order of 0.25 when raised to the nth power (for n stages), is insignificant relative to r_{so} and hence r_{ss} (as by Eq. (59)).

Thus we gain an insight as to what is actually accomplished by staging, namely that the payload factor can be reduced to insignificance with respect to the structural factor, as shown in Eq. (57). This does not occur if $n = 1$, i.e., without staging, as shown by Eq. (32).

While in the foregoing analysis condition (b), that $(r_{\ell s})^n$ be negligible compared to 1, is used to obtain the result $I_{spo} = nI_{sp}$, if $(r_{\ell s})^n$ is small but not negligible, then a similar result can be obtained with $I_{spo} = knI_{sp}$ where now k is a parameter less than 1.

From the foregoing it is evident that I_{spo} (which equals approximately nI_{sp}) is partly a measure of payload downgrading as well as propulsive system effectiveness. It seems in the interest of clarity to keep these separate by retaining Eq. (35) as the measure of propulsive system effectiveness and Eq. (39) and its subsequent implications as the measure of payload downgrading introduced by staging.

CONCLUSION

Introduction of the concept of I_{ss} as a measure of the dynamic effectiveness of a space system makes it possible to rate systems of various types of propulsion with different I_{sp}, δ, v_r and r_ℓ on a single common scale for direct comparison.

It further makes it possible (1) to separate the system net effectiveness I_{ss} of a multistage vehicle from the downgrading of payload-carrying capabilities introduced by the staging, and (2) to recognize that the system effectiveness for each stage is equal to the over-all system I_{ss} if staging is proportionate, or that it is a simple summation if the staging is non-proportionate.

Thus space vehicle analysis involving staging can be more directly concentrated on the characteristics of the individual stage with little more than adding or "stacking" of the stages to obtain the necessary $v_r = n\Delta v_r$ where Δv_r is the limited velocity characteristic per stage.

Following this procedure, it is observed from Fig. 3 that for all curves for single stage vehicles, i.e., with n = 1, the "knee" of the logarithmic curves all occur at approximately the same value of G, i.e., G_s. This effect is brought out clearly by representing v_r non-dimensionally by combining Eqs. (11), (15) and (31) as

$$\frac{v_r}{(v_r)_{max}} = \frac{\log \left[\delta + G_s^{-1}(1 - \delta)\right]}{\log \delta} \tag{60}$$

This equation is independent of I_{sp} and v_r and shows that the relative velocity increment per stage is a function only of δ and G. A graph of Eq. (60) given in Fig. 7 further shows the rather limited effect introduced by δ when $v_r/(v_r)_{max}$ is considered as a function of G_s, even for a wide range of values of δ.

Thus it becomes evident that regardless of magnitude of I_{sp}, whether as low as 250 sec as for some chemical systems or as high as 15,000 for a nuclear system, because of the logarithmic nature of the basic dynamic equation of an accelerating propellant reaction system, the payload-carrying capability ($r_{\ell s} = G_s^{-1}$) is limited, for reasonably practical magnitudes of I_{ss} (as indicated from Fig. 1), to a narrow range of values approximately given by the inequality

$$3 < G_{s1} < 6 \tag{61}$$

Recalling that for a vehicle of n stages the over-all mass to payload ratio, by Eq. (32), is

$$G = (G_{s1})^n \tag{62}$$

it is evident that using Eq. (62), the inequality (61) becomes

$$(3)^n < G < (6)^n \tag{63}$$

where $n = v_r/\Delta v_r$.

From this it becomes evident where the main advantage of massive, high I_{sp} systems, such as potential nuclear types, lies; and we reach the important conclusion that while nuclear systems will have about the same magnitude for the stage mass factor, G_s, with them it is possible to reduce the number of stages by an order of magnitude and hence the over-all mass ratio G by an <u>exponential</u> order of magnitude. Expressed numerically for a more extensive mission (v_r = 120,000 ft/sec), this reduces G (if $G_s = 4$) from 4^8 to 4 or from 4^{10} to 4^2. This is a reduction in the gross takeoff mass, for a particular payload mass, of more than 10^5.

APPENDIX

Non-proportionate Staging

The preceding derivations represented by Eq. (32) and those following are based on the assumption of proportionate staging of the Malina-Summerfield type[4] where v_r , δ , I_{sp} and $r_{\lambda's}$ are the same for each stage.

If these latter quantities are not the same per stage, Eq. (44) instead of being determined by an n-factor stretching of ordinates and abscissas will be obtained from addition of a number of unequal portions, thus

$$v_r = \sum_{i=0}^{n} \Delta v_{ri} \tag{64}$$

From (1) Eq. (64) which shows v_r to be the resultant of a sum of Δv_{ri} , and (2) the simple rule for addition of logarithms $\log G_1 + \log G_2 = \log G_1 G_2$ we have, since Eq. (20) gives $v_r = f(\log G)$, that the resultant of a typical design point such as D on Fig. 4 can be represented as a resultant vector with components $(v_r)_D$ and $(\log G)_D$. Thus $(gI_{ss})_D$ is the argument of this resultant vector, which in terms of its components is given by

$$I_{ss} = \frac{\sum_{i=0}^{n} \Delta v_{ri}}{g \sum_{i=0}^{n} \log G_i} \tag{65}$$

where

$$G_i = 1/r_{si}$$

and

$$\log G = \sum_{i=0}^{n} \log G_i \tag{66}$$

Eqs. (64), (65) and (66) show that both v_r and I_{ss} are obtainable from vector addition processes and that any combination which results in a particular vector is equally effective. This is illustrated by the two different stagings by which Point D on Fig. 4 is obtained, one with proportionate staging and one without.

FWR:db

TABLE I

Minimum Number of Stages

δ	log 1/δ	0.5	1.0	2.0	5.0	10.0
0.05	3.00	1	1	1	2	4
0.1	2.30	1	1	1	3	5
0.2	1.61	1	1	2	4	7
0.3	1.21	1	1	2	5	9
0.4	.917	1	2	3	6	11
0.5	.694	1	2	4	8	15
0.6	.511	1	2	4	10	20

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TABLE II

Typical Magnitudes of c $(=v_r/gI_{sp})$*

I_{sp}	v_r = 30,000	60,000	90,000	130,000	200,000 ft/sec
400	2.33	4.65	7.00	10.1	15.5
600	1.55	3.10	4.65	6.70	10.3
800	1.17	2.33	3.50	5.05	7.70
1000	.93	1.86	2.80	4.03	6.22
2000	.47	.93	1.40	2.02	3.11
4000	.23	.47	.70	1.01	1.55
6000	.16	.31	.47	.67	1.03

*If I_{ss} replaces I_{sp}, c becomes c^s.

LIST OF FIGURES

1. Variation of I_{ss}/I_{sp} for typical values of structural mass factor and stage mass ratio.

2a. Reduction of I_{ss} introduced by r_s for v_r = 12,000 ft/sec

b. Reduction of I_{ss} introduced by r_s for v_r = 36,000 ft/sec

c. Reduction of I_{ss} introduced by r_s for v_r = 120,000 ft/sec

3. Typical curves illustrating the significance of I_{ss} as a single measure for performance representing combinations of I_{sp}, δ and n.

4. Vector summation for proportionate and non-proportionate staging.

5. Influence of number of stages on I_{ss}/I_{sp}.

6. Mass increase factor for $n < \infty$

7. Non-dimensional stage performance.

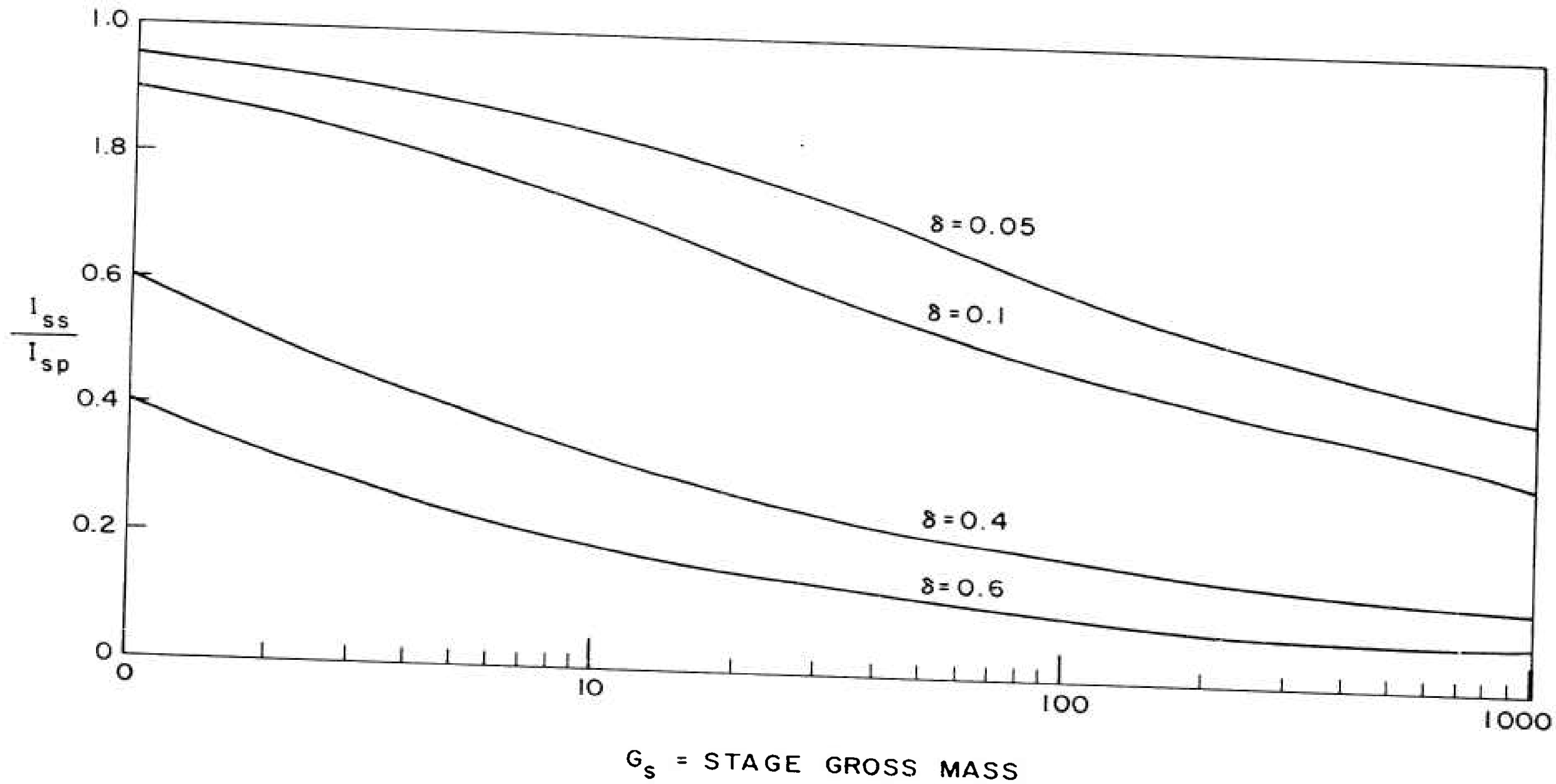
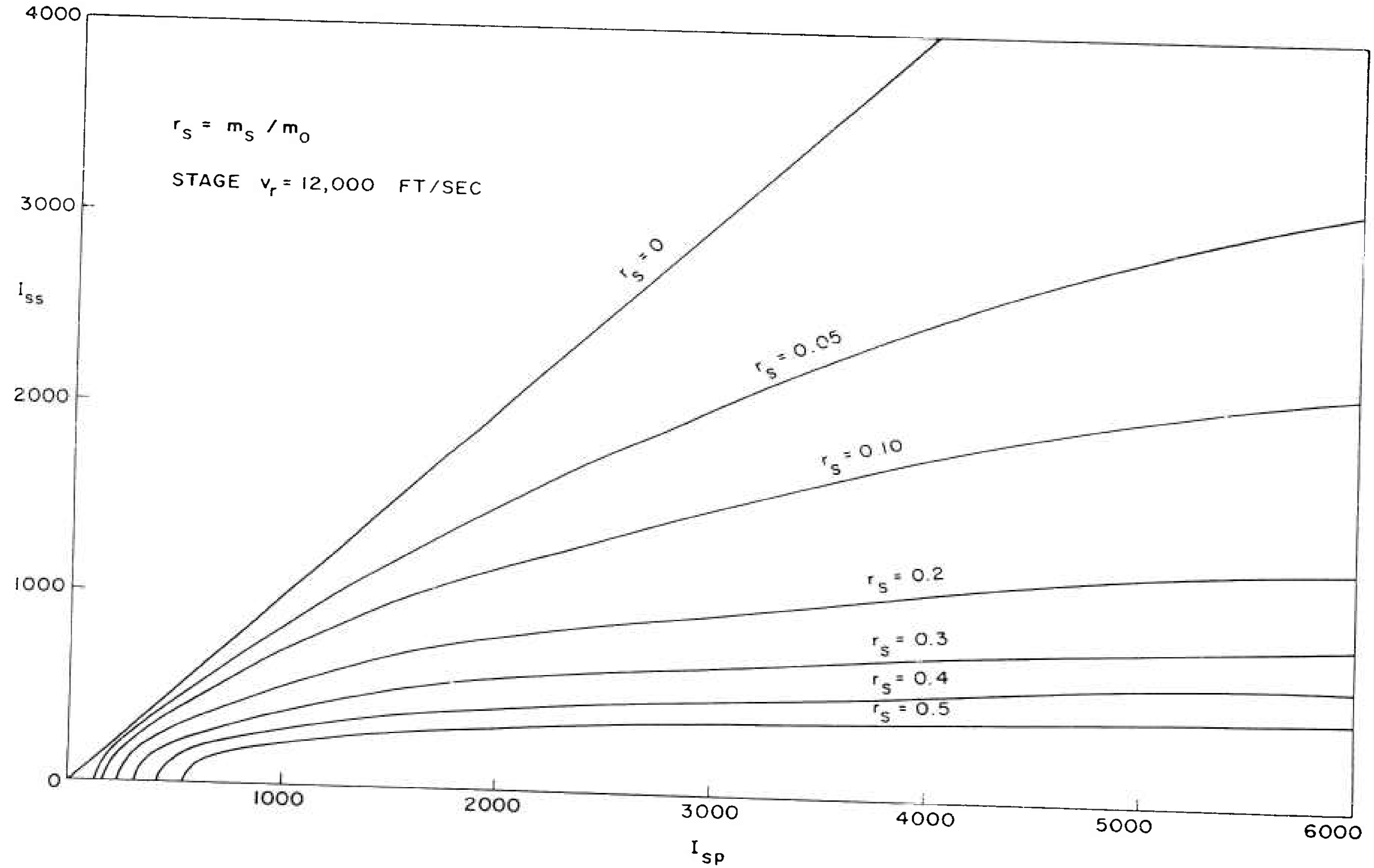
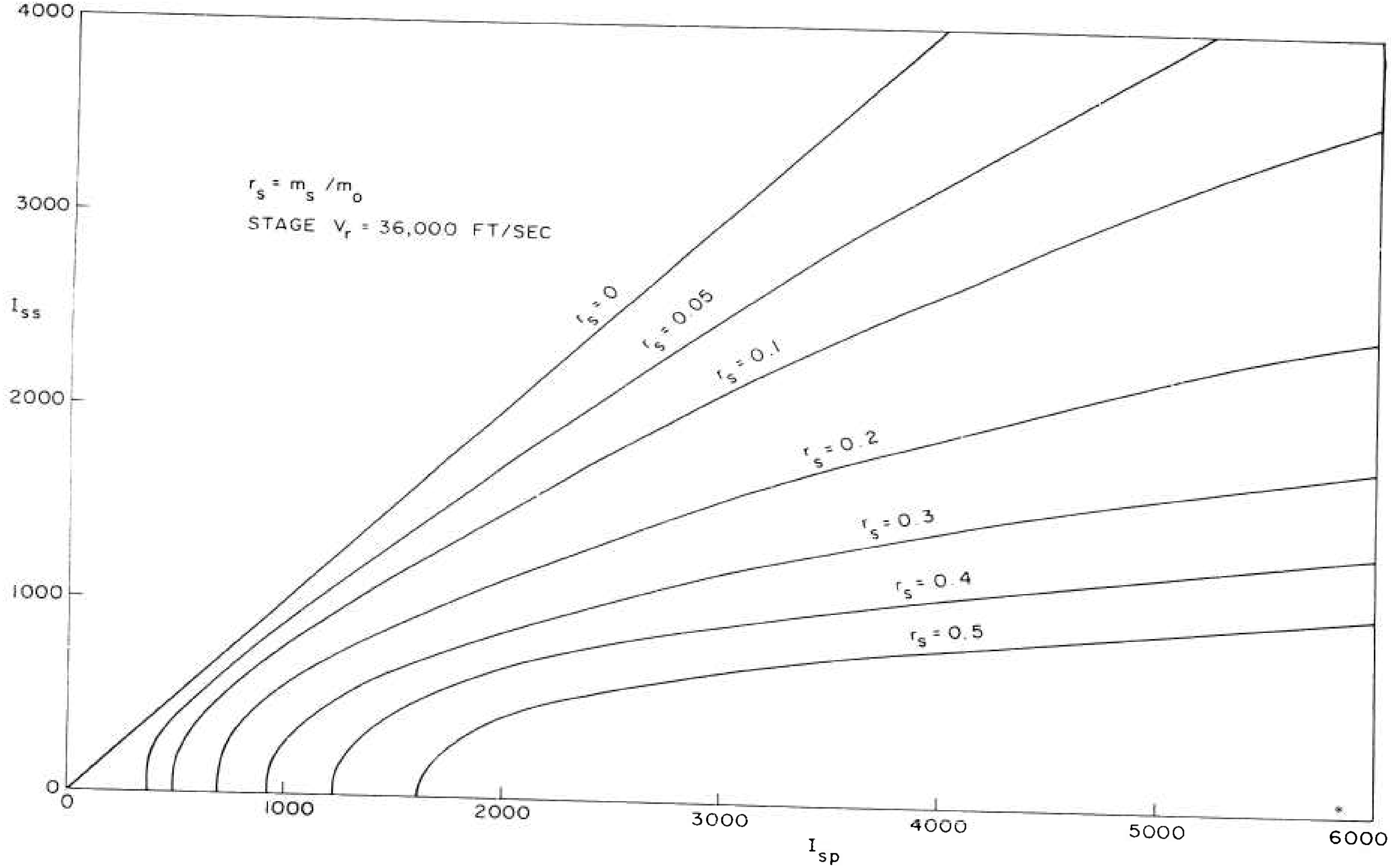

1.0
1.8
0.6
0.4
0.2
0
I_{ss} / I_{sp}
δ = 0.05
δ = 0.1
δ = 0.4
δ = 0.6
0
10
100
1000
G_s = STAGE GROSS MASS

Fig. 1



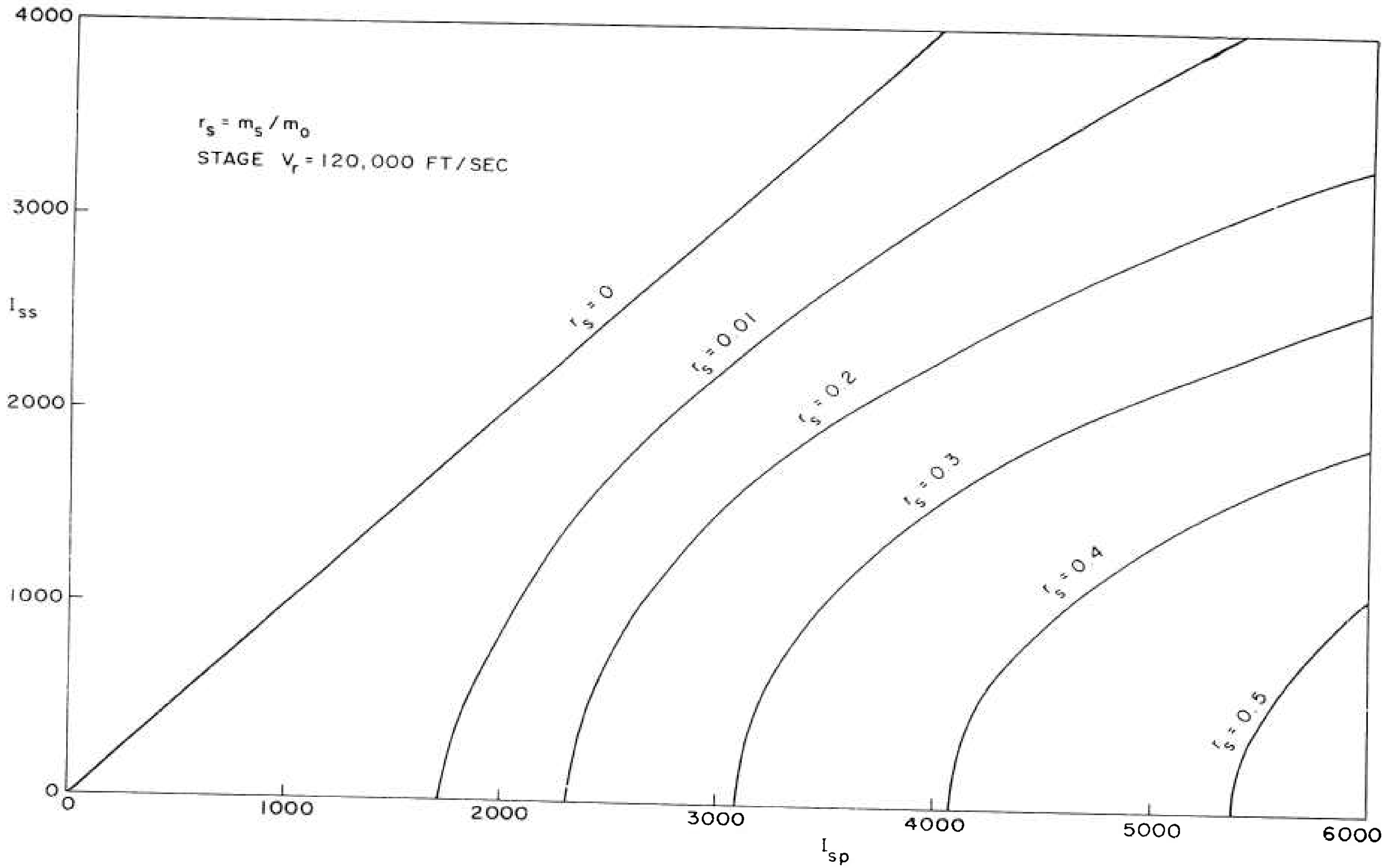
$r_s = m_s / m_0$
STAGE v_r = 12,000 FT/SEC
4000
3000
2000
1000
0
I_{ss}
$r_s = 0$
$r_s = 0.05$
$r_s = 0.10$
$r_s = 0.2$
$r_s = 0.3$
$r_s = 0.4$
$r_s = 0.5$
1000
2000
3000
4000
5000
6000
I_{sp}

Fig. 2a



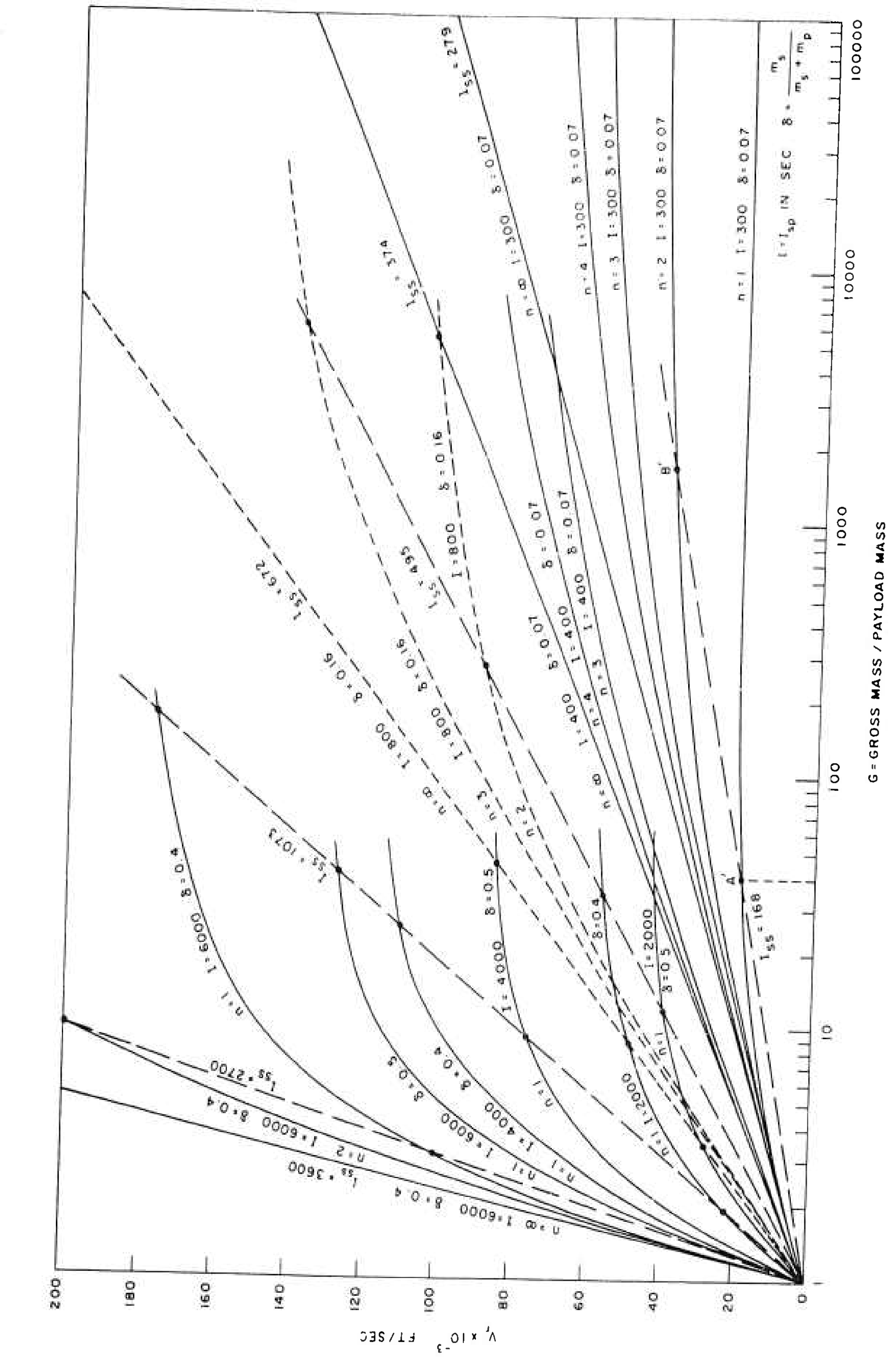
r_s = m_s / m_o
STAGE V_r = 36,000 FT/SEC
r_s = 0
r_s = 0.05
r_s = 0.1
r_s = 0.2
r_s = 0.3
r_s = 0.4
r_s = 0.5
I_ss
I_sp
0
1000
2000
3000
4000
5000
6000

Fig. 2b



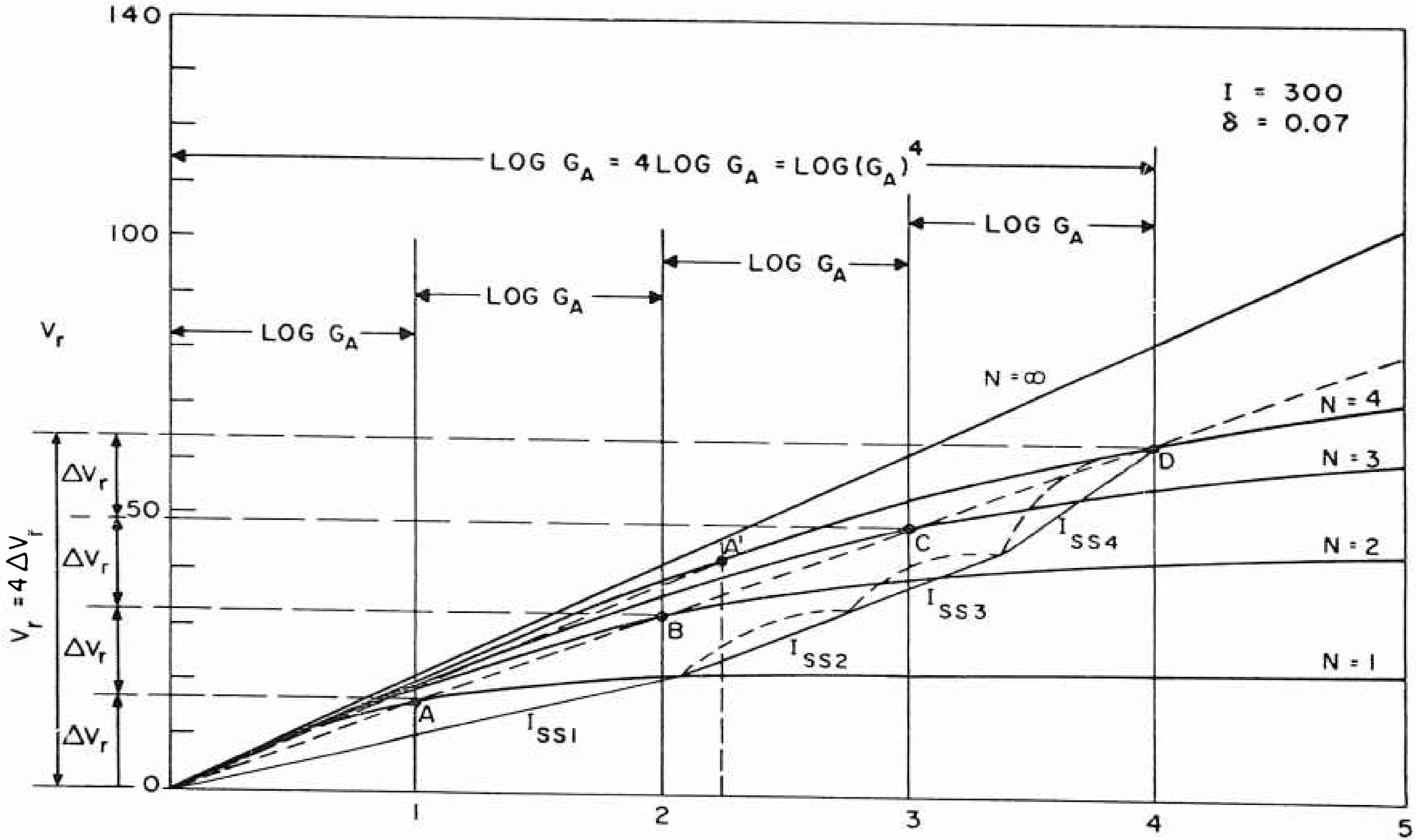
$r_s = m_s / m_o$
STAGE V_r = 120,000 FT/SEC
4000
3000
2000
1000
0
I_{ss}
0
1000
2000
3000
4000
5000
6000
I_{sp}
$r_s = 0$
$r_s = 0.01$
$r_s = 0.2$
$r_s = 0.3$
$r_s = 0.4$
$r_s = 0.5$

Fig. 2c



G = GROSS MASS / PAYLOAD MASS
V, x 10^-3 FT/SEC
I = I_sp IN SEC δ = m_s / (m_s + m_p)
0
20
40
60
80
100
120
140
160
180
200
10
100
1000
10000
100000
n = ∞ I = 6000 δ = 0.4
I_ss = 3600
n = 2 I = 6000 δ = 0.4
I_ss = 2700
n = 1 I = 6000 δ = 0.4
n = 1 I = 6000
δ = 0.5
n = 1 I = 4000
δ = 0.4
n = 1
I_ss = 1073
I = 4000 δ = 0.5
n = 1 I = 2000
n = 1
I = 2000
δ = 0.4
δ = 0.5
I_ss = 168
A'
B'
n = ∞ I = 800 δ = 0.16
I_ss = 672
n = 3 I = 800 δ = 0.16
n = 2
I_ss = 495
I = 800 δ = 0.16
I_ss = 374
n = ∞ I = 400 δ = 0.07
n = 4 I = 400 δ = 0.07
n = 3 I = 400 δ = 0.07
n = ∞ I = 300 δ = 0.07
I_ss = 279
n = 4 I = 300 δ = 0.07
n = 3 I = 300 δ = 0.07
n = 2 I = 300 δ = 0.07
n = 1 I = 300 δ = 0.07

Fig. 3



LOG G

$$G = \frac{\text{GROSS MASS}}{\text{PAYLOAD MASS}}$$

Fig. 4

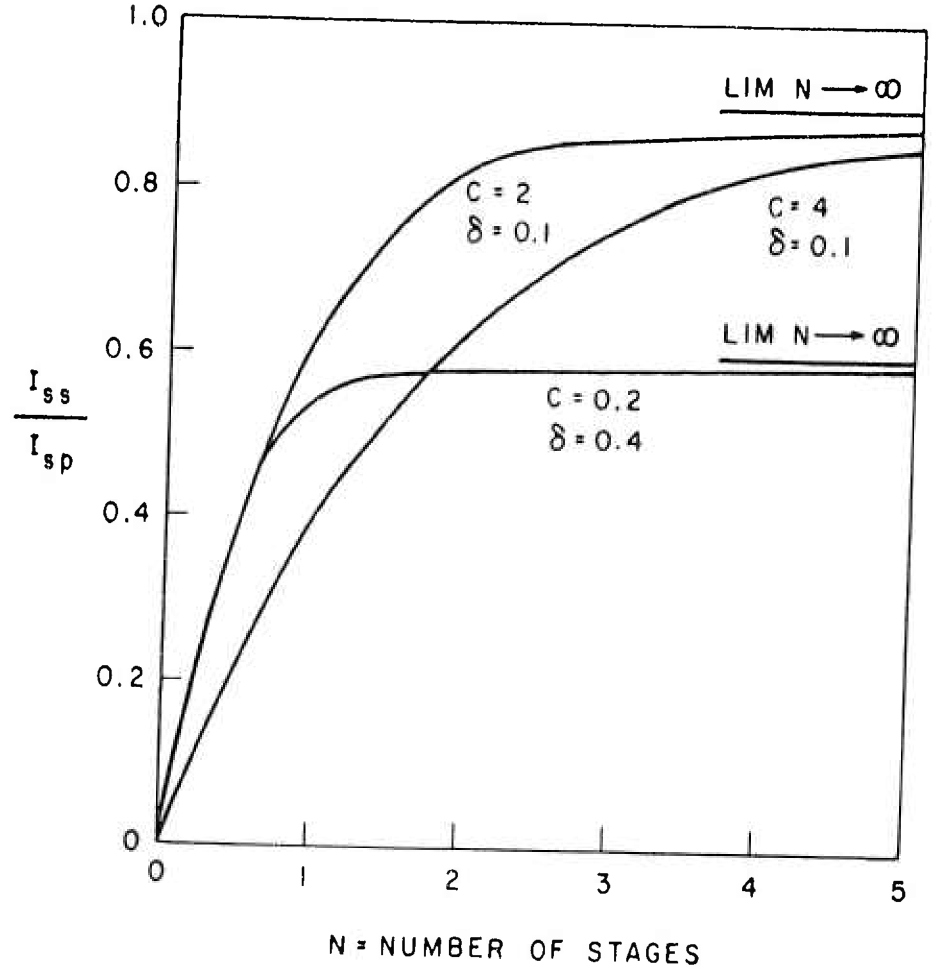
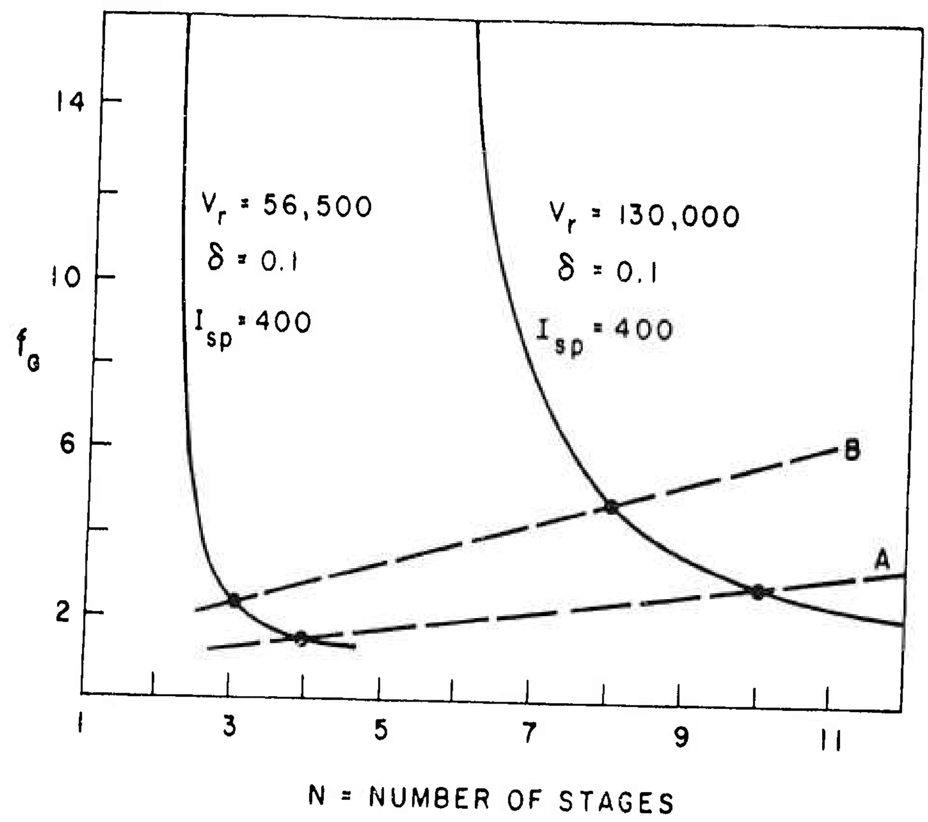

1.0
0.8
0.6
0.4
0.2
0
Iss
Isp
LIM N → ∞
C = 2
δ = 0.1
C = 4
δ = 0.1
LIM N → ∞
C = 0.2
δ = 0.4
0
1
2
3
4
5
N = NUMBER OF STAGES

Fig. 5



14
10
6
2
f_G
V_r = 56,500
δ = 0.1
I_{sp} = 400
V_r = 130,000
δ = 0.1
I_{sp} = 400
B
A
1
3
5
7
9
11
N = NUMBER OF STAGES

Fig. 6

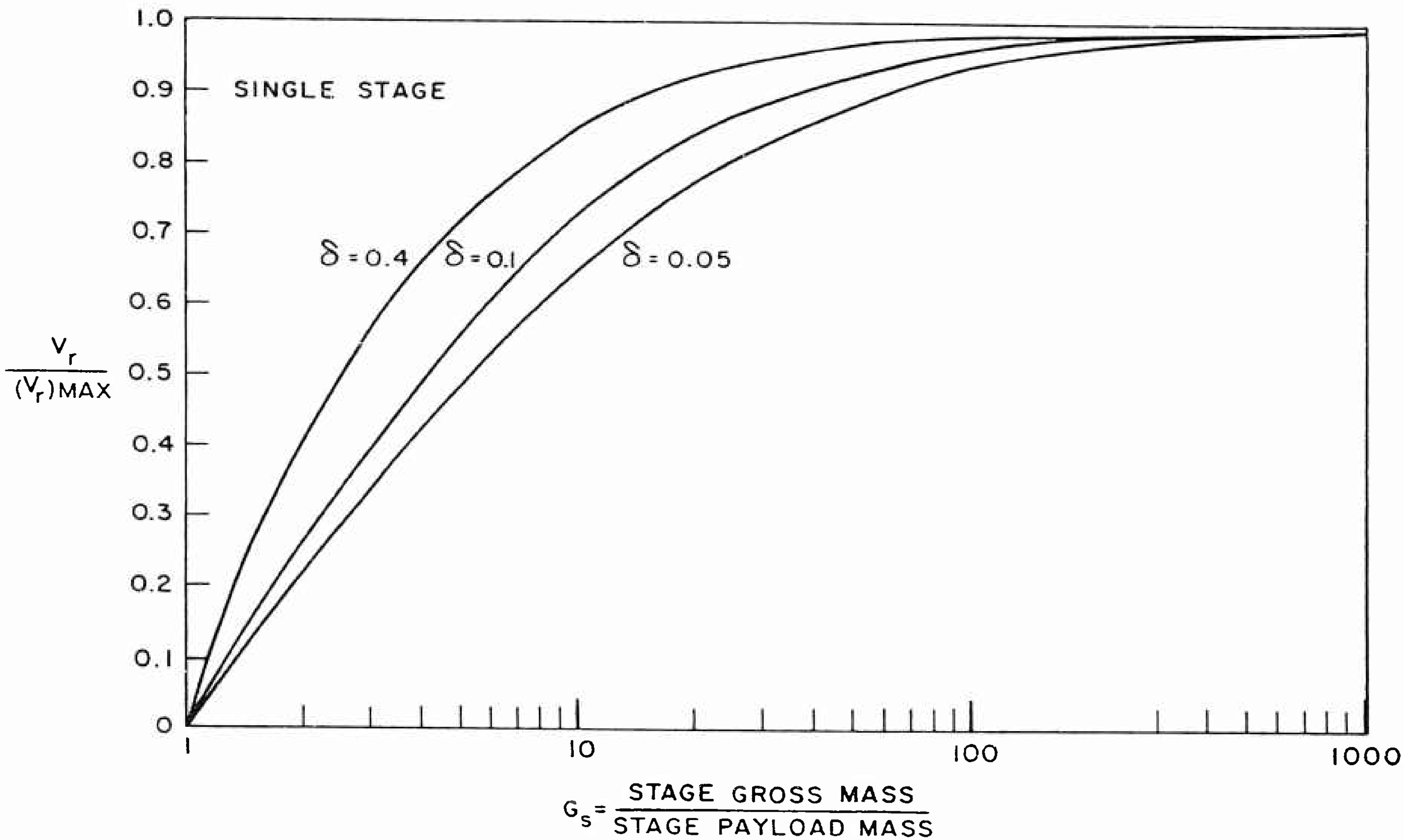

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$\frac{V_r}{(V_r)MAX}$
SINGLE STAGE
δ = 0.4
δ = 0.1
δ = 0.05
1
10
100
1000
$G_s = \frac{\text{STAGE GROSS MASS}}{\text{STAGE PAYLOAD MASS}}$

Fig. 7

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