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George B. Dantzig

OPERATIONS RESEARCH CENTER

INSTITUTE OF ENGINEERING RESEARCH

UNIVERSITY OF CALIFORNIA - BERKELEY

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COMPACT BASIS TRIANGULARIZATION FOR THE SIMPLEX METHOD

by George B. Dantzig

Alex Orden was the first to point out that the inverse of the basis in the simplex method serves no function except as a means for obtaining the representation of the vector entering the basis and for determining the new price vector. For this purpose one of the many forms of "substitute inverses" (such as the well known product form of the inverse) would do just as well, in fact may have certain advantages in computation.

Harry Markowitz was interested in developing, for a sparce matrix, a substitute inverse with as few nonzero entries as possible. He suggested several ways to do this approximately. For example, the basis could be reduced to triangular form by successively selecting for pivot position that row and that column whose product of nonzero entries (excluding the pivot) is minimum. He also pointed out that, for bases whose nonzeros appear in a band on staircase about the diagonal, proper selection of pivots could result in a compact substitute inverse with no more nonzeros than the original basis.

We shall adopt Markowitz's suggestion. However, instead of recording the successive transformations of one basis to the next in product form, we shall show that it is efficient to generate each substitute inverse in turn from its predecessor. The substitute inverse remains compact, of fixed size. Thus "reinversions" are unnecessary (except in so far as they are needed to restore loss of accuracy due to cumulative round off error).

The procedure which we shall give can be applied to a general $m \times m$ basis without special structure. As such, it is probably competitive with the standard product form, for it may have all of its advantages and none of its disadvantages. With certain matrix structures, moreover, it appears to be particularly attractive.

We shall focus our remarks on <u>staircase structures</u>. The reader will find no difficulty in finding an equally efficient way to compact block-angular structures. Letting B_{ij} be a submatrix of the basis, a basis B with staircase structure has, for example, the form:

$$(1) \qquad B = \begin{bmatrix} B_{11} & & & \\ B_{21} & B_{22} & & \\ & B_{32} & B_{33} & \\ & & B_{43} & B_{44} \end{bmatrix}$$

In (2), the marks x, *, $\textcircled{x}$ indicate the staircase pattern of nonzero entries in the basis-matrix B. P_s is some column of coefficients not in the basis. The asterisks along the diagonal mark the successive pivot positions. It is assumed (and this need not be true) that the basis can be reduced to triangular form by pivoting successively on the lower right-hand element of each submatrix formed by deleting the preceding pivot row and column. Each pivot operation consists in using the assumed nonzero diagonal term to eliminate the column variable from all nonzero terms above the diagonal only. The symbol $\textcircled{x}$ indicates the resulting position of zero coefficients above the diagonal.

$$
(2) \qquad B = \begin{bmatrix} * & \textcircled{x} & \textcircled{x} & & & \\ x & * & \textcircled{x} & & & \\ x & x & * & \textcircled{x} & & \\ x & x & x & * & & \\ & & & x & * & \textcircled{x} \\ & & & x & x & * \end{bmatrix} ; \quad P_s = \begin{bmatrix} \\ \\ x \\ x \\ x \\ x \end{bmatrix}
$$

(drop) (enter)

Let T be the resulting triangularized matrix; it has the form (3). Note particularly that the pattern of nonzeros in T is precisely the same as the pattern of nonzeros on and below the main diagonal of the original basis B and that P_s^*, the transform of P_s under the same row operations, may have nonzeros in its leading components.

$$
(3) \quad T = \begin{bmatrix} * & & & & & \\ x & * & & & & \\ x & x & * & & & \\ x & x & x & * & & \\ & & & x & * & \\ & & & x & x & * \end{bmatrix} ; \quad P_s^* = \begin{bmatrix} x \\ x \\ x \\ x \\ x \\ x \end{bmatrix}
$$

(drop) (enter)

Since T is obtained from B by a sequence of operations on rows, this is the same as multiplying B on the left by a succession of elementary matrices so that

$$
(4) \qquad T = E_1 E_2, \ldots, E_m B .
$$

Here $E_m = E_6$ represents an elementary matrix corresponding to a pivot in row 6. Thus the first pivot operation is the same as multiplying B on the left by

$$(5) \qquad E_6 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & p_{56} \\ & & & & & 1 \end{bmatrix}$$

where p_{56} is selected so that row 6, when multiplied by p_{56} and added to row 5, will cause the element (5, 6) of the matrix to vanish. Since no eliminations are required in column 5, E_5 is an identity matrix. Next, E_4 will be similar to E_6 except with one nonzero entry p_{34} for element (3, 4). E_3 will have at most two nonzero entries above the diagonal p_{13}, p_{23}, corresponding to the factors required to eliminate elements (1, 3) and (2, 3) from the matrix using row 3. Similarly E_2 will have an entry p_{12}, and E_1 will be an identity matrix. Since each elementary matrix E_i is an identity matrix except for nonzero entries above the diagonal of column i, we may, for purposes of compact recording, simply list side by side the entries in column 1 of E_1, in column 2 of E_2, etc. We shall refer to this typical product form record of the transformations as the E-structure. For our example

$$(6) \qquad E\ (\text{Structure}) = \begin{bmatrix} 1 \\ \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} p_{12} \\ 1 \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} p_{13} \\ p_{23} \\ 1 \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ p_{34} \\ 1 \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ 1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ p_{56} \\ 1 \end{bmatrix}$$

Note again that the pattern of nonzeros in the E-structure (excluding the units on the diagonal) is precisely the same as the pattern of nonzeros above the main diagonal of the original basis B. Thus the statement in product form of the nonzero coefficients in the transformations E_i necessary to reduce a basis to triangular form T and the record of nonzeros in T have as compact a representation as the original basis.

We give the formulae for the determination of the set of simplex multipliers (or pricing vector) π and the representation $\overline{P}_s$ of the vector P_s entering the basis, when E_i and T are known. Let γ be the vector of coefficients of the cost form for the basic variables, then by definition

(7) $$\pi B = \gamma \ .$$

If now we define π^* by the relation

(8) $$\pi^* T = \gamma$$

then, it is easy to see, by (4), that

(9) $$\pi = \pi^* E_1 E_2, \ldots, E_m \ .$$

Because T is triangular, π^* can be directly computed from (8) and π from (9) by applying to π^* the transformations $E_1, E_2, \ldots$ in turn on the right.

Having obtained π, we can by the usual "pricing out" procedure determine the vector P_s to enter the basis by

(10) $$\pi P_s = \min_j \pi P_j < 0 \ .$$

By definition, the representation $\overline{P}_s$ of P_s in terms of the basis satisfies

$$(11) \qquad B\overline{P}_s = P_s \ .$$

If now we define P_s^* by the relation

$$(12) \qquad P_s^* = E_1E_2, \ldots, E_mP_s$$

then, it is again easy to see, by (4) and (11), that

$$(13) \qquad T\overline{P}_s = P_s^* \ .$$

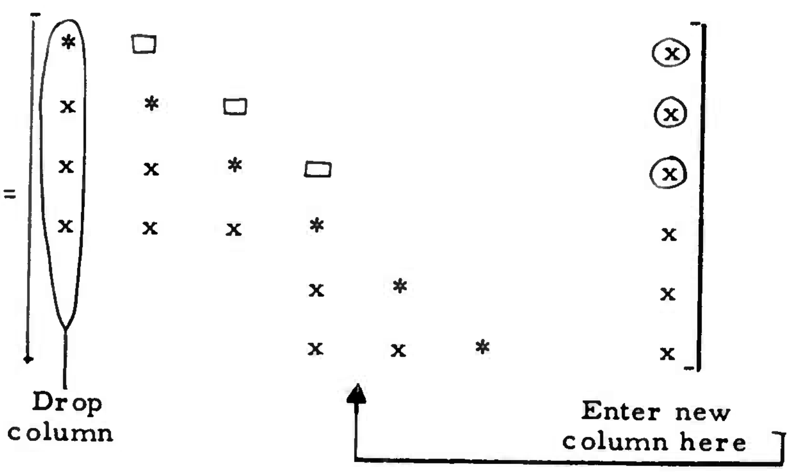
P_s^* is easily computed by (12) and, because T is triangular, $\overline{P}_s$ is computed by direct solution of (13).

Given $\overline{P}_s$ and the basic feasible solution, the usual rules are next applied to determine the vector P_r to drop from the basis and to determine the basic feasible solution for the next iteration. We shall omit these steps assuming they are known to the reader.

Our problem now becomes one of "up-dating" our substitute inverse. This of course could be done by succession of pivot operations above the diagonal such as we described earlier. But this is not very efficient computationally. We shall show instead an efficient procedure for easily modifying the E-structure and T matrix of one iteration to obtain those of the next.

Let us assume in our example, see (2), that the vector P_s entering the basis, if entered in its proper position in the staircase array, would be located, say, between either vectors P_4 and P_5 of the basis (or vectors P_3 and P_4), and let us suppose that vector P_1 is to be dropped. Starting with the columns of B and P_s <u>after</u> they have been transformed by the row operations $E_1E_2, \ldots, E_m$, namely with T and P_s^* as shown in (3), our objective is to triangularize the matrix formed by deleting the first column and introducing P_s^*, say, between columns 4 and 5 (actually between columns 3 and 4 would be less work). The row operations that accomplish this are to create zeros in the first three rows of P_s^* column by successively adding first a multiple of row 2 to row 1, next a multiple of row 3 to row 2, and then a multiple of row 4 to row 3. We shall denote these single row transformations by E_2^1, E_3^2, E_4^3. For the present we have assumed above that the second, third and fourth components of P_s^* are nonvanishing (this need not be the case). The results of these operations are shown in (14) where * indicate the elements of the previous diagonal and ▭ those of the new diagonal.

(14) New T =



The relationship between the new T and the new B may be written

(15) $$(\text{New } T) = E_4^3 E_3^2 E_2^1 E_1 E_2 E_3 E_4 E_5 E_6 \ (\text{New } B) .$$

If, however, the column to be dropped were $j = 4$ (instead of $j = 1$), it would be necessary to eliminate the ▭ element in column 3 and then the ones in column 2 by additional transformations of the type E_i^{i-1} , say $\bar{E}_3^2$, $\bar{E}_2^1$, in this case

$$(\text{New } T) = \bar{E}_2^1\bar{E}_3^2E_4^3E_3^2E_2^1E_1E_2E_3E_4E_5E_6 \; (\text{New } B) \; .$$

We have shown that the new T can be obtained by applying to the previous product of the E_i a succession of row operations of the form E_i^{i-1} where, in general, we have denoted by E_i^k an elementary matrix corresponding to adding a multiple of row i to row k. Our objective, however, has not been accomplished until we have shown how to obtain easily the new T directly from the new B by a succession of new pivot operations E_i^* . This is easy to accomplish if we observe the following rules:

I. If E_i and $\bar{E}_i$ are two elementary matrices representing adding a multiple of row i to other rows, then their product $E_i\bar{E}_i$ can be replaced by an elementary matrix of the same type, say, E_i^* . For example

$$\begin{bmatrix} 1 & & p_{13} \\ & 1 & p_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & p_{13} \\ & 1 & p_{23} \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & p_{13} + \bar{p}_{13} \\ & 1 & p_{23} + \bar{p}_{23} \\ & & 1 \end{bmatrix}$$

II. "Near commutativity" of adjacent-indexed matrices E_i^{i-1} and E_{i-1} holds; thus the product $E_i^{i-1} E_{i-1}$ can be replaced by $E_{i-1}\bar{E}_i$. For example

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & p_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & q_{13} & \\ & 1 & q_{23} & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & q_{13} & \\ & 1 & q_{23} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -q_{13}p_{34} \\ & 1 & & -q_{23}p_{34} \\ & & 1 & p_{34} \\ & & & 1 \end{bmatrix}$$

III. Nonadjacent-indexed matrices can be commuted; thus

$$E_i^{i-1} E_k = E_k E_i^{i-1} \qquad \text{if} \quad k < i - 1 .$$

For our example let us denote the new E_i by E_i^*, so that we are interested in obtaining the relation

$$(\text{New } T) = E_1^* E_2^* E_3^* E_4^* E_5^* E_6^* \; (\text{New } B)$$

by applying the above rules to (15). In this case

$$E_1^* = E_1 \quad \text{(the identity)}$$

$$E_2^* = E_2^1 E_2$$

$$E_3^* = \bar{E}_3 E_3 \quad \text{where} \quad E_3^2 E_2^* = E_2^* \bar{E}_3$$

Note particularly that the formation of each E_i^*, from a computational point of view, consists essentially of multiplying most of the elements of column $i - 1$ of E_{i-1}^* by a constant and adding it to the corresponding elements of column i of E_i.

The process described above of reducing to triangular form the matrix formed by dropping a column of T and inserting P_s^* was based on the assumption that certain coefficients of P_s^* were nonzero. If, for example, the second component is zero but the first component is not, it would not be possible to use a row operation E_2' to cause the first component of P_s^* to vanish.

Let us suppose that the position of column P_s in the new basis is k columns from the left (we assume that pivoting is done by starting always with the lower right-hand element of each submatrix).

Let the column being dropped from the basis be $r \leq k$. In the process of computation of P_s^* by (12), we obtain the vector

$$P_s' = E_k E_{k+1}, \ldots, E_m P_s .$$

Now P_s' must have its first nonzero component for some index $h \leq k$ since the new B is nonsingular. We assume for the moment that the k^{th} component is not zero. Accordingly, starting with P_s^*, the elimination of its first nonzero component with index s_1 can be effected by using its second nonzero component with index s_2, etc. until its nonzero component with index $s_t = k$ is used. This corresponds to row operations of the form $E_{s_2}^{s_1}$ followed by $E_{s_3}^{s_2}$, etc. The remaining components with indices $k, k+1, \ldots, m$ are unaffected by the above operations and hence remain the same as those of P_s^*. Thus the result is the same, as far as columns $P_k, P_{k+1}, \ldots, P_m$ are concerned, as if triangularization had been effected directly using these columns. If $s_1 + 1 - s_2 = \Delta > 0$, it will be necessary to cyclically permute certain of the rows by relabeling rows $s_1, s_1 + 1, \ldots, s_2 - 1$ as rows $s_2 - 1, s_1, s_1 + 1, \ldots, s_2 - 2$. In a similar manner, rows $s_2, s_2 + 1, \ldots, s_3 - 1$ are permuted if $s_2 + 1 - s_3 > 0$, etc. Allowing such permutations, it is no longer necessary to assume above that the k^{th} component of P_s^* (or P_s') was not zero.

It is important to note that such permutations would have been required if direct triangularization of all columns had been effected initially. Moreover, as far as staircase-structured systems are concerned, these permutations would not have affected the below-diagonal-staircase-form of T or the above-diagonal-staircase-form of the E-structure because, if direct eliminations were used, the eliminations and row interchanges would have been confined only to rows where the components of P_s are nonzero.

Let us now turn our attention to the column P_r to be dropped from the basis. Suppose first $r < k$. Deletion of the corresponding column of T followed by the necessary eliminations to restore triangularity discussed earlier will also require permutations if the indicated pivot position along the diagonal has a zero coefficient. For example, a two-cycle permutation will be required in order to lower to the diagonal the nonzero coefficient just above the diagonal. If $r \geq k$, it appears to be necessary first to drop the column corresponding to P_r from T and to retriangularize columns $k, k+1, \ldots, r-1$ (omitting r), and next to insert the column corresponding to P_s by performing the eliminations described above to P_s^*.

Since, in general, row permutations are required to obtain the triangular arrangement in standard form, it is necessary to replace (4) by

$$\text{(4')} \qquad T = E_1 E_2, \ldots, E_m JB$$

where J represents a permutation matrix. Each new cyclic permutation C introduced in the process of elimination to a new triangular form can be accounted for by appropriately relabeling the row designations of coefficients in E_i and J.

Finally, it is necessary to restate the rules given earlier for up-dating the substitute inverse when elementary matrices of the type E_i^{ℓ}, where $\ell < i$, appear on the left instead of E_i^{i-1} discussed earlier. In this case the rules are:

II': $E_i^{\ell} E_{\ell} = E_{\ell} \bar{E}_i$ where $\ell < i$ and where, letting $p_{i\ell}$ be the ℓ, i component of E_i^{ℓ}, the i^{th} column of $\bar{E}_i$ is formed by multiplying by $-p_{\ell i}$ the corresponding coefficients in column ℓ of E_{ℓ} with the exception of rows

ℓ and i. For row ℓ, the coefficient is $p_{\ell i}$, and for row i, the coefficient is unity.

III': $E_k^\ell E_i = E_i E_k^\ell$ if $i < \ell < k$

IV: $\bar{E}_i E_q = E_q \bar{E}_i$ if $\ell < q < i$ where ℓ is the same ℓ as

that which generated $\bar{E}_i$ in II'. Note that the commutativity of the matrices holds because $\bar{E}_i$ has a zero coefficient in column i for row q and E_q has a zero coefficient in column q for row i.

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