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THE DISSIPATIVITY OF A CERTAIN NONLINEAR SYSTEM OF
DIFFERENTIAL EQUATIONS. I

By

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THE DISSIPATIVITY OF A CERTAIN NONLINEAR SYSTEM OF
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 DIFFERENTIAL EQUATIONS. I

B. P. Demidovich

1. Introduction

In the article sufficient conditions are given such that each solution $\vec{x} = \vec{x}(t)$ of a nonlinear system of ordinary differential equations when $t \geq T(\vec{x})$ belongs to a certain fixed bounded domain D , that is, Levinson's D-property holds [1]. At the same time, some earlier results of the author [2, 3] and S. A. Samedova [4] are generalized.

2. The principal lemma

Lemma. Let $\vec{x} = (x_1, \dots, x_n) \in E^n$, $\vec{f}(\vec{x}) = (f_1(\vec{x}), \dots, f_n(\vec{x})) \in C^1(D)$, where $f_1(\vec{x})$ ($i = 1, \dots, n$) are real and D is a convex set of a material Euclidian space E^n . Let $\lambda(\vec{x})$ and $\Lambda(\vec{x})$ be the lowest and highest characteristic numbers of the symmetrized Jacobi matrix

$$J_s(\vec{x}) = \frac{1}{2} (\vec{f}'(\vec{x}) + \vec{f}'^*(\vec{x})) = \frac{1}{2} \left[\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \right]$$

Then for any points $\vec{x} \in D$, $\vec{x} + \Delta\vec{x} \in D$ the inequality

$$\lambda_m(\Delta\vec{x}, \Delta\vec{x}) < \langle \Delta\vec{f}(\vec{x}), \Delta\vec{x} \rangle < \Lambda_M(\Delta\vec{x}, \Delta\vec{x})^* \tag{1}$$

is satisfied, where $\Delta \vec{f}(\vec{x}) = \vec{f}(\vec{x} + \Delta \vec{x}) - \vec{f}(\vec{x})$; $\lambda_m = \min \lambda(\vec{\xi})$ and $\Delta_M = \max \Lambda(\vec{\xi})$
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 segment $\vec{\xi} = \vec{x} + t\Delta \vec{x} (0 \leq t \leq 1)$, connecting the points \vec{x} and $\vec{x} + \Delta \vec{x}$ **.

Proof. Let the points \vec{x} and $\vec{x} + \Delta \vec{x}$ be fixed. Starting from the obvious equality and applying the rule of differentiation of a complex vector function, we have

$$\Delta \vec{f}(\vec{x}) = \int_0^1 \frac{d}{dt} \vec{f}(\vec{x} + t\Delta \vec{x}) dt = \int_0^1 \vec{f}'(\vec{\xi}) \Delta \vec{x} dt,$$

where $\vec{\xi} = \vec{x} + t\Delta \vec{x}$. Hence

$$(\Delta \vec{f}(\vec{x}), \Delta \vec{x}) = \left(\int_0^1 \vec{f}'(\vec{\xi}) \Delta \vec{x} dt, \Delta \vec{x} \right) = \int_0^1 (\vec{f}'(\vec{\xi}) \Delta \vec{x}, \Delta \vec{x}) dt. \quad (2)$$

Since

$$(\vec{f}'(\vec{\xi}) \Delta \vec{x}, \Delta \vec{x}) = (J_{\vec{f}}(\vec{\xi}) \Delta \vec{x}, \Delta \vec{x}),$$

then we obviously have

$$(\vec{f}'(\vec{\xi}) \Delta \vec{x}, \Delta \vec{x}) > \lambda(\vec{\xi})(\Delta \vec{x}, \Delta \vec{x}) > \lambda_m(\Delta \vec{x}, \Delta \vec{x})$$

and

$$(\vec{f}'(\vec{\xi}) \Delta \vec{x}, \Delta \vec{x}) \leq \Lambda(\vec{\xi})(\Delta \vec{x}, \Delta \vec{x}) \leq \Delta_M(\Delta \vec{x}, \Delta \vec{x}) \quad [2].$$

Therefore, inequality (1) follows from formula (2).

3. The criterion of the fixed-sign property of a matrix

Definition. The symmetric $n \times n$ matrix $A = A(\vec{x})$, which is a function of the vector $\vec{x} \in E^m$, let us call uniformly of fixed sign (uniformly positive or uniformly negative) in a given domain D if the lower bound of the eigenvalues of this matrix is positive in D , or, correspondingly, the upper bound of the eigenvalues is negative in D .

Theorem 1. The symmetric $n \times n$ matrix $A = A(\vec{x})$ ($\vec{x} \in D$) is uniformly

* (\vec{x}, \vec{y}) is understood to be the scalar product of the real vectors $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$, that is, $(\vec{x}, \vec{y}) = \sum_{i=1}^n x_i y_i$.

** For the validity of the lemma it is sufficient to assume only that the segment $\vec{\xi} = \vec{x} + t\Delta \vec{x} (0 < t < 1)$ is entirely contained in domain D .

positively fixed in D if:

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- 1) the Sylvester conditions are satisfied

$$\Delta_1(\vec{x}) > 0, \dots, \Delta_n(\vec{x}) > 0,$$

where $\Delta_1(\vec{x})$ ($i = 1, \dots, n$) are the principal diagonal minors of the determinant $\det A(\vec{x})$;

- 2) there exists a positive number h such that

$$\frac{\Delta_n(\vec{x})}{[\text{sp } A(\vec{x})]^{n-1}} > h > 0 \quad \text{when } \vec{x} \in D. \quad (3)$$

The proof of this theorem is given in an earlier work of the author [2].

Corollary. The symmetric $n \times n$ matrix $A = A(\vec{x})$ ($\vec{x} \in D$) is uniformly negatively fixed in D if:

- 1) $(-1)^i \Delta_i(\vec{x}) > 0, \quad i = 1, \dots, n;$

$$2) \quad \frac{\Delta_n(\vec{x})}{[\text{Sp } A(\vec{x})]^n} < -h < 0 \quad \text{when } \vec{x} \in D, \quad (4)$$

where h is a positive number.

These conditions follow immediately from Theorem 1 if it is borne in mind that the uniform negative definiteness of matrix $A(\vec{x})$ follows from the uniform positive definiteness of matrix $-A(\vec{x})$.

4. Sufficient conditions for the dissipativity of the system

Theorem 2. Let

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t), \quad (5)$$

where $\vec{f}(\vec{x}, t) \in C(E^n \times I^+)$, $I^+ = (t < t < +\infty)$ and $\vec{f}(\vec{x}, t_0) \in C^1(E^n)$ for each $t_0 \in I^+$; where t (number or symbol) is ∞ .

If:

- 1) a constant symmetric positively definite $n \times n$ matrix $A = [a_{ij}]$

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is found such that the symmetric matrix

$$\tilde{J}_s(\vec{x}, t) = \frac{1}{2} [(A'_{\vec{x}}(\vec{x}, t) + A_{\vec{x}}(\vec{x}, t))^*]$$

is uniformly negatively definite in $E^n \times I^+$, that is, its highest characteristic number $\Lambda(\vec{x}, t)$ satisfies in $E^n \times I^+$ the inequality

$$\Lambda(\vec{x}, t) \leq -\alpha < 0, \quad (6)$$

where α is a positive constant which is not a function of \vec{x} and t ;

2) $\vec{f}(\vec{0}, t)$ is bounded, that is,

$$\|\vec{f}(\vec{0}, t)\| = (\vec{f}(\vec{0}, t), \vec{f}(\vec{0}, t))^{\frac{1}{2}} \leq c < +\infty \quad \text{when } t \in I^+; \quad (7)$$

then:

1) System (5) is dissipative, and there exist a closed sphere $K_R = \{\|\vec{x}\| \leq R\} \subset E^n$, such that each solution $\vec{x} = \vec{x}(t) (t_0 \in I^+, \vec{x}(t_0) \in E^n)$ of System (5) possesses the property $\vec{x}(t) \in K_R$ when $T(\vec{x}) \leq t < +\infty$, and, therefore, $\vec{x}(t)$ is bounded by the interval $[t_0, +\infty)$;

2) all solutions $\vec{x}(t)$ are asymptotically stable in the large when $t \rightarrow +\infty$; the stability is of the exponential type.

Remark. If the symmetrized Jacobi matrix

$$J_s(\vec{x}, t) = \frac{1}{2} [\vec{f}'_{\vec{x}}(\vec{x}, t) + \vec{f}''_{\vec{x}}(\vec{x}, t)]$$

is uniformly negatively defined, then, obviously, it is possible to take $A = E$, where E is a unit $n \times n$ matrix, and, therefore, in this case

$$\tilde{J}_s(\vec{x}, t) = J_s(\vec{x}, t).$$

In the particular case when System (5) is linear

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t).$$

* It is said that the solution $\vec{x}(t)$ ($t_0 \leq t < +\infty$) is asymptotically stable in the large when $t \rightarrow +\infty$ if: 1) it is according to Lyapunov when $t \rightarrow +\infty$ and 2) for any solution $\vec{y}(t)$, determined by the initial condition $\vec{y}(t_0) \in E^n$, the relation $\lim_{t \rightarrow +\infty} \|\vec{y}(t) - \vec{x}(t)\| = 0$ is valid.

where $P(t) \in C(I^+)$, $\vec{f}(t) \in C(I^+)$ and $\|\vec{f}(t)\| \leq c$ over I^+ , then for the validity of theorem 2 it is sufficient that the matrix

$$J_s(t) = \frac{1}{2} |P(t) + P^*(t)|$$

be uniformly negatively defined over I^+ .

Condition (6) was used by N. N. Krasovskiy [5], who, assuming $\vec{f}(\vec{0}, t) \equiv \vec{0}$, proved the asymptotic stability in the large of the trivial solution $\vec{x} \equiv \vec{0}$.

Proof 1!. Let us note first of all that the derivative $f'(\vec{x}, t)$ is continuous and, therefore, bound in any compact domain of the set $E^n \times I^+$; therefore, the local conditions of existence and uniqueness of the solutions are satisfied for System (5).

Let us examine the quadratic form

$$v = v(\vec{x}) = (A\vec{x}, \vec{x}).$$

Since matrix A is positively defined and symmetric, then, letting

$$a = \lambda_{\min}(A) \quad \text{and} \quad b = \lambda_{\max}(A)$$

be the lowest and highest characteristic numbers of matrix A, we have

$$a(\vec{x}, \vec{x}) \leq v(\vec{x}) \leq b(\vec{x}, \vec{x}). \quad (8)$$

For any solution $x = x(t)$ of System (5), taking the symmetry of matrix A into account, we obtain

$$\frac{dv}{dt} = \left(A \frac{d\vec{x}}{dt}, \vec{x} \right) + \left(A\vec{x}, \frac{d\vec{x}}{dt} \right) = 2 \left(A \frac{d\vec{x}}{dt}, \vec{x} \right).$$

or

$$\frac{dv}{dt} = 2(A\vec{f}(\vec{x}, t), \vec{x}) = 2((A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)), \vec{x}) + 2(A\vec{f}(\vec{0}, t), \vec{x}). \quad (9)$$

Since condition (6) is satisfied for the derivative $\frac{\partial}{\partial \vec{x}} [A\vec{f}(\vec{x}, t)]$, then according to the principal lemma in Section 2 we have

$$((A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)), \vec{x}) < -a(\vec{x}, \vec{x}) < -\frac{a}{b} v. \quad (10)$$

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In addition, using inequality (7) and the Cauchy inequality, we have

$$(\vec{A}\vec{f}(\vec{0}, t), \vec{x}) \leq \|\vec{A}\vec{f}(\vec{0}, t)\| \|\vec{x}\| \leq \|A\| \|\vec{f}(\vec{0}, t)\| \|\vec{x}\| \leq \rho c \sqrt{\frac{v}{a}}. \quad (11)$$

where

$$\rho = \|A\| = \max_i \sqrt{\sum_{j=1}^n a_{ij}^2}.$$

Therefore, from formula (9), taking inequalities (10) and (11) into account, we have

$$\frac{dv}{dt} \leq -\frac{2c}{b}v + 2c \sqrt{\frac{v}{a}} = -\frac{a}{b}v - \sqrt{v} \left(\frac{a}{b} \sqrt{v} - \frac{2c}{\sqrt{a}} \right).$$

Therefore, if

$$v > \frac{4c^2 b^2 a^2}{a^2 a} = v_0$$

and, that means, if

$$\|\vec{x}\| > \frac{2bc}{a} = R, \quad (12)$$

then

$$\frac{dv}{dt} < -\frac{a}{b}v < 0 \quad \text{when } t > t_0. \quad (13)$$

Let us consider the closed sphere $K_R \{ \|\vec{x}\| \leq R \}$. Let $v[\vec{x}(t_0)] < v_0$. Then, using inequality (13), we have, obviously, $v[\vec{x}(t)] < v_0$ when $t > t_0$. But since the ellipsoid $v(\vec{x}) = v_0$ is situated within the sphere $\|\vec{x}\| = R$, then $\vec{x}(t) \in K_R$ for $t \geq t_0$; the solution $\vec{x}(t)$ is defined when $t_0 \leq t < +\infty$.

If $v[\vec{x}(t_0)] > v_0$, then, integrating inequality (13) from t_0 to t , when $t > t_0$ we have

$$v[\vec{x}(t)] < v[\vec{x}(t_0)] e^{-\frac{a}{b}(t-t_0)} < v[\vec{x}(t_0)].$$

Therefore, solution $\vec{x}(t)$ is infinitely extendable as t increases and cannot constantly be located outside of the ellipsoid $v(\vec{x}) \leq v_0$, that is, at some $T > t_0$ the first time we shall have $v[\vec{x}(T)] = v_0$ and, therefore, $\vec{x}(T) \in K_R$. But then $\vec{x}(t) \in K_R$ when $T \leq t < +\infty$, where

$$T < t_0 + \frac{b}{a} \ln \frac{v[\vec{x}(t_0)]}{v_0}.$$

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2'. Let $\vec{x} = \vec{x}(t)$ and $\vec{y} = \vec{y}(t)$ be two solutions of System (5).

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Assuming

$$u = (A(\vec{x} - \vec{y}), (\vec{x} - \vec{y})),$$

we have

$$\frac{du}{dt} = 2((\vec{A}f(\vec{x}, t) - \vec{A}f(\vec{y}, t)), (\vec{x} - \vec{y})).$$

Hence, using inequality (6) and the principal lemma, we obtain

$$\frac{du}{dt} \leq -2\alpha \|\vec{x}(t) - \vec{y}(t)\|^2 \leq -\frac{2\alpha}{b} u.$$

Therefore

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$$u(t) \leq u(t_0) e^{-\frac{2\alpha}{b}(t-t_0)} \quad \text{when } t > t_0.$$

or

$$\|\vec{y}(t) - \vec{x}(t)\| \leq \sqrt{\frac{b}{2\alpha}} \|\vec{y}(t_0) - \vec{x}(t_0)\| e^{-\frac{\alpha}{b}(t-t_0)},$$

if $t \geq t_0$. Therefore, each solution $\vec{x}(t)$ is asymptotically stable in the large when $t \rightarrow +\infty$; the stability has an exponential nature.

Corollary. At the points of the ellipsoid

$$v(\vec{x}) = v_0,$$

where $v_0 = aR^2$, for the solutions $\vec{x}(t)$ of System (5) such that $v[\vec{x}(t_0)] = v_0$, the inequality is fulfilled

$$\frac{dv}{dt} < 0 \quad \text{when } t = t_0$$

This assertion follows immediately from formulas (12) and (13).

Theorem 2a (generalization). Let the vector function $\vec{f}(\vec{x}, t)$

have the properties indicated at the beginning of the formulation of theorem 2.

If:

1a) the symmetric matrix $\vec{J}_f(\vec{x}, t)$ possesses a highest characteristic number $\Lambda(\vec{x}, t)$ such that

$$\Lambda(\vec{x}, t) < -\alpha < 0 \tag{6a}$$

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when $\|\vec{x}\| \geq R_0 > 0$ and $\underline{t} < t < +\infty$,

$$\Lambda(\vec{x}, t) \leq \beta < +\infty \quad (6b)$$

when $\|\vec{x}\| < R_0$ and $\underline{t} < t < +\infty$, where α and β are positive numbers;

2a) the inequality

$$\|\vec{f}(\vec{0}, t)\| \leq c < +\infty,$$

is fulfilled, then System (5) is dissipative.

Proof. Retaining the symbols of theorem 2, let

$$v(\vec{x}) = (A\vec{x}, \vec{x}),$$

and let

$$\vec{x}_p = \frac{R_0}{\|\vec{x}\|} \vec{x}$$

be the nearest projection of the point \vec{x} ($\vec{x} \neq 0$) on the sphere $\|\vec{x}\| = R_0$. For the solution $\vec{x} = \vec{x}(t)$ when $\|\vec{x}(t)\| \geq 2R_0$ we have

$$\begin{aligned} \frac{1}{2} \frac{dv}{dt} &= (A\vec{f}(\vec{x}, t), \vec{x}) = \\ &= ((A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{x}_p, t)), (\vec{x} - \vec{x}_p)) + ((A\vec{f}(\vec{x}_p, t) - A\vec{f}(\vec{0}, t)), (\vec{x} - \vec{x}_p)) + \\ &\quad + ((A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)), \vec{x}_p) + (A\vec{f}(\vec{0}, t), \vec{x}). \end{aligned}$$

Hence, using the principal lemma, we have

$$\begin{aligned} ((A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{x}_p, t)), (\vec{x} - \vec{x}_p)) &\leq \max_0 \Lambda(\vec{x}_p + \theta(\vec{x} - \vec{x}_p), t) \|\vec{x} - \vec{x}_p\|^2 < \\ &< -\alpha \|\vec{x} - \vec{x}_p\|^2 = -\alpha (\|\vec{x}\| - R_0)^2 = -\alpha \left(\frac{\|\vec{x}\|}{2} + \frac{\|\vec{x}\|}{2} - R_0 \right)^2 < -\frac{\alpha}{4} \|\vec{x}\|^2, \end{aligned}$$

where

$$0 < \theta < 1 \text{ and } \|\vec{x}\| > 2R_0.$$

Similarly, using inequality (6b), we have

$$\begin{aligned} ((A\vec{f}(\vec{x}_p, t) - A\vec{f}(\vec{0}, t)), (\vec{x} - \vec{x}_p)) &= ((A\vec{f}(\vec{x}_p, t) - A\vec{f}(\vec{0}, t)), \vec{x}_p) \left(\frac{\|\vec{x}\|}{R_0} - 1 \right) < \\ &< \beta \|\vec{x}_p\|^2 \left(\frac{\|\vec{x}\|}{R_0} - 1 \right) < \beta R_0 \|\vec{x}\|. \end{aligned}$$

Further, bearing in mind that

$$\Lambda(\vec{x}, t) < \max(-\alpha, \beta) = \beta$$

follows from (6a) and (6b) when $\vec{x} \in E^n$ and $t \in I^+$, we find

$$\begin{aligned} (A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t), \vec{x}_p) &= (A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t), \vec{x}) \frac{R_0}{\|\vec{x}\|} < \\ &< \beta \|\vec{x}\|^2 \frac{R_0}{\|\vec{x}\|} = \beta R_0 \|\vec{x}\|. \end{aligned}$$

Finally,

$$(A\vec{f}(\vec{0}, t), \vec{x}) \leq \|A\| \|\vec{f}(\vec{0}, t)\| \|\vec{x}\| \leq \rho c \|\vec{x}\|,$$

where $\rho = \|A\|$. Therefore,

$$\frac{1}{2} \frac{dv}{dt} < -\frac{a}{4} \|\vec{x}\|^2 + (2\beta R_0 + \rho c) \|\vec{x}\|,$$

or

$$\frac{dv}{dt} < -\frac{a}{4} \|\vec{x}\|^2 - \left\{ \frac{a}{4} \|\vec{x}\| - 2(2\beta R_0 + \rho c) \right\} \|\vec{x}\| < -\frac{a}{4} \|\vec{x}\|^2, \quad (14)$$

$$\text{if only } \|\vec{x}\| \geq \max\left[2R_0, \frac{8(2\beta R_0 + \rho c)}{a}\right] = R_1.$$

Since

$$a \|\vec{x}\|^2 \leq v(\vec{x}) \leq b \|\vec{x}\|^2,$$

where $a = \lambda_{\min}(A)$ and $b = \lambda_{\max}(A)$, then from inequality (14) when $v \geq v_0 = bR_1^2$ we obtain

$$\frac{dv}{dt} \leq -\frac{a}{4b} v.$$

Hence

$$v[\vec{x}(t)] \leq v[\vec{x}(t_0)] e^{-\frac{a}{4b}(t-t_0)}, \quad (15)$$

if $v[\vec{x}(t)] \geq v_0$.

Therefore, when $t \geq T(\vec{x})$ for each solution $\vec{x}(t)$ the inequality $v[\vec{x}(t)] \leq v_0$ will be satisfied continuously, that is,

$$\|\vec{x}(t)\| \leq \sqrt{\frac{v[\vec{x}(t)]}{a}} \leq \sqrt{\frac{v_0}{a}} = \sqrt{\frac{b}{a}} R_1 = R$$

for $T(\vec{x}) \leq t < +\infty$.

Theorem 2b. If instead of inequalities (6a) and (6b) the inequality

$$\Lambda(\vec{x}, t) \leq -\varepsilon(r) < 0 \quad (6c)$$

is satisfied when $\|\vec{x}\| \geq r > 0$ and $t_0 < t < +\infty$, where $\alpha(r)$ is a
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 monotone nondecreasing function of r ($t_0 < t < +\infty$) and, in addition,

$$\vec{f}(0, t) \equiv \vec{0},$$

then the trivial solution $\vec{x} \equiv 0$ of System (5) is stable in the large when $t \rightarrow +\infty$.

Proof. Let

$$v(\vec{x}) = (A\vec{x}, \vec{x}).$$

Let $\varepsilon > 0$ arbitrarily and $\|\vec{x}(t)\| \geq \varepsilon$. On the basis of the reasoning of theorem 2a, taking into account that we may take $\beta = 0$ and $c = 0$, we have

$$\frac{dv}{dt} < -\frac{1}{2} \alpha\left(\frac{\varepsilon}{2}\right) \|\vec{x}(t)\|^2 < 0$$

when $\|\vec{x}(t)\| \geq \varepsilon$.

Thus $v(\vec{x})$ is a positively defined function having a negatively defined derivative $\frac{dv}{dt}$ on the strength of System (5). Therefore, on the basis of Lyapunov's theorem, the solution $\vec{x} \equiv 0$ is asymptotically stable when $t \rightarrow +\infty$.

The asymptotic stability of the solution $\vec{x} \equiv 0$ holds for any initial perturbations, since from an inequality similar to (15) for each solution $\vec{x}(t)$ we obtain

$$\|\vec{x}(T)\| = \varepsilon \quad \text{at some } T > t_0.$$

and, therefore, $\lim_{t \rightarrow +\infty} \vec{x}(t) = 0$.

Remark. Theorem 2b generalizes Krasovskiy's result [5].

REFERENCES

1. N. Levinson. Transformation Theory of Non-linear Differential Equations of the Second Order, Ann. Math., 45, No. 4, 723-737, 1944.
2. B. P. Demidovich. The Existence of Limiting Conditions for a Certain Nonlinear System of Ordinary Differential Equations, Uch. zap. MGU, VIII, Issue 181, 3-12, 1956.

* When $r = 0$ we have $\Lambda(0, t) \leq 0$.

3. B. P. Demidovich. One Case of Quasiperiodicity of the Solution of an Ordinary Differential Equation of the First Order, Usp. Matem. nauk, VIII, Issue 6(58), 103-106, 1953.

4. S. A. Samedova. The Existence of a Periodic Limiting Condition of a Nonlinear System of Ordinary Differential Equations, Izv. AN AzSSR, ser. fiz.-matem. i tekhn. nauk, No. 2, 27-33, 1960.

5. N. N. Krasovskiy. Stability in the Presence of Large Initial Perturbations, Prikl. matem. i mekh., XXI, Issue 3, 307-319, 1957.

Department of Mathematical Analysis

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