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# A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING

Richard Bellman and Robert Kalaba

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PREFACE

This Memorandum results from RAND's continuing study of computational aspects of the solution of the equations of mathematical physics. A new basic partial differential equation of particle mechanics is derived.

SUMMARY

Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian  $H = H(q,p)$ . The equations of motion are

$$(1) \quad \dot{q} = H_p,$$

$$(2) \quad -\dot{p} = H_q, \quad 0 \leq t \leq T.$$

As boundary conditions we specify

$$(3) \quad q(0) = 0,$$

$$(4) \quad p(T) = c.$$

If our aim is to solve the system of equations (1) and (2), subject to the boundary conditions in (3) and (4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time  $T$ ,  $q(T)$ , so that we shall have a complete set of conditions at time  $T$ . This would enable us to integrate the system (1) and (2) numerically, subject to initial values. Toward this end we introduce the function

$$(5) \quad r(c,T) = \text{the displacement at time } T, \text{ the} \\ \text{initial displacement being zero and} \\ \text{the momentum at time } T \text{ being } c.$$

We derive a partial differential equation for this function.

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## A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING

### 1. INTRODUCTION

Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian  $H = H(q,p)$ . The equations of motion are

$$(1.1) \quad \dot{q} = H_p,$$

$$(1.2) \quad -\dot{p} = H_q, \quad 0 \leq t \leq T.$$

As boundary conditions we specify

$$(1.3) \quad q(0) = 0,$$

$$(1.4) \quad p(T) = c.$$

If our aim is to solve the system of equations (1.1) and (1.2), subject to the boundary conditions in (1.3) and (1.4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time  $T$ ,  $q(T)$ , so that we shall have a complete set of conditions at time  $T$ . This would enable us to integrate the system (1.1) and (1.2) numerically, subject to initial values. Toward this end we introduce the function

(1.5)  $r(c,T)$  = the displacement at time  $T$ , the initial displacement being zero and the momentum at time  $T$  being  $c$ .

We wish to derive an equation for  $r(c,T)$ .

## 2. INVARIANT IMBEDDING

In earlier papers [1,2], we showed the following:

If

$$(2.1) \quad \frac{du}{dz} = F(u,v), \quad u(0) = 0,$$

$$(2.2) \quad -\frac{dv}{dz} = G(u,v), \quad v(x) = c,$$

for

$$(2.3) \quad 0 \leq z \leq x,$$

and if

$$(2.4) \quad r(c,x) = u(z) \Big|_{z=x},$$

then the function  $r(c,x)$  satisfies the partial differential equation

$$(2.5) \quad r_x = F(r,c) + G(r,c)r_c$$

and the initial condition

$$(2.6) \quad r(c,0) = 0.$$

Upon applying this result to the system of equations in (1.1) and (1.2), we find that the function  $r(c,T)$  satisfies the equation

$$(2.7) \quad \frac{\partial r}{\partial T} = H_p(r,c) + H_q(r,c) \frac{\partial r}{\partial c},$$

or, more elegantly,

$$(2.8) \quad \frac{\partial r}{\partial T} = \frac{\partial H(r,c)}{\partial c}.$$

This is the desired result, apparently a new equation of mechanics.

### 3. AN EXAMPLE

Consider the case of harmonic oscillations for which

$$(3.1) \quad H(q,p) = \frac{p^2}{2m} + \frac{1}{2} kq^2,$$

and

$$(3.2) \quad q(0) = 0, \quad r(T) = c.$$

For the unknown displacement at time  $T$ ,  $r(c,T)$ , equation (2.8) becomes

$$(3.3) \quad \frac{\partial r}{\partial T} = \frac{\partial}{\partial c} \left[ \frac{c^2}{2m} + \frac{1}{2} kr^2 \right] = \frac{c}{m} + kr \frac{\partial r}{\partial c}.$$

The solution of this equation, subject to the condition (2.6) is

$$(3.4) \quad r(c,T) = \frac{c}{\sqrt{km}} \tan \left( \sqrt{\frac{k}{m}} T \right).$$

This provides the desired displacement at time  $T$  and also shows that the function  $r(c,T)$  may become infinite for a finite value of  $T$ .

#### 4. DISCUSSION

The previous discussion may be generalized to the case of many particles moving in three-dimensional space using the theorem in [1]. Furthermore, various combinations of the  $q$ 's and  $p$ 's may be specified at the ends. There are immediate applications to perturbation analysis of results of this nature; see [3].

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