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THEORY OF ELASTIC STABILITY AND POST-BUCKLING BEHAVIOUR

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This report contains the Appendix of a projected book
on the theory of elastic stability
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THEORY OF ELASTIC STABILITY AND POST-BUCKLING BEHAVIOUR*

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W. T. Koiter**

October 1962

* The Appendix of a projected book "Theory of Elastic Stability and Post-Buckling Behaviour."

**This report was prepared while the author was a Visiting Professor of Applied Mathematics at Brown University.

THEORY OF ELASTIC STABILITY AND POST-BUCKLING BEHAVIOUR

Appendix

NOTES ON TENSOR ANALYSIS IN THREE-DIMENSIONAL EUCLIDEAN SPACE

Foreword

Cartesian tensors have now become the generally accepted shorthand in modern presentations of the classical theory of elasticity. They are entirely adequate for this purpose, and attractively simple, if Cartesian coordinates or orthogonal curvilinear coordinates are employed. The restriction to Cartesian tensors becomes awkward, however, whenever general curvilinear coordinates are more appropriate, in particular in the theory of thin shells and in the theory of finite deformations.

The analysis of general tensors in Euclidean space is the obvious tool for the discussion of fields in terms of general coordinates. Tensor analysis is now widely used in current research in the theory of elasticity, and it has also been employed successfully in several advanced text books. The treatise by A. E. Green and W. Zerna [1] is a notable and early example. In spite of the abundance of mathematical text books on tensor analysis, these authors have deemed it desirable to include a brief discussion of the basic concepts and theorems of tensor analysis in their book. The present writer fully agrees with this view. With few exceptions (in particular the book by McConnell [2]), the text books on tensor analysis are more likely to dishearten than to encourage the application of tensor analysis by physicists, and more in particular by engineers.

Moreover, most engineering curricula do not (yet) include tensor analysis in their regular courses even at a post-graduate level. Finally, the few text books on tensor theory at the appropriate unsophisticated level, still contain far more material than is actually needed.

The present notes are based on the discussion of tensors in a graduate course on elastic stability and post-buckling behaviour, given at Brown University in the academic year 1961-62. They are intended as an appendix to a projected book on the same topic. The discerning reader will observe that we have drawn heavily on McConnell's book, but he will also notice that we often start our discussion from somewhat different basic concepts. More in particular, the chapter on surfaces and shells has been largely influenced by intended applications to shell theory. The basic tools here are a special coordinate system in space, related to the surface under discussion, and consistent use of the fact that the spatial Riemann-Christoffel tensor vanishes in Euclidean space.

The notes open with a chapter on tensor algebra dealing with the properties of tensors at one particular point in space. In chapter 2, tensor analysis, we discuss properties of tensor fields, in particular the concept of covariant differentiation. The discussion of deformations in chapter 3 aims, of course, at the application in the theory of elasticity, but it is of sufficiently general context to include it in this appendix on tensor analysis. A similar discussion of deformations of surfaces has been omitted in chapter 4, because its application to the deformation of shells would require approximating physical assumptions which are out of place in these notes. The references at the end give a selection of books which the author has found convenient.

1. TENSOR ALGEBRA

1.1 Geometric and physical quantities.

In tensor analysis we deal with geometric or physical quantities which are described in any coordinate system in three-dimensional space by a set of 3^p elements, where p(=0,1,2,...) is called the order of the quantity. A quantity of order 1 or 2 is sometimes called a simple or double quantity, etc. We shall denote a quantity by its kernel letter and p indices, which may each take independently any of the values 1,2,3. We shall distinguish between upper indices or superscripts and lower indices or subscripts. Hence we have (p+1) types of quantities of the order p, accordingly as the number of superscripts varies from 0 to p.

By interchanging two superscripts or two subscripts we may define a new quantity of the same order and type. If all elements of the new quantity have the same values as the corresponding elements of the original quantity, the quantity is called symmetric with respect to these superscripts or subscripts. If all elements take opposite values on the interchange of two superscripts or subscripts, the quantity is called skew-symmetric with respect to these indices. For example, a symmetric double quantity a_{ij} and a skew-symmetric double quantity b_{ij} are characterized by

$$a_{ji} = a_{ij} , \quad b_{ji} = -b_{ij} . \tag{1.1}$$

Evidently, a symmetric double system has a 6 independent elements, and a skew-symmetric double system has 3 independent elements. A quantity is called completely symmetric in all its superscripts (or subscripts), if it is invariant upon any interchange of superscripts (or subscripts). It is called completely skew-symmetric, if each interchange results in a change of sign. Obviously, the number of superscripts (or subscripts) cannot exceed 3 in a non-vanishing completely skew-symmetric system. The two e-systems are defined as the completely skew-symmetric triple systems with three superscripts or subscripts with values +1 for the elements e^{123} and e_{123}. Hence we have for the non-vanishing elements of the e-systems

$$\left.\begin{aligned} e^{123} &= e^{231} = e^{312} = -e^{132} = -e^{213} = -e^{321} = 1 \; ; \\ e_{123} &= e_{231} = e_{312} = -e_{132} = -e_{213} = -e_{321} = 1 \; . \end{aligned}\right\} \quad (1.2)$$

Two quantities of the same order and type may be added by adding the corresponding elements, for example

$$c^{ij}_{k} = a^{ij}_{k} + b^{ij}_{k} \; . \qquad (1.3)$$

The sum is evidently a quantity of the same order and type. The product of two quantities of orders p and q is defined as a quantity of order (p+q) whose generic element is the product of arbitrary elements in the original quantities, for example

$$c^{ijmn}_{kpqr} = a^{ij}_{k} \, b^{mn}_{pqr} \qquad (1.4)$$

A contraction of a quantity with both superscripts and subscripts is obtained by taking a superscript equal to a subscript and summing over this suffix from 1 to 3, for example

$$a^{r}_{mr} = a^{1}_{m1} + a^{2}_{m2} + a^{3}_{m3} \qquad (1.5)$$

We apply the summation convention for a repeated index, if it appears once as a superscript and once as a subscript.*) A contraction results in a reduction of the order of the quantity by two; both the number of superscripts and subscripts are each reduced by one. The index with respect to which the summation is taken is called a "dead" or "dummy" index.

The Kronecker delta of order 6 is defined by the product of two e-systems

$$\delta^{ijk}_{mnr} = e^{ijk}e_{mnr} \; . \qquad (1.6)$$

Its elements are obviously zero if either two superscripts or two subscripts are equal. The elements of (1.6) are +1 if i,j,k and m,n,r differ by an even number of permutations, and -1 if i,j,k and m,n,r differ by an odd permutation. By contraction we obtain the Kronecker delta of order 4

$$\delta^{ij}_{mn} = \delta^{ijk}_{mnk} = e^{ijk}e_{mnk} \; . \qquad (1.7)$$

Its elements are evidently zero unless i,j and m,n are the same pair of different indices. The elements are +1 if $i=m$, $j=n\neq m$, and -1 if $i=n$, $j=m\neq n$. Finally, we define the Kronecker delta of

*) There is no need to distinguish between superscripts and subscripts in the analysis of Cartesian tensors. Only subscripts are usually employed in the discussion of such tensors. The summation convention is then also applied to a repeated subscript. In general tensor analysis, however, the summation convention is meaningful only if it is applied to one superscript and one subscript.

order 2, which is usually called the Kronecker delta without any further qualification, by a repeated contraction and a division by 2

$$\delta^i_m = \frac{1}{2}\delta^{ij}_{mj} = \begin{matrix} 0 & \text{if} & i \neq m \, , \\ 1 & \text{if} & i = m \, . \end{matrix} \qquad (1.8)$$

We note some simple and useful formulae involving the Kronecker delta's

$$\delta^i_i = 3 \, , \quad \delta^{ijk}_{mjk} = 2\delta^i_m \, , \quad \delta^{ijk}_{ijk} = 6 \, ; \qquad (1.9)$$

$$\delta^{rst}_{mnt} = \delta^{rst}_{tmn} = \delta^{rst}_{ntm} = -\delta^{rst}_{mtn} = -\delta^{rst}_{tnm} = -\delta^{rst}_{nmt} = \delta^{rs}_{mn} \, ; \qquad (1.10)$$

$$\delta^i_j a^j = a^i \, , \quad \delta^i_j b_i = b_j \, ; \qquad (1.11)$$

$$\delta^{ij}_{mn} a^{mn} = a^{ij} - a^{ji} \, , \quad \delta^{ij}_{mn} b_{ij} = b_{mn} - b_{nm} \, ; \qquad (1.12)$$

$$\left.\begin{aligned} \delta^{ijk}_{mnr} a^{mnr} &= a^{ijk} + a^{jki} + a^{kij} - a^{ikj} - a^{kji} - a^{jik} \, , \\ \delta^{ijk}_{mnr} b_{ijk} &= b_{mnr} + b_{nrm} + b_{rmn} - b_{mrn} - b_{rnm} - b_{nmr} \, . \end{aligned}\right\} \qquad (1.13)$$

1.2 Determinants.

The e-systems are convenient in the evaluation of determinants of the third order. The determinant a of a mixed double quantity a^r_s is defined by

$$a = \left| a^r_s \right| = \begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix} \, . \qquad (1.14)$$

The formula for the evaluation of this determinant may be written in the form

$$a = e_{ijk}a^i_1 a^j_2 a^k_3 = e^{mnr}a^1_m a^2_n a^3_r \ . \qquad (1.15)$$

Equivalent formulae are

$$ae_{mnr} = e_{ijk}a^i_m a^j_n a^k_r \ , \quad ae^{ijk} = e^{mnr}a^i_m a^j_n a^k_r \ ; \qquad (1.16)$$

$$a = \frac{1}{6}\, e^{mnr}e_{ijk}a^i_m a^j_n a^k_r \ . \qquad (1.17)$$

Similar formulae may be given for the determinant b of a double quantity with two subscripts b_{rs} and for the determinant c of a double quantity with two superscripts c^{rs}, viz.

$$be_{mnr} = e^{ijk}b_{im}b_{jn}b_{kr} = e^{ijk}b_{mi}b_{nj}b_{rk} \ ; \qquad (1.18)$$

$$ce^{mnr} = e_{ijk}c^{im}c^{jn}c^{kr} = e_{ijk}c^{mi}c^{nj}c^{rk} \ ; \qquad (1.19)$$

$$b = \frac{1}{6}\, e^{ijk}e^{mnr}b_{im}b_{jn}b_{kr} \ ; \ c = \frac{1}{6}\, e_{ijk}e_{mnr}c^{im}c^{jn}c^{kr} . \qquad (1.20)$$

The e-systems are also convenient for the evaluation of the product of two determinants of the third order. Let a and b denote the determinants of two double mixed quantities a^r_s and b^r_s. Multiplying both members in the first formula (1.16) by $b^m_p b^n_q b^r_s$ (and applying of course the summation convention with respect to the repeated indices m,n,r), we obtain

$$abe_{pqs} = e_{ijk}a^i_m a^j_n a^k_r b^m_p b^n_q b^r_s \ . \qquad (1.21)$$

Introducing the contracted product

$$a^i_m b^m_p = c^i_p \ , \qquad (1.22)$$

whose determinant is denoted by c, we may rewrite (1.21) in the form

$$ab = c \ , \qquad (1.23)$$

which expresses the product rule for determinants.

A similar result is obtained for the product of the determinants a and b of a double quantity a^{rs} and a double quantity b_{rs}. Defining a mixed double quantity c^r_s by the contracted product

$$c^r_s = a^{rt} b_{ts} \ , \qquad (1.24)$$

and denoting the determinant of (1.24) by c, we obtain from (1.18) the identity c=ab.

1.3 Coordinate transformations.

Orthogonal transformations of Cartesian coordinates are the basis for the definition of Cartesian tensors. Let x^i (i=1,2,3) and $x^{i'}$ (i' = 1',2',3') denote the original and transformed coordinates. The orthogonal transformation and its inverse transformation are given by

$$x^{i'} = A^{i'}_i x^i + b^{i'} \ , \qquad (1.25)$$

$$x^i = A^i_{i'} x^{i'} + b^i \ , \qquad (1.26)$$

where the constant transformation coefficients $A^{i'}_i = A^i_{i'}$ are given by the cosines of the angles between the x^i- and $x^{i'}$-axes.

These coefficients and the translation terms in (1.25), (1.26) satisfy the relations

$$A^{i'}_{i} A^{i}_{j'} = \delta^{i'}_{j'} , \quad A^{i'}_{i} A^{j}_{i'} = \delta^{j}_{i} , \quad |A^{i'}_{i}| \cdot |A^{j}_{j'}| = 1 ; \qquad (1.27)$$

$$|A^{i'}_{i}|^2 = 1 ; \quad b^{i'} = -A^{i'}_{i} b^{i} , \quad b^{i} = -A^{i}_{i'} b^{i'} . \qquad (1.28)$$

If the coordinate systems have the same orientation, the transformation determinant is +1.

Our formulation of orthogonal transformations differs from the more conventional formulation in two respects. First we have employed the same kernel letter x for both coordinate systems. We distinguish between the two systems by attaching a prime to the indices in one system. At the same time we have written the transformation coefficients as a mixed quantity, and we have written the index for the coordinates as a superscript. Obviously, we have to write our transformation coefficients with one superscript and one subscript, if we wish to adhere to our summation convention explained in section 1.1. The choice of a superscript for the coordinates is purely conventional in order to achieve agreement with current practice in tensor analysis.

General tensors are characterized by their transformation properties for more general transformations from one triple of coordinates to another triple of coordinates. We shall consider the group of continuously differentiable functional transformations

$$x^{\alpha'} = x^{\alpha'}(x^{\alpha}) , \qquad (1.29)$$

where α and α' run independently from 1 to 3 and from 1' to 3', for which the <u>Jacobian does not vanish</u>

$$\left| \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \right| \neq 0 \ . \tag{1.30}$$

The <u>inverse</u> transformation

$$x^{\alpha} = x^{\alpha}(x^{\alpha'}) \tag{1.31}$$

then also exists, and its Jacobian is related to (1.30) by the formula

$$\left| \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \right| \cdot \left| \frac{\partial x^{\beta'}}{\partial x^{\beta}} \right| = 1 \ . \tag{1.32}$$

We employ again the same kernel letter x for both coordinate systems, and we distinguish again between them by attaching a prime to one set of indices. In order to emphasize that our coordinates are not necessarily Cartesian, we have used <u>Greek</u> indices. Latin indices will henceforward always refer to Cartesian coordinates in the absence of an explicit statement to the contrary.*)

The partial derivatives of the functional transformation (1.29) and its inverse (1.31) are denoted by

$$\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = A^{\alpha'}_{\alpha} \ , \quad \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} = A^{\alpha}_{\alpha'} \ . \tag{1.33}$$

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*) It seems appropriate to mention here already that we shall adopt a different convention in chapter 4 of this appendix, which deals with the geometry of surfaces and shells. Latin indices will be used in that chapter to refer to the triple of space coordinates, whereas Greek indices will then be employed to refer to the pair of surface coordinates.

The transformation law for the differentials of the coordinates

$$dx^{\alpha'} = A^{\alpha'}_{\alpha} dx^{\alpha} \ , \quad dx^{\alpha} = A^{\alpha}_{\alpha'} dx^{\alpha'} \tag{1.34}$$

is a linear transformation which is formally identical with the corresponding transformation of coordinate differentials in an orthogonal transformation. We also have the relations

$$A^{\alpha'}_{\alpha} A^{\alpha}_{\beta'} = \delta^{\alpha'}_{\beta'} \ , \quad A^{\alpha'}_{\alpha} A^{\beta}_{\alpha'} = \delta^{\beta}_{\alpha} \ . \tag{1.35}$$

These relations and (1.32) are the counterpart of (1.27) for orthogonal transformations. The crucial difference is that in our general functional transformation the derivatives (1.33) are not constants but functions of the coordinates.

The most important property of a group of transformations is that the result of two consecutive transformations in this group is again a transformation belonging to the group. This group property is easily established by the chain rule of partial differentiation. Let

$$x^{\alpha'} = x^{\alpha'}(x^{\alpha}) \ , \quad x^{\alpha''} = x^{\alpha''}(x^{\alpha'}) \tag{1.36}$$

denote two consecutive transformations of our group. We have

$$A^{\alpha''}_{\alpha} = \frac{\partial x^{\alpha''}}{\partial x^{\alpha}} = \frac{\partial x^{\alpha''}}{\partial x^{\alpha'}} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = A^{\alpha''}_{\alpha'} A^{\alpha'}_{\alpha} \ , \tag{1.37}$$

$$|A^{\alpha''}_{\alpha}| = |A^{\alpha''}_{\alpha'}| \cdot |A^{\alpha'}_{\alpha}| \neq 0 \ , \tag{1.38}$$

where the product rule for determinants has also be used.

It will be convenient in the sequel to restrict our attention to functional transformations with a positive Jacobian. No essential loss in generality is incurred here, because we can always achieve this result by an additional transformation $x^{1''} = -x^{1'}$, $x^{2''} = x^{2'}$, $x^{3''} = x^{3'}$ with Jacobian -1, if our first transformation $x^{\alpha'}(x^{\alpha})$ would have a negative Jacobian.

1.4 Invariants, vectors and tensors.

An invariant or scalar is a quantity specified by a single element which takes the same value in all coordinate systems. Examples of scalars are the density, the temperature, an energy density etc.

In a Cartesian coordinate system the coordinate differentials dx^i are the components of a vector $d\underline{x}$ at a point x^j in space. Their transformation law for orthogonal transformations is

$$dx^{i'} = A^{i'}_{i} dx^{i} \ , \quad dx^{i} = A^{i}_{i'} dx^{i'} \ , \qquad (1.39)$$

which is a special case of the more general transformation law (1.34) for coordinate differentials. We may therefore define the coordinate differentials in (1.34) as components of the vector $d\underline{x}$ in terms of our general coordinates. We generalize this concept of a vector by the following definition: a quantity of the first order with a superscript u^{α} at an arbitrary point x^{β} of space is a contravariant vector, if its elements or components transform in the same way as the differentials of the coordinates

$$u^{\alpha'} = A^{\alpha'}_{\alpha} u^{\alpha} \tag{1.40}$$

This definition of a contravariant vector obviously implies the well-known transformation rules for the components of a vector in Cartesian coordinates. We have added the adjective "contravariant" because we shall presently discuss a second possible generalization of the transformation rule for Cartesian vector components. We emphasize that the definition of a contravariant vector is meaningful only if we also specify the particular point in space where the vector is defined. Although a physical vector u may be transplanted in Euclidean space, and the Cartesian components are invariant in this process, the associated contravariant vectors are not invariant due to the fact that the transformation coefficients $A^{\alpha'}_{\alpha}$ are functions of position in space. This space-dependent character of the concept of a contravariant vector in general coordinates constitutes a basic complication in the analysis.

If we consider a scalar field, i.e. an invariant φ which is a continuously differentiable function of the coordinates, we may introduce the quantity of the first order whose elements are the partial derivatives of this scalar with respect to the coordinates. We shall denote partial differentiation with respect to a coordinate x^{α} by a subscript α, preceded by a comma

$$\frac{\partial\varphi}{\partial x^{\alpha}} = \varphi_{,\alpha} \quad . \tag{1.41}$$

The chain rule of partial differentiation provides the transformation law for the quantity (1.41)

$$\varphi_{,\alpha'} = A^{\alpha}_{\alpha'}\varphi_{,\alpha} \ . \qquad (1.42)$$

In Cartesian coordinates these partial derivatives $\varphi_{,i}$ at any point x^j of space are the Cartesian components of a vector grad φ. Here again we may define the partial derivatives of φ with respect to the coordinates in a general coordinate system as the components of the vector grad φ in this general frame of reference. It appears from (1.42) that these components do not obey the transformation law for a contravariant vector. We now generalize the vector concept in a second manner by the following definition: a quantity of the first order with a subscript v_α at an arbitrary point x^β of space is a <u>covariant vector</u>, if its elements or components transform in the same way as the partial derivatives of a scalar field

$$v_{\alpha'} = A^{\alpha}_{\alpha'}v_{\alpha} \ . \qquad (1.43)$$

Here again our definition implies the transformation rule for the vector components in Cartesian coordinates. In fact, in such coordinates there is no difference in behaviour between contravariant and covariant vectors because $A^{i'}_{i} = A^{i}_{i'}$, and no reason exists for a distinction between superscripts and subscripts in Cartesian coordinates. We emphasize here again the space-dependent character of the concept of a covariant vector in general coordinates.

The <u>general product</u> of p contravariant vectors and q covariant vectors is a quantity of order (p+q) with p superscripts and q subscripts

$$\underset{(1)}{u}^{\alpha_1}\,\underset{(2)}{u}^{\alpha_2}\cdots\underset{(p)}{u}^{\alpha_p}\,\overset{(1)}{v}_{\beta_1}\,\overset{(2)}{v}_{\beta_2}\cdots\overset{(q)}{v}_{\beta_q} = t^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} . \tag{1.44}$$

The transformation law for this quantity is easily seen to be

$$t^{\alpha'_1\cdots\alpha'_p}_{\beta'_1\cdots\beta'_q} = A^{\alpha'_1}_{\alpha_1}\cdots A^{\alpha'_p}_{\alpha_p}A^{\beta_1}_{\beta'_1}\cdots A^{\beta_q}_{\beta'_q}t^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} . \tag{1.45}$$

Many important geometric and physical quantities obey the transformation law (1.45), but it does not necessarily follow that such quantities may be written as a general vector product. In fact, a quantity of order (p+q) has, in general, $3^{(p+q)}$ independent elements or components, whereas the general vector product has no more than 3(p+q) independent elements. Hence we are led to consider a more general quantity of order (p+q) which obeys the transformation law (1.45), and which we shall call a <u>tensor of contravariant order</u> p <u>and covariant order</u> q. Such a tensor is therefore <u>defined</u> by the transformation law (1.45). If we wish, we may now also call a contravariant vector a tensor of contravariant order one and covariant order zero. Similarly, a covariant vector is a tensor of contravariant order zero and covariant order one. Finally, we may call an invariant a tensor of order zero.

Our previous definition of the addition and multiplication of quantities (addition being limited to quantities of the same order and type) may now be applied to tensors. It follows immediately from (1.45) that the sum of two tensors of equal contravariant and covariant orders is again a tensor of the same order. Likewise isthe product of two tensors again a tensor, whose contravariant and covariant orders are the sum of the corresponding orders of the factors. It also follows from (1.45) that the contraction of a tensor with respect to any pair of superscript and subscript results in a tensor whose contravariant and covariant orders are each reduced by one. The contracted product of a contravariant vector u^α and a covariant vector v_β is in particular an invariant

$$u^{\alpha'} v_{\alpha'} = u^\alpha v_\alpha \,, \tag{1.46}$$

which is called the scalar product of the vectors u^α and v_β .

An invariant property of tensors is their symmetry or skew-symmetry with respect to a pair of superscripts or subscripts. We prove this result for a contravariant tensor of order two by means of the transformation law (1.45)

$$\begin{aligned} t^{\alpha'\beta'} \pm t^{\beta'\alpha'} &= A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} t^{\alpha\beta} \pm A^{\beta'}_{\beta} A^{\alpha'}_{\alpha} t^{\beta\alpha} = \\ &= A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} (t^{\alpha\beta} \pm t^{\beta\alpha}) . \end{aligned} \tag{1.47}$$

If an invariant is a function of a tensor with continuous partial derivatives with respect to the components of this tensor, the quantity formed by these partial derivatives is again a tensor. For example, if $F(t_{\alpha\beta})$ is a function of the double covariant tensor $t_{\alpha\beta}$, its partial derivatives constitute a double contravariant tensor. This results follows again from the chain rule of partial differentiation

$$\frac{\partial F}{\partial t_{\alpha'\beta'}} = \frac{\partial F}{\partial t_{\alpha\beta}} \frac{\partial t_{\alpha\beta}}{\partial t_{\alpha'\beta'}} = A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} \frac{\partial F}{\partial t_{\alpha\beta}} \quad . \tag{1.48}$$

The quotient law of tensors may be expressed in different forms. A first version may be stated for a general product. If the general product of two quantities is a tensor, and if one of the factors is a non-vanishing tensor, then the other factor is also a tensor. This theorem is again an immediate consequence of the transformation law for tensors. The most important version of the quotient law for a contracted product is the following theorem. If the contracted product of a quantity with p superscripts and q subscripts with p arbitrary covariant vectors and q arbitrary contravariant vectors is an invariant, then this quantity is a tensor of contravariant order p and covariant order q. In other words, a necessary and sufficient condition for the tensor character of a quantity $t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ is that contracted product

$$t_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \cdot \overset{(1)}{u}_{\alpha_1} \dots \overset{(p)}{u}_{\alpha_p} \underset{(1)}{v}^{\beta_1} \dots \underset{(q)}{v}^{\beta_q} = a \tag{1.49}$$

is invariant for arbitrary vectors $\overset{(1)}{u}_{\alpha_1}, \dots, \overset{(p)}{u}_{\alpha_p}, \underset{(1)}{v}^{\beta_1}, \dots, \underset{(q)}{v}^{\beta_q}$.

The proof of this theorem is easily obtained from the more general theorem: if the contracted product of a quantity of order m with an arbitrary contravariant or covariant vector is a tensor (of order (m-1)), then the quantity of order m is also a tensor. The proof of this theorem is again an immediate consequence of the transformation law (1.45). Repeated application of the general theorem yields the theorem expressed by (1.49). The result (1.49) is sometimes taken as a starting-point for the definition of a tensor, from which the transformation law (1.45) is then a consequence.

It is easily verified that the Kronecker delta's introduced in section 1.1, are tensors. Indeed, we have for the sixth-order delta

$$A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}A^{\gamma}_{\gamma'}A^{\lambda'}_{\lambda}A^{\mu'}_{\mu}A^{\nu'}_{\nu}\delta^{\lambda\mu\nu}_{\alpha\beta\gamma} =$$

$$= A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}A^{\gamma}_{\gamma'}e_{\alpha\beta\gamma} \; A^{\lambda'}_{\lambda}A^{\mu'}_{\mu}A^{\nu'}_{\nu}e^{\lambda\mu\nu} =$$

$$= e_{\alpha'\beta'\gamma'} \; |A^{\rho}_{\rho'}| \cdot e^{\lambda'\mu'\nu'} \; |A^{\sigma'}_{\sigma}| = \delta^{\lambda'\mu'\nu'}_{\alpha'\beta'\gamma'} \quad . \tag{1.50}$$

By contraction we may establish the tensor character of the other Kronecker delta's. It should be noted that the e-systems are not tensors. For these quantities we have the transformation laws

$$A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}A^{\gamma}_{\gamma'}e_{\alpha\beta\gamma} = e_{\alpha'\beta'\gamma'} \; |A^{\rho}_{\rho'}| \; , \tag{1.51}$$

$$A^{\alpha'}_{\alpha}A^{\beta'}_{\beta}A^{\gamma'}_{\gamma}e^{\alpha\beta\gamma} = e^{\alpha'\beta'\gamma'} \; |A^{\rho'}_{\rho}| \; . \tag{1.52}$$

Quantities of this type, where the Jacobian of the transformation enters explicitly in the transformation rule, are often called pseudo-tensors, relative tensors, weighted tensors or tensor densities. We shall avoid the use of these quantities, as far as possible, and we do not need to discuss them in detail.

1.5 The metric tensor and related quantities.

The metric tensor plays an important role in tensor analysis. It may be introduced in a variety of manners. For our purpose it is convenient to start from Cartesian coordinates in our Euclidean space. As we have seen, the Cartesian components of any physical vector $\underline{u}$ satisfy both transformation laws for contravariant and covariant vectors. Hence we may write $u^i=u_i$ in Cartesian coordinates, but this relationship is not preserved in arbitrary coordinates. It is therefore more convenient to write

$$u_i = g_{ij}u^j \;, \quad u^i = g^{ij}u_j \;, \qquad (1.53)$$

where

$$g_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases} \;, \quad g^{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases} \;. \qquad (1.54)$$

In this notation we obtain the free index in the relations (1.53) at the same location in both members.*)

It is immediately verified that the quantities (1.54) behave as symmetric covariant and contravariant double tensors under orthogonal transformations, and they are called the

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*) In the discussion of Cartesian tensors the quantities (1.54) are often called unit tensors and denoted as Kronecker delta's. In general tensor analysis this practice might lead to misunderstandings because the properties (1.54) are not preserved in general coordinates. For this reason it is more convenient to introduce a separate kernel letter for the quantities g_{ij} g^{ij}.

Cartesian metric tensors. They satisfy the identities

$$g_{ij}g^{jh} = \delta_j^h \, . \tag{1.55}$$

Introducing general coordinates x^α, we may regard the contravariant vector

$$u^\alpha = A_i^\alpha u^i \tag{1.56}$$

and the covariant vector

$$u_\alpha = A_\alpha^i u_i \tag{1.57}$$

at the same arbitrary point in space as equivalent representations of the same physical vector u. This equivalence is also implied by our notation, which employs the same kernel letter for the contravariant vector and the covariant vector. The vector character of (1.56) and (1.57) is of course an immediate consequence of the group property of our coordinate transformations

$$u^{\alpha'} = A_i^{\alpha'} u^i = A_\alpha^{\alpha'} A_i^\alpha u^i = A_\alpha^{\alpha'} u^\alpha \quad , \tag{1.58}$$

and similar relations for the transformation of (1.57). We also note the inverse relations

$$u^i = A_\alpha^i u^\alpha \, , \quad u_i = A_i^\alpha u_\alpha \quad . \tag{1.59}$$

Combining (1.53), (1.56), (1.57) and (1.59), we obtain relations between the contravariant vector u^α and the covariant vector u_β in the form

$$u^\alpha = A_i^\alpha A_j^\beta g^{ij} u_\beta \quad , \tag{1.60}$$

$$u_\alpha = A_\alpha^i A_\beta^j g_{ij} u^\beta \quad . \tag{1.61}$$

We now define the (symmetric) <u>covariant metric tensor</u> by

$$g_{\alpha\beta} = A^i_\alpha A^j_\beta g_{ij} , \qquad (1.62)$$

and the (symmetric) <u>contravariant metric tensor</u> by

$$g^{\alpha\beta} = A^\alpha_i A^\beta_j g^{ij} . \qquad (1.63)$$

The tensor character of these quantities is of course again a consequence of the group property of the coordinate transformations. The tensor character of the Kronecker delta's entails that (1.55) may be generalized to

$$g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha . \qquad (1.64)$$

The determinants of the metric tensors are positive. For the covariant metric tensor we have

$$|g_{\alpha\beta}| = |A^i_\alpha A^j_\beta g_{ij}| = |A^i_\alpha|^2 |g_{ij}| = |A^i_\alpha|^2 > 0 , \qquad (1.65)$$

and a similar proof holds for the contravariant metric tensor. Equations (1.64) may therefore also be used to define the contravariant metric tensor if the covariant metric tensor is already available, and vice versa.

The relationship between the contravariant and covariant vectors u^α and u_β in (1.60) and (1.61) is now written in terms of the metric tensors

$$u^\alpha = g^{\alpha\beta} u_\beta , \quad u_\alpha = g_{\alpha\beta} u^\beta . \qquad (1.66)$$

The processes involved in (1.66) are often called the raising and lowering of indices. The same process may also be employed to raise or lower an index in any tensor. The new tensor obtained in this process is called an associated tensor of the original tensor. Raising and lowering of indices is always performed by means of the metric tensors, and we shall express the relationship by retaining the same kernel letter for all associated tensors.*) It is necessary to indicate in some way which index is raised or lowered. For example, the tensors

$$g_{\alpha\beta}t^{\beta\gamma\delta} = t_{\alpha}^{\cdot\gamma\delta} \quad \text{and} \quad g_{\alpha\gamma}t^{\beta\gamma\delta} = t^{\beta\ \ \cdot\delta}_{\cdot\ \alpha} \tag{1.67}$$

are distinct unless the original tensor $t^{\beta\gamma\delta}$ is symmetric with respect to the superscripts β and γ. The dots in appropriate places in the mixed tensors indicate uniquely which index in the contravariant tensor has been lowered.

It is easily verified that a contracted product remains invariant if we raise the dummy index in one factor and lower it in the other factor. For example, we have

$$t^{\alpha\beta\gamma}s_{\lambda\mu\gamma} = g^{\gamma\delta}t^{\alpha\beta}_{\cdot\cdot\delta}s_{\lambda\mu\gamma} = t^{\alpha\beta}_{\cdot\cdot\delta}s_{\lambda\mu}^{\cdot\cdot\delta} \tag{1.68}$$

The <u>scalar product</u> was originally defined for a contravariant vector and a covariant vector. We may extend this concept to a pair of contravariant vectors or a pair of covariant vectors. This contracted product is given by any of the equivalent formulae

*) An exception is conventionally made for the metric tensor itself by the use of the Kronecker delta in the right-hand member of (1.64) instead of the kernel letter g.

$$\underline{u}\cdot\underline{v} = u_\alpha v^\alpha = \delta^\alpha_\beta u_\alpha v^\beta = g_{\alpha\beta}u^\alpha v^\beta = g^{\alpha\beta}u_\alpha v_\beta \quad . \tag{1.69}$$

This equivalence is of course an immediate consequence of the fact that the contravariant vector and the covariant vector are equivalent representations of the same physical vector.

Two vectors are <u>orthogonal</u> if their scalar product vanishes. In terms of contravariant vectors u^α, v^β or covariant vectors u_α, v_β orthogonality is therefore expressed by

$$g_{\alpha\beta}u^\alpha v^\beta = 0 \ , \quad g^{\alpha\beta}u_\alpha v_\beta = 0 \ . \tag{1.70}$$

If we take coordinate differentials along the coordinate lines 1 and 2, we have

$$\underset{1}{dx^\alpha} = \underset{1}{dx^1},0,0; \quad \underset{2}{dx^\alpha} = 0, \underset{2}{dx^2},0 \ , \tag{1.71}$$

and the condition of orthogonality between these two contravariant vectors reduces to $g_{12} = 0$. Hence we have for <u>orthogonal</u> curvilinear coordinates the relation

$$g_{\alpha\beta} = 0 \ , \quad \text{if } \alpha \neq \beta \ . \tag{1.72}$$

The square of a vector is given by the invariant

$$\underline{u}\cdot\underline{u} = u_\alpha u^\alpha = \delta^\alpha_\beta u_\alpha u^\beta = g_{\alpha\beta}u^\alpha u^\beta = g^{\alpha\beta}u_\alpha u_\beta \quad . \tag{1.73}$$

In particular we have the basic formula for the <u>square of a line element</u> ds, defined by the coordinate differentials dx^α

$$(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta \ . \tag{1.74}$$

The cosine of the angle $(\underline{u},\underline{v})$ between two vectors $\underline{u}$ and $\underline{v}$ may be given in terms of the scalar product. We have the formula

$$\cos(\underline{u},\underline{v}) = \frac{\underline{u}\cdot\underline{v}}{(\underline{u}\cdot\underline{u})^{\frac{1}{2}}(\underline{v}\cdot\underline{v})^{\frac{1}{2}}} = \frac{g_{\alpha\beta}u^{\alpha}v^{\beta}}{(g_{\varkappa\lambda}u^{\varkappa}u^{\lambda})^{\frac{1}{2}}(g_{\mu\nu}v^{\mu}v^{\nu})^{\frac{1}{2}}} . \quad (1.75)$$

We observed at the end of the previous section that the e-systems are no tensors. We may easily obtain tensors from these systems, however, by multiplying or dividing by $\sqrt{g}$. Introducing the quantities

$$\varepsilon_{\alpha\beta\gamma} = e_{\alpha\beta\gamma}\sqrt{g} \ , \quad \varepsilon^{\alpha\beta\gamma} = e^{\alpha\beta\gamma}/\sqrt{g} \ , \qquad (1.76)$$

we may prove their tensor character by means of (1.51), (1.52) and the formula for the transformation of the determinant of the metric tensor

$$g' = |g_{\alpha'\beta'}| = |A^{i}_{\alpha'}|^{2} = |A^{\alpha}_{\alpha'}|^{2}|A^{i}_{\alpha}|^{2} = |A^{\alpha}_{\alpha'}|^{2} g . \quad (1.77)$$

Taking square roots, and applying (1.51), we obtain indeed*)

$$\varepsilon_{\alpha'\beta'\gamma'} = A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}A^{\gamma}_{\gamma'}\varepsilon_{\alpha\beta\gamma} , \qquad (1.78)$$

$$\varepsilon^{\alpha'\beta'\gamma'} = A^{\alpha'}_{\alpha}A^{\beta'}_{\beta}A^{\gamma'}_{\gamma}\varepsilon^{\alpha\beta\gamma} . \qquad (1.79)$$

The use of the same kernel letter for these so-called <u>ε-tensors</u> is justified by the relations

$$\varepsilon_{\alpha\beta\gamma} = g_{\alpha\varkappa}g_{\beta\lambda}g_{\gamma\mu}\varepsilon^{\varkappa\lambda\mu} , \quad \varepsilon^{\alpha\beta\gamma} = g^{\alpha\varkappa}g^{\beta\lambda}g^{\gamma\mu}\varepsilon_{\varkappa\lambda\mu} , \qquad (1.80)$$

which relations are easily verified by means of formula (1.18) for determinants. Some further useful formulae involving the ε-tensors are

*) Our restriction to coordinate transformations which preserve the orientation is essential here.

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\varkappa\lambda\mu}\, g^{\alpha\varkappa} = g_{\beta\lambda}g_{\gamma\mu} - g_{\beta\mu}g_{\gamma\lambda} \quad , \tag{1.81}$$

$$\varepsilon^{\alpha\beta\gamma}\varepsilon^{\varkappa\lambda\mu}\, g_{\alpha\varkappa} = g^{\beta\lambda}g^{\gamma\mu} - g^{\beta\mu}g^{\gamma\lambda} \quad , \tag{1.82}$$

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\varkappa\lambda\mu}\, g^{\beta\lambda}g^{\gamma\mu} = 2g_{\alpha\varkappa} \quad , \tag{1.83}$$

$$\varepsilon^{\alpha\beta\gamma}\varepsilon^{\varkappa\lambda\mu}\, g_{\beta\lambda}g_{\gamma\mu} = 2g^{\alpha\varkappa} \quad . \tag{1.84}$$

A verification of these formulae may again be left to the reader.

The ε-tensor is convenient in the definition of the <u>vector product or cross product</u> $\underline{w}$ of two vectors $\underline{u}$ and $\underline{v}$. In an appropriate generalization of the definition of the Cartesian components of the vector product we write

$$w_\alpha = \varepsilon_{\alpha\beta\gamma}u^\beta v^\gamma \quad , \quad w^\alpha = \varepsilon^{\alpha\beta\gamma}u_\beta v_\gamma \quad . \tag{1.85}$$

The vector product is orthogonal to both factors. Its sense is such that the vectors $\underline{u}, \underline{v}, \underline{w}$, in this order, show the same orientation as the axes 1,2,3 of a Cartesian coordinate system. The magnitude of $\underline{w}$ is the area of the parallelogram spanned by $\underline{u}$ and $\underline{v}$, as is easily confirmed by calculation

$$\begin{aligned}\underline{w}\cdot\underline{w} &= g^{\alpha\varkappa}w_\alpha w_\varkappa = g^{\alpha\varkappa}\varepsilon_{\alpha\beta\gamma}u^\beta v^\gamma \varepsilon_{\varkappa\lambda\mu}u^\lambda v^\mu = \\ &= (g_{\beta\lambda}g_{\gamma\mu} - g_{\beta\mu}g_{\gamma\lambda})u^\beta u^\lambda v^\gamma v^\mu = \\ &= (\underline{u}\cdot\underline{u})(\underline{v}\cdot\underline{v}) - (\underline{u}\cdot\underline{v})^2 = (\underline{u}\cdot\underline{u})(\underline{v}\cdot\underline{v})\sin^2(\underline{u},\underline{v}) \quad .\end{aligned} \tag{1.86}$$

The surface area dS of the parallelogram, spanned by two infinitesimal vectors $\underset{1}{dx^\alpha}, \underset{2}{dx^\alpha}$, is given by

$$n_\alpha dS = \varepsilon_{\alpha\beta\gamma}\underset{1}{dx^\beta}\underset{2}{dx^\gamma} \quad , \tag{1.87}$$

where n_α is the covariant unit vector of the normal to the surface element dS.

The <u>triple product</u> of three vectors $\underline{u},\underline{v},\underline{w}$, taken in this order, is defined by the invariant

$$\varepsilon_{\alpha\beta\gamma} u^\alpha v^\beta w^\gamma = \varepsilon^{\alpha\beta\gamma} u_\alpha v_\beta w_\gamma \quad . \tag{1.88}$$

Its magnitude is the volume of the parallelepiped spanned by the three vectors, and it is positive if the three vectors $\underline{u},\underline{v},\underline{w}$ show the same orientation as the axes 1,2,3 of a Cartesian coordinate system. In particular, we have for the volume of an infinitesimal parallelepiped spanned by coordinate differentials $\underset{1}{dx^\alpha} = dx^1,0,0$; $\underset{2}{dx^\alpha} = 0,dx^2,0$; $\underset{3}{dx^\alpha} = 0,0,dx^3$

$$dv = \sqrt{g}\, dx^1 dx^2 dx^3 \quad . \tag{1.89}$$

2. TENSOR ANALYSIS

2.1 Covariant differentiation of vectors.

The partial derivatives of a scalar field constitute a covariant vector field. We have indeed based our definition of a covariant vector on the transformation law (1.42) for the partial derivative of a scalar field.

The behaviour of the partial derivatives of a vector field is less simple. We shall denote partial differentiation with respect to a coordinate x^α by an additional subscript α, preceded by a comma. If we apply a transformation to coordinates $x^{\alpha'}$ we may derive the following transformation laws for the partial derivative of a contravariant vector and a covariant vector

$$u^{\alpha'}_{,\beta'} = (A^{\alpha'}_{\alpha} u^{\alpha})_{,\beta'} = (A^{\alpha'}_{\alpha} u^{\alpha})_{,\beta} A^{\beta}_{\beta'} = A^{\beta}_{\beta'} \frac{\partial^2 x^{\alpha'}}{\partial x^{\alpha} \partial x^{\beta}} u^{\alpha} + A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} u^{\alpha}_{,\beta} \,, \tag{2.1}$$

$$u_{\alpha',\beta'} = (A^{\alpha}_{\alpha'} u_{\alpha})_{,\beta'} = \frac{\partial^2 x^{\alpha}}{\partial x^{\alpha'} \partial x^{\beta'}} u_{\alpha} + A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} u_{\alpha,\beta} \,. \tag{2.2}$$

The terms with the second derivatives of the coordinate transformation and its inverse preclude a tensor character of the quantities constituted by the partial derivatives. They are absent only in <u>linear</u> transformations. In particular, if we restrict our attention to Cartesian coordinate systems, whose orthogonal transformations are a sub-group of linear transformations, the partial derivatives of the components of a vector field are the components of a <u>Cartesian tensor</u>.

The absence of a tensor character in the partial derivatives of a vector field in general coordinates is a serious complication. It is therefore convenient to introduce a modified concept of differentiation and derivatives. We define the covariant derivative of a contravariant or covariant vector field as the mixed double tensor or covariant double tensor which reduces to the partial derivatives in a Cartesian coordinate system. Denoting covariant differentiation with respect to x^α by an additional subscript α preceded by a vertical line, our definition of covariant derivatives is thus expressed by

$$u^\alpha\Big|_\beta = A^\alpha_i A^j_\beta u^i_{,j} \,, \quad u^i_{,j} = A^i_\alpha A^\beta_j u^\alpha\Big|_\beta \,; \tag{2.3}$$

$$u_{\alpha|\beta} = A^i_\alpha A^j_\beta u_{i,j} \,, \quad u_{i,j} = A^\alpha_i A^\beta_j u_{\alpha|\beta} . \tag{2.4}$$

We now identify the primed coordinate system in (2.1) with a Cartesian system, and we obtain from (2.3) an expression for the covariant derivative

$$u^\alpha\Big|_\beta = A^\alpha_i A^j_\beta [A^\lambda_j \frac{\partial^2 x^i}{\partial x^\varkappa \partial x^\lambda} u^\varkappa + A^i_\varkappa A^\lambda_j u^\varkappa_{,\lambda}] \tag{2.5}$$

Applying (1.35), and introducing the so-called Christoffel symbols of the second kind

$$\Gamma^\alpha_{\varkappa\beta} = A^\alpha_i \frac{\partial^2 x^i}{\partial x^\varkappa \partial x^\beta} \,, \tag{2.6}$$

we obtain a formula for the covariant derivative of a contravariant vector

$$u^\alpha\Big|_\beta = u^\alpha_{,\beta} + \Gamma^\alpha_{\varkappa\beta} u^\varkappa \,. \tag{2.7}$$

Likewise, if we identify the unprimed coordinate system in (2.2) with a Cartesian system, we obtain (omitting the primes)

$$u_{\alpha,\beta} = \frac{\partial^2 x^i}{\partial x^\alpha \partial x^\beta} u_i + u_{\alpha|\beta} \; . \tag{2.8}$$

Expressing u_i in general covariant components, we have therefore

$$u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^{\varkappa}_{\alpha\beta} u_{\varkappa} \quad , \tag{2.9}$$

where the <u>same</u> Christoffel symbols (2.6) reappear.

2.2 <u>The Christoffel symbols</u>.

Since the partial derivatives of a contravariant or covariant vector do not constitute a tensor it follows immediately from (2.7) or (2.9) that the Christoffel symbols do not constitute a tensor either. Their transformation law involves the second derivatives of the coordinate transformation. It is easily verified from (2.6) that their <u>transformation law</u> is actually

$$\Gamma^{\varkappa'}_{\alpha'\beta'} = A^{\varkappa'}_{\varkappa} A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} \Gamma^{\varkappa}_{\alpha\beta} + A^{\varkappa'}_{\varkappa} \frac{\partial^2 x^{\varkappa}}{\partial x^{\alpha'} \partial x^{\beta'}} \quad . \tag{2.10}$$

It is also easily verified that (2.6) is invariant under linear transformations of the Cartesian coordinates x^i. Finally, we note that $\Gamma^{\varkappa}_{\alpha\beta}$ is <u>symmetric in the subscripts α and β</u>.

The Christoffel symbols are closely connected with the metric tensor. Differentiating (1.62) partially with respect to x^λ, we have

$$g_{\alpha\beta,\lambda} = g_{ij}\left(A^i_\alpha \frac{\partial^2 x^j}{\partial x^\beta \partial x^\lambda} + A^j_\beta \frac{\partial^2 x^i}{\partial x^\alpha \partial x^\lambda}\right) \; . \tag{2.11}$$

From (2.6) we obtain

$$\frac{\partial^2 x^i}{\partial x^\alpha \partial x^\beta} = A^i_\varkappa \Gamma^\varkappa_{\alpha\beta} \quad , \tag{2.12}$$

and substituting from (2.12) into (2.11), we find

$$g_{\alpha\beta,\lambda} = g_{ij}(A^i_\alpha A^j_\varkappa \Gamma^\varkappa_{\beta\lambda} + A^j_\beta A^i_\varkappa \Gamma^\varkappa_{\alpha\lambda}) = g_{\alpha\varkappa}\Gamma^\varkappa_{\beta\lambda} + g_{\beta\varkappa}\Gamma^\varkappa_{\alpha\lambda} \ . \tag{2.13}$$

By interchanging α and λ, and by interchanging β and λ we obtain similar results for $g_{\lambda\beta,\alpha}$ and $g_{\alpha\lambda,\beta}$. Introducing the Christoffel symbols of the first kind, in which the superscript $\varkappa$ is lowered to the first subscript

$$\Gamma_{\lambda\alpha\beta} = g_{\lambda\varkappa}\Gamma^\varkappa_{\alpha\beta} \quad , \tag{2.14}$$

we obtain from (2.13)

$$\Gamma_{\lambda\alpha\beta} = \frac{1}{2}[g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}] \ , \tag{2.15}$$

$$\Gamma^\varkappa_{\alpha\beta} = \frac{1}{2} g^{\varkappa\lambda}[g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}] \ . \tag{2.16}$$

The Christoffel symbols of the first kind are again no tensors, but they are symmetric in the last pair of subscripts.

All Christoffel symbols vanish in a Cartesian coordinate system, and even in any rectilinear coordinate system (cf.(2.6)). It should be noted that the Christoffel symbols in a general coordinate system can therefore not be specified arbitrarily as functions of the coordinates. Their specification is subject to the restriction that a transformation to Cartesian coordinates is possible, in which the Christoffel symbols vanish. In other words, the equations (2.12), rewritten in the explicit form

$$\frac{\partial^2 x^i}{\partial x^\alpha \partial x^\beta} = \Gamma^\varkappa_{\alpha\beta} \frac{\partial x^i}{\partial x^\varkappa} \tag{2.17}$$

must be integrable in order to obtain Cartesian coordinates x^i. Since the Christoffel symbols may be expressed in terms of the metric tensor, it follows also that the metric tensor cannot be specified arbitrarily as a function of the general coordinates. The restrictions on the specification of the metric tensor may also be stated in the equivalent form that a transformation of coordinates shall be possible in such a way that the metric tensor is specified by the constants (1.54) in the new (Cartesian) coordinates.*)

We shall derive the necessary and sufficient conditions to be imposed on the Christoffel symbols in order that Cartesian coordinatesmay exist in a simply-connected domain in space. We start from an arbitrary fixed point A in space with coordinates $(x^\alpha)_A$, and we assume a set of quantities $\left(\frac{\partial x^i}{\partial x^\alpha}\right)_A$ in such a way that

$$g_{ij}\left(\frac{\partial x^i}{\partial x^\alpha}\right)_A \left(\frac{\partial x^j}{\partial x^\beta}\right)_A = (g_{\alpha\beta})_A \ . \tag{2.18}$$

This purpose may be achieved in many ways because $g = |g_{\alpha\beta}| > 0$. We now take a smooth curve $x^\alpha(t)$ from A to an arbitrary point B with coordinates $(x^\alpha)_B$. We may integrate (2.17) along this curve,

*) The existence of Cartesian coordinates is a basic characteristic of Euclidean space. Several generalizations of the concept of space are considered in higher geometry. The interested reader is referred to the literature [e.g. 3-6].

and we obtain the equations

$$\left(\frac{\partial x^i}{\partial x^\alpha}\right)_B = \left(\frac{\partial x^i}{\partial x^\alpha}\right)_A + \int_A^B \Gamma^\varkappa_{\alpha\beta} \frac{\partial x^i}{\partial x^\varkappa} \frac{dx^\beta}{dt}\, dt \qquad (2.19)$$

Equations (2.19) form a complete set of Volterra integral equations for the quantities $\frac{\partial x^i}{\partial x^\alpha}$ along the curve $x^\alpha(t)$, and these equations have a unique solution. The result for $\left(\frac{\partial x^i}{\partial x^\alpha}\right)_B$ depends of course, in general, on the path along which the integration from A to B is performed. The result is independent of the path, if and only if the differentials

$$\Gamma^\varkappa_{\alpha\beta} \frac{\partial x^i}{\partial x^\varkappa}\, dx^\beta \qquad (2.20)$$

are total differentials, i.e. if

$$\frac{\partial}{\partial x^\gamma}\left(\Gamma^\varkappa_{\alpha\beta} \frac{\partial x^i}{\partial x^\varkappa}\right) = \frac{\partial}{\partial x^\beta}\left(\Gamma^\varkappa_{\alpha\gamma} \frac{\partial x^i}{\partial x^\varkappa}\right) . \qquad (2.21)$$

Substituting from (2.17) into (2.21), we reduce the necessary and sufficient conditions for a path-independent result of the integration in (2.19) to

$$\Gamma^\varkappa_{\alpha\gamma,\beta} - \Gamma^\varkappa_{\alpha\beta,\gamma} + \Gamma^\lambda_{\alpha\gamma}\Gamma^\varkappa_{\lambda\beta} - \Gamma^\lambda_{\alpha\beta}\Gamma^\varkappa_{\lambda\gamma} = 0 . \qquad (2.22)$$

Once the quantities $\frac{\partial x^i}{\partial x^\alpha}$ have been obtained, the second integration to coordinates x^i poses no difficulty because the symmetry of the Christoffel symbols in (2.17) already implies that the integrability conditions for the second integration are satisfied. The metric tensor in the coordinate system x^i is necessarily constant because the Christoffel symbols vanish in these coordinates. Its components are g_{ij} since they have these values at point A. This completes the proof that (2.22) supplies the necessary and sufficient conditions for the existence of Cartesian coordinates in a simply-connected domain. We shall see later that the left-hand member of (2.22) is a tensor, the so-called Riemann-Christoffel tensor. The vanishing of the Riemann-Christoffel tensor is thus a necessary and sufficient condition in order that space be Euclidean.*)

2.3 Covariant differentiation of tensors.

The partial derivatives of the components of a tensor field have evidently no tensor character under general coordinate transformations. Here again, we may define a modified derivative, to be called the covariant derivative, by a tensor whose components reduce to the partial derivatives in Cartesian coordinates. It is slightly more convenient, however, to start from the quotient law. For the sake of brevity we shall consider a tensor field of the fourth order with two contravariant and two covariant indices. The contracted product

$$t^{\alpha\beta}_{\lambda\mu}\, u_\alpha v_\beta w^\lambda y^\mu = a \qquad (2.23)$$

is a scalar field for arbitrary vector fields $u_\alpha, v_\beta, w^\lambda, y^\mu$.

Differentiating (2.23) partially with respect to x^ρ, we obtain a covariant vector field

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*) A direct proof of the tensor character of the left-hand member of (2.22) may of course already be given here by investigating the transformation law of this quantity, but the required analysis is laborious.

$$t^{\alpha\beta}_{\lambda\mu,\rho} u_\alpha v_\beta w^\lambda y^\mu + t^{\alpha\beta}_{\lambda\mu} u_{\alpha,\rho} v_\beta w^\lambda y^\mu +$$
$$+ t^{\alpha\beta}_{\lambda\mu} u_\alpha v_{\beta,\rho} w^\lambda y^\mu + t^{\alpha\beta}_{\lambda\mu} u_\alpha v_\beta w^\lambda_{,\rho} y^\mu +$$
$$+ t^{\alpha\beta}_{\lambda\mu} u_\alpha v_\beta w^\lambda y^\mu_{,\rho} = a_{,\rho} \qquad (2.24)$$

We now express the partial derivatives of the arbitrary vectors in terms of the corresponding covariant derivatives. We collect all terms which contain all four vectors as factors. The result of our rewriting (2.24) is

$$[t^{\alpha\beta}_{\lambda\mu,\rho} + t^{\varkappa\beta}_{\lambda\mu}\Gamma^\alpha_{\varkappa\rho} + t^{\alpha\varkappa}_{\lambda\mu}\Gamma^\beta_{\varkappa\rho} - t^{\alpha\beta}_{\varkappa\mu}\Gamma^\varkappa_{\lambda\rho} - t^{\alpha\beta}_{\lambda\varkappa}\Gamma^\varkappa_{\mu\rho}] u_\alpha v_\beta w^\lambda y^\mu +$$
$$+ t^{\alpha\beta}_{\lambda\mu}[u_\alpha|_\rho v_\beta w^\lambda y^\mu + u_\alpha v_\beta|_\rho w^\lambda y^\mu + u_\alpha v_\beta w^\lambda|_\rho y^\mu + u_\alpha v_\beta w^\lambda y^\mu|_\rho] = a_{,\rho} \qquad (2.25)$$

The second line in this equation is a contracted product of vectors and tensors with one free index in the subscript ρ . Hence it represents a covariant vector. The first line is then also a covariant vector. Since the four vector factors are arbitrary, it follows that the quantity between brackets must be a tensor of contravariant order two and covariant order three. It is called the <u>covariant derivative</u> of our tensor field

$$t^{\alpha\beta}_{\lambda\mu}\big|_\rho = t^{\alpha\beta}_{\lambda\mu,\rho} + \Gamma^\alpha_{\varkappa\rho} t^{\varkappa\beta}_{\lambda\mu} + \Gamma^\beta_{\varkappa\rho} t^{\alpha\varkappa}_{\lambda\mu} - \Gamma^\varkappa_{\lambda\rho} t^{\alpha\beta}_{\varkappa\mu} - \Gamma^\varkappa_{\mu\rho} t^{\alpha\beta}_{\lambda\varkappa} . \qquad (2.26)$$

The structure of the covariant derivative of an arbitrary tensor with respect to x^ρ will now be clear from this example. In addition to the partial derivative of the components, we have terms for each superscript α of the type $\Gamma^\alpha_{\varkappa\rho}$ multiplied by the tensor component in which the superscript α has been replaced by $\varkappa$, and terms for each subscript λ of the type $-\Gamma^\varkappa_{\lambda\rho}$

multiplied by the tensor component in which the subscript λ has been replaced by $\varkappa$.

Since the covariant derivatives obviously reduce to the partial derivatives in Cartesian coordinates, it follows immediately that the ordinary rules for partial differentiation of a sum or a product are preserved in covariant differentiation of a sum or a product of tensors.

2.4 Some general theorems.

A theorem of fundamental importance is Ricci's lemma that the covariant derivative of the metric tensor vanishes identically. It is an immediate consequence of the fact that the metric tensor components are constants in a Cartesian frame of reference. The same argument applies to the ε-tensor. Hence we may always treat the metric tensor, the ε-tensor and the Kronecker delta's as constants for the purpose of covariant differentiation.

From the fact that the covariant derivative of the ε-tensor vanishes we may obtain a useful formula for the contracted Christoffel symbol of the second kind. Taking $\lambda=1$, $\mu=2$, $\nu=3$ in the identity

$$\varepsilon_{\lambda\mu\nu|\alpha} = \varepsilon_{\lambda\mu\nu,\alpha} - \Gamma^{\varkappa}_{\lambda\alpha}\varepsilon_{\varkappa\mu\nu} - \Gamma^{\varkappa}_{\mu\alpha}\varepsilon_{\lambda\varkappa\nu} - \Gamma^{\varkappa}_{\nu\alpha}\varepsilon_{\lambda\mu\varkappa} = 0 , \quad (2.27)$$

we find

$$\sqrt{g}\,\Gamma^{\varkappa}_{\varkappa\alpha} = (\sqrt{g})_{,\alpha} \quad \text{or} \quad \Gamma^{\varkappa}_{\varkappa\alpha} = \frac{(\sqrt{g})_{,\alpha}}{\sqrt{g}} = \tfrac{1}{2}(\log g)_{,\alpha} \quad . \quad (2.28)$$

The divergence of a contravariant vector field is defined by its contracted covariant derivative. This invariant may be evaluated by means of (2.28)

$$u^{\alpha}\big|_{\alpha} = u^{\alpha}_{,\alpha} + \Gamma^{\alpha}_{\varkappa\alpha} u^{\varkappa} = u^{\alpha}_{,\alpha} + u^{\varkappa} \frac{(\sqrt{g})_{,\varkappa}}{\sqrt{g}} = \frac{1}{\sqrt{g}}(\sqrt{g}\; u^{\alpha})_{,\alpha} \quad . \qquad (2.29)$$

Similarly, we may define the divergence of a contravariant tensor field. We have

$$t^{\alpha\beta}\big|_{\beta} = t^{\alpha\beta}_{,\beta} + \Gamma^{\alpha}_{\varkappa\beta} t^{\varkappa\beta} + \Gamma^{\beta}_{\varkappa\beta} t^{\alpha\varkappa} =$$

$$= \frac{1}{\sqrt{g}}(\sqrt{g}\; t^{\alpha\beta})_{,\beta} + \Gamma^{\alpha}_{\varkappa\beta} t^{\varkappa\beta} , \qquad (2.30)$$

$$t^{\beta\alpha}\big|_{\beta} = t^{\beta\alpha}_{,\beta} + \Gamma^{\beta}_{\varkappa\beta} t^{\varkappa\alpha} + \Gamma^{\alpha}_{\varkappa\beta} t^{\beta\varkappa} =$$

$$= \frac{1}{\sqrt{g}}(\sqrt{g}\; t^{\beta\alpha})_{,\beta} + \Gamma^{\alpha}_{\varkappa\beta} t^{\beta\varkappa} . \qquad (2.31)$$

These results are of course identical for a symmetric tensor $t^{\alpha\beta}$, and of opposite sign for a skew-symmetric tensor.

The gradient of a scalar field φ is defined by the covariant vector field $\varphi_{,\alpha} = \varphi|_{\alpha}$. Its divergence is defined as the divergence of the associated contravariant vector field

$$\text{div. grad. } \varphi = (g^{\alpha\beta}\varphi|_{\alpha})|_{\beta} = g^{\alpha\beta}\varphi_{|\alpha\beta} , \qquad (2.32)$$

and it is sometimes called the Laplacian of the scalar field φ.

The rotation tensor $\alpha_{\lambda\mu}$ of a covariant vector field u_{λ} is defined by

$$2\alpha_{\lambda\mu} = u_{\mu|\lambda} - u_{\lambda|\mu} = u_{\mu,\lambda} - u_{\lambda,\mu} \quad . \qquad (2.33)$$

The associated rotation vector ω^{α} is defined by

$$2\omega^{\alpha} = \varepsilon^{\alpha\lambda\mu}\alpha_{\lambda\mu} = \varepsilon^{\alpha\lambda\mu}\; u_{\mu,\lambda} \; . \qquad (2.34)$$

In tensor analysis twice this rotation vector is conventionally called the curl of the vector field u. The rotation tensor, the rotation vector and the curl of the gradient of a scalar field vanish because the sequence of repeated partial differentiation is immaterial. For the same reason the divergence of the curl of a vector field must also vanish in Cartesian coordinates

$$(\varepsilon^{ijk}u_{k|j})|_i = \varepsilon^{ijk}u_{k,ji} = 0 , \qquad (2.35)$$

and it follows from its tensor character that this divergence also vanishes in general coordinates

$$(\varepsilon^{\alpha\beta\gamma}u_{\gamma|\beta})|_\alpha = \varepsilon^{\alpha\beta\gamma}u_{\gamma|\beta\alpha} = 0 . \qquad (2.36)$$

The well-known theorems of Green and Stokes are also easily formulated in terms of general coordinates. The <u>divergence theorem or Green's theorem</u> reads in Cartesian coordinates

$$\int_V u^i_{,i}\, dv = \int_S u^i n_i dS , \qquad (2.37)$$

where the volume integral is extended over the volume V enclosed by the surface S, and n_i is the unit outward normal vector to the surface. We may rewrite this theorem in terms of general coordinates because both the divergence in the left-hand member and the scalar product in the right-hand member are invariants

$$\int_V u^\alpha|_\alpha\, dv = \int_S u^\alpha n_\alpha dS . \qquad (2.38)$$

In Cartesian coordinates <u>Stokes's theorem</u> for the curl of a vector field is expressed by

$$\int_S \varepsilon^{ijk} u_{k,j} n_i dS = \int_C u_i dx^i \ , \tag{2.39}$$

where the surface integral is extended over any smooth surface within the closed contour C, and the direction of the unit normal vector n_i and the sense of the contour integration along C correspond to the right-hand screw rule in a right-handed coordinate system. We may again rewrite (2.39) in terms of general coordinates

$$\int_S \varepsilon^{\alpha\beta\gamma} u_{\gamma|\beta} n_\alpha dS = \int_C u_\alpha dx^\alpha \ , \tag{2.40}$$

on account of the invariance of both integrands.

On the other hand, Green's theorem for the divergence of a (contravariant) tensor field t^{ij} in Cartesian coordinates

$$\int_V t^{ij}_{,j} dv = \int_S t^{ij} n_j dS \tag{2.41}$$

has no equivalent formulation in terms of general coordinates. The reason for this is that the integrands in (2.41) are (contravariant) vectors, and not invariants. The integral of a contravariant vector in Cartesian coordinates has the physical meaning that it is the contravariant representation of the resultant vector. No such meaning can be attached to the integral of a contravariant vector in general coordinates. The absence in general coordinates of a counterpart to (2.41) is no serious drawback since the formulation of equations involving integrals of a vector can usually be avoided without difficulty.

2.5 The Riemann-Christoffel tensor.

Repeated covariant differentiation of a scalar field φ yields a double covariant tensor

$$\varphi_{|\alpha\beta} = (\varphi_{|\alpha})_{|\beta} = \varphi_{,\alpha\beta} - \Gamma^{\varkappa}_{\alpha\beta}\varphi_{,\varkappa} \ . \tag{2.42}$$

This tensor is obviously symmetric by the symmetry of the Christoffel symbols, as should also be expected from the fact that (2.42) reduces to the second partial derivatives in a Cartesian coordinate system.

Repeated covariant differentiation of a covariant vector field u_α results similarly in a triple covariant tensor field $u_{\alpha|\beta\gamma}$. This tensor must be symmetric in the last pair of subscripts, since it reduces to repeated partial derivatives in a Cartesian system. On the other hand, we obtain by the formalism of covariant differentiation after some algebra

$$u_{\alpha|\beta\gamma} - u_{\alpha|\gamma\beta} = [\Gamma^{\varkappa}_{\alpha\gamma,\beta} - \Gamma^{\varkappa}_{\alpha\beta,\gamma} + \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\varkappa}_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\varkappa}_{\lambda\gamma}]u_{\varkappa} \ . \tag{2.43}$$

We observe that the left-hand member is a tensor for any covariant vector field $u_\varkappa$. It follows that the quantity between brackets is a tensor, the so-called Riemann-Christoffel tensor

$$R^{\varkappa}_{\cdot\alpha\beta\gamma} = \Gamma^{\varkappa}_{\alpha\gamma,\beta} - \Gamma^{\varkappa}_{\alpha\beta,\gamma} + \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\varkappa}_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\varkappa}_{\lambda\gamma} \ . \tag{2.44}$$

We have already indicated in (2.44) that lowering of the superscript $\varkappa$ will presently be performed to the first subscript in the associated quadruple covariant tensor. Since (2.43) must be identically zero in view of the interchangeability of the order of covariant differentiation, all components of the Riemann-

Christoffel tensor must vanish identically, viz.

$$R^{\varkappa}_{\cdot\alpha\beta\gamma} = 0 \; . \tag{2.45}$$

The tensor character of (2.44) confirms our previous statement that the left-hand member of (2.22) is a tensor. Moreover, the necessary and sufficient conditions for the existence of Cartesian coordinates (2.22) are identical in form to (2.45), which equation expresses the vanishing of the Riemann-Christoffel tensor. Our present discussion confirms that (2.45) is a <u>necessary</u> condition for the Euclidean character of space. We know from our previous analysis in section 2.3 that it is also a <u>sufficient</u> condition in a simply-connected domain.

The associated Riemann-Christoffel tensor is defined by

$$R_{\rho\alpha\beta\gamma} = g_{\varkappa\rho} R^{\varkappa}_{\cdot\alpha\beta\gamma} \; , \tag{2.46}$$

and its first index may conversely be raised in order to obtain (2.44) again

$$R^{\varkappa}_{\cdot\alpha\beta\gamma} = g^{\varkappa\rho} R_{\rho\alpha\beta\gamma} \; . \tag{2.47}$$

From (2.13), (2.14) and (2.46) we obtain a convenient expression for the associated tensor

$$R_{\rho\alpha\beta\gamma} = \Gamma_{\rho\alpha\gamma,\beta} - \Gamma_{\rho\alpha\beta,\gamma} + g^{\lambda\mu}[\Gamma_{\lambda\rho\gamma}\Gamma_{\mu\alpha\beta} - \Gamma_{\lambda\rho\beta}\Gamma_{\mu\alpha\gamma}] \; . \tag{2.48}$$

A more revealing result is obtained, if we now substitute from (2.15) into the first two terms of (2.48). We find

$$R_{\rho\alpha\beta\gamma} = \tfrac{1}{2}[g_{\rho\gamma,\alpha\beta} + g_{\alpha\beta,\rho\gamma} - g_{\rho\beta,\alpha\gamma} - g_{\alpha\gamma,\rho\beta}] +$$

$$+ g^{\lambda\mu}[\Gamma_{\lambda\rho\gamma}\Gamma_{\mu\alpha\beta} - \Gamma_{\lambda\rho\beta}\Gamma_{\mu\alpha\gamma}] \; . \tag{2.49}$$

This form of the associated tensor shows that it is not only skew-symmetric in the subscripts β and γ, but also in the subscripts ρ and α. In fact, (2.49) is symmetric in the pairs of subscripts ρ, α and β, γ,

$$R_{\rho\alpha\beta\gamma} = R_{\beta\gamma\rho\alpha} = -R_{\alpha\rho\beta\gamma} = -R_{\rho\alpha\gamma\beta} \, . \tag{2.50}$$

It follows that no more than 6 of the 81 components of the Riemann-Christoffel tensor are independent.

The fact that the Riemann-Christoffel tensor has only six independent components permits its description in terms of a symmetric double contravariant tensor $S^{\varkappa\lambda}$, defined by

$$S^{\varkappa\lambda} = \frac{1}{4} \varepsilon^{\varkappa\rho\alpha} \varepsilon^{\lambda\beta\gamma} R_{\rho\alpha\beta\gamma} = \frac{1}{2} \varepsilon^{\varkappa\rho\alpha} \varepsilon^{\lambda\beta\gamma} [g_{\rho\gamma,\alpha\beta} + g^{\mu\nu} \Gamma_{\mu\rho\gamma} \Gamma_{\nu\alpha\beta}] . \tag{2.51}$$

Multiplying $S^{\varkappa\lambda}$ by $\varepsilon_{\varkappa\mu\nu} \varepsilon_{\lambda\rho\sigma}$, we obtain the inversion of (2.51) by means of (2.50) in the form

$$R_{\mu\nu\rho\sigma} = \varepsilon_{\varkappa\mu\nu} \varepsilon_{\lambda\rho\sigma} S^{\varkappa\lambda} \, . \tag{2.52}$$

Contraction of the Riemann-Christoffel tensor (2.44) with respect to its superscript and its last subscript results in a symmetric double covariant tensor which is called the <u>Ricci tensor</u>

$$R^{\varkappa}_{\cdot\alpha\beta\varkappa} = Q_{\alpha\beta} \, . \tag{2.53}$$

It is closely related to the associated tensor $S_{\alpha\beta}$ of $S^{\varkappa\lambda}$, obtained by lowering both superscripts. It is indeed easily verified by means of (1.81) that

$$Q_{\alpha\beta} = S_{\alpha\beta} - g_{\alpha\beta} g_{\varkappa\lambda} S^{\varkappa\lambda} \; . \tag{2.54}$$

The inversion of this relation is

$$S_{\alpha\beta} = Q_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\varkappa\lambda} Q_{\varkappa\lambda} \; . \tag{2.55}$$

The necessary and sufficient conditions for the Euclidean character of space may therefore be expressed by the vanishing of any of the tensors $R^{\varkappa}_{\cdot\alpha\beta\gamma}$, $R_{\rho\alpha\beta\gamma}$, $S^{\varkappa\lambda}$ or $Q_{\alpha\beta}$.

3. DEFORMATION

3.1 Deformation of a continuous medium.

In the discussion of the deformation of a continuous medium it is convenient to introduce first a fixed Cartesian frame of reference. Let x^i denote the Cartesian coordinates of a material point in some suitably chosen initial configuration of the medium which will serve as a reference configuration for the description of deformations. Let $\bar{x}^i$ denote the coordinates of the same material point, referred to the same Cartesian frame, in a second or final configuration of the medium. We may write

$$\bar{x}^i = x^i + u^i \ , \tag{3.1}$$

where u^i are the Cartesian displacement components in the deformation from the initial to the final configuration. The components u^i of the displacement vector $\underline{u}$ are assumed to be continuously differentiable functions of the initial Cartesian coordinates x^i up to any order required in the analysis. Furthermore we assume that the Jacobian of (3.1) is positive,

$$\left|\frac{\partial \bar{x}^i}{\partial x^j}\right| = |\delta^i_j + u^i_{,j}| > 0 \ . \tag{3.2}$$

Equation (3.1) represents a so-called point transformation, in contrast to the coordinate transformations which have been discussed previously. Whereas coordinate transformations bear on a transformation of the independent variables which specify one and the same point in space, point transformations express the change in position in space of one and the same material point.

This different character of the point transformation appears in (3.1) through the change in kernel letter from x to $\bar{x}$. On the other hand, the original superscripts are retained in a point transformation because all quantities in (3.1) are referred to the same fixed Cartesian coordinate system.

We now introduce general coordinates x^α in the reference configuration x^i by means of a functional coordinate transformation with positive Jacobian and a unique inverse transformation

$$x^i = x^i(x^\alpha) \ , \ x^\alpha = x^\alpha(x^i) \ , \ \frac{\partial x^i}{\partial x^\alpha} = A^i_\alpha \ , \ \frac{\partial x^\alpha}{\partial x^i} = A^\alpha_i \ . \qquad (3.3)$$

We have the identities from section (1.3)

$$A^i_\alpha A^\alpha_j = \delta^i_j \ , \ A^i_\alpha A^\beta_i = \delta^\beta_\alpha \ , \ |A^i_\alpha| \cdot |A^\beta_j| = 1 \ . \qquad (3.4)$$

It is now convenient to identify a material point by its general coordinates irrespective of the deformation which the medium undergoes. This implies that the Cartesian coordinates (3.1) with respect to the fixed frame of reference, and the displacement components referred to this Cartesian system, are considered as functions of the same general coordinates, viz.

$$\bar{x}^i(x^\alpha) = x^i(x^\alpha) + u^i(x^\alpha) \ . \qquad (3.5)$$

This mode of description is conveniently expressed by the statement that the general coordinates x^α are material coordinates, <u>embedded in the medium</u>.

The coordinate transformation (3.3) permits us to describe the physical displacement vector <u>u</u> from the reference configuration to the final configuration under consideration in terms of a contravariant vector u^α or a covariant vector u_α

$$u^\alpha = A^\alpha_i u^i \ , \quad u_\alpha = A^i_\alpha u_i \ , \tag{3.6}$$

where the covariant Cartesian components $u_i = g_{ij} u^j$ are equal to the corresponding contravariant Cartesian components u^i. The contravariant and covariant displacement vectors are related by

$$u^\alpha = g^{\alpha\beta} u_\beta \ , \quad u_\alpha = g_{\alpha\beta} u^\beta \ , \tag{3.7}$$

where the metric tensors in the reference configuration are specified by

$$g_{\alpha\beta} = A^i_\alpha A^j_\beta g_{ij} \ , \ g^{\alpha\beta} = A^\alpha_i A^\beta_j g^{ij} \ . \tag{3.8}$$

We emphasize that the contravariant and covariant displacement vectors (3.6) describe the physical displacement vector $\underline{u}$ from the viewpoint of the initial reference configuration of the medium. They are specified by the transformation (3.3) from Cartesian coordinates x^i to general coordinates x^α. An equivalent description of the physical displacement vector in terms of contravariant and covariant vectors, different from (3.6), would be obtained if we interpreted (3.5) as the basic transformation from the Cartesian coordinates $\bar{x}^i$ to general coordinates x^α. We shall not pursue such an alternative description of the displacement vector from the viewpoint of the final configuration $\bar{x}^i$. We are entirely free to choose the reference configuration to suit our convenience, and this freedom is adequate to obtain the simplest possible description of deformations appropriate for any particular application.

The (symmetric) covariant metric tensor in the second configuration $\bar{x}^i$ is obtained from the formula for the square of a line element $d\bar{s}$ in this configuration

$$(d\bar{s})^2 = g_{ij} d\bar{x}^i d\bar{x}^j \quad . \tag{3.9}$$

Substituting from (3.5) we obtain

$$(d\bar{s})^2 = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta \quad , \tag{3.10}$$

where the tensor $\bar{g}_{\alpha\beta}$ is defined by

$$\bar{g}_{\alpha\beta} = g_{ij}(A^i_\alpha + \frac{\partial u^i}{\partial x^\alpha})(A^j_\beta + \frac{\partial u^j}{\partial x^\beta}) \quad . \tag{3.11}$$

The tensor character of (3.11) is an immediate consequence of the chain rule for partial differentiation. From the inversion of (3.6)

$$u^i = A^i_\alpha u^\alpha \quad , \tag{3.12}$$

we obtain by partial differentiation

$$\frac{\partial u^i}{\partial x^\alpha} = \frac{\partial^2 x^i}{\partial x^\beta \partial x^\alpha} u^\beta + A^i_\beta u^\beta_{,\alpha} \quad . \tag{3.13}$$

Introducing the Christoffel symbols in our reference configuration

$$\Gamma^\varkappa_{\alpha\beta} = A^\varkappa_i \frac{\partial^2 x^i}{\partial x^\alpha \partial x^\beta} \quad , \tag{3.14}$$

we may reduce (3.13) to

$$\frac{\partial u^i}{\partial x^\alpha} = A^i_\beta [u^\beta_{,\alpha} + \Gamma^\beta_{\lambda\alpha} u^\lambda] = A^i_\beta u^\beta |_\alpha \quad , \tag{3.15}$$

where covariant differentiation is defined by means of the Christoffel symbols (3.14). The final result for the covariant metric tensor in the final configuration is

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha|\beta} + u_{\beta|\alpha} + g^{\lambda\mu}u_{\lambda|\alpha}u_{\mu|\beta} , \tag{3.16}$$

where free use has been made of Ricci's lemma for covariant differentiation of the metric tensor in the reference configuration.

The contravariant metric tensor in the final configuration is defined by equations similar to (1.64)

$$\bar{g}_{\alpha\beta}\bar{g}^{\beta\gamma} = \delta_{\alpha}^{\gamma} . \tag{3.17}$$

This contravariant metric tensor in the second configuration (3.5) will not occur very frequently in the analysis. Raising and lowering of indices will always be performed by means of the metric tensors in the reference configuration, in the absence of an explicit statement to the contrary. If the equations require a raising or lowering of indices by the metric tensors $\bar{g}^{\alpha\beta}$ or $\bar{g}_{\alpha\beta}$, this will be stipulated. The Christoffel symbols of the first kind in the second configuration are defined by equations similar to (2.15)

$$\bar{\Gamma}_{\lambda\alpha\beta} = \frac{1}{2}[\bar{g}_{\lambda\alpha,\beta} + \bar{g}_{\lambda\beta,\alpha} - \bar{g}_{\alpha\beta,\lambda}] , \tag{3.18}$$

and the Christoffel symbols of the second kind in this configuration are specified by formulae similar to (2.16)

$$\bar{\Gamma}^{\varkappa}_{\alpha\beta} = \bar{g}^{\varkappa\lambda}\bar{\Gamma}_{\lambda\alpha\beta} = \frac{1}{2}\bar{g}^{\varkappa\lambda}[\bar{g}_{\lambda\alpha,\beta}+\bar{g}_{\lambda\beta,\alpha}-\bar{g}_{\alpha\beta,\lambda}] . \tag{3.19}$$

The deformation of the medium in the transition from the reference configuration $x^i(x^\alpha)$ to the second configuration $\bar{x}^i(\alpha)$ is described completely by the change in the covariant metric tensor. The strain tensor is defined by

$$\gamma_{\alpha\beta} = \frac{1}{2}(\overline{g}_{\alpha\beta}-g_{\alpha\beta}) = \frac{1}{2}[u_{\alpha|\beta}+u_{\beta|\alpha}+g^{\lambda\mu}u_{\lambda|\alpha}u_{\mu|\beta}] \ . \tag{3.20}$$

If the general coordinates x^α coincide with the Cartesian coordinates x^i this tensor reduces to the well-known Lagrangian strain tensor

$$\gamma_{ij} = \frac{1}{2}[u_{i,j}+u_{j,i}+g^{hk}u_{h,i}u_{k,j}] \ . \tag{3.21}$$

It is sometimes convenient to employ the final configuration as the reference configuration. The original configuration is then specified by the displacement field -u from the reference configuration. The strain tensor which describes the deformation in the inverse transition from the final configuration to the original configuration is now specified by (3.20) if u_α is replaced by $-u_\alpha$. The actual strain tensor, which specifies the deformation in the transition from the original configuration to the final reference configuration, is opposite in sign. Hence we have for the actual strain tensor in this description

$$\gamma_{\alpha\beta} = \frac{1}{2}[u_{\alpha|\beta}+u_{\beta|\alpha}-g^{\lambda\mu}u_{\lambda|\alpha}u_{\mu|\beta}] \ . \tag{3.22}$$

It must be emphasized that the covariant displacement vector u_α and covariant differentiation are again defined with respect to the reference configuration, which now coincides with the final configuration. If the general coordinates x^α coincide with Cartesian coordinates x^i in this final reference configuration, (3.22) reduces to the well-known Eulerian strain tensor

$$\gamma_{ij} = \frac{1}{2}[u_{i,j}+u_{j,i}-g^{hk}u_{h,i}u_{k,j}] \ . \tag{3.23}$$

In the theory of elastic stability it is often necessary to consider three different configurations, viz. the undeformed configuration of the medium, a fundamental state obtained by a displacement field U from the undeformed configuration, and an adjacent state obtained from the fundamental state by an additional displacement field u. The strain tensor in the deformation from the undeformed state to the fundamental state will be indicated by the kernel letter Γ, the strain tensor which describes the deformation from the fundamental state to the adjacent state will be denoted by the kernel letter γ.

If the undeformed state is taken as the reference configuration, the strain tensor in the transition to the fundamental state is specified by

$$\Gamma_{\alpha\beta} = \frac{1}{2}[U_{\alpha|\beta}+U_{\beta|\alpha}+g^{\lambda\mu}U_{\lambda|\alpha}U_{\mu|\beta}] \ , \tag{3.24}$$

and the strain tensor for the deformation from the fundamental state to the adjacent state is given by

$$\gamma_{\alpha\beta} = \frac{1}{2}[u_{\alpha|\beta}+u_{\beta|\alpha}+g^{\lambda\mu}U_{\lambda|\alpha}u_{\mu|\beta}+g^{\lambda\mu}U_{\mu|\beta}u_{\lambda|\alpha}+g^{\lambda\mu}u_{\lambda|\alpha}u_{\mu|\beta}] \ . \tag{3.25}$$

Both the covariant displacement vectors U_α, u_α and the covariant derivatives are here defined with respect to the undeformed reference configuration.

On the other hand, if the fundamental state is taken as the reference configuration, the strain tensor describing the deformation from the undeformed state to the fundamental state is given by

$$\Gamma_{\alpha\beta} = \frac{1}{2}[U_{\alpha|\beta}+U_{\beta|\alpha}-g^{\lambda\mu}U_{\lambda|\alpha}U_{\mu|\beta}] , \tag{3.26}$$

and the strain tensor which describes the deformation from the fundamental state to the adjacent state is

$$\gamma_{\alpha\beta} = \frac{1}{2}[u_{\alpha|\beta}+u_{\beta|\alpha}+g^{\lambda\mu}u_{\lambda|\alpha}u_{\mu|\beta}] . \tag{3.27}$$

The covariant displacement vectors U_α, u_α and the covariant derivatives are now defined with respect to the fundamental state as the reference configuration.

The advantage of the second description, in terms of the fundamental state as the reference configuration, is obvious. Expression (3.27) for the strain tensor in the transition from the fundamental state to the adjacent state is much simpler than expression (3.25) in terms of the undeformed state as the reference configuration. On the other hand, there is little to choose between expressions (3.24) and (3.26) for the description of the deformation from the undeformed state to the fundamental state.

3.2 <u>The compatibility conditions</u>.

Since the deformation of our medium occurs in Euclidean space, the Riemann-Christoffel tensor vanishes both in the initial reference configuration and in the second or final configuration. For the reference configuration this fact is expressed by putting (2.51) equal to zero

$$S^{\varkappa\lambda} = \frac{1}{2}\,\varepsilon^{\varkappa\rho\alpha}\varepsilon^{\lambda\beta\gamma}[g_{\rho\gamma,\alpha\beta}+g^{\mu\nu}\Gamma_{\mu\rho\gamma}\Gamma_{\nu\alpha\beta}] = 0 . \tag{3.28}$$

The ε-tensor in the final configuration is given by

$$\bar{\varepsilon}^{\varkappa\rho\alpha} = e^{\varkappa\rho\alpha}(\bar{g})^{-\frac{1}{2}} , \tag{3.29}$$

where $\bar{g}$ is the determinant of the covariant metric tensor $\bar{g}_{\alpha\beta}$ in the final configuration. The vanishing of the Riemann-Christoffel tensor in this configuration is then expressed by the counterpart of (3.28)

$$\bar{S}^{\varkappa\lambda} = \frac{1}{2}\,\bar{\varepsilon}^{\varkappa\rho\alpha}\bar{\varepsilon}^{\lambda\beta\gamma}[\bar{g}_{\rho\gamma,\alpha\beta}+\bar{g}^{\mu\nu}\bar{\Gamma}_{\mu\rho\gamma}\bar{\Gamma}_{\nu\alpha\beta}] = 0 , \tag{3.30}$$

where the barred Christoffel symbols in the final configuration are specified by (3.18). This equation remains obviously valid if we replace the barred ε-tensor by the ε-tensor in the reference configuration.

If the general coordinates x^α coincide with the Cartesian coordinates x^i in the reference configuration, each term in (3.28) vanishes separately. The first term within the brackets in (3.30) and the barred Christoffel symbols reduce to

$$\bar{g}_{hk,ij} = 2\gamma_{hk,ij} , \quad \bar{\Gamma}_{mhk} = \gamma_{mh,k}+\gamma_{mk,h}-\gamma_{hk,m} . \tag{3.31}$$

Equation (3.30) then takes the form

$$\varepsilon^{phi}\varepsilon^{qjk}[\gamma_{hk,ij}+\frac{1}{2}\bar{g}^{mn}(\gamma_{mh,k}+\gamma_{mk,h}-\gamma_{hk,m})(\gamma_{ni,j}+\gamma_{nj,i}-\gamma_{ij,n})] = 0 , \tag{3.32}$$

where the tensor $\bar{g}^{mn}$ may be expressed in terms of the strain tensor by means of the equations

$$\bar{g}^{mn}(g_{nr}+2\gamma_{nr}) = \delta^m_r \tag{3.33}$$

Equations (3.32) express the so-called compatibility conditions for the Lagrangian strain tensor (3.21). They represent the six necessary and sufficient conditions in a simply-connected domain for the existence of Cartesian coordinates $\bar{x}^i$ in the second configuration of the medium, specified by (3.1). In other words, equations (3.32) are the necessary and sufficient conditions for the integrability of equations (3.21), if the strain components are specified and these equations have to be solved for the Cartesian displacement components u_i. This solution is of course not unique. We may always superimpose an arbitrary rigid-body displacement of the medium.

The compatibility conditions (3.32) are easily transformed to general coordinates. We observe that $\gamma_{\rho\sigma|\alpha\beta}$ and $\gamma_{\mu\rho|\sigma}$ are tensors, where the covariant differentiations are again defined in terms of the Christoffel symbols in the reference configuration. These tensors reduce to $\gamma_{hk,ij}$ and $\gamma_{mh,k}$ in Cartesian coordinates. It follows that the left-hand member of

$$\varepsilon^{\varkappa\rho\alpha}\varepsilon^{\lambda\beta\sigma}[\gamma_{\rho\sigma|\alpha\beta} + \tfrac{1}{2}\bar{g}^{\mu\nu}(\gamma_{\mu\rho|\sigma}+\gamma_{\mu\sigma|\rho}-\gamma_{\rho\sigma|\mu})(\gamma_{\nu\alpha|\beta}+\gamma_{\nu\beta|\alpha}-\gamma_{\alpha\beta|\nu})] = 0 \tag{3.34}$$

is a tensor which reduces to the left-hand member of (3.32) in Cartesian coordinates, and our equation (3.34) is therefore equivalent to (3.32). The tensor $\bar{g}^{\mu\nu}$ may again be expressed in terms of the strain tensor by means of the equations

$$\bar{g}^{\mu\nu}(g_{\nu\rho} + 2\gamma_{\nu\rho}) = \delta^{\mu}_{\rho} \, . \tag{3.35}$$

Similar compatibility conditions hold, if the final configuration is taken as the reference configuration with respect to which the covariant displacement vector and covariant differentiation are defined. The contravariant metric tensor in the initial configuration is then defined by

$$g_o^{\mu\nu}(g_{\nu\rho}-2\gamma_{\nu\rho}) = \delta_\rho^\mu \quad , \tag{3.36}$$

and the compatibility conditions for the strain tensor (3.22) of Eulerian type read

$$\varepsilon^{\varkappa\rho\alpha}\varepsilon^{\lambda\beta\sigma}[\gamma_{\rho\sigma|\alpha\beta} - \tfrac{1}{2}g_o^{\mu\nu}(\gamma_{\mu\rho|\sigma}+\gamma_{\mu\sigma|\rho}-\gamma_{\rho\sigma|\mu})(\gamma_{\nu\alpha|\beta}+\gamma_{\nu\beta|\alpha}-\gamma_{\alpha\beta|\nu})] = 0. \tag{3.37}$$

4. GEOMETRY OF SURFACES AND SHELLS

4.1 Surface invariants, vectors and tensors.

An arbitrary point in Euclidean space may be specified by its coordinates $y^i (i=1,2,3)$ in a fixed Cartesian frame of reference.*) A surface in space may then be described by the three Cartesian coordinates of a generic point as functions of two parameters $x^\alpha (\alpha=1,2)$, which are called surface coordinates, viz.

$$y^i = y^i(x^\alpha), \quad i=1,2,3. \tag{4.1}$$

Henceforward we shall employ Greek indices exclusively to refer to the pair of surface coordinates. The summation convention is also applied to Greek indices. A repeated Greek index, occurring once as a superscript and once as a subscript, implies summation over this dummy suffix from 1 to 2.

A coordinate line is defined as a curve on the surface for which either of the surface coordinates has a constant value. The x^1-line is defined by x^2=const., and vice versa. The partial derivatives of (4.1) with respect to x^α may be regarded as the Cartesian components of a tangent vector to to the x^α-line

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*) Heretofore we have employed Latin indices to imply reference to a Cartesian system. In the sequel Latin indices will also be used to refer to a triple of curvilinear coordinates in space. Hence we need now a special kernel letter for the indication of Cartesian coordinates and we reserve the letter y for this purpose.

$$t^i_\alpha = \frac{\partial y^i}{\partial x^\alpha} . \tag{4.2}$$

We shall restrict our attention to regular surfaces with continuous partial derivatives of (4.1) up to any order required in the analysis. We also assume that the tangent vectors to the coordinate lines, defined by (4.2), are always linearly independent. Their vector product

$$\varepsilon_{hij} t^i_1 t^j_2 = e_{hij} t^i_1 t^j_2 \tag{4.3}$$

then defines a unique positive direction of the normal to the surface. The tangential vectors t^i_1, t^i_2 and the positive normal have the same orientation in space as the fixed Cartesian frame of reference.

A point in space may now be specified by the pair of surface coordinates x^α of its projection on the surface and by its distance $z = x^3$ to this projection, taken positive if the point lies on the positive normal to the surface at its projection.*) Latin indices will henceforward be employed to refer to the triple of space coordinates $x^i (i=1,2,3)$. Although this coordinate system is not Cartesian or rectilinear, it has some special properties. The x^3-lines (x^1=const. and x^2=const.) are straight lines normal to all surfaces $z = x^3$ = const. These surfaces are therefore equidistant, and they will be called parallel to the original surface. The special nature of the

*) This description may become ambiguous for arbitrary points in space. It is entirely adequate, however, for the specifications of an arbitrary material point in a thin shell whose middle surface coincides with the surface under consideration.

coordinate system x^i appears also in its metric tensors which are characterized by the relations

$$g_{\alpha 3} = 0,\ g_{33} = 1;\ g^{\alpha 3} = 0,\ g^{33} = 1;\ g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma \ . \qquad (4.4)$$

From all functional coordinate transformations in space we now single out in particular those which preserve the special nature of our original coordinate system characterized by (4.4). In other words, we consider transformations of the type

$$x^{\alpha'} = x^{\alpha'}(x^\alpha),\ x^{3'} = x^3;\ x^\alpha = x^\alpha(x^{\alpha'}),\ x^3 = x^{3'}, \qquad (4.5)$$

which may be called <u>transformations of surface coordinates only</u>. The partial derivatives of (4.5) are denoted as before

$$\frac{\partial x^{\alpha'}}{\partial x^\alpha} = A_\alpha^{\alpha'},\ \frac{\partial x^\alpha}{\partial x^{\alpha'}} = A_{\alpha'}^\alpha \ . \qquad (4.6)$$

They are obviously <u>independent of $z=x^3$</u>, and they satisfy the relations

$$A_\alpha^{\alpha'}A_{\beta'}^\alpha = \delta_{\beta'}^{\alpha'},\ A_\alpha^{\alpha'}A_{\alpha'}^\beta = \delta_\alpha^\beta,\ |A_\alpha^{\alpha'}||A_{\beta'}^\beta| = 1. \qquad (4.7)$$

Moreover, we shall restrict our attention to transformations with a <u>positive Jacobian</u>.

A quantity of order zero which remains invariant under transformations of surface coordinates is called a <u>surface invariant</u>. A quantity with one Greek superscript or one Greek subscript is called a contravariant or covariant <u>surface vector</u> u^α or v_α, if its components transform according to the law

$$u^{\alpha'} = A^{\alpha'}_{\alpha} u^{\alpha},\ u^{\alpha} = A^{\alpha}_{\alpha'} u^{\alpha'} \ ;\ v_{\alpha'} = A^{\alpha}_{\alpha'} v_{\alpha},\ v_{\alpha} = A^{\alpha'}_{\alpha} v_{\alpha'} . \quad (4.8)$$

A similar definition is employed for <u>surface tensors</u>. A quantity $t^{\alpha\beta\cdots}_{\lambda\mu\cdots}$ is called a surface tensor if its transformation law is specified by

$$t^{\alpha'\beta'\cdots}_{\lambda'\mu'\cdots} = A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} \ \cdots\ A^{\lambda}_{\lambda'} A^{\mu}_{\mu'} \ \cdots\ t^{\alpha\beta\cdots}_{\lambda\mu\cdots} . \quad (4.9)$$

It may be left to the reader to verify that all concepts of addition, multiplication and contraction of tensors and the quotient law for tensors, discussed in section 1.4, may be carried over to surface tensors. The single modification required is a reduction of the range of Greek indices from 3 to 2.

Although the transformation law for surface tensors is independent of the third coordinate $z=x^3$, we emphasize that such tensors themselves need not be independent of this coordinate. If such a surface tensor is a function of x^3, however, its partial derivative with respect to x^3 is again a surface tensor of the same order and type. This important property will be used repeatedly in the sequel.

Geometric and physical quantities in space have been classified heretofore according to their behaviour under <u>general</u> functional transformations of all three coordinates. Where necessary, we shall indicate this behaviour by the adjective "spatial" in the description of such quantities. A spatial invariant is obviously also a surface invariant. On the other hand, a spatial contravariant vector u^i has in our special coordinates one component u^3 which is a surface invariant, and two components u^α which constitute a contravariant surface vector. Likewise, a spatial covariant vector v_i may be decomposed into a surface invariant v_3 and a covariant surface vector v_α. Similar representations hold for tensors of higher order. For instance, a double contravariant spatial tensor t^{ij} may be decomposed into a surface invariant t^{33}, two contravariant surface vectors $t^{\alpha 3}$ and $t^{3\alpha}$, and a double contravariant surface tensor $t^{\alpha\beta}$.

An important example of surface tensors is provided by $g_{\alpha\beta}$ and $g^{\alpha\beta}$, which are called the covariant and contravariant metric surface tensors, because they specify the metrics in the equidistant surfaces $z=x^3$=const. These tensors are, of course, functions of the normal coordinate, and this functional dependence on x^3 will be analyzed in the next section. The appropriate metric surface tensors $g^{\alpha\beta}$ and $g_{\alpha\beta}$ will always be used to raise and lower indices in any surface tensor specified at a surface $z=x^3$=const.

The middle surface of a shell plays a fundamental role in shell theory. It will be identified with the basic surface $z=x^3=0$ of our family of parallel surfaces. The shell domain is then specified by $|z| = |x^3| \leqq \frac{1}{2}h$, where h is the shell thickness which may be a continuously differentiable function of the surface coordinates $h(x^\alpha)$. If the shell thickness is constant the shell faces $z=x^3= \pm \frac{1}{2}h$ are also surfaces parallel to the middle surface.

Many concepts and properties are equally applicable to all surfaces in the family of equidistant surfaces $z=x^3=\text{const}$. Nevertheless it is often convenient to confine our attention to a particular surface, and it is appropriate for our purposes to single out the shell middle surface. In this connection we introduce a special kernel letter for the metric tensor of the middle surface by the definitions

$$g_{\alpha\beta}(x^\lambda,0) = a_{\alpha\beta}(x^\lambda); \; g^{\alpha\beta}(x^\lambda,0) = a^{\alpha\beta}(x^\lambda). \qquad (4.10)$$

Raising and lowering of indices in middle surface tensors will therefore always be performed by means of the metric tensors $a^{\alpha\beta}$ and $a_{\alpha\beta}$.

4.2 The fundamental tensors of a surface.

The special nature of our spatial coordinates, appearing in the properties (4.4) of the metric tensor, leads also to a considerable simplification of formulae (2.15) and (2.16) for the Christoffel symbols. All such symbols which have two or three indices 3 vanish identically, viz.

$$\Gamma_{33\alpha} = \Gamma_{3\alpha 3} = \Gamma_{\alpha 33} = \Gamma_{333} = \Gamma^3_{3\alpha} = \Gamma^3_{\alpha 3} = \Gamma^\alpha_{33} = \Gamma^3_{33} = 0. \qquad (4.11)$$

An important consequence of (4.11) is that the partial derivatives of an invariant with respect to x^3, of any order, coincide with the covariant derivatives with respect to x^3. The Christoffel symbols with one index 3 are given by simplified formulae

$$\Gamma_{3\alpha\beta} = \Gamma^3_{\alpha\beta} = -\frac{1}{2} g_{\alpha\beta,3}; \quad \Gamma_{\alpha\beta 3} = \Gamma_{\alpha 3\beta} = \frac{1}{2} g_{\alpha\beta,3};$$

$$\Gamma^\alpha_{3\beta} = \Gamma^\alpha_{\beta 3} = \frac{1}{2} g^{\alpha\lambda} g_{\lambda\beta,3} . \tag{4.12}$$

These formulae also show that these Christoffel symbols constitute surface tensors. No simplification arises if all indices are either 1 or 2, viz.

$$\Gamma_{\varkappa\alpha\beta} = \frac{1}{2}[g_{\varkappa\alpha,\beta} + g_{\varkappa\beta,\alpha} - g_{\alpha\beta,\varkappa}] ; \quad \Gamma^\lambda_{\alpha\beta} = g^{\lambda\varkappa}\Gamma_{\varkappa\alpha\beta} . \tag{4.13}$$

The transformation law of the mixed symbols (4.13) is the exact counterpart of (2.10)

$$\Gamma^{\lambda'}_{\alpha'\beta'} = A^{\lambda'}_{\lambda} A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} \Gamma^{\lambda}_{\alpha\beta} + A^{\lambda'}_{\lambda} \frac{\partial^2 x^\lambda}{\partial x^{\alpha'} \partial x^{\beta'}} . \tag{4.14}$$

We shall now prove that the covariant metric surface tensor $g_{\alpha\beta}$ is a quadratic function of the normal coordinate $z=x^3$, viz.

$$g_{\alpha\beta} = a_{\alpha\beta} - 2zb_{\alpha\beta} + z^2 c_{\alpha\beta} , \tag{4.15}$$

where the symmetric surface tensors $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $c_{\alpha\beta}$ are independent of z. These tensors are called the covariant fundamental tensors of the middle surface. The first fundamental tensor is of course identical with the metric tensor of the middle surface. The factor -2 in the linear term in (4.15) is introduced for a more convenient geometric interpretation of the second fundamental tensor.

A simple proof of (4.15) is obtained from the fact that the Riemann-Christoffel tensor vanishes identically in

Euclidean space. In our coordinates $x^i = x^\alpha, x^3$ we have from (2.49)

$$R_{hijk} = \frac{1}{2}[g_{hk,ij} + g_{ij,hk} - g_{hj,ik} - g_{ik,hj}] +$$
$$+ g^{mn}[\Gamma_{mhk}\Gamma_{nij} - \Gamma_{mhj}\Gamma_{nik}] = 0. \quad (4.16)$$

We now consider <u>three</u>*) independent components for which $h=\alpha$, $k=\beta$, $i=j=3$. From (4.4) we have

$$g_{33,\alpha\beta} = g_{\alpha 3,3\beta} = g_{3\beta,\alpha 3} = 0, \quad (4.17)$$

and by means of (4.11) and (4.12) we reduce (4.16) to

$$g_{\alpha\beta,33} = 2g^{\lambda\mu}\Gamma_{\lambda\alpha 3}\Gamma_{\mu\beta 3} = \frac{1}{2} g^{\lambda\mu} g_{\lambda\alpha,3} g_{\mu\beta,3} . \quad (4.18)$$

We now differentiate (4.18) partially with respect to $x^3 = z$. The partial derivative of the contravariant metric surface tensor may be evaluated from the equation

$$g^{\lambda\mu}{}_{,3} g_{\mu\nu} + g^{\lambda\mu} g_{\mu\nu,3} = 0, \quad (4.19)$$

obtained by partial differentiation of the last identity (4.4). We find

$$g^{\lambda\mu}{}_{,3} = - g^{\lambda\rho} g^{\mu\sigma} g_{\rho\sigma,3} . \quad (4.20)$$

The second derivatives occurring in the differentiated right-hand member of (4.18) are removed by a repeated substitution from (4.18) itself. The result of these elementary calculations is

*) We shall return to the three remaining components in section 4.5.

$$g_{\alpha\beta,333} = -\frac{1}{2} g^{\lambda\rho}g^{\mu\sigma}g_{\rho\sigma,3}g_{\lambda\alpha,3}g_{\mu\beta,3} + $$
$$+ \frac{1}{4} g^{\lambda\mu}g^{\rho\sigma}g_{\lambda\alpha,3}g_{\rho\mu,3}g_{\sigma\beta,3} + $$
$$+ \frac{1}{4} g^{\lambda\mu}g^{\rho\sigma}g_{\rho\lambda,3}g_{\sigma\alpha,3}g_{\mu\beta,3} = 0. \qquad (4.21)$$

The final result zero in the right-hand member of (4.21) is achieved by a suitable renaming of the dummy indices in the second and third terms. In the second term we replace μ by ρ, ρ by σ, and σ by μ. In the third term we interchange λ and σ.

Equation (4.21) proves that $g_{\alpha\beta}$ is indeed a quadratic function of z and admits therefore the representation (4.15). At the same time our proof shows that the three fundamental tensors are not independent. In fact, (4.18) implies the relation

$$c_{\alpha\beta} = g^{\lambda\mu}(b_{\lambda\alpha} - zc_{\lambda\alpha})(b_{\mu\beta} - zc_{\mu\beta}), \qquad (4.22)$$

holding for all z. At the middle surface z=0 we obtain therefore from (4.22)

$$c_{\alpha\beta} = a^{\lambda\mu}b_{\lambda\alpha}b_{\mu\beta} = b_{\lambda\alpha}b_{\beta}^{\lambda}, \qquad (4.23)$$

expressing the third fundamental tensor of the middle surface in terms of the metric tensor and the second fundamental tensor.

The skew-symmetric contravariant and covariant ε-tensors of the middle surface are defined by

$$\varepsilon^{\alpha\beta}(x^{\lambda}) = \varepsilon^{3\alpha\beta}(x^{\lambda},0) \ ; \ \varepsilon_{\alpha\beta}(x^{\lambda}) = \varepsilon_{3\alpha\beta}(x^{\lambda},0), \qquad (4.24)$$

where $\varepsilon^{3\alpha\beta}$ and $\varepsilon_{3\alpha\beta}$ specify the spatial ε-tensors. We have for the non-vanishing components

$$\varepsilon^{12} = -\varepsilon^{21} = \frac{1}{\sqrt{a}} \; ; \; \varepsilon_{12} = -\varepsilon_{21} = \sqrt{a} \; , \qquad (4.25)$$

where a is the determinant of the covariant middle surface metric tensor $a_{\alpha\beta}$. It may be left to the reader to verify the formulae

$$\varepsilon_{\alpha\beta} = a_{\alpha\lambda} a_{\beta\mu} \varepsilon^{\lambda\mu} \; , \; \varepsilon^{\alpha\beta} = a^{\alpha\lambda} a^{\beta\mu} \varepsilon_{\lambda\mu} \; ; \qquad (4.26)$$

$$a_{\alpha\beta} \varepsilon^{\alpha\lambda} \varepsilon^{\beta\mu} = a^{\lambda\mu} \; , \; a^{\alpha\beta} \varepsilon_{\alpha\lambda} \varepsilon_{\beta\mu} = a_{\lambda\mu} \; , \qquad (4.27)$$

which are the counterpart of formulae (1.80)-(1.84) for the spatial ε-tensors.

A symmetric surface tensor has two independent invariants. In the case of the second fundamental tensor it is convenient to introduce the invariants

$$\frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = H \; , \qquad (4.28)$$

$$\frac{1}{2} \varepsilon^{\alpha\lambda} \varepsilon^{\beta\mu} b_{\alpha\beta} b_{\lambda\mu} = \frac{b}{a} = K \; , \qquad (4.29)$$

where b is the determinant of $b_{\alpha\beta}$. It will appear presently that the second fundamental tensor is closely related to the curvature of the middle surface, and for this reason the invariants H and K are called the <u>mean</u> and <u>Gaussian</u> curvatures respectively. From (4.29) we may obtain an alternative expression for the third fundamental tensor. Multiplying both members first by $\varepsilon_{\nu\rho}\varepsilon_{\sigma\tau}$, we have the important formula

$$b_{\nu\sigma} b_{\rho\tau} - b_{\nu\tau} b_{\rho\sigma} = K \varepsilon_{\nu\rho} \varepsilon_{\sigma\tau} . \qquad (4.30)$$

Multiplying both members in (4.30) by $a^{\nu\sigma}$, we find with appropriate use of (4.23),(4.27) and (4.28)

$$2Hb_{\rho\tau} - c_{\rho\tau} = Ka_{\rho\tau} , \qquad (4.31)$$

which relation may also be written in the form of the desired expression for the third fundamental tensor

$$c_{\rho\tau} = 2Hb_{\rho\tau} - Ka_{\rho\tau} . \qquad (4.32)$$

The determinant g of the spatial metric tensor (4.4) is evaluated by means of the surface invariant

$$\frac{1}{2}\, \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu} g_{\alpha\beta} g_{\lambda\mu} = \frac{g}{a} . \qquad (4.33)$$

Substituting from (4.15) and (4.32) we obtain*)

$$\begin{aligned} \frac{g}{a} = \frac{1}{2}\, \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu} &[(1-Kz^2)a_{\alpha\beta} - 2z(1-Hz)b_{\alpha\beta}]\cdot \\ &\cdot[(1-Kz^2)a_{\lambda\mu} - 2z(1-Hz)b_{\lambda\mu}] = \\ &= \frac{1}{2}\,(1-Kz^2)^2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu} a_{\alpha\beta}a_{\lambda\mu} + \\ &- 2z(1-Hz)(1-Kz^2)\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu} a_{\alpha\beta}b_{\lambda\mu} + \\ &+ 2z^2(1-Hz)^2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu} b_{\alpha\beta}b_{\lambda\mu} = \\ &= (1-Kz^2)^2 - 4Hz(1-Hz)(1-Kz^2) + 4Kz^2(1-Hz)^2 = \\ &= [1-2Hz + Kz^2]^2. \end{aligned} \qquad (4.34)$$

*) Since g in (4.34) must be positive, we have to restrict our family of surfaces parallel to the middle surface to the range of values of z, containing z=0, for which $[1-2Hz+Kz^2]$ is always positive.

We turn now to a geometric interpretation of the second fundamental tensor of the middle surface. For this purpose we consider a line element ds(z) on a surface z=const. parallel to the middle surface, specified by differentials dx^α of the surface coordinates. Its length is obtained by means of the metric tensor (4.15)

$$\begin{aligned} ds(z) &= [a_{\alpha\beta}dx^\alpha dx^\beta - 2zb_{\alpha\beta}dx^\alpha dx^\beta + z^2 c_{\alpha\beta}dx^\alpha dx^\beta]^{\frac{1}{2}} = \\ &= [a_{\alpha\beta}dx^\alpha dx^\beta]^{\frac{1}{2}}\left[1-2z\,\frac{b_{\alpha\beta}dx^\alpha dx^\beta}{a_{\alpha\beta}dx^\alpha dx^\beta} + z^2\,\frac{c_{\alpha\beta}dx^\alpha dx^\beta}{a_{\alpha\beta}dx^\alpha dx^\beta}\right]^{\frac{1}{2}} = \\ &= ds(o)[1-z\,\frac{b_{\alpha\beta}dx^\alpha dx^\beta}{a_{\alpha\beta}dx^\alpha dx^\beta} + O(z^2)]\,, \end{aligned} \tag{4.35}$$

where ds(o) is the length of the line element dx^α on the middle surface, and $O(z^2)$ stands for a term which tends to zero as z^2 if z approaches zero.

Our formula (4.35) is reminiscent of a similar formula in the theory of plane curves. Consider a plane curve with a continuous radius of curvature R. An equidistant or parallel curve is defined by the locus of points on the normals of the first curve at a constant distance z to this curve. The line elements between adjacent normals, ds(o) on the original curve and ds(z) on the equidistant curve, are related by the simple formula

$$ds(z) = ds(o)[1 - \frac{z}{R}]\,, \tag{4.36}$$

where z is taken positive in the direction towards the center of curvature. Accordingly <u>we define the (normal) curvature of</u>

the middle surface along the line element dx^α by

$$\frac{1}{R} = \frac{b_{\alpha\beta}dx^\alpha dx^\beta}{a_{\alpha\beta}dx^\alpha dx^\beta} . \tag{4.37}$$

The remaining difference between formulae (4.35) and (4.36), consisting of the term of second order in z between the brackets in (4.35), arises from the circumstance that adjacent normals to the middle surface need not lie in a plane.

The quotient of binary quadratic forms in (4.37) with a positive definite denominator has two extreme values, a maximum and a minimum. These are called the principal curvatures of the middle surface $1/R_1$ and $1/R_2$. The associated principal directions on the middle surface are specified by line elements $\underset{1}{dx^\alpha}$ and $\underset{2}{dx^\alpha}$. They are the nonvanishing solutions of the equations

$$(b_{\alpha\beta} - \frac{1}{R} a_{\alpha\beta})dx^\beta = 0, \tag{4.38}$$

which express the conditions for a stationary value of the curvature (4.37). The principal curvatures themselves are given by the two(real) roots of the quadratic equation

$$\det |b_{\alpha\beta} - \frac{1}{R} a_{\alpha\beta}| = 0, \tag{4.39}$$

expressing the condition for the existence of a nontrivial solution of the equations (4.38). We may write (4.39) in the equivalent form

$$\frac{1}{2} \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}(b_{\alpha\beta} - \frac{1}{R} a_{\alpha\beta})(b_{\lambda\mu} - \frac{1}{R} a_{\lambda\mu}) = 0. \tag{4.40}$$

By means of (4.27), (4.28) and (4.29) this equation is reduced to

$$(\frac{1}{R})^2 - 2H \frac{1}{R} + K = 0, \tag{4.41}$$

establishing the connection between the invariants H and K (the mean curvature and the Gaussian curvature) and the principal curvatures

$$\frac{1}{R_1} + \frac{1}{R_2} = 2H \text{ , } \frac{1}{R_1 R_2} = K. \qquad (4.42)$$

In the case of unequal principal curvatures it is easily proved that the principal directions are orthogonal. Applying (4.38) to the first principal direction, and multiplying both members by $\underset{2}{dx}^\alpha$, we obtain by contraction the identity

$$(b_{\alpha\beta} - \frac{1}{R_1} a_{\alpha\beta})\underset{2}{dx}^\alpha \underset{1}{dx}^\beta = 0. \qquad (4.43)$$

Interchanging the roles of the two principal directions, we obtain the similar identity in which R_1 is replaced by R_2. Hence we must have

$$(\frac{1}{R_1} - \frac{1}{R_2})a_{\alpha\beta}\underset{1}{dx}^\alpha \underset{2}{dx}^\beta = 0, \qquad (4.44)$$

and the orthogonality of the principal directions on the middle surface has indeed been established. Orthogonality of the corresponding line elements on a surface parallel to the middle surface now follows by appropriate use of (4.38) from

$$g_{\alpha\beta}\underset{1}{dx}^\alpha \underset{2}{dx}^\beta = (a_{\alpha\beta} - 2zb_{\alpha\beta} + z^2 a^{\lambda\mu} b_{\lambda\alpha} b_{\mu\beta})\underset{1}{dx}^\alpha \underset{2}{dx}^\beta =$$

$$= (1 - \frac{z}{R_1})(1 - \frac{z}{R_2})\ a_{\alpha\beta}\underset{1}{dx}^\alpha \underset{2}{dx}^\beta = 0. \qquad (4.45)$$

In the case of equal principal curvatures $\frac{1}{R_2} = \frac{1}{R_1}$, every direction on the middle surface is a principal direction. The second and third fundamental tensors are now proportional to the metric tensor

$$b_{\alpha\beta} = \frac{1}{R_1} a_{\alpha\beta} , \quad c_{\alpha\beta} = \left(\frac{1}{R_1}\right)^2 a_{\alpha\beta} . \qquad (4.46)$$

It is, of course, always possible to select a pair of orthogonal principal directions in this case.

A continuous curve on the middle surface which is tangent everywhere to one of the principal directions is called a <u>line of curvature</u>. Two families of lines of curvature exist with the property that each line of curvature of one family is orthogonal to all lines of curvature of the other family. In shell theory it is often convenient to employ coordinates for which the coordinate lines coincide with the lines of curvature. Such coordinates might well be called <u>principal coordinates</u>.*) The components of the metric tensor in a surface parallel to the middle surface are in principal coordinates**)

$$g_{11} = a_{11}\left(1 - \frac{z}{R_1}\right)^2 , \; g_{12} = 0, \; g_{22} = a_{22}\left(1 - \frac{z}{R_2}\right)^2 . \qquad (4.47)$$

Occasionally it is more convenient in shell theory to employ orthogonal coordinates on the middle surface which do not coincide with principal coordinates. These coordinates are no longer orthogonal in surfaces parallel to the middle surface, and $a_{12} = 0$ is the only vanishing component of the fundamental tensors. Introducing the radii of curvature of the middle surface R_1^* and R_2^* along the x^1- and x^2-lines, and the radius of

*) It should be noted that a principal coordinate system x^1, x^2 is not unique. Any transformation of type $x^{1'} = x^{1'}(x^1)$, $x^{2'} = x^{2'}(x^2)$ with nonvanishing derivatives results in new principal coordinates $x^{1'}, x^{2'}$.

**) The range of admissible values of z for surfaces parallel to the middle surface is obviously restricted by the two-conditions $z/R_1 < 1$ and $z/R_2 < 1$.

torsion T^* of the middle surface referred to these coordinates, defined by the relations

$$b_{11} = \frac{a_{11}}{R_1^*} , \quad b_{12} = \frac{\sqrt{a}}{T^*} = \frac{\sqrt{a_{11}a_{22}}}{T^*} , \quad b_{22} = \frac{a_{22}}{R_2^*} , \qquad (4.48)$$

we obtain the components of the third fundamental tensor in the form

$$c_{11} = [(\frac{1}{R_1^*})^2 + (\frac{1}{T^*})^2]a_{11} , \quad c_{12} = \frac{1}{T^*}[\frac{1}{R_1^*} + \frac{1}{R_2^*}]\sqrt{a_{11}a_{22}} ,$$

$$c_{22} = [(\frac{1}{R_2^*})^2 + (\frac{1}{T^*})^2]a_{22} . \qquad (4.49)$$

Expressions for the mean and Gaussian curvatures are now given by

$$\frac{1}{R_1^*} + \frac{1}{R_2^*} = 2H , \quad \frac{1}{R_1^* R_2^*} - (\frac{1}{T^*})^2 = K. \qquad (4.50)$$

4.3 Covariant surface differentiation.

A scalar surface field is defined by an invariant φ as a continuously differentiable function of the pair of surface coordinates x^α. It follows immediately from the chain rule for partial differentiation that the partial derivatives of a scalar surface field constitute a covariant surface vector $\varphi_{,\alpha}$. On the other hand, the partial derivatives of a surface vector field, defined on some surface z=const. parallel to the middle surface, do not constitute a surface tensor field. Their transformation law is again expressed by (2.1) and (2.2), where the range of Greek indices is now reduced from 3 to 2. An appropriate modification of the concept of surface differentiation of a surface vector (i.e. differentiation with respect to a surface coordinate) is obviously again desirable.

In Euclidean space we achieved our similar purpose in section 2.1 by defining the covariant derivative of a spatial vector field as the tensor whose components reduce to the partial derivatives in a Cartesian coordinate system. A similar definition is not possible for the surface derivative of a surface vector field because we are not ensured a priori that Cartesian surface coordinates exist. In fact, it will appear in section 4.5 that Cartesian surface coordinates do not exist except for surfaces of zero Gaussian curvature. In spite of this difficulty, we may of course examine the explicit formulae (2.7) and (2.9) for the spatial covariant derivatives of contravariant and covariant spatial vector fields. The tensor character of these derivatives in general coordinates is evidently due to the particular transformation law (2.10) of the Christoffel symbols, and the same transformation law (except for the reduction in range of Greek indices) is obeyed by the Christoffel symbols (4.13) formed with the surface metric tensor. We are thus led to consider the quantities

$$u^{\alpha}|_{\beta} = u^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\varkappa\beta} u^{\varkappa} \ , \ u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^{\varkappa}_{\alpha\beta} u_{\varkappa} \ , \qquad (4.51)$$

obtained from a contravariant or covariant surface vector field, and it is easily verified by means of (4.14) and the two-dimensional counterpart of (2.1) and (2.2) that the quantities (4.51) are indeed tensors. They will be called the covariant surface derivatives of the contravariant or covariant surface vector field. Henceforward we shall denote covariant surface differentiation with respect to x^β as in (4.51) by an additional subscript β preceded by a single vertical line. In order to avoid confusion with covariant differentiation in space, we shall distinguish the spatial operation by a double vertical line preceding the additional subscript.

The concept of covariant surface differentiation is easily extended to differentiable surface tensor fields of any order, specified on some surface z=const. parallel to the middle surface, again by an appeal to the quotient law. The required discussion is the counterpart of section 2.3. For example, the covariant surface derivative of a surface tensor field of contravariant order 2 and covariant order 2 is given by a formula identical to (2.26), where the range of Greek indices is now, of course, reduced from 3 to 2, viz.

$$t^{\alpha\beta}_{\lambda\mu}\Big|_{\rho} = t^{\alpha\beta}_{\lambda\mu,\rho} + \Gamma^{\alpha}_{\varkappa\rho} t^{\varkappa\beta}_{\lambda\mu} + \Gamma^{\beta}_{\varkappa\rho} t^{\alpha\varkappa}_{\lambda\mu} - \Gamma^{\varkappa}_{\lambda\rho} t^{\alpha\beta}_{\varkappa\mu} - \Gamma^{\varkappa}_{\mu\rho} t^{\alpha\beta}_{\lambda\varkappa} . \qquad (4.52)$$

In section 2.3 we established the preservation of the ordinary rules for partial differentiation of a sum or a product in covariant differentiation of a sum or product of tensors, by recalling that the covariant derivatives reduce to partial derivatives in Cartesian coordinates. In the absence of Cartesian surface coordinates, we cannot apply the same argument to covariant surface differentiation. The formal identity between formulae (2.26) and (4.52), however, permits us to infer that the rules for differentiation of a sum or a product do nevertheless apply to covariant surface differentiation of a sum or a product of surface tensors.

In section 4.1 we have discussed the representation of spatial tensors in terms of a surface tensor of the same order and a number of surface tensors of lower orders. We shall now examine the relationship between covariant spatial and surface derivatives.

The spatial derivative $\varphi_{,i}$ of a scalar field is, of course, a covariant spatial vector, which may be decomposed into a covariant surface vector $\varphi_{,\alpha}$ and a surface invariant $\varphi_{,3}$.

The covariant spatial derivative $u_{i\|j}$ of a covariant spatial vector field constitutes a double covariant spatial tensor, which is represented by a double covariant surface tensor $u_{\alpha\|\beta}$, two covariant surface vectors $u_{3\|\beta}$ and $u_{\alpha\|3}$, and a surface invariant $u_{3\|3}$. Likewise, the covariant spatial derivative $v^i\|_j$ of a contravariant spatial vector field may be represented by a mixed double surface tensor $v^\alpha\|_\beta$, a contravariant surface vector $v^\alpha\|_3$, a covariant surface vector $v^3\|_\beta$, and a surface invariant $v^3\|_3$. By means of (4.11) we obtain the relations

$$u_{3\|3} = u_{3,3} \quad ; \quad v^3\|_3 = v^3{}_{,3} \quad ; \qquad (4.53)$$

$$u_{\alpha\|3} = u_{\alpha,3} - \Gamma^{\varkappa}_{\alpha 3} u_{\varkappa} \quad ; \quad v^\alpha\|_3 = v^\alpha{}_{,3} + \Gamma^\alpha_{\varkappa 3} v^{\varkappa} \quad ; \qquad (4.54)$$

$$u_{3\|\beta} = u_{3,\beta} - \Gamma^{\varkappa}_{3\beta} u_{\varkappa} \quad ; \quad v^3\|_\beta = v^3{}_{,\beta} + \Gamma^3_{\varkappa\beta} v^{\varkappa} \quad ; \qquad (4.55)$$

$$u_{\alpha\|\beta} = u_{\alpha,\beta} - \Gamma^{\varkappa}_{\alpha\beta} u_{\varkappa} - \Gamma^3_{\alpha\beta} u_3 \quad ; \quad v^\alpha\|_\beta = v^\alpha{}_{,\beta} + \Gamma^\alpha_{\varkappa\beta} v^{\varkappa} + \Gamma^\alpha_{3\beta} v^3 . \qquad (4.56)$$

The last formulae may be written in the form of relations between the double surface tensors, which form part of the representation of the spatial covariant derivatives, and the covariant surface derivatives of the surface vectors, viz.

$$u_{\alpha||\beta} = u_{\alpha|\beta} - \Gamma^{3}_{\alpha\beta} u_3 \; ; \; v^{\alpha}||_{\beta} = v^{\alpha}|_{\beta} + \Gamma^{\alpha}_{3\beta} v^3 . \qquad (4.57)$$

It may be left to the reader to derive similar formulae for the representation of covariant spatial derivatives for tensor fields in space. We mention only some additional examples for double and triple covariant spatial tensors s_{ij} and t_{ijk} for immediate application, viz.

$$s_{\alpha\beta||\lambda} = s_{\alpha\beta|\lambda} - \Gamma^{3}_{\alpha\lambda} s_{3\beta} - \Gamma^{3}_{\beta\lambda} s_{\alpha 3} \; ; \qquad (4.58)$$

$$t_{3\alpha\beta||\lambda} = t_{3\alpha\beta|\lambda} - \Gamma^{3}_{\alpha\lambda} t_{33\beta} - \Gamma^{3}_{\beta\lambda} t_{3\alpha 3} \; , \qquad (4.59)$$

where use has been made of (4.11) in the second relation. If we apply (4.58) to the metric tensor, the left-hand member vanishes on account of Ricci's lemma in space. Remembering (4.4), we obtain Ricci's lemma on the surface

$$g_{\alpha\beta|\lambda} = 0. \qquad (4.60)$$

This result may, of course, also be obtained directly from the basic formula for covariant surface differentiation by substituting for the Christoffel symbols from (4.13). Applying (4.59) to the ε-tensor in space, we find by a similar argument

$$\varepsilon_{3\alpha\beta|\lambda} = 0. \qquad (4.61)$$

Hence we may treat both the covariant surface metric tensor and the covariant surface ε-tensor as constants for the purpose of covariant surface differentiation. It may again be left to the reader to prove the same properties for the contravariant surface metric tensor and ε-tensor.

The foregoing discussion of covariant surface differentiation applies to any surface z=const. in the family of surfaces parallel to the middle surface under consideration. The Christoffel symbols involved must, of course, be formed with the metric tensor at the particular surface under discussion. Of special importance in shell theory is covariant surface differentiation on the middle surface itself. The Christoffel symbols (4.12) may then be expressed in the second fundamental tensor

$$\Gamma_{3\alpha\beta}(x^\lambda,0) = \Gamma^3_{\alpha\beta}(x^\lambda,0) = b_{\alpha\beta} \; ; \qquad (4.62)$$

$$\Gamma_{\alpha\beta3}(x^\lambda,0) = \Gamma_{\alpha3\beta}(x^\lambda,0) = -b_{\alpha\beta} \; ; \qquad (4.63)$$

$$\Gamma^\alpha_{3\beta}(x^\lambda,0) = \Gamma^\alpha_{\beta3}(x^\lambda,0) = -a^{\alpha\mu}b_{\mu\beta} = -b^\alpha_\beta \; . \qquad (4.64)$$

No simplification occurs in the formulae for the Christoffel symbols (4.13), but it is worthwhile to introduce a separate kernel letter A for these symbols on the middle surface

$$\Gamma_{\varkappa\alpha\beta}(x^\mu,0) = A_{\varkappa\alpha\beta} = \tfrac{1}{2}[a_{\varkappa\alpha,\beta}+a_{\varkappa\beta,\alpha}-a_{\alpha\beta.\varkappa}] \; ;$$

$$\Gamma^\lambda_{\alpha\beta}(x^\mu,0) = A^\lambda_{\alpha\beta} = a^{\lambda\varkappa}A_{\varkappa\alpha\beta} \; . \qquad (4.65)$$

The formula for the covariant surface derivatives of middle surface tensors is, of course, now given by (4.52), where $\Gamma^\lambda_{\alpha\beta}$ is replaced by $A^\lambda_{\alpha\beta}$. The relations (4.54), (4.55) and (4.57) have a special form on the middle surface

$$u_{\alpha}\|_3 = u_{\alpha,3} + b^{\varkappa}_{\alpha}u_{\varkappa} \; ; \; v^{\alpha}\|_3 = v^{\alpha}{}_{,3} - b^{\alpha}_{\varkappa}v^{\varkappa} \; ; \tag{4.66}$$

$$u_3\|_{\beta} = u_{3,\beta} + b^{\varkappa}_{\beta}u_{\varkappa} \; ; \; v^3\|_{\beta} = v^3{}_{,\beta} + b_{\varkappa\beta}v^{\varkappa} \; ; \tag{4.67}$$

$$u_{\alpha}\|_{\beta} = u_{\alpha}|_{\beta} - b_{\alpha\beta}u_3 \; ; \; v^{\alpha}\|_{\beta} = v^{\alpha}|_{\beta} - b^{\alpha}_{\beta}v^3 \; . \tag{4.68}$$

Ricci's lemma on a surface (4.60), and formulae (4.61) for the ε-tensor have the special form on the middle surface

$$a_{\alpha\beta}|_{\lambda} = 0 \; , \; \varepsilon_{\alpha\beta}|_{\lambda} = 0 \; . \tag{4.69}$$

4.4 Green's theorem on a surface.

In the theory of shells we need Green's theorem for the transformation of a surface integral of the divergence of a contravariant surface vector into a line integral. In view of the non-existence of Cartesian surface coordinates, the derivation of this theorem is slightly more complicated than the discussic of Green's theorem in space in section 2.4. We shall confine our discussion to the middle surface of the shell. The extension to surfaces parallel to the middle surface is fairly obvious and may be left to the reader.

The surface divergence of a contravariant middle surface vector field u^{α} is defined by the surface invariant

$$u^{\alpha}|_{\alpha} = u^{\alpha}{}_{,\alpha} + A^{\alpha}_{\varkappa\alpha}u^{\varkappa} \; . \tag{4.70}$$

From the identity (2.28) for the contracted Christoffel symbols we obtain in view of (4.11)

$$\sqrt{a}\; A^{\alpha}_{\varkappa\alpha} = (\sqrt{a})_{,\varkappa} \tag{4.71}$$

Hence we may rewrite the divergence (4.70) in the form

$$u^{\alpha}|_{\alpha} = \frac{1}{\sqrt{a}} (\sqrt{a}\ u^{\varkappa})_{,\varkappa} \tag{4.72}$$

Since the unit normal to the middle surface coincides with the x^3-direction, we have $n_3 = 1$ in the general formula (1.87) for a surface element spanned by the infinitesimal contravariant surface vectors $\underset{1}{dx}^{\alpha}$, $\underset{2}{dx}^{\alpha}$. This formula may therefore be written in the form

$$dS = \varepsilon_{3\alpha\beta}\underset{1}{dx}^{\alpha}\underset{2}{dx}^{\beta} . \tag{4.73}$$

If we consider the middle surface element spanned by $\underset{1}{dx}^{\alpha} = dx^1, 0$ and $\underset{2}{dx}^{\alpha} = 0, dx^2$, we may write in view of (4.24) and (4.25)

$$dS = \sqrt{a}\ dx^1 dx^2. \tag{4.74}$$

The integral of the divergence (4.72) over a portion S of the middle surface now takes the form

$$\int_S u^{\alpha}|_{\alpha}\ dS = \iint (\sqrt{a}\ u^{\varkappa})_{,\varkappa}\ dx^1 dx^2 . \tag{4.75}$$

Let C denote the closed simple smooth contour on the middle surface which encloses the area S. The positive sense on C and the positive direction of the normal to the middle surface correspond to the right-hand screw rule in a right-handed coordinate system. We may now apply Gauss's theorem to double integrals in the form

$$\iint \varphi_{,1} dx^1 dx^2 = \int_C \varphi dx^2, \iint \varphi_{,2} dx^1 dx^2 = - \int_C \varphi dx^1. \tag{4.76}$$

As a result we obtain Green's theorem on the middle surface in the form

$$\int_S u^\alpha|_\alpha dS = \int_C \varepsilon_{\alpha\beta} u^\alpha dx^\beta , \qquad (4.77)$$

if once more appropriate use is made of (4.25). An equivalent form of this theorem, which shows complete similarity with the theorem in space, is obtained by introducing the outward covariant unit surface vector n_α normal to the contour C. Let ds denote the arc element along the contour. The normal vector n_α is then given by

$$n_\alpha = \varepsilon_{\alpha\beta} \frac{dx^\beta}{ds} , \qquad (4.78)$$

and Green's theorem takes the form

$$\int_S u^\alpha|_\alpha dS = \int_C u^\alpha n_\alpha ds . \qquad (4.79)$$

4.5 The equations of Gauss and Codazzi.

Gauss's equation for a surface relates its Riemann-Christoffel tensor to the second fundamental tensor. The derivation of this equation must, of course, be preceded by a definition of the Riemann-Christoffel tensor of a surface and a discussion of its basic properties. For the sake of brevity we confine our attention to the middle surface.

The second covariant surface derivative of a scalar field φ on the middle surface is a covariant surface tensor

$$\varphi|_{\alpha\beta} = (\varphi_{,\alpha})|_\beta = \varphi_{,\alpha\beta} - A^{\varkappa}_{\alpha\beta}\varphi_{,\varkappa} . \qquad (4.80)$$

This tensor is obviously symmetric by the symmetry of the Christoffel symbols.

On the other hand, in the absence of Cartesian surface coordinates, we cannot assume a similar symmetry property of the second covariant surface derivative of a surface vector. By the formalism of covariant surface differentiation, however, we may derive a complete counterpart of (2.43) for repeated spatial differentiation in the case of surface differentiation

$$u_{\alpha|\beta\gamma} - u_{\alpha|\gamma\beta} = [A^{\varkappa}_{\alpha\gamma,\beta} - A^{\varkappa}_{\alpha\beta,\gamma} + A^{\lambda}_{\alpha\gamma}A^{\varkappa}_{\lambda\beta} - A^{\lambda}_{\alpha\beta}A^{\varkappa}_{\lambda\gamma}]u_{\varkappa} \quad . \qquad (4.81)$$

The quantity between brackets is evidently a surface tensor, by the quotient law for such tensors, and it is called the Riemann-Christoffel tensor of the middle surface

$$r^{\varkappa}_{\cdot\alpha\beta\gamma} = A^{\varkappa}_{\alpha\gamma,\beta} - A^{\varkappa}_{\alpha\beta,\gamma} + A^{\lambda}_{\alpha\gamma}A^{\varkappa}_{\lambda\beta} - A^{\lambda}_{\alpha\beta}A^{\varkappa}_{\lambda\gamma} \quad . \qquad (4.82)$$

It should be noted that this surface tensor is distinct from the fourth-order surface tensor $R^{\varkappa}_{\cdot\alpha\beta\gamma}$ which forms a part of the surface representation of the spatial Riemann-Christoffel tensor. For this reason we have also employed a new kernel letter for the Riemann-Christoffel tensor of the middle surface.

The associated Riemann-Christoffel tensor of the middle surface is obtained by lowering the superscript to the vacant first subscript

$$r_{\rho\alpha\beta\gamma} = a_{\rho\varkappa}r^{\varkappa}_{\cdot\alpha\beta\gamma} = A_{\rho\alpha\gamma,\beta} - A_{\rho\alpha\beta,\gamma} + a^{\lambda\mu}[A_{\lambda\rho\gamma}A_{\mu\alpha\beta} - A_{\lambda\rho\beta}A_{\mu\alpha\gamma}], \qquad (4.83)$$

again similar to (2.48). Substituting from (4.65) into the first pair of terms in (4.83), we find

$$r_{\rho\alpha\beta\gamma} = \frac{1}{2}[a_{\rho\gamma,\alpha\beta}+a_{\alpha\beta,\rho\gamma}-a_{\rho\beta,\alpha\gamma}-a_{\alpha\gamma,\rho\beta}] + $$

$$+ a^{\lambda\mu}[A_{\lambda\rho\gamma}A_{\mu\alpha\beta}-A_{\lambda\rho\beta}A_{\mu\alpha\gamma}] , \tag{4.84}$$

similar to (2.49). Here again the associated tensor is skew-symmetric in ρ and α, as well as in β and γ, and symmetric in the pairs of subscripts ρ,α and β,γ. It follows that the associated Riemann-Christoffel tensor of the middle surface has only <u>one independent component</u> r_{1212}, and the same conclusion remains, of course, valid for the mixed tensor

$$r^{\varkappa}_{\cdot\alpha\beta\gamma} = a^{\varkappa\rho}r_{\rho\alpha\beta\gamma} . \tag{4.85}$$

Hence we may, if we wish, represent the associated tensor in the form

$$r_{\rho\alpha\beta\gamma} = C\varepsilon_{\rho\alpha}\varepsilon_{\beta\gamma} , \tag{4.86}$$

where C is a <u>surface invariant</u>. We shall see presently that this invariant is actually the Gaussian curvature of the middle surface.

We note first the important property that the skew-symmetric part of the second covariant surface derivative of any middle surface tensor may be expressed in terms of the tensor itself and the Riemann-Christoffel tensor of the middle surface. For a contravariant surface vector we have

$$u^{\alpha}|_{\beta\gamma} = (a^{\alpha\lambda}u_{\lambda})|_{\beta\gamma} = a^{\alpha\lambda}u_{\lambda|\beta\gamma} , \tag{4.87}$$

and we obtain the desired result in terms of a second associated tensor of the Riemann-Christoffel tensor of the middle surface

$$u^{\alpha}\big|_{\beta\gamma} - u^{\alpha}\big|_{\gamma\beta} = r_{\varkappa\cdot\beta\gamma}^{\cdot\alpha} u^{\varkappa} = -r^{\alpha}_{\cdot\varkappa\beta\gamma} u^{\varkappa} \,. \tag{4.88}$$

The structure of the expression in the general case appears from the example for a fourth-order surface tensor of contravariant order 2 and covariant order 2

$$t^{\alpha\beta}_{\gamma\delta}\big|_{\lambda\mu} - t^{\alpha\beta}_{\gamma\delta}\big|_{\mu\lambda} = r^{\varkappa}_{\cdot\gamma\lambda\mu} t^{\alpha\beta}_{\varkappa\delta} + r^{\varkappa}_{\cdot\delta\lambda\mu} t^{\alpha\beta}_{\gamma\varkappa} + - r^{\alpha}_{\cdot\varkappa\lambda\mu} t^{\varkappa\beta}_{\gamma\delta} - r^{\beta}_{\cdot\varkappa\lambda\mu} t^{\alpha\varkappa}_{\gamma\delta} \quad . \tag{4.89}$$

The straightforward, if somewhat laborious direct proof of this relation may be left to the reader. A more convenient proof may be obtained by a consideration of the (symmetric) covariant second surface derivative of the invariant

$$t^{\alpha\beta}_{\gamma\delta} u_{\alpha} v_{\beta} w^{\gamma} s^{\delta} \quad , \tag{4.90}$$

where $u_{\alpha}, v_{\beta}, w^{\gamma}$ and s^{δ} are arbitrary twice continuously differentiable surface vector fields. By means of (4.81), (4.87), and an appeal to the quotient law, we may then verify (4.89) without excessive labour. The details are again left to the reader.

We now turn to a proof of Gauss's theorem that the surface invariant C in (4.86) is the Gaussian curvature K of the middle surface. We consider the components $R_{\rho\alpha\beta\gamma}$ of the Riemann-Christoffel tensor in space at some point of the middle surface. We have from (4.16)

$$R_{\rho\alpha\beta\gamma} = \frac{1}{2}[a_{\rho\gamma,\alpha\beta} + a_{\alpha\beta,\rho\gamma} - a_{\rho\beta,\alpha\gamma} - a_{\alpha\gamma,\rho\beta}] +$$

$$+ a^{\lambda\mu}[A_{\lambda\rho\gamma}A_{\mu\alpha\beta} - A_{\lambda\rho\beta}A_{\mu\alpha\gamma}] +$$

$$+ g^{33}[\Gamma_{3\rho\gamma}\Gamma_{3\alpha\beta} - \Gamma_{3\rho\beta}\Gamma_{3\alpha\gamma}] \quad . \tag{4.91}$$

The first two lines coincide with (4.84), and the third line may be rewritten by means of (4.4) and (4.62). Remembering that the Riemann-Christoffel tensor and all its associated tensors in Euclidean space vanish identically, we find Gauss's equation

$$r_{\rho\alpha\beta\gamma} = b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\alpha\beta} \quad . \tag{4.92}$$

In view of (4.30), an equivalent form is

$$r_{\rho\alpha\beta\gamma} = K\varepsilon_{\rho\alpha}\varepsilon_{\beta\gamma} \quad . \tag{4.93}$$

Two important consequences of Gauss's equation should be mentioned here. First of all, it proves our earlier statement that Cartesian surface coordinates, in which the left-hand member of (4.93) would vanish, cannot exist except on surfaces of zero Gaussian curvature everywhere. Secondly, if the surface is capable of inextensional deformation, that is a deformation in which the metric surface tensor remains unaltered, then its Gaussian curvature is also invariant in such a deformation.

So far we have examined the consequences of the vanishing of 4 independent components of the Riemann-Christoffel tensor in space. In section 4.2 we have exhausted the information obtainable from the three independent components in which two indices 3 occur, in the derivation of Gauss's equation we have used the vanishing of R_{1212}. The remaining two independent components with one subscript 3 must also vanish identically, and this condition will yield the so-called Codazzi equations of the middle surface.

From the general formula (4.16) we have

$$R_{3\alpha\beta\gamma} = \frac{1}{2}[g_{3\gamma,\alpha\beta} + g_{\alpha\beta,3\gamma} - g_{3\beta,\alpha\gamma} - g_{\alpha\gamma,3\beta}] + \\ + g^{\lambda\mu}[\Gamma_{\lambda3\gamma}\Gamma_{\mu\alpha\beta} - \Gamma_{\lambda3\beta}\Gamma_{\mu\alpha\gamma}] + \\ + g^{33}[\Gamma_{33\gamma}\Gamma_{3\alpha\beta} - \Gamma_{33\beta}\Gamma_{3\alpha\gamma}] = 0 . \tag{4.94}$$

The first and third terms in the first line vanish by (4.4), and the last line is zero by (4.11). Our equation is therefore simplified to

$$R_{3\alpha\beta\gamma} = \frac{1}{2}[g_{\alpha\beta,3\gamma} - g_{\alpha\gamma,3\beta}] + g^{\lambda\mu}[\Gamma_{\lambda3\gamma}\Gamma_{\mu\alpha\beta} - \Gamma_{\lambda3\beta}\Gamma_{\mu\alpha\gamma}] = 0. \tag{4.95}$$

A further simplification is achieved on the middle surface by means of (4.10), (4.63) and (4.65), and we obtain the Codazzi equations in the form

$$- b_{\alpha\beta,\gamma} + b_{\alpha\gamma,\beta} - A^{\lambda}_{\alpha\beta}b_{\lambda\gamma} + A^{\lambda}_{\alpha\gamma}b_{\lambda\beta} = 0 . \tag{4.96}$$

Adding and subtracting the term $A^{\lambda}_{\beta\gamma}b_{\lambda\alpha}$, we may write this equation in terms of covariant surface derivatives of the second fundamental tensor

$$b_{\alpha\beta|\gamma} - b_{\alpha\gamma|\beta} = 0 . \tag{4.97}$$

The number of independent equations is indeed 2, as is most easily seen by writing (4.97) in the equivalent form

$$\varepsilon^{\beta\gamma} b_{\alpha\beta|\gamma} = 0 \ , \tag{4.98}$$

where the left-hand member is a covariant surface vector.

It appears from our discussion that the equations of Gauss and Codazzi are <u>necessary</u> conditions for the vanishing of the Riemann-Christoffel tensor in space. We have not proved that they are also sufficient conditions, because we have only imposed the vanishing of the components $R_{3\alpha\beta3}$ in the entire domain of space under consideration (section 4.2), and the vanishing of the components $R_{\rho\alpha\beta\gamma}$ and $R_{3\alpha\beta\gamma}$ on the middle surface. We mention, without proof, that our conditions are actually <u>sufficient</u> for the vanishing of the spatial Riemann-Christoffel tensor in the entire domain of space under consideration.

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