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FOREWORD

This Manual was prepared by the Structural Research and Development Group, Structures Section, Research and Development Division of Republic Aviation Corporation. The work was initiated under Contract AF33(616)-6066 in the 750A Applied Research Program, the Mechanics of Flight, Project No. 1367, Structural Design Criteria, and Task No. 14002, "Structural Analysis Methods". The work is now documented under Task 136710. This work was initiated under the direction of the Structural Analysis Unit, Structures Branch, Aircraft Laboratory, Directorate of Laboratories, Wright Air Development Center.* Mr. I. Winnegrad acted initially as project engineer and was succeeded by Mr. C. Richard. The Manual was completed under the direction of the Structural Analysis Unit, Configuration Research Section, Structures Branch, Flight Dynamics Laboratory, Deputy Commander/Technology, Aeronautical Systems Division, with Mr. G. E. Maddux as Project Engineer.

The work was coordinated and supervised by Dr. R. S. Levy, Head of the Structural Research and Development Group. His valuable suggestions and criticisms are gratefully acknowledged as are those of the following personnel of the Applied Research and Development Division of Republic Aviation Corporation: Mr. A. Alberi, Acting Manager of Technical Engineering; Mr. C. Rosenkranz, Acting Chief Structures Engineer; and Mr. C. Meissner, Principal Structures Engineer.

NOTES ON USING THE MANUAL

This Manual consists of five basic sections, divided into numbered sub-sections and paragraphs. For simplicity in cross-referencing material in the text, all portions of the Manual designated with a two-tier number (e. g. , 1.1) are considered sub-sections, and all portions designated by numbers of three or more tiers (e. g. , 1.1.1 or 1.1.1.1) are considered paragraphs.

Throughout the Manual, the numbered paragraphs (or sub-sections) have been used as the basis for numbering figures, tables, and equations, with new sequences beginning with each numbered paragraph. Figure and table numbers consist of an appropriate paragraph number, followed by a sequence number for the particular figure or table. For convenience the paragraph designations have been omitted from the equation numbers. When an equation from another paragraph is cited in the text, the number of the paragraph in which that equation occurs is also cited. When a paragraph number is not given in conjunction with the citation of an equation, it is to be assumed that the equation is included in the paragraph in which the citation occurs.

References are listed at the end of those sections which have more than one reference. In addition, each section contains its own complete table of contents and list of symbols.

*Now under direction of Flight Dynamics Laboratory, Directorate of Aeromechanics, Aeronautical Systems Division.

ABSTRACT

This second volume of the Thermo-Structural Analysis Manual considers additional problems in the field of thermal and mechanical stress analysis not fully treated in Volume I. Special emphasis is given to nonlinear analysis of beams and plates and to axisymmetric thermo-elastic analysis of thin shells. Following the format of Volume I, nondimensional graphs, formulas and tables are developed where feasible. For clarification of the analytical techniques and the use of the numerical data, illustrative examples are given.

The following problems are treated in five individual sections:

- (1) Large deflection analysis of straight elastic beams with axial end restraint and axial end loads coupled with transverse loading and temperature.
- (2) Approximate determination of the axial end loads and deformations in heated beam columns with initial eccentricities.
- (3) Approximate solutions for the buckling of eccentric columns accommodating nonlinear stress-strain laws.
- (4) Axisymmetric large deflections of circular plates subjected to thermal and mechanical loads.
- (5) Axisymmetric thermo-elastic analysis of thin shells.

PUBLICATION REVIEW

This publication has been reviewed and is approved.

FOR THE COMMANDER:



R. F. HOKNER
Chief, Structures Branch
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INTRODUCTION

This Volume II of the Manual is an extension of the basic work presented in Volume I and covers the following special problem areas in heated beams, plates and shells:

- (1) Nonlinear beam analyses including the effects of initial eccentricities and nonlinear material properties.
- (2) Elastic circular plates loaded and heated axisymmetrically, considering the effects of large deflections
- (3) The axisymmetric thermoelastic analysis of shells developed in Volume I is generalized, removing the restrictions on geometry and including the effects of arbitrary temperature variations through the thickness.

The material is presented in the form of five independent reports or sections.

The emphasis has been placed on the development of analytical techniques and formulas with their corresponding physical interpretations. Where feasible, the investigations are self-contained, starting with fundamental theoretical considerations and notions in the field of static thermo-structural analysis. However, the reader may find it useful to refer to Volume I where a systematic development of the basic concepts is given.

Brief summaries of the five sections of this volume of the Manual follow.

Section 1. Beam Columns Subjected to Elevated Temperature and Mechanical Loads

This section treats beam columns with axial end restraints and loads coupled with transverse loads and temperature gradients. Numerical results, "exact" within the framework of large deflection beam theory are tabulated for certain cases.

Section 2. Approximate Solution for an Axially Restrained Column Subjected to Elevated Temperature and Lateral Load

This section presents an approximate method of solution of the heated beam column problem. The method is amenable to problems involving spanwise variations of load thermal gradients and stiffness and permits treatment of initial eccentricities.

Section 3. Approximate Solution for the Buckling of Eccentric Columns

This section develops and presents nondimensional curves to predict column buckling loads. Nonlinear stress-strain relationships are accommodated as well as lateral loads and initial eccentricities or thermally induced deformations.

Section 4. Axisymmetric Large Deflections of Circular Plates Subjected to Thermal and Mechanical Loads

The interaction of membrane stresses and bending is considered for axially restrained circular plates subjected to heat and load. The differential equations governing the axisymmetric case are derived and an iterative digital scheme is used to obtain numerical results for a special case over a wide range of temperature and load parameters.

Section 5. Axisymmetric Stresses and Deflections in Shells Due to Thermal and Mechanical Loads

The general equation for linear elastic analysis of axisymmetric shell problems is developed including the effects of temperature and load. The special cases of conical and cylindrical shells are developed and a numerical example is given for a cylinder with temperature gradients through the thickness and along the length.

SECTION 1
BEAM COLUMNS SUBJECTED TO ELEVATED TEMPERATURE
AND MECHANICAL LOADS

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1.0

SECTION 1
BEAM COLUMNS SUBJECTED TO ELEVATED TEMPERATURE
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SECTION 1
BEAM COLUMNS SUBJECTED TO ELEVATED TEMPERATURE
AND MECHANICAL LOADS

1.1. SUMMARY

This report considers the nonlinear analysis (large deflections) of beams with axial end restraints and axial end loads coupled with transverse loading and temperature.

In order to keep the number of parameters within practical limits the following conditions are investigated:

- (1) Distributed transverse loads are uniform over the span while concentrated loads are at the midspan.
- (2) The temperature varies linearly through the depth and is constant in a span-wise direction.
- (3) The beam is assumed to be simply supported at its ends for bending and elastically restrained axially (Figure 1.3-1).

Tables of numerical results in nondimensional form are presented for the cases of zero and full axial end restraint in rectangular beams. These tables may be used to determine maximum deflections and bending moments.

1.2 INTRODUCTION

In structural analysis beam columns differ from simple beams in that the addition of axial end loads and restraints interact with transverse loads in a nonlinear manner, thus invalidating the principle of superposition (Reference 1-1).

When the axial end loads are specified and the beam is unrestrained axially, the solution for bending moments and deflections are obtained, in general, by solving a linear, non-homogeneous differential equation with constant coefficients subject to appropriate boundary conditions. If, in addition, the beam is restrained axially, the total end load is an unknown and an additional compatibility relation must be employed.

The analysis presented considers a beam of constant cross section for which the Bernoulli-Euler assumption of classical beam theory is employed. This implies that plane sections perpendicular to the centroidal axis before bending remain plane and perpendicular to the deflected centroidal axis. It is further assumed that the material behavior is linearly elastic.

It can be shown that the seemingly approximate technique used below (Sub-section 1.3) yields results which are identical to those obtained by a moderately large deflection analysis in which nonlinear strain-displacement relations are employed.

1.2 (Cont'd)

The following symbols are used throughout this section:

b	Width of rectangular beam
h	Depth of beam
x	Spanwise coordinate
y	Vertical deflection
\bar{y}	Nondimensional central deflection
A	Cross sectional area
E	Young's modulus
H	Magnitude of axial load in beam
I	Moment of inertia
$2K$	Spring stiffness of axial end restraint
L	Half beam length
M	Bending moment
\bar{M}	Nondimensional central bending moment
P	Known applied axial end load
$2Q$	Concentrated midspan transverse load
\bar{Q}, \bar{Q}	Nondimensional concentrated midspan load
T_i, T_o	Temperatures at lower and upper beam faces, respectively
T_d, \bar{T}_d	Nondimensional temperature differences between upper and lower beam faces
\bar{T}	Nondimensional average temperature
\bar{W}	Intensity of uniformly distributed transverse load
W, \bar{W}	Nondimensional uniformly distributed load
α	Coefficient of linear thermal expansion
β	Ratio of distance from lower beam face to centroidal axis to the total depth
λ	$\pm \sqrt{\frac{H}{EI}}$
$\bar{\lambda}$	λL , nondimensional
σ	stress

SUBSCRIPTS

M	Due to mechanical loads
T	Due to temperature

1.3 DERIVATION OF BASIC EQUATIONS

The beam (Figure 1.3-1) is referred to a rectangular coordinate system in which transverse deflections y and distances along the centroidal axis are measured from a point on the undeflected centroidal axis at mid-span. Specified axial end loads are denoted by P and the elastic axial restraints are shown schematically as springs with stiffness $2K$. We assume uniform distributed loads of intensity W and a mid-span concentrated load of magnitude $2Q$. The temperature variation through the thickness is linear, varying from T_i at the lower extreme fiber to T_o at the upper extreme fiber.

Since the temperature distribution varies linearly with respect to Cartesian coordinates it produces no stresses in an externally unrestrained beam (Reference 1-2). In this case, the stress-free thermal curvature is given by

$$y_T'' = \frac{\alpha(T_o - T_i)}{h} \quad (1)$$

1.3 (Cont'd)

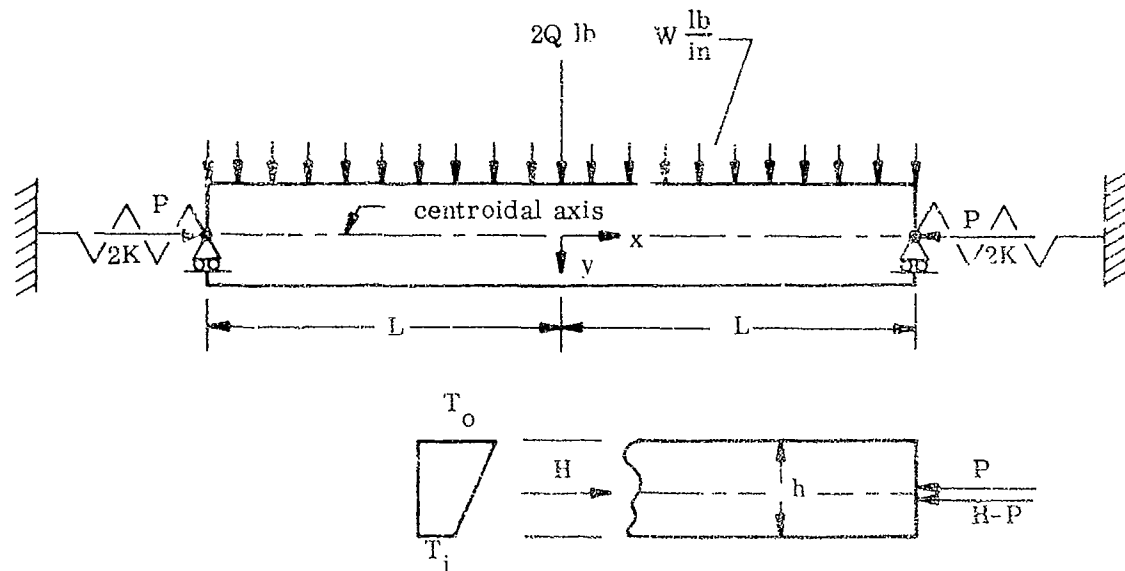


FIGURE 1.3-1 BEAM COLUMN - MODEL USED FOR ANALYSIS

We now consider that the external specified and redundant mechanical loads are applied to this thermally bent beam. Designating the additional curvatures produced by these mechanical loads as y''_M , it follows that

$$y''_M = \frac{-M}{EI} = \frac{1}{EI} \left[\pm Hy - (WL + Q)(L - x) + \frac{W}{2}(L - x)^2 \right], \quad (2)$$

where positive moments cause compression in the outer fibers, y is the total deflection, and H is the magnitude of the axial load in beam*. The negative and positive signs in the first term of the right hand side of Eq. (2) refer to compression and tension respectively. Adding Eqs. (1) and (2) and rearranging yields

$$y'' \pm \frac{Hy}{EI} = \frac{W}{2EI}(L - x)^2 - \frac{(WL + Q)(L - x)}{EI} + \frac{\alpha}{h}(T_o - T_i). \quad (3)$$

Due to symmetry it is only necessary to consider one half of the beam, so that the differential equation given by Eq. (3) applies in the interval $0 \leq x \leq L$ and $y(x)$ is an even function, i. e., $y(x) = y(-x)$. The boundary conditions are given by

$$y'(0) = y(L) = 0. \quad (4)$$

1.4 SOLUTION OF EQUATIONS

The complete solutions of Eqs. (3) and (4) of Sub-section 1.3 for the cases of axial compression and tension, respectively, are written as follows.

* H is always taken as positive, regardless of whether the axial load is tensile or compressive.

1.4 (Cont'd)

(1) Axial Compression

$$y = \frac{Q}{EI\lambda^3} \frac{\sin \lambda(L-x)}{\cos \lambda L} + \frac{\cos \lambda x}{\lambda^2 \cos \lambda L} \left[\frac{W}{EI\lambda^2} - \frac{\alpha}{h} (T_o - T_i) \right] + \frac{Wx^2}{2EI\lambda^2} + \frac{Qx}{EI\lambda^2} + \frac{1}{\lambda^2} \left[\frac{\alpha}{h} (T_o - T_i) - \frac{QL}{EI} - \frac{WL^2}{2EI} - \frac{W}{EI\lambda^2} \right] \quad (1a)$$

(2) Axial Tension

$$y = \frac{Q}{EI\lambda^3} \frac{\sinh \lambda(x-L)}{\cosh \lambda L} + \frac{\cosh \lambda x}{\lambda^2 \cosh \lambda L} \left[\frac{W}{EI\lambda^2} + \frac{\alpha}{h} (T_o - T_i) \right] - \frac{Wx^2}{2EI\lambda^2} - \frac{Qx}{EI\lambda^2} - \frac{1}{\lambda^2} \left[\frac{\alpha}{h} (T_o - T_i) - \frac{QL}{EI} - \frac{WL^2}{2EI} + \frac{W}{EI\lambda^2} \right] \quad (1b)$$

where

$$\lambda = -\sqrt{\frac{H}{EI}} \quad \text{for compression,} \quad (2a)$$

$$\lambda = +\sqrt{\frac{H}{EI}} \quad \text{for tension.}$$

The solution is not yet complete since, in general the axial end load H and hence λ is unknown. An additional compatibility relationship must be employed to evaluate this quantity. Such a relationship is obtained by noting that the change in the spanwise distance between the beam ends due to bending, thermal expansion and mechanical axial strains must be equal to the total change in length of the axial end restraints. This condition can be expressed by

$$\frac{1}{2} \int_{-L}^L (v')^2 dx + \frac{HL}{AE} - \alpha L \left\{ \beta T_o + (1 - \beta) T_i \right\} + \frac{(HL - PL)}{2K} = 0 \quad (3)$$

where positive and negative signs associated with H refer to compression and tension respectively and β defines the location of the centroidal axis (Figure 1.4-1).

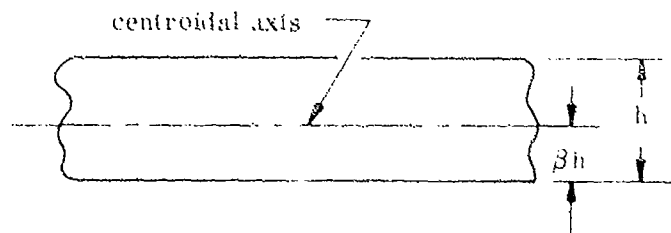


FIGURE 1.4-1 PARAMETER β , DEFINING CENTROIDAL AXIS LOCATION

1.4 (Cont'd)

Substitution of Eqs. (1a) and (1b) into (3) and simplifying yields for the compression case

$$\begin{aligned}
 & \frac{\bar{Q}^2}{\bar{\lambda}^5} \left[(\sin \bar{\lambda}) \left(\cos^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] \\
 & + \left[\frac{1}{2} - \frac{\sin 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\cos^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^3} + \frac{\bar{Q} \sin \bar{\lambda}}{\bar{\lambda}^2} - \frac{\bar{T}_d}{\bar{\lambda}} \right]^2 \\
 & + \frac{\bar{W}^2}{3\bar{\lambda}^4} - \left[\frac{\bar{W}}{\bar{\lambda}^3} + \frac{\bar{Q} \sin \bar{\lambda}}{\bar{\lambda}^2} - \frac{\bar{T}_d}{\bar{\lambda}} \right] \left[\frac{4\bar{Q}}{\bar{\lambda}^3 \cos \bar{\lambda}} \sin^4 \left(\frac{\bar{\lambda}}{2} \right) \right] \\
 & + \frac{\bar{Q}\bar{W}}{\bar{\lambda}^4} \left[1 - 2 \left(\frac{\sin \bar{\lambda}}{\bar{\lambda}} + \frac{\cos \bar{\lambda}}{\bar{\lambda}^2} - \frac{1}{\bar{\lambda}^2} \right) \right] \\
 & - \frac{2\bar{W}}{\bar{\lambda}^4 \cos \bar{\lambda}} (\sin \bar{\lambda} - \bar{\lambda} \cos \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^3} + \frac{\bar{Q} \sin \bar{\lambda}}{\bar{\lambda}^2} - \frac{\bar{T}_d}{\bar{\lambda}} \right] \\
 & + \left[\frac{2I}{AL^2} + \frac{EI}{KL^3} \right] \bar{\lambda}^2 = \frac{P}{KL} + 2\alpha \left[\beta T_o + (1 - \beta) T_i \right] \quad , \quad (4a)
 \end{aligned}$$

and for the case of axial tension:

$$\begin{aligned}
 & \frac{\bar{Q}^2}{\bar{\lambda}^5} \left[(\sinh \bar{\lambda}) \left(\cosh^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] \\
 & - \left[\frac{1}{2} - \frac{\sinh 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\cosh^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^3} - \frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}^3} + \frac{\bar{T}_d}{\bar{\lambda}} \right]^2 \\
 & + \frac{\bar{W}^2}{3\bar{\lambda}^4} + \left[\frac{4\bar{Q} \sinh^4 \frac{\bar{\lambda}}{2}}{\bar{\lambda}^3 \cosh \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^3} - \frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}^3} + \frac{\bar{T}_d}{\bar{\lambda}} \right] \\
 & + \frac{\bar{Q}\bar{W}}{\bar{\lambda}^4} \left[1 - 2 \left(\frac{\sinh \bar{\lambda}}{\bar{\lambda}} - \frac{\cosh \bar{\lambda}}{\bar{\lambda}^2} + \frac{1}{\bar{\lambda}^2} \right) \right] \\
 & + \frac{2\bar{W}}{\bar{\lambda}^4 \cosh \bar{\lambda}} (\sinh \bar{\lambda} - \bar{\lambda} \cosh \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^3} - \frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}^3} + \frac{\bar{T}_d}{\bar{\lambda}} \right] \\
 & - \left(\frac{2I}{AL^2} + \frac{EI}{KL^3} \right) \bar{\lambda}^2 = \frac{P}{KL} + 2\alpha \left[\beta T_o + (1 - \beta) T_i \right] \quad , \quad (4b)
 \end{aligned}$$

1.4 (Cont'd)

where

$$\frac{QL^2}{EI} = \tilde{Q}$$

$$\frac{WL^3}{EI} = \tilde{W}$$

$$\begin{aligned}\lambda L = \bar{\lambda} &= + \sqrt{\frac{HL^2}{EI}} \quad \text{for tension} \\ &= - \sqrt{\frac{HL^2}{EI}} \quad \text{for compression}\end{aligned}$$

$$\frac{\alpha L (T_o - T_i)}{h} = \tilde{T}_d$$

Equations (4a) and (4b) are transcendental equations from which $\bar{\lambda}$ may be determined. Equations (1a) and (1b) then yield the deflections for the known values of $\bar{\lambda}$. In general, Eqs. (4) are difficult to solve except by graphical or trial and error procedures. In addition, presentation of numerical results for the large number of parameters appearing in this general problem is cumbersome. For these reasons, the number of parameters to be used in the presentation of numerical results have been reduced by restricting considerations to the following:

- (1) The beam cross section is rectangular and therefore $\beta = \frac{1}{2}$.
- (2) The beam is prevented from moving axially at its ends ($K \rightarrow \infty$) or free to move axially ($K = 0$).
- (3) The transverse load is either uniform over the span or concentrated at the mid-span.

1.5 NUMERICAL RESULTS FOR BEAMS OF RECTANGULAR CROSS SECTION

CASE A - Uniform transverse load over the span with ends rigidly restrained axially:

The beam is shown schematically in Figure 1.5-1. For this case, we substitute $\tilde{Q} = 0$, $\beta = \frac{1}{2}$, and $\frac{1}{K} = 0$ into Eqs. (1) and (4) of Sub-section 1.4. Defining nondimensional quantities:

$$\begin{aligned}\tilde{W} &= \frac{12W}{Eb^3} \left(\frac{L}{h} \right)^4 \\ \bar{\lambda} &= + \sqrt{\frac{HL^2}{EI}} \quad (\text{tension positive})\end{aligned}$$

1.5 (Cont'd)

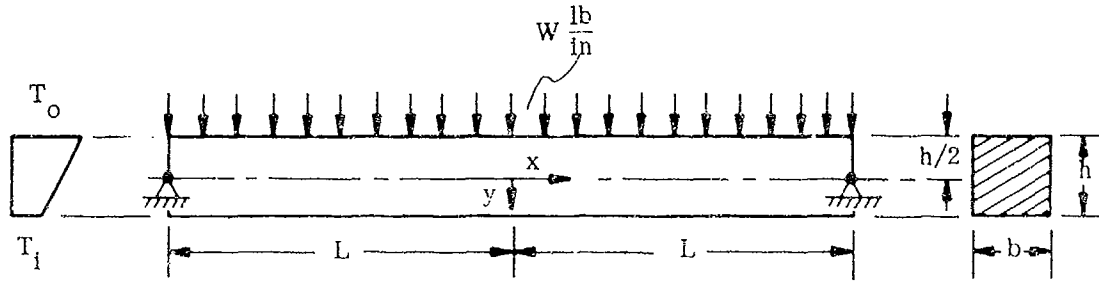


FIGURE 1.5-1 RECTANGULAR BEAM WITH UNIFORM LOAD OVER THE SPAN AND ENDS RIGIDLY RESTRAINED AXIALLY

$$\begin{aligned} \bar{T}_d &= \alpha \left(\frac{L}{h} \right)^2 (T_0 - T_1) \\ \bar{T} &= \alpha \left(\frac{L}{h} \right)^2 (T_0 + T_1) \\ \bar{y} &= \left[\frac{y}{h} \right]_{x=0} \\ \bar{M} &= \left[\frac{12ML^2}{Ebh^4} \right]_{x=0} \end{aligned} \tag{1}$$

These equations yield

$$\begin{aligned} &\left[\frac{1}{2} - \frac{\sin 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\bar{\lambda}^2 \cos^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right]^2 + \frac{\bar{W}^2}{3\bar{\lambda}^4} \\ &- \frac{2\bar{W}}{\bar{\lambda}^5 \cos \bar{\lambda}} (\sin \bar{\lambda} - \bar{\lambda} \cos \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right] + \frac{\bar{\lambda}^2}{6} = \bar{T} \end{aligned} \tag{2a}$$

$$\begin{aligned} \bar{y} &= \frac{1}{\bar{\lambda}^2} \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right] \left[\frac{1}{\cos \bar{\lambda}} - 1 \right] - \frac{\bar{W}}{2\bar{\lambda}^2} \\ \bar{M} &= \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right] \left[\frac{1}{\cos \bar{\lambda}} - 1 \right] \end{aligned} \tag{3a}$$

1.5 (Cont'd)

for the case of compressive end loads, while for the tensile case

$$\begin{aligned}
 & - \left[\frac{1}{2} - \frac{\sinh 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\bar{\lambda}^2 \cosh^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right]^2 + \frac{\bar{W}^2}{3\bar{\lambda}^4} \\
 & + \frac{2\bar{W}}{\bar{\lambda}^5 \cosh \bar{\lambda}} (\sinh \bar{\lambda} - \bar{\lambda} \cosh \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right] - \frac{\bar{\lambda}^2}{6} = \bar{T}
 \end{aligned}
 \tag{2b}$$

$$\begin{aligned}
 \bar{y} &= \frac{1}{\bar{\lambda}^2} \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right] \left[\frac{1}{\cosh \bar{\lambda}} - 1 \right] + \frac{\bar{W}}{2\bar{\lambda}^2} \\
 \bar{M} &= - \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right] \left[\frac{1}{\cosh \bar{\lambda}} - 1 \right] .
 \end{aligned}
 \tag{3b}$$

Values of $\bar{\lambda}$, \bar{y} , and \bar{M} for various combinations of \bar{T}_d , \bar{T} and \bar{W} are given in Table 1.5-1.

This table permits the determination, either directly or by interpolation, of the axial end loads, maximum (central) deflection and maximum bending moment corresponding to specified temperatures and transverse loading. A typical case, extracted from the table is plotted in Figure 1.5-2, showing the variations in end load, central bending moment and deflection with average temperature when the temperature difference and transverse load are held constant.

The figure shows that extremely large values of \bar{T} are required to raise the compressive end load to values in the neighborhood of the critical Euler value, $\bar{\lambda}_{CR} = -\frac{\pi}{2} = -1.57$. This is due to the fact that additional beam expansions caused by increasing the average temperature are accommodated by further bending with very little change of compressive end load.

Thus, although the theoretical end load in a perfectly straight axially restrained column reaches the Euler buckling load when $\bar{T} = \frac{\pi^2}{24} = .41$, values of \bar{T} several orders of magnitude higher than this can actually be achieved without excessive stresses or deflections occurring. In this respect thermal buckling differs significantly from buckling caused by deadweight loads, since stresses and deflections tend to become excessively large when mechanical loads exceed the Euler buckling value.

1.5 (Cont'd)

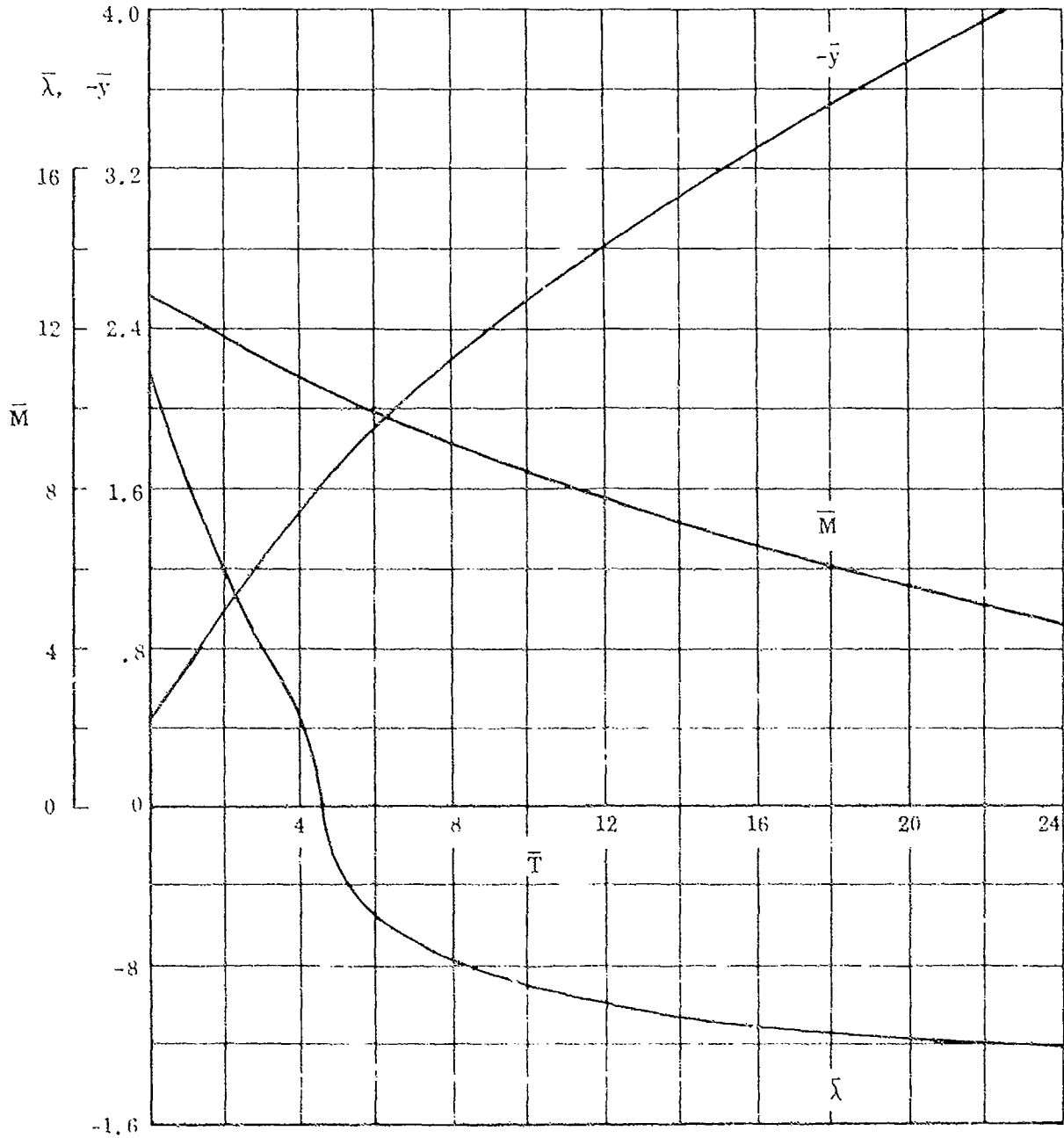


FIGURE 1.5-2 NONDIMENSIONAL END LOAD, CENTRAL DEFLECTION AND MOMENT VS. NONDIMENSIONAL AVERAGE TEMPERATURE; $\bar{W}=21$; $\bar{T}_d=12$

1.5 (Cont'd)

CASE B - Concentrated transverse load at midspan with ends rigidly restrained axially

For this case, we substitute $\bar{W} = 0$, $\beta = \frac{1}{2}$ and $\frac{1}{K} = 0$ into Eqs. (1) and (4) of Sub-section 1.4. Nondimensional quantities are as defined by Eq. (1) and in addition we define

$$\bar{Q} = \frac{12QL^3}{Ebh^4} \quad (4)$$

$$\frac{\bar{Q}^2}{\lambda^5} \left[(\sin \bar{\lambda}) \left(\cos^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] + \left[\frac{1}{2} - \frac{\sin 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\lambda^2 \cos^2 \bar{\lambda}} \right] \left[\frac{Q \sin \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right]^2 \quad (5a)$$

$$- \frac{4\bar{Q} \sin^4 \frac{\bar{\lambda}}{2}}{\bar{\lambda}^4 \cos \bar{\lambda}} \left[\frac{Q \sin \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right] + \frac{\bar{\lambda}^2}{6} = \bar{T}$$

$$\bar{v} = \frac{\bar{Q}}{\lambda^2} \left(\frac{\tan \bar{\lambda}}{\bar{\lambda}} - 1 \right) - \frac{\bar{T}_d}{\lambda^2} \left(\frac{1}{\cos \bar{\lambda}} - 1 \right)$$

$$\bar{M} = \bar{Q} \frac{\tan \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \left[\frac{1}{\cos \bar{\lambda}} - 1 \right] \quad (6a)$$

for the case of compressive end loads, while for the tensile case

$$\frac{\bar{Q}^2}{\lambda^5} \left[(\sinh \bar{\lambda}) \left(\cosh^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] + \left[\frac{1}{2} - \frac{\sinh 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\lambda^2 \cosh^2 \bar{\lambda}} \right] \left[\frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right]^2 \quad (5b)$$

$$- \frac{4\bar{Q} \sinh^4 \frac{\bar{\lambda}}{2}}{\bar{\lambda}^4 \cosh \bar{\lambda}} \left[\frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right] - \frac{\bar{\lambda}^2}{6} = \bar{T}$$

$$\bar{v} = - \frac{\bar{Q}}{\lambda^2} \left(\frac{\tanh \bar{\lambda}}{\bar{\lambda}} - 1 \right) + \frac{\bar{T}_d}{\lambda^2} \left(\frac{1}{\cosh \bar{\lambda}} - 1 \right)$$

$$\bar{M} = \frac{\bar{Q} \tanh \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \left[\frac{1}{\cosh \bar{\lambda}} - 1 \right] \quad (6b)$$

Values of $\bar{\lambda}$, \bar{v} , and \bar{M} for various combinations of \bar{T}_d , \bar{T} and \bar{Q} are given in Table 1.5-2.

TABLE 1.5-1
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}
(Pages 1.12 through 1.37)

TABLE 1.5-1
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

$\bar{T}_D = 0.$				$\bar{T}_D = 0.$			
\bar{T}	\bar{y}	\bar{M}	\bar{T}	\bar{y}	\bar{M}	\bar{T}	\bar{y}
COMPRESSIVE END LOADS							
0.	0.	0.	0.	0.	0.	0.	0.
0.1500000E-01	0.	0.	0.4085714E-01	0.6250000E 00	0.1500000E 01	0.	0.
0.2666667E-01	0.	0.	0.5381333E 00	0.6877200E 00	0.1558800E 01	0.	0.
0.4166667E-01	0.	0.	0.5820746E 00	0.6884880E 00	0.1606600E 01	0.	0.
0.4000000E-01	0.	0.	0.6430699E 00	0.6957085E 00	0.1675927E 01	0.	0.
0.0166667E-01	0.	0.	0.7258127E 00	0.7521367E 00	0.1765589E 01	0.	0.
0.1066667E-00	0.	0.	0.8178860E 00	0.7804156E 00	0.1882406E 01	0.	0.
0.1350000E-00	0.	0.	0.9221592E 01	0.8446597E 00	0.2004582E 01	0.	0.
0.1666667E-00	0.	0.	0.1211592E 01	0.9315309E 01	0.2254540E 01	0.	0.
0.1666667E-00	0.	0.	0.1539828E 01	0.1052447E 01	0.2525447E 01	0.	0.
0.2016667E-00	0.	0.	0.2071176E 01	0.1278614E 01	0.2786622E 01	0.	0.
0.2400000E-00	0.	0.	0.3041085E 01	0.1504201E 01	0.3666666E 01	0.	0.
0.2816667E-00	0.	0.	0.573538E 01	0.1988872E 01	0.4866666E 01	0.	0.
0.3266667E-00	0.	0.	0.1181068E 02	0.3048531E 01	0.7474729E 01	0.	0.
0.3524167E-00	0.	0.	0.1455000E 01	0.2255509E 02	0.1041421E 02	0.	0.
0.3750000E-00	0.	0.	0.6293117E 02	0.7118123E 01	0.1751578E 02	0.	0.
0.4004167E-00	0.	0.	0.1550000E 01	0.2395009E 02	0.5879905E 02	0.	0.
TENSILE END LOADS							
-0.1500000E-01	0.	0.	0.3000000E-00	0.4375623E-00	0.6029302E 00	0.1465736E 01	0.3000300E-00
-0.2666667E-01	0.	0.	0.4000000E-00	0.4016713E-00	0.5868110E 00	0.1406110E 01	0.4000000E-00
-0.4166667E-01	0.	0.	0.5000000E-00	0.3587850E-00	0.5673065E 00	0.1358171E 01	0.5000000E-00
-0.6000000E-01	0.	0.	0.6000000E-00	0.3099202E-00	0.5451549E 00	0.1305734E 01	0.6000000E-00
-0.8166666E-01	0.	0.	0.7000000E-00	0.2566499E-00	0.5211012E 00	0.1246666E 01	0.7000000E-00
-0.1016667E-00	0.	0.	0.8000000E-00	0.1996001E-00	0.4904901E-00	0.1182657E 01	0.8000000E-00
-0.1350000E-00	0.	0.	0.9000000E-00	0.1172460E-00	0.4709258E-00	0.1118574E 01	0.9000000E-00
-0.1666667E-00	0.	0.	0.1000000E 01	0.1735E-01	0.4444628E-00	0.1055870E 01	0.1000000E 01
-0.2016667E-00	0.	0.	0.1100000E 01	0.1703170E-01	0.4188097E-00	0.9953866E 00	0.1100000E 01
-0.2400000E-00	0.	0.	0.1200000E 01	0.4628358E-01	0.3937525E-00	0.9527372E 00	0.1200000E 01
-0.2816667E-00	0.	0.	0.1300000E 01	0.1105412E-00	0.3701328E-00	0.8754756E 00	0.1300000E 01
-0.3266667E-00	0.	0.	0.1400000E 01	0.1757703E-00	0.3478505E-00	0.8189669E 00	0.1400000E 01
-0.3750000E-00	0.	0.	0.1500000E 01	0.242073E-00	0.3259828E-00	0.7665306E 00	0.1500000E 01
-0.4266667E-00	0.	0.	0.1600000E 01	0.3096274E-00	0.3057767E-00	0.7172151E 00	0.1600000E 01
-0.4816667E-00	0.	0.	0.1700000E 01	0.3785967E-00	0.2868301E-00	0.6710374E 00	0.1700000E 01
-0.5400000E-00	0.	0.	0.1800000E 01	0.4449177E-00	0.2691486E-00	0.6279584E 00	0.1800000E 01
-0.6016667E-00	0.	0.	0.1900000E 01	0.5215618E-00	0.2528665E-00	0.5878719E 00	0.1900000E 01
-0.6666667E-00	0.	0.	0.2000000E 01	0.5956263E-00	0.2371199E-00	0.5508443E 00	0.2000000E 01
-0.7350000E-00	0.	0.	0.2100000E 01	0.6774780E-00	0.2211006E-00	0.5161262E 00	0.2100000E 01
-0.8066666E-00	0.	0.	0.2200000E 01	0.754671E-00	0.2098891E-00	0.4841414E 00	0.2200000E 01
-0.8816666E-00	0.	0.	0.2300000E 01	0.8324251E-00	0.1976125E-00	0.4545243E-00	0.2300000E 01
-0.9600000E-00	0.	0.	0.2400000E 01	0.9162757E-00	0.1862652E-00	0.4271068E 00	0.2400000E 01
-0.1041667E 01	0.	0.	0.2500000E 01	0.1002714E 01	0.1757232E-00	0.4017258E 00	0.2500000E 01
-0.1126667E 01	0.	0.	0.2600000E 01	0.1091898E 01	0.1659438E-00	0.3782254E-00	0.2600000E 01
-0.1215000E 01	0.	0.	0.2700000E 01	0.1163920E 01	0.1568644E-00	0.3566822E-00	0.2700000E 01
-0.1306667E 01	0.	0.	0.2800000E 01	0.1228796E 01	0.1484329E-00	0.3362862E-00	0.2800000E 01
-0.1401667E 01	0.	0.	0.2900000E 01	0.1276618E 01	0.1405968E-00	0.3175111E-00	0.2900000E 01
-0.1500000E 01	0.	0.	0.3000000E 01	0.1317478E 01	0.1333094E-00	0.3002740E-00	0.3000000E 01
TENSILE END LOADS							
-0.1500000E-01	0.	0.	0.3000000E-00	0.4375623E-00	0.6029302E 00	0.1465736E 01	0.3000300E-00
-0.2666667E-01	0.	0.	0.4000000E-00	0.4016713E-00	0.5868110E 00	0.1406110E 01	0.4000000E-00
-0.4166667E-01	0.	0.	0.5000000E-00	0.3587850E-00	0.5673065E 00	0.1358171E 01	0.5000000E-00
-0.6000000E-01	0.	0.	0.6000000E-00	0.3099202E-00	0.5451549E 00	0.1305734E 01	0.6000000E-00
-0.8166666E-01	0.	0.	0.7000000E-00	0.2566499E-00	0.5211012E 00	0.1246666E 01	0.7000000E-00
-0.1016667E-00	0.	0.	0.8000000E-00	0.1996001E-00	0.4904901E-00	0.1182657E 01	0.8000000E-00
-0.1350000E-00	0.	0.	0.9000000E-00	0.1172460E-00	0.4709258E-00	0.1118574E 01	0.9000000E-00
-0.1666667E-00	0.	0.	0.1000000E 01	0.1735E-01	0.4444628E-00	0.1055870E 01	0.1000000E 01
-0.2016667E-00	0.	0.	0.1100000E 01	0.1703170E-01	0.4188097E-00	0.9953866E 00	0.1100000E 01
-0.2400000E-00	0.	0.	0.1200000E 01	0.4628358E-01	0.3937525E-00	0.9527372E 00	0.1200000E 01
-0.2816667E-00	0.	0.	0.1300000E 01	0.1105412E-00	0.3701328E-00	0.8754756E 00	0.1300000E 01
-0.3266667E-00	0.	0.	0.1400000E 01	0.1757703E-00	0.3478505E-00	0.8189669E 00	0.1400000E 01
-0.3750000E-00	0.	0.	0.1500000E 01	0.242073E-00	0.3259828E-00	0.7665306E 00	0.1500000E 01
-0.4266667E-00	0.	0.	0.1600000E 01	0.3096274E-00	0.3057767E-00	0.7172151E 00	0.1600000E 01
-0.4816667E-00	0.	0.	0.1700000E 01	0.3785967E-00	0.2868301E-00	0.6710374E 00	0.1700000E 01
-0.5400000E-00	0.	0.	0.1800000E 01	0.4449177E-00	0.2691486E-00	0.6279584E 00	0.1800000E 01
-0.6016667E-00	0.	0.	0.1900000E 01	0.5215618E-00	0.2528665E-00	0.5878719E 00	0.1900000E 01
-0.6666667E-00	0.	0.	0.2000000E 01	0.5956263E-00	0.2371199E-00	0.5508443E 00	0.2000000E 01
-0.7350000E-00	0.	0.	0.2100000E 01	0.6774780E-00	0.2211006E-00	0.5161262E 00	0.2100000E 01
-0.8066666E-00	0.	0.	0.2200000E 01	0.754671E-00	0.2098891E-00	0.4841414E 00	0.2200000E 01
-0.8816666E-00	0.	0.	0.2300000E 01	0.8324251E-00	0.1976125E-00	0.4545243E-00	0.2300000E 01
-0.9600000E-00	0.	0.	0.2400000E 01	0.9162757E-00	0.1862652E-00	0.4271068E 00	0.2400000E 01
-0.1041667E 01	0.	0.	0.2500000E 01	0.1002714E 01	0.1757232E-00	0.4017258E 00	0.2500000E 01
-0.1126667E 01	0.	0.	0.2600000E 01	0.1091898E 01	0.1659438E-00	0.3782254E-00	0.2600000E 01
-0.1215000E 01	0.	0.	0.2700000E 01	0.1163920E 01	0.1568644E-00	0.3566822E-00	0.2700000E 01
-0.1306667E 01	0.	0.	0.2800000E 01	0.1228796E 01	0.1484329E-00	0.3362862E-00	0.2800000E 01
-0.1401667E 01	0.	0.	0.2900000E 01	0.1276618E 01	0.1405968E-00	0.3175111E-00	0.2900000E 01
-0.1500000E 01	0.	0.	0.3000000E 01	0.1317478E 01	0.1333094E-00	0.3002740E-00	0.3000000E 01

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

$\bar{T}_D = 0, \bar{W} = 0.1200000E 02$				$\bar{T}_D = 0, \bar{W} = 0.1500000E 02$			
\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS							
0.777 71 01	0.2500000E 01	0.6000000E 01	0.	0.4286E 02	0.3125000E 01	0.7500000E 01	0.
0.8385 01	0.2599900E 01	0.6233587E 01	-0.1000000E -00	0.1329381E 02	0.3263710E 01	0.7791984E 01	-0.1000000E -00
0.8913194E 01	0.2673952E 01	0.6427832E 01	-0.4000000E -00	0.1507189E 02	0.3382440E 01	0.8037900E 01	-0.4000000E -00
0.9464118E 01	0.2720281E 01	0.66895703E 01	-0.5000000E 00	0.1507670E 02	0.3478542E 01	0.8169638E 01	-0.5000000E 00
0.1071300E 02	0.2928547E 01	0.7054277E 01	-0.6000000E 00	0.1670550E 02	0.3660481E 01	0.8617884E 01	-0.6000000E 00
0.1218119E 02	0.3121662E 01	0.7529615E 01	-0.7000000E 00	0.1898718E 02	0.3902076E 01	0.9412018E 01	-0.7000000E 00
0.1427387E 02	0.3478619E 01	0.8162329E 01	-0.8000000E 00	0.2224291E 02	0.4273799E 01	0.1020219E 02	-0.8000000E 00
0.1735728E 02	0.3726124E 01	0.9018180E 01	-0.9000000E 00	0.2704481E 02	0.4657658E 01	0.1127770E 02	-0.9000000E 00
0.2213724E 02	0.4209789E 01	0.1029579E 02	-0.1000000E 01	0.3409569E 02	0.5262736E 01	0.1276224E 02	-0.1000000E 01
0.1012327E 02	0.4918458E 01	0.1194004E 02	-0.1100000E 01	0.4695417E 02	0.6181068E 01	0.1493111E 02	-0.1100000E 01
0.4505736E 02	0.6016803E 01	0.1466420E 02	-0.1200000E 01	0.7026712E 02	0.7521004E 01	0.1835024E 02	-0.1200000E 01
0.7853800E 02	0.7984907E 01	0.1944319E 02	-0.1300000E 01	0.1225888E 03	0.9983633E 01	0.2430474E 02	-0.1300000E 01
0.1840708E 03	0.1219333E 02	0.2993827E 02	-0.1400000E 01	0.2874269E 03	0.1524866E 02	0.3737368E 02	-0.1400000E 01
0.4556252E 03	0.1695909E 02	0.4155649E 02	-0.1500000E 01	0.5558672E 03	0.2119886E 02	0.5107518E 02	-0.1500000E 01
0.1001274E 04	0.2887249E 02	0.7026311E 02	-0.1500000E 01	0.1564279E 04	0.3559062E 02	0.8157889E 02	-0.1500000E 01
0.1123115E 04	0.9520317E 02	0.2351929E 03	-0.1500000E 01	0.1754885E 05	0.1192505E 03	0.2933992E 03	-0.1500000E 01
TENSILE END LOADS							
0.7225997E 01	0.2411721E 01	0.5782945E 01	0.1000000E -00	0.1128912E 02	0.1014651E 01	0.7226681E 01	0.1000000E -00
0.6820740E 01	0.2367246E 01	0.5628448E 01	0.4000000E -00	0.1068180E 02	0.2914055E 01	0.7030551E 01	0.4000000E -00
0.6365360E 01	0.23289228E 01	0.5452693E 01	0.5000000E 00	0.9969464E 01	0.2816453E 01	0.6790687E 01	0.5000000E 00
0.5858724E 01	0.22180020E 01	0.5214977E 01	0.6000000E 00	0.9188004E 01	0.27254775E 01	0.6518773E 01	0.6000000E 00
0.5328199E 01	0.2080050E 01	0.4970647E 01	0.7000000E 00	0.8371243E 01	0.2605506E 01	0.6223502E 01	0.7000000E 00
0.4793602E 01	0.1943390E 01	0.4733026E 01	0.8000000E 00	0.7500000E 01	0.2490000E 01	0.5913281E 01	0.8000000E 00
0.4270191E 01	0.1800103E 01	0.4477116E 01	0.9000000E 00	0.6788135E 01	0.2350127E 01	0.5582396E 01	0.9000000E 00
0.3769078E 01	0.1776651E 01	0.4223389E 01	0.1000000E 01	0.5982953E 01	0.2220914E 01	0.5278186E 01	0.1000000E 01
0.3297508E 01	0.1674755E 01	0.3973587E 01	0.1100000E 01	0.5265794E 01	0.2093444E 01	0.4966933E 01	0.1100000E 01
0.2899463E 01	0.1575730E 01	0.3730949E 01	0.1200000E 01	0.4602910E 01	0.1949666E 01	0.4663686E 01	0.1200000E 01
0.2450131E 01	0.1480533E 01	0.3497202E 01	0.1300000E 01	0.3996470E 01	0.1850668E 01	0.4427237E 01	0.1300000E 01
0.2087674E 01	0.1399802E 01	0.3275288E 01	0.1400000E 01	0.3445473E 01	0.1757253E 01	0.4094985E 01	0.1400000E 01
0.1751723E 01	0.1303933E 01	0.3066154E 01	0.1500000E 01	0.2948005E 01	0.1629914E 01	0.3832636E 01	0.1500000E 01
0.1465961E 01	0.1223103E 01	0.2868052E 01	0.1600000E 01	0.2599314E 01	0.1528081E 01	0.3586065E 01	0.1600000E 01
0.1167460E 01	0.1147153E 01	0.2684349E 01	0.1700000E 01	0.2095094E 01	0.1444191E 01	0.3351187E 01	0.1700000E 01
0.9131500E 00	0.1076498E 01	0.2511819E 01	0.1800000E 01	0.1730553E 01	0.1365743E 01	0.3139772E 01	0.1800000E 01
0.6800167E 00	0.1010466E 01	0.2351880E 01	0.1900000E 01	0.1400000E 01	0.1281332E 01	0.2919170E 01	0.1900000E 01
0.4651786E 00	0.9493517E 00	0.2212593E 01	0.2000000E 01	0.1101882E 01	0.1186490E 01	0.2753242E 01	0.2000000E 01
0.2659913E -00	0.8924024E 00	0.2064505E 01	0.2100000E 01	0.8790486E 00	0.1115503E 01	0.2586063E 01	0.2100000E 01
0.8005517E -01	0.8395248E 00	0.1936585E 01	0.2200000E 01	0.5706162E 00	0.1049444E 01	0.2426267E 01	0.2200000E 01
-0.9477098E -01	0.7905290E 00	0.1818097E 01	0.2300000E 01	0.3478578E -00	0.9881623E 00	0.2272621E 01	0.2300000E 01
-0.2603793E -00	0.7450647E 00	0.1708477E 01	0.2400000E 01	0.1531574E -00	0.9313104E 00	0.2135514E 01	0.2400000E 01
-0.4784202E -00	0.7028955E 00	0.1606903E 01	0.2500000E 01	-0.4785459E -01	0.8786191E 00	0.2000529E 01	0.2500000E 01
-0.5705633E 00	0.6617720E 00	0.1512203E 01	0.2600000E 01	0.2574627E -00	0.8297150E 00	0.1821177E 01	0.2600000E 01
-0.7178392E 00	0.6227878E 00	0.1427631E 01	0.2700000E 01	-0.4375413E -00	0.7883223E 00	0.1762927E 01	0.2700000E 01
-0.8607147E 00	0.5871335E 00	0.1349445E 01	0.2800000E 01	0.6099038E 00	0.7421644E 00	0.1681813E 01	0.2800000E 01
-0.1001208E 01	0.5423871E 00	0.1270314E 01	0.2900000E 01	-0.7159508E 00	0.7029859E 00	0.1587906E 01	0.2900000E 01
-0.1194484E 01	0.5112318E 00	0.1200684E 01	0.3000000E 01	-0.9144508E 00	0.6664422E 00	0.1501170E 01	0.3000000E 01
TENSILE END LOADS							
TENSILE END LOADS							
TENSILE END LOADS							
TENSILE END LOADS							

TABLE 1.5-1 (Cont'd)
VALUES OF λ , γ AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

\bar{T}_D	\bar{T}	COMPRESSION END LOADS		\bar{T}_D	\bar{T}	COMPRESSION END LOADS	
		λ	γ			λ	γ
0.67333736 01	0.21375000 01	3.16532070 02	0.00000000 00	0.10819054 02	0.35000000 01	0.12000000 02	0.00000000 00
0.17618576 01	0.24633550 01	0.10721701 02	-0.10700000 00	0.11662301 02	0.31120851 01	0.12220000 02	-0.10000000 00
0.17201192 01	0.25360000 01	0.10209594 02	-0.10000000 00	0.12392574 02	0.32052294 01	0.12512858 02	-0.10000000 00
0.18085771 01	0.26190000 01	0.11157511 02	-0.10000000 00	0.13620450 02	0.33177451 01	0.12833444 02	-0.10000000 00
0.19275817 01	0.27135311 01	0.11529971 02	-0.10000000 00	0.14675876 02	0.34505600 01	0.13262094 02	-0.10000000 00
0.10540531 02	0.29510101 01	0.11946774 02	-0.10000000 00	0.16903580 02	0.37336571 01	0.13829391 02	-0.10000000 00
0.12364701 02	0.31710671 01	0.12582746 02	-0.10000000 00	0.17760170 02	0.40365711 01	0.14583361 02	-0.10000000 00
0.15000270 02	0.35166811 01	0.13366894 02	-0.10000000 00	0.24061711 02	0.44467194 01	0.15600152 02	-0.10000000 00
0.19117611 02	0.39230471 01	0.14368171 02	-0.10000000 00	0.30677491 02	0.50163341 01	0.17016311 02	-0.10000000 00
0.25995081 02	0.44018131 01	0.15638794 02	-0.10000000 00	0.41736071 02	0.58467451 01	0.19049456 02	-0.10000000 00
0.34895271 02	0.50841801 01	0.17195751 02	-0.10000000 00	0.62633401 02	0.71635601 01	0.22269501 02	-0.10000000 00
0.67678391 02	0.71402711 01	0.21715101 02	-0.10000000 00	0.16806791 03	0.94908491 01	0.27918251 02	-0.10000000 00
0.15040041 03	0.11372211 02	0.25271251 02	-0.10000000 00	0.22551591 03	0.13442341 02	0.40261871 02	-0.10000000 00
0.10600501 03	0.15792711 02	0.24379401 02	-0.10000000 00	0.19101981 03	0.20032451 02	0.54118171 02	-0.10000000 00
0.18515271 03	0.24472491 02	0.27035111 02	-0.10000000 00	0.13642071 03	0.33590821 02	0.67570891 02	-0.10000000 00
0.26624701 04	0.40550451 02	0.27328794 02	-0.10000000 00	0.15572831 03	0.52402431 02	0.81500001 02	-0.10000000 00

TABLE 1.5-1 (Cont'd)

VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF T_D , T AND \bar{W}

T_D	0.1200000E 02	\bar{W}	0.1500000E 02	T_D	0.1200000E 02	\bar{W}	0.1800000E 02
T	\bar{y}	\bar{M}	$\bar{\lambda}$	T	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS							
0.1214286E 02	-0.2675000E 01	0.7500000E 01	0.	0.7885715E 01	-0.2750000E 01	0.9000000E 01	0.
0.1304939E 02	-0.2989888E 01	0.7230915E 01	-0.3000000E -00	0.8464307E 01	-0.2341076E 01	0.8789302E 01	-0.3000000E -00
0.1383018E 02	-0.3085393E 01	0.7006337E 01	-0.4000000E -00	0.8943726E 01	-0.2416904E 01	0.8613275E 01	-0.4000000E -00
0.1499038E 02	-0.3217166E 01	0.6695708E 01	-0.5000000E 00	0.9672099E 01	-0.2521458E 01	0.8169636E 01	-0.5000000E 00
0.1649172E 02	-0.3393593E 01	0.6278505E 01	-0.6000000E 00	0.1046035E 02	-0.2661457E 01	0.8004184E 01	-0.6000000E 00
0.1864908E 02	-0.3627537E 01	0.5722507E 01	-0.7000000E 00	0.1204537E 02	-0.2847219E 01	0.7800911E 01	-0.7000000E 00
0.2176156E 02	-0.3939010E 01	0.4979021E 01	-0.8000000E 00	0.1401026E 02	-0.3094371E 01	0.7019030E 01	-0.8000000E 00
0.2623261E 02	-0.4360405E 01	0.3967991E 01	-0.9000000E 00	0.1690479E 02	-0.3428975E 01	0.6222531E 01	-0.9000000E 00
0.3340647E 02	-0.4967555E 01	0.2552447E 01	-0.1000000E 01	0.2139260E 02	-0.3895106E 01	0.5104934E 01	-0.1000000E 01
0.4523745E 02	-0.5805422E 01	0.4778598E -00	-0.1100000E 01	0.2988046E 02	-0.4574808E 01	0.3464842E 01	-0.1100000E 01
0.6736586E 02	-0.7143192E 01	-0.2766197E 01	-0.1200000E 01	0.4207251E 02	-0.5638991E 01	0.8798524E 00	-0.1200000E 01
0.1170112E 03	-0.9500159E 01	-0.8555268E 01	-0.1300000E 01	0.7424220E 02	-0.7511432E 01	-0.3694320E 01	-0.1300000E 01
0.2713835E 03	-0.1465276E 02	-0.2122823E 02	-0.1400000E 01	0.1730076E 03	-0.1160893E 02	-0.1375350E 02	-0.1400000E 01
0.5276361E 03	-0.2045722E 02	-0.3551215E 02	-0.1500000E 01	0.3335699E 03	-0.1621785E 02	-0.2509803E 02	-0.1500000E 01
0.1484528E 04	-0.3467249E 02	-0.7006311E 02	-0.1500000E 01	0.9378210E 03	-0.2735437E 02	-0.5754731E 02	-0.1500000E 01
0.1664587E 05	-0.1159489E 03	-0.2710873E 03	-0.1552000E 01	0.1051162E 05	-0.9709886E 02	-0.2122674E 03	-0.1550000E 01
TENSILE END LOADS							
0.1133877E 02	-0.2768293E 01	0.7749166E 01	0.3000000E -00	0.7380255E 01	-0.2165162E 01	0.9194988E 01	0.3000000E -00
0.1274753E 02	-0.2690386E 01	0.7939042E 01	0.4000000E -00	0.6991468E 01	0.1703574E 01	0.9358572E 01	0.4000000E -00
0.1006708E 02	-0.2596161E 01	0.8149904E 01	0.5000000E 00	0.6555028E 01	0.2028854E 01	0.9527214E 01	0.5000000E 00
0.9319633E 01	-0.2489202E 01	0.8396113E 01	0.6000000E 00	0.6074882E 01	0.1944047E 01	0.9697957E 01	0.6000000E 00
0.8517166E 01	-0.2371554E 01	0.8662816E 01	0.7000000E 00	0.5570988E 01	0.1842036E 01	0.9807497E 01	0.7000000E 00
0.7750542E 01	-0.2251381E 01	0.8940886E 01	0.8000000E 00	0.5028308E 01	0.1755532E 01	0.9932538E 01	0.8000000E 00
0.6960824E 01	-0.2126988E 01	0.9222460E 01	0.9000000E 00	0.4508154E 01	0.1658962E 01	0.10035214E 02	0.9000000E 00
0.6240255E 01	-0.2002535E 01	0.9502135E 01	0.1000000E 01	0.4019049E 01	0.1558372E 01	0.1055837E 02	0.1000000E 01
0.5553954E 01	-0.1880191E 01	0.9774924E 01	0.1100000E 01	0.36141529E 01	0.1463141E 01	0.1107851E 02	0.1100000E 01
0.4911608E 01	-0.1761266E 01	0.1001287E 02	0.1200000E 01	0.3209705E 01	0.1367354E 01	0.1102949E 02	0.1200000E 01
0.4326075E 01	-0.1647238E 01	0.1021414E 02	0.1300000E 01	0.2819817E 01	0.1277136E 01	0.1111651E 02	0.1300000E 01
0.3790742E 01	-0.1538751E 01	0.1051542E 02	0.1400000E 01	0.2461010E 01	0.1191285E 01	0.1113442E 02	0.1400000E 01
0.3302084E 01	-0.1436260E 01	0.1077515E 02	0.1500000E 01	0.2131179E 01	0.1110257E 01	0.1114940E 02	0.1500000E 01
0.2865423E 01	-0.1339971E 01	0.1093251E 02	0.1600000E 01	0.1812750E 01	0.1034195E 01	0.1116475E 02	0.1600000E 01
0.2466731E 01	-0.1249959E 01	0.1111248E 02	0.1700000E 01	0.1506933E 01	0.0963197E 01	0.1118745E 02	0.1700000E 01
0.2104749E 01	-0.1166691E 01	0.1131781E 02	0.1800000E 01	0.1210513E 01	0.0896421E 01	0.1121020E 02	0.1800000E 01
0.1768191E 01	-0.1088163E 01	0.1154267E 02	0.1900000E 01	0.1036759E 01	0.0835966E 01	0.1123161E 02	0.1900000E 01
0.1457519E 01	-0.1015204E 01	0.1178961E 02	0.2000000E 01	0.8852913E 00	0.788656E 00	0.1124940E 02	0.2000000E 01
0.1168421E 01	-0.9469215E 00	0.1211695E 02	0.2100000E 01	0.8092291E 00	0.7500237E 00	0.1126512E 02	0.2100000E 01
0.9319633E 00	-0.8971295E 00	0.1249949E 02	0.2200000E 01	0.7458723E 00	0.6773664E 00	0.1127762E 02	0.2200000E 01
0.7162531E 00	-0.8629974E 00	0.1293044E 02	0.2300000E 01	0.6927020E 00	0.6321024E 00	0.1128348E 02	0.2300000E 01
0.5496251E 00	-0.7770963E 00	0.1337257E 02	0.2400000E 01	0.6501751E 00	0.5928102E 00	0.1128160E 02	0.2400000E 01
0.4229164E 00	-0.7262654E 00	0.1382517E 02	0.2500000E 01	0.5974326E 01	0.5525603E 00	0.1127455E 02	0.2500000E 01
0.3266647E 01	-0.6815965E 00	0.1428781E 02	0.2600000E 01	0.5319431E 00	0.5127257E 00	0.1126234E 02	0.2600000E 01
0.2607502E 01	-0.6415104E 00	0.1477051E 02	0.2700000E 01	0.453415E 00	0.4696657E 00	0.1124513E 02	0.2700000E 01
0.2284425E 00	-0.6072928E 00	0.1527277E 02	0.2800000E 01	0.3629969E 00	0.4255454E 00	0.1122368E 02	0.2800000E 01
0.2044519E 00	-0.5671461E 00	0.1579453E 02	0.2900000E 01	0.2673137E 00	0.3762659E 00	0.1119892E 02	0.2900000E 01
0.1871160E 00	-0.5344549E 00	0.1633578E 02	0.3000000E 01	0.1724279E 00	0.3234447E 00	0.1117166E 02	0.3000000E 01

TABLE 1.5-1 (Cont'd)
VALUES OF λ, γ AND M̄ FOR SPECIFIED VALUES OF T̄_D, T̄ AND W̄

Table with columns for T̄_D, T̄, W̄, λ, γ, and M̄. It is divided into four sections: two for 'COMPRESSIVE END LOADS' and two for 'TENSILE END LOADS'. Each section contains numerical data for various parameter values.

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

\bar{T}_d	$\bar{W} = 0.20000000$			$\bar{W} = 0.90000000$			\bar{T}_d	$\bar{W} = 0.20000000$			$\bar{W} = 0.12000000$		
	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{y}	\bar{M}	$\bar{\lambda}$		\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS													
0.89704768	-0.81250000	0.45000000	0.	0.77104768	-0.75000000	0.40000000	0.	0.96509535	-0.84450190	0.37401280	-0.30000000	0.	
0.10236046	-0.87275900	0.31007800	-0.40000000	0.84291870	-0.77942770	0.35298515	-0.30000000	0.10236046	-0.87275900	0.31007800	-0.40000000	-0.40000000	
0.11070800	-0.90725890	0.22319010	-0.50000000	0.91512580	-0.83766800	0.32059300	-0.50000000	0.11070800	-0.90725890	0.22319010	-0.50000000	-0.50000000	
0.12239520	-0.95407160	0.10581420	-0.60000000	0.98513040	-0.88285810	0.28217110	-0.60000000	0.12239520	-0.95407160	0.10581420	-0.60000000	-0.60000000	
0.13879130	-1.02081100	-0.50197410	-0.70000000	0.11914400	-0.94278950	0.21380429	-0.70000000	0.13879130	-1.02081100	-0.50197410	-0.70000000	-0.70000000	
0.16221500	-0.11069700	-0.25847370	-0.80000000	0.13926190	-0.10225290	-0.54415510	-0.80000000	0.16221500	-0.11069700	-0.25847370	-0.80000000	-0.80000000	
0.19681000	-0.12215070	-0.54108900	-0.90000000	0.16891070	-0.11304140	-0.31563560	-0.90000000	0.19681000	-0.12215070	-0.54108900	-0.90000000	-0.90000000	
0.25055410	-0.13958070	-0.94589750	-0.10000000	0.21692690	-0.12805550	-0.06805760	-0.10000000	0.25055410	-0.13958070	-0.94589750	-0.10000000	-0.10000000	
0.34951300	-0.16224970	-0.15132270	-0.10000000	0.29205750	-0.14996300	0.12145600	-0.11000000	0.34951300	-0.16224970	-0.15132270	-0.10000000	-0.11000000	
0.50894350	-0.19927720	-0.24195920	-0.12000000	0.41862580	-0.18421550	0.20529970	-0.12000000	0.50894350	-0.19927720	-0.24195920	-0.12000000	-0.12000000	
0.68714410	-0.26440140	-0.40181840	-0.14000000	0.76061670	-0.24413410	0.35322890	-0.14000000	0.68714410	-0.26440140	-0.40181840	-0.14000000	-0.14000000	
0.92797850	-0.40686550	-0.75245610	-0.16000000	0.17823400	-0.37638700	-0.67770800	-0.16000000	0.92797850	-0.40686550	-0.75245610	-0.16000000	-0.16000000	
0.40190410	-0.58708100	-0.11472890	-0.18000000	0.34435940	-0.52648190	-0.10431780	-0.18000000	0.40190410	-0.58708100	-0.11472890	-0.18000000	-0.18000000	
0.13318070	-0.95417490	0.21014910	-0.15000000	0.96979140	-0.88294150	-0.19267150	-0.15000000	0.13318070	-0.95417490	0.21014910	-0.15000000	-0.15000000	
0.12869710	-0.12344470	0.27631780	-0.15000000	0.13879040	-0.12065700	0.15520000	-0.15000000	0.12869710	-0.12344470	0.27631780	-0.15000000	-0.15000000	
TENSILE END LOADS													
0.81607750	0.78296970	0.42045400	0.10000000	0.71983490	-0.72265710	0.36650180	0.10000000	0.81607750	0.78296970	0.42045400	0.10000000	0.10000000	
0.72010100	0.75116190	0.37181820	0.20000000	0.68164100	-0.73268240	0.37262420	0.20000000	0.72010100	0.75116190	0.37181820	0.20000000	0.20000000	
0.62628550	0.71525720	0.34541420	0.30000000	0.64816070	-0.74852810	0.37676160	0.30000000	0.62628550	0.71525720	0.34541420	0.30000000	0.30000000	
0.68687000	0.70506190	0.37240210	0.40000000	0.59286710	-0.85110080	0.38343930	0.40000000	0.68687000	0.70506190	0.37240210	0.40000000	0.40000000	
0.62684700	0.71874320	0.47724720	0.50000000	0.58276850	-0.82133310	0.33845320	0.50000000	0.62684700	0.71874320	0.47724720	0.50000000	0.50000000	
0.70484660	0.69594590	0.60515770	0.60000000	0.58202700	-0.92009410	0.37768290	0.60000000	0.70484660	0.69594590	0.60515770	0.60000000	0.60000000	
0.81400180	0.66257700	0.74261470	0.70000000	0.58202700	-0.95501790	0.37024220	0.70000000	0.81400180	0.66257700	0.74261470	0.70000000	0.70000000	
0.91619280	0.62855970	0.91269430	0.80000000	0.58202700	-0.97922990	0.36262790	0.80000000	0.91619280	0.62855970	0.91269430	0.80000000	0.80000000	
0.30685820	0.62815880	0.91269430	0.90000000	0.58202700	-0.99518190	0.35682390	0.90000000	0.30685820	0.62815880	0.91269430	0.90000000	0.90000000	
0.12282520	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.12282520	0.62815880	0.91269430	0.90000000	0.90000000	
0.22858200	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22858200	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	
0.22222500	0.62815880	0.91269430	0.90000000	0.58202700	-1.00262470	0.35159310	0.90000000	0.22222500	0.62815880	0.91269430	0.90000000	0.90000000	

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, $\bar{\gamma}$ AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

\bar{T}_d	\bar{W}	$\bar{\lambda}$	$\bar{\gamma}$	\bar{M}	\bar{T}_d	\bar{W}	$\bar{\lambda}$	$\bar{\gamma}$	\bar{M}
0.4201905E 02	-0.4000000E 01	0.9000000E 01	-0.0000000E 01	0.9000000E 01	0.5153333E 02	-0.4000000E 01	0.1050000E 02	0.0000000E 00	0.1050000E 02
0.4526657E 02	0.5970301E 01	0.9537373E 01	-0.3000000E 00	0.3000000E 00	0.5551355E 02	0.6610042E 01	0.1109571E 02	-0.3000000E 00	0.3000000E 00
0.4806130E 02	0.6152159E 01	0.9845666E 01	-0.4000000E 00	0.4000000E 00	0.5894039E 02	0.6822072E 01	0.1159152E 02	-0.4000000E 00	0.4000000E 00
0.5204661E 02	0.6808154E 01	0.1060154E 02	-0.5000000E 00	0.5000000E 00	0.6382609E 02	0.7121862E 01	0.1227576E 02	-0.5000000E 00	0.5000000E 00
0.5767465E 02	0.6744746E 01	0.1142793E 02	-0.6000000E 00	0.6000000E 00	0.7066531E 02	0.7476382E 01	0.1319150E 02	-0.6000000E 00	0.6000000E 00
0.6545005E 02	0.7192365E 01	0.1252426E 02	-0.7000000E 00	0.7000000E 00	0.8026096E 02	0.7972780E 01	0.1440666E 02	-0.7000000E 00	0.7000000E 00
0.7662910E 02	0.7780754E 01	0.1398479E 02	-0.8000000E 00	0.8000000E 00	0.9197016E 02	0.8653194E 01	0.1602577E 02	-0.8000000E 00	0.8000000E 00
0.9113756E 02	0.8595239E 01	0.1596214E 02	-0.9000000E 00	0.9000000E 00	0.1142175E 03	0.9526770E 01	0.1821668E 02	-0.9000000E 00	0.9000000E 00
0.1107826E 03	0.9717966E 01	0.1871795E 02	-1.0000000E 00	1.0000000E 00	0.1366733E 03	0.1077039E 02	0.2127039E 02	-1.0000000E 00	1.0000000E 00
0.1617063E 03	0.1135384E 02	0.2273815E 02	-1.1000000E 00	1.1000000E 00	0.1988270E 03	0.1258746E 02	0.2572477E 02	-1.1000000E 00	1.1000000E 00
0.2420924E 03	0.1391327E 02	0.2920351E 02	-1.2000000E 00	1.2000000E 00	0.2938166E 03	0.1543747E 02	0.3272036E 02	-1.2000000E 00	1.2000000E 00
0.4276095E 03	0.1841362E 02	0.4011902E 02	-1.3000000E 00	1.3000000E 00	0.5180416E 03	0.2090235E 02	0.4443307E 02	-1.3000000E 00	1.3000000E 00
0.9915535E 03	0.2825629E 02	0.6438233E 02	-1.4000000E 00	1.4000000E 00	0.1216394E 04	0.3130662E 02	0.7105766E 02	-1.4000000E 00	1.4000000E 00
0.1216771E 04	0.3932433E 02	0.9167899E 02	-1.5000000E 00	1.5000000E 00	0.2351516E 04	0.4356190E 02	0.1020211E 03	-1.5000000E 00	1.5000000E 00
0.5199192E 04	0.6608311E 02	0.1576820E 03	-1.5000000E 00	1.5000000E 00	0.6827637E 04	0.7318121E 02	0.1951576E 03	-1.5000000E 00	1.5000000E 00
0.6057755E 04	0.2215003E 03	0.5411546E 03	-1.5000000E 00	1.5000000E 00	0.7427107E 04	0.2245350E 03	0.5829544E 03	-1.5000000E 00	1.5000000E 00
0.5912062E 07	0.5545251E 01	0.8500929E 01	0.3000000E 00	0.3000000E 00	0.7978310E 07	0.6186161E 01	0.3946665E 01	0.3000000E 00	0.3000000E 00
0.5704092E 02	0.5395800E 01	0.8136691E 01	0.4000000E 00	0.4000000E 00	0.8583192E 07	0.5982622E 01	0.3847491E 01	0.4000000E 00	0.4000000E 00
0.3461780E 02	0.5214717E 01	0.7876316E 01	0.5000000E 00	0.5000000E 00	0.9276204E 07	0.5782044E 01	0.3769489E 01	0.5000000E 00	0.5000000E 00
0.3199276E 02	0.5039235E 01	0.7596088E 01	0.6000000E 00	0.6000000E 00	0.9920266E 07	0.5584413E 01	0.3653082E 01	0.6000000E 00	0.6000000E 00
0.2919550E 02	0.4876154E 01	0.7334784E 01	0.7000000E 00	0.7000000E 00	0.1058198E 08	0.5392756E 01	0.3529455E 01	0.7000000E 00	0.7000000E 00
0.2642069E 02	0.4551970E 01	0.6886719E 01	0.8000000E 00	0.8000000E 00	0.1242122E 08	0.5047917E 01	0.3269336E 01	0.8000000E 00	0.8000000E 00
0.2372072E 02	0.4312527E 01	0.6500853E 01	0.9000000E 00	0.9000000E 00	0.1491153E 08	0.4702553E 01	0.3024132E 01	0.9000000E 00	0.9000000E 00
0.2155531E 02	0.4072760E 01	0.6272420E 01	0.1000000E 01	0.1000000E 01	0.1797563E 08	0.4316923E 01	0.2790377E 01	0.1000000E 01	0.1000000E 01
0.1976424E 02	0.3836688E 01	0.6105760E 01	0.1100000E 01	0.1100000E 01	0.2105003E 08	0.4025537E 01	0.2551916E 01	0.1100000E 01	0.1100000E 01
0.1836831E 02	0.3607765E 01	0.6005568E 01	0.1200000E 01	0.1200000E 01	0.2420368E 08	0.4001177E 01	0.2471805E 01	0.1200000E 01	0.1200000E 01
0.1657532E 02	0.3386744E 01	0.5763691E 01	0.1300000E 01	0.1300000E 01	0.2742950E 08	0.3756897E 01	0.2310994E 01	0.1300000E 01	0.1300000E 01
0.1278275E 02	0.3176699E 01	0.5736870E 01	0.1400000E 01	0.1400000E 01	0.3157807E 08	0.3524150E 01	0.2259267E 01	0.1400000E 01	0.1400000E 01
0.1118121E 02	0.2977948E 01	0.5299168E 01	0.1500000E 01	0.1500000E 01	0.3671710E 08	0.3303931E 01	0.2206619E 01	0.1500000E 01	0.1500000E 01
0.9157091E 01	0.2789941E 01	0.1055197E 01	0.1600000E 01	0.1600000E 01	0.4302912E 08	0.3087117E 01	0.2157804E 01	0.1600000E 01	0.1600000E 01
0.8694319E 01	0.2615724E 01	0.1840094E 01	0.1700000E 01	0.1700000E 01	0.5051597E 08	0.2902596E 01	0.2111513E 01	0.1700000E 01	0.1700000E 01
0.7372755E 01	0.2452170E 01	0.1054270E 01	0.1800000E 01	0.1800000E 01	0.9158706E 08	0.2721318E 01	0.1688299E 01	0.1800000E 01	0.1800000E 01
0.6188689E 01	0.2299811E 01	0.6976104E 00	0.1900000E 01	0.1900000E 01	0.1960220E 09	0.2552497E 01	0.1285848E 01	0.1900000E 01	0.1900000E 01
0.5513454E 01	0.2158225E 01	0.5670980E 00	0.2000000E 01	0.2000000E 01	0.6901535E 09	0.2395566E 01	0.9177872E 00	0.2000000E 01	0.2000000E 01
0.4757269E 01	0.2026772E 01	0.4191510E 01	0.2100000E 01	0.2100000E 01	0.9965132E 09	0.2289075E 01	0.5780613E 00	0.2100000E 01	0.2100000E 01
0.4047202E 01	0.1908051E 01	0.2194774E 00	0.2200000E 01	0.2200000E 01	0.5135181E 10	0.2178179E 01	0.2646680E 00	0.2200000E 01	0.2200000E 01
0.3631709E 01	0.1791827E 01	0.8787655E 00	0.2300000E 01	0.2300000E 01	0.1397400E 10	0.1989660E 01	0.2872124E 01	0.2300000E 01	0.2300000E 01
0.2800592E 01	0.1687075E 01	0.7175394E 00	0.2400000E 01	0.2400000E 01	0.1739395E 10	0.1873319E 01	0.2994324E 01	0.2400000E 01	0.2400000E 01
0.2104910E 01	0.1589978E 01	0.9173602E 00	0.2500000E 01	0.2500000E 01	0.1510556E 10	0.1768701E 01	0.3356344E 01	0.2500000E 01	0.2500000E 01
0.1936871E 01	0.1499950E 01	0.1139731E 01	0.2600000E 01	0.2600000E 01	0.1665901E 10	0.1665901E 01	0.3838977E 01	0.2600000E 01	0.2600000E 01
0.1720972E 01	0.1418448E 01	0.1386059E 01	0.2700000E 01	0.2700000E 01	0.1160907E 10	0.1535297E 01	0.9695621E 00	0.2700000E 01	0.2700000E 01
0.1157597E 01	0.1318979E 01	0.1657398E 01	0.2800000E 01	0.2800000E 01	0.1356001E 10	0.1487812E 01	0.1161108E 01	0.2800000E 01	0.2800000E 01
0.8154615E 00	0.1267922E 01	0.1655454E 01	0.2900000E 01	0.2900000E 01	0.1107729E 10	0.1407619E 01	0.1448070E 01	0.2900000E 01	0.2900000E 01
0.9490651E 00	0.1200196E 01	0.1801344E 01	0.3000000E 01	0.3000000E 01	0.9423344E 10	0.1335848E 01	0.1568112E 01	0.3000000E 01	0.3000000E 01
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00
0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00	0.8000000E 01	0.4000000E 01	0.7400000E 02	0.3000000E 00	0.3000000E 00

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

$\bar{T}_d = -0.1200000E 02$				$\bar{W} = 0.2400000E 02$			
\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS							
0.1558857E 03	0.1100000E 02	0.1200000E 02	0.	0.8533333E 02	0.8000000E 01	0.	0.
0.1676674E 03	0.1152348E 02	0.1102811E 02	-0.3000000E -03	0.9184519E 02	0.8311396E 01	0.7480757E 00	-0.3000000E -00
0.1781756E 03	0.1177591E 02	0.1386412E 02	-0.4000000E -00	0.9784352E 02	0.8570441E 01	0.1371271E 01	-0.4000000E -00
0.1928837E 03	0.1226118E 02	0.1506534E 02	-0.5000000E -00	0.1034325E 03	0.8927611E 01	0.2231903E 01	-0.5000000E 00
0.2134776E 03	0.1291137E 02	0.1664809E 02	-0.6000000E -00	0.1106165E 03	0.9405702E 01	0.3386053E 01	-0.6000000E 00
0.2423784E 03	0.1377295E 02	0.1874374E 02	-0.7000000E -00	0.1132084E 03	0.1003949E 02	0.4919348E 01	-0.7000000E 00
0.2836796E 03	0.1491726E 02	0.2154855E 02	-0.8000000E -00	0.1547291E 03	0.1088310E 02	0.6965187E 01	-0.8000000E 00
0.3446918E 03	0.1647341E 02	0.2534103E 02	-0.9000000E -00	0.1878403E 03	0.1202921E 02	0.9739613E 01	-0.9000000E 00
0.4395021E 03	0.1862937E 02	0.3062737E 02	-0.1200000E 01	0.2392845E 03	0.1361305E 02	0.1361305E 02	-0.1000000E 01
0.5982366E 03	0.2177540E 02	0.3834023E 02	-0.1100000E 01	0.3253595E 03	0.1592865E 02	0.1927130E 02	-0.1100000E 01
0.8955770E 03	0.2669780E 02	0.5046483E 02	-0.1200000E 01	0.4466848E 03	0.1955226E 02	0.2815526E 02	-0.1200000E 01
0.1563402E 04	0.3535366E 02	0.7147591E 02	-0.1300000E 01	0.6488909E 03	0.2592508E 02	0.4381335E 02	-0.1300000E 01
0.3683614E 04	0.5428557E 02	0.1151177E 03	-0.1400000E 01	0.1999472E 04	0.3986522E 02	0.7813588E 02	-0.1400000E 01
0.7091748E 04	0.7547467E 02	0.1708957E 03	-0.1450000E 01	0.1847030E 04	0.5554198E 02	0.1167770E 03	-0.1450000E 01
0.1997694E 05	0.1270001E 03	0.2977807E 03	-0.1500000E 01	0.1035481E 05	0.9341746E 02	0.2101893E 03	-0.1500000E 01
0.2241410E 06	0.4280001E 03	0.1035465E 04	-0.1550000E 01	0.1215549E 06	0.3135797E 03	0.7534220E 03	-0.1550000E 01
TENSILE END LOADS							
0.1451682E 03	0.1060639E 02	0.1104542E 02	0.3000000E -00	0.7749444E 02	0.7710731E 01	-0.6932534E 00	0.3000000E -00
0.1375351E 03	0.1031493E 02	0.1034897E 02	0.4000000E -00	0.7536021E 02	0.7499255E 01	0.1199891E 01	0.4000000E 00
0.1286060E 03	0.9971146E 01	0.9507214E 01	0.5000000E 00	0.7050505E 02	0.7243591E 01	-0.1010498E 01	0.5000000E 00
0.1188268E 03	0.9576216E 01	0.8552562E 01	0.6000000E 00	0.6516618E 02	0.6953102E 01	-0.2303189E 01	0.6000000E 00
0.1086424E 03	0.9124511E 01	0.7517749E 01	0.7000000E 00	0.5946547E 02	0.6638189E 01	0.3752712E 01	0.7000000E 00
0.9845314E 02	0.8697419E 01	0.6433652E 01	0.8000000E 00	0.5462916E 02	0.6307502E 01	-0.4033840E 01	0.8000000E 00
0.8811984E 02	0.8237121E 01	0.5323739E 01	0.9000000E 01	0.4858667E 02	0.5833132E 01	0.5957112E 01	0.9000000E 01
0.7910820E 02	0.7776651E 01	0.4223389E 01	0.1000000E 01	0.4107820E 02	0.5298042E 01	0.6303655E 01	0.1000000E 01
0.7034875E 02	0.7323056E 01	0.3139102E 01	0.1100000E 01	0.3439398E 02	0.4948549E 01	-0.7136423E 01	0.1100000E 01
0.6232071E 02	0.6882409E 01	0.2089131E 01	0.1200000E 01	0.2839255E 02	0.4661972E 01	0.7781194E 01	0.1200000E 01
0.5505225E 02	0.6458965E 01	0.1084350E 01	0.1300000E 01	0.2405644E 02	0.4387284E 01	-0.9561748E 01	0.1300000E 01
0.4853140E 02	0.6055592E 01	0.1310126E -00	0.1400000E 01	0.2057139E 02	0.4088236E 01	-0.9194561E 01	0.1400000E 01
0.4273133E 02	0.5674017E 01	-0.7665106E 00	0.1500000E 01	0.1708412E 02	0.3825136E 01	-0.7722349E 01	0.1500000E 01
0.3759391E 02	0.5315061E 01	-0.1606557E 01	0.1600000E 01	0.1435208E 02	0.3578661E 01	-0.1354292E 02	0.1600000E 01
0.3306339E 02	0.4978856E 01	-0.2188933E 01	0.1700000E 01	0.1161971E 02	0.3349112E 01	-0.1085112E 02	0.1700000E 01
0.2907867E 02	0.4665021E 01	-0.3136674E 01	0.1800000E 01	0.0922779E 02	0.3133452E 01	0.1333452E 02	0.1800000E 01
0.2577927E 02	0.4372828E 01	-0.4189498E 01	0.1900000E 01	0.0724989E 02	0.2926797E 01	-0.1174738E 02	0.1900000E 01
0.2280827E 02	0.4102974E 01	-0.5405181E 01	0.2000000E 01	0.0569095E 02	0.2727678E 01	0.1111971E 02	0.2000000E 01
0.1981259E 02	0.3849110E 01	-0.6475661E 01	0.2100000E 01	0.0455577E 02	0.2542794E 01	0.1284739E 02	0.2100000E 01
0.1764457E 02	0.3615617E 01	-0.7439868E 01	0.2200000E 01	0.0379592E 02	0.2424134E 01	0.1328356E 02	0.2200000E 01
0.1516150E 02	0.3399157E 01	-0.8081519E 01	0.2300000E 01	0.0329848E 02	0.2327933E 01	0.1331207E 02	0.2300000E 01
0.1352564E 02	0.3198557E 01	-0.8421686E 01	0.2400000E 01	0.0296062E 02	0.2242634E 01	0.1333086E 02	0.2400000E 01
0.1190399E 02	0.3012894E 01	-0.8829319E 01	0.2500000E 01	0.0276211E 02	0.2170202E 01	0.1336626E 02	0.2500000E 01
0.1046769E 02	0.2840454E 01	-0.9201613E 01	0.2600000E 01	0.0267127E 02	0.2100131E 01	-0.1389099E 02	0.2600000E 01
0.9191761E 01	0.2680748E 01	-0.9542654E 01	0.2700000E 01	0.0264155E 02	0.2039326E 01	-0.1406129E 02	0.2700000E 01
0.8055561E 01	0.2532600E 01	-0.9855656E 01	0.2800000E 01	0.0263173E 02	0.1987979E 01	-0.1426557E 02	0.2800000E 01
0.7037408E 01	0.2395099E 01	-0.0142780E 01	0.2900000E 01	0.0263797E 02	0.1945136E 01	-0.1448371E 02	0.2900000E 01
0.6124175E 01	0.2278164E 01	-0.0406275E 01	0.3000000E 01				

TABLE 1.5-2
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{m} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}
(Pages 1.38 through 1.55)

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{Q}

$\bar{T}_D = 0$				$\bar{T}_D = 0.1200000E 02$				$\bar{T}_D = 0$				$\bar{T}_D = 0.1500000E 02$			
\bar{y}		\bar{M}		\bar{y}		\bar{M}		\bar{y}		\bar{M}		\bar{y}		\bar{M}	
COMPRESSIVE END LOADS															
0.1920000E 02	0.4000000E 01	0.1220000E 02	0.	0.3300000E 02	0.5000000E 01	0.1500000E 02	0.	0.2090000E 02	0.2420000E 01	0.0620000E 02	0.1200000E 02	0.3000000E 01	0.0800000E 02	0.1000000E 02	0.1500000E 01

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{Q}

$\bar{T}_D = -0.1200000E 02$				$\bar{T}_D = 0.1200000E 02$				$\bar{T}_D = -0.1200000E 02$				$\bar{T}_D = 0.1500000E 02$			
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS				COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}	\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}	\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}	\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}
0.127200E 05	0.1000000E 07	0.1200000E 02	0.	0.	0.1000000E 07	0.1200000E 02	0.	0.1100000E 02	0.1500000E 02	0.1500000E 02	0.	0.	0.1000000E 07	0.1200000E 02	0.
0.136995E 03	0.105859E 07	0.129348E 02	0.	0.	0.105859E 07	0.129348E 02	0.	0.116787E 03	0.157255E 02	0.157255E 02	0.	0.	0.105859E 07	0.129348E 02	0.
0.145818E 05	0.110705E 07	0.1371225E 02	0.	0.	0.110705E 07	0.1371225E 02	0.	0.127201E 03	0.167201E 02	0.167201E 02	0.	0.	0.110705E 07	0.1371225E 02	0.
0.157433E 05	0.1144075E 07	0.1437851E 02	0.	0.	0.1144075E 07	0.1437851E 02	0.	0.138987E 03	0.177520E 02	0.177520E 02	0.	0.	0.1144075E 07	0.1437851E 02	0.
0.174261E 05	0.117285E 07	0.1492272E 02	0.	0.	0.117285E 07	0.1492272E 02	0.	0.150943E 03	0.18711E 02	0.18711E 02	0.	0.	0.117285E 07	0.1492272E 02	0.
0.197877E 05	0.120076E 07	0.1541879E 02	0.	0.	0.120076E 07	0.1541879E 02	0.	0.164069E 03	0.196443E 02	0.196443E 02	0.	0.	0.120076E 07	0.1541879E 02	0.
0.221625E 05	0.1234488E 07	0.160097E 02	0.	0.	0.1234488E 07	0.160097E 02	0.	0.178397E 03	0.205597E 02	0.205597E 02	0.	0.	0.1234488E 07	0.160097E 02	0.
0.241473E 05	0.1269462E 07	0.1667269E 02	0.	0.	0.1269462E 07	0.1667269E 02	0.	0.193975E 03	0.21468E 02	0.21468E 02	0.	0.	0.1269462E 07	0.1667269E 02	0.
0.269949E 05	0.1309968E 07	0.1740896E 02	0.	0.	0.1309968E 07	0.1740896E 02	0.	0.210802E 03	0.223697E 02	0.223697E 02	0.	0.	0.1309968E 07	0.1740896E 02	0.
0.298854E 05	0.1356913E 07	0.1822999E 02	0.	0.	0.1356913E 07	0.1822999E 02	0.	0.228867E 03	0.232647E 02	0.232647E 02	0.	0.	0.1356913E 07	0.1822999E 02	0.
0.316619E 05	0.1399103E 07	0.1913796E 02	0.	0.	0.1399103E 07	0.1913796E 02	0.	0.248081E 03	0.241547E 02	0.241547E 02	0.	0.	0.1399103E 07	0.1913796E 02	0.
0.327273E 04	0.1427176E 07	0.1991184E 02	0.	0.	0.1427176E 07	0.1991184E 02	0.	0.268454E 03	0.250397E 02	0.250397E 02	0.	0.	0.1427176E 07	0.1991184E 02	0.
0.3297198E 04	0.1451386E 07	0.1942940E 02	0.	0.	0.1451386E 07	0.1942940E 02	0.	0.289987E 03	0.259197E 02	0.259197E 02	0.	0.	0.1451386E 07	0.1942940E 02	0.
0.329634E 04	0.1481757E 07	0.1957831E 02	0.	0.	0.1481757E 07	0.1957831E 02	0.	0.313672E 03	0.267947E 02	0.267947E 02	0.	0.	0.1481757E 07	0.1957831E 02	0.
0.326226E 05	0.1518897E 07	0.2045536E 02	0.	0.	0.1518897E 07	0.2045536E 02	0.	0.339507E 03	0.276647E 02	0.276647E 02	0.	0.	0.1518897E 07	0.2045536E 02	0.
0.311443E 06	0.185119E 07	0.2572470E 02	0.	0.	0.185119E 07	0.2572470E 02	0.	0.4220425E 06	0.4220425E 06	0.4220425E 06	0.	0.	0.185119E 07	0.2572470E 02	0.
TENSILE END LOADS				TENSILE END LOADS				TENSILE END LOADS				TENSILE END LOADS			
0.1184199E 03	0.968600E 01	0.111320E 02	0.	0.	0.968600E 01	0.111320E 02	0.	0.1528532E 03	0.1000000E 02	0.1000000E 02	0.	0.	0.1184199E 03	0.968600E 01	0.111320E 02
0.112705E 05	0.958691E 01	0.1062856E 02	0.	0.	0.958691E 01	0.1062856E 02	0.	0.1549552E 03	0.1092490E 02	0.1092490E 02	0.	0.	0.112705E 05	0.958691E 01	0.1062856E 02
0.1069075E 05	0.908987E 01	0.9732530E 01	0.	0.	0.908987E 01	0.9732530E 01	0.	0.1576722E 03	0.1184722E 02	0.1184722E 02	0.	0.	0.1069075E 05	0.908987E 01	0.9732530E 01
0.9891470E 02	0.871270E 01	0.948630E 01	0.	0.	0.871270E 01	0.948630E 01	0.	0.1608584E 03	0.1286354E 02	0.1286354E 02	0.	0.	0.9891470E 02	0.871270E 01	0.948630E 01
0.885937E 02	0.8528478E 01	0.7921056E 01	0.	0.	0.8528478E 01	0.7921056E 01	0.	0.1646596E 03	0.1396409E 02	0.1396409E 02	0.	0.	0.885937E 02	0.8528478E 01	0.7921056E 01
0.8025162E 02	0.7917265E 01	0.693292E 01	0.	0.	0.7917265E 01	0.693292E 01	0.	0.1690574E 03	0.1514925E 02	0.1514925E 02	0.	0.	0.8025162E 02	0.7917265E 01	0.693292E 01
0.7318510E 02	0.7501025E 01	0.5924174E 01	0.	0.	0.7501025E 01	0.5924174E 01	0.	0.1741565E 03	0.1642404E 02	0.1642404E 02	0.	0.	0.7318510E 02	0.7501025E 01	0.5924174E 01
0.686542E 02	0.7084217E 01	0.4991970E 01	0.	0.	0.7084217E 01	0.4991970E 01	0.	0.1799476E 03	0.1779476E 02	0.1779476E 02	0.	0.	0.686542E 02	0.7084217E 01	0.4991970E 01
0.5729108E 02	0.6671791E 01	0.3924759E 01	0.	0.	0.6671791E 01	0.3924759E 01	0.	0.1873865E 03	0.1926819E 02	0.1926819E 02	0.	0.	0.5729108E 02	0.6671791E 01	0.3924759E 01
0.5027206E 02	0.6275076E 01	0.2781947E 01	0.	0.	0.6275076E 01	0.2781947E 01	0.	0.1964781E 03	0.2084781E 02	0.2084781E 02	0.	0.	0.5027206E 02	0.6275076E 01	0.2781947E 01
0.4478561E 02	0.5891268E 01	0.2082913E 01	0.	0.	0.5891268E 01	0.2082913E 01	0.	0.2073421E 03	0.2253421E 02	0.2253421E 02	0.	0.	0.4478561E 02	0.5891268E 01	0.2082913E 01
0.3985516E 02	0.5526887E 01	0.1387722E 01	0.	0.	0.5526887E 01	0.1387722E 01	0.	0.2199817E 03	0.2434817E 02	0.2434817E 02	0.	0.	0.3985516E 02	0.5526887E 01	0.1387722E 01
0.3673495E 02	0.5181181E 01	0.5621358E 00	0.	0.	0.5181181E 01	0.5621358E 00	0.	0.2346629E 03	0.2640629E 02	0.2640629E 02	0.	0.	0.3673495E 02	0.5181181E 01	0.5621358E 00
0.3052581E 02	0.4586183E 01	0.2843156E 00	0.	0.	0.4586183E 01	0.2843156E 00	0.	0.2528699E 03	0.2878699E 02	0.2878699E 02	0.	0.	0.3052581E 02	0.4586183E 01	0.2843156E 00
0.267977E 02	0.4285382E 01	0.1358570E 00	0.	0.	0.4285382E 01	0.1358570E 00	0.	0.2748008E 03	0.3148008E 02	0.3148008E 02	0.	0.	0.267977E 02	0.4285382E 01	0.1358570E 00
0.235852E 02	0.4287472E 01	0.1898301E 00	0.	0.	0.4287472E 01	0.1898301E 00	0.	0.3003359E 03	0.3443359E 02	0.3443359E 02	0.	0.	0.235852E 02	0.4287472E 01	0.1898301E 00
0.2006930E 02	0.4002613E 01	0.2844525E 00	0.	0.	0.4002613E 01	0.2844525E 00	0.	0.3297202E 03	0.3767202E 02	0.3767202E 02	0.	0.	0.2006930E 02	0.4002613E 01	0.2844525E 00
0.181518E 02	0.3756521E 01	0.3228709E 00	0.	0.	0.3756521E 01	0.3228709E 00	0.	0.363068E 03	0.413068E 02	0.413068E 02	0.	0.	0.181518E 02	0.3756521E 01	0.3228709E 00
0.1584098E 02	0.3559171E 01	0.355228E 00	0.	0.	0.3559171E 01	0.355228E 00	0.	0.4023686E 03	0.4523686E 02	0.4523686E 02	0.	0.	0.1584098E 02	0.3559171E 01	0.355228E 00
0.1329711E 02	0.3318988E 01	0.359779E 00	0.	0.	0.3318988E 01	0.359779E 00	0.	0.4479598E 03	0.4949598E 02	0.4949598E 02	0.	0.	0.1329711E 02	0.3318988E 01	0.359779E 00
0.1228587E 02	0.3197989E 01	0.3088137E 00	0.	0.	0.3197989E 01	0.3088137E 00	0.	0.49993E 03	0.54193E 02	0.54193E 02	0.	0.	0.1228587E 02	0.3197989E 01	0.3088137E 00
0.107583E 02	0.2937983E 01	0.2627315E 00	0.	0.	0.2937983E 01	0.2627315E 00	0.	0.5585417E 03	0.5935417E 02	0.5935417E 02	0.	0.	0.107583E 02	0.2937983E 01	0.2627315E 00
0.9639294E 01	0.2739183E 01	0.2532732E 00	0.	0.	0.2739183E 01	0.2532732E 00	0.	0.6248866E 03	0.6428866E 02	0.6428866E 02	0.	0.	0.9639294E 01	0.2739183E 01	0.2532732E 00
0.8783129E 01	0.2571272E 01	0.9888252E 00	0.	0.	0.2571272E 01	0.9888252E 00	0.	0.7000912E 03	0.7180912E 02	0.7180912E 02	0.	0.	0.8783129E 01	0.2571272E 01	0.9888252E 00
0.7199238E 01	0.2384782E 01	0.3549892E 00	0.	0.	0.2384782E 01	0.3549892E 00	0.	0.7862954E 03	0.7962954E 02	0.7962954E 02	0.	0.	0.7199238E 01	0.2384782E 01	0.3549892E 00
0.6257193E 01	0.2253334E 01	0.252131E 00	0.	0.	0.2253334E 01	0.252131E 00	0.	0.8842991E 03	0.8842991E 02	0.8842991E 02	0.	0.	0.6257193E 01	0.2253334E 01	0.252131E 00
0.541675E 01	0.2120012E 01	0.3527277E 00	0.	0.	0.2120012E 01	0.3527277E 00	0.	0.9951234E 03	0.9951234E 02	0.9951234E 02	0.	0.	0.541675E 01	0.2120012E 01	0.3527277E 00
0.4651985E 01	0.2001991E 01	0.3587788E 00	0.	0.	0.2001991E 01	0.3587788E 00	0.	1.1190005E 03	1.1190005E 02	1.1190005E 02	0.	0.	0.4651985E 01	0.2001991E 01	0.3587788E 00
TENSILE END LOADS				TENSILE END LOADS				TENSILE END LOADS				TENSILE END LOADS			
2.1500000E 02								2.1500000E 02							
\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}	\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}	\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}	\bar{y}	$\bar{\lambda}$	\bar{M}	\bar{Q}
TENSILE END LOADS				TENSILE END LOADS				TENSILE END LOADS				TENSILE END LOADS			
0.985315E 02	0.1000000E 07	0.	0.	0.	0.1000000E 07	0.	0.	0.1000000E 02	0.1500000E 02	0.1500000E 02	0.	0.	0.1000000E 07	0.1500000E 02	0.
0.2166579E 02	0.1041317E 07	0.	0.	0.	0.1041317E 07	0.	0.	0.116787E 03	0.157255E 02	0.157255E 02	0.	0.	0.1041317E 07	0.157255E 02	0.
0.4078872E 02	0.1082788E 07	0.	0.	0.	0.1082788E 07	0.	0.	0.127201E 03	0.167201E 02	0.167201E 02	0.	0.	0.1082788E 07	0.167201E 02	0.
0.6134375E 02	0.1124075E 07	0.	0.	0.	0.1124075E 07	0.	0.	0.138987E 03	0.177520E 02	0.177520E 02	0.	0.	0.1124075E 07	0.177520E 02	0.
0.7758124E 02	0.1165285E 07	0.	0.	0.	0.1165285E 07	0.	0.	0.150943E 03	0.18711E 02	0.18711E 02	0.	0.	0.1165285E 07	0.18711E 02	0.
0.9127378E 02	0.1206488E 07	0.	0.	0.	0.1206488E 07	0.	0.	0.164069E 03	0.196443E 02	0.196443E 02	0.	0.	0.1206488E 07	0.196443E 02	0.
0.1047273E 03	0.1247691E 07	0.	0.	0.	0.1247691E 07	0.	0.	0.178397E 03	0.205597E 02	0.205597E 02	0.	0.	0.1247691E 07	0.205597E 02	0.
0.1195266E 03															

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{Q}

\bar{T}_D	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}_D	\bar{y}	\bar{M}	$\bar{\lambda}$
0.1000000E 02	0.0000000E 02	0.0000000E 02	0.0000000E 02	-0.1000000E 02	0.0000000E 02	0.0000000E 02	0.0000000E 02
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.1301333E 03	0.1000000E 02	0.6000000E 01	0.0000000E 02	0.1501333E 03	0.1100000E 02	0.9000000E 01	0.0000000E 02
0.1401115E 03	0.1018812E 02	0.6254751E 01	-0.1000000E 02	0.1681220E 03	0.1162348E 02	0.1002811E 02	-0.1000000E 02
0.1486909E 03	0.1070731E 02	0.7731698E 01	-0.4000000E 00	0.1784304E 03	0.1177574E 02	0.1028812E 02	-0.4000000E 00
0.1509354E 03	0.1115015E 02	0.8707531E 01	-0.5000000E 00	0.1931433E 03	0.1226139E 02	0.1206535E 02	-0.5000000E 00
0.1780794E 03	0.1174785E 02	0.1022774E 02	-0.6000000E 00	0.2137441E 03	0.1291140E 02	0.1364310E 02	-0.6000000E 00
0.2021373E 03	0.1252350E 02	0.1213876E 02	-0.7000000E 00	0.2426547E 03	0.1377900E 02	0.1574877E 02	-0.7000000E 00
0.2365353E 03	0.1357519E 02	0.1466749E 02	-0.8000000E 00	0.2839680E 03	0.1491372E 02	0.1854867E 02	-0.8000000E 00
0.2872769E 03	0.1498946E 02	0.1814007E 02	-0.1000000E 01	0.3449978E 03	0.1647061E 02	0.2235119E 02	-0.9000000E 00
0.3667072E 03	0.1695750E 02	0.2295753E 02	-0.1000000E 01	0.4398124E 03	0.1827272E 02	0.2762722E 02	-0.1000000E 01
0.4983094E 03	0.1982489E 02	0.2999584E 02	-0.1100000E 01	0.5986124E 03	0.2117603E 02	0.3514894E 02	-0.1100000E 01
0.7457561E 03	0.2451669E 02	0.4101641E 02	-0.1200000E 01	0.8790331E 03	0.2469384E 02	0.4744539E 02	-0.1200000E 01
0.1107403E 04	0.3221299E 02	0.5043943E 02	-0.1300000E 01	0.1156402E 04	0.3353561E 02	0.5875079E 02	-0.1300000E 01
0.1555165E 04	0.4040152E 02	0.1029713E 03	-0.1400000E 01	0.1566957E 04	0.4542877E 02	0.1154079E 03	-0.1400000E 01
0.2102149E 04	0.5090161E 02	0.1503657E 03	-0.1500000E 01	0.2047477E 04	0.7550145E 02	0.1679100E 03	-0.1500000E 01
0.2862438E 04	0.1158200E 03	0.2605750E 03	-0.1500000E 01	0.2798227E 05	0.1270233E 03	0.2947728E 03	-0.1500000E 01
0.3865127E 04	0.1580537E 03	0.4917512E 03	-0.1500000E 01	0.3721479E 05	0.4260507E 03	0.1035532E 04	-0.1500000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.1211077E 03	0.0964123E 02	0.5132729E 01	0.1000000E 02	0.1451450E 03	0.1126049E 02	0.4905427E 01	0.1000000E 02
0.1140544E 03	0.9379719E 01	0.4692454E 01	0.1000000E 02	0.1377775E 03	0.1031424E 02	0.4544771E 01	0.1000000E 02
0.1070194E 03	0.2061270E 01	0.3744029E 01	0.1000000E 02	0.1244444E 03	0.2971158E 01	0.3653221E 01	0.1000000E 02
0.9227464E 02	0.4073775E 01	0.2267557E 01	0.1000000E 02	0.1122065E 03	0.2770207E 01	0.3555255E 01	0.1000000E 02
0.9078185E 02	0.3431016E 01	0.1327733E 01	0.1000000E 02	0.1011111E 03	0.2187849E 01	0.4531773E 01	0.1000000E 02
0.8229126E 02	0.7270493E 01	0.2634785E 01	0.1000000E 02	0.9265462E 02	0.2697891E 01	0.3433512E 01	0.1000000E 02
0.7422867E 02	0.2763340E 01	0.5222568E 01	0.1000000E 02	0.8478446E 02	0.2857806E 01	0.2327634E 01	0.1000000E 02
0.6819685E 02	0.2333357E 01	0.1301557E 01	0.1000000E 02	0.7722866E 02	0.2777679E 01	0.1323000E 01	0.1000000E 02
0.5595949E 02	0.3586177E 01	0.2706427E 01	0.1000000E 02	0.7056950E 02	0.2195130E 01	0.1346261E 01	0.1000000E 02
0.5212407E 02	0.6024661E 01	0.2225183E 01	0.1000000E 02	0.6512771E 02	0.2052287E 01	0.2193119E 01	0.1000000E 02
0.4613132E 02	0.3465421E 01	0.3202475E 01	0.1000000E 02	0.6057447E 02	0.2683206E 01	0.1813126E 01	0.1000000E 02
0.4009141E 02	0.5823330E 01	0.4766304E 01	0.1000000E 02	0.5719371E 02	0.2575277E 01	0.2128271E 01	0.1000000E 02
0.3544245E 02	0.1345773E 02	0.5577470E 01	0.1000000E 02	0.5424211E 02	0.2576677E 01	0.2128271E 01	0.1000000E 02
0.3155465E 02	0.2401870E 02	0.6155222E 01	0.1000000E 02	0.5170505E 02	0.2158071E 01	0.4841276E 01	0.1000000E 02
0.2777212E 02	0.4552567E 02	0.7414787E 01	0.1000000E 02	0.4934807E 02	0.2272571E 01	0.5151917E 01	0.1000000E 02
0.2444325E 02	0.8220495E 02	0.7625132E 01	0.1000000E 02	0.4722547E 02	0.2688572E 01	0.2121702E 01	0.1000000E 02
0.2146211E 02	0.1326203E 03	0.8524885E 01	0.1000000E 02	0.4530585E 02	0.2770498E 01	0.2121702E 01	0.1000000E 02
0.1872912E 02	0.1310722E 03	0.9185213E 01	0.1000000E 02	0.4358070E 02	0.2471922E 01	0.1874001E 01	0.1000000E 02
0.1627431E 02	0.1544643E 03	0.9486806E 01	0.1000000E 02	0.4204113E 02	0.2170000E 01	0.1874001E 01	0.1000000E 02
0.1400257E 02	0.1827159E 03	0.9535195E 01	0.1000000E 02	0.4064203E 02	0.1870000E 01	0.1874001E 01	0.1000000E 02
0.1242498E 02	0.2107573E 03	0.9275375E 01	0.1000000E 02	0.3936202E 02	0.1582000E 01	0.1874001E 01	0.1000000E 02
0.1100154E 02	0.2392471E 03	0.8286153E 01	0.1000000E 02	0.3828295E 02	0.1338729E 01	0.2633261E 01	0.1000000E 02
0.1023954E 02	0.2703838E 03	0.6322722E 02	0.1000000E 02	0.3735762E 02	0.1127206E 01	0.2633261E 01	0.1000000E 02
0.8722648E 01	0.2758774E 03	0.3335131E 02	0.1000000E 02	0.3658041E 02	0.1000000E 01	0.2633261E 01	0.1000000E 02
0.7702630E 01	0.2427054E 03	0.1352893E 02	0.1000000E 02	0.3593941E 02	0.2252744E 01	0.2633261E 01	0.1000000E 02
0.6833228E 01	0.2199754E 03	0.1246331E 02	0.1000000E 02	0.3541111E 02	0.2254551E 01	0.2633261E 01	0.1000000E 02
0.5986842E 01	0.2102854E 03	0.2121397E 02	0.1000000E 02	0.3497444E 02	0.2197124E 01	0.2633261E 01	0.1000000E 02
0.5205949E 01	0.2108874E 03	0.2378234E 02	0.1000000E 02	0.3462444E 02	0.2162571E 01	0.2633261E 01	0.1000000E 02
TENSILE END LOADS				TENSILE END LOADS			
0.1000000E 02	0.0000000E 02	0.0000000E 02	0.0000000E 02	0.1000000E 02	0.0000000E 02	0.0000000E 02	0.0000000E 02

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

\bar{T}_d	\bar{Q}	\bar{T}	$\bar{\lambda}$	\bar{y}	\bar{M}	\bar{T}_d	\bar{Q}	\bar{T}	$\bar{\lambda}$	\bar{y}	\bar{M}
0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02
COMPRESSIVE END LOADS											
0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02
TENSILE END LOADS											
0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02	0.1000000E 02
TENSILE END LOADS											

TABLE 1.5-2 (Cont'd)
VALUES OF \bar{Y} , \bar{M} AND \bar{N} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

\bar{T}_d	\bar{Y}	\bar{M}	\bar{N}	\bar{T}	\bar{T}_d	\bar{Y}	\bar{M}	\bar{N}	\bar{T}
-0.200000E 02	0.120000E 02	0.120000E 02	0.120000E 02	0.120000E 02	-0.200000E 02	0.120000E 02	0.120000E 02	0.120000E 02	0.120000E 02
COMPRESSIVE END LOADS					COMPRESSIVE END LOADS				
0.252533E 03	0.140000E 02	0.120000E 02	0.120000E 02	0.120000E 02	0.209135E 03	0.150000E 02	0.150000E 02	0.150000E 02	0.150000E 02
0.271020E 03	0.145184E 02	0.131682E 02	0.131682E 02	-0.300000E -03	0.210687E 03	0.155760E 02	0.164018E 02	0.164018E 02	-0.300000E -00
0.288590E 03	0.149847E 02	0.134970E 02	0.134970E 02	-0.400000E -00	0.229530E 03	0.160557E 02	0.175466E 02	0.175466E 02	-0.400000E -00
0.317385E 03	0.156245E 02	0.159011E 02	0.159011E 02	-0.500000E -00	0.256720E 03	0.167158E 02	0.191709E 02	0.191709E 02	-0.500000E -00
0.345703E 03	0.164113E 02	0.177157E 02	0.177157E 02	-0.600000E -00	0.294608E 03	0.175996E 02	0.213359E 02	0.213359E 02	-0.600000E -00
0.392443E 03	0.175275E 02	0.205881E 02	0.205881E 02	-0.700000E -00	0.344245E 03	0.187180E 02	0.241982E 02	0.241982E 02	-0.700000E -00
0.419269E 03	0.189040E 02	0.231506E 02	0.231506E 02	-0.800000E -00	0.324617E 03	0.204515E 02	0.280127E 02	0.280127E 02	-0.800000E -00
0.550011E 03	0.209544E 02	0.274976E 02	0.274976E 02	-0.900000E -00	0.374458E 03	0.224409E 02	0.331771E 02	0.331771E 02	-0.900000E -00
0.711426E 03	0.237052E 02	0.357021E 02	0.357021E 02	-0.100000E 01	0.412787E 03	0.253774E 02	0.403774E 02	0.403774E 02	-0.100000E 01
0.968294E 03	0.277075E 02	0.455258E 02	0.455258E 02	-0.110000E 01	0.410636E 04	0.296564E 02	0.508326E 02	0.508326E 02	-0.110000E 01
0.146946E 04	0.319671E 02	0.601558E 02	0.601558E 02	-0.120000E 01	0.155632E 04	0.363513E 02	0.673459E 02	0.673459E 02	-0.120000E 01
0.255020E 04	0.409801E 02	0.880108E 02	0.880108E 02	-0.130000E 01	0.289150E 04	0.47391E 02	0.963290E 02	0.963290E 02	-0.130000E 01
0.593666E 04	0.690625E 02	0.147365E 03	0.147365E 03	-0.140000E 01	0.678506E 04	0.72744E 02	0.159790E 03	0.159790E 03	-0.140000E 01
0.114767E 05	0.981677E 02	0.214146E 03	0.214146E 03	-0.150000E 01	0.131172E 05	0.107826E 03	0.231192E 03	0.231192E 03	-0.150000E 01
0.325292E 05	0.131574E 03	0.375480E 03	0.375480E 03	-0.150000E 01	0.546510E 05	0.172782E 03	0.403750E 03	0.403750E 03	-0.150000E 01
0.362742E 06	0.141954E 03	0.413720E 04	0.413720E 04	-0.150000E 01	0.414593E 06	0.579419E 03	0.140705E 04	0.140705E 04	-0.150000E 01
TENSILE END LOADS					TENSILE END LOADS				
0.235154E 03	0.114794E 02	0.107850E 02	0.107850E 02	0.300000E -00	0.268473E 03	0.144645E 02	0.136981E 02	0.136981E 02	0.100000E -00
0.227852E 03	0.131316E 02	0.939616E 01	0.939616E 01	0.400000E -00	0.254612E 03	0.140735E 02	0.127482E 02	0.127482E 02	0.400000E -00
0.202409E 03	0.126917E 02	0.887139E 01	0.887139E 01	0.500000E -00	0.237912E 03	0.136004E 02	0.115928E 02	0.115928E 02	0.500000E -00
0.192591E 03	0.121889E 02	0.761200E 01	0.761200E 01	0.600000E -00	0.219839E 03	0.130611E 02	0.102977E 02	0.102977E 02	0.600000E -00
0.176171E 03	0.116416E 02	0.629670E 01	0.629670E 01	0.700000E -00	0.201020E 03	0.124799E 02	0.898384E 01	0.898384E 01	0.700000E -00
0.159622E 03	0.110710E 02	0.491450E 01	0.491450E 01	0.800000E -00	0.182169E 03	0.118676E 02	0.740608E 01	0.740608E 01	0.800000E -00
0.143585E 03	0.104857E 02	0.350653E 01	0.350653E 01	0.900000E -00	0.163847E 03	0.112417E 02	0.589419E 01	0.589419E 01	0.900000E -00
0.128372E 03	0.999779E 01	0.210075E 01	0.210075E 01	0.100000E 01	0.148687E 03	0.106150E 02	0.438499E 01	0.438499E 01	0.100000E 01
0.114715E 03	0.942812E 01	0.719577E 00	0.719577E 00	0.110000E 01	0.132298E 03	0.999787E 01	0.290257E 01	0.290257E 01	0.110000E 01
0.101244E 03	0.876711E 01	0.617357E 00	0.617357E 00	0.120000E 01	0.115481E 03	0.919832E 01	0.186636E 01	0.186636E 01	0.120000E 01
0.895043E 02	0.827373E 01	0.199007E 01	0.199007E 01	0.130000E 01	0.102073E 03	0.882216E 01	0.905340E -01	0.905340E -01	0.130000E 01
0.789804E 02	0.771062E 01	0.311731E 01	0.311731E 01	0.140000E 01	0.900568E 02	0.827370E 01	0.121564E 01	0.121564E 01	0.140000E 01
0.696165E 02	0.727528E 01	0.425689E 01	0.425689E 01	0.150000E 01	0.795664E 02	0.775404E 01	0.244659E 01	0.244659E 01	0.150000E 01
0.613307E 02	0.676872E 01	0.532792E 01	0.532792E 01	0.160000E 01	0.699095E 02	0.726554E 01	0.359979E 01	0.359979E 01	0.160000E 01
0.540294E 02	0.634107E 01	0.632576E 01	0.632576E 01	0.170000E 01	0.615779E 02	0.68071E 01	0.467504E 01	0.467504E 01	0.170000E 01
0.476131E 02	0.594123E 01	0.725186E 01	0.725186E 01	0.180000E 01	0.542580E 02	0.618001E 01	0.567385E 01	0.567385E 01	0.180000E 01
0.412645E 02	0.557027E 01	0.810877E 01	0.810877E 01	0.190000E 01	0.478602E 02	0.598308E 01	0.659982E 01	0.659982E 01	0.190000E 01
0.370509E 02	0.527494E 01	0.899770E 01	0.899770E 01	0.200000E 01	0.422166E 02	0.561347E 01	0.745374E 01	0.745374E 01	0.200000E 01
0.327267E 02	0.498445E 01	0.967867E 01	0.967867E 01	0.210000E 01	0.372903E 02	0.527036E 01	0.824210E 01	0.824210E 01	0.210000E 01
0.289351E 02	0.468071E 01	0.102995E 02	0.102995E 02	0.220000E 01	0.329728E 02	0.495273E 01	0.948834E 01	0.948834E 01	0.220000E 01
0.250065E 02	0.433149E 01	0.107160E 02	0.107160E 02	0.230000E 01	0.291852E 02	0.465739E 01	0.963721E 01	0.963721E 01	0.230000E 01
0.226800E 02	0.407682E 01	0.116425E 02	0.116425E 02	0.240000E 01	0.256574E 02	0.438612E 01	0.102522E 02	0.102522E 02	0.240000E 01
0.201021E 02	0.384042E 01	0.120028E 02	0.120028E 02	0.250000E 01	0.229290E 02	0.413102E 01	0.109144E 02	0.109144E 02	0.250000E 01
0.173243E 02	0.362159E 01	0.124406E 02	0.124406E 02	0.260000E 01	0.203600E 02	0.389616E 01	0.113494E 02	0.113494E 02	0.260000E 01
0.150121E 02	0.341879E 01	0.127193E 02	0.127193E 02	0.270000E 01	0.180625E 02	0.367577E 01	0.113142E 02	0.113142E 02	0.270000E 01
0.140245E 02	0.322950E 01	0.132424E 02	0.132424E 02	0.280000E 01	0.162387E 02	0.347689E 01	0.122580E 02	0.122580E 02	0.280000E 01
0.124333E 02	0.305625E 01	0.138927E 02	0.138927E 02	0.290000E 01	0.142388E 02	0.328944E 01	0.126655E 02	0.126655E 02	0.290000E 01
0.110123E 02	0.289259E 01	0.146332E 02	0.146332E 02	0.300000E 01	0.126344E 02	0.311515E 01	0.130431E 02	0.130431E 02	0.300000E 01

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1.5 (Cont'd)

CASE C - Transverse load uniformly distributed or concentrated at midspan - specified axial loads (zero axial end restraint):

The beam is shown schematically in Figure 1.5-3. Since the ends are unrestrained axially, the beam ends are free to move due to the action of temperature, transverse loads and specified end loads. Hence the compatibility Eqs. (2) and (5) do not apply, and since λ is now a known quantity, the maximum deflections and bending moments may be determined directly from Eqs. (3) and (6). These quantities are independent of the average temperature in the beam.

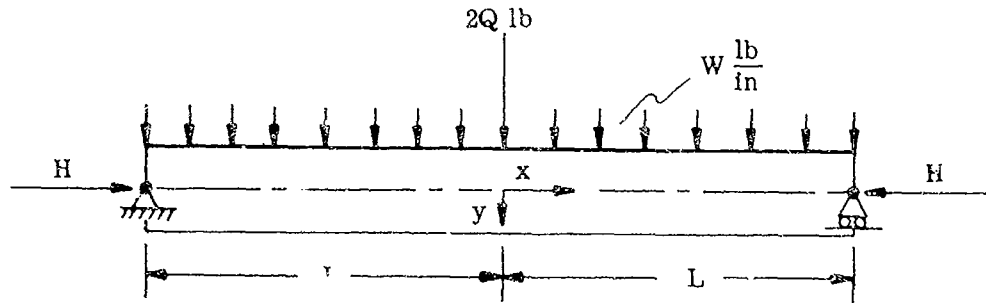
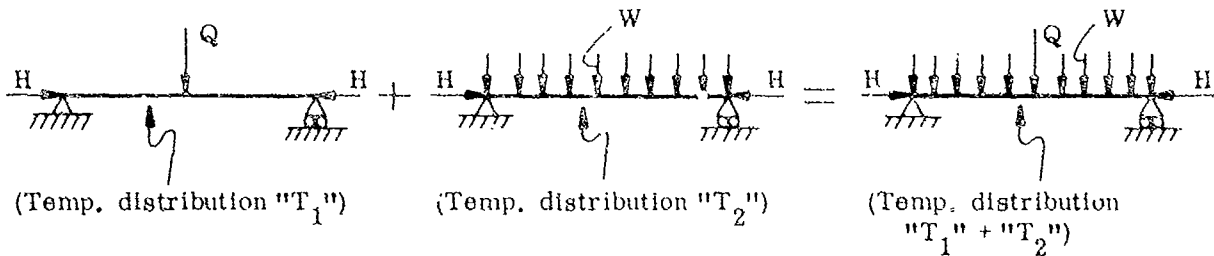
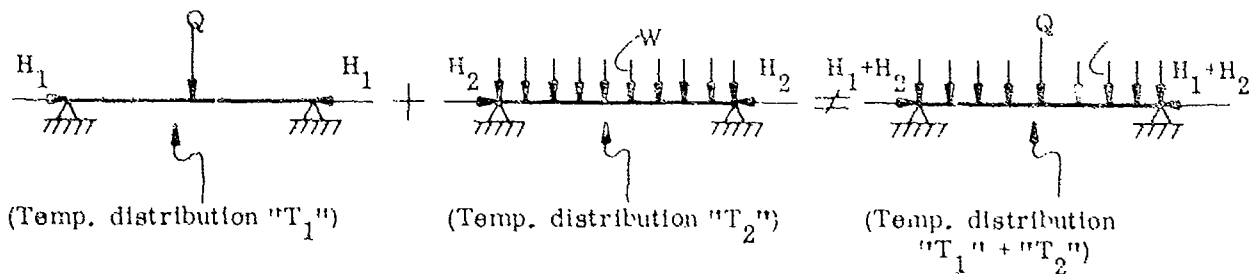


FIGURE 1.5-3. RECTANGULAR BEAM WITH ZERO AXIAL END RESTRAINT AND SPECIFIED AXIAL LOADS

It is important to note that when the end loadings are specified then a modified principle of superposition as shown by Figure 1.5-4a may be used. The figure shows that the resultant effect of several transverse loads and temperature distributions acting simultaneously



(a) Superposition valid for an axially unrestrained beam with specified end loads "H"



(b) Superposition not valid for an axially restrained beam

FIGURE 1.5-4 MODIFIED SUPERPOSITION PRINCIPLE

1.5 (Cont'd)

in the presence of a specified axial load can be obtained by superposing the effects of the individual transverse loads and temperatures acting with the axial load.

However, this superposition principle is not valid when the ends are restrained axially (e.g., Cases A and B). In such cases the axial loads depend on the transverse loads and temperatures nonlinearly.

1.6 USE OF THE TABLES

Tables 1.5-1 and 1.5-2 are reproductions of IBM 7090 digital computer numerical solutions for the beam-column problem. Table 1.5-1 presents results for the case of uniform transverse loading and Table 1.5-2 applies for concentrated midspan loads. Each table is first subdivided into sections corresponding to given temperature differences and transverse loads and then further subdivided into compressive and tensile end loading cases. Since the quantities \bar{W} , \bar{T}_d , and \bar{T} specified in a given problem will not in general coincide with those listed in the tables, interpolation must be employed. The important quantity to be evaluated is the nondimensional end loading parameter, $\bar{\lambda}$ and the spacing of the tabulated values has been made sufficiently close so as to allow reasonably accurate interpolation for engineering purposes. A systematic interpolation formula will be given as part of an illustrative problem in Sub-section 1.7.

Numerical values are given in terms of a floating decimal number system and are to be interpreted as shown by the following examples

$$0.2456582E 00 = 0.2456582 \times 10^0 = 0.2456582$$

$$0.2456582E 02 = 0.2456582 \times 10^2 = 24.56582$$

$$0.2456582E-01 = 0.2456582 \times 10^{-1} = 0.02456582$$

1.7 NUMERICAL EXAMPLES

EXAMPLE I - Beam with Full Axial End Restraint:

Figure 1.7-1 shows a simply supported strip with immovable ends subjected to a uniformly distributed load of $2 \frac{\text{lb}}{\text{in}}$. The temperature varies linearly through the thickness from 100° F at the upper face to 150° F at the lower face. Young's modulus and the linear coefficient of thermal expansion are taken to be

$$E = 30 \times 10^6 \text{ psi}$$

$$\alpha = 6 \times 10^{-6} \text{ in/in} - ^\circ \text{F.}$$

Find the axial end load, midspan deflection, bending moment and the maximum stress.

1.7 (Cont'd)

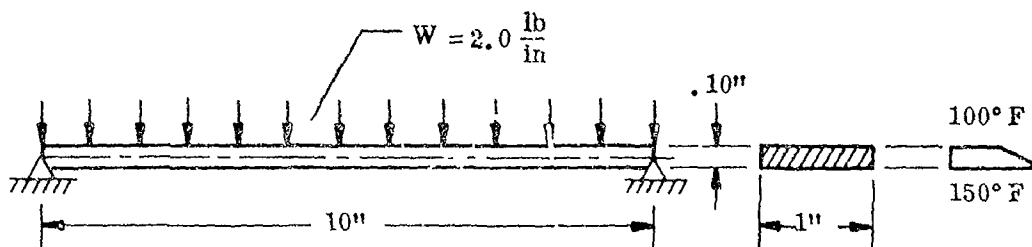


FIGURE 1.7-1 ILLUSTRATIVE PROBLEM; STRIP WITH IMMOVABLE ENDS

SOLUTION

From Figure 1.7-1 and the given data:

$b = 1''$	$T_o = 100^\circ \text{ F}$
$h = 0.10''$	$T_i = 150^\circ \text{ F}$
$L = 5''$	$E = 30 \times 10^6 \text{ psi}$
$W = 2.0 \frac{\text{lb}}{\text{in}}$	$\alpha = 6 \times 10^{-6} \text{ in/in} - ^\circ \text{ F}$

Therefore

$$\bar{W} = \frac{12W}{Eb} \left(\frac{L}{h} \right)^4 = \frac{12(2.0)}{(30)(10^6)(1)} \left(\frac{5}{0.10} \right)^4 = 5.0$$

$$\bar{T}_d = \alpha \left(\frac{L}{h} \right)^2 (T_o - T_i) = 6(10)^{-6} \left(\frac{5}{0.10} \right)^2 (100-150) = -0.75$$

$$\bar{T} = \alpha \left(\frac{L}{h} \right)^2 (T_o + T_i) = 6(10)^{-6} \left(\frac{5}{0.10} \right)^2 (100+150) = 3.75$$

Using the above nondimensional quantities we must now utilize Table 1.5-1 to determine the nondimensional axial loading parameter $\bar{\lambda}$. Since the table does not give $\bar{\lambda}$ directly for the above combination of quantities the following interpolation procedure is recommended:

- (1) Determine the next lower and higher values of \bar{W} that are given in the table. Designate these values as \bar{W}_0 and \bar{W}_1 respectively. For this example

$$\bar{W}_0 = 3.0 \text{ and}$$

$$\bar{W}_1 = 6.0 .$$

1.7 (Cont'd)

- (2) Determine the next lower and higher values of \bar{T}_d that are given in the table. Designate these values as \bar{T}_{d0} and \bar{T}_{d1} , respectively.

For this example

$$\bar{T}_{d0} = -4.0 \quad \text{and}$$

$$\bar{T}_{d1} = 0.$$

- (3) For each of the four combinations $(\bar{W}_0, \bar{T}_{d0})$, $(\bar{W}_0, \bar{T}_{d1})$, $(\bar{W}_1, \bar{T}_{d0})$, $(\bar{W}_1, \bar{T}_{d1})$ the table lists $\bar{\lambda}$'s corresponding to \bar{T} 's on either side of the given value for \bar{T} . Denote the values of \bar{T} listed in the tables for the combination $(\bar{W}_i, \bar{T}_{dj})$ by $(\bar{T}_{ij}^-, \bar{T}_{ij}^+)$ and the corresponding values of $\bar{\lambda}$ by $(\bar{\lambda}_{ij}^-, \bar{\lambda}_{ij}^+)$ where $\bar{T}_{ij}^- < \bar{T} < \bar{T}_{ij}^+$

For this example

$$\bar{T}_{00}^- = 3.397$$

$$\bar{\lambda}_{00}^- = 1.2$$

$$\bar{T}_{00}^+ = 3.896$$

$$\bar{\lambda}_{00}^+ = 1.1$$

$$\bar{T}_{01}^- = 3.041$$

$$\bar{\lambda}_{01}^- = -1.2$$

$$\bar{T}_{01}^+ = 5.174$$

$$\bar{\lambda}_{01}^+ = -1.3$$

$$\bar{T}_{10}^- = 3.416$$

$$\bar{\lambda}_{10}^- = 1.5$$

$$\bar{T}_{10}^+ = 3.969$$

$$\bar{\lambda}_{10}^+ = 1.4$$

$$\bar{T}_{11}^- = 3.648$$

$$\bar{\lambda}_{11}^- = -0.8$$

$$\bar{T}_{11}^+ = 4.441$$

$$\bar{\lambda}_{11}^+ = -0.9$$

- (4) The value of $\bar{\lambda}$ may now be obtained from the following interpolation formula.

$$\bar{\lambda} = \left[\frac{\bar{W} - \bar{W}_0}{\bar{W}_1 - \bar{W}_0} \right] A + \left[\frac{\bar{W}_1 - \bar{W}}{\bar{W}_1 - \bar{W}_0} \right] B$$

1.7 (Cont'd)

where

$$\begin{aligned}
 A &= \left[\frac{\bar{T}_d - \bar{T}_{d0}}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{11}) \bar{\lambda}'_{11} + (\bar{T}'_{11} - \bar{T}) \bar{\lambda}_{11}}{\bar{T}'_{11} - \bar{T}_{11}} \right] \\
 &+ \left[\frac{\bar{T}_{d1} - \bar{T}_d}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{10}) \bar{\lambda}'_{10} + (\bar{T}'_{10} - \bar{T}) \bar{\lambda}_{10}}{\bar{T}'_{10} - \bar{T}_{10}} \right] \\
 B &= \left[\frac{\bar{T}_d - \bar{T}_{d0}}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{01}) \bar{\lambda}'_{01} + (\bar{T}'_{01} - \bar{T}) \bar{\lambda}_{01}}{\bar{T}'_{01} - \bar{T}_{01}} \right] \\
 &+ \left[\frac{\bar{T}_{d1} - \bar{T}_d}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{00}) \bar{\lambda}'_{00} + (\bar{T}'_{00} - \bar{T}) \bar{\lambda}_{00}}{\bar{T}'_{00} - \bar{T}_{00}} \right]
 \end{aligned}$$

Substituting the known quantities into the above formula yields

$$\bar{\lambda} = -.524$$

Thus the axial end load is given by

$$H = \frac{E\bar{\lambda}^2}{L^2} = \frac{(30)(10)^6 [(1)(.10)^3 / 12] (-.524)^2}{(5)^2} = 27.4 \text{ lb. (compression)}$$

The nondimensional deflection and bending moment at midspan can be determined from Table 1.5-1 using an identical interpolation procedure. However, a simpler and more accurate method is to substitute the known values of $\bar{\lambda}$, \bar{W} , and \bar{T}_d into Eqs. (3a) of Subsection 1.5*). Direct substitution yields

$$\bar{y} = \left[\frac{y}{h} \right]_{x=0} = 1.60 \text{ and}$$

$$\bar{M} = \left[\frac{12M_1 L^2}{Ebh^4} \right]_{x=0} = 2.94,$$

so that

$$\left. \begin{aligned} y \\ \end{aligned} \right\}_{x=0} &= h \bar{y} \\ &= (.10)(1.60) = .16 \text{ in. (downward)}$$

$$\begin{aligned}
 \left. \begin{aligned} M \\ \end{aligned} \right\}_{x=0} &= \frac{Ebh^4}{12L^2} \bar{M} \\ &= \frac{30(10)^6 (.10)^4}{12(5)^2} (2.94) \\ &= 29.4 \text{ in.-lb (compression in upper fiber)}
 \end{aligned}$$

* Eqs. (3b) are not to be used here since they apply only for $\bar{\lambda} > 0$ (tension)

1.7 (Cont'd)

The maximum stress is thus

$$\sigma_{\max} = \frac{H}{bh} + \frac{M(6)}{bh^2} = 17,900 \text{ psi.}$$

The effect of axially restraining the beam is evidenced by comparing these results with those for an axially unrestrained and simply supported beam subjected to the same transverse loading and temperature but with $H = 0$. In such a case the central deflection, bending moment and maximum stress are given by (Section 4 of Reference 1-2):

$$y \Big|_{x=0} = \frac{5W(2L)^4}{384EI} + \frac{\alpha (T_i - T_o) L^2}{2h}$$

$$= .104 + .036$$

$$= .142 \text{ in}$$

$$M \Big|_{x=0} = \frac{W(2L)^2}{8} = 25 \text{ in-lb.}$$

$$\sigma_{\max} = 15,000 \text{ psi.}$$

Thus, in this case, neglecting axial end restraint which may be present yields unconservative results for both the central deflection and maximum stress.

EXAMPLE II - Unrestrained Beam Column With Prescribed Axial Load:

Figure 1.7-2 shows a simply supported strip with movable ends and a prescribed 20 lb tensile load subjected to a concentrated midspan load of 10 lbs. The temperature varies linearly through the thickness from 200° F at the upper face to 150° F at the bottom face. Young's modulus and the linear coefficient of thermal expansion are given as

$$E = 30 \times 10^6 \text{ psi}$$

$$\alpha = 6 \times 10^{-6} \text{ in./in.} - ^\circ \text{F}$$

Find the midspan deflection, bending moment and the maximum stress.

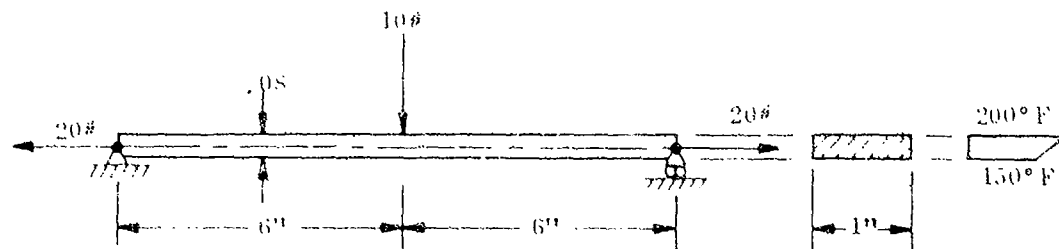


FIGURE 1.7-2 ILLUSTRATIVE PROBLEM; STRIP WITH MOVABLE ENDS

1.7 (Cont'd)

SOLUTION

From Figure 1.7-2 and the given data:

$$b = 1''$$

$$T_o = 200^\circ \text{F}$$

$$h = .08''$$

$$T_i = 150^\circ \text{F}$$

$$L = 6''$$

$$Q = 5 \text{ lb}$$

$$E = 30 \times 10^6 \text{ psi}$$

$$H = 20 \text{ lb}$$

$$\alpha = 6 \times 10^{-6} \text{ in/in} - ^\circ \text{F}$$

Therefore

$$\bar{Q} = \frac{12QL^3}{Ebh^4} = \frac{(12)(5)(6)^3}{30(10)^6(1)(.08)^4} = 10.55$$

$$\bar{T}_d = \alpha \left(\frac{L}{h} \right)^2 (T_o - T_i) = 6(10)^{-6} \left(\frac{6}{.08} \right)^2 (200 - 150) = 1.69$$

$$\bar{\lambda} = \sqrt{\frac{HL^2}{EI}} = \sqrt{\frac{20(6)^2}{(30)(10)^6(1)(.08)^3/12}} = .75$$

As discussed in Sub-section 1.5, since the ends are free to move, the bending response of the beam is independent of the average temperature. The central deflection* and bending moment can thus be obtained by direct substitution of the nondimensional parameters into Eqs. (6b) of Sub-section 1.5. This yields

$$\bar{y} = \left[\frac{y}{h} \right]_{x=0} = .674$$

$$M = \left[\frac{12ML^2}{Ebh^4} \right]_{x=0} = 2.94$$

so that

$$y \Big|_{x=0} = .054 \text{ in (downward)}$$

$$M \Big|_{x=0} = 26.5 \text{ in-lb (compression in upper fiber).}$$

* This is not always the maximum deflection (see Sub-section 1.8)

1.8 ADDITIONAL CONSIDERATIONS

(1) Application of Basic Equations to the Case of Unequal Elastic End Restraints.

The general formulas presented in Sub-section 1.3 have been developed for the case of equal elastic end restraints of stiffness $2K$ (Figure 1.3-1). As is frequently the case, these restraints may be unequal. This situation can be accommodated by replacing the quantity $2K$

in Sub-section 1.3 with the equivalent quantity $\frac{4K_1 K_2}{K_1 + K_2}$ where $2K_1$ and $2K_2$ are the unequal spring stiffnesses. Thus for example in the special case where one end is held ($K_1 = \infty$) then

$\lim_{K_1 \rightarrow \infty} \frac{4K_1 K_2}{K_1 + K_2} = 4K_2$; and this quantity is to be substituted for $2K$ in Eqs. (3) and (4) of

Sub-section 1.4. This procedure will yield the correct results for the deflection and bending moment.

(2) Maximum Bending Moments and Deflections.

This report presents results for the determination of the central deflection and bending moment for a symmetrically loaded beam-column. It can be shown by the usual maximum-minimum procedure that the central bending moment is always numerically the largest bending moment in the beam. If the temperatures and loads tend to produce curvatures in the same direction then the central deflection is a maximum. However where temperature and loads tend to relieve each other, the central deflection may not be a maximum. If this quantity is desired it may be found by considering the full deflection formula (Eq. (1) of Sub-section 1.4).

1.9 REFERENCES

- 1-1 Timoshenko, S., "Theory of Elastic Stability," McGraw Hill Book Company, Inc. pp. 6-8, 1936.
- 1-2 Switzky, H., Forray, M., and Newman, M., "Thermo-Structural Analysis Manual"-Volume I, Sections 1, 2, and 4 of Republic Aviation Corporation Report No. RAC 679-1, September 1960, revised November 1961 (to be published as WADD TR 60-517, Vol. I).

SECTION 2

APPROXIMATE SOLUTION FOR AN AXIALLY RESTRAINED COLUMN
SUBJECTED TO ELEVATED TEMPERATURES AND LATERAL LOADS

by

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SECTION 2

APPROXIMATE SOLUTION FOR AN AXIALLY RESTRAINED COLUMN SUBJECTED TO ELEVATED TEMPERATURES AND LATERAL LOADS

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SECTION 2

APPROXIMATE SOLUTION FOR AN AXIALLY RESTRAINED COLUMN SUBJECTED TO ELEVATED TEMPERATURES AND LATERAL LOADS

2.1 SUMMARY

Approximate solutions are obtained for the axial compressive load and the lateral deflections of columns which are restrained by an axial spring and subjected to an increase in temperature. The solutions are presented graphically in terms of nondimensional parameters.

The problem considered is a column of arbitrary cross section with pinned or clamped ends subjected to an arbitrary temperature and lateral load distribution in addition to finite initial eccentricities. The effects of the thermal gradients, lateral loads, non-linear axial springs, and plasticity of the material are discussed. The simpler problem of constant bending stiffness is explored to illustrate the evaluation of the nondimensional parameters.

2.1.1 Definition of Symbols

The following symbols are used throughout this section:

a	Amplitude of lateral deflection, inches
b	Axial extension parameter ($b = \alpha T / \epsilon_1$)
b	Width of rectangular cross section, inches
d	Axial shortening parameter $\left(d_i = \frac{\Delta_i}{\epsilon_1} \right)$
h	Depth of cross section of column, inches
i	Integer indicating order of the deformation mode
k	Axial stiffness of end restraint, pounds per inch
l	Length of column, inches
m _j	Coefficients expressing the lateral deformation as a polynomial, inches
q	Lateral load, pounds per inch
r	Force function. Ratio of axial load in column to a reference load ($r(x) = F(x) / F_0$)
w	Lateral deflection of column, inches
x	Axial coordinate of column, inches
z	Lateral coordinate of column cross section, inches
α	Coefficient of thermal (linear) expansion, inches per inch \cdot $^{\circ}$ F
Δ	Deflection of ends of column, inches
Δ	Incremental change
ϵ	Axial strain due to axial load or temperature, inches per inch
ϵ_i	Buckling strain corresponding to i th mode $\left(\epsilon_i = \lambda_i \sigma^2 \right)$
n	Load parameter $\left(n_i = \frac{F_i}{F_1} \right)$
κ	Curvature of column, 1/inches
λ_i	Eigenvalue for which non-trivial solutions of the differential equilibrium equation exist $\left(\lambda_i = \frac{F_i}{EI} \right)$, 1/sq. inches

2.1.1 (Cont'd)

μ_i	Eigenvalue ($\mu_i = \tan \mu_i$)
ξ	Nondimensional axial coordinate ($\xi = x/l$)
ρ	Effective radius of gyration = $\sqrt{\overline{EI}/\overline{EA}}$, inches
σ	Axial stress ($\sigma = F/A$), psi
c	Stiffness function. Ratio of bending stiffness of cross section to a reference bending stiffness ($c(x) = \frac{EI(x)}{E_o I_o}$)
ϕ_i	Additional axial shortening function ($\phi_i = \left(\frac{1}{1-\eta_i}\right)^2 - 1$)
Φ	Nondimensional linear axial shortening term ($\Phi = 1 + \frac{EA}{kL} + 2 \sum \frac{\eta_i d_i}{\eta_i}$)
A	Cross sectional area, sq. inches
C	Amplitude of curvature ($v = \sum C_i x_i$)
E_s	Secant modulus ($E_s = \sigma/\epsilon$ for a linear material), psi
\overline{EA}	Axial stiffness ($\overline{EA} = \int E_s dA$, note $\int E_s z dA = 0$), lb
\overline{EI}	Bending stiffness ($\overline{EI} = \int E_s z^2 dA$), pounds square inches
F	Axial load in column, lb
I	Moment of inertia of cross sections, inches ⁴
l	Half length of column ($l = l/2$), inches
M	Moment ($M = EI w''$), pounds per inch
P	Redundant transverse load, lb
T	Temperature increment from datum, °F

SUBSCRIPTS

o	Condition before application of axial load
i	i th mode
q	Due to lateral mechanical loads
T	Due to temperature
s. s.	Simple-simple boundary conditions
s. c.	Simple-clamped boundary conditions
c. c.	Clamped-clamped boundary conditions
s	Symmetrical mode
a	Anti-symmetrical mode

2.1.2 GENERAL APPROACH

The analysis of the problem is approached by examining the deformation characteristics of the column with a linear material to obtain a solution which satisfies compatibility and equilibrium. The following steps are employed to solve the column shown in Figure 2.1.1-1.

(1) The column deforms in characteristic modes which are dependent upon the distribution of the bending stiffness and axial load in the column as well as the boundary (end) con

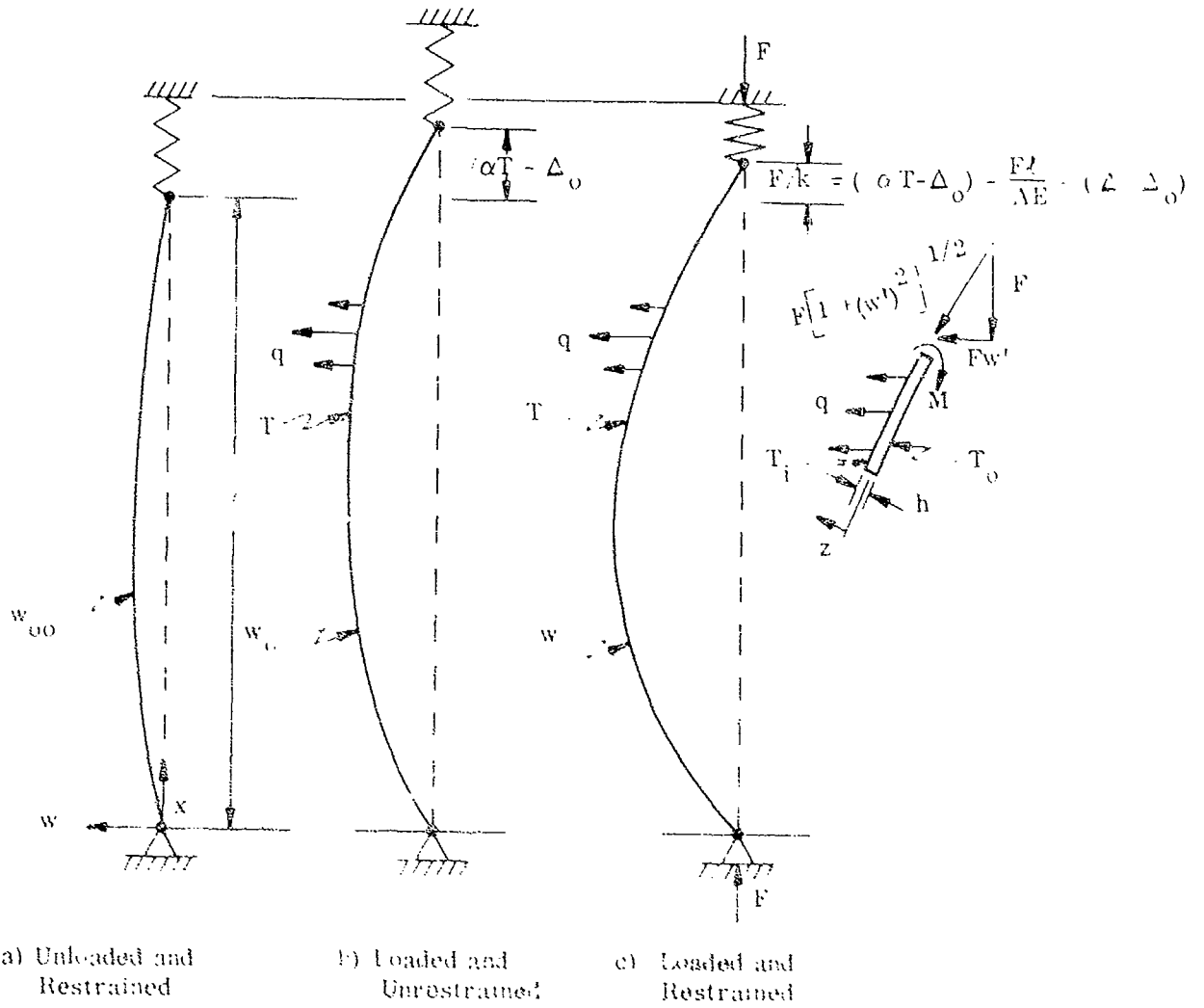


FIGURE 2.1.2-1 RESTRAINED COLUMN

2.1.2 (Cont'd)

ditions. Basic orthogonality relationships exist among the characteristic deformation modes which are used in obtaining a solution for the column in a thermal and mechanical environment.

(2) The deformations of a column subjected to lateral loadings and temperature gradients in addition to an axial load are then determined by solving the equilibrium equation. The analysis results in a solution in terms of the "initial" deformations caused by the lateral loads, thermal gradients, and manufacturing and loading eccentricities acting on a column with no axial load. These "initial" deformations grow with the magnitude of the axial load. Each characteristic deformation mode is magnified as a rectangular hyperbola, by a factor

$\left(\frac{1}{1 - r_c} \right)$ which is a function of the applied load and the characteristic "buckling" load corresponding to the deformation mode. The method of determining deformation modes and corresponding buckling loads is illustrated for columns of constant bending stiffness with pinned or clamped ends.

(3) The axial load is then expressed as the solution of a compatibility equation which considers the thermal expansion, the lateral deflection, and the axial deformation of a restraining spring. The axial load determined by this method satisfies both the equilibrium and compatibility conditions of the structure. The solution is, therefore, correct for a reversible (one to one relationship of load and deformation) structure even though a column deforms non-linearly with load.

(4) The compatibility equation is quite difficult to solve but a simple and reasonable approximation of the additional axial shortening of the column due to the axial load reduces the compatibility equation to a form which can be readily solved by graphical means.

(5) The solutions of the axial load and deformation of columns are presented for pinned (simple) or clamped ends simplified by applicable formulae and graphs together with a computational procedure and illustrative problems. The techniques to obtain the "initial" deformations due to lateral load and thermal gradients, and the effective stiffness of the restraining spring are also presented. The effects of nonlinearity in the spring or material are discussed.

2.2 ANALYSIS

The mathematical relationships necessary to solve the problem are derived in Reference 2-1. The results are summarized below.

2.2.1 Relationships of Deformation Modes

Various orthogonality relationships exist between the characteristic deflection modes and their derivatives whenever the boundary conditions are natural (e.g., free, clamped, simple, etc.). These relationships are useful in evaluating the initial and final lateral deformations as well as the axial shortening in terms of these characteristic modes.

Orthogonality relationships, (see Section 1A of Reference 2-1) are obtained by solving the homogeneous differential equilibrium equation

$$(\phi w'')'' + \lambda(rw')' = 0 \quad (1)$$

2.2.1 (Cont'd)

where

- ϕ is the ratio of bending stiffness to a reference bending stiffness ($E_0 I_0$)
- w is the lateral deflection
- r is the ratio of the axial load in the cross section to the reference load (F_0)
- λ is the eigenvalue $\frac{F_0}{E_0 I_0}$.

The following orthogonality relationships are derived

$$\int_0^l (r w_i')' w_k dx = 0 \quad i \neq k \quad (2)$$

Thus the characteristic modes, which are the solutions to the differential equilibrium equation, are orthogonal to the derivative of the weighted slope $[(r w)']$. For end loads ($r = 1$), the deflection and curvature modes are orthogonal. This relationship permits the determination of the amplitudes of the deflection modes for the lateral deflections caused by the lateral load and thermal gradients and permits the determination of the growth of these modes when the column is compressed.

$$\int_0^l r w_i' w_k dx = 0 \quad i \neq k \quad (3)$$

Similarly, the slopes of the deflection modes are orthogonal with respect to a load weighting factor (r). This relationship permits the rapid determination of the axial shortening of the column in terms of the magnitude of the deformation modes.

$$\int_0^l \phi w_i'' w_k'' dx = 0 \quad i \neq k \quad (4)$$

The orthogonality of the curvature with respect to the stiffness weighting function (ϕ), permits the solution of the non-homogeneous differential equilibrium equation resulting from the initial deformations due to the lateral load and thermal gradients. The effect of the lateral load and temperature can be expressed as initial curvatures which grow as a rectangular hyperbola with an increase in axial load as indicated in Figure 2.2.1-1.

2.2.2 Solution of Non-Homogeneous Differential Equilibrium Equation

The homogeneous differential equilibrium equation can be rewritten as a function of the curvature ($\chi = w''$)

$$\text{i. e.} \quad (\phi \chi)'' + \lambda r \chi = 0 \quad (1)$$

The solutions of this equation results in eigenvectors of the curvature which are orthogonal with respect to the weighting function ϕ (Eq. (4) of Paragraph 2.2.1). It is assumed that these curvature modes form a closed set for the "natural" boundary conditions (columns of constant EI ($r = 1$) result in Fourier expansions). Thus, any curvature can be expressed as a weighted sum of the eigenvectors.

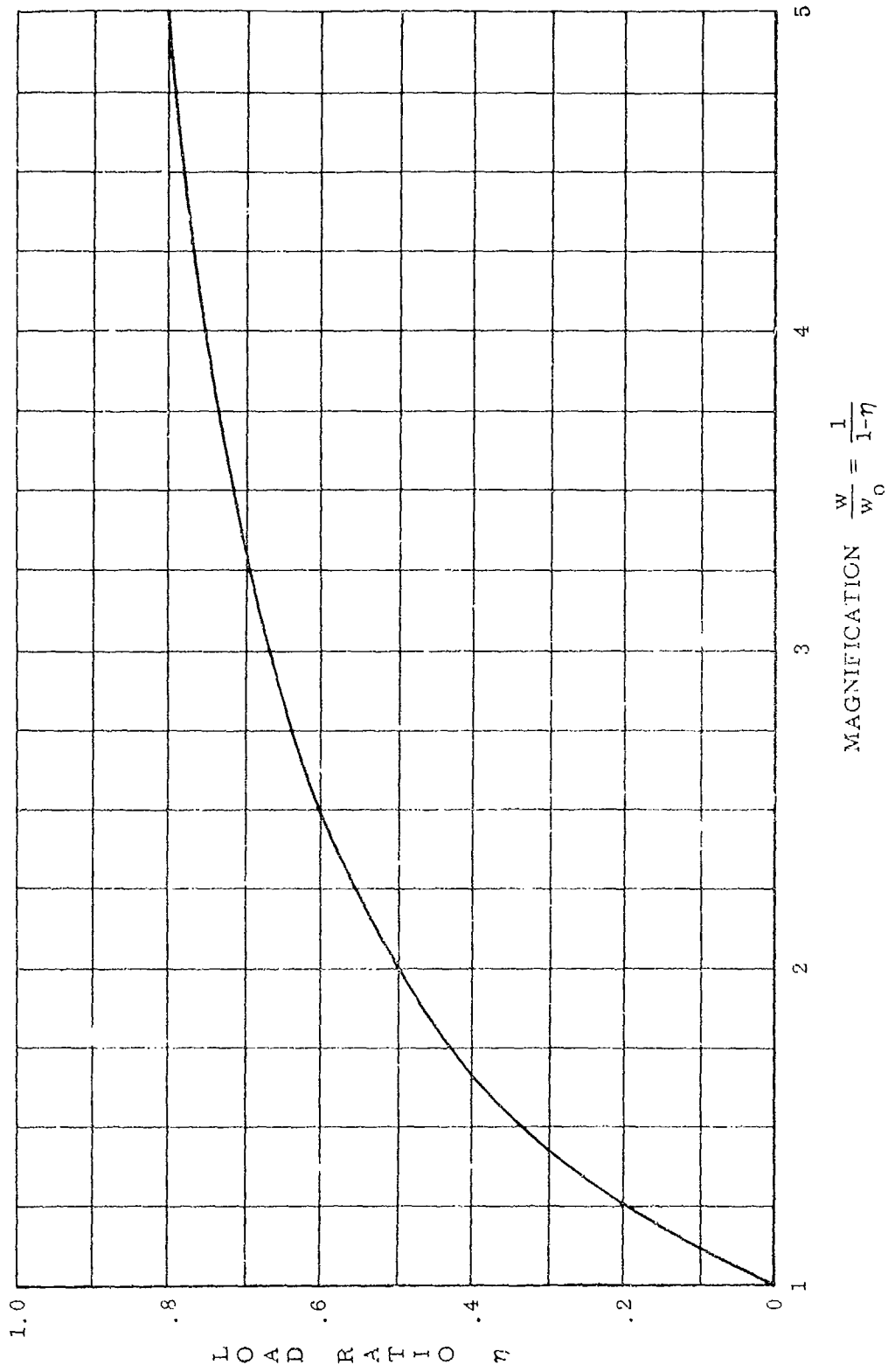


FIGURE 2.2.1-1 MAGNIFICATION OF DEFORMA ION MODES

2.2.2 (Cont'd)

$$\text{i. e.} \quad \kappa = \sum C_i \kappa_i \quad (2a)$$

$$\text{where} \quad (\phi \kappa_i)'' + \lambda_i r \kappa_i = 0 \quad (2b)$$

If the unrestrained column is not straight because of initial eccentricities, thermal gradients, and lateral loads, then the moment acting on the column is proportional to the change in curvature and the equilibrium equation becomes, for an end loaded column ($r = 1$),

$$(\phi \kappa)'' + \lambda \kappa = \left[\phi (\kappa_L + \kappa_T) \right]'' \quad (3)$$

where κ_L is the curvature caused by the lateral load and κ_T is the curvature caused by the thermal gradient.

Expressing the solution as a weighted sum of the characteristic curvatures (Eq. (2a)) which satisfy the homogeneous equation and manipulating Eq. (3) results in the solution for the curvature of an axially loaded column in terms of the initial curvature and the load ratio.

$$\kappa = \sum \frac{b_i}{1 - \eta_i} \kappa_i \quad (4)$$

where $b_i \kappa_i$ corresponds to the i^{th} component of the initial curvature of the column due to the lateral load and temperature. Thus the column deforms under axial load by having each component of the initial curvature increase by a magnification factor $1/(1-\eta_i)$. Correspondingly the slopes and lateral deflection modes increase by this same factor.

Since the initial deformations are expressible as an infinite series of the characteristic curvatures, it becomes expeditious to employ the initial curvatures and to integrate this infinite series to obtain the expression for the slope which is employed in obtaining the axial shortening. Employing the initial slope or lateral deflection will result in a slower convergence of the series expressing the axial shortening and would reduce the accuracy of the approximate procedure employed to solve the compatibility equation.

2.2.3 Determination of Deformation Modes and Buckling Loads

The deflection modes (w) of the column are obtained by solving the homogeneous equilibrium Eq. (1) of Paragraph 2.2.1. The buckling loads (F_i) are those values of the axial load which result in non-trivial solutions of the equation. The technique of obtaining these items is illustrated for a column of constant bending stiffness ($\phi = 1$).

The homogeneous equilibrium equation for constant bending stiffness (EI) is

$$\frac{d^4 w}{dx^4} + \frac{F}{EI} \frac{d^2 w}{dx^2} = w^{IV} + \lambda w'' = 0 \quad (1)$$

The general solution is

$$w = c_1 + c_2 x + c_3 \cos \sqrt{\lambda} x + c_4 \sin \sqrt{\lambda} x \quad (2)$$

where c_1, c_2, c_3, c_4 are constants to be determined by the boundary conditions.

2.2.3 (Cont'd)

Solutions for various boundary conditions can be found in various texts. The simple-simple, clamped-clamped, and simple-clamped are summarized below. In addition the method of determining the magnitude of the initial curvature modes is indicated.

(1) Simple-Simple Column

The boundary conditions are

$$w(0) = w(l) = 0$$

$$w''(0) = w''(l) = 0$$

$$\therefore w_i = a_i \sin \frac{i \pi x}{l} = a_i \sin \sqrt{\lambda_i} x \quad (3a)$$

$$\text{where } w = \sum w_i$$

$$\frac{F_i}{EI} = \lambda_i = \left(\frac{i \pi}{l} \right)^2 \quad (4a)$$

$$x_i = \sin \frac{i \pi x}{l} \quad (5a)$$

$$\int_0^l x_i^2 dx = l/2$$

$$C_i = \frac{\int_0^l w'' x_i dx}{\int_0^l x_i^2 dx} = \frac{2}{l} \int_0^l w'' \sin \frac{i \pi x}{l} dx \quad (6a)$$

$$\text{where } w'' = C_i x_i$$

(2) Clamped-Clamped Column

The boundary conditions are

$$w(0) = w(l) = 0$$

$$w'(0) = w'(l) = 0$$

This results in two different types of modes. Modes which are symmetrical about the mid-length of the column and modes which are anti-symmetrical.

For symmetrical modes

$$w_{is} = a_i \left(1 - \cos \frac{2 i \pi x}{l} \right) \quad (3b)$$

2.2.3 (Cont'd)

$$\frac{F_i}{EI} = \lambda_i = \frac{4 i^2 \pi^2}{l^2} \quad (4b)$$

$$x_{is} = \cos \frac{2 i \pi x}{l} \quad (5b)$$

$$C_{is} = \frac{\int_0^l w'' x_i dx}{\int_0^l x_i^2 dx} = \frac{2}{l} \int_0^l w'' \cos \frac{2 i \pi x}{l} dx \quad (6b)$$

For anti-symmetrical modes

$$w_i = a_i \left(\frac{2\mu_i \frac{x}{l} - \sin 2\mu_i \frac{x}{l}}{2\mu_i - \sin 2\mu_i} - \frac{1 - \cos 2\mu_i \frac{x}{l}}{1 - \cos 2\mu_i} \right)$$

$$x_i = \frac{\sin 2\mu_i \frac{x}{l}}{2\mu_i - \sin 2\mu_i} - \frac{\cos 2\mu_i \frac{x}{l}}{1 - \cos 2\mu_i} \quad \text{where } \mu_i = \tan \mu_i$$

An alternate form, employing the mid-length of the column as an origin, results in a simpler result which is identical to the joining of two simple-clamped columns.

$$w_{ia} = a_i \left(\frac{\sin \frac{\mu_i}{2} \frac{x}{l}}{\sin \frac{\mu_i}{2}} - \frac{x}{2l} \right) \quad (3c)$$

$$\frac{F_{ia}}{EI} = \lambda_i = \left(\frac{\mu_i}{l/2} \right)^2 = \frac{4\mu_i^2}{l^2} \quad (4c)$$

where $\tan \mu_i = \mu_i$

$$\text{and } \mu_1 = 1.43 \quad \mu_n = \left(\frac{2n+1}{2} \right) \pi$$

$$x_{ia} = \sin \frac{\mu_i}{2} \frac{x}{l} \quad (5c)$$

$$C_{ia} = \frac{\int_0^{l/2} w'' x_i dx}{\int_0^{l/2} x_i^2 dx} = \frac{l}{l \left(\frac{1}{4} - \frac{\cos \mu_i}{\mu_i} \right)} \int_0^{l/2} w'' \sin \frac{\mu_i}{2} \frac{x}{l} dx \quad (6c)$$

2.2.3 (Cont'd)

(3) Simple Clamped Column

The boundary conditions are

$$w(0) = w''(0) = 0$$

$$w(l) = w'(l) = 0$$

$$w_i = a_i \left(\frac{\sin \mu_i \frac{x}{l}}{\sin \mu_i} - \frac{x}{l} \right) \quad (3d)$$

where $\tan \mu_i = \mu_i$

$$\frac{F_i}{EI} = \lambda_i = \frac{\mu_i^2}{l^2} \quad (4d)$$

$$\kappa_i = \sin \mu_i \frac{x}{l} \quad (5d)$$

$$C_{ia} = \frac{\int_0^l w'' \kappa_i dx}{\int_0^l \kappa_i^2 dx} = \frac{1}{l \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)} \int_0^l w'' \sin \mu_i \frac{x}{l} dx \quad (6d)$$

The deformation modes and buckling loads are required in the determination of the end load and deflection of a restrained column as indicated in Subsection 2.3. If the boundary conditions or stiffness distribution do not conform to the case summarized above, then the buckling loads and modes must be determined for the column considered. Special cases may be found in various text and recourse must be taken to approximate solutions when the column becomes more involved.

The method of solution for a column with a linear material, proposed in this report, is quite general and can handle the more complex configurations provided the initial deflection can be described by characteristic eigenvectors with known eigenvalues.

2.2.4 Compatibility Equation

The axially unrestrained column (Figure 2.1-1b) will deform when subjected to lateral loads and temperature. This will cause an axial motion of the ends of the column of Δ_{00} .

$$\Delta_{00} = l\alpha T - \Delta_0 \quad (1a)$$

where $\alpha T = \epsilon_T$ is the unrestrained axial strain due to temperature. If the strains are not uniform then

$$\alpha T = \frac{1}{l} \int_0^l \frac{\int \alpha T dA}{\int dA} dx \quad (1b)$$

2.2.4 (Cont'd)

Δ_o = axial shortening of the column due to the lateral load and temperature gradients through the cross sections

$$\Delta_o = \frac{1}{2} \int_0^l (w_T' + w_q')^2 dx \quad (1c)$$

The application of the axial load causes the column to shorten due to the axial strain $\frac{F\ell}{AE}$ and additional axial shortening $(\Delta - \Delta_o)$. The final movement of the ends must be equal to the deformation of the axial spring F/k .

$$\therefore \alpha T - \Delta_o - \frac{F\ell}{AE} - (\Delta - \Delta_o) = \frac{F}{k} \quad (2a)$$

Dividing by the quantity $\ell\epsilon_1$ results in

$$\frac{\alpha T - \frac{\Delta_o}{\ell}}{\epsilon_1} - \frac{F}{AE \epsilon_1} - \frac{\frac{\Delta}{\ell} - \frac{\Delta_o}{\ell}}{\epsilon_1} - \frac{F}{k\ell \epsilon_1} = 0 \quad (2b)$$

$$\text{where } \epsilon_1 = \frac{F\ell}{AE} = \frac{\lambda_1 EI}{AE} = \lambda_1 \rho^2 \quad (2c)$$

Noting that

$$\frac{F}{AE \epsilon_1} = \frac{\sigma A}{AE \epsilon_1} = \frac{E \epsilon A}{AE \epsilon_1} = \frac{\epsilon}{\epsilon_1} = \eta_1 \quad (3a)$$

and

$$\frac{F}{k\ell \epsilon_1} = \frac{E \epsilon A}{k\ell \epsilon_1} = \frac{EA}{k\ell} \eta_1 \quad (3b)$$

and letting

$$\frac{\alpha T - \Delta_o / \ell}{\epsilon_1} = b \quad (4)$$

and evaluating the axial shortening

$$\Delta_o = \frac{1}{2} \int_0^l (w_T' + w_q')^2 dx = \frac{1}{2} \int_0^l (w_o')^2 dx = \frac{1}{2} \int_0^l (\Sigma w_{1o}')^2 dx \quad (5a)$$

but because of the orthogonality of the characteristic slopes (Eq. (3) of Paragraph 2.2.1) we obtain

$$\Delta_o = \frac{1}{2} \int_0^l \Sigma (w_{1o}')^2 dx = \frac{1}{2} \Sigma \int_0^l (w_{1o}')^2 dx \quad (5b)$$

2.2.4 (Cont'd)

and

$$\begin{aligned} \Delta - \Delta_0 &= \frac{1}{2} \int_0^l [(w')^2 - (w'_0)^2] dx = \frac{1}{2} \int_0^l \Sigma (w'_{i0})^2 \left[\left(\frac{1}{1 - \eta_1} \right)^2 - 1 \right] dx \\ &= \frac{1}{2} \Sigma \left[\left(\frac{1}{1 - \eta_1} \right)^2 - 1 \right] \int_0^l (w'_{i0})^2 dx \end{aligned} \quad (5c)$$

letting

$$\frac{\Delta_i}{l \epsilon_1} = \frac{\frac{1}{2} \int_0^l (w'_{i0})^2 dx}{l \epsilon_1} = d_i \quad (6a)$$

and

$$\left(\frac{1}{1 - \eta_1} \right)^2 - 1 = c_i \quad (6b)$$

we obtain

$$\frac{\Delta_0}{l \epsilon_1} = \Sigma d_i \quad (6c)$$

and

$$\frac{\Delta - \Delta_0}{l \epsilon_1} = \Sigma d_i c_i \quad (6d)$$

Substituting Eqs. (3a), (3b), (4), and (6d) into Eq. (2b) results in

$$b - \eta_1 - \Sigma d_i c_i \frac{EA}{kL} \eta_1 = 0 \quad (7)$$

The above equation is quite difficult to solve, but the solution can be simplified if it is noted that the expression $d_i c_i$ denotes the increase in axial deformation due to the growth of the i^{th} mode of deformation. The first mode is magnified to a much greater degree than the higher modes when the axial load is compression. In fact the first mode becomes predominant as the axial load approaches the first buckling load. The approximate method utilizes this fact in approximating the value of $\Sigma d_i c_i$ by expanding $d_i c_i$ (for $i = 1$) as a power series and only employing the most significant term and by approximating the infinite series by a finite series. This approximation method is not recommended for tension loads, since the approximation for c_i and the infinite series may be incorrect.

2.2.4 (Cont'd)

Noting that

$$\varphi_i = \left(\frac{1}{1 - \eta_i} \right)^2 - 1 = 1 + 2\eta_i + 3\eta_i^2 + \dots - 1 \sim 2\eta_i \quad (8a)$$

and that $1 > \eta_1 > \eta_i \quad (i \geq 2)$

$$\therefore \sum_{i=1}^{\infty} d_i \varphi_i = d_1 \varphi_1 + \sum_{i=2}^{\infty} d_i \varphi_i \text{ is approximately } d_1 \varphi_1 + 2 \sum_{i=2}^n \eta_i d_i \quad (8b)$$

where n is a sufficiently large integer.

Equation (7) can thus be approximated by

$$b - d_1 \varphi_1 - \left(1 + \frac{EA}{k} + 2 \sum_{i=2}^n \frac{\eta_i}{\eta_1} d_i \right) \eta_1 = 0 \quad (9a)$$

Letting

$$\Phi = 1 + \frac{EA}{k} + 2 \sum_{i=2}^n \frac{\eta_i}{\eta_1} d_i \quad (9b)$$

we obtain

$$\frac{b}{\Phi} - \frac{d_1}{\Phi} \varphi_1 - \eta_1 = 0 \quad (9c)$$

or

$$\bar{b} - \bar{d}_1 \varphi_1 - \eta_1 = 0 \quad (9d)$$

where

$$\bar{b} = \frac{b}{\Phi} \quad (9e)$$

and

$$\bar{d}_1 = \frac{d_1}{\Phi} \quad (9f)$$

The value of the axial strain parameter (η_1) can be determined from a graphical plot (Figure 2.2.4-1 and -2) of the variation of \bar{b} with η_1 for various \bar{d}_1 . This value of η_1 can then

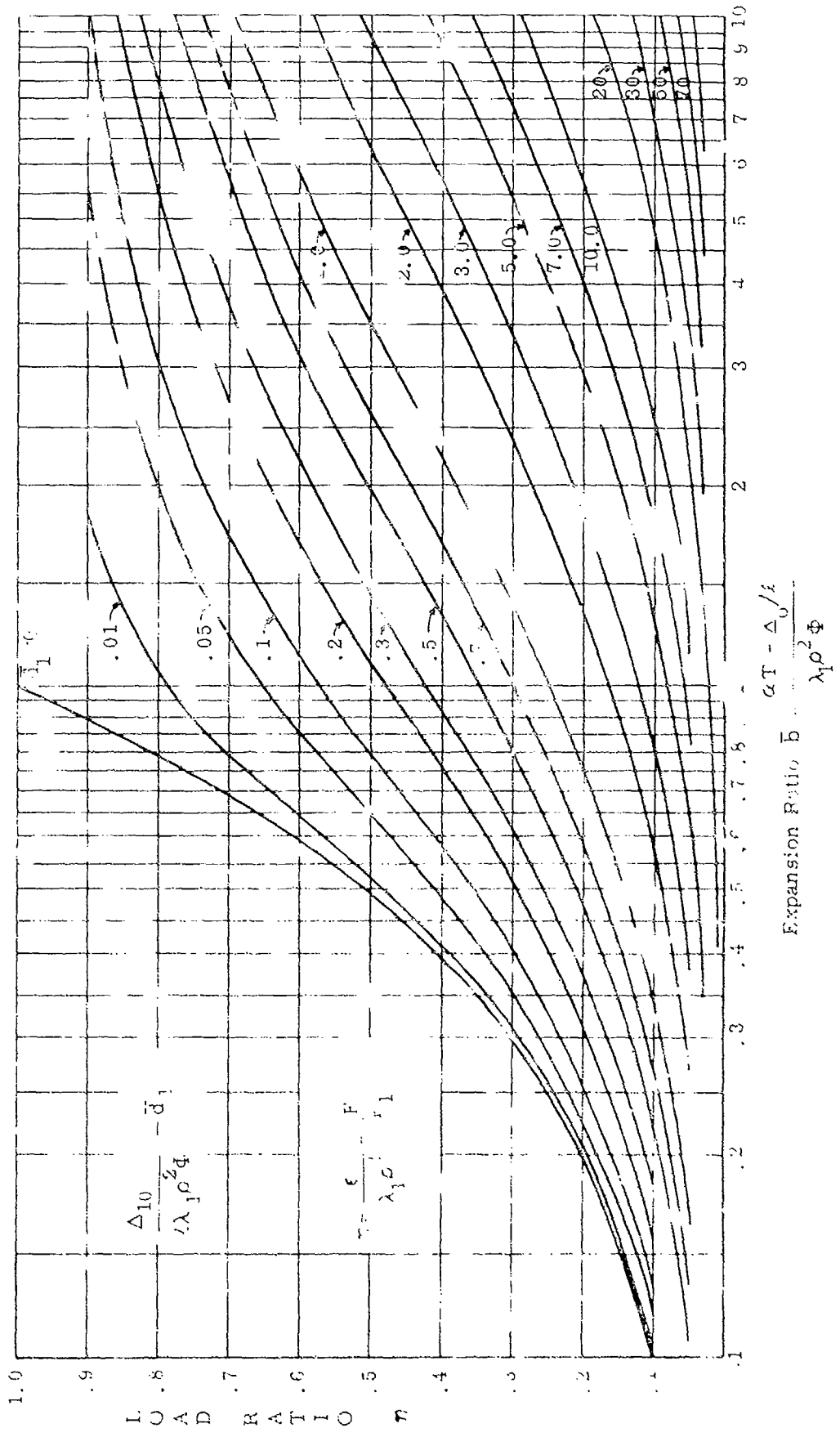


FIGURE 2.2.4-1 AXIAL LOAD IN A RESTRAINED COLUMN

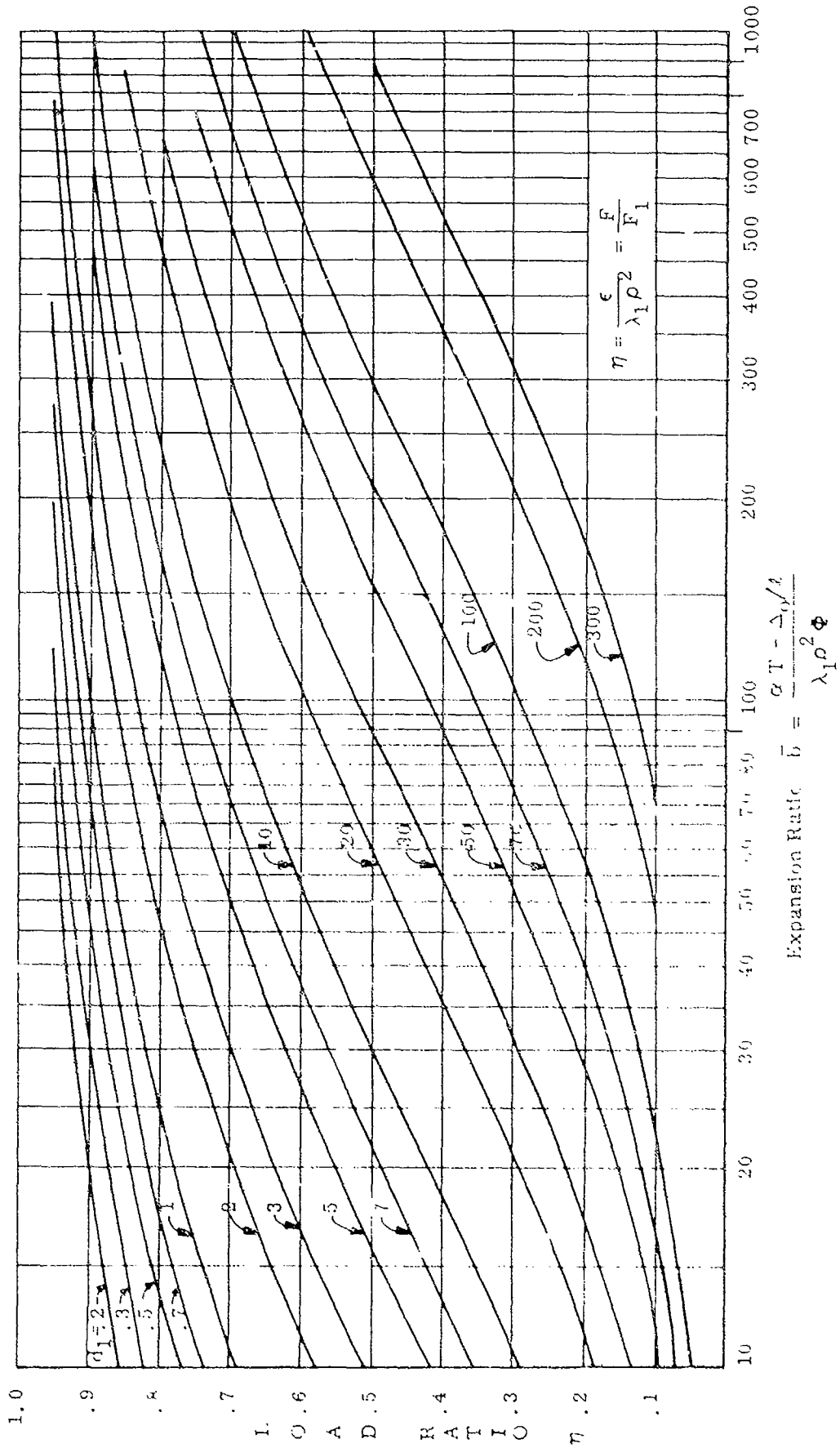


FIGURE 2.2.4-2 AXIAL LOAD IN A RESTRAINED COLUMN

2.2.4 (Cont'd)

be employed to obtain the axial load acting on the column and the deformations of the column. It should be noted that the approximate method results in an upper bound on the axial load since it ignores a small part of the change of axial motion due to the higher modes.

2.3 PROCEDURE

The information and technique needed to solve the problem are enumerated in the following sections.

2.3.1 Determination of Nondimensional Parameters

By the definition Eq. (6a) of Paragraph 2.2.4:

$$d_1 = \frac{\frac{1}{2} \int_0^l (w'_{10})^2 dx}{l \epsilon_1} = \frac{\Delta_1}{l \lambda_1 \rho^2}$$

Equation (9.1.2 - 3c) of Reference 2-2 indicates that

$$\lambda_1 = \frac{F_1}{EI} = \frac{\int_0^l \phi (w''_1)^2 dx}{\int_0^l (w'_1)^2 dx} = \frac{\int_0^l \phi C_1^2 x_1^2 dx}{2 \Delta_1} \quad (1a)$$

$$\therefore \Delta_1 = \frac{C_1^2 \int_0^l \phi x_1^2 dx}{2 \lambda_1} \quad (1b)$$

and

$$d_1 = \frac{C_1^2 \int_0^l \phi x_1^2 dx}{2 l \lambda_1 \lambda_1 \rho^2} \quad (2a)$$

Noting that

$$\frac{\eta_1}{\eta_1} = \frac{\epsilon / \epsilon_1}{\epsilon / \epsilon_1} = \frac{\epsilon_1}{\epsilon_1} = \frac{\lambda_1 \rho^2}{\lambda_1 \rho^2} = \frac{\lambda_1}{\lambda_1} \quad (3)$$

$$\therefore \frac{\eta_1}{\eta_1} d_1 = \frac{C_1^2 \int_0^l \phi x_1^2 dx}{(\lambda_1^2 \rho^2)} \quad (4a)$$

2.3.1 (Cont'd)

d_i and $\frac{\eta_1 d_i}{\eta_1}$ will be evaluated for the chosen boundary conditions and for constant EI ($\phi = 1$) utilizing Eqs. (5) and (6) of Paragraph 2.2.3.

(1) Simple-Simple Column

$$d_i = \frac{C_i^2 \ell/2}{2\ell \left(\frac{\pi}{\ell}\right)^2 \left(\frac{i\pi}{\ell}\right)^2 \rho^2} = \frac{\ell^4}{4\pi^4 \rho^2} \left(\frac{C_i^2}{i^2}\right) \quad (2b)$$

$$2 \frac{d_i \eta_1}{\eta_1} = \frac{C_i^2 \ell/2}{\ell \left(\frac{i\pi}{\ell}\right)^4 \rho^2} = \frac{\ell^4}{2\pi^4 \rho^2} \left(\frac{C_i^2}{i^4}\right) \quad (4b)$$

(c) Clamped-Clamped Column

Symmetrical

$$d_{1s} = \frac{C_{1s}^2 \ell/2}{2\ell \rho^2 \left(\frac{2\pi}{\ell}\right)^2 \left(\frac{2i\pi}{\ell}\right)^2} = \frac{\ell^4}{64\pi^4 \rho^2} \left(\frac{C_{1s}^2}{i^2}\right) \quad (2c)$$

$$2 \frac{d_{1s} \eta_{1s}}{\eta_1} = \frac{C_{1s}^2 \ell/2}{\ell \rho^2 \left(\frac{2i\pi}{\ell}\right)^4} = \frac{\ell^4}{32\pi^4 \rho^2} \left(\frac{C_{1s}^2}{i^4}\right) \quad (4c)$$

Antisymmetrical

$$d_{ia} = \frac{C_{ia}^2 \ell \left(\frac{1}{2} - \frac{2 \cos \mu_1}{\mu_1}\right)}{2\ell \rho^2 \left(\frac{2\pi}{\ell}\right)^2 \left(\frac{2\mu_1}{\ell}\right)^2} = \frac{\ell^4 C_{ia}^2 \left(\frac{1}{2} - \frac{2 \cos \mu_1}{\mu_1}\right)}{32\pi^2 \rho^2 \mu_1^2} \quad (2d)$$

$$2 \frac{d_{ia} \eta_{ia}}{\eta_1} = \frac{C_{ia}^2 \ell \left(\frac{1}{2} - \frac{2 \cos \mu_1}{\mu_1}\right)}{\ell \rho^2 \left(\frac{2\mu_1}{\ell}\right)^4} = \frac{\ell^4 C_{ia}^2 \left(\frac{1}{2} - \frac{2 \cos \mu_1}{\mu_1}\right)}{16 \rho^2 \mu_1^4} \quad (4d)$$

2.3.1 (Cont'd)

(3) Clamped-Simple Column

$$d_i = \frac{C_i^2 l \left(\frac{1}{2} - \frac{\cos \mu_i}{\mu_i} \right)}{2l\rho^2 \left(\frac{1.43\pi}{l} \right)^2 \left(\frac{\mu_i}{l} \right)^2} = \frac{l^4 C_i^2 \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)}{4.1 \pi^2 \rho^2 \mu_i^2} \quad (2e)$$

$$2 \frac{d_i \eta_i}{\eta_i} = \frac{C_i^2 l \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)}{l\rho^2 \left(\frac{\mu_i}{l} \right)^2} = \frac{l^4}{\rho^2} \frac{C_i \left(\frac{1}{2} - \frac{\cos \mu_i}{\mu_i} \right)}{\mu_i^4} \quad (4e)$$

(4) Summary

Equation (9a) of Paragraph 2.2.4 can then be summarized for columns of constant EI as follows:

Simple-Simple

$$\frac{\alpha T - \Delta_o/l}{\frac{\pi^2}{l^2} \rho^2} - \left(\frac{l^4 C_1^2}{4\pi^4 \rho^2} \right) \phi_1 - \left(1 + \frac{EA}{kl} + \frac{l^4}{2\pi^4 \rho^2} \sum_{i=2}^n \frac{C_i^2}{i^2} \right) \eta_1 = 0 \quad (5a)$$

Clamped-Clamped Column

$$\frac{\alpha T - \Delta_o/l}{\frac{4\pi^2}{l^2} \rho^2} - \left(\frac{l^4 C_1^2}{64\pi^4 \rho^2} \right) \phi_1 \quad (5b)$$

$$- \left[1 + \frac{EA}{kl} + \frac{l^4}{32\pi^4 \rho^2} \sum_{i=1}^n \frac{C_{18}^2}{i^4} + \frac{l^4}{16\rho^2} \sum_{i=1}^n \frac{C_i^2 \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)}{\mu_i^4} \right] = 0$$

Simple-Clamped Column

$$\frac{\alpha T - \Delta_o/l}{\frac{2.05\pi^2}{l^2} \rho^2} - \left(\frac{.07081 l^4 C_1^2}{\pi^4 \rho^2} \right) \phi_1 - \left[1 + \frac{EA}{kl} + \frac{l^4}{\rho^2} \sum_{i=2}^n \frac{C_i^2 \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)}{\mu_i^4} \right] = 0 \quad (5c)$$

2.3.2 Computational Procedure

The following computational procedure is recommended to obtain the approximate solution for the axial load and lateral deflection of a restrained column.

- (1) Determine the axial expansion of the column due to temperature and the lateral load, assuming, at this step, that the restraint against actual expansion is removed.

$$l \alpha T - \Delta_0 = \int_0^l \epsilon_T dx - \int_0^l (w_0')^2 dx \quad (1)$$

where

$$w_0 = w_q + w_T$$

w_q = lateral deflection due to lateral loads (usually obtained from reference texts)

w_T = lateral deflection due to thermal gradients (method of determining w_T indicated in Section 4 of Reference 2-2).

ϵ_T = axial expansion due to temperature (method of determining ϵ_T indicated in Section 4 of Reference 2-2).

- (2) Calculate the critical axial strain from Eq. (2c) of Paragraph 2.2.4)

$$\epsilon_1 = \lambda_1 \frac{I}{A} = \lambda_1 \rho^2$$

$$\text{(e.g. } \epsilon_1 = \frac{\pi^2}{2} \rho^2 \text{ for simple-simple, } \phi = 1)$$

- (3) Expand the initial curvature ($w_0'' = \sum C_i \chi_i$), due to the lateral load and temperature, in terms of the characteristic curvatures of the column. A sufficient number of terms of the series should be taken to ensure accuracy. The coefficients of the characteristic curvatures are obtained by the general equation

$$C_i = \frac{\int_0^l w_0'' \chi_i dx}{\int_0^l c \chi_i^2 dx} \quad (2)$$

Formulas for constant EI ($c = 1$) are presented in Eqs. (6a), (6b), (6c), and (6d) of Paragraph 2.2.3. Tables to evaluate the integrals of Eqs. (6a) and (6b) of Paragraph 2.2.3 are presented in Tables 2.3.3.1, 2-1 and -2.

- (4) Evaluate the pertinent parameters

$$(a) \quad d_1 = \frac{\Delta_{10}}{\epsilon_1} = \frac{C_i^2 \int c \chi_i^2 dx}{2 \rho^2 \lambda_1 \lambda_1'} \quad \text{(Reference Eq. (2a) of Paragraph 2.3.1)}$$

2.3.2 (Cont'd)

Appropriate formulas for d_i are presented for columns of constant EI and for the chosen boundary conditions in Paragraph 2.3.1.

(e.g. $d_i = \frac{l^4}{4\pi^4 \rho^2} \frac{C_i^2}{i^2}$ for simple-simple column).

(b) $\Phi = 1 + \frac{EA}{kl} + 2 \sum_{i=2}^n \frac{\eta_i}{\eta_1} d_i$ (Reference Eq. (9b) of Paragraph 2.2.4)

Values of Φ are indicated in Eq. (5) of Paragraph 2.3.1 for constant EI.

(e.g. $\Phi = 1 + \frac{EA}{kl} + \frac{l^4}{2\pi^4 \rho^2} \sum_{i=2}^n \frac{C_i^2}{i^4}$ for simple-simple column)

(c) $\bar{d}_1 = \frac{d_1}{\Phi}$ (Reference Eq. (9f) of Paragraph 2.2.4)

(d) $\bar{b} = \frac{\alpha T - \Delta_{o/l}}{\epsilon_1 \Phi} = \frac{\alpha T - \Delta_{o/l}}{\lambda_1 \rho^2 \Phi}$ (Reference Eq. (9e) of Paragraph 2.2.4)

(5) Enter Figure 2.2.4-1 or -2 with the appropriate values of \bar{b} and \bar{d}_1 and determine η_1 .

(6) The axial load is then determined from Eq. (3a) of Paragraph 2.2.4.

$$F = \sigma A = E \frac{\epsilon}{\epsilon_1} \epsilon_1 A = EA \lambda_1 \rho^2 \eta_1 \quad (3)$$

(7) The lateral deflection of the column could likewise be determined as

$$w = \sum \frac{w_{oi}}{1 - \eta_i} \quad (4a)$$

where the value of $\eta_i = r_1^2 \frac{\lambda_1}{\lambda_i}$ (4b)

and the value of $\frac{1}{1 - \eta_1}$ can be calculated or determined with the aid of Figure 2.2.1-1.

(e.g. $w = \frac{2}{\pi^2} \sum \left(\frac{l^2 C_i}{2} \right) \frac{\sin (i\pi x/l)}{i^2 - \eta_1}$ for pinned end column).

2.3.2 (Cont'd)

(8) A slightly better approximation to the value of η_1 can be obtained by employing

$$\bar{b} = \frac{\frac{\alpha T - \Delta_o/t}{\lambda_1 \rho^2} - \left(\frac{\Delta_o}{\lambda_1 \rho^2} - \sum_{i=1}^n d_i \right)}{\Phi} \quad (5)$$

which corrects for the error in approximating the initial shortening by a finite series. The additional shortening of the higher modes is still approximated by the first term of the series in order to simplify the solution of the compatibility equation.

(9) The graphs can also be employed to determine the approximate average temperature rise (with no cross-sectional gradient) required to produce a maximum permissible lateral deflection. The permissible ratio of maximum to initial lateral deflection $\left(\frac{w_{\max}}{w_o} \right)$ can be employed with Figure 2.2.1-1 to estimate a value of η_1 which can be utilized with \bar{d}_1 in Figures 2.2.4 to determine a value of \bar{b} which could then be employed to calculate αT . (If the higher modes of the initial deflection are significant, then they must be included as indicated in Eq. (4a)). The initial eccentricity will permit the calculation of Φ and \bar{d}_1 . Assuming a value of η_1 or of αT will permit the determination of the other from Figure 2.2.4. The value of η_1 could then be used to determine w which could be plotted against αT ; αT could also be plotted against the axial load by employing Eq. (3a) of Paragraph 2.2.4. A typical plot is shown in Figure 2.3.2-1.

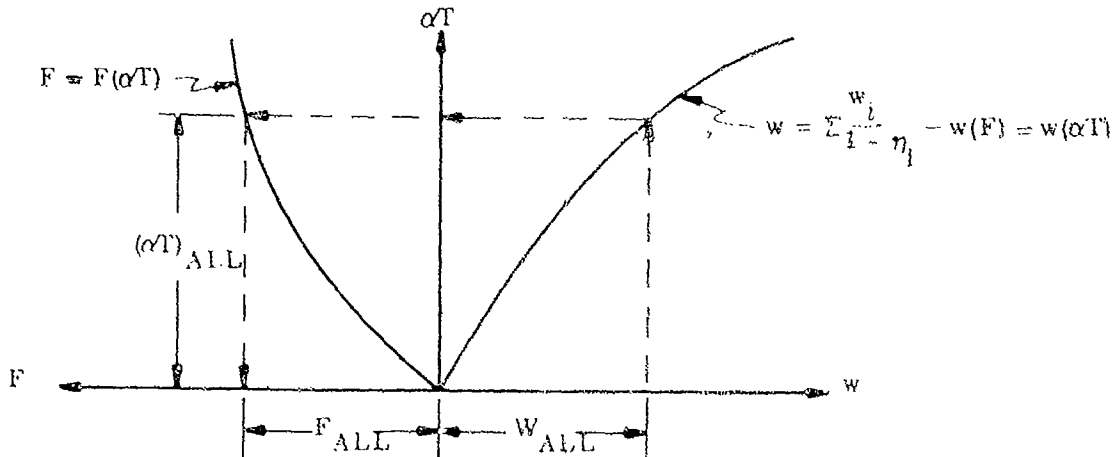


FIGURE 2.3.2-1 AXIAL LOAD AND LATERAL DEFLECTION VS AVERAGE TEMPERATURE RISE

2.3.3 Initial Deformations

The initial deformations before the end load is applied must be determined to a fair degree of accuracy. Two methods are presented herein. The first procedure assumes that an analytical expression for the initial lateral deflections is available as a polynomial, while the second method assumes that the lateral deflection is known only at a finite number of points (e.g., determined experimentally or by computations).

2.3.3.1. Polynomial Representation

The lateral deformation of a beam can be determined by integrating the equilibrium equation (Eq. (3) of Paragraph 2.2.2) with $\lambda = 0$ and employing the given boundary conditions. Solutions for lateral loading are available in literature for the type of boundary conditions con-

sidered (e.g., $w_q = \frac{q l^4}{24 EI} \left[\left(\frac{x}{l}\right) - 2 \left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4 \right]$ for a simple-simple beam with uni-

form lateral load). The slopes and curvatures can be obtained by differentiating these polynomial expressions. The lateral deflection due to thermal gradients is not as readily available but can be derived quite simply by utilizing the equations of Section 4 of Reference 2-2. The technique is illustrated for the simple-simple and clamped-clamped beams.

2.3.3.1.1 Lateral Deformation of Beams by Temperature Gradients

Let the thermal gradient through the depth of the beam be expressed as a polynomial in the spanwise dimension,

$$\text{e.g.} \quad \frac{\alpha T_o - \alpha T_i}{h} = - \frac{\Delta \alpha T}{h} = \frac{1}{l^2} \sum m_j \left(\frac{x}{l}\right)^j = \frac{1}{l^2} \sum m_j \xi^j \quad (1)$$

where T_i and T_o are the temperatures on the positive and negative side of the column with a linear gradient (see Figure 2.1-1).

If the structure is statically determinate, as in a simple-simple beam, then the thermal gradient represents the total curvature (w''_T) since no redundant forces are produced by the temperature. The slope and deflection can then be obtained by integrating this curvature and employing the given boundary conditions.

$$\therefore w = \frac{1}{l^2} \int_0^x \int_0^{x_0} \sum m_j \left(\frac{x}{l}\right)^j dx dx = \int_0^{\xi} \int_0^{\xi_0} \sum m_j \xi^j d\xi d\xi \quad (2)$$

$$w = \sum \frac{m_j \xi^{j+2}}{(j+1)(j+2)} + c_1 \xi + c_2$$

For simple supports $w(0) = w(l) = 0$

$$\therefore w = \sum \frac{m_j}{(j+1)(j+2)} (\xi^{j+2} - \xi) \quad (3a)$$

$$\& w' = \frac{1}{l} \sum \frac{m_j}{(j+1)(j+2)} \left[(j+2) \xi^{j+1} - 1 \right] \quad (3b)$$

2.3.3.1.1 (Cont'd)

If the structure is statically indeterminate, as in the clamped-clamped beam, then the curvature is affected by the curvature caused by the redundant loads. Equations (4.2.2.5 - 7 and -8) of Reference 2-2 can be employed to determine the deformation of the clamped beams where subjected to a thermal gradient. The thermal gradient $\frac{\Delta \alpha T}{h} =$

$-m_j \xi^j / l^2$ induces redundant loads P_0 and M_0 and the following total deformations result:

$$w_T'' = \frac{1}{l^2} \sum m_j \left[\xi^j - \frac{6j\xi}{(j+1)(j+2)} - \frac{2(1-j)}{(j+1)(j+2)} \right] \quad (4a)$$

$$w_T' = \frac{1}{l} \sum m_j \left[\frac{\xi^{j+1}}{j+1} - \frac{3j\xi^2}{(j+1)(j+2)} - \frac{2(1-j)\xi}{(j+1)(j+2)} \right] \quad (4b)$$

$$w_T = \sum \frac{m_j}{(j+1)(j+2)} \left[\xi^{j+2} - j\xi^3 - (1-j)\xi^2 \right] \quad (4c)$$

2.3.3.1.2. Expansion in Characteristic Curvatures

The polynomial definition of curvature can be converted to a eigenvector expansion by means of Eqs. (2a) of Paragraph 2.2.2 and (2) of Paragraph 2.3.2.

$$w_T'' + w_q'' = w_0'' = \sum C_i x_i \quad (\text{Refer to Eq. (2a) of Paragraph 2.2.2})$$

$$C_i = \frac{\int_0^l \phi w_0'' x_i dx}{\int_0^l \phi x_i^2 dx} \quad (\text{Refer to Eq. (2) of Paragraph 2.3.2})$$

For the simple-simple column with constant EI, this results in the simple Fourier sine expansion of the initial curvature.

$$C_i = \frac{2}{l} \int_0^l w_0'' \sin \frac{i\pi x}{l} dx = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \sin i\pi \xi d\xi \quad (1a)$$

where $w_0'' = \frac{1}{l^2} \sum m_j \xi^j$

The clamped-clamped column with constant EI and symmetrical loading results in a Fourier cosine expansion.

$$C_{is} = \frac{2}{l} \int_0^l w_{0s}'' \cos \frac{2i\pi x}{l} dx = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \cos 2i\pi \xi d\xi \quad (1b)$$

The curvature coefficients for the clamped-simple and the clamped-clamped columns with anti-symmetrical load must be similarly evaluated.

2.3.3.1.2 (Cont'd)

Tables 2.3.3.1.2-1 and -2 present the values of the integrals for the simple-simple and the symmetrical clamped-clamped columns.

TABLE 2.3.3.1.2-1 FOURIER EXPANSION OF MONOMIAL FOR SIMPLE-SIMPLE COLUMN

		S(j, i)					
j \ i		1	2	3	4	5	6
0		.6366	0	.2122	0	.12732	0
1		.3183	-.1591	.1061	-.07958	.06366	-.05305
2		.1893	-.1591	.1013	-.07958	.06262	-.05303
3		.1248	-.1350	.09894	-.07655	.06211	-.05215
4		.08814	-.1108	.09241	-.07353	.06061	-.05125
5		.06541	-.09078	.08383	-.06988	.05862	-.05011
6		.05038	-.07497	.07489	-.06560	.05629	-.04872
7		.03995	-.06258	.06647	-.06099	.05368	-.04712
8		.03243	-.05280	.05889	-.05631	.05088	-.04537
9		.02683	-.04503	.05223	-.05176	.04799	-.04350
10		.02256	-.03877	.04644	-.04748	.04510	-.04155

$$w'' = \sum C_i \sin \frac{i\pi x}{l} = \frac{1}{l^2} \sum m_j \left(\frac{x}{l}\right)^j = \frac{1}{l^2} \sum m_j \xi^j$$

$$C_i = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \sin i\pi \xi d\xi = \frac{2}{l^2} \sum_j \sum_i m_j S(j, i) \quad (2a)$$

$$S(j, i) = \int_0^1 \xi^j \sin i\pi \xi d\xi$$

$$S(j, i) = \frac{[1 + (-1)^j] (-1)^{j-2} j!}{2 (i\pi)^{j+1}} (i)^i \sum_{n=0, 2, 4}^{j+1} \frac{[(-1)^n - 1]}{i^n} \frac{(-1)^{\frac{n}{2}+1} j!}{(i\pi)^{n+1} (j-n)!} \quad (2b)$$

2.3.3.1.2 (Cont'd)

TABLE 2.3.3.1.2-2 FOURIER EXPANSION OF MONOMIAL FOR CLAMPED-CLAMPED COLUMN

		C(j, i)					
j \ i	1	2	3	4	5	6	
1	0	0	0	0	0	0	
2	.05066	.01267	.00563	.00317	.00203	.00141	
3	.07599	.01900	.00844	.00475	.00304	.00211	
4	.08592	.02437	.01107	.00627	.00403	.00280	
5	.08815	.02926	.01360	.00777	.00500	.00349	
6	.08669	.03337	.01595	.00920	.00596	.00416	
7	.08353	.03655	.01809	.01057	.00688	.00482	
8	.07967	.03883	.02000	.01185	.00777	.00546	
9	.07563	.04033	.02166	.01304	.00862	.00609	
10	.07167	.04120	.02308	.01414	.00942	.00669	

$$w'' = \sum C_i \cos \frac{2\pi i x}{l} = \frac{1}{l^2} \sum m_j \left(\frac{x}{l}\right)^j = \frac{1}{l^2} \sum m_j \xi^j$$

$$C_i = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \cos 2i\pi\xi d\xi = \frac{2}{l^2} \sum_j \sum_i m_j (C(j, i)) \quad (3a)$$

$$C(j, i) = \int_0^1 \xi^j \cos 2i\pi\xi d\xi$$

$$C(j, i) = \frac{[1 + (-1)^j] (-1)^{j/2} j!}{2 (2i\pi)^{j+1}} + \sum_{n=1,3,5}^{j-\frac{1}{2}[(-1)^j+1]} \frac{j!}{(2i\pi)^{1+n} (j-n)!} \quad (3b)$$

2.3.3.2 Discrete Lateral Deformation

In some instances the lateral deflection is not known as a continuous function, but is determined analytically (see Paragraph 4.2.1 of Reference 2-2) or experimentally at a discrete number of points. The deformation of "n" discrete points can be employed to obtain an approximating expansion of "n" characteristic deflection shapes. The amplitude of the "n" characteristic deflections are determined by matching the displacements at the known points. The technique is illustrated for a simple-simple column with known deflections at the 1/6 points (n = 5).

$$w(\xi_j) = w\left(\frac{x_j}{l}\right) = w_j = \sum_{i=1}^5 a_i \sin \frac{i\pi x}{l} \quad (1a)$$

The symmetry of the odd characteristic deflections and the anti-symmetry of the even characteristics can be employed to reduce the order of the simultaneous equations necessary to solve for the amplitudes (a_i).

$$\begin{aligned} \therefore \quad \frac{w_1 + w_5}{2} &= a_1 \sin \frac{\pi}{6} + a_3 \sin \frac{3\pi}{6} + a_5 \sin \frac{5\pi}{6} \\ \frac{w_2 + w_4}{2} &= a_1 \sin \frac{2\pi}{6} + a_3 \sin \frac{6\pi}{6} + a_5 \sin \frac{10\pi}{6} \\ w_3 &= a_1 \sin \frac{3\pi}{6} + a_3 \sin \frac{9\pi}{6} + a_5 \sin \frac{15\pi}{6} \\ \frac{w_1 - w_5}{2} &= a_2 \sin \frac{2\pi}{6} + a_4 \sin \frac{4\pi}{6} \\ \frac{w_2 - w_4}{2} &= a_2 \sin \frac{4\pi}{6} + a_4 \sin \frac{8\pi}{6} \end{aligned} \quad (1)$$

The solution is

$$\begin{pmatrix} a_1 \\ a_3 \\ a_5 \end{pmatrix} = \begin{pmatrix} .333 & .577 & .333 \\ .667 & 0 & .333 \\ .333 & .577 & .333 \end{pmatrix} \begin{pmatrix} \frac{w_1 + w_5}{2} \\ \frac{w_2 + w_4}{2} \\ w_3 \end{pmatrix} \quad (1c)$$

$$a_2 = \frac{1}{3.464} (w_1 + w_2 - w_4 - w_5)$$

$$a_4 = \frac{1}{3.464} (w_1 - w_2 + w_4 - w_5)$$

2.3.3.2 (Cont'd)

It should be noted that $C_1 = -\frac{i^2 \pi^2}{l^2} a_1$ (2)

Similar solutions can be obtained for different boundary conditions and for different types of known deformations.

A rapid solution can be obtained if one assumes that the lateral deflection is primarily in the fundamental mode (e.g., uniform thermal gradient on a simple-simple column). Thus the values of the nondimensional parameters can be simply determined with a minimum amount of computation. As an example, the simple-simple column equation can be expressed as

$$\frac{\alpha T - \Delta_0/l}{\frac{\pi^2 \rho}{l^2}} - \left(\frac{\Delta_0/l}{\frac{\pi^2 \rho}{l^2}} \quad \frac{w^2(l/2)}{4 \rho^2} \right) - \frac{w^2(l/2)}{4 \rho^2} \varphi_1 - \left(1 + \frac{EA}{k l} \right) \tau_{11} = 0 \quad (3)$$

Since

$$w(l/2) \sim \frac{C_1 l^2}{\pi^2}$$

Similar expressions can be derived for the other boundary conditions.

2.3.3.3 Axial Shortening

The initial axial shortening due to the application of temperature and lateral load is determined as a function of the polynomial coefficients expressing the lateral deflection. The axial shortening is determined as a function of the actual lateral deflection rather than by a function of the amplitude of the characteristic modes, in order to improve the accuracy of the solution (see Eq. 5 of Paragraph 2.3.2).

Assume that the lateral deflection (w_0) is readily available and is expressible as a polynomial of the fourth degree or less. Higher degrees of polynomials can be treated in a manner similar to that indicated below. Differentiating the deflection results in the slope (w_0') which can be squared and integrated over the length of the column to obtain twice the axial shortening

$$\text{i. e.} \quad w_0 = a + b \xi + c \xi^2 + d \xi^3 + e \xi^4 \quad (1a)$$

$$w_0' = \frac{1}{l} [b + 2c \xi + 3d \xi^2 + 4e \xi^3] \quad (1b)$$

$$2\Delta_0 = \int_0^l (w_0')^2 dx = \frac{1}{l} \left[b^2 + 2bc + \frac{6bd+4c^2}{3} + 2be + 3cd \right. \\ \left. + \frac{16ce+9d^2}{5} + 4de + \frac{16e^2}{7} \right] \quad (1c)$$

2.3.3.3 (Cont'd)

Typical examples for simple-simple columns are presented.

For Constant Thermal Gradient $\left(\frac{\Delta\alpha T}{h}\right)$

$$-\frac{\Delta\alpha T}{h} = \frac{1}{l^2} m_o \xi^0$$

$$j = 0 \text{ and } w_o = \frac{m_o}{(0+1)(0+2)} (\xi^{0+2} - \xi) \text{ from Eq. (3a) of Paragraph 2.3.3.1.1}$$

$$w_o = \frac{m_o}{2} (\xi^2 - \xi) = -\frac{l^2 \Delta\alpha T}{2h} (\xi^2 - \xi) = m_T (\xi - \xi^2)$$

where

$$m_T = \frac{l^2 \Delta\alpha T}{2h}$$

Substituting $b = m_T$ and $c = -m_T$ in Eq. (1c)

results in

$$2l \Delta_o = \frac{1}{3} m_T^2 \tag{2a}$$

For Uniform Lateral Load (q)

$$w_o = \frac{q l^4}{24EI} (\xi - 2\xi^3 + \xi^4) = m_q (\xi - 2\xi^3 + \xi^4)$$

$$\text{where } m_q = \frac{q l^4}{24EI}$$

substituting $b = m_q$, $d = -2m_q$, and $e = m_q$

$$\text{results in } 2l \Delta_o = .486 m_q^2 \tag{2b}$$

For Combined Uniform Load and Thermal Gradient

$$w_o = (m_T + m_q) \xi - m_T \xi^2 - 2m_q \xi^3 + m_q \xi^4 \tag{2c}$$

results in

$$2l \Delta_o = .333 m_T^2 + .800 m_T m_q + .486 m_q^2 \tag{2d}$$

2.3.3.3 (Cont'd)

It should be noted that any manufacturing or loading eccentricities will augment the lateral deflection parameters (C_i and d_i) by causing an axial shortening when the column is loaded, but should not be included directly in the Δ_0 (initial axial shortening because of thermal gradients and lateral loads) of the compatibility equation. The lateral eccentricities due to the various causes are additional and are included in the additional axial motion of the ends. The axial shortening however, is not linear but is proportional to the square of the deformations. Thus the initial axial shortening (Δ_0) must be computed as the difference of the axial shortening between the column including the manufactured eccentricities and the column containing only the manufactured eccentricities (w_{00}).

$$\text{i. e. } \Delta_0 = \int_0^L (w_q' + w_T' + w_{00}')^2 dx - \int_0^L (w_{00}')^2 dx \quad (3)$$

2.3.4 Spring Constant (k)

The value of k to be employed in the solution of the problem can significantly effect the magnitude of the resulting axial load. A column with zero axial restraint ($k = 0$) does not develop any axial load. The load increases with increasing axial restraint until it reaches a maximum for complete restraint ($k = \infty$).

The value of k is the value of the load produced by moving one end of the column a unit distance in the axial direction relative to the other end (assuming that the column offers no resistance). This is the stiffness coefficient defined in Section 4.2.5 of Reference 2-2. If both ends of the column are spring mounted, then the value of k is determined by putting the two end springs in series

$$\text{i. e. } \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\therefore k = \frac{k_1 k_2}{k_1 + k_2} \quad (1)$$

where k - effective axial restraint

k_1, k_2 - axial restraint at the ends of the column

The method of determining the value of k in a composite structure is illustrated below for a truss joint.

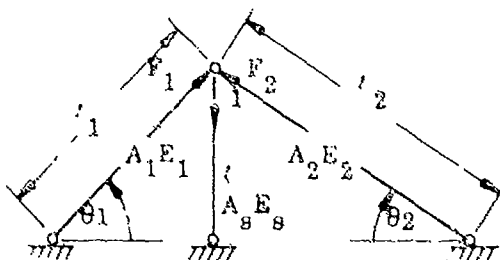


FIGURE 2.3.4-1 DETERMINATE TRUSS JOINT

The equilibrium equations are:

$$F_1 \sin \theta_1 + F_2 \sin \theta_2 = 1 \quad (2a)$$

$$F_1 \cos \theta_1 = F_2 \cos \theta_2 \quad (2b)$$

2.3.4 (Cont'd)

For symmetrical geometry ($\theta_1 = \theta_2 = \theta$; $A_1 E_1 = A_2 E_2 = AE$)

$$F_1 = F_2 = \frac{1}{2 \sin \theta} \quad (2c)$$

$$\frac{1}{k} = \Delta = \frac{F \ell / \sin \theta}{AE \sin \theta} = \frac{\ell}{2AE \sin^3 \theta} \quad (3a)$$

$$\frac{E_s A_s}{k \ell} = \frac{1}{2 \sin^3 \theta} \left(\frac{E_s A_s}{EA} \right) \quad \text{where } E_s A_s = \text{axial stiffness of strut} \quad (4a)$$

For unsymmetrical geometry

The load will cause a non-vertical motion of center strut and change the angles. For small deformations the following relationships result:

$$F_1 = \frac{1}{\sin \theta_1 (1 + \cot \theta_1 \tan \theta_2)} \quad ; \quad F_2 = \frac{1}{\sin \theta_2 (1 + \cot \theta_2 \tan \theta_1)} \quad (2d)$$

$$\begin{aligned} \frac{1}{k} = \Delta &\sim \frac{1}{2} \left(\frac{F_1 \ell_1}{A_1 E_1 \sin \theta_1} + \frac{F_2 \ell_2}{A_2 E_2 \sin \theta_2} \right) \\ &= \frac{\ell}{2} \left(\frac{1/A_1 E_1}{\sin^3 \theta_1 (1 + \cot \theta_1 \tan \theta_2)} + \frac{1/A_2 E_2}{\sin^3 \theta_2 (1 + \cot \theta_2 \tan \theta_1)} \right) \end{aligned} \quad (3b)$$

$$\text{Thus } \frac{E_s A_s}{k \ell} \sim \frac{E_s A_s}{2} \left(\frac{1}{A_1 E_1 \sin^3 \theta_1 \left(1 + \frac{\tan \theta_2}{\tan \theta_1} \right)} + \frac{1}{A_2 E_2 \sin^3 \theta_2 \left(1 + \frac{\tan \theta_1}{\tan \theta_2} \right)} \right)$$

This reduces to $\frac{E_s A_s}{k \ell} \sim \frac{1}{2 \sin^3 \theta} \left(\frac{E_s A_s}{EA} \right)$ for symmetrical geometry.

2.3.5 Effect of Non-Linearity in Spring or Material

Solutions to the axial load in the column can be obtained even when the axial restraint and/or stiffness of the column varies with the axial load.

The solution of a column with a variable axial restraint can be obtained by superimposing a plot of the flexibility parameter $\left[EA/\ell k = EA/\ell k (\eta_1) \right]$ as a function of the axial load parameter (η_1) upon a plot of the solution of the compatibility equation for the load parameter $\left[\eta_1 = \eta_1 \left(\frac{EA}{\ell k} \right) \right]$ as a function of the flexibility parameter. The first plot is obtained directly from the spring characteristics whereas the second plot is obtained by varying the value of $EA/\ell k$ to obtain different values of η_1 . The intersection of the two plots, as illustrated in Figure 2.3.5-1, will result in a compatible solution.

2.3.5 (Cont'd)

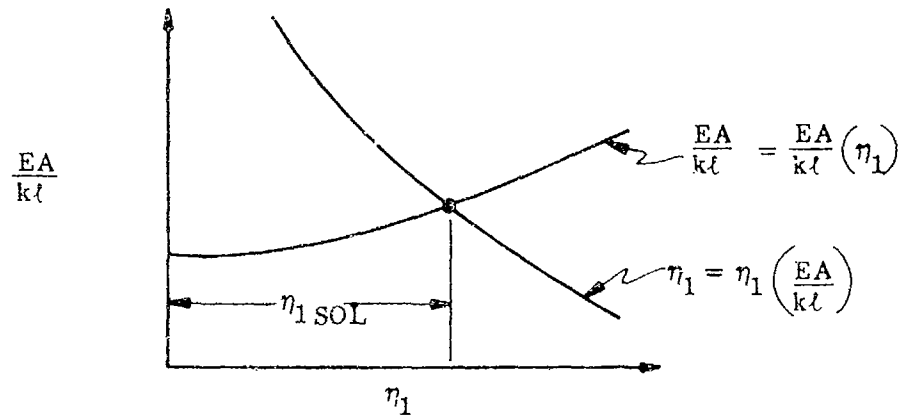


FIGURE 2.3.5-1 FLEXIBILITY VS AXIAL LOAD

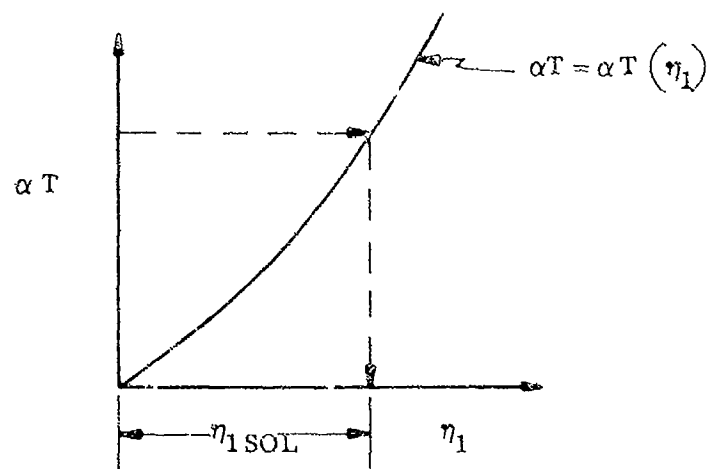


FIGURE 2.3.5-2 AXIAL LOAD VS AVERAGE THERMAL EXPANSION

2.3.5 (Cont'd)

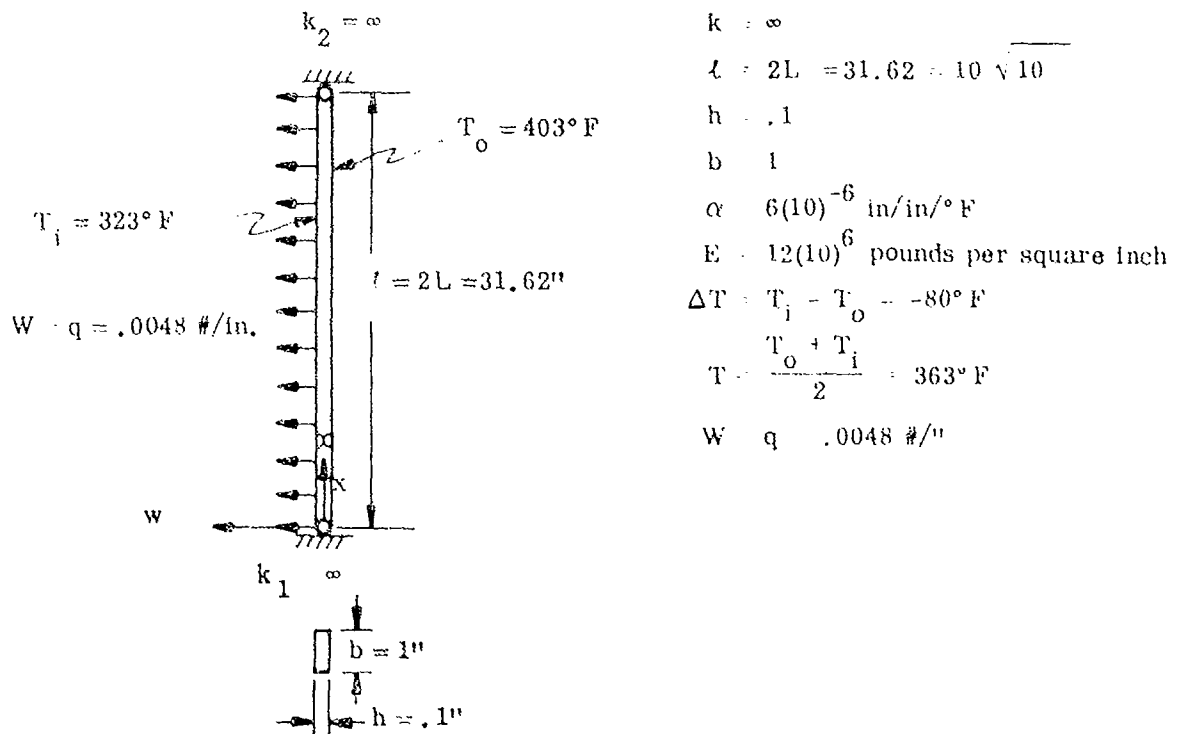
The solution of a column with a load dependent material stiffness (E_S) due to the plasticity of the material can similarly be obtained by a trial and error procedure. Assuming a value of η_1 will determine a value of E_S . This will determine a value of the flexibility parameter $\frac{E_S A}{k\ell}$ which varies linearly with E_S ; values of the lateral deflection parameters (C_i) whose lateral load component varies inversely with E_S and whose thermal component is unaffected; and the axial shortening parameters (d_i) which varies as the square of the C_i . Corresponding to these values of η_1 , values of \bar{d}_1 can be computed and employed in Figures 2.2.4 to obtain values of \bar{b} which can be utilized to calculate αT . The actual value of η_1 results when the calculated value of αT corresponds to the correct value of αT . A graphical approach is illustrated in Figure 2.3.5-2.

This above method can be utilized to solve a problem with a load-dependent axial restraint and a load dependent material stiffness. The above technique assumes that the axial and bending stiffness of the column is not significantly affected by the variation of the stress through the thickness of the cross section but is determined primarily by the mean stress (F/A).

2.3.6 Illustrative Problem

The computation procedure for the approximate solution is relatively simple and is illustrated for a problem which can be directly compared to an "exact" solution obtained from Table 1.5-1.

The temperature on each face is constant along the length with a linear gradient through the thickness.



2.3.6 (Cont'd)

AXIAL EXPANSION

From Eq. (1b) of Paragraph 2.2.4

$$\alpha T = \alpha \frac{T_o + T_i}{2} = 6(10)^{-6} \cdot 323 = .0021780$$

AXIAL SHORTENING

From Eq. (2d) of Paragraph 2.3.3.3

$$\frac{\Delta_o}{l} = \frac{1}{l^2} \left[\frac{1}{6} m_T^2 + .400 m_T m_q + .243 m_q^2 \right]$$

but

$$m_T = \frac{l^2 \Delta \alpha T}{2h} = \frac{1000 \cdot 6(10)^{-6} \cdot (-80)}{2(.1)} = -2.4''$$

and

$$m_q = \frac{q l^4}{24EI} = \frac{+.0048 (10)^6}{(24)12(10)^6 \cdot \frac{1}{12} (.1)^3} = +.2''$$

$$\therefore \frac{\Delta_o}{l} = \frac{1}{1000} \left[\frac{1}{6} (2.4)^2 + .4(2.4)(-.2) + .243(.2)^2 \right] = 7.777 (10)^{-4}$$

From Eq. (2c) of Paragraph 2.2.4

$$\epsilon_1 = \lambda_1 \rho^2 = \lambda_1 \left(\frac{l}{A} \right)^2 = \frac{\pi^2}{l^2} \frac{h^2}{12} = \frac{\pi^2}{1000} \frac{(.1)^2}{12} = 8.225 (10)^{-6}$$

$$\therefore \frac{\alpha T}{\epsilon_1} = 264.8 \quad \text{and} \quad \frac{\Delta_o}{l \epsilon_1} = 94.56$$

EXPANSION OF CURVATURE

From Eq. (2c) of Paragraph 2.3.3.3

$$w_o = (m_T + m_q) \xi - m_T \xi^2 - 2m_q \xi^3 + m_q \xi^4 \quad (1a)$$

$$\therefore w_o' = \frac{1}{l} \left[m_T + m_q - 2m_T \xi - 6m_q \xi^2 + 4m_q \xi^3 \right] \quad (1b)$$

$$\therefore w_o'' = \frac{1}{l^2} \left[-2m_T - 12m_q \xi + 12m_q \xi^2 \right] \quad (1c)$$

2.3.6 (Cont'd)

Direct substitution in the Fourier sine expansion or the use of Table 2.3.3.1.2-1 will enable the determination of the magnitude of the curvatures. Direct substitution is employed in this illustration to obtain general equations for the case of uniform load and thermal gradients. The Fourier coefficients defined by Eq. (2b) of Paragraph 2.3.3.1.2 are

$$S(0, i) = \frac{1 - (-1)^i}{i\pi}$$

$$S(1, i) = -\frac{(-1)^i}{i\pi}$$

and

$$S(2, i) = \frac{-(-1)^i}{i\pi} + \frac{2}{(i\pi)^3} \left[(-1)^i - 1 \right]$$

From Eq. (2a) of Paragraph 2.3.3.1.2

$$\therefore C_i = \frac{2}{l^2} \sum_j \sum_i m_j S(j, i)$$

$$C_i = \frac{2}{l^2} \left[(-2 m_T) \left(\frac{1 - (-1)^i}{i\pi} \right) + (-12 m_Q) \left(\frac{-(-1)^i}{i\pi} \right) + (12 m_Q) \left(\frac{-(-1)^i}{i\pi} + \frac{2}{(i\pi)^3} \left[(-1)^i - 1 \right] \right) \right]$$

$$C_i = \frac{2}{l^2} \left[(-2 m_T) \frac{(1 - (-1)^i)}{i\pi} - \frac{24 m_Q}{(i\pi)^3} (1 - (-1)^i) \right]$$

$$C_i = -\frac{4}{l^2} (1 - (-1)^i) \left[\frac{m_T}{i\pi} + \frac{12 m_Q}{(i\pi)^3} \right]$$

$$\therefore \text{for } C_{i(\text{odd})} \quad \frac{-l^2 C_{2k+1}}{2} = \frac{4}{2k+1} m_T + \frac{(48/\pi^3) m_Q}{(2k+1)^3} \quad (2a)$$

$$\text{and for } C_{i(\text{even})} \quad C_{2k} = 0 \quad (2b)$$

2.3.6 (Cont'd)

The values of C_i are then calculated. The same values would be obtained from Table

$$2.3.3.1.2-1 \left[\frac{C_i t^2}{2} = \sum \sum m_j S(j, i) \right]$$

$$\therefore -\frac{t^2 C_1}{2} \begin{cases} = 4/\pi (m_T) + 48/\pi^3 (m_Q) \\ = 1.27324(-2.4) + 1.548076(+.2) \\ = -3.05578 + .30962 = -2.746 \end{cases}$$

$$-\frac{t^2 C_3}{2} = -\frac{1}{3} (3.05578) + \frac{1}{9} (.30962) = -1.007$$

$$-\frac{t^2 C_5}{2} = -\frac{1}{5} (3.05578) + \frac{1}{25} (.30962) = -.609$$

$$-\frac{t^2 C_7}{2} = -\frac{1}{7} (3.05578) + \frac{1}{49} (.30962) = -.435$$

PERTINENT PARAMETERS

From Eqs. (2b) and (4b) of Paragraph 2.3.1

$$d_1 = \left(\frac{t^2 C_1}{2} \right)^2 \frac{1}{\pi^4} \rho^2 t^2 \quad \text{and} \quad \frac{d_1 \eta_1}{\eta_1} = \frac{d_1}{t^2}$$

$$\therefore d_1 = \frac{(2.746)^2}{\pi^4 \left(\frac{.1}{12} \right)^2} = 92.895$$

similarly

$$d_3 = 1.388 \quad \text{and} \quad \frac{d_3 \eta_3}{\eta_1} = d_3/3^2 = .154$$

$$d_5 = .183 \quad \frac{d_5 \eta_5}{\eta_1} = d_5/5^2 = .007$$

$$d_7 = .047 \quad \frac{d_7 \eta_7}{\eta_1} = d_7/7^2 = .001$$

2.3.6 (Cont'd)

$$\therefore \sum_{i=1}^8 d_i = d_2 + d_3 + d_5 + d_7 = 94.513$$

and

$$\frac{\sum_{i=2}^8 \eta_i d_i}{\eta_1} = d_3/3 + d_5/5 + d_7/7 = .162$$

Employing Eq. (9a) of Paragraph 2.24 and including correction for finite sum approximation of the initial shortening (Eq. (5) of Paragraph 2.3.2) results in

$$0 = \frac{\alpha T}{\epsilon_1} - \frac{\Delta_o}{l\epsilon_1} - \left(\frac{\Delta_o}{l\epsilon_1} - \sum_{i=1}^8 d_i \right) - d_1 \varphi_1 - \left(1 + \frac{EA}{kl} + 2 \sum_{i=2}^8 \frac{\eta_i d_i}{\eta_1} \right) \eta_1 \quad (3)$$

Substituting the calculated values results in

$$0 = (264.8 - 94.555) - (94.555 - 94.513) - 92.895 \varphi_1 - (1.234) \eta_1$$

$$0 = 170.203 - 92.895 \varphi_1 - (1.234) \eta_1$$

$$0 = 137.93 - 75.23 \varphi_1 - \eta_1 = \bar{b} - \bar{d}_1 \varphi_1 - \eta_1$$

Entering Figure 2.2.4-2 with $\bar{b} = 137.93$ and $\bar{d}_1 = 75.28$ results in the solution $\eta_1 = .407$.

AXIAL LOAD

From Eq. (3) of Paragraph 2.3.2

$$F = AE \epsilon_1 \eta_1 = (1.1) (1) (12) (10)^6 8.225 (10)^{-6} .407 = 4.017\#$$

LATERAL DEFLECTION

From Eq. (4a) of Paragraph 2.3.2

$$w_{\max} = w(l/2) = \sum \frac{w_{oi}}{1-\eta_i} = -\frac{l^2}{\pi^2} \sum \frac{C_i}{i^2(1-\eta_i)} = -\frac{2}{\pi^2} \sum \frac{l^2 C_i}{2i^2(1-\eta_i)}$$

2.3.6 (Cont'd)

$$\therefore w_{\max} \sim \frac{2}{\pi^2} \left[\frac{-2.74616}{1-.407} + \frac{-1.00712}{9(1-.407/9)} + \frac{-.60867}{25(1-.407/25)} + \frac{-.43523}{49(1-.407/49)} \right]$$

$$w_{\max} \sim .97''$$

The geometry and loads employed in this illustrative problem results in parameters which correspond to an exact solution found in Table 1.5-1.

These parameters are

$$\bar{T}_D = \alpha \left(\frac{L}{h} \right)^2 (T_0 - T_1) = 12.0$$

$$\bar{W} = \frac{12W}{Eb} \left(\frac{L}{h} \right)^4 = 3.0$$

$$\bar{T} = \alpha \left(\frac{L}{h} \right)^2 (T_0 + T_1) = 108.9$$

A comparison of the exact and approximate solutions is presented below.

	Exact Solution	Approximate Solution
$\bar{\lambda}$	- 1.000	- 1.005
η_1	.405	.407
F	4.01#	4.02#
w_{\max}	.92''	.97''

Note that the approximate solution results in slightly larger values. The approximate method, however can be applied to more general types of loadings and boundary conditions.

It should be noted that the illustrated example shown in Figure 1.7-1 of Section 1 is solved by an interpolation of the "exact" tabular values. The approximate graphical procedure as well as the interpolation of tables procedure is subjected to relatively greater errors in the vicinity of low load ratios. This is because of the greater relative significance of the higher modes in the graphic approach and the large slope of the \bar{T} vs $\bar{\lambda}$ (as illustrated in Figure 1.5-2 of Section 1 of the tabular solution which permits large variations in $\bar{\lambda}$ for small variations in \bar{T}). Fortunately the design of the column is not determined by this condition of low axial load. The interpolation and graphical solutions are not subject to significant errors, however, when the load ratios are higher.

A comparison of the interpolation solution and of the approximate graphical solution of this report is presented as follows:

	Interpolated Sol.	Approx. Graphical Sol.
$\bar{\lambda}$	- .524	- .607
η_1	.111	.155
F	27.4	37.2
w_{\max}	.16	.18

2.4 REFERENCES

- 2-1. Switzky, H., "Approximate Solution For an Axially Restrained Column Subjected to Elevated Temperatures and Lateral Loads," Republic Aviation Corporation Report No. ARD-679-4, September 1961.
- 2-2. Switzky, H., Forray, M., and Newman, M. "Thermo-Structural Analysis Manual"- Volume I, Republic Aviation Corporation Report No. RAC 679-1, September 1960, revised November 1961 (to be published as WADD TR 60-517, Vol. I).

SECTION 3
APPROXIMATE SOLUTION FOR THE BUCKLING OF
ECCENTRIC COLUMNS

by

H. Switzky

SECTION 3
APPROXIMATE SOLUTION FOR THE BUCKLING OF
ECCENTRIC COLUMNS

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SECTION 3 - APPROXIMATE SOLUTION FOR THE BUCKLING OF
ECCENTRIC COLUMNS

3.1 SUMMARY

An approximate solution $\left(\bar{F}_1 = \frac{K}{l^2} E_{So} I_o \left[1 - \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right]^{2/3} \right)$ is obtained

for the buckling of an eccentric column which includes the effects of the stress level, the stress-strain relationship, and the initial eccentricity.

The analysis technique is presented in a nondimensional form to permit the analyst to consider various cross sections, boundary conditions, materials and degrees of eccentricity. The type of cross section is reflected in the I/A ratio which defines the square of the radius of gyration (r). The boundary conditions are contained in the value of K/l^2

which consider the end fixity and length of the column. The material characteristics are reflected in material parameters (E_A, σ_o, β) which represent the stress-strain relationship.

The nondimensional analysis graphs are presented for several values of the eccentricity

ratio $\left(\frac{a_{10}}{r} \right)$ which may occur because of initial waviness, eccentric loading, lateral loads, or thermal gradients. Analysis for intermediate values of the eccentricity can be conducted by interpolation.

Illustrative problems are presented to indicate the computation procedure and the effect of the initial eccentricity upon the stability of the structure.

3.1.1 Definition of Symbols

The following symbols are used throughout this section:

a_{10}	Original amplitude (eccentricity) of fundamental mode
h	Depth of cross section of the column
l	Length of column
m_j	Coefficient expressing initial deformations of the column as a power series
q	Lateral load acting on column
r	Radius of gyration of cross section
w	Lateral deflection of column
x	Axial coordinate of column
z'	Distance from reference axis
\bar{z}'	Distance from reference axis to bending axis
z_o'	Distance from reference axis to centroidal axis $\bar{z}_o' = \int z' dA / \int dA$
z	Distance from bending axis
A	Area of cross section
$C(j, 1)$	Fourier expansion coefficient $(C(j, 1) = \int_0^1 \xi^j \cos 2\pi \xi d\xi)$
$E = E_A$	Initial slope of the σ vs ϵ_σ curve
E_S	Secant modulus of the σ vs ϵ_σ curve $(E_S = \sigma/\epsilon_\sigma)$
E_{So}	Secant modulus at the bending axis
E_T	Tangent modulus, slope of the σ vs ϵ_σ curve $(E_T = d\sigma/d\epsilon_\sigma)$
E_{To}	Tangent modulus at the bending axis

3.1.1 (Cont'd)

- \overline{EA} Axial stiffness of cross section ($\overline{EA} = \int E_S dA$)
- \overline{EI} Bending stiffness of cross section ($\overline{EI} = \int E_S z^2 dA$)
- $\overline{\overline{EI}}$ Buckling stiffness of cross section ($\overline{\overline{EI}} = \overline{EI} + \kappa \frac{\partial \overline{EI}}{\partial \kappa} = \overline{F}_1 / \frac{K}{l^2}$)
- F Axial load on structure (compression positive)
- F_1 Buckling load of column with zero eccentricity or a linear material
- \overline{F}_1 Buckling load of column ($\overline{F}_1 = \frac{K}{l^2} E_{So} I_o \left[1 - \left[\sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right]^{2/3} \right]$)
- F_E Euler Buckling load ($F_E = \frac{K}{l^2} EI_o$)
- I_o Second moment of area (inertia) of cross section ($I_o = \int (z' - \bar{z}_o')^2 dA$)
- K Stability constant depending upon boundary conditions and the bending stiffness

distribution along the columns $\left(K = l^2 \frac{\delta \kappa}{\delta w} = \frac{l^2 \int_0^l \phi (w_1'')^2 dx}{\int_0^l (w_1')^2 dx} \right)$

M Moment acting on cross sections (Positive M causes compression in positive fibers).

Q_o Third moment of area of cross section ($Q_o = \int (z' - \bar{z}_o')^3 dA$)

S(j, 1) Fourier expansion coefficient ($S(j, 1) = \int_0^1 \xi^j \sin \pi \xi d\xi$)

T Temperature rise above datum

α Coefficient of linear thermal expansion

α Parameter expressing variation of secant modulus in the cross section

$\left(\alpha = \frac{\kappa}{\epsilon_o} \left(1 - \frac{E_{To}}{E_{So}} \right) \right)$

β Nondimensional parameter employed in mathematical definition of stress-strain relationship (Section 3 of Reference 3-1)

γ Nondimensional parameter expressing the average variation of the tangent

modulus in the cross section ($\gamma = \frac{\int_0^z E_T dz}{z E_{To}} = \frac{z E_{To}}{z E_{To}}$)

$\Delta \alpha T$ Difference in thermal expansion

δ Operator denoting a small variation

ϵ Axial strain of cross section

ϵ_σ Axial strain caused by stress ($\epsilon_\sigma = \epsilon - \alpha T$)

ϵ_1 Average axial strain corresponding to an axial load F_1

ϵ_o Axial strain at the bending axis

$\overline{\epsilon}_1$ Average axial strain corresponding to an axial load \overline{F}_1

η Shift of bending axis ($\eta = \bar{z}_o' - \bar{z}^1$)

κ Curvature

ξ Nondimensional axial coordinate ($\xi = x/l$)

σ Axial stress of cross section (compression positive)

3. 1. 1 (Cont'd)

- σ_o Axial stress at bending axis
 σ_o Reference stress in nondimensional stress-strain relationship (Section 3 of Reference 3-1)
 ϕ Nondimensional parameter expressing the variation of the bending stiffness along the length of the column ($\phi(x) = \bar{EI}/E_o I_o$)
 Φ Nondimensional parameter $\left(\Phi = \left(\frac{E A}{\sigma_o} \right) \left(\frac{KI}{Ab^2} \right) = \frac{E A \epsilon_1}{\sigma_o} \right)$
 $\bar{\Phi}$ Nondimensional parameter $\bar{\Phi} = \left(\frac{E A}{\sigma_o} \right) \left(\frac{KI}{Ab^2} \right) \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{\bar{E}_T}{E_1} \right) \right\}^{2/3} \right]$
 $= \frac{E A \bar{\epsilon}_1}{\sigma_o}$

SUBSCRIPTS

- M Caused by mechanical load
T Caused by temperature
1 Pertaining to the first (fundamental) mode
o Pertaining to initial or datum

3. 1. 2 Discussion of the Problem

The stability of a structure is a very complex problem. Exact solutions exist only for very few special cases. Approximations must be attempted when the solution is complicated by variations (reductions) of the stiffness with the applied load because of the resulting non-linearity of the equation. The reduction in the stiffness is caused by a non-linear structural material. The non-linear stress strain relationship reduces the stiffness of the structure by lowering the secant modulus and shifting the neutral axis.

Temperature, time, and the eccentricities of the structure tend to reduce the allowable magnitude of the applied load on the structure. Elevated temperatures reduce the stiffness of the structure, increase the non-linearity of the material because of plasticity at the stresses caused by the applied and thermal stresses, and usually increase the eccentricity of the structure. The axial forces acting through the eccentricity of the structure imposes moments upon the structure which increase at a greater rate than the applied load. These moments cause a variation of stresses through the cross sections which can reduce the bending stiffness. The larger the eccentricity the earlier the initiation of the non-linearity of the material and the lower the stability of the structure. The bending stiffness can also decrease with time because of creep of the material.

The approach, employed in this section, to examine the stability of a column (Figure 3. 1. 2-1) is to determine when the increase of external moment acting on the column cross sections tend to become greater than the increase of internal moment that can be generated by the cross sections. The axial load acting on the column at this time is the buckling load and is a function of the boundary conditions, the bending stiffness of the cross sections, and the rate that the bending stiffness is decreasing. The bending stiffness is a function of the cross-section, the magnitude of the stresses and their distribution and the stress-strain relationship. The bending stiffness usually decreases with an increase in the applied load, temperature, time, or eccentricity of the column. Thus the column becomes unstable when the applied load becomes equal to the buckling load which is determined by the bending stiffness distribution and its rate of decrease. This can occur by increasing the applied load and/or decreasing the magnitude of the buckling load. The buckling load is reduced by the eccentricity, the temperature, and time as indicated previously.

3.1.2 (Cont'd)

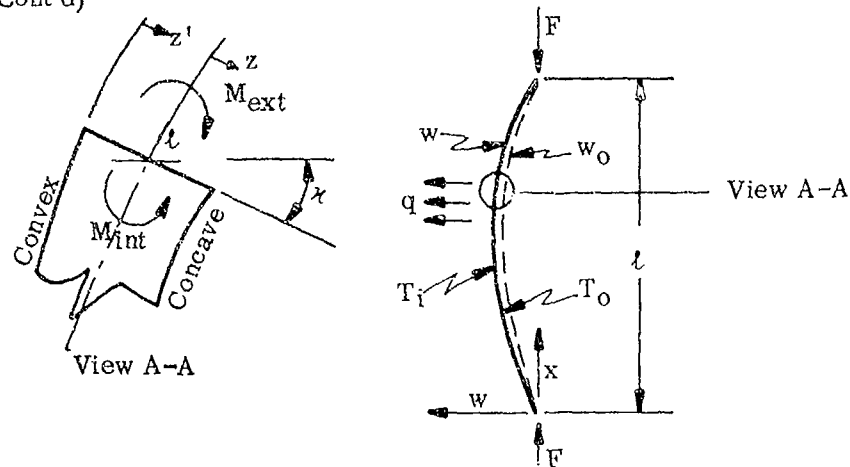


FIGURE 3.1.2-1 ECCENTRIC COLUMN SUBJECTED TO AXIAL LOADS

The evaluation of the buckling load requires the determination of the stiffness of the column and its rate of change. This is complicated by the shifting of the bending axis towards the convex side of the cross section because the higher stress and strain levels on the concave side reduce the secant modulus. The stiffness relationships, derived in Appendix A of Reference 3-2, are employed with the stability criterion described previously to obtain an approximate solution for the buckling load of the column in terms of the stress and lateral deflections. The solution is then approximated in terms of average stress level and initial eccentricity which can be readily computed. The initial eccentricities may result from the unloaded shape, eccentric loading, lateral loads, or thermal gradients. Methods of computing the initial eccentricity due to load and temperature are presented in Section 2 and summarized in this section.

An analogous approach can be employed to transform the loads or stresses to strains. Thus the column becomes unstable when the average axial strain becomes equal to a buckling strain. This approach is most convenient for a nondimensional presentation of the analysis.

The solution is predicated upon the following assumptions:

- (1) The bending strains are small compared to the axial strains in the column.
- (2) The temperature distribution can result in deformations and reductions in the moduli but do not cause significant thermal stresses.

(3) There is a one-to-one relationship between the stress (σ) and the strain (ϵ_σ) for each fiber of the column. This assumes that there is no stress reversal throughout the loading history. It is also assumed that the stress-strain relationship can be approximated to a sufficient degree of accuracy by the relationship

$$\left[\frac{E_A \epsilon}{\sigma_0} = (1-\beta) \frac{\sigma}{\sigma_0} + \beta \sinh \frac{\sigma}{\sigma_0} \right]$$

defined and described in Section 3 of Reference 3-1.

The approximate solution, which results from these assumptions, is of a form which does not violate known solutions of special cases and is indicative of the structural behavior in experiments.

3.2 ANALYSIS

A column subjected to loads and temperature (Figure 3.1.2-1) must satisfy equilibrium. Thus the change in applied moment (M_{ext}) must be equal to the change in the internal moment (M_{int}). The changes in external moment arise from changes in the axial load (F) and the lateral deflection (w). (The effect of elastic supports in introducing changes in the moment is reflected in the value of the stability constant K). The change in the internal moment is evidenced by a change in the curvature (κ_M) and a possible change in the bending stiffness (\bar{EI}).

Referring to Figure 3.1.2-1, we note the equilibrium requirements.

$$M_{ext} + M_{int} = 0 \quad (1a)$$

and

$$-\delta(M_{ext}) = \delta(M_{int}) \quad (1b)$$

evaluating the change in moments, we obtain

$$-\delta(M_{ext}) = -\delta(Fw) = -[F(\delta w) + w(\delta F)] \quad (1c)$$

and

$$\delta(M_{int}) = \delta(\bar{EI} \kappa_M) = [\bar{EI}(\delta \kappa_M) + \kappa_M(\delta \bar{EI})] \quad (1d)$$

$$\therefore -[F(\delta w) + w(\delta F)] = [\bar{EI}(\delta \kappa_M) + \kappa_M(\delta \bar{EI})] \quad (1e)$$

If the load is monotonically increasing, then the column will fail when the buckling load (\bar{F}_1) is applied. No additional incremental load can be applied so that the buckling load is defined by setting δF equal to zero. The above buckling criterion is equivalent to finding the load on a structure for which the deflection is undefined (Figure 9.1-1b of Reference 3-1). It is also equivalent to determining the load on the structure at which the ratio of $\delta F/\delta w$ is zero, where δF is the incremental load necessary to cause an incremental deflection (δw). The latter buckling criteria ($\delta F/\delta w = 0$) can also be employed to determine the stability of a column which exhibits creep.

Setting $\delta F = 0$ and $F = \bar{F}_1$ in equation (1e) results in

$$-[\bar{F}_1 \delta w + w(0)] = \bar{EI} \delta \kappa_M + \kappa_M \delta \bar{EI} \quad (2a)$$

$$\therefore \bar{F}_1 = - \left[\bar{EI} \frac{\delta \kappa_M}{\delta w} + \kappa_M \frac{\delta \bar{EI}}{\delta w} \right] = - \frac{\delta \kappa_M}{\delta w} \left(\bar{EI} + \kappa_M \frac{\delta \bar{EI}}{\delta \kappa_M} \right)$$

$$\bar{F}_1 = - \frac{\delta \kappa_M}{\delta w} \bar{EI} \quad (2b)$$

Thus the buckling load is a function of the ratio of the change of curvature to the change of lateral deflection and of the bending stiffness modified by an expression indicating the rate of change of bending stiffness with curvature.

The first expression $\delta \kappa_M/\delta w$ is relatively simple to calculate if we assume that the deformation modes of the given structure do not change as the axial load is increased to the buckling load. Thus it is assumed that a pin ended column of constant bending stiff-

3.2 (Cont'd)

ness, which buckles elastically in a sine wave, will buckle in a sine wave even if it becomes plastic. This condition is satisfied as long as the ratio of the bending stiffness distribution $(\varphi(x) = \overline{EI}(x)/E_0 I_0)$ along the column does not change significantly with the axial load. The stiffness can change due to plasticity but it is assumed that the φ ratio does not change significantly. This is consistent with our assumption that the bending stresses are small compared to the axial stresses so that the material moduli remain approximately equal (for constant cross-section area). Thus the value of $\delta x_M / \delta w$ is assumed equal to the value of the "linear" column. This is the classical stability coefficient found in various textbooks which can be expressed as the curvature to deflection ratio as indicated above or by the general expression of Eq. (1a) of Paragraph 2.3.1.

$$-\frac{\delta x_M}{\delta w} = \frac{F_1}{\overline{EI}} = \lambda_1 = \frac{\int_0^l \varphi (w_1'')^2 dx}{\int_0^l (w_1')^2 dx} = \frac{K}{l^2} \quad (3)$$

Values of K can be found in various texts (e. g. References 3-3, and 3-4).

As an example for a pin-ended column of constant EI (i. e. $\varphi = 1$)

$$-\frac{\delta x_M}{\delta w} = -\frac{\delta \left(\frac{\pi^2}{l^2} \sin \frac{\pi x}{l} \right)}{\delta \left(\sin \frac{\pi x}{l} \right)} = \frac{\pi^2}{l^2} = \frac{K}{l^2} \quad (4a)$$

$$\text{or } \lambda_1 = \frac{\int_0^l (1) \left(-\frac{\pi^2}{l^2} \sin \frac{\pi x}{l} \right)^2 dx}{\int_0^l \left(\frac{\pi}{l} \cos \frac{\pi x}{l} \right)^2 dx} = \frac{\pi^2}{l^2} = \frac{K}{l^2} \quad (4b)$$

$$\therefore K = \pi^2 \quad (4c)$$

The second expression $\left(\overline{EI} = \overline{EI} + x_M \frac{\delta \overline{EI}}{\delta x_M} \right)$ is evaluated by utilizing the definition of the bending stiffness derived in Appendix A of Reference 3-2 which approximates the effects of the bending stress.

From Eq. (A10b) of Reference 3-2

$$\overline{EI} \approx E_{S_0} I_0 (1 - 2\alpha^2 r^2) \quad (5a)$$

where

$$\alpha = \frac{x_M}{\epsilon_0} \left(1 - E_{S_0} \frac{r_0}{\gamma} \right) \quad (5b)$$

3.2 (Cont'd)

and
$$\gamma = \int_0^z E_T dz / z E_T \quad (5c)$$

Substituting Eq. (3), (5a), and (5b) in Eq. (2b) results in

$$\bar{F}_1 = \frac{K}{l^2} E_{So} I_o (1 - 6\alpha^2 r^2) \quad (6)$$

The solution for the buckling load is now expressed in terms of geometry and boundary conditions (l, I_o, r, K), the stress level at the neutral axis (E_{So}, E_{To}) and the stress distribution

$$\left(\alpha = \frac{\kappa_M}{\epsilon_o} \left(1 - \frac{E_{To}}{E_{So}} \gamma \right) \right).$$

Unfortunately the stress distribution parameter α is not specifically defined but varies in a complex way with the applied load. It varies directly with the curvature (κ_M) which increases nonlinearly with the load, inversely with the axial strain (ϵ_o) which is defined by the stress-strain relationship (Figure 3.2-1) and directly with the stress distribution factor $\left(1 - \frac{E_{To}}{E_{So}} \gamma \right)$ which increase with load and curvature. Some additional approximations are required to reduce the solution to a simple form expressible in terms of the initial conditions. Each of the approximations employed introduces small over- or under-estimates of the buckling load. It is hoped that the combined effect of all the approximations will result in a reasonably accurate solution that considers the effects of eccentricity.

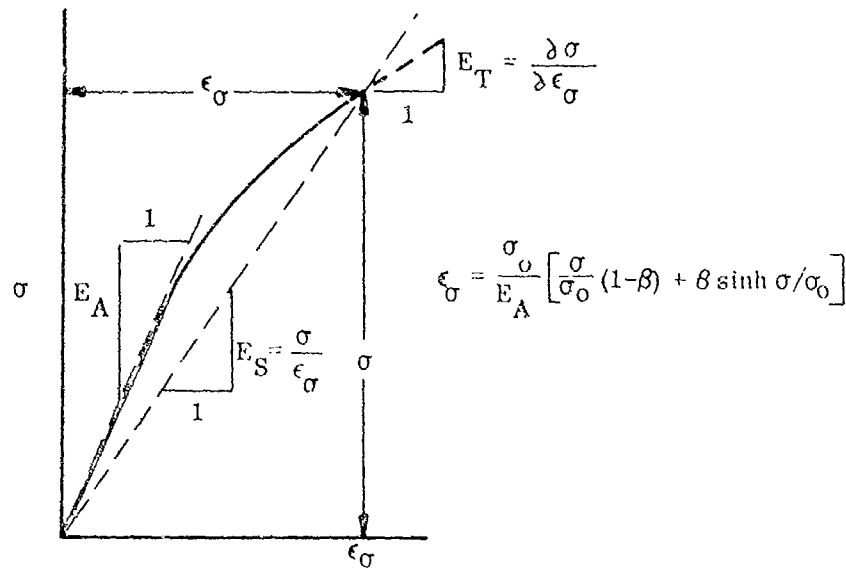


FIGURE 3.2-1 STRESS-STRAIN RELATIONSHIP

The first approximation is to assume that the lateral deflection will increase as a rectangular hyperbola of the form

$$w = \frac{a_{10} w_1}{1 - F/F_1} \quad (7)$$

3.2 (Cont'd)

where w = the lateral deflection
 w_1 = fundamental deflection mode
 a_{10} = initial amplitude of the fundamental mode

(The higher modes need not be considered in the buckling load problem since the column will buckle in the fundamental mode)

F_1 = buckling load for a linear or straight column

where
$$F_1 = \frac{K}{l^2} E_{So} I_o \quad (8)$$

The second approximation is to assume that γ has the magnitude of unity in evaluating the value of α . Under the assumption of small bending stresses, the magnitude of γ should be very close to but slightly lower than 1.

The third approximation is to assume the average axial strain (ϵ_o) to be equal to the value of $F_1/E_{So}A$ at buckling.

The above three approximations are consistent with the assumption of relatively small bending strains and result in a slight underestimation of α and a resulting slight overestimate of \bar{F}_1 .

The final approximation is to evaluate α with the stress distribution which exists at the cross section with the maximum lateral deflection. This assumption is made because the stability is primarily a function of the square of the curvature (see Eq. (3)) which is largest in the vicinity of the maximum lateral deflection (for columns of constant cross sections). This approximation of the bending to axial stress ratio overestimates α and results in an underestimate of \bar{F}_1 and compensates, to some degree, for the overestimates resulting from the small bending approximations.

The approximations result in the following equation

$$\bar{F}_1 = \frac{K}{l^2} E_{So} I_o \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (9)$$

These approximations cannot be rigorously justified but they result in an expression for the stability of the column that can be simply applied to the analysis of a very complex problem and which retains the correct sense for the effect of the parameters. A well controlled test program is recommended to evaluate the accuracy of the resulting formulation and to supply empirical correction factors, as needed.

An examination of the experimental data of Reference 3-5 was conducted in Appendix C of Reference 3-2 and indicated good agreement with the results for specimens tested with eccentricities caused by eccentric loads or thermal gradients. The predicted results were slightly unconservative. Better agreement can possibly be obtained between the experimental results and the theoretical predictions by statistically determining the best value for the expression $(\sqrt{6} a_{10}/r)$. The graphs (Figures 3.3.1-1 through -10) can still be employed by modifying the indicated value of a_{10}/r . Additional test data is necessary, however, to examine the reliability of the above approximations for greater eccentricities and larger buckling stresses than those reported in Reference 3-5.

3.2 (Cont'd)

Using Eq. (9), the average buckling stress becomes

$$\bar{\sigma}_1 = \frac{\bar{F}_1}{A} = E_{So} \frac{Kr^2}{t^2} \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (10)$$

and the average buckling strain becomes

$$\bar{\epsilon}_1 = \frac{\bar{\sigma}_1}{E_{So}} = \frac{Kr^2}{t^2} \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (11a)$$

Since
$$\epsilon_1 = \frac{\sigma_1}{E_{So}} = \frac{F_1/A}{E_{So}} = \frac{K}{t^2} \frac{E_{So} I_o}{E_{So} A} = K \frac{r^2}{t^2} \quad (11b)$$

$$\therefore \frac{\bar{\epsilon}_1}{\epsilon_1} = \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (11c)$$

Letting
$$\bar{\Phi} = \frac{E_A}{\sigma_o} \bar{\epsilon}_1 \quad \text{and} \quad \Phi = \frac{E_A}{\sigma_o} \epsilon_1$$

We obtain from Eqs. (9), (10), and (11c)

$$\frac{\bar{F}_1}{F_1} = \frac{\bar{\sigma}_1}{\sigma_1} = \frac{\bar{\epsilon}_1}{\epsilon_1} = \frac{\bar{\Phi}}{\Phi} = \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (11d)$$

The factor $\left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right]$ approximates the effect of the eccentricity and stress distribution. The effect is more pronounced with larger initial eccentricities or higher average stresses. It should be noted at this time that the factor was determined for sections which were symmetrical about the centroidal axis parallel to the bending axis ($Q_o = 0$). The stability of the column should decrease slightly with increasing values of Q_o/I_o [as indicated by a comparison of the bending stiffness in the plastic range for $Q_o \neq 0$ with $Q_o = 0$ (see Eqs. (A10a) and (A10b) of Reference 3-2)].

3.3 TECHNIQUE

The final formulation for the approximate solution for the buckling of an eccentric column is presented by the non-linear Eqs. (9), (10), or (11) of Sub-section 3.2. It is necessary to resort to graphical solutions in order to solve these equations. In order to simplify the analysis, a nondimensional solution is presented in Figure 3.3.1-1 to -10. The graphs are based upon the stress-strain relationship presented in Section 3 of Reference 3-1.

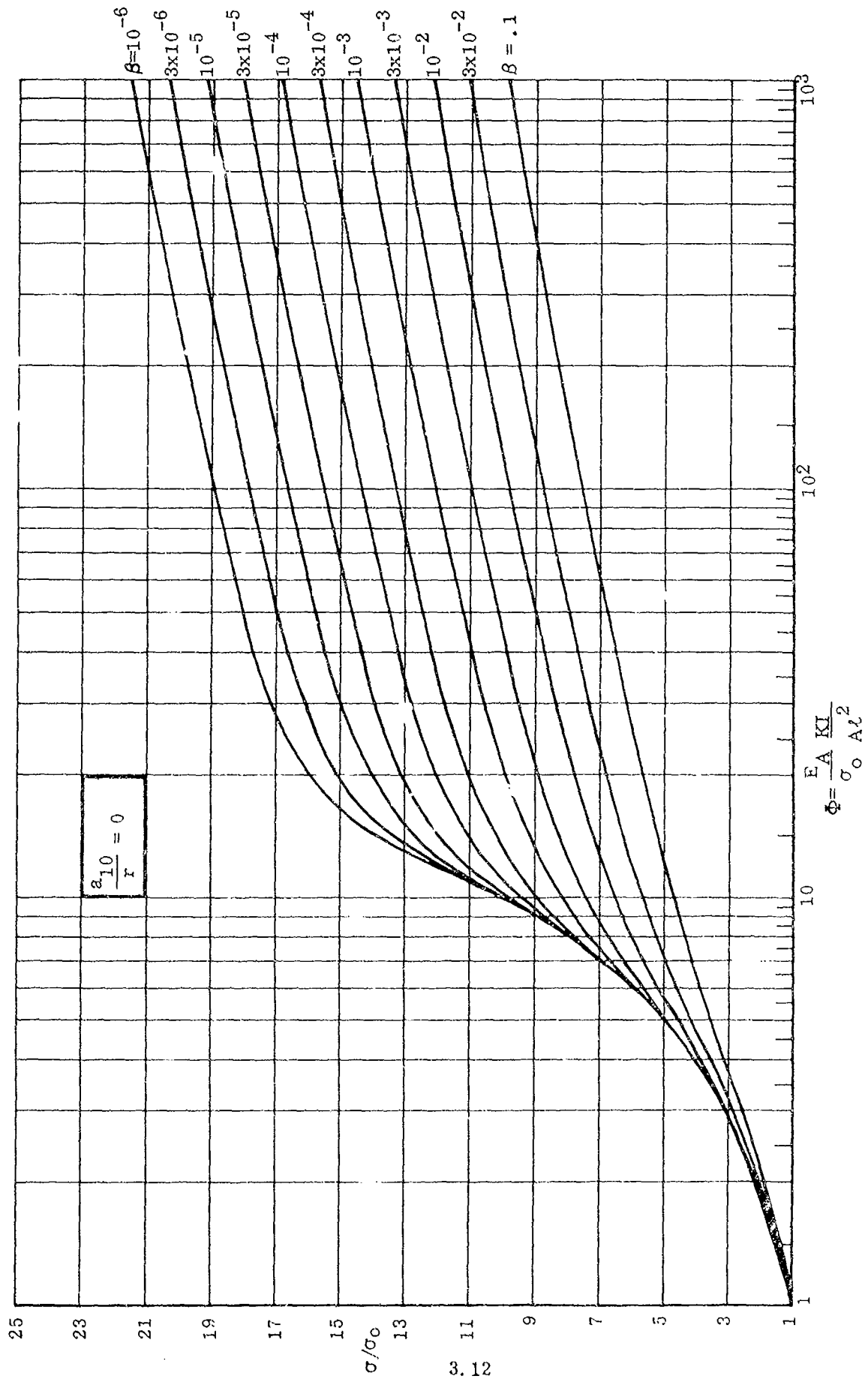
3.3.1 Nondimensional Buckling Curves

Equation (10) of Sub-section 3.2 can be transformed to

$$\left(\frac{\bar{\sigma}_1}{\sigma_o} \right) \frac{1}{\frac{E_s}{E_A} \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right]} = \frac{E_A}{\sigma_o} \frac{K r^2}{l^2} = \frac{E_A}{\sigma_o} \left(\frac{KI}{Al^2} \right) = \Phi \quad (1)$$

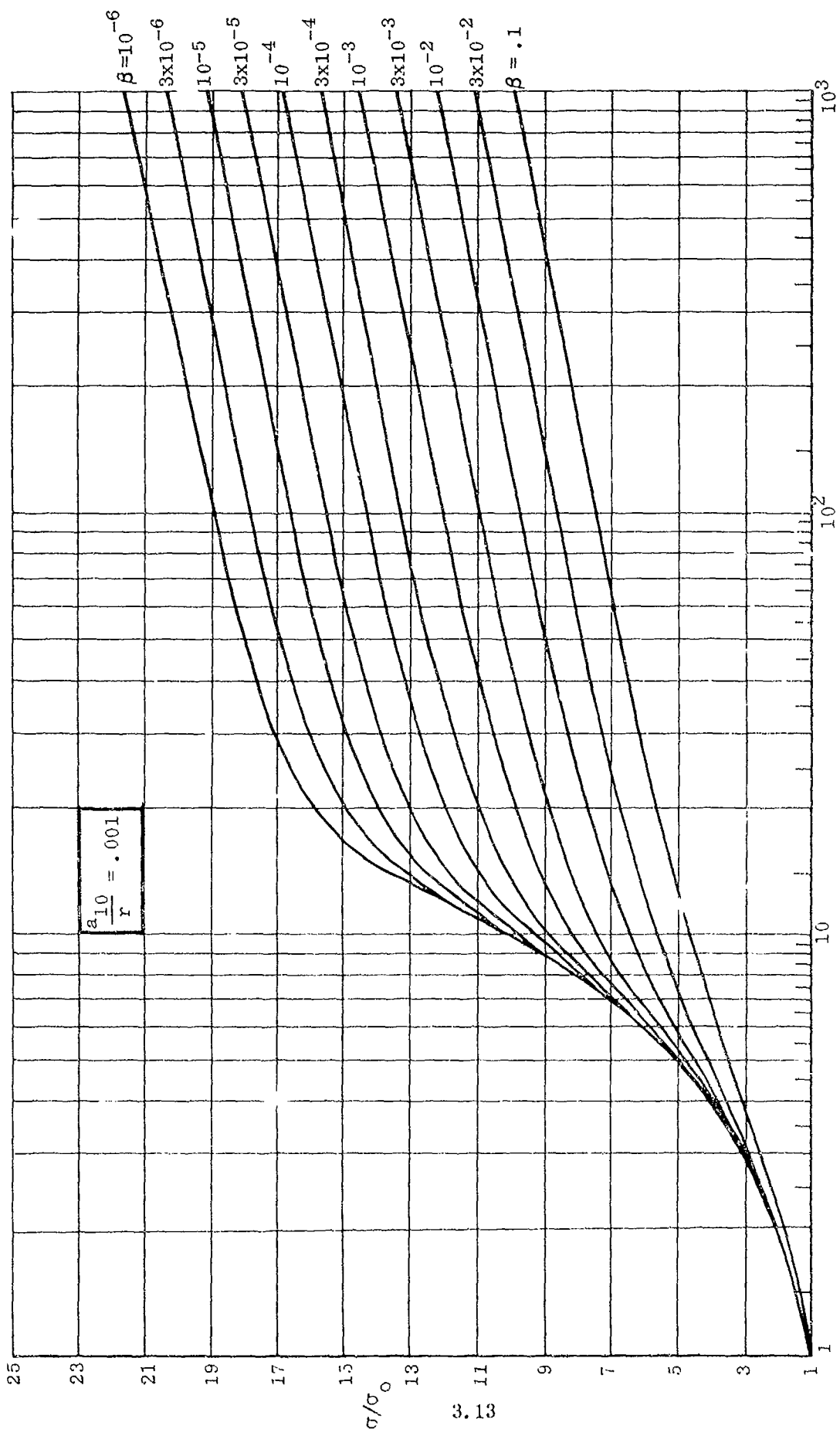
Utilizing the stress-strain relationship of Section 3 of Reference 3-1 $\left(\frac{E_A \epsilon}{\sigma_o} = (1 - \beta) \frac{\sigma}{\sigma_o} + \beta \sinh \frac{\sigma}{\sigma_o} \right)$, it is possible to calculate values of E_s/E_A and E_T/E_s for various values of σ/σ_o and given values of β . For a given value of the eccentricity ratio (a_{10}/r) it is then possible to calculate the value of Φ , which would result in a buckling stress ratio ($\bar{\sigma}_1/\sigma_o$), by direct substitution in Eq. (1). This relationship is plotted in Figures 3.3.1-1 to -10 for a large range of eccentricity ratios. The argument Φ is a function of the material properties (E_A/σ_o) and the geometry and boundary conditions (KI/Al^2). The shape of the stress-strain relationship is represented by the value of β . The method of obtaining the material parameters from a uniaxial test is presented in Section 3 of Reference 3-1. The determination of the initial eccentricity which may be caused by initial waviness, eccentric loads, lateral loading, and thermal gradients is discussed in Paragraph 3.3.2.

The graphs for small eccentricity ratios should result in fairly good agreement with experimental data since the approximations employed should be satisfactory. The graphs for large eccentricity ratios are less accurate and are expected to be unconservative.



3.12

FIGURE 3.3.1-1 NON-DIMENSIONAL BUCKLING OF ECCENTRIC COLUMNS



$$\phi = \frac{EA KI}{\sigma_0 A l^2}$$

FIGURE 3.3.1-2 BUCKLING OF ECCENTRIC COLUMNS

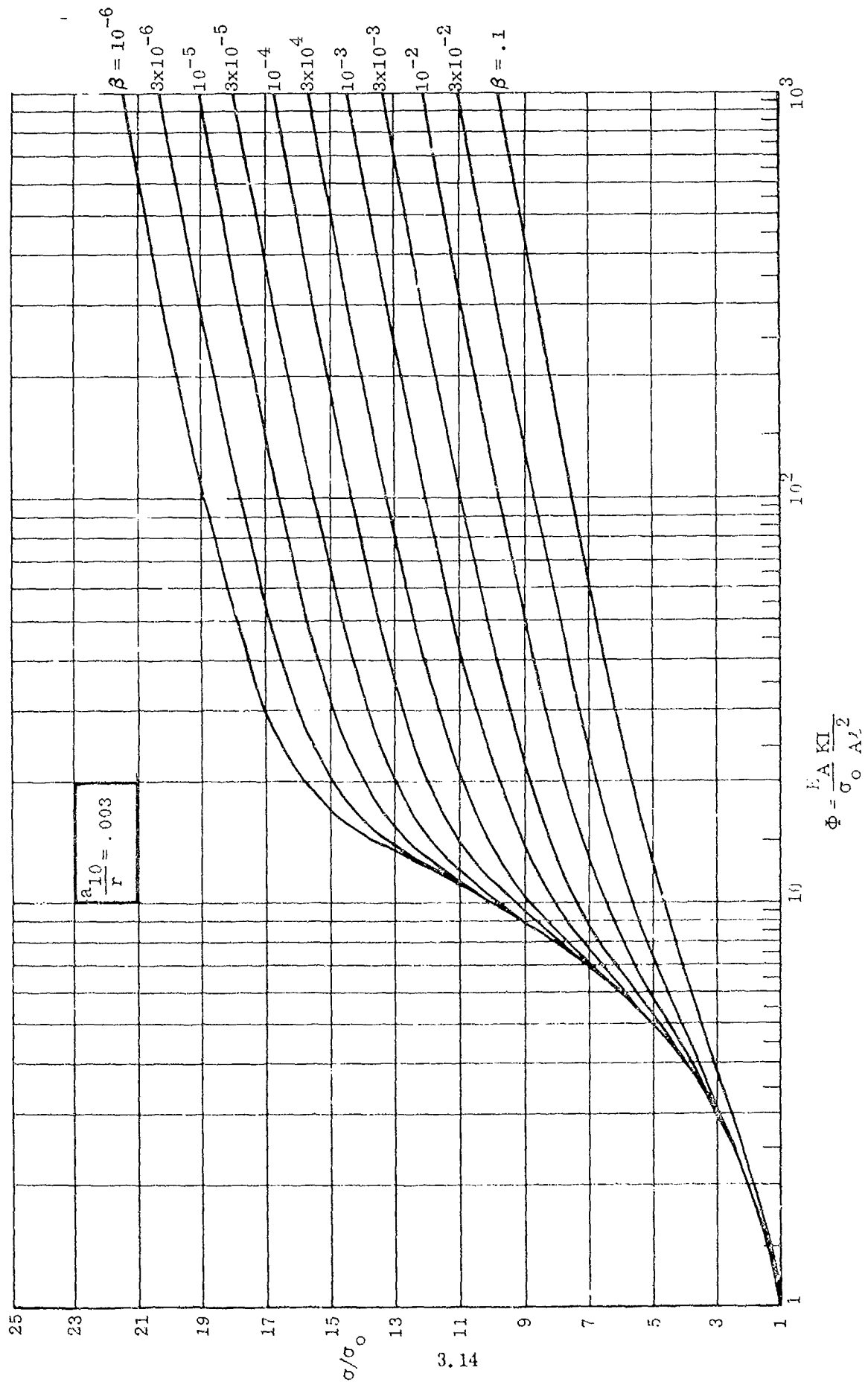


FIGURE 3.3.1-3 BUCKLING OF ECCENTRIC COLUMNS

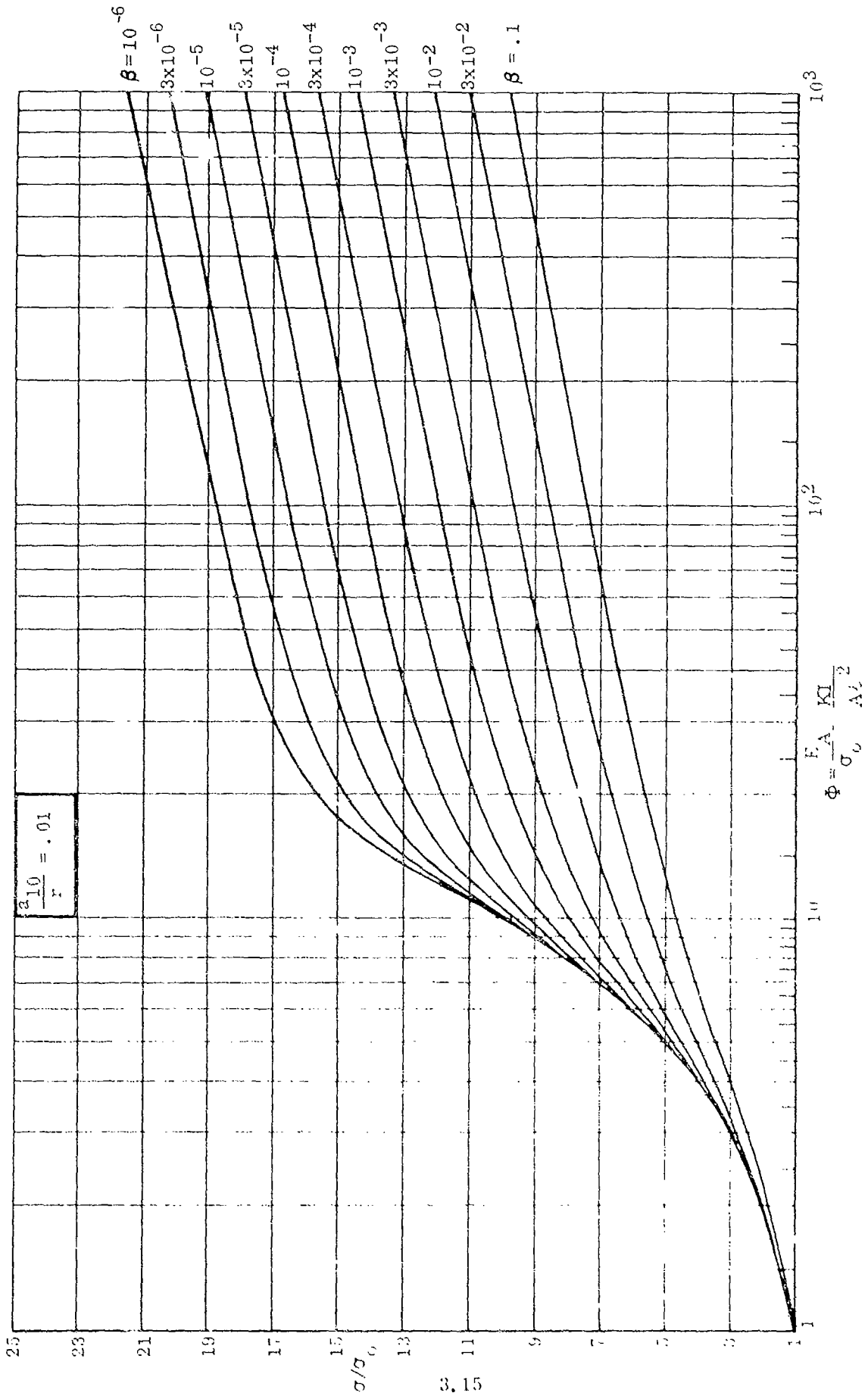


FIGURE 3.5.1-4 BUCKLING OF ECCENTRIC COLUMNS

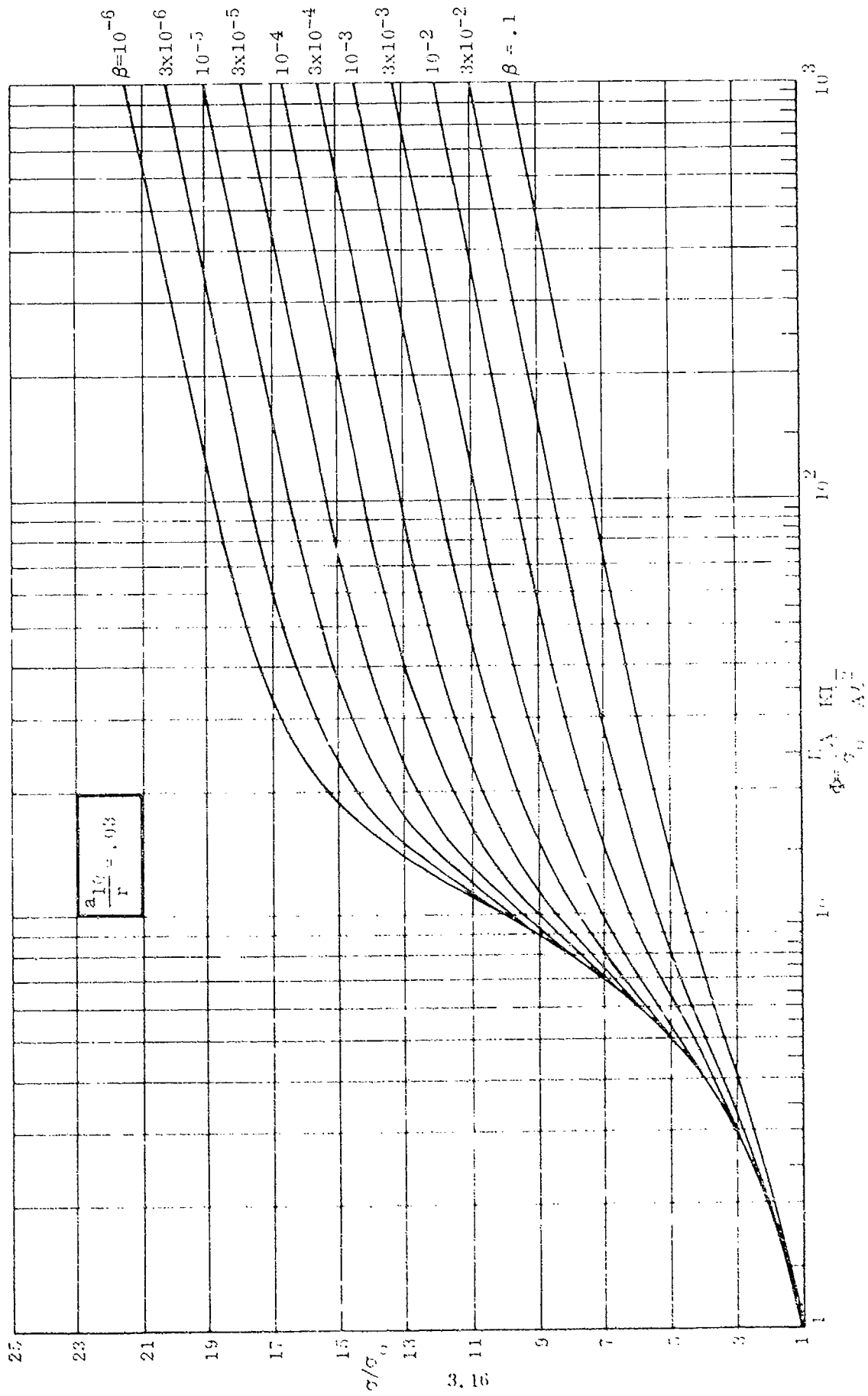


FIGURE 3.3.1-5 BUCKLING OF ECCENTRIC COLUMNS

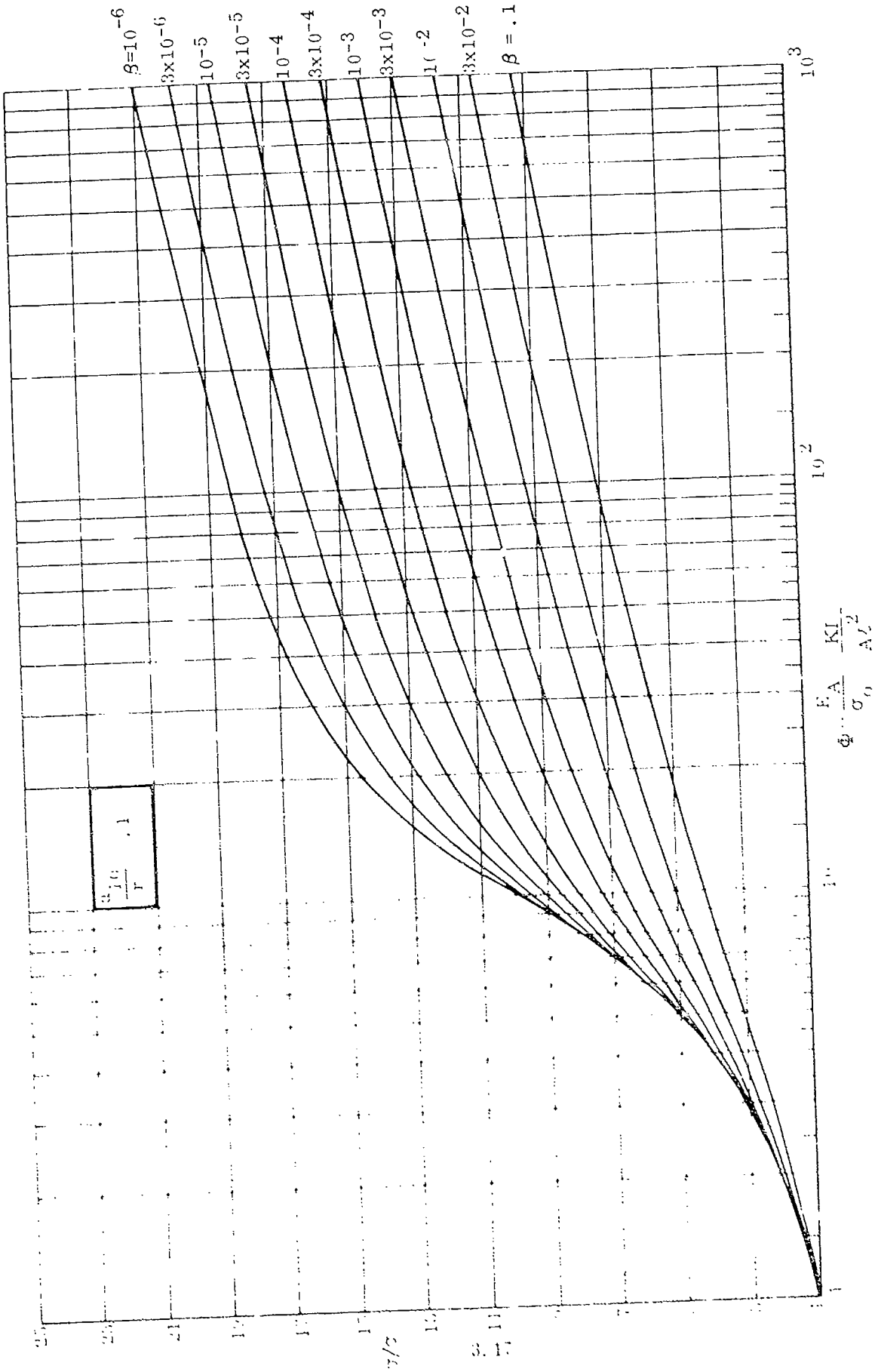


FIGURE 3.3.1-6 BECKLING OF ECCENTRIC COLUMNS

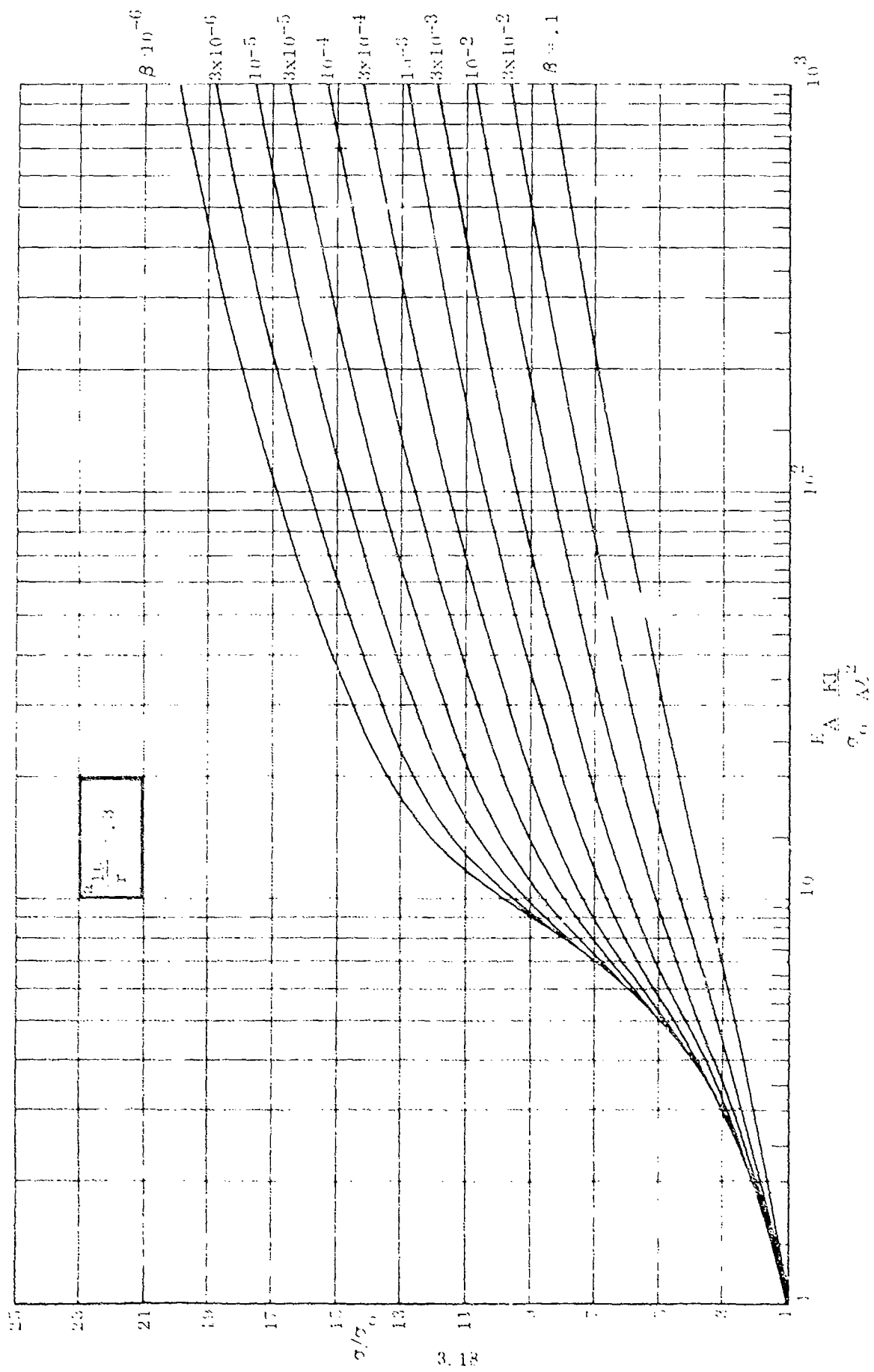


FIGURE 3.6.17 BUCKLING OF ECCENTRIC COLUMNS

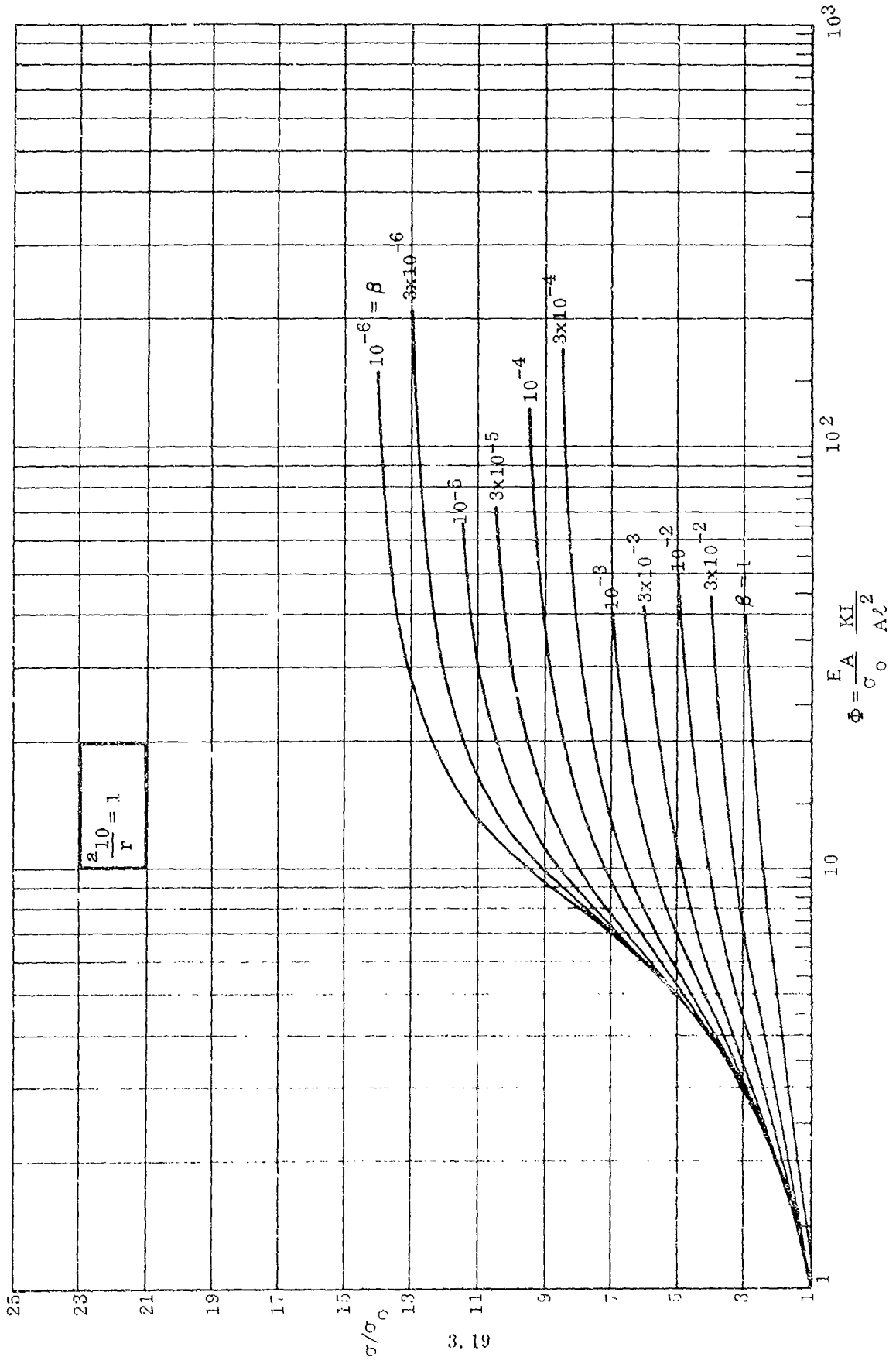


FIGURE 3.3.1-8 BUCKLING OF ECCENTRIC COLUMNS

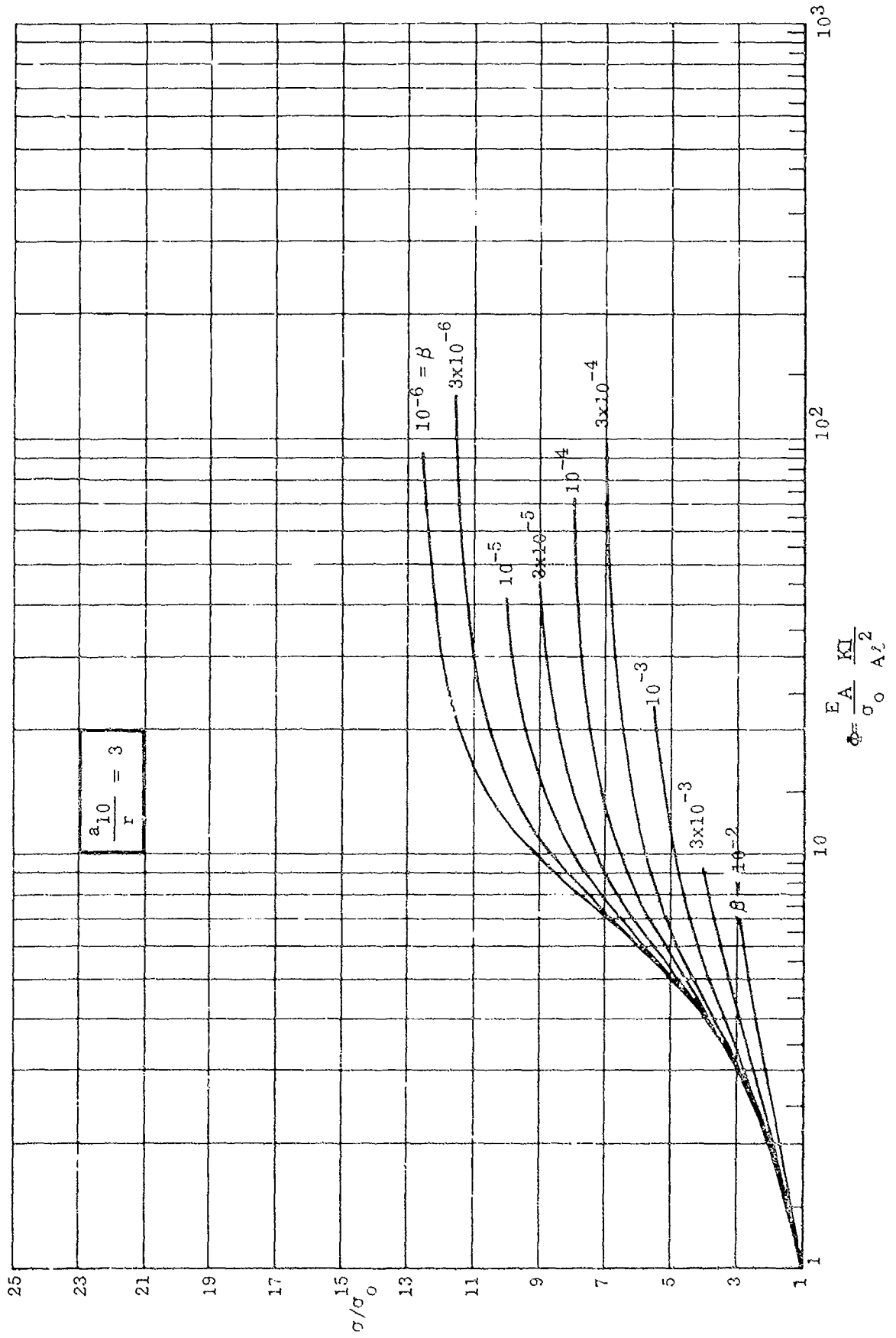


FIGURE 3.3.1-9 BUCKLING OF ECCENTRIC COLUMNS

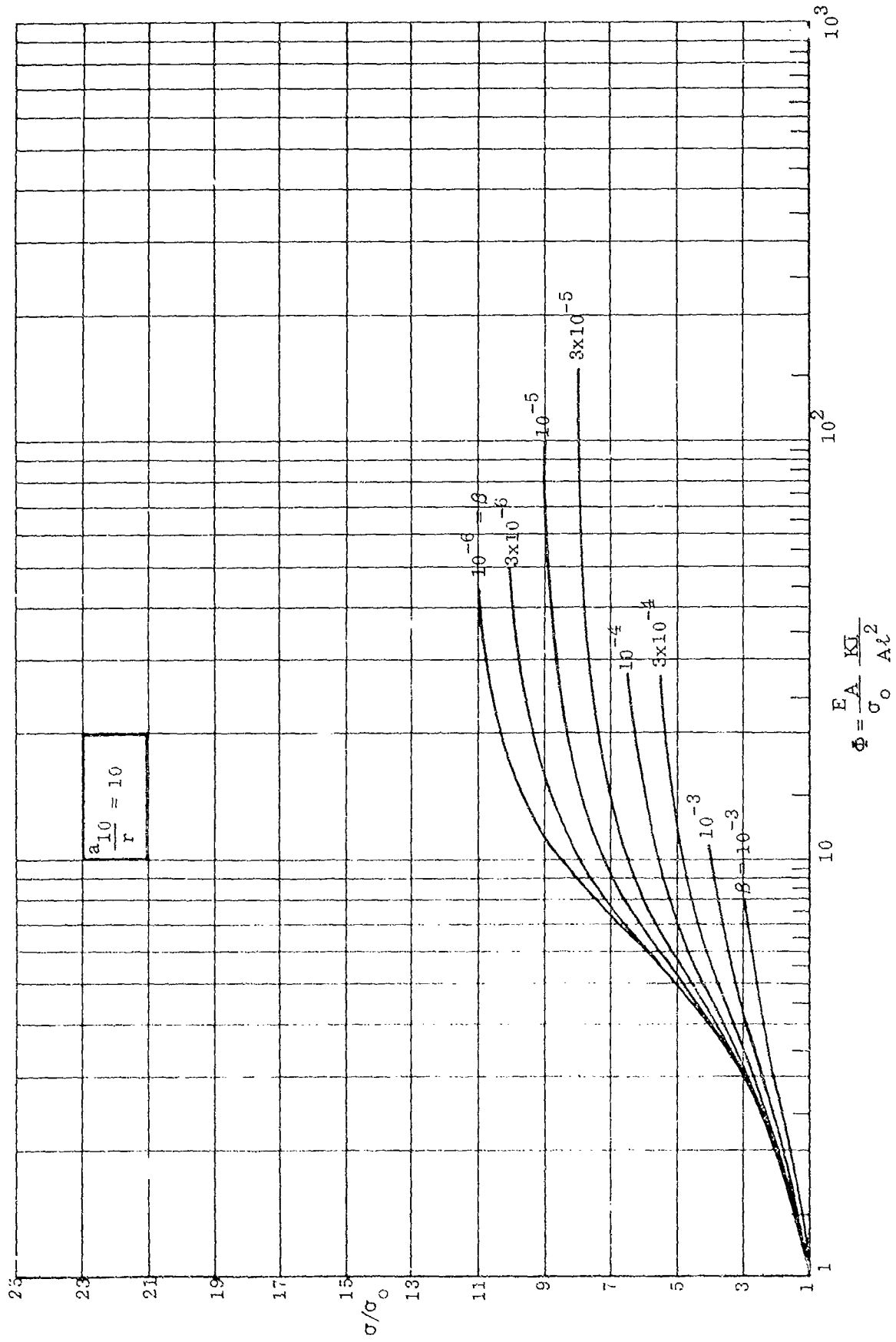


FIGURE 3.3.1-10 BUCKLING OF ECCENTRIC COLUMNS

3.3.2 Initial Eccentricity

The initial eccentricity must be evaluated in order to use the analysis curves.

Paragraph 2.3.3 indicated that the amplitude of the fundamental modes can be obtained by a weighted integration of the actual lateral deflection or by matching the deformation at discrete points.

Assuming that it is possible to express the initial eccentricity (w_0 due to shape, axial load location, lateral loads and temperatures) as an analytical expression, it is possible to determine the magnitude of the first mode by utilizing the orthogonality between the lateral deflection and curvature (Eq. (2) of Paragraph 2.2.1),

Thus, if

$$w_0 = \sum a_{10} w_i \quad (1a)$$

then from Eq. (2) of Paragraph 2.2.1 we obtain for constant bending stiffness ($C=1$)

$$a_{10} = \frac{\int_0^l w_0 \kappa_i dx}{\int_0^l w_i \kappa_i dx} \quad (1b)$$

The following formulae are applicable to columns of constant bending stiffness:

when

$$w_0 = \sum m_j \xi^j \quad (2)$$

then for pin ends

$$a_{10} = 2 \sum m_j S(j, 1) \quad (3a)$$

while for clamped ends

$$a_{10} = 2 \sum m_j C(j, 1) \quad (3b)$$

when

$$\kappa_T = \frac{1}{l^2} \sum m_j \xi^j \quad (4)$$

3.3.2 (Cont'd)

then for pin ends (Reference Eq. (3a) of Paragraph 2.3.3.1.1)

$$w_0 = \sum \frac{m_j}{(j+1)(j+2)} (\xi^{j+2} - \xi) \quad (5a)$$

and

$$a_{10} = 2 \sum \frac{m_j}{(j+1)(j+2)} [S(j+2, 1) - S(1, 1)] \quad (6a)$$

while for clamped ends (Reference Eq. (4c) of Paragraph 2.3.3.1.1)

$$w_0 = \sum \frac{m_j}{(j+2)(j+2)} (\xi^{j+2} - j \xi^3 - (1-j) \xi^2) \quad (5b)$$

and

$$a_{10} = 2 \sum \frac{m_j}{(j+2)(j+1)} [C(j+2, 1) - j C(3, 1) - (1-j) C(2, 1)] \quad (6b)$$

Values of $S(j, 1)$ and $C(j, 1)$ are found in Tables 2.3.3.1.2-1 and -2 and are summarized in Table 3.3.2-1 below.

TABLE 3.3.2-1. FOURIER COEFFICIENTS FOR POLYNOMIALS

j	S(j, 1)	C(j, 1)
0	.6366	-
1	.3183	0
2	.1893	.05066
3	.1248	.07599
4	.08814	.08592
5	.06541	.08815
6	.05038	.08669

3.3.2 (Cont'd)

For a pinned end column of constant $E\bar{I}$ and constant linear thermal gradient

$$x_T = \frac{-\Delta \alpha T}{h} = \frac{m_0 \xi^0}{l^2} \text{ we obtain from Eqs. (5a) and (6a).}$$

$$w_0 = \frac{m_0}{(0+1)(0+2)} (\xi^{0+2} - \xi) = \frac{m_0}{2} (\xi^2 - \xi)$$

$$\therefore w_0(l/2) = \frac{m_0}{2} (.5^2 - .5) = -\frac{m_0}{8} = \frac{l^2 \Delta \alpha T}{8h}$$

and
$$a_{10} = 2 \frac{m_0}{(1)(2)} [S(2, 1) - S(1, 1)]$$

$$\therefore a_{10} = 2 \frac{m_0}{2} [.1893 - .3183] = -.129 m_0 = \frac{.129 l^2 \Delta \alpha T}{h} \quad (7)$$

A second method of obtaining the initial eccentricity is by matching the displacements at a discrete number of points. This requires the solution of the set of simultaneous equations $w_0(\xi) = \sum a_{10} w_i(\xi)$ for the value of a_{10} and is described in Paragraph

2.3.3.2. The approximation $a_{10} \approx w_0(1/2)$ is a solution where only one point, the deflection at the center of the column, was matched. The accuracy would increase with the number of displacements which are matched, although the matching of the center deflection of a pin ended column with a uniform lateral load or thermal gradient results in a satisfactory determination of the initial eccentricity. The accuracy also depends upon the form of the lateral deflection. For example, a matching of the mid-length deflection for a pin-ended column results in a more accurate amplitude of the fundamental mode caused by a uniform thermal gradient (parabolic) than by a constant eccentricity.

3.4 SPECIAL CASES

There exist special situations for which the exact solution is known or for which engineering approximations have been accepted because of experimental data. The plausibility of the approximate formulation, presented in this section, is reviewed by degenerating it to those situations with which the analyst has had some experience.

3.4.1 Linear Material

A material whose modulus is independent of the stress levels ($\frac{\partial E_S}{\partial \sigma} = 0$, $E = E_T = E_S$) is described as a linear material. The value of $(1 - E_{T0}/E_{S0})$ is identically zero and no reduction of bending stiffness occurs. The eccentricity (a_{10}/r) does not affect the stability and buckling occurs when the average axial strain attains a value of $K \left(\frac{r^2}{l^2}\right)$ (with the load equal to $F_E = \frac{K}{l^2} EI$) provided no fiber is stressed to its ultimate strength.

This corresponds to the classical "Euler Column" whose buckling load and strain are well defined because of the linearity of the material beyond the buckling stress. A column with a large slenderness ratio (l/r) would behave in the manner described. Although the nondimensional stress-strain relationship includes the linear case (letting $\beta = 0$), it is recommended that the actual value of β for the structural material be used even for the case of low buckling stresses. This will permit an approximation of the effects of eccentricities in causing extreme fiber stresses that may be beyond the "proportional limit" even when the average stress is very low. The deviation of the stability of a structure composed of material which is non-linear to a slight degree from that of a linear material will be insignificant for small eccentricities but may become significant for large eccentricities.

3.4.2 Perfectly Straight Columns

The perfectly straight column does not bend when subjected to an axial load. This condition is virtually impossible to attain experimentally but has some theoretical value as a mathematical model. The column will be stable as long as the axial strain remains below the critical strain $K \frac{r^2}{l^2}$ since a_{10} is zero. Any lateral excitation of the column below this value of the critical strain will dampen out whereas it will magnify and become excessively large when the critical strain is reached or exceeded by the column.

3.4.3 Small Initial Eccentricities

The effect of the eccentricity upon the stability of the column is small when the eccentricity ratio (a_{10}/r) is small. An upper bound upon the stability load will always be F_1 which corresponds to the employment of $E_{S0} I_0$ as the effective buckling stiffness of the cross section and a critical strain of $K \frac{r^2}{l^2}$.

Under the conditions of continuous loadings ($E_{S0} \approx E_{T0}$) and small eccentricities ($1 \gg \left(\frac{\kappa M z}{\epsilon_0}\right)^2$), the lower bound of the stability load will be $F_1 \left(\frac{E_{T0}}{E_{S0}}\right) = \frac{K}{l^2} E_{T0} I_0$ which

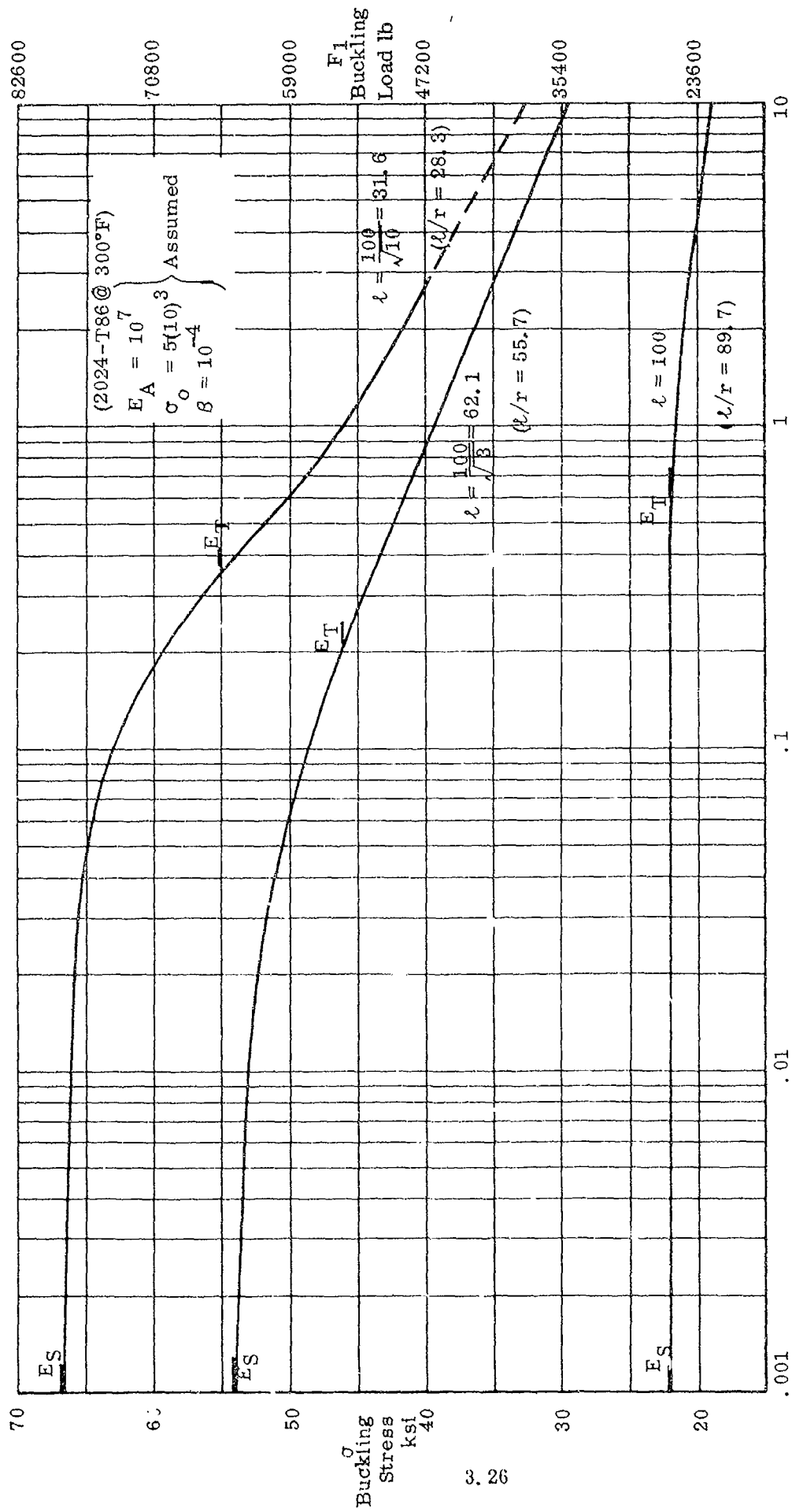


FIGURE 3.4.3-1 EFFECT OF ECCENTRICITY ON STABILITY

3.4.3 (Cont'd)

corresponds to the employment of $E_{T_0} I_0$ as the effective buckling stiffness of the cross section and a critical strain of $\left(\frac{E_{T_0}}{E_{S_0}}\right) \left(K \frac{r^2}{l^2}\right)$

This lower bound is satisfactory provided the eccentricity is not too large. The bending stresses become more significant in determining the bending stiffness as the eccentricity increases. An eccentricity ratio exists for each column beyond which the tangent modulus stability ceases to be a lower bound. This eccentricity ratio can be obtained by equating the tangent modulus load to the eccentric column load.

Equality results when
$$\frac{a_{10}}{r} = \sqrt{\frac{1 - E_{T_0}/E_{S_0}}{6}}$$

A value of (a_{10}/r) less than $\sqrt{\frac{1 - E_{T_0}/E_{S_0}}{6}}$ results in a tangent modulus load

that is conservative with respect to the eccentric column load (i. e., $E_{T_0} I_0 \leq \bar{E} I$) whereas a larger eccentricity ratio will make the tangent modulus load unconservative. The value of the critical eccentricity ratio depends upon the material and the slenderness ratio of the column. This is illustrated in Figure 3.4.3-1 which indicates the effect of eccentricity on the stability of the columns described in the illustrative problems. The tangent modulus loads stops being conservative for eccentricity ratios between .2 and .5 for the columns analyzed, and the critical eccentricity ratios are indicated in Figure 3.4.3-1.

3.5 ILLUSTRATIVE PROBLEMS

The computational techniques are illustrated in the following problems:

Find the buckling loads of the following pin ended columns of constant cross section illustrated in Figure 3.5-1.

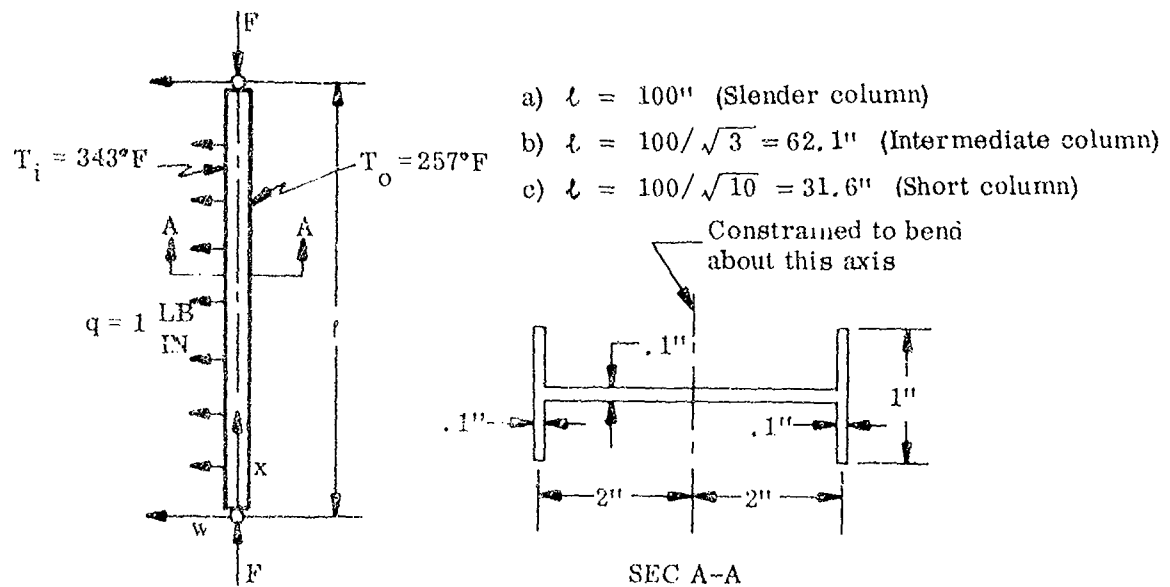


FIGURE 3.5-1 COLUMN SUBJECTED TO LOAD AND TEMPERATURE

3.5 (Cont'd)

The temperature is assumed constant on each face of the column with a linear gradient through the thickness. Material parameters are assumed for 2024-T86 at the mean temperature of 300°F to illustrate the computational technique. The actual values must be determined from standard material tests.

Material Properties

The following material properties are assumed:

$$\begin{aligned}\alpha &= 12(10)^{-6} \text{ in/in/}^\circ\text{F} \\ E_A &= 10^7 \text{ psi} \\ \sigma_o &= 5000 \text{ psi} \\ \beta &= 10^{-4}\end{aligned}$$

Initial Geometry

$$\begin{aligned}A &= 0.1(1.0 + 3.9 + 1.0) = .59 \text{ in}^2 \\ I_o &= 2(.1)(2)^2 + \frac{1}{12}(.1)(3.9)^3 = 1.30 \text{ in}^4 \\ r^2 &= I_o/A = 2.2 \text{ in}^2\end{aligned}$$

$$r = 1.15 \text{ in}$$

$$\Phi = \frac{E_A}{\sigma_o} \frac{KI}{Al^2} = \frac{E_A}{\sigma_o} K \frac{r^2}{l^2} = \frac{10^7}{5(10)^3} \pi^2 \frac{2.2}{l^2} = \frac{4.4}{(l/100)^2}$$

$$\epsilon_1 = K \frac{r^2}{l^2} = \pi^2 \frac{2.2}{l^2} = \frac{22}{l^2}$$

Initial Eccentricity

Because the temperature distribution is linear through the thickness we can utilize Eq. (4) of Paragraph 3.3.2.

$$(a_{10})_T = .129 \frac{\Delta\alpha T}{h} l^2 = .129 \frac{\alpha(T_i - T_o)}{h} l^2 \quad (1)$$

$$\therefore \frac{(a_{10})_T}{r} = \frac{.129 \alpha (T_i - T_o) l^2}{r h} = \frac{.129(12)(10)^{-6} (343-257)}{(1.15)(4.1)} = .28 \left(\frac{l}{100} \right)^2$$

3.5 (Cont'd)

From standard reference texts the lateral deflection due to a uniform load q is

$$(w_0)_q = \frac{q l^4}{24 EI} (\xi - 2\xi^2 + \xi^4)$$

From Eq. (3a) of Paragraph 3.3.2

$$(a_{10})_q = 2 \frac{q l^4}{24 EI} [S(1,1) - 2S(2,1) + S(4,1)]$$

From Table 3.3.2-1

$$(a_{10})_q = 2 \frac{q l^4}{24 EI} [.3187 - 2(.1248) + (.08814)] = .013066 \frac{q l^4}{EI} \quad (2)$$

$$\therefore \frac{(a_{10})_q}{r} = \frac{.013066 (1) l^4}{(10)^7 (1.3)(1.15)} = .08 \left(\frac{l}{100}\right)^4$$

It should be noted that

$$(w_0)_q \text{ (at } x = l/2) = \frac{5}{384} \frac{q l^4}{EI} = .01302 \frac{q l^4}{EI} \approx .013066 \frac{q l^4}{EI} = (a_{10})_q$$

and

$$(w_0)_T \text{ (at } x = l/2) = .125 \frac{\Delta \alpha T}{h} l^2 \approx .129 \frac{\Delta \alpha T}{h} l^2 = (a_{10})_T$$

This indicates the magnitude of the errors which can be introduced by approximating the amplitude of the fundamental mode by the deflection at the center of the column are quite small for a pin ended column subjected to uniform lateral load or thermal gradient.

The initial eccentricity ratio is then calculated as follows:

$$\left(\frac{a_{10}}{r}\right) = \frac{(a_{10})_T}{r} + \frac{(a_{10})_q}{r} = .28 \left(\frac{l}{100}\right)^2 + .08 \left(\frac{l}{100}\right)^4$$

For

$$l = 100 \quad ; \quad \frac{a_{10}}{r} = .28 + .08 = .36$$

$$l = 100/\sqrt{3} \quad ; \quad \frac{a_{10}}{r} = .093 + .008 = .101$$

$$l = 100/\sqrt{10} \quad ; \quad \frac{a_{10}}{r} = .028 + .0008 = .0288$$

3.5 (Cont'd)

Stability

Referring to the appropriate graph of Figures 3.3.1 results in the determination of $(\bar{\sigma}_1/\sigma_o)$ for the given values of β , Φ , and (a_{10}/r) . The effect of the eccentricity, which is expressed in Eq. (11d) of Paragraph 3.2, can be evaluated by comparing the given value of Φ to the value $\bar{\Phi}$ which corresponds to $(\bar{\sigma}_1/\sigma_o)$ for a straight column (Figure 3.3.1-1

with $a_{10}/r = 0$). This value of $\bar{\Phi} = \frac{E_A}{\sigma_o} \bar{\epsilon}_1$, corresponds to a material and geometry parameter for a straight column which would buckle at the same stress as the eccentric column and is a measure of the average axial strain in the column.

a) For $l = 100$, $E_A = 10^7$, $\sigma_o = 5000$, and $\beta = .0001$ we obtain
 $\Phi = 4.4$, $\epsilon_1 = .0022$, and $\frac{a_{10}}{r} = .36$.

From Figures 3.3.1-7 and -8

$$\frac{\bar{\sigma}_1}{\sigma_o} = 4.4$$

$$\bar{\sigma}_1 = 4.4(5000) = 22000 \text{ psi}$$

$$\bar{F}_1 = \bar{\sigma}_1 A = 22000 (.59) = 13000\#$$

From Figure 3.3.1-1

$$\bar{\Phi} = 4.4$$

$$\bar{\Phi}/\Phi = 4.4/4.4 = 1 \text{ (no effect of eccentricity because of the large slenderness ratio)}$$

and

$$\bar{\epsilon}_1 = \epsilon_1 (\bar{\Phi}/\Phi) = .0022$$

b) Similarly for $l = 100/\sqrt{3}$

$$\Phi = 13.2$$
, $\epsilon_1 = .0066$, and $\frac{a_{10}}{r} = .101$

From Figure 3.3.1-6

$$\frac{\bar{\sigma}_1}{\sigma_o} = 9.7$$

$$\bar{\sigma}_1 = 9.7(5000) = 48500 \text{ psi}$$

$$\bar{F}_1 = 48500(.59) = 28500\#$$

3.5 (Cont'd)

and from Figure 3.1.1-1

$$\bar{\Phi} = 10.5 \quad \bar{\Phi}/\Phi = \frac{10.5}{13.2} = .80$$

$$\therefore \bar{\epsilon}_1 = .0068(.80) = .0053$$

c) For $\ell = 100/\sqrt{10}$

$$\Phi = 44, \quad \epsilon_1 = .022, \quad \text{and} \quad \frac{a_{10}}{r} = .0288$$

$$\therefore \bar{\sigma}_1/\sigma_0 = 13.1$$

$$\bar{C}_1 = 13.1(5000) = 65500 \text{ psi}$$

$$\bar{F}_1 = 65500(.59) = 38500 \#$$

$$\bar{\Phi} = 38 \quad \bar{\Phi}/\Phi = 38/44 = .87$$

$$\bar{\epsilon}_1 = .022(.87) = .019$$

The eccentricity ratios a_{10}/r were sufficiently close to available values that there was no need to interpolate between graphs. Plots can be made to show the variation in the stability of a given column with the initial eccentricity and can be utilized if interpolation is required. These plots are shown in Figure 3.4.3-1 for the three (long, intermediate and short) columns analyzed above. The values corresponding to an effective modulus of E_S are obtained from Figure 3.3.1-1 ($\frac{a_{10}}{r} = 0$) and the values corresponding to an effective

modulus of E_T are obtained from Figure 9.2.1-2 of Reference 3-1. The plots indicate that $\bar{EI} = E_{S0} I_0$ is always unconservative and $\bar{EI} = E_{T0} I_0$ becomes unconservative in

the vicinity of $.2 \approx \frac{a_{10}}{r} \approx .5$ for the analyzed columns. It should be noted that the reduction in the stability of the long column is least while the reduction in the stability of the intermediate column is greatest with the short column affected to an intermediate degree. This is because the reduction in the stability of the eccentric column depends both upon the eccentricity and the stress levels attained by the column. The long column has the largest eccentricity ratio but has very low axial stresses, the short column has very high axial stresses but very small eccentricities; while the intermediate column has high stresses and moderately high eccentricities.

It is interesting to note the contributions of the various factors in reducing the stability of the column below the "Euler Load" $\left(P_E = \frac{KEI_0}{\ell^2} \right)$. The factors reducing the stability stiffness can be roughly divided into two parts. The first part represents the reduction in the axial stiffness due to the plasticity caused by average stresses which are above the proportional limit of the material. The second part represents additional reductions in

3.5 (Cont'd)

the stability stiffness because of the eccentricity causing a shift in the neutral axis and a rate of change of the bending stiffness.

Equation (2b) of Paragraph 3.2 can be transformed to the following approximate form in order to evaluate quantitatively the destabilizing effects of eccentricity and plasticity.

$$\frac{\bar{F}_1}{F_E} = \frac{E_{S_0} I_0}{EI_0} \left(\frac{\bar{EI}}{E_{S_0} I_0} + \frac{\kappa}{E_{S_0} I_0} \frac{\partial \bar{EI}}{\partial \kappa} \right) \approx \left(\frac{E_{S_0} I_0}{EI_0} \right) \left(\frac{\bar{EI}}{E_{S_0} I_0} \right) \left(1 - \frac{\kappa}{E_{S_0} I_0} \frac{\partial \bar{EI}}{\partial \kappa} \right)$$

where

\bar{F}_1 is the buckling load of the eccentric column

F_E is the Euler buckling load

$\frac{E_{S_0} I_0}{EI_0}$ is the factor representing the reduction in axial stiffness

$\frac{\bar{EI}}{E_{S_0} I_0}$ is the effect of the shifting of the neutral axis

$\left(1 - \frac{\kappa}{E_{S_0} I_0} \frac{\partial \bar{EI}}{\partial \kappa} \right)$ is the effect due to the change in the bending stiffness

The numerical calculations for the destabilizing effects are summarized in Table 3.5-1.

TABLE 3.5-1 STABILITY RATIOS

DESTABILIZING PHENOMENON	LENGTH OF COLUMN		
	$l=100''$	$l=100/\sqrt{3}$	$l=100/\sqrt{10}$
<u>Reduction in Axial Stiffness</u> (Plasticity Effect) $\frac{E_{So} I_o}{EI_o} = \frac{F_1}{F_E}$	1.00	.73	.30
<u>Further Reductions in Stability Stiffness</u> (Eccentricity Effects)			
a) Shift of Neutral Axis $\frac{\overline{EI}}{E_{So} I_o} = 1 - 2\alpha^2 r^2$	1.00	.93	.955
b) Rate of Change of Bending Stiffness $1 - \frac{\kappa}{E_{So} I_o} \frac{\partial EI}{\partial \kappa} = 1 - 4\alpha^2 r^2$	1.00	.87	.91
c) Critical Strain Ratio-Total Eccentricity Effect $\frac{\overline{\epsilon}_1}{\epsilon_1} = 1 - 6\alpha^2 r^2 = \frac{\overline{F}_1}{F_1}$	1.00	.80	.865
<u>Stability Stiffness Reduction</u> $\frac{\overline{F}_1}{F_E} = \frac{\overline{F}_1}{F_1} \frac{F_1}{F_{L_1}}$	1.00	.58	.26

3.6 REFERENCES

- 3-1 Switzky, H., Forray, M., and Newman, M., "Thermo-Structural Analysis Manual"-Volume I, Republic Aviation Report No. RAC 679-1, September 1960, revised November 1961 (to be published as WADD TR 60-517, Vol. I).
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- 3-5 Winkler, J. E., "Effect of a Transverse Gradient on Column Strength", Society for Experimental Stress Analysis Paper No. 674, May 1961.

SECTION 4
AXISYMMETRIC LARGE DEFLECTIONS OF
CIRCULAR PLATES SUBJECTED TO THERMAL AND MECHANICAL LOADS

by

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M. Forray

SECTION 4
AXISYMMETRIC LARGE DEFLECTIONS OF
CIRCULAR PLATES SUBJECTED TO THERMAL AND MECHANICAL LOADS

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SECTION 4 - AXISYMMETRIC LARGE DEFLECTIONS OF
CIRCULAR PLATES SUBJECTED TO THERMAL AND MECHANICAL LOADS

4.1 SUMMARY

This report is concerned with the nonlinear axisymmetric analysis of circular plates with in-plane edge restraint. Both temperature and mechanical loads are accommodated as an extension of investigations performed for the isothermal mechanical loading problem (References 4-1 through 4-4). An exact mathematical formulation within the framework of the v. Karman large strain-displacement relations (Reference 4-5) is developed. The equilibrium equations and boundary conditions are then derived by utilizing the calculus of variations for arbitrary axisymmetrical temperatures and normal distributed loading. The satisfaction of equilibrium and compatibility equations requires the solution of two simultaneous nonlinear ordinary differential equations subject to the prescribed boundary conditions. Analytical solutions of such equations are apparently not possible and therefore numerical procedures must be employed.

A finite difference procedure utilizing "relaxed iterations," developed by H. Keller and E. Reiss (Reference 4-4), and employed by them for the solution of isothermal problems with apparently unlimited load parameter ranges, is used here for combined thermo-mechanical problems. Numerical results are presented for the special case of a simply supported circular plate with radially immovable boundaries, subject to a uniform pressure and an arbitrary temperature variation through the thickness (no planform variation). These results have been obtained for a large range of temperature and load parameters. However, because of space limitations, only a limited amount of data is presented in this report.

4.2 INTRODUCTION

One of the basic assumptions of the classical linear theory of plates is that bending action does not induce significant midplane stretching. It is further assumed that stresses and deformations produced by loads and restraints in the midplane are superposable on the bending solution. Thus, coupling between the two effects is not accommodated by the classical theory. When the deflections are not small compared to the plate thickness, midplane stretching is no longer insignificant, resulting in a nonlinear interaction between bending and membrane stresses. Therefore, large deflection theory must be employed.

It is the purpose of this report to investigate the axisymmetric large deflection problem for circular plates.

The general formulation presented considers arbitrary axisymmetric temperature and pressure variation, where the von Karman large strain-displacement relations (Reference 4-5) are utilized. These assume infinitesimal strains and finite but small normal deflections ($(\frac{dw}{dr})^2$ is of the order of the strains, but small compared to unity). The remaining assumptions are those of classical plate theory. This formulation is more complete than the conventional linear theory in that the results are valid for deflection magnitudes several times the plate thickness. Moreover, buckling and postbuckling behavior are embodied in the analysis.

4.2 (Cont'd)

Numerical results in nondimensional tabular and graphical form are given for a simply supported circular plate, with full boundary restraint to radial movement, subjected to uniform pressure and arbitrary temperature variation through the thickness.

The following symbols are used throughout this section:

b	Plate radius
h	Plate thickness
r	Radial coordinate
\bar{r}	$\frac{r}{b}$, nondimensional radial coordinate
u	Midplane radial displacement
u_b	Midplane radial displacement at plate edge
u^*	Radial displacement
w	Normal deflection
\bar{w}	$\frac{w}{h}$, nondimensional normal deflection
z	Thickness coordinate
D	$\frac{Eh^3}{12(1-\nu^2)}$, flexural rigidity
E	Young's modulus
K	Relaxation parameter
M_r, M_t	Radial and tangential bending moments, respectively
\bar{M}_r, \bar{M}_t	Nondimensional radial and tangential bending moments, respectively
M_T	$\int_{-h/2}^{h/2} E\alpha Tz dz$
\bar{M}_T	Nondimensional form of M_T
N_r, N_t	Radial and tangential membrane forces, respectively
\bar{N}_r, \bar{N}_t	Nondimensional radial and tangential membrane forces, respectively

4.2 (Cont'd)

N_T	$\int_{-h/2}^{h/2} E\alpha T dz$
\bar{N}_T	Nondimensional form of N_T
q	Normal pressure
Q	Nondimensional normal pressure
T	Local temperature with respect to an unstressed and undeflected datum
U_0	Strain energy density
V	Total potential energy
α	Coefficient of linear thermal expansion
β	Slope
ϵ_r, ϵ_t	Radial and tangential strains, respectively
$\epsilon_r^0, \epsilon_t^0$	Midplane radial and tangential strains, respectively
λ	Elastic in-plane edge restraint
ν	Poisson's ratio
ϕ	$\frac{\psi b}{D}$, nondimensional stress function
ϕ_i	Finite difference value of ϕ at the i 'th grid point
ψ	Stress function
σ_r, σ_t	Radial and tangential stresses, respectively
θ	$\frac{b}{h} \sqrt{6(1-\nu^2)} \beta$
θ_i	Finite difference value of θ at the i 'th grid point

4.3 BASIC EQUATIONS

4.3.1 Stress-Strain - Displacement Relations

The von Kármán large strain - displacement relations for the axisymmetric strains at any point in the plate are given by

$$\epsilon_r = \frac{du^*}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2$$

$$\epsilon_t = \frac{u^*}{r}$$
(1)

where $u^*(r, z)$ is the radial displacement of the point and $w(r)$ is the deflection normal to the undeflected midplane (Figure 4.3.1-1).

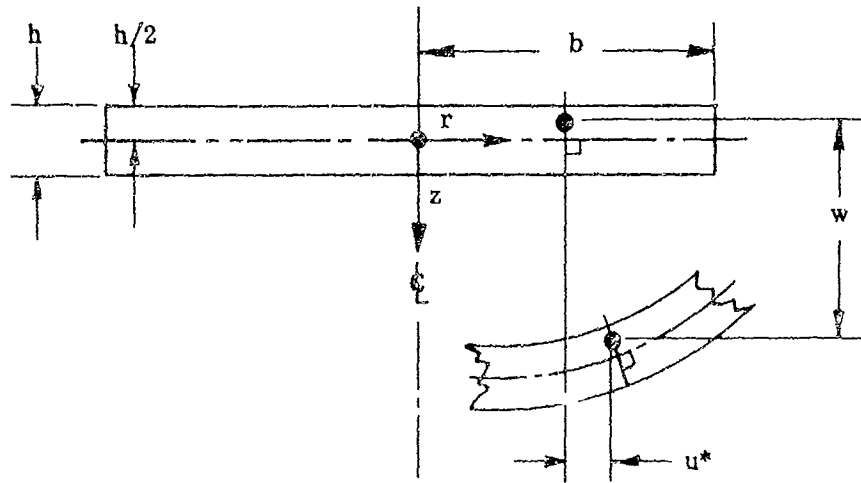


FIGURE 4.3.1-1 GENERAL AXISYMMETRIC DEFLECTION OF A TYPICAL POINT IN THE PLATE

Assuming plane sections to remain plane and normal to the deflected middle surface,

$$u^* = u - z \frac{dw}{dr}$$
(2)

4.3.1 (Cont'd)

where $u = u(r)$ is the radial displacement of the midplane. From (1) and (2),

$$\epsilon_r = \frac{du}{dr} - z \frac{d^2w}{dr^2} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \quad (3)$$

$$\epsilon_t = \frac{u}{r} - \frac{z}{r} \frac{dw}{dr}$$

The last (nonlinear) term in the first of (3) does not appear in the conventional small deflection theory. From Hooke's law (neglecting normal stresses in the thickness direction), including the temperature "T", there results

$$\sigma_r = \frac{E}{1-\nu^2} \left[\epsilon_r + \nu \epsilon_t - (1+\nu) \alpha T \right] \quad (4)$$

$$\sigma_t = \frac{E}{1-\nu^2} \left[\epsilon_t + \nu \epsilon_r - (1+\nu) \alpha T \right]$$

An integration of (4) through the plate thickness yields[†]

$$N_r = \int_{-h/2}^{h/2} \sigma_r dz = \frac{Eh}{1-\nu^2} \left[\epsilon_r^o + \nu \epsilon_t^o - \frac{(1+\nu)}{Eh} N_T \right] \quad (4a)$$

$$N_t = \int_{-h/2}^{h/2} \sigma_t dz = \frac{Eh}{1-\nu^2} \left[\epsilon_t^o + \nu \epsilon_r^o - \frac{(1+\nu)}{Eh} N_T \right]$$

where the midplane strains are given by

$$\epsilon_r^o = \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \quad (4b)$$

$$\epsilon_t^o = \frac{u}{r}$$

[†] In what follows, it is assumed that $\alpha T = \alpha T(r, z)$ but that E (and hence D) is constant.

4.3.1 (Cont'd)

and

$$N_T = \int_{-h/2}^{h/2} E\alpha T dz \quad (4c)$$

Multiplying (4) by z and then integrating through the thickness yields

$$M_r = \int_{-h/2}^{h/2} \sigma_r z dz = -D \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} + \frac{M_T}{D(1-\nu)} \right] \quad (4d)$$

$$M_t = \int_{-h/2}^{h/2} \sigma_t z dz = -D \left[\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2 w}{dr^2} + \frac{M_T}{D(1-\nu)} \right]$$

where

$$M_T = \int_{-h/2}^{h/2} E\alpha T z dz \quad (4e)$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

4.3.2 Strain and Potential Energies

The strain energy per unit of volume is given by (Reference 4-6):

$$U_o = \frac{1}{2} \left[\epsilon_r \sigma_r + \epsilon_t \sigma_t - \alpha T (\sigma_r + \sigma_t) \right] \quad (1)$$

Substitution of the stress-strain relations (4) of Paragraph 4.3.1 into (1) yields

$$U_o = \frac{E}{2(1-\nu^2)} \left[\epsilon_r^2 + \epsilon_t^2 + 2\nu \epsilon_r \epsilon_t - 2(1+\nu)(\epsilon_r + \epsilon_t) \alpha T + 2(1+\nu)(\alpha T)^2 \right] \quad (2)$$

For rotationally workless restraints at the outer boundary, the total potential of the plate and external loading system is given by the following equation:

4.3.2 (Cont'd)

$$V = 2\pi \int_0^b r \int_{-h/2}^{h/2} U_0 dz dr - 2\pi \int_0^b qwr dr + \pi b \lambda u_b^2 \quad (3)$$

where q is the normal pressure, λ is an elastic restraint per unit circumferential length to radial displacement of the boundary, and $u_b = u|_{r=b}$. From Eqs. (3) of Paragraph 4.3.1, and (2) and (3) the potential energy in terms of the displacement components and temperature becomes

$$\begin{aligned} V = & \frac{\pi E}{1-\nu} \int_0^b r \left\{ \left[\frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \right]^2 + \left(\frac{u}{r} \right)^2 + \frac{2\nu u}{r} \left[\frac{dr}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \right] \right\} h \\ & + \left\{ \left(\frac{d^2 w}{dr^2} \right)^2 + \left(\frac{1}{r} \frac{dw}{dr} \right)^2 + \frac{2\nu}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right\} \frac{h^3}{12} \\ & + 2(1+\nu) \left\{ \int_{-h/2}^{h/2} (\alpha T)^2 dz + \frac{M_T}{E} \left[\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right] \right. \\ & \left. - \frac{N_T}{E} \left[\frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 + \frac{u}{r} \right] \right\} dr - 2\pi \int_0^b qwr dr + \pi b \lambda (u_b)^2 \end{aligned} \quad (4)$$

4.3.3 Governing Differential Equations and Boundary Conditions

The equilibrium equations and "natural" boundary conditions are now obtained by making the potential energy stationary with respect to variations of the displacements w and u ; i. e. $\delta_w V = \delta_u V = 0$. The first of the variations ($\delta_w V = 0$) yields the equilibrium equation

$$D \nabla^4 w - \frac{1}{r} \frac{d}{dr} \left(r N_r \frac{dw}{dr} \right) = q - \frac{\nabla^2 M_T}{1-\nu} \quad (1)$$

and the following boundary conditions:

$$(1) \quad \underline{r=0}$$

Assumed regularity of the solution at the center requires that M_r and M_t be finite. This implies that $\frac{dw}{dr} = 0$ and $M_r = M_t$. (2a)

4.3.3 (Cont'd)

(2) $r = b$

(i) w prescribed or $-\frac{d}{dr} \left[DV^2 w + \frac{M_T}{1-\nu} \right] + N_r \frac{dw}{dr} = 0$

(2b)

(ii) w' prescribed or $D \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right] + \frac{M_T}{1-\nu} = 0$.

The second of the variations ($\delta_u V = 0$) results in

$$N_t - \frac{d}{dr} (r N_r) = 0 \quad (3)$$

with boundary conditions:

(1) $r = 0$

N_r and N_t finite. This implies that $u = 0$ (hence $N_r = N_t$).

(4a)

(2) $r = b$

$\left(u + \frac{N_r}{\lambda} \right) = 0$ or u prescribed.

(4b)

The two equations (1) and (3) contain three unknown functions of r ; i. e., w , N_r and N_t . A third equation is obtained from the necessary condition that these three quantities yield a set of single-valued displacements, u and w . A statement of this requirement is obtained by eliminating u from Equation (4b) of Paragraph 4.3.1, which results in

$$\frac{d\epsilon_t^0}{dr} + \frac{\epsilon_t^0 - \epsilon_r^0}{r} = -\frac{1}{2r} \left(\frac{dw}{dr} \right)^2 \quad (5a)$$

Substituting (4a) of Paragraph 4.3.1 into (5a),

$$\frac{d}{dr} (N_t - \nu N_r + N_T) + \frac{(1+\nu)}{r} (N_t - N_r) + \frac{Eh}{2r} \left(\frac{dw}{dr} \right)^2 = 0 \quad (5b)$$

Equation (3) is automatically satisfied by a stress function ψ defined through the relations

$$\psi = r N_r$$

$$\frac{d\psi}{dr} = N_t$$

(6)

4.3.3 (Cont'd)

Substitution of (6) into Eqs. (1) and (5b) results in the following coupled set of non-linear differential equations

$$D \nabla^4 w - \frac{1}{r} \frac{d}{dr} \left(\psi \frac{dw}{dr} \right) = q - \frac{1}{1-\nu} \nabla^2 M_T \quad (7)$$

and

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{Eh}{2r} \left(\frac{dw}{dr} \right)^2 = - \frac{d}{dr} N_T \quad (8)$$

where

$$\nabla^2 = \frac{1}{r} \frac{d}{dr} r \frac{d}{dr}$$

$$\nabla^4 = \nabla^2 \nabla^2$$

A more convenient form is obtained by introducing the slope $\beta = \frac{dw}{dr}$, which yields

$$D \frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\beta) - \frac{d}{dr} (\psi\beta) = qr - \frac{1}{1-\nu} \frac{d}{dr} \left(r \frac{dM_T}{dr} \right) \quad (9)$$

and

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{Eh}{2r} \beta^2 = - \frac{d}{dr} N_T \quad (10)$$

Equation (9) may now be integrated with respect to r , resulting in

$$D \left(\frac{d^2 \beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\beta}{r^2} \right) - \frac{\beta\psi}{r} - \frac{1}{r} \int_0^r q(\eta) \cdot \eta d\eta - \frac{1}{1-\nu} \frac{d}{dr} M_T \quad (11)$$

The Eqs. (10) and (11) are to be solved subject to the boundary conditions (obtained from (2) and (4))

(i) $\beta(0) = 0$

(ii) $\beta(b)$ prescribed (known slope)

(12)

or

$$D \left[\frac{d\beta}{dr} + \frac{\nu}{r} \beta \right]_{r=b} + \left[\frac{M_T}{1-\nu} \right]_{r=b} = 0 \quad (\text{no radial bending moment})$$

(iii) $\psi(0) = 0$ (since N_T is finite at $r = 0$)

4.3.3 (Cont'd)

$$(iv) \left[\frac{d\psi}{dr} + \left(\frac{Eh}{b^2\lambda} - \frac{\nu}{b} \right) \psi + N_T \right]_{r=b} = 0 \text{ (elastic radial restraint "\lambda" at } r=b \text{)}$$

or

$$\left[\frac{d\psi}{dr} - \frac{\nu}{r} \psi + N_T \right]_{r=b} \text{ prescribed (known radial displacement at } r=b \text{)}$$

The above differential equations and boundary conditions can be expressed in non-dimensional form as

$$\ddot{\theta} + \frac{\dot{\theta}}{\bar{r}} - \frac{\theta}{\bar{r}^2} - \frac{\theta \phi}{\bar{r}} = \bar{r} Q(\bar{r}) - \bar{M}_T$$

$$\ddot{\phi} + \frac{\dot{\phi}}{\bar{r}} - \frac{\phi}{\bar{r}^2} + \frac{\theta^2}{\bar{r}} = -\bar{N}_T \tag{13}$$

where:

$$\bar{r} = \frac{r}{b}, \quad 0 < \bar{r} < 1$$

$$\left(\cdot = \frac{d}{d\bar{r}} \right)$$

$$\phi = \frac{4b}{D}$$

$$\theta = \frac{b}{h} \sqrt{6(1-\nu^2)} \alpha$$

$$Q(\bar{r}) = \frac{2b^4}{Eh^4} \left[6(1-\nu^2) \right]^{3/2} \frac{1}{\bar{r}^2} \int_0^{\bar{r}} q(\xi) \xi d\xi \tag{14}$$

$$\bar{M}_T = \frac{M_T b^2}{Dh} \sqrt{\frac{6(1-\nu^2)}{1-\nu}}$$

$$\bar{N}_T = \frac{N_T b^2}{D}$$

4.3.3 (Cont'd)

The boundary conditions become:

(1) $\theta(0) = 0$

(2) $\phi(0) = 0$

(3) $\theta(1)$ prescribed

or

$$\theta(1) + \nu \theta(1) + \bar{M}_T(1) = 0$$

(14a)

(4) $\phi(1) + \left(\frac{Eh}{b\lambda} - \nu \right) \phi(1) + \bar{N}_T(1) = 0$

or

$$\phi(1) - \nu \phi(1) + \bar{N}_T(1) \text{ prescribed}$$

The set of nonlinear differential Equations (13) and accompanying boundary conditions (14a) are not amenable to analytic solution. Numerical procedures must be used to obtain solutions for specified values of the parameters. A finite-difference approach is presented below together with numerical results.

4.4 NUMERICAL INVESTIGATION

4.4.1 Finite Difference Procedure

The procedure developed in Reference 4-4 is employed here for the combined thermo-mechanical problem. In particular, we consider the case of radially immovable edges ($\lambda \rightarrow \infty$ in (14a) of Paragraph 4.3.3) with simple supports for bending where q , M_T , and N_T are constant (uniform pressure and temperature which varies only through the thickness). For this problem, (13) and (14a) of Paragraph 4.3.3 reduce to

$$\begin{aligned} L\theta &= \theta\phi + Q\bar{r}^2 \\ L\phi &= -\theta^2 \end{aligned} \quad (1)$$

and

$$\begin{aligned} \theta(0) &= 0 \\ \phi(0) &= 0 \\ \dot{\theta}(1) + \nu\theta(1) + \bar{M}_T &= 0 \\ \dot{\phi}(1) - \nu\phi(1) + \bar{N}_T &= 0 \end{aligned} \quad (2)$$

where

$$\begin{aligned} L &= \bar{r} \frac{d}{d\bar{r}} \frac{1}{\bar{r}} \frac{d}{d\bar{r}} \bar{r} \\ Q &= \frac{q}{E} \left(\frac{b}{h} \right)^4 \left[6(1-\nu^2) \right]^{3/2} \end{aligned}$$

and the other nondimensional quantities are as defined previously.

A central difference representation of the differential equations (1) yields the following for the interior points:

$$\begin{aligned} \bar{L}\theta_i &= \theta_i\phi_i + Q\left(\frac{i}{m}\right)^2 \\ \bar{L}\phi_i &= -\theta_i^2 \end{aligned} \quad (i = 1, 2, \dots, m-1) \quad (3)$$

where for an arbitrary function ζ ,

4.4.1 (Cont'd)

- (2) Substituting $[\theta_i]_n$, $[\phi_i]_{n+1}$ into (5b) again yields a set of linear equations from which the provisional values $[\theta_i^*]_{n+1}$ for the next set of iterates are determined.
- (3) The actual value of the next set of iterates $[\theta_i]_{n+1}$ is obtained from Eq. (5c) where a suitable value of "relaxation" parameter is employed.
- (4) Iterations to convergence are performed.

4.4.2 Numerical Results

Based on the procedure indicated, results are presented in both tabular and graphical form (Table 4.4.2-1 and Figures 4.4.2-1 through 4.4.2-6) for the simply supported solid plate with radially immovable edges.

The nondimensional deflections (\bar{w}), membrane forces (\bar{N}_r and \bar{N}_t), and bending moments (\bar{M}_r and \bar{M}_t), presented in the graphs and tables are defined as follows:

$$\begin{aligned} \bar{w} &= \frac{w}{h} \\ \bar{N}_r &= \frac{N_r b^2}{D} \\ \bar{N}_t &= \frac{N_t b^2}{D} \\ \bar{M}_r &= -\frac{M_r b^2}{Dh} \sqrt{6(1-\nu^2)} \\ \bar{M}_t &= -\frac{M_t b^2}{Dh} \sqrt{6(1-\nu^2)} \end{aligned} \quad (1)$$

and as defined previously,

$$\begin{aligned} \bar{r} &= \frac{r}{b} \\ \bar{M}_T &= \frac{Eb^2}{Dh} \sqrt{\frac{6(1+\nu)}{1-\nu}} \int_{-h/2}^{h/2} \alpha T z dz \\ \bar{N}_T &= \frac{Fb^2}{D} \int_{-h/2}^{h/2} \alpha T dz \end{aligned}$$

4.4.2 (Cont'd)

Stresses (in nondimensional form) may be obtained from these formulas (Reference 4-7):

$$\frac{b^2 h \sigma_r}{D} = \frac{12z}{h} \left[\frac{\bar{M}_T - \bar{M}_r}{\sqrt{6(1-\nu^2)}} \right] + \frac{\bar{N}_T}{1-\nu} - 12(1+\nu) \frac{b^2}{h^2} \alpha T + \bar{N}_r \quad (2)$$

$$\frac{b^2 h \sigma_t}{D} = \frac{12z}{h} \left[\frac{\bar{M}_T - \bar{M}_t}{\sqrt{6(1-\nu^2)}} \right] + \frac{\bar{N}_T}{1-\nu} - 12(1+\nu) \frac{b^2}{h^2} \alpha T + \bar{N}_t$$

A discussion of the numerical results follows.

(1) Thermal Buckling

The thermal buckling problem of a circular plate due to an average elevated temperature through the thickness (proportional to \bar{N}_T), where $\bar{M}_T = Q = 0$ and radial edge displacement is prevented, is equivalent to the mechanical buckling problem of a plate having an edge thrust corresponding to a prescribed edge displacement. It is shown in Reference 4-2 that buckling can occur only when the edge thrust exceeds the lowest eigenvalue of the linearized buckling problem. The critical thrust corresponding to this eigenvalue is given by (Reference 4-5):

$$-\left[\tau_r \right]_{cr} = \frac{(2.05)^2 D}{b^2 h} \quad (3)$$

Since, for the thermal problem, the thrust (up to buckling) is given by

$$\sigma_r = \frac{N_T}{h(1-\nu)} \quad (4)$$

then from (3), (4), and the definition of \bar{N}_T , we find that

$$\left[\bar{N}_T \right]_{cr} = 2.94$$

The postbuckling behavior of the plate is given by the nonlinear analysis; the numerical results are presented in Figure 4.4.2-1 and the first five sub-tables of Table 4.4.2-1. The table employs a floating decimal number system which is to be interpreted as shown by the following examples:

$$\begin{aligned} 0.5159E 00 & (0.5159) \times (10^0) = 0.5159 \\ 0.5159E 02 & (0.5159) \times (10^2) = 51.59 \\ 0.5159E -01 & (0.5159) \times (10^{-1}) = 0.05159 \end{aligned}$$

4.4.2 (Cont'd)

Figure 4.4.2-1 shows the variation of the nondimensional deflection, membrane stress resultants and bending moments with the nondimensional radial coordinate \bar{r} , as \bar{N}_T varies from 0 to 100. It is noted that for $0 < \bar{N}_T < 2.94$ the plate is undeflected, and the two-dimensional linear elastic solution holds, yielding stresses which are constant over the planform and zero bending moments. For higher values of \bar{N}_T , Figure 4.4.2-1a shows nonzero deflections that increase monotonically with increasing \bar{N}_T . For low values of \bar{N}_T , the quantities \bar{N}_r and \bar{N}_t are compressive (Figures 4.4.2-1b and c) throughout the plate. With higher values of \bar{N}_T (25, 50, and 100, for example) deflections in the central region appear to be restrained by tensile membrane stresses, while in the vicinity of the plate edge the membrane stresses become compressive. Figures 4.4.2-1d and e indicate that for small \bar{N}_T (but above that causing initial buckling), the maximum moments in the radial and tangential directions occur at the center of the plate. With larger \bar{N}_T , maximum moments are away from the center and approach the outer edge as \bar{N}_T increases. The radial moment, in particular, exhibits a strong "boundary layer" effect (sharp gradients near the edge) in meeting the condition $\left[\bar{M}_r \right]_{\bar{r}=1} = 0$.

(2) Bending Due To Combined Thermal And Mechanical Effects

Starting with the sixth sub-table of Table 4.4.2-1, nondimensional numerical results are listed for the thermal bending problem ($Q = 0$), corresponding to the following combinations of thermal parameters:

$$\bar{M}_T = 50, 100, 1000, 2000$$

$$\bar{N}_T = 0, 2.94, 5, 10, 25, 100$$

For a qualitative discussion of the results, we refer to the graphs of Figures 4.4.2-2 through 4.4.2-6 in which mechanical loads ($Q \neq 0$) as well as temperature are considered.

(a) Deflections (Figure 4.4.2-2)

For low pressures (quantities proportional to Q) the deflection in the plate interior becomes constant for large temperature differences (quantities proportional to \bar{M}_T). This flat deflected shape is maintained almost to the plate edges where the boundary requirement of zero deflection causes a boundary layer effect in which the deflections drop precipitously. This effect becomes less pronounced as the pressure loading increases. However, it is interesting to note that even for high pressure loading (Figure 4.4.2-2d), the plate still tends to flatten out in the central region as the temperature difference increases and, contrary to what would be expected, the deflection at the center does not in all cases increase monotonically with the temperature difference.

(b) Membrane Stresses (Figures 4.4.2-3 and 4.4.2-4)

With increasing temperature difference between the plate faces, the radial and tangential stresses tend to become constant and equal to each other in the plate interior,

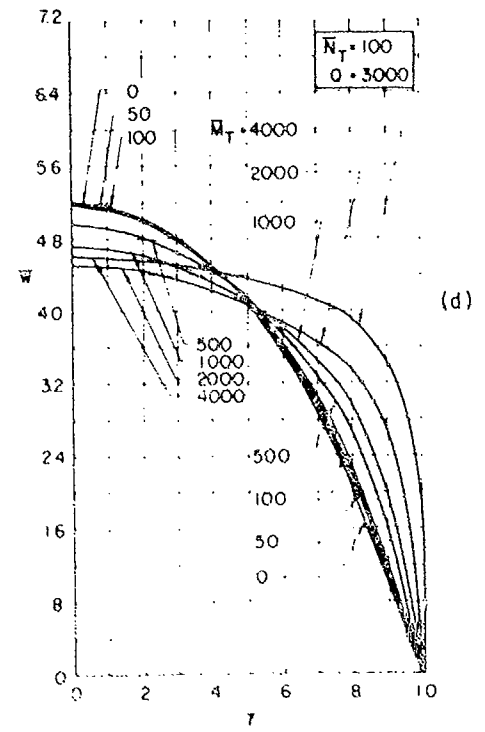
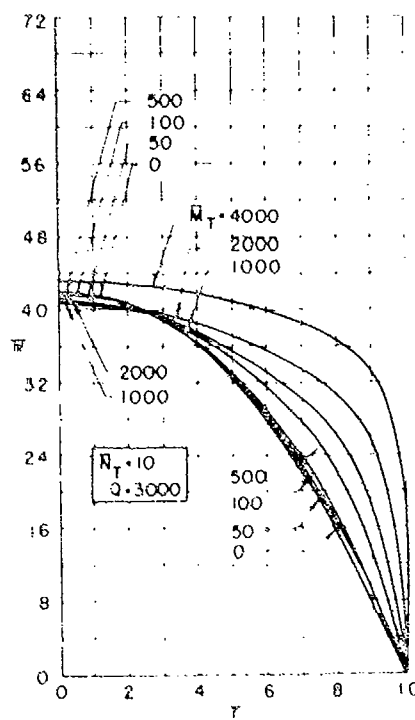
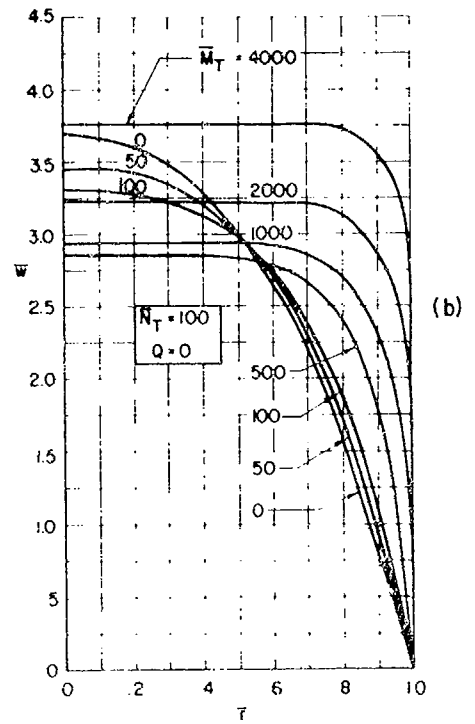
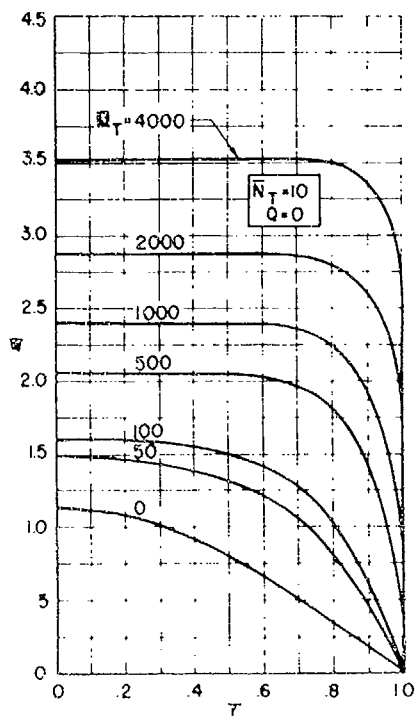


FIGURE 4.4.2-2 NONDIMENSIONAL DEFLECTIONS DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

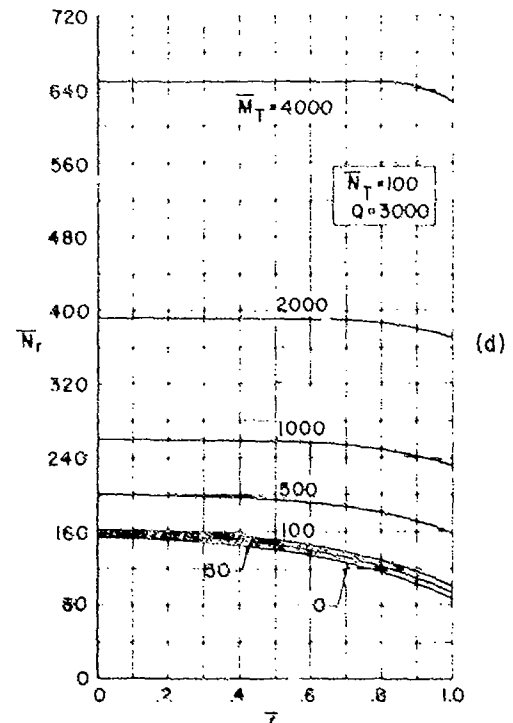
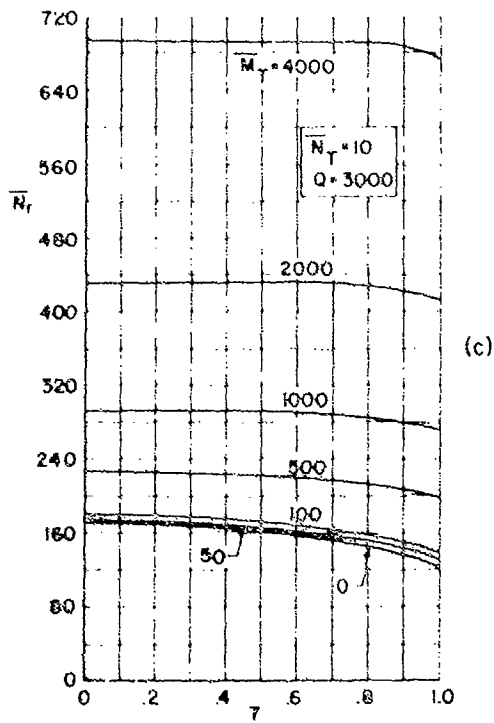
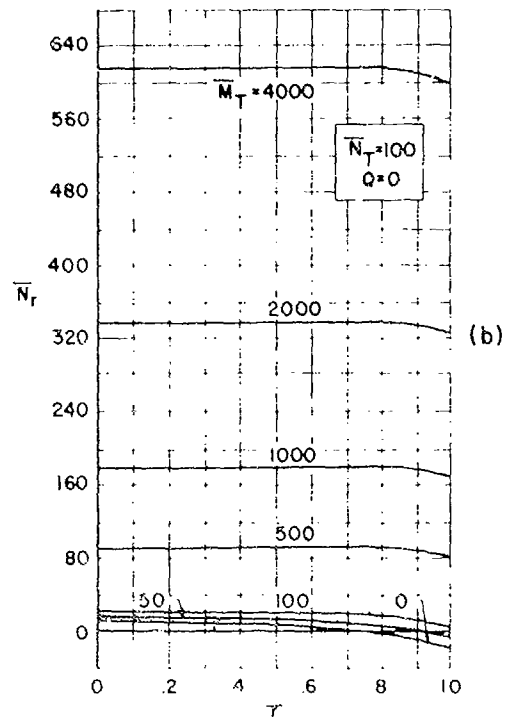
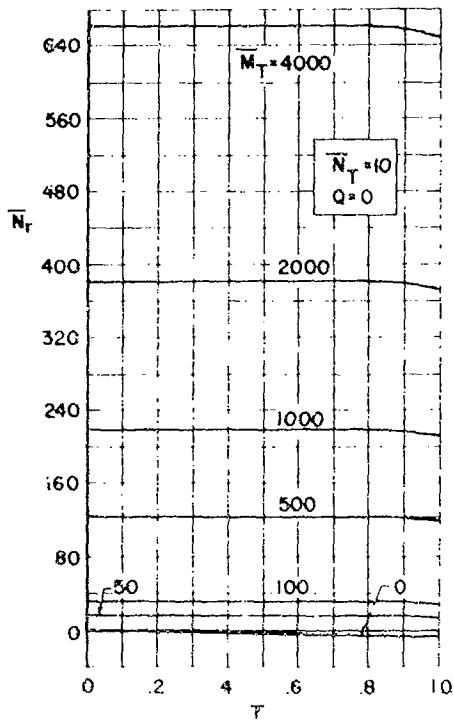
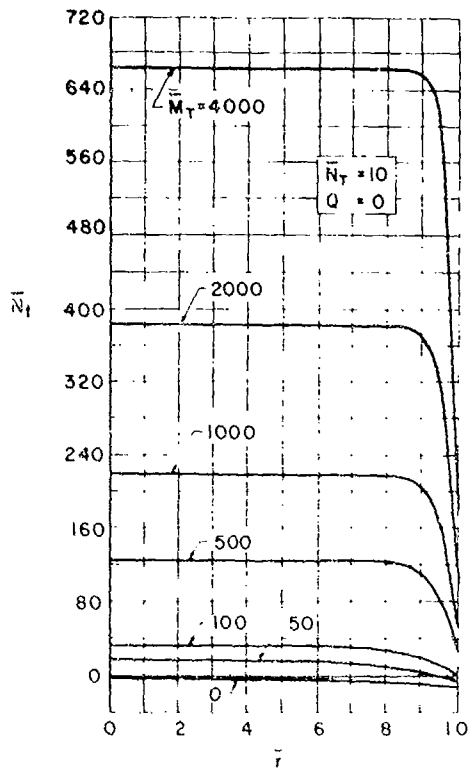
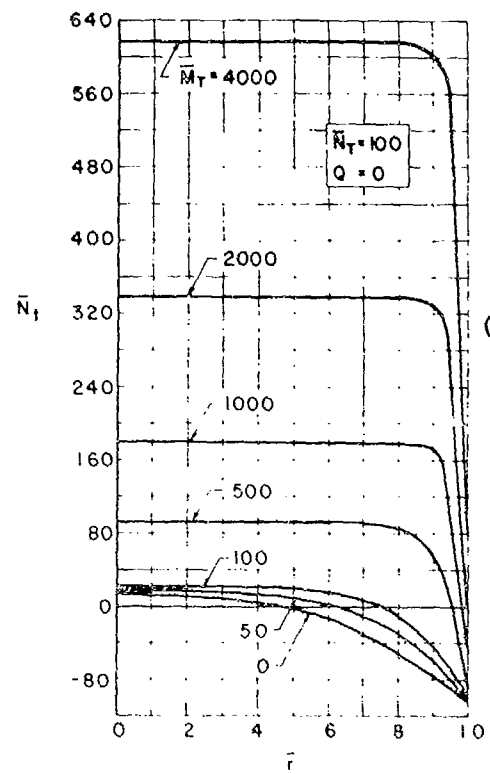


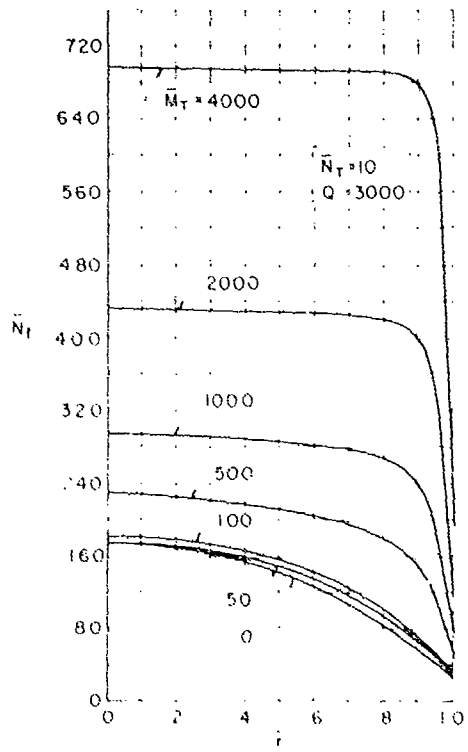
FIGURE 4.4.2-3 NONDIMENSIONAL RADIAL MEMBRANE FORCES DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING



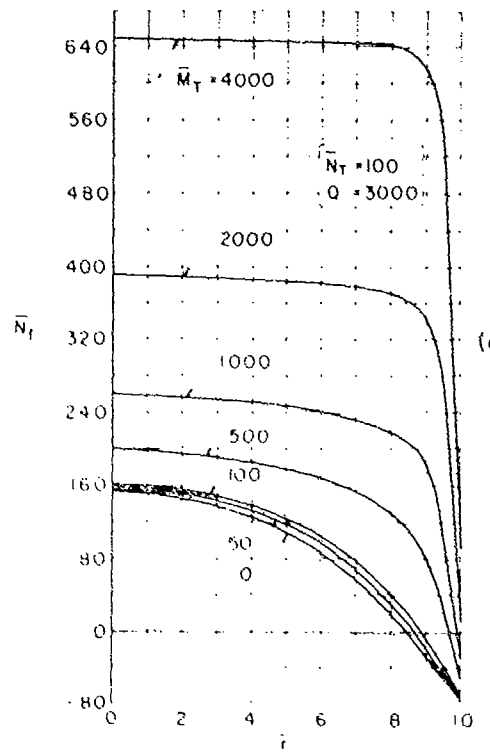
(a)



(b)



(c)



(d)

FIGURE 4.4.2-4 NONDIMENSIONAL TANGENTIAL MEMBRANE FORCES DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

4.4.2 (Cont'd)

as in a pure membrane. This effect is evident even for high pressure loading, (Figures 4.4.2-3d and 4d). In addition, the magnitudes of these tensile membrane stress resultants increase with increasing temperature difference and normal pressure while they decrease with increasing average temperature (quantities proportional to \bar{N}_T). This is to be expected, since increasing the pressure and temperature difference each cause additional middle plane stretching while \bar{N}_T tends to neutralize this effect. The membrane tension \bar{N}_r , decreases locally as the outer boundary is approached, where the gradients are most pronounced. This negative increment $\Delta\bar{N}_r$, may be responsible for the abrupt reduction in tensile hoop stresses (\bar{N}_t) in the vicinity of the edge (Figure 4.4.2-4). For the larger values of average temperature, the high tensile stress resultants \bar{N}_t , in the plate interior reduce sharply in a boundary layer. Low temperature differences permit transition to compression near the boundary (Figures 4.4.2-4b and d).

(c) Bending Moments (Figures 4.4.2-5 and -6)

The bending moments in the radial and tangential directions are constant and essentially equal over the major central area of the plate. However, radial moments decrease radically near the boundary, satisfying the zero moment boundary condition. Due to the predominantly flat profile of the deflected plate in the interior for large temperature differences (Figures 4.4.2-2a and b), it may be conjectured that the constant and equal interior moments are the same as would occur in a clamped plate subjected to the same temperature gradient (since such a plate will remain flat). To show that this is the case, we proceed as follows:

The bending moments in a fully clamped plate subjected to a thermal gradient through the thickness are given by (Reference 4-8)

$$M_r = M_t = -\frac{M_T}{1-\nu}$$

while

$$w \equiv 0$$

or, in nondimensional form, using the notation of (14) of Paragraph 4.3.3 and (1)

$$\bar{M}_r = \bar{M}_t = \bar{M}_T$$

This result is readily verified by Figures 4.4.2-5 and -6.

ACKNOWLEDGEMENT

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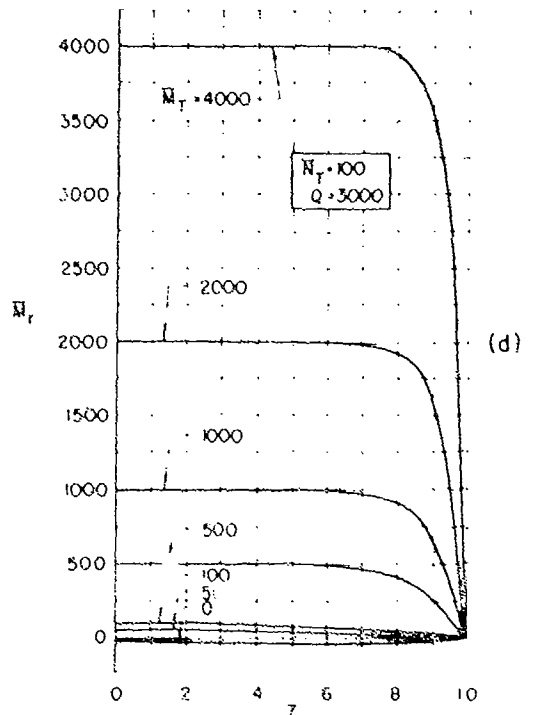
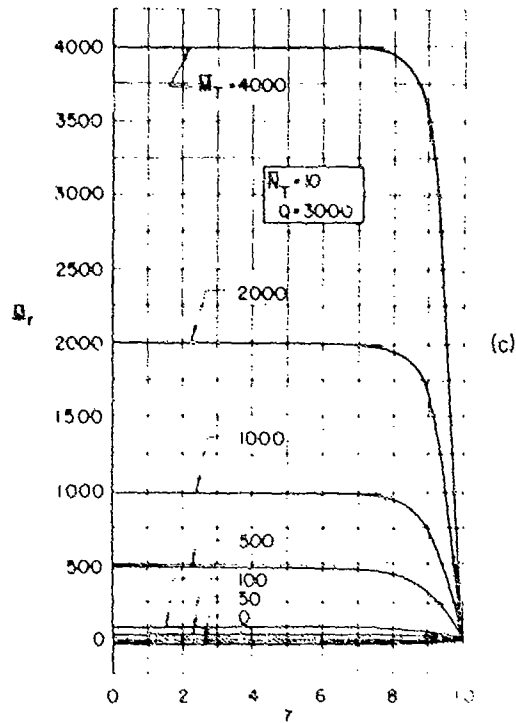
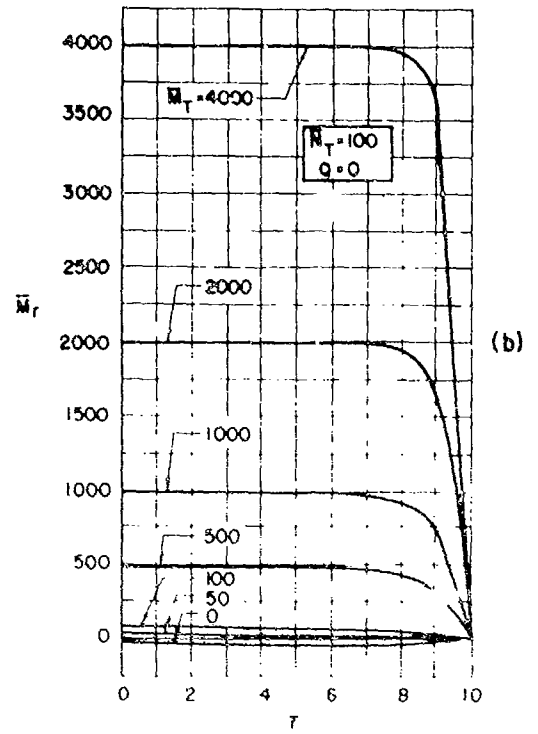
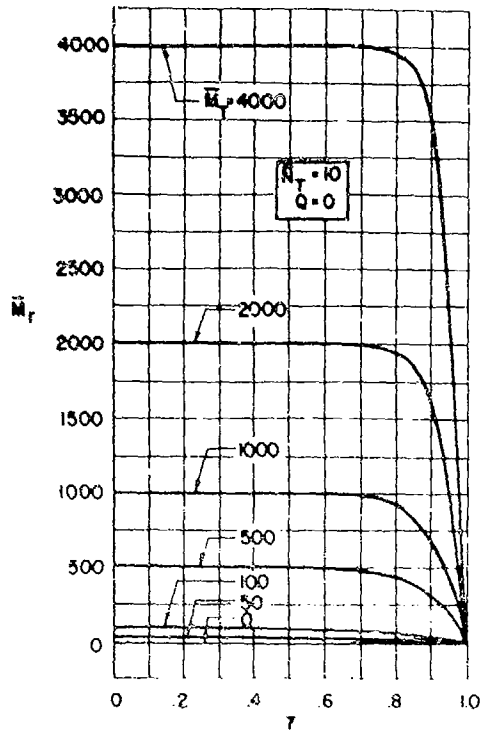
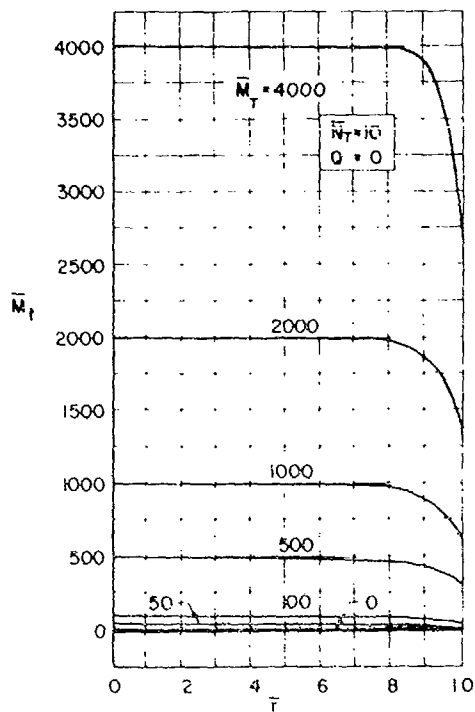
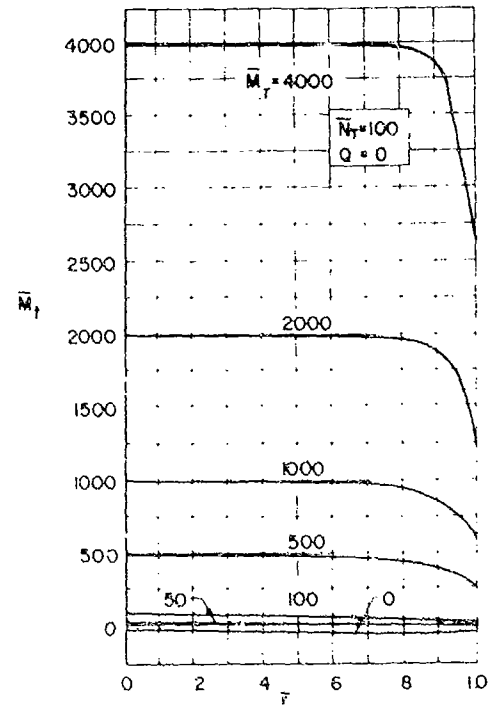


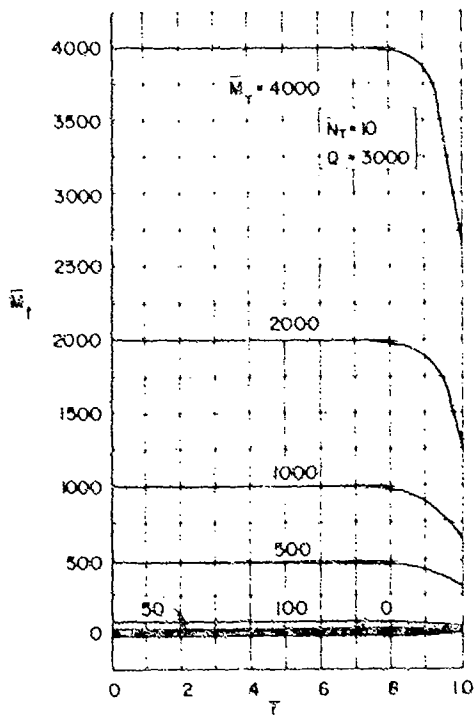
FIGURE 4.4.2-5 NONDIMENSIONAL RADIAL BENDING MOMENTS DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING



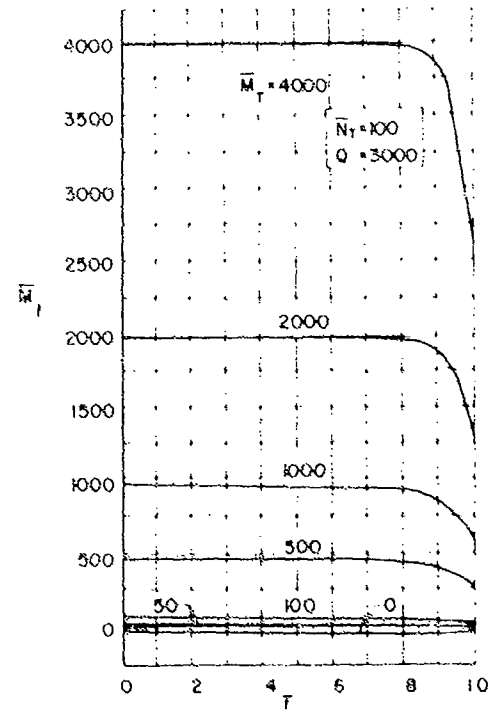
(a)



(b)



(c)



(d)

FIGURE 4. 4. 2-6 NONDIMENSIONAL TANGENTIAL BENDING MOMENTS DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

4.4.2 (Cont'd)

TABLE 4.4.2-1

NONDIMENSIONAL DEFLECTIONS, MEMBRANE FORCES, AND
BENDING MOMENTS FOR A SIMPLY SUPPORTED CIRCULAR
PLATE WITH RADIALLY IMMOVABLE EDGE

(Pages 4.25 through 4.35)

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

\bar{r}	$\bar{N}_T = 0.$		$\bar{N}_T = 0.2940E 01$		
	\bar{w}	\bar{N}_T	\bar{N}_t	\bar{N}_T	\bar{N}_t
0.	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.2072E-00	-0.2072E-00
0.1000E-00	0.2566E-01	-0.4193E 01	-0.4193E 01	-0.2043E-00	-0.2055E-00
0.2000E-00	0.2466E-01	-0.4193E 01	-0.4193E 01	-0.1980E-00	-0.2007E-00
0.3000E-00	0.2300E-01	-0.4193E 01	-0.4193E 01	-0.1824E-00	-0.1959E-00
0.4000E-00	0.2077E-01	-0.4193E 01	-0.4193E 01	-0.1682E-00	-0.1830E-00
0.5000E-00	0.1803E-01	-0.4193E 01	-0.4193E 01	-0.1547E-00	-0.1698E-00
0.6000E-00	0.1486E-01	-0.4193E 01	-0.4193E 01	-0.1420E-00	-0.1574E-00
0.7000E-00	0.1135E-01	-0.4193E 01	-0.4193E 01	-0.1300E-00	-0.1457E-00
0.8000E-00	0.0767E-01	-0.4193E 01	-0.4193E 01	-0.1186E-00	-0.1345E-00
0.9000E-00	0.0311E-01	-0.4193E 01	-0.4193E 01	-0.1086E-00	-0.1237E-00
1.0000E-00	0.	-0.4193E 01	-0.4193E 01	-0.1000E-00	-0.1133E-00
	$\bar{N}_T = 0.$		$\bar{N}_T = 0.4000E 01$		
0.	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.2072E-00	-0.2072E-00
0.1000E-00	0.2566E-01	-0.4193E 01	-0.4193E 01	-0.2043E-00	-0.2055E-00
0.2000E-00	0.2466E-01	-0.4193E 01	-0.4193E 01	-0.1980E-00	-0.2007E-00
0.3000E-00	0.2300E-01	-0.4193E 01	-0.4193E 01	-0.1824E-00	-0.1959E-00
0.4000E-00	0.2077E-01	-0.4193E 01	-0.4193E 01	-0.1682E-00	-0.1830E-00
0.5000E-00	0.1803E-01	-0.4193E 01	-0.4193E 01	-0.1547E-00	-0.1698E-00
0.6000E-00	0.1486E-01	-0.4193E 01	-0.4193E 01	-0.1420E-00	-0.1574E-00
0.7000E-00	0.1135E-01	-0.4193E 01	-0.4193E 01	-0.1300E-00	-0.1457E-00
0.8000E-00	0.0767E-01	-0.4193E 01	-0.4193E 01	-0.1186E-00	-0.1345E-00
0.9000E-00	0.0311E-01	-0.4193E 01	-0.4193E 01	-0.1086E-00	-0.1237E-00
1.0000E-00	0.	-0.4193E 01	-0.4193E 01	-0.1000E-00	-0.1133E-00
	$\bar{N}_T = 0.$		$\bar{N}_T = 0.5000E 01$		
0.	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.2072E-00	-0.2072E-00
0.1000E-00	0.2566E-01	-0.4193E 01	-0.4193E 01	-0.2043E-00	-0.2055E-00
0.2000E-00	0.2466E-01	-0.4193E 01	-0.4193E 01	-0.1980E-00	-0.2007E-00
0.3000E-00	0.2300E-01	-0.4193E 01	-0.4193E 01	-0.1824E-00	-0.1959E-00
0.4000E-00	0.2077E-01	-0.4193E 01	-0.4193E 01	-0.1682E-00	-0.1830E-00
0.5000E-00	0.1803E-01	-0.4193E 01	-0.4193E 01	-0.1547E-00	-0.1698E-00
0.6000E-00	0.1486E-01	-0.4193E 01	-0.4193E 01	-0.1420E-00	-0.1574E-00
0.7000E-00	0.1135E-01	-0.4193E 01	-0.4193E 01	-0.1300E-00	-0.1457E-00
0.8000E-00	0.0767E-01	-0.4193E 01	-0.4193E 01	-0.1186E-00	-0.1345E-00
0.9000E-00	0.0311E-01	-0.4193E 01	-0.4193E 01	-0.1086E-00	-0.1237E-00
1.0000E-00	0.	-0.4193E 01	-0.4193E 01	-0.1000E-00	-0.1133E-00

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

	$\bar{I}_T = 0.$			$\bar{N}_T = 0.2500E 02$		
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t	
0.	0.1936E 01	0.2361E 01	0.2361E 01	0.1179E 02	-0.1179E 02	
0.1000E-00	0.1916E 01	0.2250E 01	0.2039E 01	-0.1188E 02	-0.1164E 02	
0.2000E-00	0.1857E 01	0.1938E 01	0.1698E 01	-0.1211E 02	-0.1198E 02	
0.3000E-00	0.1759E 01	0.1411E 01	-0.5052E 00	-0.1212E 02	-0.1217E 02	
0.4000E-00	0.1619E 01	0.6603E 00	-0.2012E 01	-0.1266E 02	-0.1236E 02	
0.5000E-00	0.1437E 01	-0.3210E 00	-0.5852E 01	-0.1269E 02	-0.1245E 02	
0.6000E-00	0.1212E 01	-0.1500E 01	-0.9613E 01	-0.1238E 02	-0.1232E 02	
0.7000E-00	0.9491E 00	-0.3007E 01	-0.1399E 02	-0.1005E 02	-0.1182E 02	
0.8000E-00	0.6514E 00	-0.4677E 01	-0.1878E 02	-0.9402E 01	-0.1050E 02	
0.9000E-00	0.3298E-00	-0.6514E 01	-0.2359E 02	-0.4600E 01	-0.2149E 01	
1.0000E-00	0.	-0.8445E 01	-0.2753E 02	-0.5334E-05	-0.6265E 01	
	$\bar{M}_T = 0.$			$\bar{N}_T = 0.1000E 03$		
0.	0.3684E 01	0.1306E 02	0.1306E 02	-0.1357E 02	-0.1357E 02	
0.1000E-00	0.3662E 01	0.1292E 02	0.1263E 02	-0.1416E 02	-0.1392E 02	
0.2000E-00	0.3592E 01	0.1248E 02	0.1128E 02	-0.1587E 02	-0.1490E 02	
0.3000E-00	0.3468E 01	0.1170E 02	0.8718E 01	-0.1877E 02	-0.1656E 02	
0.4000E-00	0.3272E 01	0.1046E 02	0.4373E 01	-0.2281E 02	-0.1889E 02	
0.5000E-00	0.2998E 01	0.8602E 01	-0.2625E 01	-0.2773E 02	-0.2178E 02	
0.6000E-00	0.2639E 01	0.5893E 01	-0.1350E 02	-0.3274E 02	-0.2492E 02	
0.7000E-00	0.2151E 01	0.2042E 01	-0.2968E 02	-0.3601E 02	-0.2757E 02	
0.8000E-00	0.1534E 01	-0.3253E 01	-0.5210E 02	-0.3417E 02	-0.2833E 02	
0.9000E-00	0.7997E 00	-0.1018E 02	-0.7981E 02	-0.2940E 02	-0.2517E 02	
1.0000E-00	0.	-0.1859E 02	-0.1055E 03	0.2771E-04	-0.1406E 02	
	$\bar{M}_T = 0.5000E 02$			$\bar{N}_T = 0.$		
0.	0.1166E 01	0.2166E 02	0.2166E 02	0.4794E 02	0.4794E 02	
0.1000E-00	0.1177E 01	0.2169E 02	0.2169E 02	0.4772E 02	0.4770E 02	
0.2000E-00	0.1166E 01	0.2164E 02	0.2164E 02	0.4733E 02	0.4729E 02	
0.3000E-00	0.1145E 01	0.2162E 02	0.2164E 02	0.4687E 02	0.4710E 02	
0.4000E-00	0.1113E 01	0.2159E 02	0.2161E 02	0.4634E 02	0.4680E 02	
0.5000E-00	0.1062E 01	0.2153E 02	0.2158E 02	0.4573E 02	0.4620E 02	
0.6000E-00	0.9841E 00	0.2143E 02	0.2064E 02	0.4504E 02	0.4548E 02	
0.7000E-00	0.8206E 00	0.2125E 02	0.1963E 02	0.4428E 02	0.4492E 02	
0.8000E-00	0.6007E 00	0.2094E 02	0.1797E 02	0.4345E 02	0.4415E 02	
0.9000E-00	0.4129E 00	0.2036E 02	0.1333E 02	0.4258E 02	0.4316E 02	
1.0000E-00	0.	0.1927E 02	0.5783E 01	0.2110E-01	0.2429E 02	

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{M}_T = 0.5000E 02$				$\bar{N}_T = 0.2940E 01$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{N}_r	\bar{N}_t
0.	0.1270E 01	0.2027E 02	0.2027E 02	0.4760E 02	0.4760E 02
0.1000E-00	0.1266E 01	0.2027E 02	0.2026E 02	0.4744E 02	0.4751E 02
0.2000E-00	0.1254E 01	0.2025E 02	0.2021E 02	0.4694E 02	0.4722E 02
0.3000E-00	0.1230E 01	0.2022E 02	0.2012E 02	0.4603E 02	0.4671E 02
0.4000E-00	0.1193E 01	0.2018E 02	0.1995E 02	0.4454E 02	0.4590E 02
0.5000E-00	0.1136E 01	0.2010E 02	0.1962E 02	0.4224E 02	0.4468E 02
0.6000E-00	0.1049E 01	0.1998E 02	0.1899E 02	0.3875E 02	0.4288E 02
0.7000E-00	0.9187E 00	0.1976E 02	0.1778E 02	0.3352E 02	0.4025E 02
0.8000E-00	0.7239E 00	0.1938E 02	0.1532E 02	0.2575E 02	0.3644E 02
0.9000E-00	0.4330E-00	0.1871E 02	0.1060E 02	0.1438E 02	0.3098E 02
1.0000E-00	0.	0.1747E 02	0.2329E 01	-0.7152E-05	0.2385E 02
$\bar{M}_T = 0.5000E 02$				$\bar{N}_T = 0.5000E 01$	
0.	0.1334E 01	0.1943E 02	0.1943E 02	0.4736E 02	0.4736E 02
0.1000E-00	0.1330E 01	0.1942E 02	0.1941E 02	0.4718E 02	0.4726E 02
0.2000E-00	0.1316E 01	0.1940E 02	0.1936E 02	0.4666E 02	0.4696E 02
0.3000E-00	0.1291E 01	0.1937E 02	0.1925E 02	0.4592E 02	0.4642E 02
0.4000E-00	0.1250E 01	0.1932E 02	0.1904E 02	0.4493E 02	0.4557E 02
0.5000E-00	0.1188E 01	0.1925E 02	0.1866E 02	0.4378E 02	0.4431E 02
0.6000E-00	0.1094E 01	0.1908E 02	0.1794E 02	0.3944E 02	0.4246E 02
0.7000E-00	0.9461E 00	0.1883E 02	0.1657E 02	0.3290E 02	0.3978E 02
0.8000E-00	0.7507E 00	0.1840E 02	0.1392E 02	0.2513E 02	0.3594E 02
0.9000E-00	0.4472E-00	0.1766E 02	0.8734E 01	0.1322E 02	0.3052E 02
1.0000E-00	0.	0.1631E 02	-0.6811E-01	-0.4735E-06	0.2355E 02
$\bar{M}_T = 0.5000E 02$				$\bar{N}_T = 0.1000E 02$	
0.	0.1470E 01	0.1776E 02	0.1776E 02	0.4675E 02	0.4675E 02
0.1000E-00	0.1464E 01	0.1775E 02	0.1774E 02	0.4663E 02	0.4663E 02
0.2000E-00	0.1467E 01	0.1773E 02	0.1766E 02	0.4637E 02	0.4630E 02
0.3000E-00	0.1436E 01	0.1768E 02	0.1749E 02	0.4590E 02	0.4570E 02
0.4000E-00	0.1388E 01	0.1760E 02	0.1720E 02	0.4531E 02	0.4577E 02
0.5000E-00	0.1314E 01	0.1747E 02	0.1667E 02	0.4406E 02	0.4430E 02
0.6000E-00	0.1205E 01	0.1727E 02	0.1569E 02	0.3992E 02	0.4143E 02
0.7000E-00	0.1046E 01	0.1693E 02	0.1390E 02	0.2443E 02	0.3864E 02
0.8000E-00	0.7155E 00	0.1636E 02	0.1097E 02	0.2364E 02	0.3474E 02
0.9000E-00	0.4372E-00	0.1541E 02	0.4325E 01	0.1283E 02	0.2941E 02
1.0000E-00	0.	0.1376E 02	-0.5825E 01	-0.8136E-05	0.2282E 02

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_T	\bar{I}_t
0.	0.1936E 01	0.1510E 02	0.1510E 02	0.1498E 02	0.1498E 02
0.1000E-00	0.1928E 01	0.1508E 02	0.1504E 02	0.1473E 02	0.1483E 02
0.2000E-00	0.1902E 01	0.1502E 02	0.1486E 02	0.1429E 02	0.1440E 02
0.3000E-00	0.1855E 01	0.1491E 02	0.1449E 02	0.1426E 02	0.14363E 02
0.4000E-00	0.1783E 01	0.1474E 02	0.1384E 02	0.14051E 02	0.14247E 02
0.5000E-00	0.1676E 01	0.1446E 02	0.1273E 02	0.3743E 02	0.14080E 02
0.6000E-00	0.1522E 01	0.1402E 02	0.1082E 02	0.3311E 02	0.3848E 02
0.7000E-00	0.1305E 01	0.1335E 02	0.7521E 01	0.2719E 02	0.3537E 02
0.8000E-00	0.1001E 01	0.1230E 02	0.1813E 01	0.1939E 02	0.3131E 02
0.9000E-00	0.47.7E 00	0.1001E 02	-0.8014E 01	0.2722E 01	0.2625E 02
1.0000E-00	0.	0.0000E 01	-0.2234E 02	-0.1232E-04	0.2074E 02

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_T	\bar{I}_t
0.	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.1000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.2000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.3000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.4000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.5000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.6000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.7000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.8000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
0.9000E-00	0.3474E 01	0.1679E 02	0.1600E 02	0.1600E 02	0.1600E 02
1.0000E-00	0.	0.0000E 01	-0.1000E 03	-0.2000E-05	0.1000E 02

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_T	\bar{I}_t
0.	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.1000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.2000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.3000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.4000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.5000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.6000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.7000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.8000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
0.9000E-00	0.1386E 01	0.3719E 02	0.3719E 02	0.3719E 02	0.3719E 02
1.0000E-00	0.	0.3706E 02	0.1015E 02	0.1015E 02	0.5397E 02

$$\bar{M}_T = 0.1000E 03$$

$$\bar{N}_T = 0.2940E 01$$

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.11450E 01	0.3557E 02	0.3557E 02	0.9876E 02	0.9876E 02
0.1000E-00	0.11448E 01	0.3557E 02	0.3557E 02	0.9861E 02	0.9867E 02
0.2000E-00	0.11442E 01	0.3557E 02	0.3556E 02	0.9843E 02	0.9840E 02
0.3000E-00	0.11438E 01	0.3556E 02	0.3552E 02	0.9819E 02	0.9809E 02
0.4000E-00	0.11434E 01	0.3554E 02	0.3545E 02	0.9791E 02	0.9790E 02
0.5000E-00	0.11430E 01	0.3551E 02	0.3536E 02	0.9759E 02	0.9751E 02
0.6000E-00	0.11426E 01	0.3547E 02	0.3526E 02	0.9723E 02	0.9704E 02
0.7000E-00	0.11422E 01	0.3543E 02	0.3515E 02	0.9684E 02	0.9669E 02
0.8000E-00	0.9661E 00	0.3497E 02	0.3106E 02	0.6374E 02	0.8202E 02
0.9000E-00	0.6110E 00	0.3420E 02	0.2376E 02	0.3795E 02	0.7052E 02
1.0000E-00	0.	0.3235E 02	0.7190E 01	0.1476E-02	0.5319E 02

$$\bar{M}_T = 0.1000E 03$$

$$\bar{N}_T = 0.5000E 01$$

0.	0.11495E 01	0.3462E 02	0.3462E 02	0.9867E 02	0.9867E 02
0.1000E-00	0.11493E 01	0.3462E 02	0.3461E 02	0.9851E 02	0.9858E 02
0.2000E-00	0.11485E 01	0.3461E 02	0.3460E 02	0.9801E 02	0.9800E 02
0.3000E-00	0.11471E 01	0.3460E 02	0.3456E 02	0.9703E 02	0.9776E 02
0.4000E-00	0.11466E 01	0.3458E 02	0.3448E 02	0.9588E 02	0.9663E 02
0.5000E-00	0.11402E 01	0.3455E 02	0.3429E 02	0.9226E 02	0.9529E 02
0.6000E-00	0.1328E 01	0.3447E 02	0.3382E 02	0.8707E 02	0.9274E 02
0.7000E-00	0.1202E 01	0.3431E 02	0.3267E 02	0.7819E 02	0.8854E 02
0.8000E-00	0.9886E 00	0.3395E 02	0.2771E 02	0.6303E 02	0.8157E 02
0.9000E-00	0.6232E 00	0.3312E 02	0.2198E 02	0.3735E 02	0.7007E 02
1.0000E-00	0.	0.3116E 02	0.4734E 01	0.2535E-03	0.5223E 02

$$\bar{M}_T = 0.1000E 03$$

$$\bar{N}_T = 0.1000E 02$$

0.	0.1604E 01	0.3253E 02	0.3253E 02	0.9847E 02	0.9847E 02
0.1000E-00	0.1602E 01	0.3253E 02	0.3253E 02	0.9824E 02	0.9831E 02
0.2000E-00	0.1593E 01	0.3253E 02	0.3251E 02	0.9769E 02	0.9800E 02
0.3000E-00	0.1576E 01	0.3251E 02	0.3246E 02	0.9660E 02	0.9740E 02
0.4000E-00	0.1547E 01	0.3249E 02	0.3235E 02	0.9469E 02	0.9639E 02
0.5000E-00	0.1497E 01	0.3244E 02	0.3209E 02	0.9145E 02	0.9473E 02
0.6000E-00	0.1414E 01	0.3234E 02	0.3151E 02	0.8597E 02	0.9202E 02
0.7000E-00	0.1275E 01	0.3213E 02	0.3009E 02	0.7678E 02	0.8763E 02
0.8000E-00	0.1042E 01	0.3170E 02	0.2660E 02	0.6144E 02	0.8222E 02
0.9000E-00	0.6523E 00	0.3072E 02	0.1764E 02	0.3598E 02	0.6901E 02
1.0000E-00	0.	0.2851E 02	-0.1004E 01	0.5435E-04	0.5229E 02

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{M}_T = 0.1000E 03$				$\bar{M}_T = 0.2500E 02$	
\bar{r}	\bar{w}	\bar{N}_T	\bar{N}_t	\bar{N}_T	\bar{M}_t
0.	0.1934E 01	0.2797E 02	0.2797E 02	0.2756E 02	0.9756E 02
0.1000E-00	0.1930E 01	0.2796E 02	0.2795E 02	0.2733E 02	0.9713E 02
0.2000E-00	0.1917E 01	0.2795E 02	0.2790E 02	0.2661E 02	0.9702E 02
0.3000E-00	0.1892E 01	0.2792E 02	0.2779E 02	0.2524E 02	0.9626E 02
0.4000E-00	0.1850E 01	0.2786E 02	0.2756E 02	0.2282E 02	0.9500E 02
0.5000E-00	0.1780E 01	0.2776E 02	0.2703E 02	0.1902E 02	0.9299E 02
0.6000E-00	0.1689E 01	0.2757E 02	0.2601E 02	0.1327E 02	0.8985E 02
0.7000E-00	0.1489E 01	0.2720E 02	0.2366E 02	0.7273E 02	0.6496E 02
0.8000E-00	0.1200E 01	0.2643E 02	0.1831E 02	0.5680E 02	0.7711E 02
0.9000E-00	0.7377E 00	0.2499E 02	0.6061E 01	0.3213E 02	0.6595E 02
1.0000E-00	0.	0.2189E 02	-0.1798E 02	-0.2956E-04	0.5045E 02

$\bar{M}_T = 0.1000E 03$			$\bar{M}_T = 0.1000E 03$		
0.	0.3313E 01	0.2268E 02	0.2268E 02	0.9410E 02	0.9117E 02
0.1000E-00	0.3303E 01	0.2266E 02	0.2260E 02	0.9365E 02	0.9384E 02
0.2000E-00	0.3272E 01	0.2257E 02	0.2232E 02	0.9227E 02	0.9305E 02
0.3000E-00	0.3211E 01	0.2240E 02	0.2174E 02	0.8971E 02	0.9163E 02
0.4000E-00	0.3119E 01	0.2211E 02	0.2059E 02	0.8552E 02	0.8934E 02
0.5000E-00	0.2973E 01	0.2161E 02	0.1853E 02	0.7903E 02	0.8590E 02
0.6000E-00	0.2741E 01	0.2075E 02	0.1395E 02	0.6932E 02	0.8086E 02
0.7000E-00	0.2393E 01	0.1823E 02	0.5233E 01	0.5536E 02	0.7374E 02
0.8000E-00	0.1771E 01	0.1653E 02	-0.1201E 02	0.3661E 02	0.6418E 02
0.9000E-00	0.1101E 01	0.1172E 02	-0.4537E 02	0.1500E 02	0.5270E 02
1.0000E-00	0.	0.3266E 01	-0.9855E 02	-0.3814E-04	0.4256E 02

$\bar{M}_T = 0.1000E 04$			$\bar{M}_T = 0.$		
0.	0.2344E 01	0.2247E 03	0.2243E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2340E 01	0.2246E 03	0.2246E 03	1.0000E 03	1.0000E 03
0.2000E-00	0.2335E 01	0.2246E 03	0.2246E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2345E 01	0.2246E 03	0.2240E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2348E 01	0.2246E 03	0.2245E 03	0.9997E 03	0.9999E 03
0.5000E-00	0.2343E 01	0.2246E 03	0.2245E 03	0.9990E 03	0.9990E 03
0.6000E-00	0.2337E 01	0.2245E 03	0.2245E 03	0.9967E 03	0.9984E 03
0.7000E-00	0.2313E 01	0.2243E 03	0.2244E 03	0.9943E 03	0.9939E 03
0.8000E-00	0.2241E 01	0.2245E 03	0.2235E 03	0.9870E 03	0.9750E 03
0.9000E-00	0.1700E 01	0.2242E 03	0.2107E 03	0.9727E 03	0.8900E 03
1.0000E-00	0.	0.2170E 03	0.6405E 02	0.5314E 00	0.6289E 03

$\bar{M}_T = 0.1000E 04$				$\bar{N}_T = 0.2940E 01$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2362E 01	0.2229E 03	0.2229E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2362E 01	0.2229E 03	0.2229E 03	0.9999E 03	1.0000E 03
0.2000E-00	0.2362E 01	0.2229E 03	0.2229E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2361E 01	0.2229E 03	0.2229E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2361E 01	0.2229E 03	0.2229E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2360E 01	0.2229E 03	0.2229E 03	0.9990E 03	0.9996E 03
0.6000E-00	0.2354E 01	0.2229E 03	0.2229E 03	0.9961E 03	0.9984E 03
0.7000E-00	0.2329E 01	0.2229E 03	0.2228E 03	0.9840E 03	0.9938E 03
0.8000E-00	0.2226E 01	0.2228E 03	0.2220E 03	0.9735E 03	0.9749E 03
0.9000E-00	0.1797E 01	0.2223E 03	0.2090E 03	0.7215E 03	0.8275E 03
1.0000E-00	0.	0.2153E 03	0.6208E 02	0.1335E-01	0.6285E 03

$\bar{M}_T = 0.1000E 04$				$\bar{N}_T = 0.5000E 01$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2373E 01	0.2217E 03	0.2217E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9999E 03	1.0000E 03
0.2000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2371E 01	0.2217E 03	0.2217E 03	0.9990E 03	0.9996E 03
0.6000E-00	0.2365E 01	0.2217E 03	0.2217E 03	0.9961E 03	0.9984E 03
0.7000E-00	0.2340E 01	0.2217E 03	0.2217E 03	0.9838E 03	0.9937E 03
0.8000E-00	0.2236E 01	0.2217E 03	0.2208E 03	0.9330E 03	0.9747E 03
0.9000E-00	0.1804E 01	0.2212E 03	0.2076E 03	0.7206E 03	0.8271E 03
1.0000E-00	0.	0.2141E 03	0.5981E 02	0.3875E-02	0.6281E 03

$\bar{M}_T = 0.1000E 04$				$\bar{N}_T = 0.1000E 02$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2401E 01	0.2191E 03	0.2191E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9999E 03	1.0000E 03
0.2000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2399E 01	0.2191E 03	0.2191E 03	0.9990E 03	0.9995E 03
0.6000E-00	0.2393E 01	0.2191E 03	0.2191E 03	0.9959E 03	0.9983E 03
0.7000E-00	0.2367E 01	0.2191E 03	0.2190E 03	0.9834E 03	0.9935E 03
0.8000E-00	0.2260E 01	0.2190E 03	0.2181E 03	0.9318E 03	0.9743E 03
0.9000E-00	0.1820E 01	0.2185E 03	0.2044E 03	0.7182E 03	0.8961E 03
1.0000E-00	0.	0.2112E 03	0.5386E 02	0.1136E-02	0.6280E 03

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{M}_T = 0.1000E 04$			$\bar{N}_T = 0.2500E 02$		
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2486E 01	0.2114E 03	0.2114E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9999E 03	0.9999E 03
0.2000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9988E 03	0.9995E 03
0.6000E-00	0.2477E 01	0.2114E 03	0.2113E 03	0.9955E 03	0.9981E 03
0.7000E-00	0.2448E 01	0.2113E 03	0.2113E 03	0.9821E 03	0.9930E 03
0.8000E-00	0.2332E 01	0.2113E 03	0.2102E 03	0.9283E 03	0.9728E 03
0.9000E-00	0.1869E 01	0.2107E 03	0.1952E 03	0.7113E 03	0.8932E 03
1.0000E-00	0.	0.2009E 03	0.3657E 02	0.7047E-03	0.6268E 03

$\bar{M}_T = 0.1000E 04$			$\bar{N}_T = 0.1000E 03$		
0.	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.1000E-00	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.2000E-00	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2924E 01	0.1794E 03	0.1794E 03	0.9998E 03	0.9999E 03
0.5000E-00	0.2920E 01	0.1794E 03	0.1794E 03	0.9990E 03	0.9991E 03
0.6000E-00	0.2907E 01	0.1794E 03	0.1794E 03	0.9990E 03	0.9990E 03
0.7000E-00	0.2861E 01	0.1794E 03	0.1792E 03	0.9950E 03	0.9980E 03
0.8000E-00	0.2698E 01	0.1793E 03	0.1773E 03	0.9410E 03	0.9690E 03
0.9000E-00	0.2111E 01	0.1783E 03	0.1536E 03	0.6574E 03	0.7400E 03
1.0000E-00	0.	0.1670E 03	-0.4804E 02	0.4877E-04	0.6210E 03

$\bar{M}_T = 0.2000E 04$			$\bar{N}_T = 0.$		
0.	0.2831E 01	0.3874E 03	0.3874E 03	0.9999E 03	0.9999E 03
0.1000E-00	0.2831E 01	0.3874E 03	0.3874E 03	0.9999E 03	0.9999E 03
0.2000E-00	0.2831E 01	0.3874E 03	0.3874E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2831E 01	0.3874E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2831E 01	0.3874E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.5000E-00	0.2831E 01	0.3873E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.6000E-00	0.2830E 01	0.3873E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.7000E-00	0.2821E 01	0.3873E 03	0.3872E 03	0.9999E 03	0.9999E 03
0.8000E-00	0.2768E 01	0.3873E 03	0.3869E 03	0.9999E 03	0.9999E 03
0.9000E-00	0.2600E 01	0.3870E 03	0.3766E 03	0.9999E 03	0.9999E 03
1.0000E-00	0.	0.3768E 03	0.1100E 02	0.1000E 01	0.1000E 01

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$$\bar{M}_T = 0.2000E 04$$

$$\bar{N}_T = 0.2940E 01$$

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2841E 01	0.3856E 03	0.3856E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2832E 01	0.3856E 03	0.3856E 03	0.1991E 04	0.1996E 04
0.8000E-00	0.2778E 01	0.3856E 03	0.3854E 03	0.1944E 04	0.1980E 04
0.9000E-00	0.2416E 01	0.3853E 03	0.3750E 03	0.1631E 04	0.1870E 04
1.0000E-00	0.	0.3751E 03	0.1101E 03	0.5632E-01	0.1286E 04

$$\bar{M}_T = 0.2000E 04$$

$$\bar{N}_T = 0.5000E 01$$

0.	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2849E 01	0.3845E 03	0.3845E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2840E 01	0.3845E 03	0.3845E 03	0.1991E 04	0.1996E 04
0.8000E-00	0.2785E 01	0.3845E 03	0.3842E 03	0.1944E 04	0.1980E 04
0.9000E-00	0.2471E 01	0.3842E 03	0.3738E 03	0.1630E 04	0.1870E 04
1.0000E-00	0.	0.3732E 03	0.1078E 03	0.1512E-01	0.1286E 04

$$\bar{M}_T = 0.2000E 04$$

$$\bar{N}_T = 0.1000E 02$$

0.	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2867E 01	0.3817E 03	0.3817E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2859E 01	0.3817E 03	0.3817E 03	0.1991E 04	0.1996E 04
0.8000E-00	0.2803E 01	0.3817E 03	0.3814E 03	0.1943E 04	0.1979E 04
0.9000E-00	0.2434E 01	0.3814E 03	0.3708E 03	0.1627E 04	0.1869E 04
1.0000E-00	0.	0.3710E 03	0.1025E 03	0.3662E-02	0.1286E 04

$\bar{N}_T = 0.2000E 04$

$\bar{N}_T = 0.2500E 02$

\bar{N}_1	\bar{V}	\bar{N}_r	\bar{N}_t	\bar{N}_r	\bar{N}_t
0.	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2925E 01	0.3738E 03	0.3738E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2924E 01	0.3738E 03	0.3738E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2915E 01	0.3738E 03	0.3738E 03	0.1990E 04	0.1996E 04
0.8000E-00	0.2855E 01	0.3738E 03	0.3734E 03	0.1941E 04	0.1979E 04
0.9000E-00	0.2473E 01	0.3734E 03	0.3620E 03	0.1620E 04	0.1866E 04
1.0000E-00	0.	0.3626E 03	0.8473E 02	0.5188E-03	0.1285E 04

$\bar{N}_T = 0.2000E 04$

$\bar{N}_T = 0.1000E 03$

0.	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.1999E 04	0.2000E 04
0.5000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.3213E 01	0.3378E 03	0.3378E 03	0.1997E 04	0.1999E 04
0.7000E-00	0.3199E 01	0.3378E 03	0.3378E 03	0.1988E 04	0.1995E 04
0.8000E-00	0.3122E 01	0.3378E 03	0.3373E 03	0.1929E 04	0.1974E 04
0.9000E-00	0.2669E 01	0.3373E 03	0.3216E 03	0.1587E 04	0.1853E 04
1.0000E-00	0.	0.3242E 03	-0.9233E 00	0.2807E-02	0.1280E 04

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SECTION 5
AXISYMMETRIC STRESSES AND DEFLECTIONS IN SHELLS
DUE TO THERMAL AND MECHANICAL LOADS

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SECTION 5

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SECTION 5 - AXISYMMETRIC STRESSES AND DEFLECTIONS IN SHELLS

DUE TO THERMAL AND MECHANICAL LOADS

5.1 SUMMARY

This section considers the axisymmetric stresses and deflections in heated shells under load. Basic equations, applicable to general shells of revolution, are derived in Sub-section 5.3. These are discussed and applied specifically to conical shells (Sub-section 5.4) and cylindrical shells (Sub-section 5.5) where solutions to the differential equations are given.

5.2 INTRODUCTION

The purpose of this section is to present the theory for the linear elastic analysis of shells of revolution subjected to axisymmetric mechanical loads and temperatures. The formulation used is the linearized version of the development given by E. Reissner (Reference 5-1), extended to include the effects of temperature.

The present work is exact within the framework of linear shell theory and removes the following restrictions of the more elementary presentation in Section 7, Volume I (Reference 5-2):

- (1) The material in Volume I was limited to the cases of truncated conical and cylindrical shells.
- (2) The development in that volume was approximate for the conical shells and,
- (3) The temperature was constant through the thickness, i.e., only meridional variation was permitted.

Since the main objective of this section is to develop the general equations governing the analysis of axisymmetric shells, minimal emphasis is placed upon obtaining specific numerical results. However, problems involving interaction between shells and bulkheads are discussed and, for the purposes of illustration, the general solution for conical and cylindrical shells are derived. Numerical results are given for an unrestrained cylindrical shell subjected to a prescribed temperature variation.

The following symbols are used throughout this section:

h Shell thickness

$$t = \frac{\sqrt{Rh}}{\sqrt{3(1-\nu)^2}}$$

r Distance from a general point on the meridian to the axis of revolution

s Meridional coordinate

5.2 (Cont'd)

u	Displacement in the r direction
w	Axial displacement
z	Axial coordinate
D	Flexural rigidity, $\frac{Eh^3}{12(1-\nu^2)}$
E	Young's modulus
G(s), H(s)	Load - temperature functions defined by Eqs. (1b) of Paragraph 5.3.2 and (1b) of Paragraph 5.4.2, respectively
H	Horizontal force per unit of circumferential length
K	Curvature
L	Length
M	Moment per unit of length
N	Force per unit of length
M_T, N_T	$\int_h E\lambda T \zeta d\zeta, \int_h E\lambda T d\zeta$, respectively
P	Surface traction (force per unit of area)
P_H, P_V	Horizontal and vertical components of P, respectively
Q	Shear force per unit of circumferential length perpendicular to shell mid-plane
R	Radius of cylinder
T	Temperature rise above unstressed undeflected datum
V	Vertical force per unit of circumferential length
α	$\sqrt{(r')^2 + (z')^2}$
β	Rotation of meridional element
ϵ	Strain
ζ	Thickness coordinate
θ	Azimuth or hoop angle
λ	Coefficient of linear thermal expansion
ν	Poisson's ratio
ξ	Meridional coordinate
ρ	$\frac{4\sqrt{Eh \tan^2 \phi}}{D}$
σ	Stress
ϕ	Angle between tangent to meridian and the horizontal
ψ	$\frac{1}{2\rho s}$

SUBSCRIPTS

o, 1	Refer to shell edges
c, p	Complementary and particular solution, respectively
i, f	Initial and final respectively
s	In the meridional direction
θ	In the azimuth or hoop direction
ξ	In the meridional direction

5.3 BASIC EQUATIONS

5.3.1 Configuration of a Typical Shell Element

Since this study is concerned with axisymmetric problems for a shell of revolution it is sufficient to consider a typical meridional element before and after deformation due to loads and temperature (Figure 5.3.1-1). Positive directions for loads and displacements are as shown in the figure. The equation of the meridional curve is expressed parametrically by

$$\begin{aligned} r &= r(\xi) \\ z &= z(\xi) \end{aligned} \tag{1a}$$

where ξ is the independent variable which defines the position of a general point on the meridional mid-surface curve. However, the quantity ξ may not have the units of length (for example, ξ may denote an angle). In order to define length along the curve in a general manner, we introduce the quantity $\alpha = \alpha(\xi)$, such that a differential arc length ds is given by

$$ds = \alpha d\xi. \tag{1b}$$

But,

$$\begin{aligned} \cos \varphi &= \frac{dr}{ds} = \frac{dr}{\alpha d\xi} = \frac{r'}{\alpha} \\ \sin \varphi &= \frac{dz}{ds} = \frac{dz}{\alpha d\xi} = \frac{z'}{\alpha} \end{aligned} \tag{1c}$$

Therefore,

$$(r')^2 + (z')^2 = \alpha^2, \tag{1d}$$

which defines α .

We consider that the tangent to the meridional curve before deformation makes an angle φ with a radial line normal to the axis of revolution; the angle after deformation is designated by $(\varphi + \beta)$. Other quantities in the figure are self-explanatory or indicated in the nomenclature, and moment vectors are obtained using the right-hand rule.

5.3.2 Equilibrium Equations

In the following derivation the forces per unit of circumferential length are designated by the components (N_ξ, Q) or the statically equivalent components (V, H) . The relationships between these two sets of forces are given by

$$\begin{aligned} Q &= V \cos \varphi - H \sin \varphi \\ N_\xi &= V \sin \varphi + H \cos \varphi \end{aligned} \tag{1}$$

where, in the linear theory, the initial configuration is employed in writing the force equilibrium equations (terms in β neglected).

A free body diagram showing equilibrium of forces (Figure 5.3.2-1) yields the following equations:

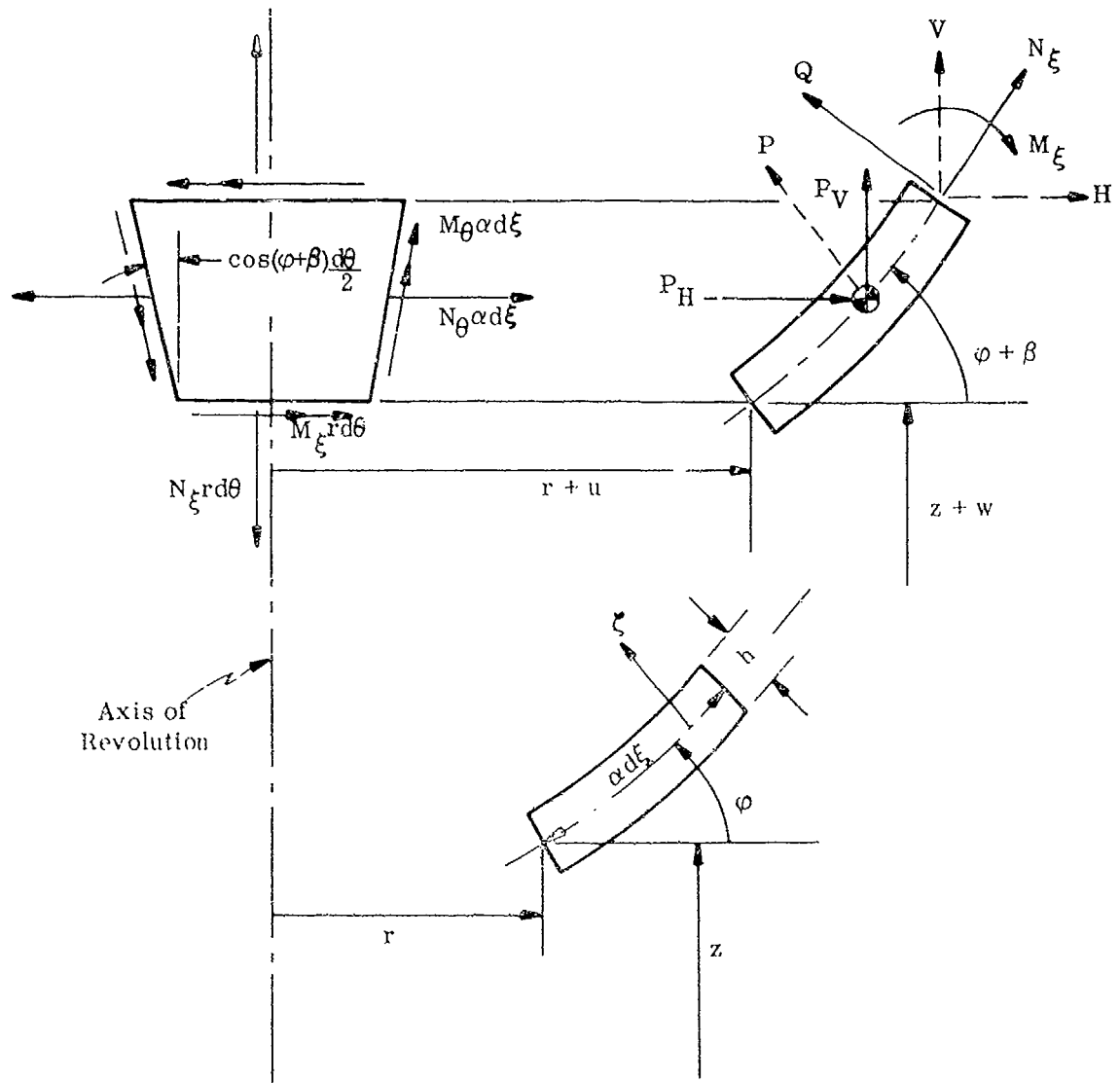


FIGURE 5. 3.1-1 CONFIGURATION OF A TYPICAL MERIDIONAL ELEMENT BEFORE AND AFTER DEFORMATION

5.3.2 (Cont'd)

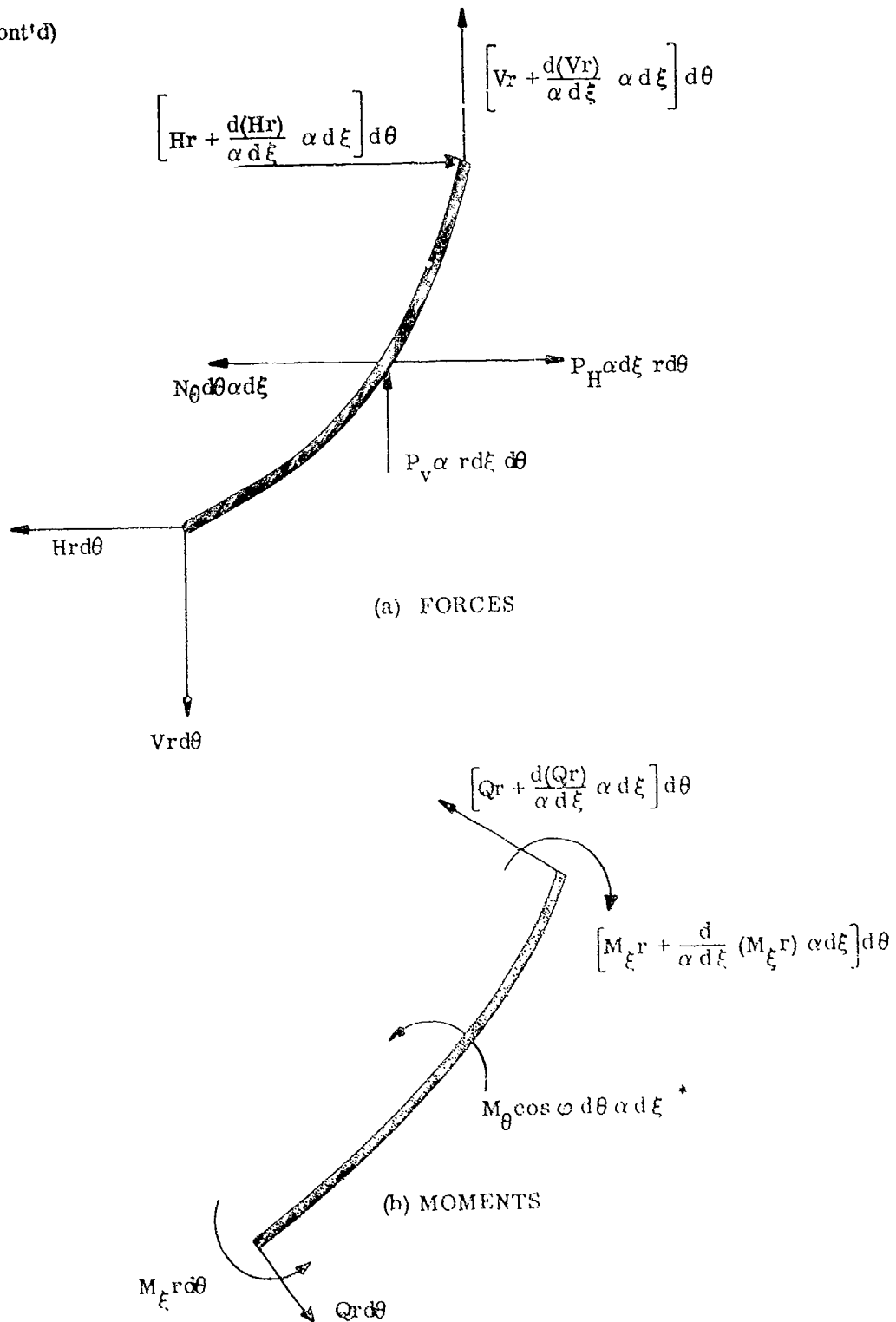


FIGURE 5.3.2-1 EQUILIBRIUM OF FORCES AND MOMENTS

* This represents the vector sum of the hoop moments shown in Figure 5.3.1-1. Since the vectors representing the hoop moments form an angle $\phi d\theta$ (neglecting terms in β), their resultant is of magnitude $M_\theta \cos \phi d\theta \alpha d\xi$ with the direction as indicated in this figure.

5.3.2 (Cont'd)

$$(rV)' + r\alpha P_V = 0 \quad (2a)$$

$$(rH)' - \alpha N_\theta + r\alpha P_H = 0 \quad (2b)$$

From the free body diagram of Figure 5.3.2-1b, moment equilibrium results in

$$(rM_\xi)' - \alpha \cos \varphi M_\theta - \alpha rQ = 0 \quad (3)$$

5.3.3 Curvature Changes and Mid-Surface Strains in Terms of Displacements

Consider a meridional element of the shell mid-surface before deformation (Figure 5.3.1-1). The length of this element is given by

$$dL_i = \frac{dr}{d\xi} d\xi = \frac{r'}{\cos \varphi} d\xi$$

After deformation, the new length is

$$dL_f = \frac{(r' + u') d\xi}{\cos(\varphi + \beta)}$$

where β is the rotation of the element. Thus, the meridional strain of the mid-surface is

$$\begin{aligned} \epsilon_\xi &= \frac{dL_f - dL_i}{dL_i} = \frac{\frac{r' + u'}{\cos(\varphi + \beta)} - \frac{r'}{\cos \varphi}}{\frac{r'}{\cos \varphi}} \\ &= \left(1 + \frac{u'}{r'}\right) \frac{\cos \varphi}{\cos(\varphi + \beta)} - 1 \\ &= \left(1 + \frac{u'}{r'}\right) \cos \varphi \left[\frac{1}{\cos \varphi} + \frac{\beta \sin \varphi}{\cos^2 \varphi} + \dots \right] - 1 \end{aligned}$$

If we neglect terms higher than order β , the linear meridional midsurface strain becomes*

$$\epsilon_\xi = \frac{u'}{r'} + \frac{\sin \varphi}{\cos \varphi} \beta$$

or

$$\epsilon_\xi = \frac{u'}{r'} + \frac{z'}{r'} \beta \quad (1)$$

* It should be noted that the derivations in this section involve division by quantities r and r' . Subsequent derivations also involve division by z' . Therefore, at points where these quantities become zero, the resulting derivations and equations become valid only when appropriate limiting processes and regularity conditions are imposed (see Paragraphs 5.4 and 5.5).

5.3.3 (Cont'd)

The hoop strain of the mid-surface ϵ_θ is a measure of the increase in circumferential length per unit of length and is therefore given by

$$\epsilon_\theta = \frac{u}{r}. \quad (2)$$

In order to obtain bending moments in terms of deformations, it is necessary to determine the changes in principal curvatures of the shell mid-surface. From the definition of curvature, the change in meridional curvature due to deformation is

$$K_\xi = \frac{d(\varphi + \beta)}{\alpha d\xi} = \frac{d\varphi}{\alpha d\xi} = \frac{\beta'}{\alpha} \quad (3a)$$

The hoop principal radius of curvature is defined by the length of the normal line to the surface which is bounded by the surface and the axis of revolution. Thus, the change in hoop curvature due to deformation is

$$K_\theta = \frac{\sin(\varphi + \beta) - \sin\varphi}{r}$$

Again, neglecting terms of order higher than β yields

$$K_\theta = \frac{\beta \cos\varphi}{r} \quad (3b)$$

5.3.4 Stress-Strain and Moment-Curvature Relations

The stress-strain relations for the shell mid-surface including temperature are given by

$$\begin{aligned} \sigma_\xi &= \frac{E}{1-\nu^2} (\epsilon_\xi + \nu\epsilon_\theta) - \frac{E\lambda T}{1-\nu} \\ \sigma_\theta &= \frac{E}{1-\nu^2} (\epsilon_\theta + \nu\epsilon_\xi) - \frac{E\lambda T}{1-\nu} \end{aligned} \quad (1)$$

where λ is the coefficient of linear thermal expansion. Assuming that the strains are linear through the thickness of the shell, the bending strains are odd functions of ζ and therefore integration of the stress-strain relations through the thickness yields

$$\begin{aligned} N_\xi &= \int_{-h}^h \sigma_\xi d\zeta = \frac{Eh}{1-\nu^2} (\epsilon_\xi + \nu\epsilon_\theta) - \frac{N_T}{1-\nu} \\ N_\theta &= \int_{-h}^h \sigma_\theta d\zeta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu\epsilon_\xi) - \frac{N_T}{1-\nu} \end{aligned} \quad (2)$$

where

$$N_T = \int_{-h}^h E\lambda T d\zeta.$$

5.3.4 (Cont'd)

For purposes of obtaining expressions for the moments, it is further assumed that normals to the mid-surface before deformation remain normal after deformation, and that mid-surface stretching due to bending is negligible. Then, employing the stress-strain law, integration of the first moments of the stresses through the thickness results in

$$M_{\xi} = \int_h \sigma_{\xi} \zeta d\zeta = -D \left[K_{\xi} + \nu K_{\theta} \right] - \frac{M_T}{1-\nu} \quad (3)$$

$$M_{\theta} = \int_h \sigma_{\theta} \zeta d\zeta = -D \left[K_{\theta} + \nu K_{\xi} \right] - \frac{M_T}{1-\nu}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$M_T = \int_h E\lambda T \zeta d\zeta .$$

Substitution of Eqs. (1) - (3) of Paragraph 5.3.3 into Eqs. (2) and (3) gives the force resultants and moments in terms of slopes and displacements:

$$N_{\xi} = \frac{Eh}{1-\nu^2} \left[\frac{u' + z' \beta}{r'} + \frac{\nu u}{r} \right] - \frac{N_T}{1-\nu} \quad (4a)$$

$$N_{\theta} = \frac{Eh}{1-\nu^2} \left[\frac{u}{r} + \frac{\nu(u' + z' \beta)}{r'} \right] - \frac{N_T}{1-\nu}$$

$$M_{\xi} = -D \left[\frac{\beta'}{\alpha} + \frac{\nu \beta r'}{r\alpha} \right] - \frac{M_T}{1-\nu} \quad (4b)$$

$$M_{\theta} = -D \left[\frac{\beta r'}{r\alpha} + \frac{\nu \beta'}{\alpha} \right] - \frac{M_T}{1-\nu} ,$$

where $\cos \varphi = \frac{r'}{\alpha}$ has been employed.

A compatibility condition in terms of force resultants and slope may now be obtained. Elimination of the displacement u between Eqs. (1) and (2) of Paragraph 5.3.3 results in

$$\epsilon_{\xi} = \frac{(r'_{,\theta})'}{r'} + \frac{z'}{r'} \beta . \quad (5a)$$

The above may be combined with Eqs. (2) to yield

5.3.4 (Cont'd)

$$N_{\xi} - \nu N_{\theta} + N_T = \frac{[r(N_{\theta} - \nu N_{\xi} + N_T)]'}{r'} + \frac{Eh z' \beta}{r'} \quad (5b)$$

5.3.5 Formulation of the Boundary Value Problem

(1) Differential Equation

In the following development, it will be assumed that V is known at one of the shell edges ($V = V_0$ at $\xi = \xi_0$) so that from static equilibrium Eq. (2a) of Paragraph 5.3.2,

$$rV = r_0 V_0 - \int_{\xi_0}^{\xi} r \alpha P_V d\xi. \quad (1)$$

This implies that V is a known quantity throughout the shell.

Then the quantities Q , N_{ξ} , N_{θ} , H , M_{ξ} , M_{θ} may be eliminated from among the seven equations (1), (2a), (2b), (3) of Paragraph 5.3.2, and (4b) and (5b) of Paragraph 5.3.4, to yield a differential equation for the unknown slope β . After detailed calculation and considerable simplification, this equation may be written in the form:

$$\begin{aligned} & \left(\frac{r}{r'}\right)^2 [L(\beta)]'' + \frac{1}{\alpha r} \left[\left(\frac{\alpha r}{r'}\right)^2 \left(\frac{r}{\alpha}\right)'\right]' [L(\beta)]' + \left[\nu \frac{1}{\alpha} \left(\frac{\alpha r}{r'}\right)'\right. \\ & \left. - i' - 1 + \frac{1}{r'} \left(\frac{r}{\alpha} \left(\frac{\alpha r}{r'}\right)'\right)'\right] L(\beta) + \frac{Eh z'}{D r'} \beta'' - \frac{r^2}{\alpha z'} \frac{V''}{D} + \left[\nu \frac{r}{r'} \left(\frac{z'}{\alpha}\right)'\right. \\ & \left. - \frac{z'}{r r'^2} \left(\frac{r^3}{\alpha} \left(\frac{r'}{z'}\right)'\right)'\right] \frac{V'}{D} + \left[\left(\frac{r}{r'}\right)^2 \left(\frac{r'^2}{\alpha z'}\right)'' - \frac{1}{\alpha r} \left(\frac{\alpha r}{r'}\right)^2 \left(\frac{r}{\alpha}\right)'\left(\frac{r'^2}{\alpha z'}\right)'\right. \\ & \left. - \left(\frac{r}{\alpha}\right) \left(\frac{\alpha r}{r'}\right)'\left(\frac{r'}{\alpha z'} + \frac{\alpha}{z'} + \nu \frac{1}{r'} \left(\frac{r z'}{\alpha}\right)'\right) - \frac{r r'^2}{\alpha^2 z'} \left(\frac{\alpha}{r'}\right)'\right] \frac{V}{D} \\ & + \frac{1}{D} \left[-\nu r P_H - \frac{1}{r'} \left(r^2 P_H\right)'\right] - \left(\frac{r}{r'}\right)^2 \left(\frac{r'}{z' \alpha}\right) \frac{M_T'}{D(1-\nu)} \\ & - \frac{1}{\alpha r} \left[\left(\frac{\alpha r}{r'}\right)^2 \left(\frac{r}{\alpha}\right)'\left(\frac{r'}{z' \alpha}\right) \frac{M_T'}{D(1-\nu)}\right]' - \left[\nu \frac{1}{\alpha} \left(\frac{\alpha r}{r'}\right)'\right] - i' - 1 \\ & + \frac{1}{r'} \left[\frac{r}{\alpha} \left(\frac{\alpha r}{r'}\right)'\right]' \left(\frac{r'}{z' \alpha}\right) \frac{M_T'}{D(1-\nu)} \end{aligned} \quad (2)$$

5.3.5 (Cont'd)

where

$$L(\beta) = \frac{r'}{z'\alpha^2} \left[\beta'' + \frac{\alpha}{r} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \nu \frac{\alpha}{r} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{r} \right)^2 \right\} \beta \right]$$

The Equation (2) is the general differential equation governing the linear analysis for stresses, and deformations in arbitrary shells of revolution due to axisymmetric loads and temperature (see, however, remarks made in footnote, page 5.7).

(2) Force Resultant, Moment, and Displacement Boundary Conditions in Terms of the Slope β .

Since the differential equation (2) is in terms of the dependent variable β , it is advantageous to express the force resultants, moments and displacements, as well as the boundary conditions, as functions of this variable. The derived quantities are given by

$$N_\theta = \frac{D}{\alpha} \left[\frac{r}{\alpha z'} \beta'' + \frac{1}{z'} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \frac{\nu}{z'} \left(\frac{r'}{\alpha} \right)' - \frac{r'^2}{\alpha r z'} \right\} \beta \right] + \left\{ \frac{r r' V}{z'} + \frac{r M_T'}{z' (1-\nu)} \right\} + r P_H \quad (3a)$$

$$N_\xi = D \left[\frac{r'}{\alpha^2 z'} \beta'' + \frac{r'}{r \alpha z'} \left(\frac{r}{\alpha} \right)' \beta' + \frac{r'}{r \alpha} \left\{ \frac{\nu}{z'} \left(\frac{r'}{\alpha} \right)' - \frac{r'^2}{\alpha r z'} \right\} \beta \right] + \frac{\alpha}{z'} V + \frac{r'}{\alpha z'} \frac{M_T'}{1-\nu} \quad (3b)$$

$$Q = -D \left[-\frac{1}{\alpha^2} \beta'' + \frac{1}{r \alpha} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \frac{\nu}{r \alpha} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{\alpha r} \right)^2 \right\} \beta \right] - \frac{M_T'}{\alpha (1-\nu)} \quad (3c)$$

$$H = \frac{D}{\alpha z'} \left[\beta'' + \frac{\alpha}{r} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \frac{\nu \alpha}{r} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{r} \right)^2 \right\} \beta + \frac{\alpha r' V}{D} + \frac{\alpha M_T'}{D(1-\nu)} \right] \quad (3d)$$

$$M_\xi = -D \left[\frac{\beta'}{\alpha} + \frac{\nu r'}{r \alpha} \beta \right] - \frac{M_T'}{1-\nu} \quad (3e)$$

5.3.5 (Cont'd)

$$M_{\theta} = -D \left[\frac{r'}{r\alpha} \beta + \frac{\nu\beta'}{\alpha} \right] - \frac{M_T}{1-\nu} \quad (3f)$$

$$u = \frac{r}{Eh} (N_{\theta} - \nu N_{\xi}) + \frac{N_T r}{Eh} \quad (3g)$$

$$w = \int_{\xi_0}^{\xi} \left[\frac{z'}{Eh} (N_{\xi} - \nu N_{\theta} + N_T) + r' \beta \right] d\xi. \quad (3h)$$

The stresses may then be obtained from

$$\sigma_{\xi} = \frac{1}{h} \left(N_{\xi} + \frac{N_T}{1-\nu} \right) - \frac{E\lambda T}{1-\nu} + \frac{12\zeta}{h^3} \left(M_{\xi} + \frac{M_T}{1-\nu} \right) \quad (4a)$$

$$\sigma_{\theta} = \frac{1}{h} \left(N_{\theta} + \frac{N_T}{1-\nu} \right) - \frac{E\lambda T}{1-\nu} + \frac{12\zeta}{h^3} \left(M_{\theta} + \frac{M_T}{1-\nu} \right). \quad (4b)$$

Typical boundary conditions can be expressed in terms of the above derived quantities as shown by the following examples:

- (a) Clamped Edge at $\xi = \xi_0$

$$\left[\beta \right]_{\xi = \xi_0} = \left[u \right]_{\xi = \xi_0} = 0 \quad (5a)$$

- (b) Pinned Edge at $\xi = \xi_0$

$$\left[u \right]_{\xi = \xi_0} = \left[M_{\xi} \right]_{\xi = \xi_0} = 0 \quad (5b)$$

- (c) Free Edge at $\xi = \xi_0$

$$\left[Q \right]_{\xi = \xi_0} = \left[M_{\xi} \right]_{\xi = \xi_0} = 0 \quad (5c)$$

- (d) Specified Radial Load " H_0 " and Meridional Moment " M_0 " at edge $\xi = \xi_0$

$$\left[\frac{D}{\alpha z'} \left(\nu'' + \frac{\alpha}{r} \left(\frac{r'}{\alpha} \right)' \beta' + \left\{ \frac{\nu\alpha}{r} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{r} \right)^2 \right\} \beta \right. \right. \\ \left. \left. + \frac{\alpha r' \nu}{D} + \frac{\alpha M_T'}{D(1-\nu)} \right) \right]_{\xi = \xi_0} = H_0 \quad (5d)$$

$$\left[-D \left(\frac{\beta'}{\alpha} + \frac{\nu r'}{r\alpha} \beta \right) - \frac{M_T}{1-\nu} \right]_{\xi = \xi_0} = M_0$$

5.3.5 (Cont'd)

In order to apply the differential equation (2) and boundary conditions (3) to a specific shell geometry, the quantities r , α , z must be determined from the equation of the shell meridian. Specific examples for the cases of conical and cylindrical shells are presented in the following paragraphs. In general, the major problem encountered in solving Eq. (2) for a given shell of revolution involves obtaining the complementary solution for a fourth order differential equation with variable coefficients. Usually closed form solutions are not possible. However, as may be observed from an inspection of the right hand side of Eq. (2), the introduction of temperature does not complicate matters, since it enters in a form analogous to terms resulting from mechanical loads.

The boundary conditions (Eq. (5d)) are of importance in considering the interaction of the shell with both transverse circular bulkheads and other shells of revolution. Once the linear response to edge loads and moments is determined from a solution of Eq. (2) subject to boundary conditions of the type (5d), influence coefficients are known and internal loads at the junctions of shells and bulkheads may be evaluated by imposing compatibility and equilibrium conditions on slopes and deflections. A description of the procedure to be followed is given in detail in Volume 1, Section 8.

5.4 CONICAL SHELLS

5.4.1 Basic Equations

A meridional section of the cone and the coordinate system is shown in Figure 5.4.1-1.

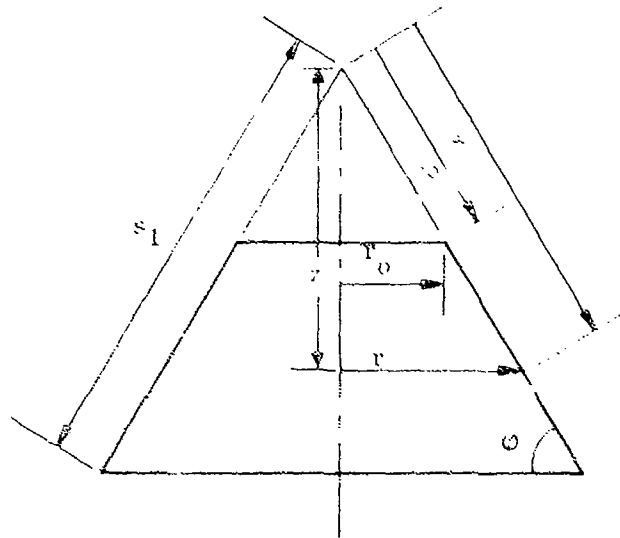


FIGURE 5.4.1-1 CONE GEOMETRY

Referring to Eqs. (1) of Paragraph 5.3.1, we choose the independent variable as

$$\xi = s \tag{13}$$

5.4.1 (Cont'd)

from which it follows that

$$\left. \begin{aligned} r &= s \cos \phi \\ z &= s \sin \phi \\ r' &= \cos \phi \\ z' &= \sin \phi \\ \alpha &= 1, \end{aligned} \right\} \quad (1b)$$

where $\phi = \text{constant}$ has been employed and primed quantities indicate differentiation with respect to s .

The above quantities may be substituted into the general differential equation (2) of Paragraph 5.3.5. After simplification, the resulting differential equation for the linear analysis of conical shells becomes

$$\begin{aligned} s^3 IV + 4\beta^{III} + D^4 \frac{\beta}{s} &= \frac{1}{D} \left[\frac{1}{s} (P_V s^2)' - \nu \tan^2 \phi P_V' \cos \phi \right. \\ &\left. - \frac{1}{\cos \phi s^2} \int_{s_0}^s s P_V ds - \frac{1}{s^2} (P_H s^2)' + \nu P_H' \sin \phi \right] \\ &\left[\frac{r^2 V}{s^2 \cos^2 \phi} - \tan \phi N_T' - \frac{1}{s^2} \left(\frac{s^3 M_T''}{1-\nu} \right)' \right] \end{aligned} \quad (2)$$

where $\phi^4 = \frac{Eh \tan^2 \phi}{D}$.

We will consider the special case of normal surface pressure. Then, from Figure 5.3.1-1:

$$P_V = P \cos \phi$$

$$P_H = P \sin \phi,$$

and Eq. (2) reduces to

$$\begin{aligned} s^3 IV + 4\beta^{III} + D^4 \frac{\beta}{s} &= \frac{1}{D} \left[\frac{1}{s} (P s^2)' - \frac{1}{s^2} \int_{s_0}^s P s ds + \frac{r^2 V}{s^2 \cos^2 \phi} \right. \\ &\left. - \tan \phi N_T' - \frac{1}{s^2} \left(\frac{s^3 M_T''}{1-\nu} \right)' \right] \end{aligned} \quad (3)$$

5.4.1 (Cont'd)

Correspondingly, the expressions for the force resultants and moments are obtained from Eqs. (3) of Paragraph 5.3.5 and are given by

$$N_{\theta} = \frac{1}{\tan \phi} \left[D \left(s\beta'' + \beta' - \frac{\beta}{s} \right)' - P_s + \frac{(sM_T)'}{1-\nu} \right] \quad (4a)$$

$$N_s = \frac{1}{\tan \phi} \left[D \left(\beta'' + \frac{\beta'}{s} - \frac{\beta}{s^2} \right) + \frac{r_0 V_0}{s \cos^2 \phi} - \frac{1}{s} \int_0^s P_s ds + \frac{M_T}{1-\nu} \right] \quad (4b)$$

$$Q = -D \left[\beta'' + \frac{\beta'}{s} - \frac{\beta}{s^2} \right] - \frac{M_T}{1-\nu} \quad (4c)$$

$$H = \frac{D}{\sin \phi} \left[\beta'' + \frac{\beta'}{s} - \frac{\beta}{s^2} \right] + \frac{V_0 r_0}{s \sin \phi} - \frac{\cos^2 \phi}{s \sin \phi} \int_0^s P_s ds + \frac{1}{\sin \phi} \frac{M_T}{1-\nu} \quad (4d)$$

$$M_s = -D \left[\beta' + \frac{\nu \beta}{s} \right] - \frac{M_T}{1-\nu} \quad (4e)$$

$$M_{\phi} = -D \left[\frac{\beta}{s} + \nu \beta' \right] - \frac{M_T}{1-\nu} \quad (4f)$$

5.4.2 Solution of the Differential Equation*

When Eq. (3) of Paragraph 5.4.1 is multiplied by s^2 , we have

$$s^2 \beta^{IV} + 4s \beta''' + 6 \beta'' - H(s) = \frac{r_0 V_0}{D s \cos^2 \phi} \quad (1a)$$

where

$$H(s) = \frac{1}{D} \left[(P_s s^2)' + \frac{1}{s} \int_0^s P_s ds + \frac{1}{s} \left(s^3 \frac{M_T}{1-\nu} \right)' - \tan \phi + N_{\theta} s \right] \quad (1b)$$

Introduce a new independent variable ξ defined by

$$\xi = 2\alpha s^2 \quad (1c)$$

* This differential equation for cones of revolution subjected to axisymmetric mechanical loads and temperature was solved by Birkhoff (Reference 5-5) where reference is given to previous work on the subject. However, the equations did not proceed from any general development on shells of revolution. For laminated, doubly linear temperature variations through the thickness are accommodated in this reference.

5.4.2 (Cont'd)

Then the homogeneous form of Eq. (1) becomes

$$\frac{d^4 \beta_c}{d\psi^4} + \frac{2}{\psi} \frac{d^3 \beta_c}{d\psi^3} - \frac{9}{\psi^2} \frac{d^2 \beta_c}{d\psi^2} + \frac{9}{\psi^3} \frac{d\beta_c}{d\psi} + \beta_c = 0. \quad (3a)$$

This equation is now factorable as

$$\left(\frac{d^2}{d\psi^2} + \frac{1}{\psi} \frac{d}{d\psi} - i - \frac{4}{\psi^2} \right) \left(\frac{d^2}{d\psi^2} + \frac{1}{\psi} \frac{d}{d\psi} - i - \frac{4}{\psi^2} \right) \beta_c = 0, \quad (3b)$$

for which the solution is

$$\beta_c = c_1 J_2 \left(i \frac{1}{2} \psi \right) + c_2 Y_2 \left(i \frac{1}{2} \psi \right) + c_3 J_2 \left(\frac{3}{2} \psi \right) + c_4 Y_2 \left(\frac{3}{2} \psi \right), \quad (4)$$

where J_2 and Y_2 are Bessel functions of the first and second kind of order 2. After much manipulation Eq. (4) may be written in terms of Kelvin functions as

$$\begin{aligned} \beta_c = & A \left(\text{ber} \psi - \frac{2}{\psi} \dot{\text{bei}} \psi \right) + B \left(\text{bei} \psi + \frac{2}{\psi} \dot{\text{ber}} \psi \right) \\ & + C \left(\text{ker} \psi - \frac{2}{\psi} \dot{\text{kei}} \psi \right) + F \left(\text{kei} \psi + \frac{2}{\psi} \dot{\text{ker}} \psi \right) \end{aligned} \quad (5)$$

where $\dot{} = \frac{d}{d\psi}$.

The particular solution of Eq. (1) can be easily obtained if the function $H(s)$ is expressed as a polynomial of the form

$$H(s) = \sum_{p=0}^N A_p s^p \quad (6)$$

Then, selecting the solution on the form of series involving integral powers of s , there results

$$\begin{aligned} \beta_p = & \frac{r_0 V_0}{E h s \sin^2 \varphi} + \frac{A_0}{\rho^4} \\ & + \sum_{p=1}^l \sum_{m=1}^p \left[\frac{(-1)^{p-m} + 1}{2} \right] a_m s^m, \end{aligned} \quad (7)$$

5.4.2 (Cont'd)

where the a_m are given by

$$a_m = (-1)^{\frac{p-m}{2}} \frac{A (p+1) (p) (m+1) (m)}{\rho^2 (p-m+2)} \left[\frac{(p-1)!}{(m+1)!} \right]^2 \quad (8)$$

The total solution to the cone problem is $\beta = \beta_c + \beta_p$ and the constants A, B, C, F must be determined from the four boundary conditions (two at each edge). For the special case of a full cone ($s_0 = 0$), finiteness of stresses at the apex requires that $V_0 = 0$. Further, regularity conditions on stress resultants and shears, require that "C" = "F" = 0. Specific solutions for the full cone then resolve themselves into the determination of the two constants "A" and "B" from the boundary conditions at the base.

The Kelvin functions and their first derivatives appearing in Eq. (5) are extensively and accurately tabulated in Reference 5-4 for a wide range of the argument. The form of solution developed above is readily adaptable to numerical computation using digital computers. Parametric studies to determine stresses and deflections can be made.

5.5 CYLINDRICAL SHELLS

5.5.1 Basic Equations

The cylinder and the coordinate system is shown in Figure 5.5.1-1.

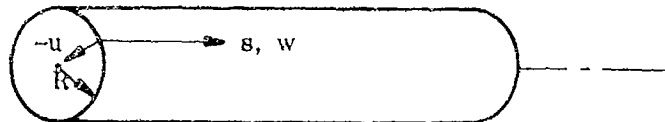


FIGURE 5.5.1-1 CYLINDRICAL GEOMETRY

We choose s again as the independent variable, however, since $r' = 0$ for the cylinder (see Figure 5.3.1-1), then as indicated previously Eq. (2) of Paragraph 5.3.5 is not applicable in the form given. The equations for the cylindrical shell are more readily derivable from the conical shell equations (which were developed from Eq. (2) of Paragraph 5.3.5). Proceeding in this manner, if the equation (3) of Paragraph 5.4.1 is multiplied by $\frac{D}{B}$, the quantity $\frac{1}{B}$ replaced by $\frac{\cos \phi}{r}$ and the appropriate limit as $\phi \rightarrow \frac{\pi}{2}$, $r \rightarrow R$, is taken, there results, for the case of normal pressures,

$$D\beta^{IV} + \frac{Eh\beta}{R^2} = \left[P - \frac{M_T''}{1-\nu} - \frac{N_T}{R} \right]' \quad (1)$$

5.5.1 (Cont'd)

Similarly, the limiting process applied to Eqs. (4) of Paragraph 5.4.1 for the force and moment resultants yields

$$\begin{aligned}
 N_{\theta} &= R \left[D\beta''' + \frac{M_T''}{1-\nu} - P \right] \\
 N_s &= V_o \\
 Q &= - \left[D\beta'' + \frac{M_T'}{1-\nu} \right] = -H \\
 M_s &= -D\beta' - \frac{M_T}{1-\nu} \\
 M_{\theta} &= -\nu D\beta' - \frac{M_T}{1-\nu} .
 \end{aligned} \tag{2}$$

An alternate form of the equilibrium equation in terms of the radial displacement component "u" can be obtained by integrating Eq. (1) once, noting, that $\beta = -u'$. This results in

$$Du^{IV} + \frac{Eh}{R^2} u = -P + \frac{M_T''}{1-\nu} + \frac{N_T}{R} - \frac{\nu V_o}{R} , \tag{3}$$

where the constant of integration is given by the last term on the right. Substituting $\beta = -u'$ into Eqs. (2) the force and moment resultants are

$$\begin{aligned}
 N_{\theta} &= R \left[-Du^{IV} + \frac{M_T''}{1-\nu} - P \right] \\
 N_s &= V_o \\
 Q &= Du''' - \frac{M_T'}{1-\nu} = -H \\
 M_s &= Du'' - \frac{M_T}{1-\nu} \\
 M_{\theta} &= \nu Du' - \frac{M_T}{1-\nu} .
 \end{aligned} \tag{4}$$

Using the formulation of Eq. (1), for all sets of boundary conditions involving β , the solution may be integrated once to obtain u. The constant of integration is determined from

5.5.1 (Cont'd)

the prescribed value of V_0 . When using the formulation of Eq. (3), however, V_0 appears explicitly in the differential equation.

5.5.2 Solution of the Differential Equation

Equation (3) may be written as

$$\frac{t^4}{4} u^{IV} + u = G(s) \quad (1a)$$

where

$$t = \frac{\sqrt{Rh}}{4\sqrt{3(1-\nu)^2}} \quad (= .778\sqrt{Rh} \text{ for } \nu = .30).$$

and

$$G(s) = \frac{R^2}{Eh} \left[-P + \frac{M_T''}{1-\nu} + \frac{N_T}{R} - \frac{\nu V_0}{R} \right]. \quad (1b)$$

The complementary solution of (1a) is given by

$$u_c = \sinh \frac{s}{t} \left(A_1 \sin \frac{s}{t} + A_2 \cos \frac{s}{t} \right) + \cosh \frac{s}{t} \left(A_3 \sin \frac{s}{t} + A_4 \cos \frac{s}{t} \right). \quad (2)$$

A particular solution can easily be obtained if $G(s)$ is expressed as a polynomial with terms of the form

$$G_k(s) = C_k \left(\frac{s}{R} \right)^k; \quad k = 0, 1, 2 \dots M. \quad (3)$$

Then a particular solution corresponding to $G(s) = C_k \left(\frac{s}{R} \right)^k$ can be determined by assuming this solution in the form of a polynomial in (s/R) of degree K and substituting in Eq. (1). There results

$$(u_p)_k = \begin{cases} C_k \left(\frac{s}{R} \right)^k & ; k \leq 3 \\ \sum_{j=0}^{[N]} A_{(k-4j)} \left(\frac{s}{R} \right)^{(k-4j)} & ; k > 3, \end{cases} \quad (4)$$

5.5.2 (Cont'd)

where

$$A_{(k-4j)} = (-1)^j C_k \frac{(k)!}{(k-4j)!} \left[\frac{1}{4} \left(\frac{l}{R} \right)^4 \right]^j,$$

and

$$[N] = \text{Greatest integer} \leq \frac{k}{4}.$$

The complete solution corresponding to $G(s) = \sum_{k=0}^M C_k \left(\frac{s}{R} \right)^k$ is then given by

$$u = u_c + \sum_{k=0}^M (u_p)_k. \quad (5)$$

The constants A_1, A_2, A_3, A_4 must be determined from the specified boundary conditions at the ends of the cylinder. This requires the solution of four simultaneous, linear algebraic equations. In order to eliminate the necessity of solving four equations an approximate method of solution in which the less tedious procedure of solving two pairs of simultaneous equations, each for two unknown constants, is used extensively. The basis of this method is the assumption that edge shears and moments (which are self equilibrating) applied at one end of the shell negligibly affect the stresses and deflections at the opposite end. This assumption is valid if the length of the shell "L" satisfies the condition $L > 3l$, or equivalently, for $\nu = 0.30$, when

$$\frac{L}{R} \geq \frac{7}{3} \left(\frac{h}{R} \right)^{1/2}. \quad \text{The inequality is satisfied in all but very short/thick cylindrical shells.}$$

The method, which appears in Reference 5-2, may be described as follows:

(1) Determine the deflections and stresses corresponding to the particular solution

$$u_p = \sum_{k=0}^M (u_p)_k \quad \text{where the } (u_p)_k \text{ are given by Eq. (4). In general this solution will not satisfy the boundary conditions.}$$

(2) Edge moments and loads are then applied to the semi-infinite cylinder such that when their effects are superposed on the solutions of (1) the boundary conditions at each edge are satisfied. Since there are only two conditions to be met at each edge and there is no assumed interaction of edge effects then, for each edge, the solution of two simultaneous equations in two unknowns (one shear and one bending moment) is required.

5.5.3 NUMERICAL EXAMPLE

A free cylindrical shell is subjected to an elevated temperature as shown in Figure 5.5.3-1, i. e., the inside surface of the cylinder is at a uniform elevated temperature T_i and the outside temperature varies linearly from T_i at $s=0$ to a temperature $T_i(1+A)$ at $s=L$. The

5.5.3 (Cont'd)

variation through the thickness of the shell is assumed to be linear. It is required to determine the stresses and deflections.

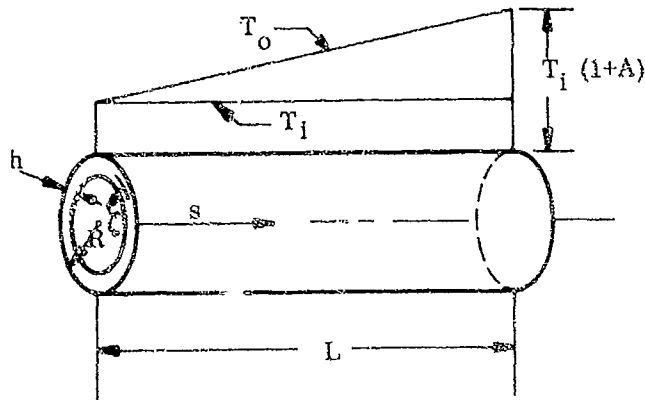


FIGURE 5.5.3-1 ILLUSTRATIVE EXAMPLE

Procedure:

The temperature distribution is given by

$$T(s, z) = T_1 \left[1 + \frac{As}{2L} \left(1 - \frac{2z}{h} \right) \right]$$

and therefore

$$N_T = E\lambda \int_{-\frac{h}{2}}^{\frac{h}{2}} T dz = E\lambda T_1 h \left(1 + \frac{As}{2L} \right)$$

$$M_T = E\lambda \int_{-\frac{h}{2}}^{\frac{h}{2}} T z dz = -E\lambda T_1 \frac{As}{L} \frac{h^2}{12}$$

Referring to Eq. (1b) of Paragraph 5.5.2, since no mechanical loads are applied, $P = V_0 = 0$, and Eq. (1a) of Paragraph 5.5.2 becomes $\frac{1}{4} u^{IV} + u = \frac{R^2}{Eh} \left[\frac{M_T''}{1-\nu} + \frac{N_T}{R} \right]$
 $= R\lambda T_1 \left(1 + \frac{As}{2L} \right)$.

The right hand side can be written in the form $C_0 + C_1 \frac{s}{R}$,

where

$$C_0 = R\lambda T_1$$

$$C_1 = \frac{R^2 \lambda T_1 A}{2L}$$

5.5.3 (Cont'd)

Then from Eqs. (2), (4), and (5) of Paragraph 5.5.2 the solution for the displacement is given by

$$u = \sinh \frac{s}{\ell} \left(A_1 \sin \frac{s}{\ell} + A_2 \cos \frac{s}{\ell} \right) + \cosh \frac{s}{\ell} \left(A_3 \sin \frac{s}{\ell} + A_4 \cos \frac{s}{\ell} \right) + R\lambda T_i \left(1 + \frac{As}{2L} \right) \quad (1)$$

The boundary conditions for free edges are

$$M_s = Q = 0 \quad \text{at } s = 0, L,$$

which from Eqs. 4 of Paragraph 5.5.1 can be written as

$$Du'' - \frac{M_T}{1-\nu} = Du''' - \frac{M_T}{1-\nu} = 0 \quad \text{at } s = 0, L. \quad (2)$$

The constants $A_1 - A_4$ in Eq. (1) are determined from the boundary conditions.

Force resultants and moments are then found by substituting into Eqs. (2) of Paragraph 5.5.1. Since the calculation details are straightforward, only the results are shown. Figure 5.5.3-2 gives nondimensional deflections, force and moment resultants for both a cylinder with $L/\ell = 10$ and a longer cylinder corresponding to $L/\ell = 50$. The graphs show that as the cylinder becomes longer, the peak deflections and stresses approach the end of the surface subjected to the higher thermal gradient. In general, these peak values tend to increase in magnitude for the longer cylinders, resulting in sharp gradients in the vicinity of the edge.

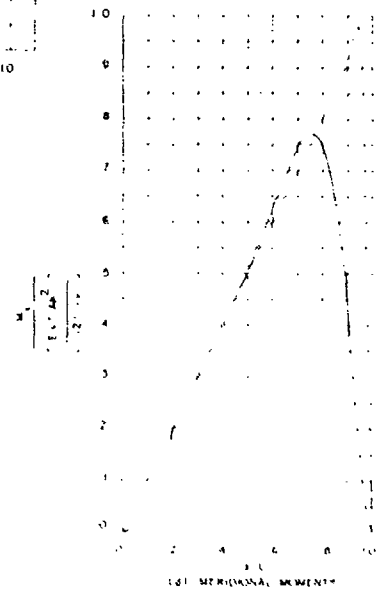
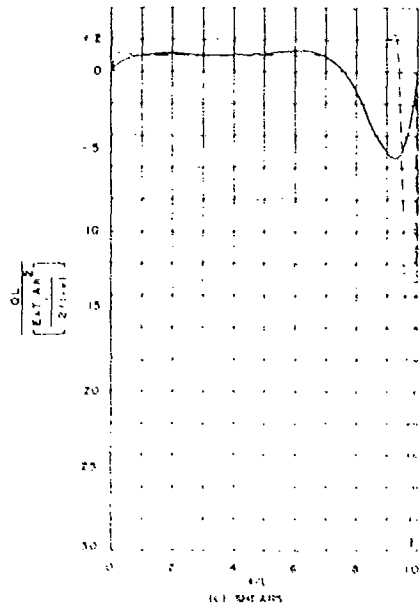
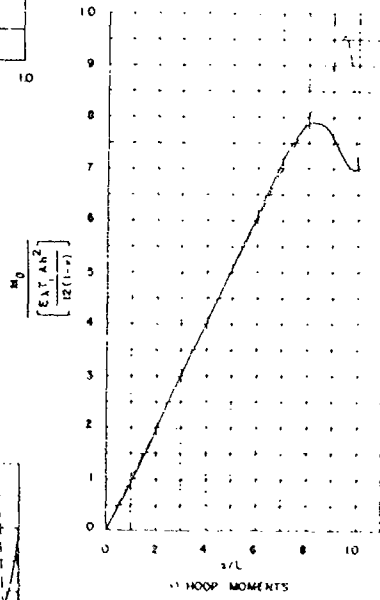
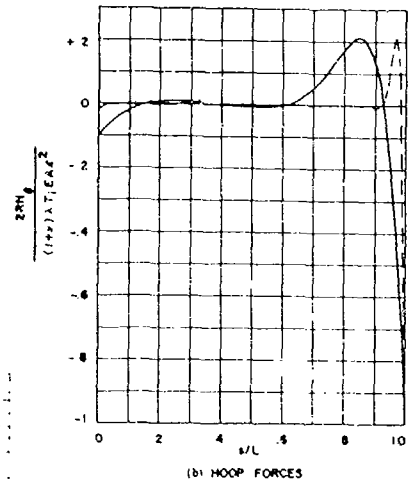
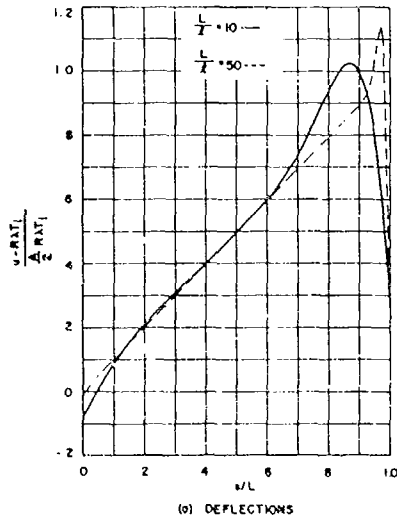


FIGURE 5.5.3-2 NONDIMENSIONAL DEFLECTIONS, FORCE AND MOMENT RESULTANTS FOR THE FREE CYLINDER HEATED AS SHOWN IN FIGURE 5.5.3-1: $l/t = 10, 50$.

5.6 REFERENCES

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2. Structures
3. Structural shells
4. Thermal stresses
5. Mechanical stresses
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Unclassified Report
- SECTION 1 - BEAM COLUMNS
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