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THE OPTIMUM ALLOCATION OF WEAPONS TO TARGETS: A DYNAMIC PROGRAMMING APPROACH

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Department of the Army Project Nos. 503-05-005 and 503-06-002

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

The findings in this report are not to be construed as an official Department of the Army position.

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ABSTRACT

The Optimality Principle of Dynamic Programming is used to determine the optimum solution of a weapons allocation problem for the case involving a total weight constraint on the amount of ammunition employed.

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INTRODUCTION

In many military combat situations one is confronted with the task of determining how the available resources of weapons (or actions) may be allocated to a given target complex so as to minimize the expected number or the "worth" of surviving targets. Since each weapon (or action) has associated with it a certain probability of destruction which is dependent upon the type or class of targets against which the weapon is directed, then one will need, in general, different amounts of each weapon available in inventory. Thus, the planning stage of such an activity requires (among other things) the determination of the number of weapons of a certain type which will be needed to carry out the given objective. That is, given a complex of m targets and n types of weapons that may be directed against the m-targets how often should weapon i be directed against target j so as to accomplish the mission? If, after analysis, it is determined that weapon i should be directed against target j, K_j times, then it is clear that one needs $K = \sum_{j=1}^{m} K_j$ rounds of type i on hand. However, in practice, it may not be possible to fulfill the demands for weapon i due to restrictions placed upon our resources. That is, due to limited production, size and weight restrictions, shipping, etc., it may not be possible to achieve the level of damage desired or to totally destroy the target complex (which would probably be possible with unlimited resources) so that our problem reduces to one of doing the best possible job with the available resources. That is, we wish to minimize the expected "worth" of the surviving targets where we have limited resources available. In general, this problem may be described as a multi-dimensional allocation process, but with the particular constraint that will be used in the problem to be described in the following pages, it will be possible to reduce this problem to a sequence of simpler problems.

THE FORMULATION OF THE PROBLEM

The problem that we shall discuss is a weapon allocation problem related to problems of current interest to the Weapon Systems Laboratory of BRL. We shall use the following notations and definitions:

p_{ij}: the conditional probability that target T_j is destroyed by action A_i, given that the target survives all other actions

q_{ij}: $1 - p_{ij}$

a_j: the worth of target T_j

w_i: the weight of a "missile" from action A_i

x_{ij}: the number of times that action A_i is directed against target T_j

An interpretation of such words as action, worth, missile, etc., may be provided by thinking of an action as the firing of a particular type of projectile, "missile", shell, etc., against a target. The weight would be the weight of the given shell. Since each type of shell may have different blast properties, fragmentation characteristics or different type warheads one may consider action A_i as the firing of shell of type i, i.e., each shell in this group has similar properties. The x_{ij} would be the number of times that shells of type i (action A_i) are directed against target T_j.

The worth of a target is a measure of the importance the offense has assigned to that particular target. The constraint on our resources will be a total weight constraint, W. That is, if K_i is the weight of all projectiles included in A_i and we have n actions then $W = \sum_{i=1}^{n} K_i$ where W is a fixed preassigned number. The problem is: given n actions and m targets; and a fixed total weight, W, determine the set of values $\{x_{ij}\}$ which minimizes the expected worth of the surviving targets.

The expected worth of the surviving targets is a function of nm variables and is given by

$$(1) \qquad F(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} q_{ij}^{x_{ij}}$$

where

$$x_1 = (x_{11}, x_{12}, \ldots, x_{1m})$$

$$\vdots$$

$$x_n = (x_{n1}, x_{n2}, \ldots, x_{nm})$$

The problem then is to select the set of nm numbers x_{ij} $i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$ (or the n m-dimensional vectors) which gives the minimum value of F subject to the conditions:

(a) $0 \leq q_{ij} \leq 1$ for all i and j

(b) $x_{ij} \geq 0$; in particular x_{ij} is to be a non-negative integer,

and (c) $\sum_{i=1}^{n} (w_i \sum_{j=1}^{m} x_{ij}) \leq W$

We shall call x_{ij} an allocation (an allocation of a "missile" from action A_i to the jth target) and a sequence of allocations $x_{11},\ldots,x_{1m}$, $x_{21},\ldots,x_{2m},\ldots,x_{n1},\ldots,x_{nm}$ which satisfies the constraints (b) and (c) we shall refer to as a policy. We can then speak of finding an optimal policy, that is, a policy which results in a minimum value of F.

DYNAMIC PROGRAMMING FORMULATION

Examination of equation 1, subject to the constraints a, b, and c and the dimension involved quickly convinces one that naive approaches such as enumeration of policies or the use of classical methods to obtain the minimum value of F are not feasible because of the unreasonable computing times involved and the complexity of the problem. For example, if $n = m = 4$, and each of the x_{ij} are permitted to run over ten values, the computing time involved in enumeration of F would exceed by several fold the life time of any modern day computing machine.

In order to circumvent this dimensional difficulty, we will reformulate this problem as a multi-stage process and apply the techniques of dynamic programming to obtain a feasible computational scheme. Each stage of the process will consist of determining a value for a specific allocation, x_{ij}. Our goal shall be the reduction of this mn dimensional problem to a sequence of one-dimensional problems. Then at each stage of the process it will be possible to determine an allocation from a search over a one-dimensional space. To attain this simplification we imbed this particular problem within a family of analogous problems, that is, instead of considering a particular weight of resources and a fixed number of actions and targets, we consider the entire family of such problems where W may assume any positive value and n and m may be any natural number. This approach, as we will see, has many computational advantages and furthermore, we shall be able to obtain vital information about the change in optimal policies as the basic parameters, W, n and m change, i.e., we can assess the effect on the optimal policy of adding additional targets and additional actions or of varying the total weight constraint.

In the analysis which follows, the reduction in dimensions takes place in two steps, the two steps come about through the analysis of the structure of the problem. Our problem is to determine the policy $\{x_{11}, x_{12}, \ldots, x_{1m}; x_{21}, x_{22}, \ldots, x_{2m}; \ldots; x_{n1}, \ldots, x_{nm}\}$ which minimizes the expression F, i.e.,

$$\underset{\{x_{11},\ldots,x_{1m};x_{21},\ldots,x_{2m};\ldots;x_{n1},\ldots,x_{nm}\}}{\text{Min}} \sum_{j=1}^{m} a_j \prod_{i=1}^{n} q_{ij}^{x_{ij}}$$

subject to constraints (a), (b) and (c). We can write this as follows:

$$\underset{\{x_{i1}\}}{\text{Min}}\ a_1 \prod_{i=1}^{n} q_{i1}^{x_{i1}} + \underset{\{x_{i2}\}}{\text{Min}}\ a_2 \prod_{i=1}^{n} q_{i2}^{x_{i2}} + \ldots + \underset{\{x_{im}\}}{\text{Min}}\ a_m \prod_{i=1}^{n} q_{im}^{x_{im}}$$

The jth term in this sum is a minimization over the set of values $\{x_{ij}, x_{2j}, \ldots, x_{nj}\}$ and since we can write the constraint (c) as

$$\text{(c)} \quad \sum_{i=1}^{n} \left(w_i \sum_{j=1}^{m} x_{ij}\right) = \sum_{i=1}^{n} w_i\, x_{ij} + \sum_{i=1}^{n} \left(w_i \sum_{\substack{k=1 \\ k\neq j}}^{m} x_{ik}\right) \leq W$$

The x_{ij}, $i = 1, \ldots, n$, involved in the jth summand must satisfy a constraint of the form $\sum_{i=1}^{n} w_i\, x_{ij} = z \leq W$.

Thus the j-th term in the sum is a function of the number, k, of actions directed against target T_j and the combined weight, z_j, of the "missiles", which are allocated to T_j. We denote the minimum value of the j-th term by $f_k^j(z_j)$, i.e.,

$$f_k^j(z_j) = \underset{\{x_{1j},\ldots,x_{kj}\}}{\text{Min}} (a_j q_{1j}^{x_{1j}} \; q_{2j}^{x_{2j}} \; \ldots \; q_{kj}^{x_{kj}})$$

where

$$0 \leq \sum_{i=1}^{k} x_{ij} w_i = z_j \leq W \; .$$

The minimization involved in the above equation can be accomplished in k one-dimensional minimization processes by employing Bellman's simple but powerful Principle of Optimality.[1] We then have

(2) $$f_k^j (z_j) = \underset{x_{kj}}{\text{Min}} \; (q_{kj}^{x_{kj}} \; f_{k-1}^j \; (z_j - w_k x_{kj})); \; k = 2,3,\ldots,n$$

where x_{kj} can assume the values $0,1,2,\ldots, \left[z_j/w_k \right]$. The symbol $[y]$ is defined to be the greatest integer which does not exceed y. When k = 1,

(3) $$f_1^j \; (z_j) = \underset{x_{1j}}{\text{Min}} \;\; (a_j q_{1j}^{x_{1j}})$$

where $x_{1j} \;\; \epsilon \; \left\{ 0,1,2,\ldots \left[z_j/w_1 \right] \right\}$, thus

(4) $$f_1^j \; (z_j) = a_j q_{1j}^{\left[z_j/w_1 \right]} \text{ for all } j = 1,2,\ldots,m.$$

Equations (2) and (3) hold for all $j = 1,2,\ldots,m$; however, it should be noted that although z_j is unknown, we do know that

$$0 \leq z_1 + z_2 + \ldots + z_m = z \leq W, \text{ and hence that } 0 \leq z_j \leq W.$$

This implies that we must tabulate $f_k^j \; (z_j)$ over some grid of z_j values. That is, we obtain from eq. (2) the sequence $\left\{ f_k^{(j)} \;\; (z_j) \right\}$ and the corresponding optimal allocations $\left\{ x_{kj}(z_j) \right\}$. The former set of values must be stored in the computer memory in order to determine $f_{k+1}^{(j)}$. At this stage of the process we have obtained some vital information as to the effect on the optimal policy with respect to the j-th target, resulting from a change of total weight allocated to the j-th target and also the effect resulting from a change in the number of actions directed against the j-th target.

Finally we define a function g depending on the total weight allocated to a total of N targets, and the number, N, of targets against which n actions are directed and where an optimal allocations of weights, z, to each of the targets T_j is used. That is,

$$(5) \quad g_N(z,n) = \underset{\{z_1,z_2,\ldots,z_N\}}{\mathrm{Min}} (f_n^1(z_1) + f_n^2(z_2) + \ldots + f_n^N(z_N)) ;$$

$$N = 1,2,\ldots,m.$$

Where $z_1 + z_2 + \ldots + z_N = z \leq W$.

When N = m and we have obtained the optimal allocation of weights $\{z_1,z_2,\ldots,z_m\}$, i.e., the sequence $\{z_i\}$ which minimizes the expression (5) then the process is complete and the solution to our problem is obtained. Again, employing Bellman's Optimality Principle equation (5) becomes

$$(6) \quad g_N(z,n) = \underset{z_N}{\mathrm{Min}} (f_n^N(z_N) + g_{N-1}(z - z_N,n)); \; N = 2,3,\ldots,m$$

and where $0 \leq z_N \leq z$. For N = 1, $g_1(z,n)$ becomes

$$(7) \quad g_1(z,n) = f_n^1(z).$$

Since equations (6) and (7) are true for all n, m, and z we can assess the effect on the optimal policy $\{x_{11},\ldots,x_{1m};\ldots;x_{n1},\ldots x_{nm}\}$ which results from a variation in any one of the parameters. However if we are interested, always, in a particular value of n, say, n = k, then considerable reduction in computation is achieved since one needs to compute $g_N(z,k)$, only, instead of $g_N(z,1),\ldots,g_N(z,k),\ldots g_N(z,n)$.

COMPUTATIONAL PROCEDURE

The dynamic programming formulation automatically imbeds the original problem within a family of analogous problems in which the basic parameters n, m and W assume sets of values which permit us to obtain, in the course of the computation, the solution to a variety of sub-problems. Thus, great savings in time and labor can be obtained by careful analysis of the class of particular problems which are of interest. This is of great importance because of the very structure of the process, i.e., we use the information obtained from sub-problems of the original to obtain the solution of the original problem. For example, consider the following two sets of actions, targets and weights:

$$\text{I:} \quad \begin{Bmatrix} T_1, T_2 \\ A_1, A_2 \\ W_1 \end{Bmatrix} \qquad\qquad \text{II:} \quad \begin{Bmatrix} T_1, T_2, T_3, T_4 \\ A_1, A_2, A_3 \\ W_2 \end{Bmatrix}; \quad W_2 > W_1$$

Then, it is clear that we should obtain the solution to problem II, since by adding actions and targets, according to subscript order; we have solved problem I in an intermediate stage (provided we include W_1 in the grid of z values). This results from the recursive relationships involved in equations 2,3,6 and 7.

Now let us examine the computational procedure, in detail, and determine what information is needed to start the computation. The input information that is required is the knowledge of the values of n, m, W and the values of the q_{ij}'s. Again if we desire solutions for a set of values of n, m and W, the proper indexing of the actions and the targets is essential since we wish to determine solutions for as wide a subclass of problems as possible since we can use these solutions to obtain solutions for a larger class of problems of interest. We shall illustrate this, later, by a numerical example.

Given the above information we select the largest W for which we desire a solution. Now equations 2 and 3 involve weights z_j where $0 \leq z_j \leq W$, that is, we compute $f_k^j(z_j)$ for a set of z_j's. The choice of the values of z_j's that we must select is restricted by available computer memory, time and cost. If we are interested in a set of W's then these values should be included in the set of z_j's and the remaining values of z_j's are to be chosen in such a way as to eliminate the need for sophisticated methods of interpolation, if this is possible, and the grid should be chosen coarse enough so that time requirements are not excessive. Unfortunately, there seems to be no *a priori* method of optimizing this selection of grid points, rather, one must rely on experience, ingenuity or divine guidance.

Once the grid of z_j values has been chosen the calculation begins with equation (4). That is, we compute and store (and print) the following values for each target T_j:

Table I

$$\begin{cases} f_1^j(z_1) \; ; \; x_{1j}(z_1) \\ f_1^j(z_2) \; ; \; x_{1j}(z_2) \\ \cdot \\ \cdot \\ \cdot \\ f_1^j(z_\ell) \; ; \; x_{1j}(z_\ell) \end{cases}$$

where $x_{1j}(z_r)$ is the allocation of weapon of type A_1 to the j-th target, given a weight constraint z_r. We compute the table for each j separately, since in general, the memory capacity of the computing machine may not be great enough to accommodate the storage of the $2m \times \ell$ values of $\left\{f_1^j(z_r) \; ; \; x_{1j}(z_r)\right\}$, $j = 1,\ldots,m$ and $r = 1,\ldots,\ell$, which would be required for the simultaneous computation of the m targets. After the set of values $\left\{f_1^j(z_r) \; ; \; x_{1j}(z_r)\right\}$; $r = 1,2,\ldots, \ell$ is obtained for the j-th target then the set of values $\left\{f_2^j(z_r), x_{2j}(z_r)\right\}$ is computed from equation (3). The calculation of $f_2^j(z_i)$ involves, of course, a knowledge of $f_1^j(z_k)$ where $z_k \leq z_r$. This works to our advantage since when $x_{2j}(z_r)$ is determined then x_{1j} can be determined from the set of values $\left\{f_1^j(z_r - x_{2j}w_2) \; ; \; x_{1j}(z_r - x_{2j}w_2)\right\}$ by interpolation of Table I, thus we obtain the optimal policy $\left\{x_{1j}, x_{2j}\right\}$ for the j-th target corresponding to the two actions and the weight restriction z_r. We continue this calculation (eq. 2) until the desired number, n, of actions is obtained. At this stage of the calculation we have the following sets of values for the j-th target: $\left\{f_n^j(z_r)\right\}$ and the allocations $\left\{x_{1j}, x_{2j},\ldots,x_{nj}\right\}$ where

$$x_{nj} = x_{nj}(z_r) \text{ and } x_{n-k,j} = x_{n-k,j}\Big(z_r - \sum_{s=n-k+1}^{n} x_{sj}w_s\Big); \; k=0,1,2,\ldots,n-1.$$

We now have the necessary information to compute the g_N functions defined by equations 5 - 7; in fact, at this stage we have $g_1(z,n)$ since this is $f_n^1(z)$. Thus, we start with equation (6) and compute $g_2(z,n)$, i.e.,

$$(7) \qquad g_2(z,n) = \min_{z_2} \left\{f_n^2(z_2) + g_1(z - z_2, n)\right\} \text{ for } 0 \leq z \leq W.$$

In computing $g_2(z,n)$, we are again confronted with the problem of our choice of grid values for z. Now, z_2 represents an allocation of weight resulting from the assignment of n actions against target T_2, thus it is clear, that the values of z_2 should run over values which are linear combinations (with non-negative integral coefficients) of the weights w_1, $w_2, \ldots, w_n$ and only those linear combinations of the weights which are less than z. Any other assignment of weight to target 2 would result in a waste of weapons since we only consider the case where the x_{ij} are non-negative integers. In general, it will not be possible to include all such linear combinations of the weights because of restricted machine memory and excessive computing times.

Once the value of z_2, which minimizes the expression in (6) above, has been determined, then we can determine the allocation to target 2 (and target 1) from our knowledge of $f_n^2(z)$ and the corresponding allocations which were determined previously, i.e., we have $g_2(z,n)$; $f_n^2(z_2)$ and $\left\{x_{12}, x_{22}, \ldots, x_{n2}\right\}$ and also $g_1(z - z_2,n) = f_n^1(z - z_2)$ from which we obtain $\left\{x_{11}, x_{21}, \ldots, x_{n1}\right\}$. These sets of x's; $\left\{x_{12}, \ldots, x_{n2}\right\}$ and $\left\{x_{11}, x_{21}, \ldots, x_{n1}\right\}$ represent the optimal policy for the case where we have n actions, 2 targets and a total weight constraint z. We continue the above process for all choices of z and then $g_3(z,n)$ can be computed in the same fashion obtaining the optimal allocation for the case of 3 targets, n-actions with a total weight constraint z. Thus for similar values of z we can observe the change in optimum policy as more targets are added to the process - we can observe, also, the change in optimum policy at any stage as a function of z. The above procedure is continued until the desired number of targets has been included, i.e., until $g_m(z,n)$ is obtained. A similar calculation is made for each n that is desired.

We conclude this discussion with a simple numerical example which we hope illustrates some of the salient features of this approach - although - one which eliminates many of the difficulties which may arise in any realistic problem.

A NUMERICAL EXAMPLE

Suppose we have n = 2 actions, A_1, A_2, m = 3 targets. Let $a_1 = 0.2$, $a_2 = 0.3$, and $a_3 = 0.5$ and let us assume that $q_{11} = 0.1$, $q_{12} = 0.4$, $q_{13} = 0.6$, $q_{21} = 0.5$, $q_{22} = 0.3$ and $q_{23} = 0.2$. Also, $w_1 = 2$ lbs., $w_2 = 3$ lbs. and W = 10 lbs. We wish to minimize equation (1) with those values; i.e., we wish to determine the optimum policy $\left\{ x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \right\}$.

We begin the calculation with equation (4) and compute the value of $f_1^j(z_k)$, j = 1,2,3 for values of $z_k = 0,1,2,\ldots,10$. For each j and z_k we record the values of $f_1^j(z_k)$ and the value of x_{1j} which gives this value of f_1^j. That is, we construct Tables 2 and 3:

TABLE 2

$f_1^j(z_k)$

$z_k \backslash j$	1	2	3
0	0.20	0.30	0.50
1	0.20	0.30	0.50
2	0.02	0.12	0.30
3	0.02	0.12	0.30
4	0.002	0.048	0.18
5	0.002	0.048	0.18
6	0.0002	0.0192	0.108
7	0.0002	0.0192	0.108
8	0.00002	0.00768	0.0648
9	0.00002	0.00768	0.0648
10	0.000002	0.003072	0.03888

TABLE 3

$x_{1j}(z_k)$

$z_k \backslash j$	1	2	3
0	0	0	0
1	0	0	0
2	1	1	1
3	1	1	1
4	2	2	2
5	2	2	2
6	3	3	3
7	3	3	3
8	4	4	4
9	4	4	4
10	5	5	5

The next step of the calculation is to compute $f_2^j(z_k)$, j = 1,2,3. Again, we record the value of x_{2j} which gives f_2^j. Since $f_2^j(z_k)$ can be interpreted as the return from the direction of 2 actions against the jth target, then we must obtain x_{1j} also. Now x_{1j} is obtained from table 3 and is found in the row corresponding to $z_k - 3x_{2j}$ and in the jth column. Thus we have tables 4 and 5:

TABLE 4

$f_2^j(z_k)$

$z_k \backslash j$	1	2	3
0	0.2	0.3	0.5
1	0.2	0.3	0.5
2	0.02	0.12	0.3
3	0.02	0.09	0.1
4	0.002	0.048	0.1
5	0.002	0.036	0.06
6	0.0002	0.0192	0,020
7	0.0002	0.0144	0.020
8	0.00002	0.00768	0.012
9	0.00002	0.00576	0.004
10	0.000002	0.003072	0.004

TABLE 5

$(x_{2j}(z_k),\ x_{1j}(z_k - 3x_{2j}))$

$z_k \backslash j$	1	2	3
0	(0,0)	(0,0)	(0,0)
1	(0,0)	(0,0)	(0,0)
2	(0,1)	(0,1)	(0,1)
3	(0,1)	(1,0)	(1,0)
4	(0,2)	(0,2)	(1,0)
5	(0,2)	(1,1)	(1,1)
6	(0,3)	(0,3)	(2,0)
7	(0,3)	(1,2)	(2,0)
8	(0,4)	(0,4)	(2,1)
9	(0,4)	(1,3)	(3,0)
10	(0,5)	(0,5)	(3,0)

The calculation of $f_n^j(z)$ is now complete and we are now able to compute the g functions. Since $g_1(z,2) = f_2^1(z)$ we can proceed to the calculation of $g_2(z,2)$ (Equation 6). Since equation 6, for N =2, is

$$g_2(z_k,n) = \text{Min}\left\{ f_2^2(z_2) + g_1(z_k - z_2,n)\right\}$$

we are to find the value of z_2 which minimizes the expression on the right for the given z_k. The values; z_2 and $z_k - z_2$ we use to enter the Table 5 and obtain the policy $\{x_{22}(z_k),\ x_{12}(z_k - 3x_{22}),\ x_{21}(z_k - z_2),\ x_{11}(z_k - z_2 - 3x_{21})\}$. For example; when $z_k = 6$ lbs., $g_2(6,2) = 0.068$ and the value of z_2 which gives this value of $g_2(6,2)$ is $z_2 = 4$ lbs. Thus, entering Table 5 at row $z_k = 4$ lbs., we find in the j = 2 column: $x_{22} = 0$, $x_{12} = 2$. In the same table in row $z_k - z_2 = 6 - 4 = 2$ lbs. We find in the j = 1 column $x_{21} = 0$ and $x_{11} = 1$. Thus the optimal policy for two targets, two actions and $z_k = 6$ lbs. is $\{x_{11}, x_{12}, x_{21}, x_{22}\} = \{1,2,0,0\}$.

We have, now, Table 6:

TABLE 6

z_k	$g_2(z_k,2)$	$\{x_{11},x_{12},x_{21},x_{22}\}$
0	0.5	0, 0, 0, 0
1	0.5	0, 0, 0, 0
2	0.32	1,0,0,0 or 0,1,0,0
3	0.29	0, 0, 0, 1
4	0.14	1, 1, 0, 0
5	0.11	1, 0, 0, 1
6	0.068	1, 2, 0, 0
7	0.056	1, 1, 0, 1
8	0.0392	1, 3, 0, 0
9	0.0344	1, 2, 0, 1
10	0.0212	2, 3, 0, 0

Finally, we compute $g_3(10,2)$. We need z = 10 only since we do not need to compute $g_4(z,2)$.

Proceeding as above, we find that $g_3(10,2) = 0.156$ and $\{x_{11},x_{12},x_{13},x_{21},x_{22},x_{23}\} = \{1,1,0,0,1,1\}$. Examining the structure of the above calculation, it is readily seen that, we have actually solved many simpler type problems in the process. For example, we have solved the problem of 2 targets (index 1 and 2), two actions and a total weight constraint of 7 lbs. The optimal policy for this problem is given in Table 6. Furthermore, if one is interested in the same targets and actions but with a total weight constraint of, say, 9 lbs., then $g_3(9,2)$ must be computed. For this example, $g_3(9,2) = 0.168$ and the optimal policy is $\{1,2,0,0,0,1\}$.

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BIBLIOGRAPHY

[1] Bellman, R. Dynamic Programming. Princeton University Press: 1957.

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