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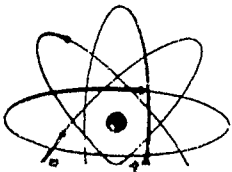
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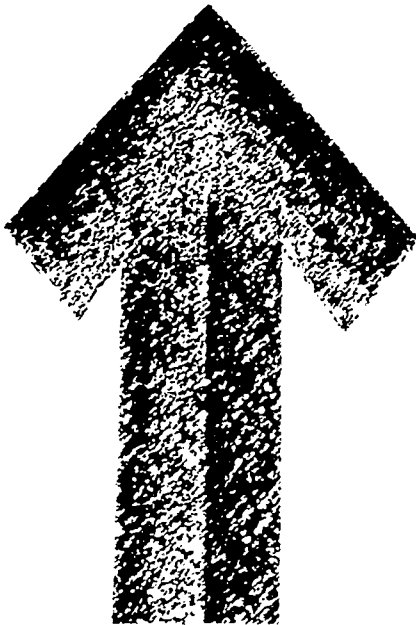
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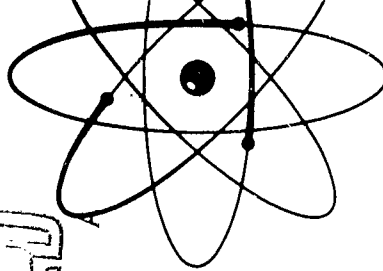
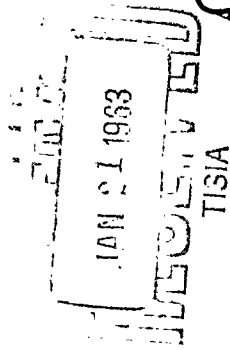
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PLASMA BOUNDARY LAYERS

by

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SUMMARY

A review is given of recent work on boundary layers in electrically-conducting fluids. This includes a discussion of the differences between ordinary boundary layers and plasma boundary layers with special reference to the new dimensionless parameters for the magnetic case, including those which arise from the microscopic structure of the fluid. The convection of current and vorticity along magnetic field lines by the Alfvén wave mechanism is shown to be the principal new propagation mechanism in magnetohydrodynamic boundary layers. Recent work in inertial boundary layers, channel flows, wakes, and electrode boundary layers is summarized.

LIST OF SYMBOLS

$\vec{B}$	magnetic induction
$\vec{B}_0$	magnetic induction due to external currents
$\vec{B}'$	magnetic induction due to induced currents
B_y	component of B normal to wall
$\vec{E}$	electric field
e	charge of electron
h	Debye distance (Eq. 2.12)
$\vec{j}$	current density
K	magnetic diffusivity, $(\mu_0 \sigma)^{-1}$
k	Boltzmann's constant
ℓ	mean free path
ℓ_i	ion mean free path
ℓ_e	electron mean free path
L	characteristic length
m_e	electron mass
M	Mach number
M_A	flow velocity divided by Alfvén velocity, Eq. (2.8)
M_H	Hartmann number, $LB\sqrt{\sigma/\rho\nu}$
n	distance normal to plane of Alfvén wave
n_e	electron number density
n_i	ion number density
n_a	atom number density
p	pressure

P_r	Prandtl number
r	Coulomb radius, Eq. (2.9)
R_V	viscous Reynolds number, VL/ν
R_M	magnetic Reynolds number, $\mu_o \sigma VL$
S	interaction parameter, $\sigma B^2 L / \rho V$
T	temperature ($^{\circ}K$)
T_e	electron temperature
T_A	atom temperature
t	time
ΔU_A	velocity change across Alfven wave
ΔU_H	velocity change across Hartmann layer
$\bar{V}$	mass averaged velocity
$\bar{V}_e$	mean electron velocity
$\bar{V}_i$	mean ion velocity
$\bar{V}_a$	mean atom velocity
$\bar{V}_A$	Alfven velocity, $B/\sqrt{\rho \mu_o}$
∇	vector differential operator
∇^B	vector differential operator which operates only on $\underline{B}$
δ_H	Hartmann layer thickness, Eq. (3.8)
δ_A	Alfven wave thickness
δ_V	viscous layer thickness, $\sqrt{\nu t}$
ϵ	magnetic Prandtl number, Eq. (2.7)
ϵ_o	vacuum electric permittivity
λ	thermal conductivity
μ_o	vacuum magnetic permeability

ν	Kinematic viscosity
ω	vorticity
ρ	mass density
ρ_e	charge density
σ	electrical conductivity
τ_i	mean time of ion elastic collisions
τ_e	mean time of electron elastic collisions
τ_E	mean time of electron collisions for energy equilibration

I. Introduction

While the physical conditions under which a plasma may be created or maintained cover many orders of magnitude change in the particle density and temperature, the plasma boundary layers which we shall discuss can exist only under the limited conditions for which the plasma may be treated as a continuum in the usual fluid-mechanical sense. This limitation reflects in part the tendency of aerodynamicists to move from the familiar to the less familiar by the extrapolation of existing understanding and in part the possibilities of the practical application of magnetohydrodynamics (MHD). The plasma jet, MHD generator, MHD propulsion motor, MHD flow meter and other devices are already sufficiently practical to require a more complete understanding of their operation, an understanding which can probably be acquired by the extrapolation of known techniques (such as boundary layer theory). This should not blind us to the fact that only a limited region of the plasma state is being subject to scientific and engineering scrutiny--a state of affairs which cannot be considered to be satisfactory.

In the classical sense, a boundary layer is a region of the flow into which disturbances diffuse from boundaries, carrying mass, momentum and energy (including magnetic energy). From a practical point of view, a boundary layer is the region which "insulates" the flow from its boundaries, and in many plasma devices the boundary layer provides the mechanism for containment of the high temperature gases. In this respect, a magnetic field may be considered to be a bounding surface, and therefore, a source of disturbances as well as a mechanism for containment. It is this point of view which will be found most helpful in understanding MHD boundary layers.

In some plasma devices an electric current must pass from a solid boundary into the fluid. The detailed mechanism by which electrical charges flow from the fluid to the wall are not well understood and appear to be highly specific to the wall properties and chemical state of the plasma. Thus, boundary layers on electrode surfaces have received very little attention, reflecting the generally unsatisfactory state of our understanding of electrical discharges in continuum flows. In one sense this lack of concern on the part of aerodynamicists for the problem of current emission is justified, for the plasma sheath which forms at bounding surfaces is not a boundary layer in the ordinary sense of fluid dynamics, but is more aptly described as a problem in surface electro-chemistry. Nevertheless, it is to be hoped that this problem will not continue to be ignored.

The major part of this paper will be devoted to a discussion of MHD boundary layers. Section II deals with the scaling laws appropriate to viscous MHD flows, while Section III is concerned with the basic mechanism of diffusion of vorticity and current. Section IV is a summary of present understanding concerning the principal types of MHD boundary layer problems, including turbulent boundary layers. In Section V we discuss some plasma boundary layers in which magnetic effects are absent.

II. Scaling Laws

The interaction of a conducting fluid with an electromagnetic field must be characterized by a larger number of dimensionless parameters than is needed for an ordinary fluid flow. The bewildering variety of these new parameters is the result, not of an attempt to inflate the currency of

dimensional analysis, but to select those groups which most aptly describe the principal effects in a particular problem. As an example, consider the conservation of mass and momentum of an incompressible, viscous, electrically conducting fluid:

$$\nabla \cdot \underline{V} = 0 \quad (2.1)$$

$$\rho \left\{ \frac{\partial}{\partial t} + \underline{V} \cdot \nabla \right\} \underline{V} = - \nabla p + \underline{j} \times \underline{B} + \rho \nu \nabla^2 \underline{V} \quad (2.2)$$

together with Maxwell's equations for the electromagnetic field:*

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \quad (2.3)$$

$$\nabla \times \underline{B} = \mu_0 \underline{j} \quad (2.4)$$

and Ohm's law for the conduction of charge:

$$\underline{j} = \sigma (\underline{E} + \underline{V} \times \underline{B}) \quad (2.5)$$

Following the usual arguments, the viscous Reynolds number, $R_v \equiv VL/\nu$, is the ratio of inertia force to viscous force in a steady flow as determined from

* Rationalized mks units will be used throughout.

the second and last terms of Eq. (2.2). By a similar argument, the ratio of magnetic force ($\mathbf{j} \times \mathbf{B}$) to inertia force is found to be $\sigma B^2 L / \rho V \equiv S$ when Eq. (2.5) is substituted into Eq. (2.2). S is often termed the magnetic interaction parameter. Alternatively, one might compare the ratio of magnetic force to viscous force, thereby defining the Hartmann number, $M_H \equiv LB \sqrt{\sigma / \rho \nu}$. Naturally, R_v , S and M_H are not independent, but are related by $M_H^2 = R_v S$.

In an inviscid flow the interaction parameter, S , measures the effect of the magnetic forces on the flow. For an MHD boundary layer, however, it would seem more appropriate to compare the relative effects of the magnetic and viscous forces, in which case the Hartmann number is more appropriate, especially for fully-developed laminar channel flow for which the inertia force is zero. Nevertheless, in some boundary layers where the free stream flow disturbance is measured by S , it is customary to retain this parameter as descriptive of the boundary layer flow as well.

In inviscid flows it is often important to distinguish between the component ($\underline{B}_0$) of magnetic field caused by currents external to the flow and the component ($\underline{B}'$) induced by currents flowing within the fluid. If one retains this distinction and combines Eqs. (2.4) and (2.5) to write Ohm's law as:

$$\nabla \times \underline{B}' = \mu_0 \sigma \left[\underline{E} + \underline{V} \times (\underline{B}' + \underline{B}_0) \right] \quad (2.6)$$

then the condition that $\underline{B}'$ be much less than $\underline{B}_0$ (and thus that $\underline{B}_0$ should appear in the definitions of S and M_H), is that $\mu_0 \sigma VL \equiv R_m \ll 1$. The magnetic Reynolds number, R_m , thereby describes the effect of induced currents on the applied magnetic field, and specifies the condition under which the magnetic

field may be considered to be only slightly perturbed, irrespective of whether or not the flow is viscous.

With respect to MHD boundary layers, it can be expected that three independent parameters, such as R_v , R_m , and S , will be pertinent in specifying the nature of the flow. In some degenerate cases, such as the semi-infinite flat plate or the Rayleigh problem where no characteristic length, L , exists, one must use the two ratios:

$$\epsilon \equiv \frac{R_m}{R_v} = \sigma \mu_o \nu \quad (2.7)$$

$$M_A^2 \equiv \frac{R_m}{S} = \frac{\mu_o \rho U^2}{B^2} \quad (2.8)$$

ϵ , called the magnetic Prandtl number, is the ratio of the viscous diffusivity (ν) to the magnetic diffusivity ($K \equiv (\mu_o \sigma)^{-1}$) and M_A is the ratio of the flow velocity to the Alfvén velocity ($V_A \equiv B/\sqrt{\mu_o \rho}$). It will be seen below in Section III that both of these parameters arise naturally when the MHD equations are recast in the form of diffusion equations, and that they can be interpreted physically as the ratio of diffusion times and flow times, respectively.

In all the preceding we have ignored the effect of compressibility, whose existence would introduce another degree of freedom and the usual parameter, Mach number. With the benefit of hindsight, it appears that the effect of compressibility within the boundary layer is not greatly different from what is ordinarily the case, and does not greatly alter our previous conclusions concerning the effect of the magnetic field.

We have patently ignored the electric field in our order-of-magnitude analysis, and it plays no part in the dimensionless parameters so far considered. This is a result of the fact that $\underline{E}$ may be eliminated between Eqs. (2.3) and (2.5) and will only influence the flow through conditions at the boundary. For example, in channel flows $\underline{E}$ appears in the "loading parameter", E/VB , and in other problems determines the boundary values of $\underline{j}$ or $\frac{\partial \underline{B}}{\partial t}$.

The point of view so far maintained is one which describes the flow in terms of continuum conservation laws and phenomenological transport effects. It is equally important to delineate the limitations imposed by the microscopic structure of the plasma. To do this without an extensive discussion of kinetic theory is difficult, and we shall be content to emphasize only the salient points which are most pertinent to boundary layers.

If we adopt as the basic microscopic length the Coulomb radius, r ,

$$r \equiv \frac{e^2}{4\pi\epsilon_0 kT} \quad (2.9)$$

that is, the distance of separation of two atomic charges of potential energy equal to kT , then the mean free path, ℓ , of the charged particles is:

$$\ell = (nr^2)^{-1} \quad (2.10)$$

except for a logarithmic factor not ordinarily much different from unity. For a continuum description of the plasma motion, it is certainly necessary that $L \gg \ell$.

The lack of distinction between the conduction current in the Ohm's law of Eq. (2.5) and the total current of Eq. (2.4) implies the usual assumption that the charge density, ρ_e , is so small that the convection current can be neglected, that is,

$$\rho_e V \ll j \quad (2.11)$$

For this to be so, the geometric scale, L , should be greater than the Debye distance, h , which is the distance over which the maximum possible charge density ($n_e e$) may cause a potential difference kT/e :

$$h \equiv \left(\frac{\epsilon_0 kT}{n_e e^2} \right)^{1/2} \\ = (4\pi r n_e)^{-1/2} \quad (2.12)$$

By combination of Eqs. (2.12) and (2.10) we find that

$$\frac{h}{\ell} = \left\{ \frac{n_e r^3}{4\pi} \right\}^{1/2} \quad (2.13)$$

that is, the Debye distance is always less than a mean free path whenever the plasma can be considered to be a perfect gas (volume per molecule much greater than r^3). Thus, charge separation imposes a less stringent condition than that of the continuum assumption.

At electrode surfaces there may be considerably more energy than kT available to a charged particle. The sheath which forms at such surfaces will have a thickness comparable to that obtained by replacing kT/e in Eq. (2.12) by the electrode potential drop, provided this does not exceed the mean free path. Except in non-continuum flows, these sheaths will not affect the fluid motion within the boundary layer.

It is possible to replace the transport coefficients ν and σ in the expressions for R , S and M_H by their values from mean free path theory

$$\nu \approx \ell_i^2 / \tau_i$$

$$\sigma \approx \frac{n_e^2 \tau_e}{m_e} \quad (2.14)$$

so that these parameters contain explicitly a microscopic length in a fashion which emphasizes the microscopic structure of a plasma. More to the point, however, is the modification to Ohm's law (Eq. 2.5) which results from the fact that the electron, ion and neutral particle mean velocities ($\underline{V}_e$, $\underline{V}_i$ and $\underline{V}_a$, respectively) may be appreciably different. Momentum gain by the electrons due to their electric field may be equated to their momentum loss due to

collisions with the ions:

$$-n_e e (\underline{E} + \underline{V}_e \times \underline{B}) = \frac{n_e m_e (\underline{V}_e - \underline{V}_i)}{\tau_e} \quad (2.15)$$

Now the body force, $\underline{j} \times \underline{B}$, must be applied to the heavy particles, and a fraction, $n_a/(n_i + n_a)$, of this force must be transmitted from the ions to the neutral atoms by collisions between the latter:

$$\left(\frac{n_a}{n_i + n_a} \right) (\underline{j} \times \underline{B}) = \frac{n_i m_i (\underline{V}_i - \underline{V}_a)}{\tau_{ia}} \quad (2.16)$$

If the mean velocity of the heavy particles is defined by:

$$(n_i + n_a) \underline{V} \equiv n_i \underline{V}_i + n_a \underline{V}_a \quad (2.17)$$

then the revised Ohm's law obtained by combining Eqs. (2.15), (2.16) and (2.17) is:

$$\begin{aligned} \underline{j} = & \sigma (\underline{E} + \underline{V} \times \underline{B}) - \frac{\omega_e \tau_e}{B} \underline{j} \times \underline{B} \\ & + \frac{(\omega_e \tau_e)(\omega_i \tau_{ia})}{B^2} \left(\frac{n_a}{n_i + n_a} \right)^2 (\underline{j} \times \underline{B}) \times \underline{B} \end{aligned} \quad (2.18)$$

The second term on the right side is the Hall field which results from the finite value of the electron-ion collision frequency compared with the electron cyclotron frequency, ω_e . The third term is an ion slip field caused by the relative motion of the ions and atoms, but is absent when the plasma is completely ionized ($n_a = 0$). For a slightly ionized gas, the ion slip field is comparable to the Hall field when $\omega_i \tau_{ia}$ is approximately one.

In our discussion so far we have ignored energy conservation. For an incompressible fluid, the variation of temperature in the boundary layer will affect the flow pattern only through changes in the transport coefficients. The coupling in a compressible flow is more direct because of the additional variation in density. Nevertheless, the principal new effect on the macroscopic scale would be the joule heating (j^2/σ per unit volume) which must be conducted to the wall or the free stream. One may compare the ratio of the joule heating to the heat diffusion ($\lambda \nabla^2 T$) terms in the energy equation, obtaining

$$\frac{\text{joule heating}}{\text{heat diffusion}} = P_r (M_H \cdot M)^2 \quad (2.19)$$

in which the Prandtl number (P_r) and Mach number (M) are introduced as is to be expected in compressible, heat-conducting flows.

It is not our purpose to discuss the transport properties of plasmas, except insofar as they will affect the behavior of boundary layers. It is sufficient to point out that, as a gas becomes ionized, the thermal conductivity increases due to the presence of the light electrons while the viscosity decreases for high degrees of ionization because the Coulomb cross section is much larger than the neutral atom cross section. Thus, the thermal boundary layer thickness

is much greater than the viscous layer thickness in a completely ionized gas.

On the microscopic scale we can again expect the redistribution of energy to modify the classical situation in a manner similar to the modification of Ohm's law occasioned by the redistribution of momentum between the ions, electrons and neutral particles. In particular, the joule heating is an energy added to the electrons which must be shared quickly with the particles with which they collide if the electron temperature (T_e) and heavy particle temperature (T_A) are to be very nearly the same. If τ_E is the mean time for electrons to lose their thermal energy kT_e , then

$$\frac{j^2}{\sigma} = \frac{n_e k (T_e - T_A)}{\tau_E}$$

$$\text{or } \frac{T_e - T_A}{T_A} = (\omega_e r_e)^2 M^2 \left\{ \frac{m_e \tau_E}{m_i \tau_e} \right\} \quad (2.20)$$

If energy is lost by elastic collisions of frequency τ_e^{-1} , then $\tau_E = m_i \tau_e / m_e$ because of the disparity in mass, and non-equilibrium temperatures will exist only if Hall effects are also important. If inelastic collisions are effective, then usually $\tau_E < m_i \tau_e / m_e$ and temperature equilibrium is more likely to be obtained. Whether or not a two-temperature model must be used within the boundary layer therefore depends quite critically upon the magnetic field strength (through $\omega_e \tau_e$) and the ratio of elastic to inelastic electron cross sections. Thus, one additional microscopic parameter must be introduced under these circumstances.

As is the case for boundary layers in dissociated gases, cooled plasma boundary layers will recombine ions and electrons either within the boundary

layer by gas phase recombination, or at the wall by surface recombination. The diffusion of ion-electron pairs towards the wall proceeds by ambipolar diffusion in which the high diffusivity of the electrons induces an electric field which doubles the inherent diffusivity of the ions through the neutral gas in the vicinity of the wall. Since the electrical conductivity of the boundary layer gas is very sensitive to the degree of ionization when the latter is less than a few percent, there may again be strong coupling between the effects of diffusion and recombination on one hand and the distribution of electric current within the boundary layer on the other. It is possible that the joule heating can produce ionization (and hence a source of diffusing ion-electron pairs) by purely equilibrium thermal effects or by virtue of superheating of the electrons. (We will leave it to the reader to define the appropriate parameter for the ratio of ionization source to ambipolar diffusion). On the whole, one may expect that many of the techniques used in the dissociating gas boundary layer may be adopted for this case as well.

In the foregoing we have attempted to show how MHD boundary layers involve several additional similarity parameters (R_M , S , ...) which can be incorporated in the usual manner in the continuum analysis, and also microscopic parameters ($\omega\tau$, τ_E/τ_e , ...) which usually necessitate a reformulation of the classical statements such as Ohm's law, energy conservation, etc. Within this framework it is possible to treat the extremes of liquid metal flow in a tube and flow of a partially ionized gas through a strong magnetic field adjacent to a cold wall which "de-ionizes" the gas by wall recombination. While the former is more amenable to analysis, the latter is of great practical importance and must undoubtedly be understood more thoroughly than is presently the case. Most of the work

reviewed below is much more closely related to the former, rather than the latter, problem.

III. The Diffusion of Vorticity and Current

One way to characterize the difference between an ordinary boundary layer and an MHD boundary layer is to compare the processes of diffusion of momentum and magnetic field energy. To simplify this analysis, let us consider a two-dimensional incompressible flow with co-planar velocity and magnetic fields. The current, vorticity, and electric field (neglecting Hall effects) will be normal to the plane containing $\underline{V}$ and $\underline{B}$. By taking the curl of Eqs. (2.2), (2.4) and (2.5), we find:

$$\left\{ \frac{\partial}{\partial t} + \underline{V} \cdot \nabla - \nu \nabla^2 \right\} \underline{\omega} = \frac{1}{\rho} (\underline{B} \cdot \nabla) \underline{j} \quad (3.1)$$

$$\left\{ \frac{\partial}{\partial t} + \underline{V} \cdot \nabla - K \nabla^2 \right\} \underline{j} = \frac{1}{\mu_0} (\underline{B} \cdot \nabla) \underline{\omega} + \frac{2}{\mu_0} \nabla^B (\underline{B} \cdot \nabla) \times \underline{V} \quad (3.2)^*$$

The lefthand side of either equation expresses the usual balance of diffusion and convection along streamlines, while the righthand side is a source term which is proportionate to the convection of the complementary quantity ($\underline{j}$ or $\underline{\omega}$) in the direction of $\underline{B}$. It is this convection of vorticity or current along the magnetic field lines which distinguishes the MHD from an ordinary flow.

* The last term, which also equals $-2\mu_0^{-1} \nabla^V (\underline{V} \cdot \nabla) \times \underline{B}$, is identically zero for the Alfvén wave and Hartmann boundary layer discussed below, and also if $\underline{V}$ is proportional to $\underline{B}$. For small R_M and constant $\underline{B}_0$ it may be neglected. In the following discussion it will be considered negligible.

A possible solution to Eqs. (3.1) and (3.2) for the special case of $\nu = K$ is one for which $\sqrt{\mu_0} \ j = \pm \sqrt{\rho} \ \omega$. Eq. (3.1) may then be written as:

$$\left\{ \frac{\partial}{\partial t} + (\underline{V} \pm \underline{V}_A) \cdot \nabla - \nu \nabla^2 \right\} \omega = 0 \quad (3.3)$$

which expresses the propagation of a diffusing disturbance with a velocity $\underline{V} \pm \underline{V}_A$ and is therefore an Alfvén wave. In MHD boundary layers the boundary conditions on j and ω generally do not permit their being everywhere proportional in the ratio $\sqrt{\rho/\mu_0}$, so that there will be more disturbances than Alfvén waves alone. Nevertheless, we must expect that Alfvén waves can be excited within the boundary layer and thereby carry away energy and momentum.

When $K \neq \nu$, then $\sqrt{\mu_0} \ j \neq \sqrt{\rho} \ \omega$ everywhere within the Alfvén wave, but the current and vorticity are integrally related by:

$$\sqrt{\mu_0} \int_{-\infty}^{\infty} j \, dn = \sqrt{\rho} \int_{-\infty}^{\infty} \omega \, dn \quad (3.4)$$

which is equivalent to the statement that the ratio of the velocity change across the wave to the magnetic field change (both changes being parallel to the front) equals V_A/B . Thus, by Eq. (3.4) the ratio of total vorticity to total current in an Alfvén wave is $\sqrt{\mu_0/\rho}$.

From an aerodynamic point of view, the propagation of vortex and current elements along magnetic field lines can be understood by noting that the force $-\underline{V} \times \underline{\omega}$ on a vortex convecting with a velocity $\underline{V}$ can be

balanced by a force $\underline{j} \times \underline{B}$ provided $\underline{v} = \underline{v}_A$ and $\sqrt{\mu_0} \underline{j} = \sqrt{\rho} \underline{\omega}$, that is, the current and vorticity are coincident and move with the Alfvén velocity.

An interesting example of the diffusion of vorticity and current within a boundary layer is that of the Hartmann boundary layer which forms on a flat plate having $\underline{B}$ perpendicular to the plane of the plate. Assuming the flow to be steady and to depend only upon the distance y normal to the plate, Eqs. (3.1) and (3.2) become:

$$\nu \frac{d^2 \omega}{dy^2} + \frac{B_y}{\rho} \frac{dj}{dy} = 0 \quad (3.5)$$

$$K \frac{d^2 j}{dy^2} + \frac{B_y}{\mu_0} \frac{d\omega}{dy} = 0 \quad (3.6)$$

Because of the symmetry of the plane problem, B_y is constant and ω and j are given by:

$$\begin{aligned} \omega &= \omega_0 \exp \left\{ -\frac{y}{\delta_H} \right\} \\ j &= \sqrt{\frac{\rho}{\mu_0} \frac{\nu}{K}} \omega \end{aligned} \quad (3.7)$$

$$\text{where } \delta_H \equiv \sqrt{K\nu/V_A} \quad (3.8)$$

Thus, the Hartmann boundary layer thickness, δ_H , is the usual diffusion layer thickness for a stationary source in a fluid of diffusivity $\sqrt{K\nu}$ flowing with a velocity V_A . In this sense the vorticity and current are convected toward the wall with the Alfvén velocity and diffuse away from the wall with the geometric mean diffusivity $\sqrt{K\nu}$.

One may use this picture to understand the development of an MHD boundary layer on an infinite flat plate suddenly set into motion (Rayleigh problem). In the non-magnetic case, the vorticity diffuses outwards forming a boundary layer of thickness $\delta_V = \sqrt{\nu t}$. In the MHD case, there is a tendency for vorticity to convect towards and away from the wall with the Alfvén velocity as well as to diffuse away from the wall. After some time, a portion of the vorticity originated by the plate motion convects away from the wall in an Alfvén wave while another portion establishes a steady state balance between convection toward the wall and diffusion away from the wall, that is, forms a Hartmann boundary layer.

The relative strength of these two effects may be estimated by applying the rule that the total current which flows in the Hartmann boundary layer must return in the Alfvén wave (assuming an insulating plate):

$$\delta_H j_H = \delta_A j_A \quad (3.9)$$

The velocity change across the Alfvén wave (ΔU_A) is related to the velocity change across the Hartmann layer (ΔU_H) by:

$$\frac{\Delta U_A}{\Delta U_H} = \frac{\omega_A \delta_A}{\omega_H \delta_H} \quad (3.10)$$

Using Eqs. (3.4), (3.7) and (3.9), this may be evaluated as:

$$\frac{\Delta U_A}{\Delta U_H} = \sqrt{\epsilon} \quad (3.11)$$

Thus, for small magnetic Prandtl number (ϵ), the Alfvén wave propagation may be neglected when studying boundary layers on surfaces with a magnetic field normal to the surface.

Eqs. (3.1) and (3.2) are the basic boundary layer equations for plane flows. They are sometimes linearized with respect to the magnetic field by assuming $\underline{B} \cdot \nabla$ to be $\underline{B}_0 \cdot \nabla$ when $R_M \ll 1$, in which case they reduce to:

$$\left\{ \left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla - \nu \nabla^2 \right) \left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla - K \nabla^2 \right) - (\underline{v}_A \cdot \nabla)^2 \right\} \left\{ \frac{\omega}{j} \right\} = 0 \quad (3.12)$$

For $K \gg \nu$ (i.e., $\epsilon \ll 1$), within the viscous boundary layer Eq. (3.12) may be simplified by ignoring the terms $\frac{\partial}{\partial t} + \underline{v} \cdot \nabla$ compared to $K \nabla^2$. Either or both of these simplifications are commonly used in the literature.

The question of MHD wakes which propagate along the magnetic field lines, even in the upstream direction if $\underline{v}_A > \underline{v}$, has received considerable attention. In some cases the flow is considered inviscid ($\nu = 0$) and the disturbance originates from currents induced at the surface of a body which disturbs the flow. This might be classified as an inviscid magnetic boundary layer which, in the most general case, will be the source of Alfvén waves spreading into the flow. It is also possible to consider the complementary case, a viscous, superconducting boundary layer ($K = 0$) in which the roles

of $\underline{j}$ and $\underline{\omega}$ are interchanged in accordance with the symmetry of Eqs. (3.1) and (3.2).

For a plasma, the value of ϵ is found to be

$$\epsilon = \frac{3 \times 10^{-3} T^4}{n_e} \left\{ \ln \left(\frac{1.2 \times 10^4 T^{3/2}}{n_e^{1/2}} \right) \right\}^{-2} \quad (3.13)$$

where n_e is the electron density in particles per cubic centimeter.

Because of the steep temperature dependence of both ν and σ , $\epsilon \ll 1$ for low temperatures ($T < 10^5$ at $n_e = 10^{16}$) but $\epsilon \gg 1$ at high temperatures. However, the conditions under which $\epsilon \gg 1$ for a plasma generally imply such strong magnetic fields that the continuum MHD assumptions no longer hold.

IV. MHD Boundary Layers

The MHD Rayleigh problem, a particular case of which was discussed above, was first treated by Rossow⁴⁷ for the case of $\epsilon = 1$. However, the boundary conditions on the electric field used by Rossow were not physically meaningful, having arisen through a misapprehension concerning "moving" and "stationary" magnetic fields.⁴⁹ A proper solution for a superconducting plate has been obtained by Chang and Yen⁷, and a further discussion is given by Williams.⁵⁶

The corresponding Stokes problem (oscillating infinite plate, $\underline{B}$ perpendicular to the plate), was solved by Ong and Nicholls⁴⁰ for $\epsilon \ll 1$ and $R_M \ll 1$. More complete solutions were obtained by Hide and Roberts²¹ and Axford² for other limiting cases.

The incompressible flow over a semi-infinite flat plate with $\underline{B}$ parallel to the flow was shown by Greenspan and Carrier^{14, 6} to involve an upstream wake if $\underline{V} < \underline{V}_A$ so that steady solutions exist under such circumstances only for a finite plate.¹⁹ Asymptotic solutions for large or small ϵ were given by Glauert.¹¹ For this configuration, the disturbance to the magnetic field is caused by the displacement effect of the viscous boundary layer.

The magnetic Blasius problem with $\underline{B}$ perpendicular to the wall has been discussed by Fay⁹, who pointed out the significance of the Hartmann layer, and related incompressible flows have also been considered.^{41, 58} More attention has been given to variable σ hypersonic boundary layers^{4, 44, 45} and compressible boundary layers.^{31, 32, 34} In the former case, Bush⁴ found two possible solutions under some circumstances, probably due to the very strong dependence of σ upon the temperature, which peaks in the center of the hypersonic boundary layer when the wall is cooled. Kerrebrock^{26, 27} and Napolitano and Pozzi³⁸ have discussed flat plate boundary layers with $\underline{B}$ parallel to the wall but perpendicular to the flow. Kerrebrock²⁷ also considered the case of a current flowing into the fluid from the plate (electrode).

Because of its great practical interest, the MHD boundary layer at the stagnation point of a blunt body having $\underline{B}$ normal to the surface has been very thoroughly studied. Rossow⁴⁶ showed that the heat transfer to a cylinder would be reduced by the magnetic field. Meyer³⁶ pointed out that the

principal effect of the magnetic field was to alter the inviscid flow, thereby reducing the stagnation point velocity gradient and, hence, heat transfer. A discussion of the inviscid flow calculation and summary of the heat transfer effects have been given by Kemp^{24,25}, but the inviscid flow calculations have been subject to some debate.¹⁸ Recent developments include an incompressible solution,²³ extension to rarefied gas flow,³⁹ and an attempt to find solutions for large R_M .⁵⁷ Bush⁵ has shown that the reduction of σ within a cooled stagnation point boundary layer causes an overshoot in the velocity profile and a lesser reduction in heat transfer than would be the case for constant σ . This overshoot, which is caused by the fact that the pressure gradient at the stagnation point is unaffected by the magnetic field, has been discussed by Lykoudis.³⁵

The simplest channel flow is Couette flow, which was treated by Bleviss³ for a variable property hypersonic flow. Tao⁵⁴ considered the incompressible time-dependent Couette flow which is related to the Rayleigh problem. Hartmann type channel flows have been investigated by Shercliff⁵⁰ for the case of circular pipes and by Globe¹² for the case of an annulus. The time-dependent channel flow has been investigated by Lundgren, Atabek and Chang.³⁰ In the steady flow, the Hartmann number is the primary parameter defining the flow while the Reynolds number is also involved in the unsteady flow.

Very little attention has been given to Hall effects. Fay⁹ has shown that the Hall current causes a cross-flow in the Hartmann boundary layer. Sutton and Sherman⁵² considered Hall effects in Poiseuille flow and also found the same cross-flow due to Hall currents. The principal effect of

permitting Hall currents to flow appears to be that of reducing the electrical conductivity approximately by the factor, $1 + (\omega_e \tau_e)^2$.

Studies of MHD boundary layer stability have shown, in general, that the magnetic field has a stabilizing effect. For a flat plate boundary layer with $\underline{B}$ parallel to the flow, Rossow⁴³ and Arkhipov¹ found that the minimum critical Reynolds number more than doubles for an interaction parameter somewhat less than unity. For the corresponding free boundary layers, which are always unstable in the absence of a magnetic field, the flow may be stabilized for a finite wave number by a sufficiently large magnetic field⁸ and is absolutely stable¹³ for all R_V if $\epsilon > 1$ and $V < V_A$. For Poiseuille flow with $\underline{B}$ parallel to the flow, Stuart⁵¹ found a sixfold increase in the minimum critical Reynolds number for $S = 0.10$. When $\underline{B}$ is perpendicular to the wall, the minimum critical Reynolds number also increases with increasing M_H ,²⁹ approaching a value of 50,000 M_H when $M_H \gg 1$. (The experimental value of R_V/M_H at which transition occurs was found by Murgatroyd to be about 250. Since R_V/M_H is the value of the viscous Reynolds number based on the Hartmann thickness, it would appear that in the experiments the principal effect of the magnetic field is to alter the basic flow rather than to stabilize it.)

Most of the approaches to turbulent MHD flows have been speculative in nature,^{20, 37, 55} being patterned after classical analyses of turbulence. Kovaszny²⁸ has investigated the resistivity of a turbulent plasma in which the current paths are greatly lengthened by the random $\underline{V} \times \underline{B}$ field, and the turbulent energy is provided by random $\underline{j} \times \underline{B}$ forces. It was

suggested by Lykoudis³³ that transition to turbulence in channel flows will occur when the Reynolds number of the Hartmann layer exceeds that of the laminar sub-layer (about 200) in a turbulent flat-plate boundary layer. This hypothesis is in reasonable agreement with Murgatroyd's experiments.

The general discussions of wakes¹⁵⁻¹⁷ have been concerned with the propagation of Alfvén waves originating from the surface of a finite body, and make use of the magnetic equivalent of the Oseen approximation. Sears⁴⁸ has discussed magnetic boundary layers, but no solutions of such problems for R_M not small have been obtained. MHD jet flow²² and natural convection flow⁴² have recently been studied, at least within the boundary layer approximation.

V. Electronic Boundary Layers

Boundary layers in plasmas with no magnetic field present are much more nearly like ordinary boundary layers. The principal interest in such boundary layers lies in the effects of the presence of electrons on the thermal conductivity, which might considerably enhance the heat transfer.⁹ An experimental study of the increase in heat transfer due to the passage of a current through such a boundary layer has been made by Fay and Hogan,¹⁰ who found that the increase at either cathode or anode was proportional to the current. Talbot⁵ has analyzed the plasma sheath inside a stagnation point boundary layer on a Langmuir probe, but no experimental data is available.



VI. Conclusion

In surveying the progress to date, one is struck by the comparative richness of the theoretical analyses of plasma boundary layers as compared to the paucity of experimental evidence. As noted previously, this reflects the relative ease of extrapolating existing boundary layer techniques as opposed to developing an entirely new set of laboratory experiments and techniques. While one might be tempted to conclude that our theoretical understanding of the problem is far ahead of our laboratory experience, it is more likely than not that we have been analyzing the wrong problems. For the variety of physical phenomena exhibited by plasmas is certainly no less than that to be seen in ordinary fluids, and quite possibly is even more extensive and of greater practical utility. It seems highly unlikely that an adequate understanding of fluid plasma behavior can be manufactured from the ingredients of classical fluid mechanics and electromagnetic theory alone.

In addition to an increased emphasis on experimental research, one would hope for a better understanding of the microscopic processes which play such an important role in transport properties and the tendency toward maintenance of thermodynamic equilibrium. Because of the difficulty of creating and maintaining a plasma, one is inevitably forced to work under marginal circumstances which make necessary a quantitative knowledge of those processes by which energy is lost and the plasma degraded.

Finally, one must be prepared for more constructive concepts of plasma dynamics if such seem useful. Only in a very limited sense is a plasma an anisotropic, electrically-conducting fluid. It might, for example, be more convenient to consider it to be a system of hydromagnetic waves,

for which an appropriate kinetic theory and macroscopic conservation laws must be devised. Given such alternatives, the problem of boundary effects would undoubtedly acquire a different significance.

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