63-2-3

OCEANICS INC.

Report No. 63-02

November, 1962

Forces On A Hovering Slender Body of Revolution Submerged Under Waves of Moderate Wavelength

by

Theodore R. Goodman

Oceanics, Inc.
Technical Industrial Park
Plainview, L.I., New York

Forces On A Hovering Slender Body of Revolution

Submerged Under Waves of Moderate Wavelength

by

Theodore R. Goodman

Submitted to:

Department of the Navy
Bureau of Ships
Washington 25, D.C.

Under Contract No. NObs - 86514

Oceanics, Inc.
Technical Industrial Park
Plainview, L.I., New York

ABSTRACT

The forces on a slender body of revolution hovering under waves are determined. It is assumed in the analysis that the wavelength is of the same order of magnitude as the body radius, i.e. $kR=O(1)$, where k is the wave number and R is the body radius. Two cases are considered: beam seas and head seas. It is shown that for head seas conventional slender body theory is inadequate, and, for this case, a generalization of slender body theory is developed.

LIST OF PHYSICAL SYMBOLS

a	wave amplitude
$A^2(\omega)$	spectrum of random sea
c	wave propagation speed ($=\omega/k$)
C	empirical constant in Neumann spectrum ($=32.8\ ft^2/sec^5$)
g	acceleration due to gravity ($32.2\ ft/sec^2$)
h	depth of body centerline from undisturbed waterline
k	wave number ($=\omega^2/g$)
ℓ	body length
p	pressure
R	body radius
S	body cross-sectional area ($=\pi R^2$)
t	time
(u,v,w)	body velocities
V_w	wind speed
X	surge force
(x,r,θ)	inertial cylindrical coordinates
(x,y,z)	inertial cartesian coordinates - see Figure 1
Z	vertical force
β	angle between body axis and wave crests
λ	wavelength $= (2\pi/k)$
(ξ,η,ζ)	body displacements
ρ	water density
φ	velocity potential
ω	wave frequency

FORCES ON A HOVERING SLENDER BODY OF REVOLUTION SUBMERGED UNDER WAVES OF MODERATE WAVELENGTH

I. INTRODUCTION

When a slender submerged body is hovering in a velocity field caused by a sinusoidal wave train of large wavelength, the forces on the body can be calculated using conventional slender body theory. Such a calculation has been performed, for example, by Kaplan and Hu [1]. In this approach the orbital velocity potential associated with the undisturbed waves is expanded in the neighborhood of the body, in which case the problem reduces to solving a two-dimensional potential problem in the cross flow, and the force per unit length can be determined in terms of the added mass of the section. An implicit assumption which is made in conventional slender body theory is that the longitudinal variations are small in comparison with the lateral variations. It is clear that for a long slender body in long waves this assumption is satisfactory except, perhaps, locally near a blunt end or at the beginning of an appendage. Local aberrations cannot, of course, affect the total force to any great extent.

A different situation prevails when the wavelength is of the same order of magnitude as the dimensions of the cross-section and the waves are not beam-on. In this case, longitudinal variations become of the same order of magnitude as lateral variations and it is necessary, in calculating the body potential, to take this into account. A similar situation would prevail if one were to calculate the flow about a slender corrugated body with no waves if the wavelength of the corrugations were of the same order as the cross-sectional dimensions.

In the present report we will deal with the case of a slender body of revolution, and we will consistently make the assumption that the wavelength is of the same order of magnitude as the body radius, i.e. $kR = O(1)$, in which case, since the body is slender, $k\ell \gg 1$. We will then seek the leading term in the expression for the force consistent with this ordering hypothesis. In the course of the analysis, body motions will be included. However, these motions are assumed to be wave-induced, and, as a consequence, their oscillatory part will be at most of the order of the wave orbital motions; in other words, an upper bound on the oscillatory part of the body motion is given by the motion of a particle. This fact will aid us in simplifying the expressions that are obtained. Two cases will be considered: beam seas and head seas.

The velocity potential of the waves with wavelength λ , amplitude a , and wave propagation speed c is given by:

$$\varphi_w = ace^{-kh} e^{k\{z + i[x\cos\beta + y\sin\beta - ct]\}} \qquad (1)$$

where $k = 2\pi/\lambda = g/c^2$ and the real part is to be taken. The coordinates (x,y,z) are fixed with respect to an inertial system, with z pointing upward and x directed along the axis of the undisturbed body. The angle β is the angle between the x axis and the wave crests (see Figure 1.). In terms of a coordinate system fixed with respect to the body, the wave potential becomes

$$\varphi_w = ace^{-kh}\, e^{k\{z+\zeta+i[(x+\xi)\cos\beta + (y+\eta)\sin\beta - ct]\}} \tag{2}$$

where (ξ, η, ζ) represent the (x,y,z) components of the body displacements. Their time rates of change will be denoted by (u,v,w).

II. BEAM SEAS

For beam seas $\beta = \pi/2$ and Equation (2) reduces to

$$\varphi_w = ace^{-kh}\, e^{k\{z+\zeta+i[(y+\eta)-ct]\}} \tag{3}$$

There is no x variation in the wave potential for beam seas, and consequently none in the wave orbital velocities. Furthermore, the x component of orbital velocity is zero. Because of this, the longitudinal variations are indeed small compared with the lateral variations, and, if the body is slender, it is permissable to use conventional slender body theory. The body potential, therefore, satisfies the two dimensional Laplace equation in the cross flow. Let us transform to polar coordinates in the cross flow:

$$\begin{aligned} y &= r\cos\theta \\ z &= r\sin\theta \end{aligned} \tag{4}$$

The body potential can be determined by using the circle theorem [2]. The wave potential plus body potential satisfies the condition of no normal flow on the circle $r = R$, and is given by

$$\varphi_w + \varphi_s = ace^{-k(h-\zeta-r\sin\theta)}\cos k(r\cos\theta+\eta-ct) + ace^{-k(h-\zeta-\frac{R^2}{r}\sin\theta)}\cos k(\frac{R^2}{r}\cos\theta+\eta-ct) \tag{5}$$

where the real part has been taken. To this must be added the potential induced by the body motions v and w (see Figure 1.). The complete potential is thus given by

$$\varphi = ace^{-k(h-\zeta-r\sin\theta)}\cos k(r\cos\theta+\eta-ct) + ace^{-k(h-\zeta-\frac{R^2}{r}\sin\theta)}\cos k(\frac{R^2}{r}\cos\theta+\eta-ct) + \frac{vR^2}{r}\cos\theta + \frac{wR^2}{r}\sin\theta. \tag{6}$$

The radial and circumferential velocities evaluated at the body $(r = R)$ are:

$$V_\theta = \frac{-1}{R}\frac{\partial\varphi}{\partial\theta} = 2kace^{-k(h-\zeta-R\sin\theta)}\cos[\theta-kR\cos\theta-k\eta+kct] - v\sin\theta + w\cos\theta \tag{7}$$

$$V_r = -\frac{\partial\varphi}{\partial r} = v\cos\theta + w\sin\theta \tag{8}$$

The time rate of change of the velocity potential evaluated on the body is given by

$$\frac{\partial\varphi}{\partial t} = 2acke^{-k(h-\zeta-R\sin\theta)}[-(v-c)\sin k(R\cos\theta+\eta-ct) + w\cos k(R\cos\theta+\eta-ct)]. \tag{9}$$

Here the terms proportional to $\dot{v}$ and $\dot{w}$ have been ignored.

These ignored terms give rise to standard added mass terms in the force which can be added in at the end of the analysis.

The pressure on the body is obtained from the (two-dimensional) Bernoulli equation in moving coordinates:

$$\frac{p}{\rho} = \frac{\partial \varphi}{\partial t} - v\frac{\partial \varphi}{\partial y} - w\frac{\partial \varphi}{\partial z} - \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] \tag{10}$$

Transforming to polar coordinates with the aid of Equation (4), it is found that

$$\frac{p}{\rho} = \frac{\partial \varphi}{\partial t} - v\left(\frac{\partial \varphi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\sin\theta\right) - w\left(\frac{\partial \varphi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\cos\theta\right)$$
$$- \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)^2\right] \tag{11}$$

However, we will consistently ignore squares of the body motions, and since $\frac{\partial \varphi}{\partial r}$ is of the order of the body motions according to Equation (8), the pressure simplifies to

$$\frac{p}{\rho} = \frac{\partial \varphi}{\partial t} + (v\sin\theta - w\cos\theta)\frac{1}{r}\frac{\partial \varphi}{\partial \theta} - \frac{1}{2}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)^2 \tag{12}$$

The vertical (heaving) force per unit length is given by

$$\frac{dZ}{dx} = -\int_0^{2\pi} p\sin\theta\, R\, d\theta \tag{13}$$

Equations (7-9) must be substituted into Equation (12) and, thence, into Equation (13). In so doing, all quadratic terms in the body motions are to be ignored. In calculating the term $\frac{1}{2}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)^2$ which appears in Equation (12), it will be necessary to square Equation (7), and in so doing the term $\cos^2(\theta - kR\cos\theta - k\eta + kct)$ will appear. The double angle formula

$$\cos^2(\quad) = \frac{1}{2}\left[1 + \cos 2(\quad)\right] \tag{14}$$

must be used, and the higher harmonic term will be ignored. The heaving force per unit length is then given by

$$\begin{aligned}
\frac{dZ}{dx} = {} & 2ack\rho R e^{-k(h-\zeta)}\Big[(v-c)\int_0^{2\pi} e^{kR\sin\theta} \sin k(R\cos\theta + \eta - ct)\sin\theta \, d\theta \\
& - w\int_0^{2\pi} e^{kR\sin\theta}\cos k(R\cos\theta + \eta - ct)\sin\theta \, d\theta\Big] \\
& + 2ack\rho R e^{-k(h-\zeta)}\Big[w\int_0^{2\pi} e^{kR\sin\theta} \sin\theta\cos\theta\cos(\theta - kR\cos\theta - k\eta + kct)\, d\theta \\
& - v\int_0^{2\pi} \sin^2\theta \, e^{kR\sin\theta}\cos(\theta - kR\cos\theta - k\eta + kct)\, d\theta\Big] \\
& + ack\rho R e^{-k(h-\zeta)}\Big[ack e^{-k(h-\zeta)}\int_0^{2\pi} e^{2kR\sin\theta}\sin\theta \, d\theta \\
& + 2w\int_0^{2\pi} \cos\theta\sin\theta \, e^{kR\sin\theta}\cos(\theta - kR\cos\theta - k\eta + kct)\, d\theta \\
& - 2v\int_0^{2\pi}\sin^2\theta \, e^{kR\sin\theta}\cos(\theta - kR\cos\theta - k\eta + kct)\, d\theta\Big]
\end{aligned} \tag{15}$$

There are five different integrals in Equation (15) and each of them is determined in the Appendix. The final result is

$$\frac{dZ}{dx} = 2\rho S k^2 c^2\left[ka^2 e^{-2k(h-\zeta)}\frac{I_1(2kR)}{kR} + a e^{-k(h-\zeta)}\sin k(ct - \eta)\right] \tag{16}$$

where I_1 is a modified Bessel function of the first kind.

The first term in this equation is a suction force. It is of second order in wave amplitude, and attenuates twice as rapidly with depth h as the second term which is oscillatory. Nevertheless, for certain operations it is important to retain the suction force because it persistently acts upward and tends to make the body rise. Notice that explicit dependence on the body velocities (v,w) has disappeared. The result still depends on body motions, however, through the displacements (η, ζ).

A. <u>An Alternative Derivation</u>

The derivation of the heaving force given above is perfectly straightforward. We will now present an alternative derivation from a different point of view which is actually a prologue to the derivation of the heaving force for a body in a random sea.

Consider the body displacements η, ζ and the body velocities v, w. From the equations of motion of the body we can expect that the body motions will consist of a drifting motion which varies slowly with time and which is due, primarily, to the suction force, plus an oscillatory motion which varies rapidly with time and which is due, primarily, to the oscillatory force. Hence, let

$$\begin{aligned} \eta &= \eta_0 + \eta' \\ \zeta &= \zeta_0 + \zeta' \\ v &= v_0 + v' \\ w &= w_0 + w' \end{aligned} \qquad (17)$$

where the quantities with subscript zero denote the drifting motions, and the primed quantities denote the rapidly oscillating motions. The velocity potential and pressure can be similarly separated,

$$\begin{aligned} \varphi &= \varphi_0 + \varphi' \\ p &= p_0 + p' \end{aligned} \qquad (18)$$

If the relations between all quantities were linear then we could state unequivocally that the oscillating input caused the oscillating output and the drifting input caused the drifting output. However, Bernoulli's equation for the pressure is quadratic with respect to the velocity potential, and even the potential itself, Equation (6), involves the body motions (η, ζ) in a nonlinear way. It is clear, therefore, that the drifting part of the input can affect the oscillatory part of the output and vice versa. Substituting Equations (17,18) into Equation (6) there is obtained

$$\begin{aligned}\varphi_0+\varphi' &= ace^{-k(h-\zeta_0-\zeta'-r\sin\theta)}\cos k(r\cos\theta+\eta_0+\eta'-ct)\\ &+ ace^{-k(h-\zeta_0-\zeta'-\frac{R^2}{r}\sin\theta)}\cos k\left(\frac{R^2}{r}\cos\theta+\eta_0-\eta'-ct\right)\\ &+(V_0+V')\frac{R^2}{r}\cos\theta+(W_0+W')\frac{R^2}{r}\sin\theta \end{aligned} \tag{19}$$

Similarly substituting into Bernoulli's equation, Equation (12), there is obtained

$$\begin{aligned}\frac{1}{\rho}(P_0+P') &= \frac{\partial\varphi_0}{\partial t}+\frac{\partial\varphi'}{\partial t}+\left[(V_0+V')\sin\theta-(W_0+W')\cos\theta\right]\left[\frac{1}{r}\frac{\partial\varphi_0}{\partial\theta}+\frac{1}{r}\frac{\partial\varphi'}{\partial\theta}\right]\\ &-\frac{1}{2}\left[\frac{1}{r^2}\left(\frac{\partial\varphi_0}{\partial\theta}\right)^2+\frac{2}{r^2}\frac{\partial\varphi_0}{\partial\theta}\frac{\partial\varphi'}{\partial\theta}+\frac{1}{r^2}\left(\frac{\partial\varphi'}{\partial\theta}\right)^2\right]\end{aligned} \tag{20}$$

We will retain terms which may be of the order of the wave amplitude squared and ignore higher order terms. In so doing, we will assume that the oscillatory motions are, at most, of the order of the wave amplitude . This is a reasonable assumption in view of the fact that the motions are solely wave-induced. Thus in Equation (19), we expand the exponential and trigonometric terms treating ζ' and η' as small quantities, and retain only linear terms in the expansion.

$$\begin{aligned}
\varphi_0 + \varphi' = ac\Big\{ & e^{-k(h-\zeta_0-r\sin\theta)}\cos k(r\cos\theta+\eta_0-ct) \\
& + k\zeta' e^{-k(h-\zeta_0-r\sin\theta)}\cos k(r\cos\theta+\eta_0-ct) \\
& - k\eta' e^{-k(h-\zeta_0-r\sin\theta)}\sin k(r\cos\theta+\eta_0-ct) \\
& + e^{-k(h-\zeta_0-\frac{R^2}{r}\sin\theta)}\cos k\left(\frac{R^2}{r}\cos\theta+\eta_0-ct\right) \\
& + k\zeta' e^{-k(h-\zeta_0-\frac{R^2}{r}\sin\theta)}\cos k\left(\frac{R^2}{r}\cos\theta+\eta_0-ct\right)\Big\} \\
& + (V_0+V')\frac{R^2}{r}\cos\theta + (W_0+W')\frac{R^2}{r}\sin\theta
\end{aligned} \tag{21}$$

Now average Equation (21) over one period and denote the averaging with a bar. Clearly, the average of any primed quantity is zero. Drifting quantities will be assumed to be almost constant over one period and, therefore, unaffected by the averaging process. They are said to sift through the averaging integral.

$$\begin{aligned}
\varphi_0 = & acke^{-k(h-\zeta_0-r\sin\theta)}\left\{\overline{\zeta'\cos k(r\cos\theta+\eta_0-ct)} - \overline{\eta'\sin k(r\cos\theta+\eta_0-ct)}\right\} \\
& + acke^{-k(h-\zeta_0-\frac{R^2}{r}\sin\theta)}\left\{\overline{\zeta'\cos k\left(\frac{R^2}{r}\cos\theta+\eta_0-ct\right)} - \overline{\eta'\sin k\left(\frac{R^2}{r}\cos\theta+\eta_0-ct\right)}\right\} \\
& + V_0\frac{R^2}{r}\cos\theta + W_0\frac{R^2}{r}\sin\theta
\end{aligned} \tag{22}$$

Subtracting Equation (22) from Equation (21) we obtain the following equation for φ' :

$$
\begin{aligned}
\varphi' = ac\Big\{ & e^{-k(h-\zeta_0 - r\sin\theta)} \cos k(r\cos\theta + \eta_0 - ct) \\
& + k\zeta' e^{-k(h-\zeta_0 - r\sin\theta)} \cos k(r\cos\theta + \eta_0 - ct) \\
& - k\zeta' e^{-k(h-\zeta_0 - r\sin\theta)} \cos k(r\cos\theta + \eta_0 - ct) \\
& - k\zeta' e^{-k(h-\zeta_0 - r\sin\theta)} \sin k(r\cos\theta + \eta_0 - ct) \\
& + k\zeta' e^{-k(h-\zeta_0 - r\sin\theta)} \sin k(r\cos\theta + \eta_0 - ct) \\
& + e^{-k(h-\zeta_0 - \frac{R^2}{r}\sin\theta)} \cos k\left(\frac{R^2}{r}\cos\theta + \eta_0 - ct\right) \\
& + k\zeta' e^{-k(h-\zeta_0 - \frac{R^2}{r}\sin\theta)} \cos k\left(\frac{R^2}{r}\cos\theta + \eta_0 - ct\right) \\
& - k\zeta' e^{-k(h-\zeta_0 - \frac{R^2}{r}\sin\theta)} \cos k\left(\frac{R^2}{r}\cos\theta + \eta_0 - ct\right) \\
& - k\eta' e^{-k(h-\zeta_0 - \frac{R^2}{r}\sin\theta)} \sin k\left(\frac{R^2}{r}\cos\theta + \eta_0 - ct\right) \\
& + k\eta' e^{-k(h-\zeta_0 - \frac{R^2}{r}\sin\theta)} \sin k\left(\frac{R^2}{r}\cos\theta + \eta_0 - ct\right)\Big\} \\
& + v' \frac{R^2}{r}\cos\theta + w' \frac{R^2}{r}\sin\theta
\end{aligned}
\tag{23}
$$

The first and sixth term in the bracket must be retained but all other terms can be ignored since they give rise only to higher harmonics. Whence,

$$
\begin{aligned}
\varphi' = ac\Big\{ & e^{-k(h-\zeta_0 - r\sin\theta)} \cos k(r\cos\theta + \eta_0 - ct) + \\
& + e^{-k(h-\zeta_0 - \frac{R^2}{r}\sin\theta)} \cos k\left(\frac{R^2}{r}\cos\theta - \eta_0 - ct\right)\Big\} \\
& + v' \frac{R^2}{r}\cos\theta + w' \frac{R^2}{r}\sin\theta
\end{aligned}
\tag{24}
$$

To determine the mean pressure, average Equation (20) over one period.

$$
\begin{aligned}
\frac{1}{\rho} P_0 = \frac{\partial \varphi_0}{\partial t} & + (V_0 \sin\theta - W_0\cos\theta)\frac{1}{r}\frac{\partial \varphi_0}{\partial \theta} \\
& + \overline{(v'\sin\theta - w'\cos\theta)\frac{1}{r}\frac{\partial \varphi'}{\partial \theta}} \\
& - \frac{1}{2}\left[\frac{1}{r^2}\left(\frac{\partial \varphi_0}{\partial \theta}\right)^2 + \frac{1}{r^2}\overline{\left(\frac{\partial \varphi'}{\partial \theta}\right)^2}\right]
\end{aligned}
\tag{25}
$$

The term $\frac{\partial \varphi_o}{\partial t}$ can be obtained from Equation (22). In performing the differentiation, it will be assumed that $\dot{\eta}_o$, c, $\dot{\zeta}_o$ filter out of the averaging. However, since the first four terms are already of second order in the wave amplitude, and since $\dot{\eta}_o$ and $\dot{\zeta}_o$ are wave induced, any terms which arise from the filtering of $\dot{\eta}_o$ and $\dot{\zeta}_o$ will be of higher order and can be ignored. Terms which arise from the filtering of c, on the other hand, must be retained.

$$\frac{\partial \varphi_o}{\partial t} = ac^2k^2e^{-k(h-\zeta_o-r\sin\theta)}\left\{\overline{\zeta'\sin k(r\cos\theta+\eta_o-ct)}+\overline{\eta'\cos k(r\cos\theta-\eta_o-ct)}\right\}$$

$$+ac^2k^2e^{-k(h-\zeta_o-r\sin\theta)}\left\{\overline{\zeta'\sin k\left(\frac{R^2}{r}\cos\theta+\eta_o-ct\right)}+\overline{\eta'\cos k\left(\frac{R^2}{r}\cos\theta+\eta_o-ct\right)}\right\} \quad (26)$$

The terms proportional to $\dot{v}_o$ and $\dot{w}_o$ have been ignored, consistent with our policy of not considering added mass terms.

Ignoring terms of the order of the body motion squared, the mean pressure becomes

$$\frac{1}{\rho}P_o = \overline{(v'\sin\theta - w'\cos\theta)\frac{1}{r}\frac{\partial \varphi'}{\partial \theta}} - \frac{1}{2}\frac{1}{r^2}\overline{\left(\frac{\partial \varphi'}{\partial \theta}\right)^2} + \frac{\partial \varphi_o}{\partial t} \quad (27)$$

To obtain p' subtract Equation (27) from Equation (20). If all terms of second order in the body motion and all higher harmonics are ignored, there is obtained

$$\frac{1}{\rho}P' = \frac{\partial \varphi'}{\partial t} + (v_o\sin\theta - w_o\cos\theta)\frac{1}{r}\frac{\partial \varphi'}{\partial \theta} - \frac{1}{r^2}\left(\frac{\partial \varphi_o}{\partial \theta}\right)\frac{\partial \varphi'}{\partial \theta} \quad (28)$$

When substituting ϕ_o into Equation (28) it is permissible to ignore the first four terms in Equation (22) because they give rise to terms of higher order in the pressure. Then, upon evaluating at the body, Equation (28) simplifies to

$$\left(\frac{1}{\rho}p'\right)_{r=R} = \frac{\partial \varphi'}{\partial t} + 2(v_o \sin\theta - w_o \cos\theta)\frac{1}{R}\frac{\partial \varphi}{\partial \theta} \tag{29}$$

Upon substituting Equations (24,26) into Equation (27), letting $r = R$ and integrating with respect to θ according to Equation (13), the mean vertical force per unit length becomes after higher order terms are ignored:

$$\frac{dZ_o}{dx} = 2\rho S k^3 c^2 a^2 e^{-2k(h-\zeta_o)} \frac{I_1(2kR)}{kR} + 2\rho S c^2 a k^2 e^{-k(h-\zeta_o)} \left\{ \overline{\zeta' \sin k(ct-\eta_o)} - \overline{\eta' \cos k(ct-\eta_o)} \right\} \tag{30}$$

The fluctuating vertical force per unit length is obtained by integrating Equation (29) with respect to θ, and there is obtained:

$$\frac{dZ'}{dx} = 2\rho S k^2 c^2 a e^{-k(h-\zeta_o)} \sin k(ct-\eta_o) \tag{31}$$

It can easily be seen that Equation (31) is identical to the second term of Equation (16) with η_o and ζ_o replacing η and ζ respectively. The first term of Equation (30) is the same as the first term of Equation (16) with ζ_o replacing ζ. The second and third terms of Equation (30) are implicit in Equation (16) also, for if we let $\eta = \eta_o + \eta'$ and $\zeta = \zeta_o + \zeta'$ and then expand Equation (16) retaining terms of the order of the square of the wave amplitude, the second and third terms of Equation (30) are reproduced identically once higher harmonics are ignored. The oscillatory force given by Equation (31) is identical with the result which would have been obtained if the long wave approximation were made at the outset of the analysis (see [1]). It can be shown that all other first order forces and moments are, likewise, identical to their long wave limits in beam seas.

If the oscillatory part of the motion is such that the body moves like a water particle (which is very nearly the case), then

$$\zeta' = -a e^{-k(h-\zeta_0)} \sin k(ct-\eta_0)$$
$$\eta' = a e^{-k(h-\zeta_0)} \cos k(ct-\eta_0) \tag{32}$$

Substituting into Equation (30), there is obtained the following expression for the mean suction force per unit length*:

$$\frac{dZ_0}{dx} = 2\rho S k^3 a^2 c^2 e^{-2k(h-\zeta_0)} \left[\frac{I_1(2kR)}{kR} - 1 \right] \tag{33}$$

The universal function $\alpha_0(kR) = \left[I_1(2kR) / kR - 1 \right]$ is tabulated in Table I.

II.A. RANDOM SEAS

Suppose the wave potential, as given by Equation (1) for a sinusoidal wave train, is replaced by a function of a random variable with spectrum $A^2(\omega)$. Using the formulation of Pierson [4], the complete potential, Equation (6), is replaced by

$$\varphi = \int_0^\infty \left\{ c e^{-k(h-\zeta-r\sin\theta)} \cos k(r\cos\theta + \eta - ct) + c e^{-k(h-\zeta-\frac{R^2}{r}\sin\theta)} \cos k\left(\frac{R^2}{r}\cos\theta + \eta - ct\right) \right\} \sqrt{A^2(\omega)\,d\omega} + \frac{vR^2}{r}\cos\theta + \frac{wR^2}{r}\sin\theta \tag{34}$$

Now assume that all quantities can be written in the form

$$(\ \ ) = (\ \ )_0 + (\ \ )' \tag{35}$$

*This result has been obtained independently by Ogilvie [3].

where the subscript zero represents the mean value, and is a slowly varying function of time. Quantities with a prime are fluctuating random quantities with zero mean. The similarity between this notation and the notation used in the preceding section will be noted.

Continuing the parallel with the preceding section, we will operate on the potential and the pressure with the average:

$$\lim_{T\to\infty} \frac{1}{2T} \sum_{-T}^{T} \tag{36}$$

This average replaces the average over one period which was used in the preceding section. By the limit $T\to\infty$ we mean that T is large enough so that a meaningful sample of any random quantity is included, but small enough so that mean quantities are essentially constant over this time interval. Thus, quantities with subscript zero will filter through the operation represented by Equation (36), just as they did for the average over one period in the periodic case. The analysis is identical in every respect to the analysis performed for periodic fluctuations, provided only that we interpret the bar to be the statistical average represented by Equation (36).

Suppose the fluctuating part of the motion to be such that the body follows a water particle. Then the equation corresponding to Equation (33) is

$$\frac{dZ_0}{dx} = 2\rho S \int_0^\infty k^3 c^2 A^2(\omega) e^{-2k(h-\zeta_0)} \left[\frac{I_1(2kR)}{kR} - 1 \right] d\omega \tag{37}$$

The fluctuating force per unit length is

$$\frac{dZ'}{dx} = 2\rho S \int_0^\infty k^2 c^2 e^{-k(h-\zeta_0)} \sin k(ct-\eta_0) \sqrt{A^2(\omega)\, d\omega} \tag{38}$$

If the wave spectrum is a Neumann spectrum for a fully developed sea (Reference [4])

$$A^2(\omega) = \frac{\pi}{2} \frac{C}{\omega^6} e^{-2g^2/V_w^2\omega^2} \tag{39}$$

Then, letting $c = \omega/k$, $k = \omega^2/g$, Equation (37) becomes

$$\frac{dZ_0}{dx} = \frac{\pi\rho SC}{g} \int_0^\infty \frac{1}{\omega^2} \exp\left[\frac{-2g^2}{V_w^2\omega^2} - \frac{2\omega^2}{g}(h-\zeta_0)\right]\left[\frac{I_1(2kR)}{kR} - 1\right] d\omega \tag{40}$$

If the waves are long with respect to the body radius, then $I_1(2kR)/kR \approx 1 + \frac{(kR)^2}{2}$, and the ω integral can be evaluated exactly. The integrated force can then be evaluated to be

$$Z_0 = \frac{1}{4}\left(\frac{\pi}{2}\right)^{3/2} \frac{I_r \rho C}{g^4(h-\zeta_0)} \left[\sqrt{\frac{g}{2(h-\zeta_0)}} + \frac{4g}{V_w}\right] e^{-4\sqrt{g(h-\zeta_0)}/V_w} \tag{41}$$

where I_r is the volumetric polar moment of inertia. If the waves are not long, but the body is deeply submerged $\left[h - \zeta_0 \gg V_w^2/16g\right]$, the ω integral can be evaluated approximately by the method of steepest descents [5]. The result is

$$Z_0 \approx \frac{\left(\frac{\pi}{2}\right)^{3/2} \rho C V_w}{g^2} e^{-4\sqrt{g(h-\zeta_0)}/V_w} \int S(x) \left[\frac{I_1\left(\frac{2R\sqrt{\frac{g}{h-\zeta_0}}}{V_w}\right)}{\frac{R}{V_w}\sqrt{\frac{g}{h-\zeta_0}}} - 1\right] dx \tag{42}$$

where the integral is taken over the length of the body.

III. HEAD SEAS

For head seas $\beta = 0$, and Equation (2) reduces to

$$\varphi_w = ace^{-kh} e^{k\{z+\zeta+i[x+\xi-ct]\}} \tag{43}$$

There is an explicit x variation in the wave potential for head seas, but no y variation. Consequently, there are x variations in the orbital velocities, but the y-component of orbital velocity is zero. The wavelength of the orbital velocity will be assumed to be of the same order as a cross-sectional dimension, i.e. $kR = O(1)$, and, as a consequence, conventional slender body theory is not applicable. For convenience, let

$$\underline{X} = x + \xi - ct \tag{44}$$

and transform to polar coordinates according to Equation (4). The wave potential then becomes:

$$\varphi_w = ace^{-k(h-\zeta)+ik\underline{X}} e^{kr\sin\theta} \tag{45}$$

Expanding the wave potential as a Fourier series in θ, there is obtained

$$\varphi_w = ace^{-k(h-\zeta)+ik\underline{X}} \left[I_0(kr) + 2\sum_{1}^{\infty} I_n(kr)\cos n(\theta - \pi/2) \right] \tag{46}$$

where I_n is a modified Bessel function of the first kind.

The body potential must satisfy the three dimensional Laplace equation. In cylindrical coordinates, this potential can be expressed in terms of line sources and higher order line singularities, as follows:

$$\varphi_B = \int_0^{\ell} \frac{f_0(x_1, t)\,dx_1}{[(x-x_1)^2 + r^2]^{1/2}} + \sum_1^{\infty} r^n \cos n(\theta - \pi/2) \int_0^{\ell} \frac{f_n(x_1, t)\,dx_1}{[(x-x_1)^2 + r^2]^{n+1/2}} \tag{47}$$

The potential φ_B will be used to cancel the normal flow at the body, $r = R(x)$. It is, likewise, necessary to add another potential, φ_m, to account for the body motions. However, the body motions induce velocities which are slowly varying in the x direction, and, consequently, an ordinary slender body representation will be adequate for this potential. In other words, the potential φ_m is also represented by an equation of the form of (47), but we may take the limit $r \to 0$ without regard to the rapidity of the variation of f with respect to x_1. This limit yields (see Reference [6]):

$$\varphi_m = -\frac{1}{2} f_0(x) \ln r + g(x) + \sqrt{\pi} \sum_1^{\infty} \frac{f_n(x)\,\Gamma(n) \cos n(\theta - \frac{\pi}{2})}{r^n\,\Gamma(n+\frac{1}{2})} \tag{48}$$

where

$$g(x) = \frac{d}{dx} \int_0^{\ell} f_0(x_1) \operatorname{sgn}(x-x_1) \ln 2|x-x_1|\,dx_1 \tag{49}$$

The expression for φ_B can also be simplified by taking the limit $r \to 0$, but, because this potential must cancel the wave orbital velocities, the strength of the singularities will vary rapidly with x_1. Hence, let

$$f_n(x_1, t) = e^{ikx} \cdot F_n(x_1, t) \tag{50}$$

where F_n is a slowly varying function of x_1. Equation (47) becomes

$$\varphi_B = \int_0^{\ell} \frac{F_0(x_1)\, e^{ikx_1}\, dx_1}{\left[(x-x_1)^2+r^2\right]^{1/2}} + \sum_1^{\infty} r^n \cos n\left(\theta-\tfrac{\pi}{2}\right) \int_0^{\ell} \frac{F_n(x_1)\, e^{ikx_1}\, dx_1}{\left[(x-x_1)^2+r^2\right]^{n+1/2}} \tag{51}$$

We must now take the limit $k \to \infty$, which expresses the fact that the wavelength is small compared with the body length, i.e. $k\ell \gg 1$. According to a lemma of Riemann-Lebesque, Reference [7], integrals of the form found in Equation (51) will vanish as $k \to \infty$, provided the non-oscillatory part of the integrand has bounded variation. However, if we simultaneously take the limit $r \to 0$, it is clear that all the integrands will become singular in the neighborhood of $x_1 = x$. Hence, only that part of the integrand in the immediate vicinity of the singularity will contribute to the integral. We may, therefore, evaluate $F_n(x_1)$ at $x_1 = x$ and bring F_n outside the integral sign. Furthermore, we may let the limits of integration extend from $x_1 = -\infty$ to $x_1 = +\infty$ since the results of so doing will contribute a negligible amount to the value of the integral by virtue of the Riemann-Lebesque lemma. In accordance with these concepts, Equation (51) becomes

$$\varphi_B = F_0(x) \int_{-\infty}^{\infty} \frac{e^{ikx_1}\, dx_1}{\left[(x-x_1)^2+r^2\right]^{1/2}} + \sum_1^{\infty} r^n F_n(x) \cos n\left(\theta-\tfrac{\pi}{2}\right) \int_{-\infty}^{\infty} \frac{e^{ikx_1}\, dx_1}{\left[(x-x_1)^2+r^2\right]^{n+1/2}} \tag{52}$$

The integrals can be expressed in terms of modified Bessel functions of the second kind, with the result that

$$\varphi_B = 2F_0(x)\, e^{ikx} K_0(kr) + 2\sqrt{\pi} \sum_1^{\infty} \frac{F_n(x)\, e^{ikx} \left(\frac{k}{2}\right)^n K_n(kr) \cos n\left(\theta-\frac{\pi}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \tag{53}$$

In the neighborhood of the body, the quantity (kr) which appears as the argument of the Bessel functions, is postulated to be of order unity, and, as a consequence, Equation (53) cannot be simplified any further. The complete potential is given as the sum

$$\varphi = \varphi_W + \varphi_m + \varphi_B \tag{54}$$

where expressions for φ_W , φ_m and φ_B are given by Equations (46,48,54) respectively.

The boundary condition for a three-dimensional body of revolution, including the body motions surge, heave, and pitch, has been derived by Cuthbert and Kerr in Reference [8]:

$$\left[u + R\,q\cos\left(\theta - \tfrac{\pi}{2}\right) + \varphi_x\right]\frac{dR}{dx} = (w - q\,x)\cos\left(\theta - \tfrac{\pi}{2}\right) + \varphi_r\,, \quad r = R \tag{55}$$

where the subscripts x,r denote partial derivatives. By comparing the second term on the left with the first term on the right, it is clear that the second term on the left can be neglected for a slender body, and the boundary condition then simplifies to

$$\left[u + \varphi_x\right]\frac{dR}{dx} = (w - q\,x)\cos\left(\theta - \tfrac{\pi}{2}\right) + \varphi_r\,, \quad r = R \tag{56}$$

Differentiating Equations (46,48,54) with respect to x and r , and substituting into Equation (56), there is obtained

$$\left\{u + acike^{-k(h-\zeta)+ik\bar{X}}\left[I_0(kR) + 2\sum_1^\infty I_n(kR)\cos n(\theta - \tfrac{\pi}{2})\right]\right.$$
$$- \frac{1}{2}f_0'(x)\ln R + g'(x) + \sqrt{\pi}\sum_1^\infty \frac{f_n'(x)\Gamma(n)\cos n(\theta-\frac{\pi}{2})}{R^n\Gamma(n+\frac{1}{2})}$$
$$\left.+ 2K_0(kR)\frac{d}{dx}(F_0e^{ikx}) + 2\sqrt{\pi}\sum_1^\infty \frac{d}{dx}(e^{ikx}F_n(x))\left(\frac{k}{2}\right)^n\cos n(\theta-\tfrac{\pi}{2})K_n(kR)\Big/\Gamma(n+\tfrac{1}{2})\right\}\frac{dR}{dx}$$
$$= (w - qx)\cos(\theta - \tfrac{\pi}{2}) + kace^{-k(h-\zeta)+ik\bar{X}}\left[I_0'(kR) + 2\sum_1^\infty I_n'(kR)\cos n(\theta-\tfrac{\pi}{2})\right]$$
$$- \frac{1}{2R}f_0(x) - \sqrt{\pi}\sum_1^\infty \frac{f_n(x)n\Gamma(n)\cos n(\theta-\frac{\pi}{2})}{R^{n+1}\Gamma(n+\frac{1}{2})}$$
$$+ 2kF_0(x)e^{ikx}K_0'(kR) + 2k\sqrt{\pi}\sum_1^\infty \frac{e^{ikx}F_n(x)(\frac{k}{2})^n\cos n(\theta-\frac{\pi}{2})K_n'(kR)}{\Gamma(n+\frac{1}{2})} \qquad (57)$$

Consider the terms corresponding to $n = 0$:

$$\left\{u + acike^{-k(h-\zeta)+ik\bar{X}}I_0(kR) - \tfrac{1}{2}f_0'(x)\ln R + g'(x) + 2K_0(kR)\frac{d}{dx}(F_0e^{ikx})\right\}\frac{dR}{dx}$$
$$= kace^{-k(h-\zeta)+ik\bar{X}}I_0'(kR) - \frac{1}{2}\frac{f_0(x)}{R} + 2kF_0e^{ikx}K_0'(kR) \qquad (58)$$

We equate the non-oscillatory parts:

$$\left\{u - \tfrac{1}{2}f_0'(x)\ln R + g'(x)\right\}\frac{dR}{dx} = -\frac{1}{2}\frac{f_0(x)}{R} \qquad (59)$$

To the first order in body radius we obtain

$$f_0(x) = -2R\frac{dR}{dx}u \qquad (60)$$

We now equate the oscillatory parts:

$$\frac{dR}{dx}\left\{acike^{-k(h-\zeta)+ik\bar{X}}I_0(kR) + 2K_0(kR)\frac{d}{dx}(F_0e^{ikx})\right\}$$
$$= kace^{-k(h-\zeta)+ik\bar{X}}I_0'(kR) + 2kF_0e^{ikx}K_0'(kR) \qquad (61)$$

If $kR = O(1)$, then, to the first order in body radius we obtain

$$F_0 e^{ikx} = \frac{-ace^{-k(h-\zeta)+ik\bar{X}} I_0'(kR)}{2K_0'(kR)} \tag{62}$$

Now consider the non-oscillatory terms corresponding to $n = 1$. To the first order in body radius we obtain

$$f_1 = R^2 W \tag{63}$$

Finally, consider the oscillatory terms corresponding to $n \geq 1$. To the first order in body radius we obtain

$$e^{ikx} F_n(x) = -\frac{ace^{-k(h-\zeta)+ik\bar{X}} I_n'(kR)\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\left(\frac{k}{2}\right)^n K_n'(kR)} \tag{64}$$

Substituting back into the potential, we obtain after some combining and simplifying

$$\varphi = ace^{-k(h-\zeta)+ik\bar{X}} \sum_1^{\infty} \frac{I_n(kr)K_n'(kR) - K_n(kr)I_n'(kR)}{K_n'(kR)} \epsilon_n \cos n\left(\theta - \frac{\pi}{2}\right)$$
$$+ uR\frac{dR}{dx}\ln r + g(x) + \frac{R^2 W}{r}\cos\left(\theta - \pi/2\right) \tag{65}$$

where $g(x) = -2u\frac{d}{dx}\int_0^l R(x_1)R'(x_1)\,\mathrm{sgn}(x-x_1)\ln 2|x-x_1|\,dx_1$, and

$$\epsilon_n = 1 \quad n = 0$$
$$= 2 \quad n > 0$$

The partial derivatives of ϕ with respect to r, θ, x and t evaluated at the body $r = R$, are given respectively by

$$\frac{\partial \varphi}{\partial r} = u\frac{dR}{dx} - W\cos\left(\theta - \pi/2\right) \tag{66}$$

$$\frac{1}{R}\frac{\partial\varphi}{\partial\theta} = \frac{2ac}{kR^2}\,e^{-kh+ik\bar{X}}\sum_{1}^{\infty}\frac{n\sin n\left(\theta-\frac{\pi}{2}\right)}{K_n'(kR)} - w\sin\left(\theta-\frac{\pi}{2}\right) \tag{67}$$

$$\frac{\partial\varphi}{\partial x} = -\frac{aci\,e^{-k(h-\zeta)+ik\bar{X}}}{R}\sum_{0}^{\infty}\frac{\epsilon_n\cos n\left(\theta-\frac{\pi}{2}\right)}{K_n'(kR)} + u\frac{d}{dx}(RR')\ln R \tag{68}$$

$$+\frac{1}{R}\frac{d}{dx}\left(R^2 w\right)\cos\left(\theta-\frac{\pi}{2}\right) + q'(x)$$

$$\frac{\partial\varphi}{\partial t} = -\frac{ac}{kR}\,e^{-k(h-\zeta)+ik\bar{X}}\sum_{0}^{\infty}\frac{\epsilon_n\cos n\left(\theta-\frac{\pi}{2}\right)}{K_m'(kR)} \tag{69}$$

where the Wronskian identity,

$$I_n(kR)K_n'(kR) - K_n(kR)I_n'(kR) = -\frac{1}{kR} \tag{70}$$

has been employed (see Reference [9]). In addition, higher order terms in body radius have been dropped in the expression for $\frac{\partial\varphi}{\partial x}$, and terms which contribute to the added mass force have been ignored in the expression for $\frac{\partial\varphi}{\partial t}$.

The pressure on the body is obtained from the (three-dimensional) Bernoulli equation in moving coordinates. Expressed in terms of cylindrical coordinates this is

$$\frac{P}{\rho} = \frac{\partial\varphi}{\partial t} - (w-qx)\left(\frac{\partial\varphi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\cos\theta\right)$$

$$-(u+rq\sin\theta)\frac{\partial\varphi}{\partial x} - \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)^2 + \left(\frac{\partial\varphi}{\partial r}\right)^2 + \left(\frac{\partial\varphi}{\partial x}\right)^2\right] \tag{71}$$

But $\frac{\partial\varphi}{\partial r}$ is of the order of the body motions according to Equation (66), and we will consistently ignore squares of the body motions. Hence the pressure simplifies to

$$\frac{P}{\rho} = \frac{\partial\varphi}{\partial t} - (w-qx)\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\cos\theta - (u+rq\sin\theta)\frac{\partial\varphi}{\partial x}$$

$$-\frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)^2 + \left(\frac{\partial\varphi}{\partial x}\right)^2\right] \tag{72}$$

The vertical force per unit length is given by Equation (13). After taking the real part, Equations (67-69) must be substituted into Equation (72) and thence into Equation (13). In so doing, all quadratic terms in the body motions are to be ignored, and the orthogonal properties of the trigonometric terms in θ may be used. If, at the same time, we ignore higher order terms in body radius, we are finally led to the following expression for the heaving force per unit length:

$$\frac{dZ}{dx} = \frac{\pi \rho a^2 c^2 e^{-2k(h-\zeta)}}{R} \sum_{0}^{\infty} \frac{1}{K_n'(kR) K_{n+1}'(kR)} \left[\frac{n(n+1)}{(kR)^2} + 1 \right]$$
$$+ \frac{2\pi \rho a c (w - q x) e^{-k(h-\zeta)} \cos k\overline{X}}{K_1'(kR)} + \frac{2\pi \rho a c^2 e^{-k(h-\zeta)} \sin k\overline{X}}{K_1'(kR)} \qquad (73)$$

In arriving at Equation (73) it has been assumed that the surge velocity u is of the same order of magnitude as the heaving velocity w, in which case the specific dependence of the vertical force on u is negligible for a slender body.

Equation (73) can be simplified even further by taking into account the order of magnitude of the body motions (which are wave induced), and also by focusing attention on the total force instead of the force per unit length.

Consider the body motions, and let each motion be represented by a slowly varying part plus a rapidly oscillating part, in the same manner as for the beam sea case. For head seas, the oscillating part of the motion will certainly be smaller than the motion of a water particle and so the oscillatory displacements given by Equation (32) represent upper bounds. An upper bound on the oscillatory velocities will then be

$$\left.\begin{matrix} w \\ q'l \end{matrix}\right\} \leq -ack e^{-k(h-\zeta_0)} \cos k (ct - \eta_0) \qquad (74)$$

The drifting velocities w_0 , q_0 which are wave-induced are clearly much smaller than the wave speed c , and hence the drifting terms which appear in the $\cos k\bar{X}$ term of Equation (73) are negligible compared with the entire $\sin k\bar{X}$ term. Upon estimating the oscillatory velocities with the aid of Equation (74), we obtain an upper bound for the $\cos k\bar{X}$ term of Equation (73)

$$\frac{2\pi\rho a^2 c^2 k e^{-2k(h-\zeta_0)} \cos k\bar{X} \cos k(ct-\eta_0)}{K_1'(kR)} \qquad (75)$$

which is seen to be of the same order of magnitude as the first term of Equation (73), i.e. $O(\rho a^2 c^2 k)$. On the basis of this argument, we must retain the $\cos k\bar{X}$ term. However, the total force is the integral over the length of the body. Integrating the first term of Equation (73) over the length of the body, it is seen that this term is $O(\rho a^2 c^2 k \ell)$, where the ordering hypothesis $kR = O(1)$ has been utilized. On the other hand, integrating Equation (75) over the length of the body, there is obtained

$$2\pi\rho a^2 c^2 k e^{-2k(h-\zeta_0)} \cos k(ct-\eta_0) \int \frac{\cos k(x+\xi-ct)}{K_1'(kR)} dx \qquad (76)$$

According to the Riemann-Lebesque lemma (Reference [7]), as $k \to \infty$ the integral is $O(1/k)$. Hence, the $\cos k\bar{X}$ term is, in effect, $O(\rho a^2 c^2)$. The ratio of this term to the first term of Equation (73) is $O(1/k\ell)$ which may be neglected. For a closed body $1/K_1'(kR)$ vanishes at the end points of the integral,and, in this case, if the body radius has a continuous first derivative, it can be shown, by an integration by parts, that the ratio of the $\cos k\overline{X}$ term to the first term of Equation (73) actually vanishes as $(1/k\ell)^2$. A similar argument can be used to show that the oscillatory parts of ρ and ξ may be neglected in the exponentials of the remaining terms. Hence, the total heaving force becomes

$$Z = \rho a c^2 k^2 \left\{ a k e^{-2k(h-\zeta_0)} \int S(x)\alpha_1(kR)\,dx + 2e^{-k(h-\zeta_0)} \int S(x)\alpha_2(kR)\sin k\left(x+\xi_0-ct\right)dx \right. \tag{77}$$

where

$$\alpha_1(kR) = \frac{1}{kR^3}\sum_1^{\infty} \frac{1}{K_n'(kR)K_{n+1}'(kR)}\left[\frac{n(n+1)}{(kR)^2}+1\right] \tag{78}$$

$$\alpha_2(kR) = \frac{1}{(kR)^2 K_1'(kR)} \tag{79}$$

and the integrations extend over the length of the body. The universal functions α_1 and α_2 are tabulated in Table I.

The result for head seas may be contrasted with that for beam seas. In the beam sea case the oscillatory body motions contribute terms of the same order of magnitude as the suction force on a stationary body, and, furthermore, the body motions tend to reduce the suction force by reducing the relative motion of body to water. In the head sea case, the body motions are not necessarily in phase with the water motion, and hence do not necessarily reduce the suction force. But, in any event, the effect of the body motions is of higher order, and may be neglected. Because of the relief afforded the suction force in the beam sea case, we may expect the suction force to be smaller in beam seas than in head seas.

In the long wave limit the ordering hypothesis is $KR \ll 1$, $k\ell = O(1)$. The analysis has been carried out by Cuthbert and Kerr [8] for long waves, and they demonstrate that the effect of body motions is present in the force per unit length for head seas as it is in the present analysis. However, if in the long wave analysis we take the limit $k \to \infty$ and assume that body motions are bounded in accordance with Equation (74), the effect of body motions on the integrated force in head seas is negligible. If in the present analysis we let $KR \to 0$ then $\alpha_1 \to 1.5$, $\alpha_2 \to -1$ and Equation (77) reduces to the long wave solution for $k \to \infty$.

The pitching moment can easily be obtained by introducing an x into the integrand in Equation (77). The lateral force and yawing moment are, of course, zero for head seas.

In a similar fashion, and neglecting terms of the order of the square of the wave amplitude, we can derive the following expression for the surge force

$$\underline{X} = -\rho a c^2 k^2 e^{-k(h-\zeta_0)} \int S(x)\alpha_3(kR)\cos k(x+\xi_0-ct)\,dx \tag{80}$$

where

$$\alpha_3(kR) = \frac{2}{(kR)^2}\int_0^{kR}\frac{d\tau}{K_1(\tau)} \tag{81}$$

The universal function $\alpha_3(kR)$ is tabulated in Table I. In the long wave limit $\alpha_3 \longrightarrow 1$, and Equation (80) reduces to the correct long wave solution (see Reference [1]).

III.A. RANDOM SEAS

Arguments similar to that used for beam seas lead to the following expressions for the steady suction force in head seas when the waves are described by a spectrum $A^2(\omega)$:

$$Z_o = \rho \int_0^\infty A^2(\omega)\, d\omega\, c^2 k^3 e^{-2k(h-\zeta_o)} \int S(x)\alpha_1(kR)\,dx \tag{82}$$

The fluctuating force is

$$Z' = 2\rho \int_0^\infty \sqrt{A^2(\omega)\,d\omega}\, c^2 k^2 e^{-k(h-\zeta_o)} \int S(x)\alpha_2(kR)\sin k(x-\xi_o-ct)\,dx \tag{83}$$

The fluctuating surge force is

$$X' = -\rho \int_0^\infty \sqrt{A^2(\omega)\,d\omega}\; c^2 k^2 e^{-k(h-\zeta_o)} \int S(x)\alpha_3(kR)\cos k(x-\xi_o-ct)\,dx \tag{84}$$

Using a Neumann spectrum (Equation (39)), the steady suction force becomes

$$Z_o = \frac{\rho\pi C}{2g} \int S(x)\,dx \int_0^\infty \frac{1}{\omega^2} \exp\left[-\frac{2g^2}{V_w^2\omega^2} - \frac{2\omega^2}{g}(h-\zeta_o)\right]\alpha_1(kR)\,d\omega \tag{85}$$

For long waves $\alpha_1 \to 1.5$ and the ω integral can be evaluated exactly

$$Z_o \approx \frac{3\rho}{4}\left(\frac{\pi}{2}\right)^{3/2} \frac{C V_w}{g^2}\, e^{-4\sqrt{g(h-\zeta_o)}/V_w} \quad (\text{Volume}) \tag{86}$$

For the deeply submerged case, the method of steepest descents yields

$$Z_0 \approx \frac{1}{2}\left(\frac{\pi}{2}\right)^{3/2} \frac{\rho C v_w}{g^2} e^{-4\sqrt{g(h-\zeta_0)}/v_w} \int S(x)\, \alpha_1\left(\frac{R}{v_w}\sqrt{\frac{g}{h-\zeta_0}}\right) dx \tag{87}$$

TABLE I.

Tables of the functions

$$\alpha_0(kR) = \frac{I_1(2kR)}{kR} - 1$$

$$\alpha_1(kR) = \frac{1}{kR^3}\sum_{m=0}^{\infty} \frac{1 + \frac{m(m+1)}{kR^2}}{K_n'(kR).K_{n+1}'(kR)}$$

$$\alpha_2(kR) = \frac{1}{kR^2\, K_1'(kR)}$$

$$\alpha_3(kR) = \frac{2}{kR^2}\int_0^{kR} \frac{dx}{K_1(x)}$$

kR	$\alpha_0(kR)$	$\alpha_1(kR)$	$\alpha_2(kR)$	$\alpha_3(kR)$
0.0	0.00000	1.5000	-1.00000	1.000
0.1	0.00501	1.5035	-0.99044	1.009
0.2	0.02013	1.5211	-0.97532	1.026
0.3	0.04568	1.5569	-0.96124	1.050
0.4	0.08216	1.6127	-0.95051	1.080
0.5	0.13032	1.6899	-0.94400	1.115
0.6	0.19113	1.7900	-0.94197	1.154
0.7	0.26585	1.9151	-0.94442	1.198
0.8	0.35601	2.0676	-0.95125	1.247
0.9	0.46352	2.2506	-0.96234	1.300
1.0	0.59064	2.4680	-0.97758	1.358
1.1	0.74009	2.7242	-0.99689	1.422
1.2	0.91510	3.0254	-1.02024	1.490

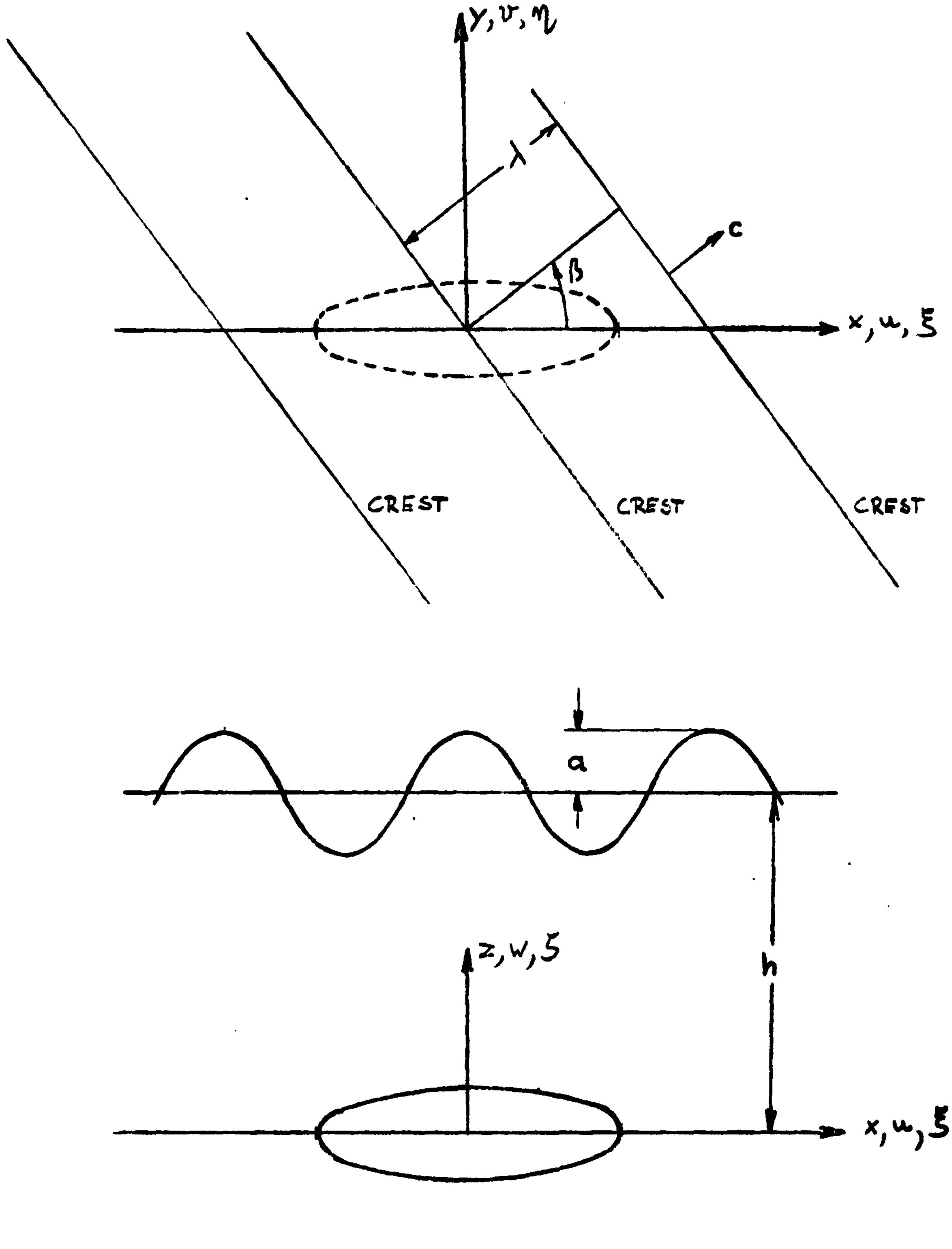

FIGURE 1. BODY IN REGULAR OBLIQUE WAVES

REFERENCES

1. Kaplan, P. and Hu, P. N.: "Virtual Mass and Slender Body Theory for Bodies in Waves," Proc., Sixth Annual Conference on Fluid Mechanics, September 1959, pp. 183-197.

2. Milne-Thomson, L. M.: "Theoretical Hydrodynamics," The MacMillan Co., New York, 2nd Edition, 1950, p. 149.

3. Ogilvie, T. F.: "First and Second Order Forces on a Submerged Cylinder Under a Free Surface," David Taylor Model Basin (No number; no date).

4. Pierson, W. J.: "Wind Generated Gravity Waves," Advances in Geophysics, Vol. II, edited by H. E. Landsberg, Academic Press, Inc., New York, 1955, pp. 93-178.

5. Erdelyi, A.: "Asymptotic Expansions," Dover Publications, New York, 1956.

6. Adams, M. C. and Sears, W. R.: "Slender Body Theory - Review and Extension," Journal Aero. Sci., Vol. 20, #2, February 1953, pp. 85-95.

7. Whittaker, E. T. and Watson, G. N.: "A Course of Modern Analysis," 4th Edition, Cambridge University Press, 1950, p. 170.

8. Cuthbert, J. W. and Kerr, K. P.: "The Effect of Waves on the Ideal Fluid Forces Acting on a Slender Axisymmetric Body Having Three Degrees of Freedom," Lockheed Missiles & Space Div'n, TM 81-73-15 LMSC/801398, March 1962.

9. Bateman, H.: "Higher Transcendental Functions," Vol. II, edited by A. Erdelyi, McGraw-Hill Book Co., New York, 1953, p. 82.

DISTRIBUTION LIST FOR REPORTS -NOBS 86514

Copies	Addressee
8	Chief, Bureau of Ships Department of the Navy Washington 25, D.C. Attn: Code 335 (2) Code 420 (1) Code 432 (1) Code 440 (2) Code 525 (1) Code 632 (1)
5	Commanding Officer and Director David Taylor Model Basin Washington 7, D.C.
2	Chief of Naval Research Department of the Navy Washington 25, D.C. Attn: Code 438
3	Director, Special Projects Department of the Navy Washington 25, D.C. Attn: SP22 (2) SP26 (1)
1	Director, National Bureau Standards Washington 25, D.C. Attn: Dr. Schubauer
2	Commander, Armed Services Technical Information Agency Arlington Hall Station Arlington 12, Virginia ATTN: TIPDR
2	Electric Boat Division General Dynamics Corporation Groton, Connecticut Attn: Mr. R. McCandliss
2	Hydronautics, Inc. 200 Monroe Street Rockville, Maryland

Copies	Addressees
1	Westinghouse Electric Corp. Hendy Avenue Sunnyvale, California Attn: Mr. M. Macovsky
1	Westinghouse Electric Corp. Air Arm Division, Attn: Mr. G. Foessler P.O. Box 746 Baltimore 3, Maryland
2	Vitro Laboratories 14000 Georgia Avenue Silver Spring, Maryland Attn: Mr. Mortimer
1	Supervisor of Shipbuilding U.S. Navy Croton, Connecticut

APPENDIX

In Equation (15) a number of definite integrals must be evaluated. We will show how these integrals can be obtained.

Consider the integral

$$\oint \sin\theta \cdot e^{2kR\sin\theta}\, d\theta \tag{A1}$$

If the substitution $\theta = \alpha + \pi/2$ is made, the integral can be reduced to

$$2\int_0^{\pi} \cos\alpha\, e^{2kR\cos\alpha}\, d\alpha$$

which, according to formula (4) on p. 181 of Reference [6], is

$$2\pi I_1(2kR) \tag{A2}$$

Consider the integral

$$\oint e^{kR\sin\theta} \cos\left[\theta - kR\cos\theta - k\eta + kct\right] \sin\theta \cos\theta\, d\theta \tag{A3}$$

If the substitution $\theta = \alpha + \pi/2$ is made, the integral can be reduced to

$$\cos k(ct-\eta)\int_0^{\pi} e^{kR\cos\alpha} \sin\left[\alpha + kR\sin\alpha\right] \sin 2\alpha\, d\alpha$$

By using the product into sum formula for trigonometric functions this can be expanded into

$$\frac{1}{2}\cos k(ct-\eta)\left\{\int_0^{\pi} e^{kR\cos\alpha} \cos\left[kR\sin\alpha - \alpha\right] d\alpha - \int_0^{\pi} e^{kR\cos\alpha}\left[\cos kR\sin\alpha + 3\alpha\right] d\alpha\right\}$$

In order to evaluate these two integrals, consider the following identity (see Reference [8]).

$$\pi\left(\frac{x+y}{x-y}\right)^{\nu/2} J_\nu\left[(x^2-y^2)^{1/2}\right] = \int_0^\pi e^{y\cos\alpha}\cos(x\sin\alpha-\nu\alpha)\,d\alpha$$

Let $y \to x$, and we obtain

$$\int_0^\pi e^{x\cos t}\cos(x\sin t-\nu t)\,dt = \pi x^\nu \qquad \nu \geq 0$$
$$= 0 \qquad \nu < 0 \tag{A4}$$

By applying Equation (A4), it is seen, finally, that the integral presented in Equation (A3) reduces to

$$\frac{\pi}{2} kR\cos k(ct-\eta) \tag{A5}$$

Similarly, the integral

$$\int_0^{2\pi} e^{kR\sin\theta}\sin^2\theta\cos\left[\theta-kR\cos\theta-k\eta+kct\right]d\theta \tag{A6}$$

can be shown to be

$$-\frac{\pi}{2}kR\sin k(ct-\eta) \tag{A7}$$

Also the terms

$$(v-c)\int_0^{2\pi} e^{kR\sin\theta}\sin k(R\cos\theta+\eta-ct)\sin\theta\,d\theta$$
$$-w\int_0^{2\pi} e^{kR\sin\theta}\cos k(R\cos\theta+\eta-ct)\sin\theta\,d\theta \tag{A8}$$

can, similarly, be shown to be

$$\pi kR\left[(v-c)\sin k(ct-\eta)+w\cos k(ct-\eta)\right] \tag{A9}$$