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Columbia University
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DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS

INERTIA TERMS ASSOCIATED WITH SMALL MOTIONS OF A SUBMERGED FLEXIBLE RECTANGULAR CYLINDER

by

Jacob Lubliner and Hans H. Bleich

Office of Naval Research
Project NR 064-428
Contract Nonr 266(86)
Technical Report No. 30
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ABSTRACT

An expression for the kinetic energy of an infinite liquid mass due to small motions of a submerged cylinder is obtained analytically. When the motion is described in terms of generalized coordinates, the corresponding inertia coefficients appear in the kinetic-energy expression. The analysis is carried out for the problem of a floating flexible rectangular box. Calculations are performed for generalized coordinates representing rigid-body motion and plastic deformation.

I. Introduction

The purpose of this paper is to provide the background for the treatment of transient problems of elastic or plastic box structures resembling a surface ship and floating on the surface of a semi-infinite fluid. Whenever the transient problem is such that the fluid may be considered incompressible, the problem can be formulated in terms of generalized coordinates; the presence of the fluid then produces only terms which can be derived from the kinetic energy of the fluid, expressed in the generalized velocities. The results of this paper are intended to be applied to the determination of the elastic or plastic response of a ship-like structure to initial velocities imparted to it by an underwater explosion. Appropriate velocity distributions may be obtained by experimental or analytical means; an analytical approach has been presented in Ref. 1.

The present problem becomes amenable to analysis if it is treated as a two-dimensional one, that is, if the box is regarded as an infinitely long, flexible rectangular cylinder whose boundary undergoes small motions perpendicular to the generatrices; the latter remain straight and parallel to the axis. The fluid is assumed at rest at large distances from the cylinder, and the motions of the cylindrical boundary are small deviations from its rest configuration.

The problem may thus be viewed, approximately, as a boundary-value problem in plane ideal-fluid flow with prescribed normal velocities along a stationary boundary. It can therefore be attacked by the classical methods of plane potential theory, and, in particular, by the conformal mapping of the given boundary curve into a circle, and subsequent contour integration. The analysis is carried out for the general case in Section II, resulting in the general expression for the inertia coefficients. The expression takes the form of an infinite series whose terms depend on the Fourier coefficients of the transformed normal-velocity distribution. These Fourier coefficients may be obtained by quadrature (numerical, in general), provided the mapping function is known. The mapping function appropriate to the present case of a floating rectangular box is developed analytically in Section III, and is tabulated at the end of the report in a form suitable for integration, for the two beam-draft ratios of 2 and 3.

Calculations were actually carried out for the special case of motion in which the degrees of freedom considered were (i) the three rigid-body motions (horizontal and vertical translations, and rotation), and (ii) three deformations of the box representing possible plastic damage. These degrees of freedom are illustrated in Figure 1. The results are tabulated in the form of inertia matrices for the two beam-draft ratios considered.

An idea of the accuracy of the calculations (which, besides numerical integrations, involve truncations of infinite series) may be had by virtue of the fact that for the two translatory degrees of freedom the inertia coefficients may be calculated exactly. This is shown in the Appendix.

The results of this paper are also applicable to vibration problems, provided the frequencies are low enough to permit the assumption of incompressibility.

II. Virtual Inertia Terms for a Closed Cylinder

The kinetic energy of an incompressible fluid in irrotational two-dimensional flow in the $x\,y$ plane is, per unit distance in the z direction,

$$T = \frac{1}{2}\,\rho \oint \phi \; d\psi \;, \tag{1}$$

Where ρ is the (constant) fluid density, ϕ the velocity potential, ψ the stream function, and the integration is carried out in the positive direction around the boundary. (See, for example, Ref. 2, p. 66.) If the fluid is bounded internally by a closed cylinder, then, in accordance with the usual convention, the direction of integration is clockwise around the projection C of the cylinder. On introducing the complex potential function,

$$w = \phi + i\psi \;,$$

one may write Equation (1) as

$$T = \frac{1}{4i} \rho \oint w^* dw \ , \tag{2}$$

(the asterisk denotes the complex conjugate), since

$$w^*dw = (\phi - i\psi)(d\phi + id\psi) = 2i\phi d\psi + \frac{1}{2} d(\phi^2 + \psi^2 - 2i\phi\psi),$$

and the integral of a perfect differential around a closed curve is zero.

The $\underline{x}$ and $\underline{y}$ components of fluid velocity, $\underline{u}$ and $\underline{v}$, are given by the relation

$$u - iv = dw/dz \ , \tag{3}$$

where

$$z = x + iy \ .$$

The normal velocity (considered positive into the fluid) at a point $z = z_c$ on the curve C is therefore

$$\begin{aligned} u_\nu &= u \frac{dy}{ds} - v \frac{dx}{ds} \\ &= \mathrm{Im}(dw/ds)_{z=z_c} \end{aligned} \tag{4}$$

where $\underline{ds}$ $(= |dz|)$ is an element of arc length on C, measured positive counterclockwise.

If the $\underline{z}$ plane is now mapped conformally into a ζ plane such that the outside of C is mapped into the outside of the unit circle,

$$\zeta = e^{i\theta} ,$$

then for a fluid which is at rest at infinity the complex potential function may be expressed, in general, in the form

$$w = \sum_{n=1}^{\infty} \frac{a_n + ib_n}{n} \zeta^{-n} , \tag{5}$$

where the coefficients a_n and b_n are real. The normal velocity on C is now given by

$$u_\nu = \frac{d\theta}{ds} \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) . \tag{6}$$

Consequently, by Fourier's theorem,

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{\pi} \int_0^{2\pi} u_\nu \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \frac{ds}{d\theta} \, d\theta . \tag{7}$$

The arc length on C, measured counterclockwise from the point corresponding to $\theta = 0$, may be treated as a function of θ, defined by

$$s(\theta) = \int_0^{\theta} \frac{ds}{d\theta} \, d\theta , \tag{8}$$

where

$$\frac{ds}{d\theta} = \left| \frac{dz}{d\zeta} \right|_{\zeta = \exp(i\theta)} . \tag{9}$$

In particular,

$$s(2\pi) \equiv p ,$$

the perimeter of C. We may further define a dimensionless variable,

$$\sigma \equiv s/p \ ,$$

and consider θ a function of σ. Equation (7) may then be written alternatively as

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{p}{\pi} \int_o^1 u_\nu(\sigma) \begin{Bmatrix} \cos n\theta(\sigma) \\ \sin n\theta(\sigma) \end{Bmatrix} d\sigma \ . \qquad (10)$$

When the expression (5) for the complex potential function is used in Equation (2) and the integration is carried out along $\zeta = \exp(i\theta)$; the resulting expression for the kinetic energy is

$$T = \frac{1}{2} \pi\rho \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{n} \ . \qquad (11)$$

A small normal displacement of C may in general be regarded as the superposition of normal displacements corresponding to different modes of motion (or degrees of freedom), each of which may be expressed, at any time $\underline{t}$, as

$$f_i(\sigma) q_i(t) \ ,$$

where $f_i(\sigma)$ describes the shape, while $q_i(t)$ gives the amplitude. (It should be noted that if q_i is a length, f_i is dimensionless, while if q_i is an angle, f_i has the dimension of length.) The general normal velocity is consequently

$$u_v = \sum_i f_i(\sigma)\dot{q}_i \quad . \tag{12}$$

The Fourier coefficients a_n, b_n, are then

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{p}{\pi} \sum_i \begin{Bmatrix} c_{in} \\ s_{in} \end{Bmatrix} \dot{q}_i \ , \tag{13}$$

where

$$\begin{Bmatrix} c_{in} \\ s_{in} \end{Bmatrix} = \oint f_i(\sigma) \begin{Bmatrix} \cos n\theta(\sigma) \\ \sin n\theta(\sigma) \end{Bmatrix} d\sigma \ . \tag{14}$$

The amplitudes q_i may now be treated as generalized coordinates of the motion, so that the kinetic energy becomes a quadratic form in the generalized velocities:

$$T = \frac{1}{2} \sum_i \sum_j m_{ij} \ \dot{q}_i \dot{q}_j \ , \tag{15}$$

where

$$m_{ij} = \frac{\rho p^2}{\pi} \sum_{n=1}^{\infty} \frac{c_{in}c_{jn} + s_{in}s_{jn}}{n} \quad . \tag{16}$$

These are the inertia coefficients of the system (cf. Ref. 2, p. 188).

III. <u>Application to a Floating Rectangle</u>

If the curve C is symmetric about the $\underline{x}$ axis, then in the conformal mapping this axis may be mapped into the real axis of the ζ plane. In such a case, a normal-velocity distribution which is an odd function of θ, that is, one for which

$$a_n = 0$$

for all n, corresponds to the condition

$$\phi = 0 \quad \text{on} \quad y = 0 \; .$$

This, however, is just the condition for the surface $y = 0$ to be a free surface. If, consequently, we are dealing with a cylinder floating on the surface of the liquid occupying the half-space $y \leq 0$, the projection of the immersed portion of the cylinder being C_1, then the problem is equivalent to the previously treated one of the submerged closed cylinder, provided C is the union of C_1 and of its reflection, and the normal-velocity distribution is antisymmetric about the x axis.

The only possible modes of motion are, therefore, those for which the coefficients a_n vanish. Furthermore, since the kinetic energy of the half-space is half of that of the equivalent full space, the inertia coefficients for the floating case are half of those given by Equation (16).

We are interested here in the motion of a rectangular box of beam $2a$ and draft b. The equivalent problem is that of the closed rectangle given by

$$- a \leq x \leq a \; ,$$
$$- b \leq y \leq b \; .$$

The six degrees of freedom which are described in the Introduction and illustrated in Figure 1 are conveniently grouped into, first, the three which are antisymmetric, and secondly, the three which are symmetric about the y axis. Consequently,

$$\left.\begin{aligned} s_{i,2m} &= 4\int_0^{\frac{1}{4}} f_i(\sigma)\sin 2m\theta(\sigma)d\sigma, \quad m = 1,2\ldots \\ s_{i,2m+1} &= 0\ , \quad m = 0,1,2\ldots \end{aligned}\right\} \quad i=1,2,3$$

$$\left.\begin{aligned} s_{i,2m} &= 0\ , \quad m = 1,2\ldots \\ s_{i,2m+1} &= 4\int_0^{\frac{1}{4}} f_i(\sigma)\sin(2m+1)\theta(\sigma)d\sigma, \ m = 0,1\ldots \end{aligned}\right\} \quad i=4,5,6 \quad .$$

It remains, then, only to obtain the mapping function $\theta(\sigma)$. The appropriate mapping is given by the following special case of the Schwarz-Christoffel transformation:

$$\frac{dz}{d\zeta} = R(1 - 2\zeta^{-2}\cos 2\alpha + \zeta^{-4})^{\frac{1}{2}} \tag{17}$$

where R is real, and the points $\zeta = \pm\exp(\pm i\alpha)$ correspond to the corners of the rectangle. In accordance with Equation (9) we have

$$\frac{ds}{d\theta} = R\,|\,2\cos 2\alpha - 2\cos 2\theta\,|^{\frac{1}{2}} \quad . \tag{18}$$

Equation (18) can be integrated with the aid of elliptic integrals. Using the notation of Ref. 3, we let

$$k = \sin\alpha \ , \ k' = \cos\alpha \ .$$

The coordinates of the rectangle are related to θ by the equations

$$\begin{aligned} x &= \pm a, \\ y &= 2R[E(k,\phi) - k'^2 F(k,\phi)], \\ x &= 2R[E(k',\phi') - k^2 F(k',\phi')], \\ y &= \pm b, \end{aligned} \tag{19}$$

where

$$\phi = \sin^{-1}(\sin\theta/k), \ \phi' = \sin^{-1}(\cos\theta/k') \ .$$

In particular, the dimensions of the rectangle are related to R and α by

$$\begin{aligned} a &= 2Rk'^2 B', \\ b &= 2Rk^2 B \ , \end{aligned}$$

so that the beam-draft ratio $2a/b$ is a function of α only, as is the scale factor γ, defined by

$$\gamma = \frac{2R}{a+b} \ .$$

Consequently, both α and γ are functions of $2a/b$. For the two ratios of 2 and 3, the values of α and γ, obtained by interpolation from the tables of Ref. 4, are given in Table 1.

Since for the closed rectangle

$$p = 4(a + b) ,$$

the relation between σ and θ is given by

$$\frac{d\sigma}{d\theta} = \frac{\gamma}{8} \mid 2 \cos 2\alpha - 2 \cos 2\theta \mid^{\frac{1}{2}} . \tag{20}$$

Though Equation (20) is, of course, integrable in terms of elliptic integrals, there remains the task of inverse interpoloation in order to obtain θ as a function of σ. If this task is to be performed by a digital computer, then it is far simpler to include the integration of Equation (20) in the same program, as well as the calculation of $\cos n\theta$ and $\sin n\theta$. The results of these computations are shown in Tables 2 to 5. The inertia coefficients for the six aforementioned degrees of freedom were calculated by numerical integration and summation up to $n = 10$, and are tabulated in matrix form in Table 6.

REFERENCES

1. R. P. Shaw and M. B. Friedman, "Diffraction of Pulses by Arbitrary Two-Dimensional Free Surfaces" (Columbia University, Office of Naval Research Project NR 064-428, Contract Nonr-266(08), Technical Report No. 29, 1961).

2. H. Lamb, <u>Hydrodynamics</u> (6th ed., Dover Publications, New York, 1945).

3. E. Jahnke and F. Emde, <u>Tables of Functions with Formulas and Curves</u> (4th ed., Dover Publications, New York, 1945), p. 52 ff.

4. P. F. Byrd and M. D. Friedman, <u>Handbook of Elliptic Integrals for Engineers and Physicists</u> (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954), pp. 322-323.

APPENDIX

For the two translatory degrees of freedom, that is $i=1$ and $i=4$ (see Figure 1), the integrals for s_{in} are expressible in closed form.

Consider, first, $i=1$. We have

$$s_{1,2m} = -\frac{\gamma}{2}\int_0^{\alpha} \sin 2m\theta(2\cos 2\theta - 2\cos 2\alpha)^{\frac{1}{2}}\, d\theta \qquad \text{(A1)}$$

but the sine of an even multiple of θ can be written as

$$\sin 2m\theta = 2\sin\theta\cos\theta \sum_{k=0}^{m-1} \frac{(-4)^k (m+k)!}{(2k+1)!(m-k-1)!} \sin^{2k}\theta . \qquad \text{(A2)}$$

The change of variable

$$\sin\theta = \sin\alpha \sin\psi$$

yields

$$s_{1,2m} = -2\gamma \sin^3\alpha \sum_{k=0}^{m-1} \frac{(-4)^k (m+k)!\sin^{2k}\alpha}{(2k+1)!(m-k-1)!} \int_0^{\pi/2} \cos^2\psi \sin^{2k+1}\psi\, d\psi$$

$$= -4\gamma \sin^3\alpha \sum_{k=0}^{m-1} \frac{(-16)^k (m+k)!k!(k+1)!\sin^{2k}\alpha}{(m-k-1)!(2k+1)!(2k+3)!} . \qquad \text{(A3)}$$

Similarly, for $i=4$,

$$s_{4,2m+1} = \frac{\gamma}{2}\int_{\alpha}^{\pi/2} \sin(2m+1)\theta\ (2\cos 2\alpha - 2\cos 2\theta)^{\frac{1}{2}} d\theta, \qquad \text{(A4)}$$

but

$$\sin(2m+1)\theta = (-1)^m \sin\theta \sum_{k=0}^{m} \frac{(-4)^k (m+k)!}{(2k)!(m-k)!} \cos^{2k}\theta . \qquad \text{(A5)}$$

On changing the variable

$$\cos\theta = \cos\alpha \sin\psi ,$$

we have

$$s_{4,2m+1} = (-1)^m \gamma \cos^2\alpha \sum_{k=0}^{m} \frac{(-4)^k (m+k) \cos^{2k}\alpha}{(2k) (m-1)} \int_0^{\pi/2} \cos^2\psi \sin^{2k}\psi \, d\psi$$

$$= (-1)^m \frac{\pi\gamma}{4} \cos^2\alpha \sum_{k=0}^{m} \frac{(-1)^k (m+k)! \cos^{2k}\alpha}{k!(k+1)!(m-1)!} . \qquad \text{(A6)}$$

In the particular case $a/b = 1$, for which

$$\cos^2\alpha = 1/2 ,$$

we have

$$\frac{8}{\pi\gamma} s_{4n} = \begin{cases} 1, & n = 1 \\ 2(-1)^r \dfrac{4^{-r}(2r-2)!}{r!(r-1)!} , & n = 4r-1,\ r = 1,2,\ldots \\ 0 & \text{all other } n . \end{cases} \qquad \text{(A7)}$$

From the closed-form expressions (A3) and (A6), the inertia coefficients m_{11} and m_{44} can be computed to any desired accuracy, for we can find, with the aid of Parseval's theorem, an upper bound to the truncation error due to summing a finite number of terms on the right-hand side of Equation (16). Defining

$$m_{ii}^{(N)} = \frac{\rho p^2}{\pi} \sum_{n=1}^{N} \frac{c_{in}^2 + s_{in}^2}{n} ,$$

we have

$$\Delta m_{ii}^{(N)} = m_{ii} - m_{ii}^{(N)} = \frac{\rho p^2}{\Pi} \sum_{n=N+1}^{\infty} \frac{c_{in}^2 + s_{in}^2}{n} .$$

Hence

$$\Delta m_{ii}^{(N)} \leq \frac{\rho p^2}{\pi(N+1)} \sum_{n=N+1}^{\infty} (c_{in}^2 + s_{in}^2). \tag{A8}$$

but

$$\frac{1}{\pi} \sum_{n=1}^{\infty} (c_{in}^2 + s_{in}^2) = \oint [f_i(\sigma)]^2 \frac{d\sigma}{d\theta}\, d\sigma . \tag{A9}$$

Calling the integral on the right-hand side of Equation (A9) K_i, we have

$$\Delta m_{ii}^{(N)} \leq \frac{\rho p^2}{N+1} [K_i - \frac{1}{\pi} \sum_{n=1}^{N} (c_{in}^2 + s_{in}^2)]. \tag{10}$$

In particular, for the cases under consideration

$$K_1 = 4(\gamma/8)^2 \int_0^{\alpha} (2 \cos 2\theta - 2 \cos 2\alpha)\, d\theta$$

$$= (\gamma/4)^2 (\sin 2\alpha - 2\alpha \cos 2\alpha),$$

and

$$K_4 = 4(\gamma/8)^2 \int_{\alpha}^{\pi/2} (2 \cos 2\alpha - 2 \cos 2\theta)\, d\theta$$

$$= (\gamma/4)^2 [\sin 2\alpha + 2(\tfrac{\pi}{2} - \alpha) \cos 2\alpha].$$

TABLE 1

Mapping Parameters

2a/b	α	γ
2	0.785398 ($=\pi/4$)	1.180341
3	0.693231	1.175372

TABLE 2

Mapping Function $\theta(\sigma)$, and $\cos n\theta$
for $2a/b = 2$

4σ	θ	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$\cos 6\theta$	$\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$\cos 10\theta$
.0000	.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.0500	.0600	.9982	.9928	.9839	.9714	.9554	.9359	.9131	.8871	.8578	.8255
.1000	.1204	.9928	.9711	.9355	.8863	.8242	.7502	.6654	.5709	.4682	.3587
.1500	.1817	.9835	.9347	.8550	.7472	.6148	.4622	.2943	.1167	-.0647	-.2440
.2000	.2445	.9703	.8828	.7428	.5586	.3411	.1034	-.1405	-.3760	-.5891	-.7672
.2500	.3095	.9525	.8144	.5990	.3266	.0232	-.2825	-.5612	-.7867	-.9373	-.9989
.3000	.3777	.9295	.7280	.4239	.0600	-.3124	-.6407	-.8787	-.9228	-.9670	-.8048
.3500	.4506	.9002	.6207	.2172	-.2296	-.6305	-.9056	-.9999	-.8946	-.6107	-.2049
.4000	.5311	.8623	.4870	-.0224	-.5256	-.8841	-.9990	-.8387	-.4474	.0671	.5632
.4500	.6258	.8105	.3138	-.3018	-.8030	-.9999	-.8179	-.3259	.2896	.7954	.9997
.5000	.7854	.7071	.0000	-.7071	-1.0000	-.7071	-.0000	.7071	1.0000	.7071	.0000
.5500	.9450	.5857	-.3138	-.9534	-.8030	.0127	.8719	.9454	.2896	-.6061	-.9997
.6000	1.0397	.5065	-.4870	-.9997	-.5256	.4673	.9990	.5446	-.4474	-.9977	-.5632
.6500	1.1202	.4355	-.6207	-.9761	-.2296	.7762	.9056	.0126	-.8946	-.7919	.2049
.7000	1.1931	.3688	-.7280	-.9057	.0600	.9500	.6407	-.4774	-.9228	-.2548	.8048
.7500	1.2613	.3046	-.8144	-.8008	.3266	.9997	.2825	-.8277	-.7867	.3484	.9989
.8000	1.3263	.2421	-.8828	-.6695	.5586	.9400	-.1034	-.9901	-.3760	.8080	.7672
.8500	1.3891	.1807	-.9347	-.5186	.7472	.7887	-.4622	-.9557	.1167	.9979	.2440
.9000	1.4504	.1201	-.9711	-.3534	.8863	.5663	-.7502	-.7465	.5709	.8836	-.3587
.9500	1.5108	.0600	-.9928	-.1790	.9714	.2954	-.9359	-.4076	.8871	.5140	-.8255
1.0000	1.5708	.0000	-1.0000	.0000	1.0000	.0000	-1.0000	.0000	1.0000	.0000	-1.0000

TABLE 3

Mapping Function $\theta(\sigma)$, and $\sin n\theta$ for $2a/b = 2$

4σ	θ	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$\sin 7\theta$	$\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$
.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.0500	.0600	.0600	.1200	.1790	.2376	.2954	.3522	.4076	.4616	.5140	.5645
.1000	.1204	.1201	.2385	.3534	.4632	.5663	.6612	.7465	.8210	.8836	.9335
.1500	.1817	.1807	.3555	.5186	.6646	.7887	.8868	.9557	.9932	.9979	.9698
.2000	.2445	.2421	.4698	.6695	.8295	.9400	.9946	.9901	.9266	.8080	.6413
.2500	.3095	.3046	.5803	.8008	.9452	.9997	.9523	.8277	.6174	.3484	.0463
.3000	.3777	.3688	.6856	.9057	.9982	.9500	.7678	.4774	.1197	-.2548	-.5935
.3500	.4506	.4355	.7841	.9761	.9733	.7762	.4241	-.0126	-.4468	-.7919	-.9788
.4000	.5311	.5065	.8734	.9997	.8507	.4673	-.0448	-.5446	-.8943	-.9977	-.8263
.4500	.6258	.5857	.9495	.9534	.5960	.0127	-.5754	-.9454	-.9571	-.6061	-.0254
.5000	.7854	.7071	1.0000	.7071	.0000	-.7071	-1.0000	-.7071	.0000	.7071	1.0000
.5500	.9450	.8105	.9495	.3018	-.5960	-.9999	-.5754	.3259	.9571	.7954	-.0254
.6000	1.0397	.8623	.8734	.0224	-.8507	-.8841	-.0448	.8387	.8943	.0671	-.8263
.6500	1.1202	.9002	.7841	-.2172	-.9733	-.6305	.4241	.9999	.4468	-.6107	-.9788
.7000	1.1931	.9295	.6856	-.4239	-.9982	-.3124	.7678	.8787	-.1197	-.9670	-.5935
.7500	1.2613	.9525	.5803	-.5990	-.9452	.0232	.9523	.5612	-.6174	-.9373	.0463
.8000	1.3263	.9703	.4698	-.7428	-.8295	.3411	.9946	.1405	-.9266	-.5891	.6413
.8500	1.3891	.9835	.3555	-.8550	-.6646	.6148	.8868	-.2943	-.9932	-.0647	.9698
.9000	1.4504	.9928	.2385	-.9355	-.4632	.8242	.6612	-.6654	-.8210	.4682	.9335
.9500	1.5108	.9982	.1200	-.9839	-.2376	.9554	.3522	-.9131	-.4616	.8578	.5645
1.0000	1.5708	1.0000	.0000	-1.0000	.0000	1.0000	.0000	-1.0000	.0000	1.0000	.0000

TABLE 4

Mapping Function $\theta(\sigma)$, and $\cos n\theta$ for $2a/b = 3$

4σ	θ	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$\cos 4\theta$	$\cos 5\theta$	$\cos 6\theta$	$\cos 7\theta$	$\cos 8\theta$	$\cos 9\theta$	$\cos 10\theta$
.0000	.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.0500	.0667	.9978	.9911	.9801	.9646	.9449	.9210	.8930	.8610	.8252	.7857
.1000	.1341	.9910	.9642	.9201	.8595	.7834	.6933	.5907	.4775	.3557	.2275
.1500	.2032	.9794	.9186	.8200	.6876	.5270	.3446	.1482	-.0544	-.2548	-.4446
.2000	.2749	.9625	.8527	.6788	.4541	.1952	-.0784	-.3460	-.5877	-.7852	-.9238
.2500	.3509	.9391	.7637	.4952	.1664	-.1827	-.5095	-.7742	-.9446	-.9999	-.9333
.3000	.4341	.9072	.6462	.2653	-.1649	-.5644	-.8593	-.9947	-.9456	-.7211	-.3628
.3500	.5313	.8622	.4867	-.0230	-.5263	-.8845	-.9990	-.8380	-.4461	.0688	.5647
.4000	.6932	.7692	.1833	-.4872	-.9328	-.9478	-.5252	.1398	.7403	.9990	.7966
.4500	.8533	.6575	-.1353	-.8355	-.9634	-.4314	.3961	.9522	.8562	.1736	-.6278
.5000	.9473	.5838	-.3182	-.9555	-.7974	.0243	.8258	.9400	.2718	-.6226	-.9988
.5500	1.0267	.5176	-.4641	-.9981	-.5692	.4088	.9925	.6186	-.3520	-.9830	-.6657
.6000	1.0982	.4522	-.5856	-.9833	-.3143	.7022	.9536	.1660	-.8025	-.8966	-.0138
.6500	1.1646	.3951	-.6877	-.9386	-.0540	.8959	.7620	-.2938	-.9942	-.4919	.6054
.7000	1.2275	.3366	-.7734	-.8573	.1963	.9894	.4698	-.6732	-.9230	.0519	.9578
.7500	1.2879	.2792	-.8441	-.7505	.4251	.9878	.1264	-.9172	-.6386	.5607	.9516
.8000	1.3464	.2226	-.9010	-.6235	.6235	.9010	-.2225	-1.0000	-.2226	.9010	.6236
.8500	1.4036	.1664	-.9446	-.4810	.7845	.7421	-.5375	-.9210	.2309	.9978	.1013
.9000	1.4598	.1108	-.9755	-.3268	.9031	.5269	-.7863	-.7011	.6310	.8409	-.4448
.9500	1.5155	.0553	-.9939	-.1653	.9756	.2732	-.9454	-.3778	.9036	.4778	-.8507
1.0000	1.5708	.0000	-1.0000	.0000	1.0000	.0000	-1.0000	.0000	1.0000	.0000	-1.0000

TABLE 5

Mapping Function $\theta(\sigma)$, and $\sin n\theta$ for $2a/b = 3$

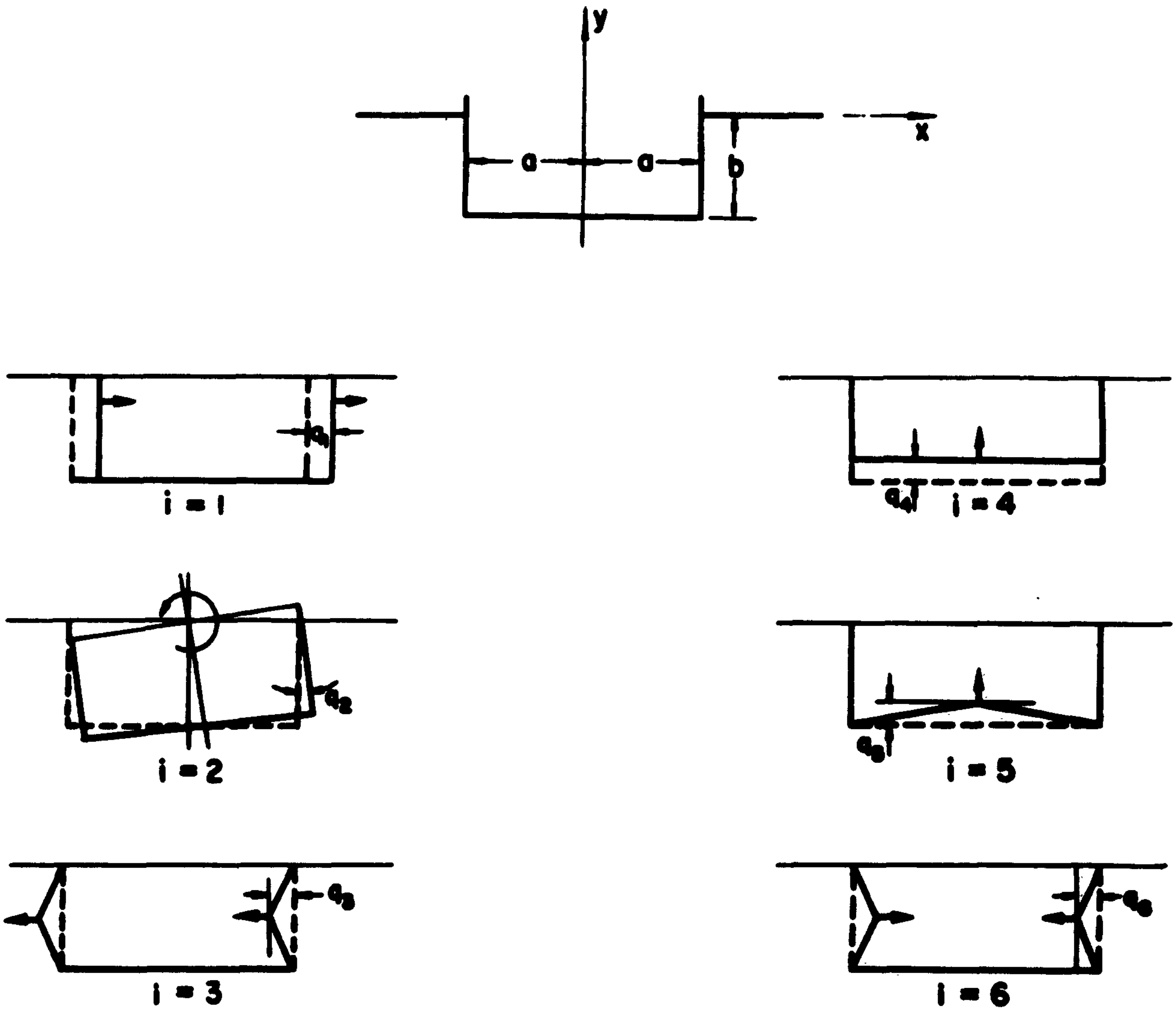
4σ	θ	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$	$\sin 5\theta$	$\sin 6\theta$	$\sin 7\theta$	$\sin 8\theta$	$\sin 9\theta$	$\sin 10\theta$
.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.0500	.0667	.0666	.1330	.1987	.2636	.3273	.3896	.4501	.5806	.5648	.6186
.1000	.1341	.1337	.2650	.3916	.5111	.6215	.7207	.8069	.8786	.9346	.9738
.1500	.2032	.2018	.3953	.5724	.7261	.8499	.9387	.9890	.9985	.9670	.8957
.2000	.2749	.2714	.5225	.7343	.8910	.9808	.9969	.9382	.8091	.6192	.3828
.2500	.3509	.3437	.6456	.8688	.9861	.9832	.8605	.6329	.3282	-.0165	-.3592
.3000	.4341	.4206	.7632	.9642	.9863	.8255	.5115	.1026	-.3253	-.6928	-.9319
.3500	.5313	.5066	.8736	.9997	.8503	.4665	-.0459	-.5457	-.8950	-.9976	-.8253
.4000	.6932	.6390	.9831	.8733	.3604	-.3184	-.8510	-.9902	-.6723	-.0441	.6045
.4500	.8533	.7534	.9908	.5495	-.2682	-.9022	-.9182	-.3053	.5167	.9848	.7784
.5000	.9473	.8119	.9480	.2951	-.6034	-.9997	-.5639	.3412	.9624	.7825	-.0486
.5500	1.0267	.8556	.8858	.0614	-.8222	-.9126	-.1226	.7857	.9360	.1834	-.7462
.6000	1.0982	.8904	.8106	-.1523	-.9493	-.7120	.3011	.9861	.5967	-.4429	-.9999
.6500	1.1646	.9186	.7259	-.3449	-.9985	-.4442	.6475	.9559	.1078	-.8707	-.7959
.7000	1.2275	.9416	.6339	-.5149	-.9805	-.1452	.8828	.7395	-.3850	-.9987	-.2873
.7500	1.2879	.9602	.5361	-.6609	-.9051	.1555	.9920	.3983	-.7696	-.8280	.3073
.8000	1.3464	.9749	.4339	-.7818	-.7818	.4339	.9749	.0417	-.9749	-.4339	.7818
.8500	1.4036	.9860	.3283	-.8768	-.6201	.6703	.8433	-.3896	-.9730	.0657	.9949
.9000	1.4598	.9938	.2202	-.9451	-.4295	.8499	.6178	-.7131	-.7758	.5412	.8956
.9500	1.5155	.9985	.1105	-.9862	-.2196	.9620	.3260	-.9259	-.4284	.8785	.5256
1.0000	1.5708	1.0000	.0000	-1.0000	.0000	1.0000	.0000	-1.0000	.0000	1.0000	.0000

TABLE 6

Inertia Matrices

($m' = \rho ab$; $r = (a + b)/\sqrt{3}$; $m_{ji} = m_{ij}$)

2a/b	2.0	3.0
$\frac{1}{m'}[m_{ij}]$	$\begin{bmatrix} 0.754 & 0.264r & -0.444 & 0 & 0 & 0 \\ & 0.265r^2 & -0.160r & 0 & 0 & 0 \\ & & 0.271 & 0 & 0 & 0 \\ & & & 2.373 & 1.270 & 0.232 \\ & & & & 0.713 & 0.122 \\ & & & & & 0.288 \end{bmatrix}$	$\begin{bmatrix} 0.510 & 0.079r & -0.296 & 0 & 0 & 0 \\ & 0.382r^2 & -0.058r & 0 & 0 & 0 \\ & & 0.177 & 0 & 0 & 0 \\ & & & 3.344 & 1.817 & 0.202 \\ & & & & 1.046 & 0.098 \\ & & & & & 0.164 \end{bmatrix}$
Exact Values:		
$\frac{1}{m'} m_{11}$	0.774	0.525
$\frac{1}{m'} m_{44}$	2.377	3.349



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