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Technical Report No. 287

Incoherent Scattering
from a Plasma

M. S. Macrakis

I. The Electronic Spectrum

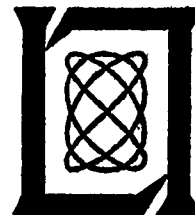
II. The Ion-Carried Electronic Spectrum

14 November 1962

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INCOHERENT SCATTERING FROM A PLASMA

- I. THE ELECTRONIC SPECTRUM
- II. THE ION-CARRIED ELECTRONIC SPECTRUM

M. S. MACRAKIS

Group 25

TECHNICAL REPORT NO. 287

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ABSTRACT

The purpose of this report is twofold: (1) To review some aspects of the theory on density fluctuations in a plasma (Part I) and (2) to introduce a novel method for the determination of the fluctuation spectrum of a multicomponent plasma (Part II). Part I attempts to follow the theory that starts from first principles (dynamics of point-charged particles) and leads to the derivation of coarse-grained properties of a plasma in order to compute the response of the plasma to an externally applied longitudinal field. By applying the Nyquist theorem to the response function of the system, we derive the fluctuation spectrum.

Part II introduces a technique by which the space and time-dispersive response function of a plasma that has an arbitrary number of charged constituents can be evaluated with little effort. The technique consists of reducing the problem to one in network theory. The network corresponding to a plasma is a bank of capacitors in parallel, with each capacitor representing one constituent of the plasma. A plier's entry at any desired location in the network determines the response function to an applied external potential.

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INCOHERENT SCATTERING FROM A PLASMA*

I. THE ELECTRONIC SPECTRUM

A. INTRODUCTION

Recent work on incoherent scattering from a plasma¹⁻³ has centered on the calculation of the ionic effect on both the spectral quality of the scattering cross section and the integrated scattering cross section. Since the cross section is proportional to the mean square of the fluctuation of the density of the system, this work effectively consisted in generalizing earlier results obtained for the fluctuations in an electron plasma by Pines and Bohm.⁴ This generalization demonstrated that the contribution to the cross section from the ion-carried electron fluctuations is far more important than the contribution from the pure electron fluctuations. However, we believe that the basic issue of what it is that we are scattering from can be discussed without carrying along the cumbersome formulas that include the effect of the ions.

In this report we attempt to follow the theory that starts from first principles (dynamics of point charged particles) and leads to the derivation of coarse-grained properties of a plasma (the plasma as a continuum) in order to compute the response of the plasma to an externally applied longitudinal field. We then compute the energy input from the external field to the plasma and show that it is nonzero, even in a plasma without collisions. However, when there is a dissipation process in a system, a fluctuation is associated with it. By applying the Nyquist theorem to the system we proceed to derive these fluctuations. We further indicate that, for large wavelengths, these fluctuations become, in effect, organized oscillations at the plasma frequency ω_p . We then conclude that in a phenomenological way, scattering of an electromagnetic wave from a plasma can be described as scattering from a density wave.

B. THE EQUATION OF MOTION FOR A PLASMA

A complete investigation of density fluctuations in a plasma would require study along the lines described by Tchen⁵ or Rostoker and Rosenbluth.⁶ Although there has never been any doubt as to the soundness of the basic dynamic principles from which we must start in order to describe a plasma and its fluctuations, the statistical processing of the basic equations pertaining to many interacting particles (to derive macroscopic properties of the system) introduces considerable difficulties. Many compromises and mental models have to be used in order to keep a clear and useful link between first principles (dynamics, statistics) and final results (macroscopic variables). By keeping track of the meaning of the compromises and of the models we may hopefully retain the connection between a microscopic model and a macroscopic result. To this end we have adopted the method of Klimontovich⁷ which, though deceptively simple, is very helpful for following the mathematical manipulations.

* This report contains notes on some aspects of the problem of incoherent scattering from a plasma. These notes were prepared in the summer of 1961.

Consider now a system consisting of N electrons. When electromagnetic interactions are neglected, the Hamiltonian of the system is

$$H = \sum_{i=1}^N \left[\frac{1}{2m} \bar{p}_i^2 + U(\bar{q}_i) \right] + \frac{1}{2} \sum_{i \neq j}^N V(\bar{q}_i - \bar{q}_j) \quad , \quad (1)$$

where $\bar{p}_i$ and $\bar{q}_i$ are the momentum and the position of the i^{th} particle, $U(\bar{q}_i)$ is an external potential arising from an external force acting on each individual particle and $V(\bar{q}_i - \bar{q}_j) = e^2/|\bar{q}_i - \bar{q}_j|$ is the electrostatic interaction potential of the particles taken in pairs. Introducing the function $N(\bar{p}\bar{q}t)$,

$$N(\bar{p}\bar{q}t) = \sum_{i=1}^N \delta(\bar{p} - \bar{p}_i) \delta(\bar{q} - \bar{q}_i) \quad , \quad (2)$$

for the number of particles at $\bar{p}\bar{q}$, we can write for any function $g(\bar{p}\bar{q})$,

$$\int g(\bar{p}\bar{q}) N(\bar{p}\bar{q}t) d\bar{p}d\bar{q} = \sum_{i=1}^N g(\bar{p}_i, \bar{q}_i) \quad .$$

Accordingly, the Hamiltonian can be written as

$$H = \int \frac{1}{2m} p^2 N(\bar{p}\bar{q}t) d\bar{p}d\bar{q} + \frac{1}{2} \int \frac{e^2}{|\bar{q} - \bar{q}'|} N(\bar{p}\bar{q}t) N(\bar{p}'\bar{q}'t) d\bar{p}d\bar{p}'d\bar{q}d\bar{q}' + \int U(\bar{q}) N(\bar{p}\bar{q}t) d\bar{p}d\bar{q} \quad . \quad (3)$$

We now note that, in a strictly formal way, the Hamiltonian density in phase space can be written as

$$\Delta H = \frac{1}{2m} p^2 N(\bar{p}\bar{q}t) + \frac{1}{2} \int \frac{e^2}{|\bar{q} - \bar{q}'|} N(\bar{p}\bar{q}t) N(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}' + U(\bar{q}) N(\bar{p}\bar{q}t) \quad ;$$

therefore, its derivative with respect to N is

$$\mathfrak{K} = \frac{\Delta H}{\Delta N} = \frac{1}{2m} p^2 + \int \frac{e^2}{|\bar{q} - \bar{q}'|} N(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}' + U(\bar{q}) \quad .$$

Here $\mathfrak{K}$ represents the Hamiltonian of one particle; therefore, we can use it to write the equations of motion for this particle,

$$\begin{aligned} \dot{\bar{q}} &= \frac{\partial \mathfrak{K}}{\partial \bar{p}} = \frac{\bar{p}}{m} \\ \dot{\bar{p}} &= -\frac{\partial \mathfrak{K}}{\partial \bar{q}} = -\frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} N(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}' - \frac{\partial U(\bar{q})}{\partial \bar{q}} \quad , \end{aligned}$$

where $\bar{p}$ and $\bar{q}$ are the canonical variables. Introducing these variables in the continuity equation

$$\frac{\partial N(\bar{p}\bar{q}t)}{\partial t} + \frac{\dot{q}}{q} \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{q}} + \frac{\dot{p}}{p} \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{p}} = 0 \quad ,$$

we finally obtain

$$\begin{aligned} \frac{\partial N(\bar{p}\bar{q}t)}{\partial t} + \frac{\bar{p}}{m} \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{q}} - \left(\frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} N(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}' \right) \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{p}} \\ - \frac{\partial U(\bar{q})}{\partial \bar{q}} \cdot \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{p}} = 0 \quad . \end{aligned} \quad (4)$$

Equation (4) is an exact equation describing the microscopic evolution of the random function $N(\bar{p}\bar{q}t)$. Gross⁸ has pointed out that the procedure used by Klimontovich poses a number of questions; nevertheless, Gross was able to derive the same equation for $N(\bar{p}\bar{q}t)$ by using a more direct method. In any case, the surprising fact is that we can obtain an exact microscopic equation for the evolution of $N(\bar{p}\bar{q}t)$ which looks so much like the well-known transport equation of Boltzmann-Vlasov. To be of any practical use, Eq. (4) has to be averaged in order to derive a macroscopically meaningful density from the highly discontinuous $N(\bar{p}\bar{q}t)$. In the macroscopic description of the system we are not interested in the complete set of mechanical variables, but only in a much more restricted number of variables such as the energy, electric currents or charges pertaining to macroscopically infinitesimal regions of the system. In effect, we are interested in a formulation in which density of matter, charge density, etc., are continuous functions of position and time. The "homogenization" of the discrete system is made through the classical statistical methods by considering an ensemble of similar systems.⁶ For our purpose we need only note that

$$\langle N(\bar{p}\bar{q}t) \rangle_{av} = f(\bar{p}\bar{q}t) \quad ,$$

where $f(\bar{p}\bar{q}t)$ is the distribution function, usually known as the one-particle momentum distribution function. With this step the individuality of the particles in the system has been lost, and we must begin to think in terms of a fluid. Consider the ensemble average of all terms in Eq. (4),

$$\begin{aligned} \frac{\partial f(\bar{p}\bar{q}t)}{\partial t} + \frac{\bar{p}}{m} \cdot \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{q}} - \frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} d\bar{p}'d\bar{q}' \langle N(\bar{p}'\bar{q}'t) \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{p}} \rangle_{av} \\ = \left\langle \frac{\partial U(\bar{q})}{\partial \bar{q}} \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{p}} \right\rangle_{av} \quad . \end{aligned} \quad (5)$$

The average inside the integral is a pair distribution function

$$\langle N(\bar{p}\bar{q}t) N(\bar{p}'\bar{q}'t) \rangle_{av} = f_2(\bar{p}\bar{p}', \bar{q}\bar{q}', t)$$

denoting the extent of the correlation at one time t of two "particles," one at q , the other at q' and with respective momenta p and p' . As a first approximation, we assume that the pair distribution function can be separated as (Ref. 7)

$$f_2(\bar{p}\bar{p}', \bar{q}\bar{q}', t) = f(\bar{p}\bar{q}t) f(\bar{p}'\bar{q}'t) + \psi(\bar{p}\bar{p}', \bar{q}\bar{q}', t) \quad , \quad (6)$$

where again ψ is a pair distribution function. It can be argued that since $\partial U(\bar{q})/\partial \bar{q}$ represents an external force, not necessarily of a statistical nature,

$$\left\langle \frac{\partial U}{\partial \bar{q}} \frac{\partial N(\bar{p}\bar{q}t)}{\partial \bar{p}} \right\rangle = \frac{\partial U}{\partial \bar{q}} \frac{\partial}{\partial \bar{p}} \langle N(\bar{p}\bar{q}t) \rangle = -\bar{F}^{\text{ext}}(\bar{q}) \cdot \frac{\partial}{\partial \bar{p}} f(\bar{p}\bar{q}t) \quad (7)$$

Introducing Eqs. (6) and (7) into Eq. (5), we obtain

$$\begin{aligned} \frac{\partial f(\bar{p}\bar{q}t)}{\partial t} + \frac{\bar{p}}{m} \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{q}} - \frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} f(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}' \cdot \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{p}} \\ = -\bar{F}^{\text{ext}}(\bar{q}) \cdot \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{p}} + \frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} \frac{\partial \psi(\bar{p}\bar{p}', \bar{q}\bar{q}', t)}{\partial \bar{p}} d\bar{p}'d\bar{q}' \quad (8) \end{aligned}$$

It is now easy to obtain the standard form of the Boltzmann-Vlasov equation from Eq. (8), and also the equation which has been used by Gasirowicz, et al.,⁹ for the test particle. First consider $\bar{F}^{\text{ext}}(\bar{q}t) = 0$ and $\psi(\bar{p}\bar{p}', \bar{q}\bar{q}', t) \neq 0$ and call

$$\bar{F}^{\text{int}}(\bar{q}t) = -\frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} f(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}' \quad ,$$

then

$$\frac{\partial f}{\partial t} + \frac{\bar{p}}{m} \cdot \frac{\partial f}{\partial \bar{q}} + \bar{F}^{\text{int}} \cdot \frac{\partial f}{\partial \bar{p}} = 0 \quad (9)$$

In the evaluation of the integral, $\bar{q} = \bar{q}'$ is excluded; therefore, $\bar{F}^{\text{int}}(\bar{q}t)$ is the internal force acting at a point $\bar{q}$ and is due to all other particles in the system. We note that $\bar{F}^{\text{int}}$ is the self-consistent force in the system and that Eq. (9) is an approximate version of Eq. (5) which does not take account of the pair distribution function. We know, however, that the two-particle distribution function (itself a function of the one- and three-particle distribution function) has important properties¹⁰ leading to the Debye screening potential. It is therefore expected that the simplification leading to Eq. (9) would require further justification. Assuming that such a justification exists, it can be extended to the case where $\bar{F}^{\text{ext}} \neq 0$. Equation (8) then becomes

$$\begin{aligned} \frac{\partial f(\bar{p}\bar{q}t)}{\partial t} + \frac{\bar{p}}{m} \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{q}} - \frac{\partial}{\partial \bar{q}} \int d\bar{p}'d\bar{q}' \frac{e^2}{|\bar{q} - \bar{q}'|} f(\bar{p}'\bar{q}'t) \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{p}} \\ = -\bar{F}^{\text{ext}}(\bar{q}t) \cdot \frac{\partial f(\bar{p}\bar{q}t)}{\partial \bar{p}} \quad (10) \end{aligned}$$

This is the equation used by Gasirowicz. It will be used here for the determination of the response of the system to an applied external force rather than for the response of the system to a test particle. Since the observable quantity in a plasma is the macroscopic density and not the distribution function, the momentum will have to be integrated out of the equations to determine the density

$$n(\bar{q}t) = \int f(\bar{p}\bar{q}t) d\bar{p} \quad .$$

Consider then, that the variations of the density from the average value are due to variations of the distribution function from its equilibrium value $f_0(p^2)$,

$$f(\bar{p}\bar{q}t) = f_0(p^2) + f_1(\bar{p}\bar{q}t) \quad .$$

Introducing the above into Eq. (10) and retaining only linear terms, we obtain*

$$\frac{\partial f_1}{\partial t} + \frac{\bar{p}}{m} \cdot \frac{\partial f_1}{\partial \bar{q}} - \frac{\partial}{\partial \bar{q}} \int d\bar{p}' d\bar{q}' \frac{e^2}{|\bar{q} - \bar{q}'|} f_1(\bar{p}' \bar{q}' t) \frac{\partial f_0}{\partial \bar{p}} = - \bar{F}^{\text{ext}}(\bar{q} t) \cdot \frac{\partial f_0}{\partial \bar{p}}. \quad (11)$$

Equations (9) and (11) are widely used equations for the distribution function f . They have deficiencies, but they represent an improved description of the plasma when compared with the classical Boltzmann equation with the collision term. The main reason for this is that, in an ionized gas, the particles are constantly under the electrostatic field set up by all the other particles; in a neutral gas, the particles are virtually moving in free space, except when they come close to each other. In an ionized gas "collisions" are not discrete events unless the collisions are with neutral particles that may happen to be in the plasma. Whatever success has been derived in the discussion of fully ionized gases can be attributed directly to the electrostatic term in Eq. (11). Despite its shortcomings, Eq. (11) can at least be used to derive a number of useful macroscopic parameters such as the charge density, the current density, and the electric fields. In the following we shall compute the charge density induced in the system by an external potential. We can picture, for example, a plasma confined between the two plates of a capacitor and the external field due to a battery connected to the plates. For all practical purposes we shall then deal with a dielectric as described in electrostatics.

C. THE LINEAR RESPONSE TO AN APPLIED FIELD

The density induced by an external force can be computed in a straightforward manner if we Fourier-transform Eq. (11). Let

$$f_1(\bar{p} \bar{q} t) = \int f_1(\bar{p} \bar{k} \omega) \exp[i\bar{k} \cdot \bar{q} - i\omega t] \frac{d\bar{k} d\omega}{(2\pi)^4},$$

$$f_1(\bar{p} \bar{k} \omega) = \int f_1(\bar{p} \bar{q} t) \exp[-i\bar{k} \cdot \bar{q} + i\omega t] d\bar{q} dt.$$

Multiplying Eq. (11) with $\exp[-i\bar{k} \cdot \bar{q} + i\omega t]$, integrating over $\bar{q}$ and t , and noting that with $\bar{q} - \bar{q}' = \bar{Q}$

$$V(\bar{k}) = \int \exp[-i\bar{k} \cdot \bar{Q}] V(\bar{Q}) d\bar{Q} = \int \frac{e^2}{|\bar{Q}|} \exp[-i\bar{k} \cdot \bar{Q}] d\bar{Q} = \frac{4\pi e^2}{k^2},$$

we obtain

$$\begin{aligned} (-i\omega + i\bar{k} \cdot \bar{p}/m) f_1(\bar{p} \bar{k} \omega) - i\bar{k} \frac{4\pi e^2}{k^2} \int f_1(\bar{p}' \bar{q}' \omega) \exp[-i\bar{k} \cdot \bar{q}'] d\bar{p}' d\bar{q}' \cdot \frac{\partial f_0}{\partial \bar{p}} \\ = - \bar{F}^{\text{ext}}(\bar{k} \omega) \cdot \frac{\partial f_0}{\partial \bar{p}}, \end{aligned}$$

or, since $\int f_1(\bar{p}' \bar{q}' \omega) \exp[-i\bar{k} \cdot \bar{q}'] d\bar{p}' d\bar{q}' = \delta n(\bar{k} \omega)$, it follows that

* The term inside the integral with f_0 has been neglected since it is canceled by the neutralizing positive charge in the system.

$$(-i\omega + i\bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m) f_1(\bar{\mathbf{p}}\bar{\mathbf{k}}\omega) - i\bar{\mathbf{k}} \frac{4\pi e^2}{k^2} \delta n(\bar{\mathbf{k}}\omega) \frac{\partial f_0}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{F}}^{\text{ext}}(\bar{\mathbf{k}}\omega) \cdot \frac{\partial f_0}{\partial \bar{\mathbf{p}}} .$$

Dividing by $(-i\omega + i\bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m)$ and integrating the resulting expression with respect to $\bar{\mathbf{p}}$, we have

$$\delta n(\bar{\mathbf{k}}\omega) - \delta n(\bar{\mathbf{k}}\omega) i\bar{\mathbf{k}} \frac{4\pi e^2}{k^2} \int \frac{(\partial/\partial \bar{\mathbf{p}}) f_0}{-i\omega + i\bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} d\bar{\mathbf{p}} = -\bar{\mathbf{F}}^{\text{ext}}(\bar{\mathbf{k}}\omega) \cdot \int \frac{(\partial/\partial \bar{\mathbf{p}}) f_0}{-i\omega + i\bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} d\bar{\mathbf{p}} ,$$

or, solving for $\delta n(\bar{\mathbf{k}}\omega)$,

$$\delta n(\bar{\mathbf{k}}\omega) = \frac{-i\bar{\mathbf{F}}^{\text{ext}}(\bar{\mathbf{k}}\omega) \cdot \int \frac{(\partial/\partial \bar{\mathbf{p}}) f_0}{\omega - \bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} d\bar{\mathbf{p}}}{1 + \frac{4\pi e^2}{k^2} \int \frac{\bar{\mathbf{k}} \cdot (\partial/\partial \bar{\mathbf{p}}) f_0}{\omega - \bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} d\bar{\mathbf{p}}} . \quad (12)$$

Equation (12) gives the desired expression for the response of the system to external force. If this force is of electric origin, then we can write $\bar{\mathbf{F}} = -e\bar{\mathbf{E}} = e\nabla_{\bar{\mathbf{q}}} \varphi$ or, in $\bar{\mathbf{k}}\omega$ -space, $\bar{\mathbf{F}}(\bar{\mathbf{k}}\omega) = e i\bar{\mathbf{k}}\varphi(\bar{\mathbf{k}}\omega)$. The induced electric charge $\rho(\bar{\mathbf{k}}\omega) = -e\delta n(\bar{\mathbf{k}}\omega)$ can then be written as

$$\rho(\bar{\mathbf{k}}\omega) = C(\bar{\mathbf{k}}\omega) \varphi^{\text{ext}}(\bar{\mathbf{k}}\omega) , \quad (13)$$

where the response function $C(\bar{\mathbf{k}}\omega)$ has the form

$$C(\bar{\mathbf{k}}\omega) = \frac{-e^2 \int \frac{\bar{\mathbf{k}} \cdot (\partial/\partial \bar{\mathbf{p}}) f_0}{\omega - \bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} d\bar{\mathbf{p}}}{1 + \frac{4\pi e^2}{k^2} \int \frac{\bar{\mathbf{k}} \cdot (\partial/\partial \bar{\mathbf{p}}) f_0}{\omega - \bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} d\bar{\mathbf{p}}} . \quad (14)$$

The use of C has been introduced in order to indicate that what we have computed is a capacitance. This point underscores the fact that, at this stage of our calculations, we are dealing with a model of a plasma that is completely divorced from any concept of granularity in the microscopic scale.

Before discussing the properties of the response function, we must proceed with computations that will give a more appealing form to $C(\bar{\mathbf{k}}\omega)$. For this purpose we adopt the notation that has been introduced for the first time by Lindhard.¹¹ This notation has the advantage of retaining some of the concepts of electrostatics in continuous media. Lindhard introduces the space and time-dispersive dielectric constant $\epsilon(\bar{\mathbf{k}}\omega)$ in the following way. He begins with the linearized version of Eq. (9)

$$\frac{\partial f_1}{\partial t} + \frac{\bar{\mathbf{p}}}{m} \cdot \frac{\partial f_1}{\partial \bar{\mathbf{q}}} + \bar{\mathbf{F}}^{\text{int}}(\bar{\mathbf{q}}t) \cdot \frac{\partial f_0}{\partial \bar{\mathbf{p}}} = 0 , \quad (15)$$

which he proceeds to solve along the lines used earlier in this section. Fourier-transforming Eq. (15) and solving for $f_1(\bar{\mathbf{p}}\bar{\mathbf{k}}\omega)$, he finds

$$f_1(\bar{\mathbf{p}}\bar{\mathbf{k}}\omega) = -i\bar{\mathbf{F}}^{\text{int}}(\bar{\mathbf{k}}\omega) \cdot \frac{(\partial/\partial \bar{\mathbf{p}}) f_0}{\omega - \bar{\mathbf{k}} \cdot \bar{\mathbf{p}}/m} ;$$

therefore,

$$\delta n(\bar{k}\omega) = -i\bar{F}^{\text{int}}(\bar{k}\omega) \cdot \int \frac{(\partial/\partial \mathbf{p}) f_0}{\omega - \bar{k} \cdot \bar{p}/m} d\bar{p} \quad . \quad (16a)$$

However,

$$\bar{F}^{\text{int}}(\bar{k}\omega) = e i \bar{k} \varphi^{\text{int}}(\bar{k}\omega)$$

and

$$\delta \rho(\bar{k}\omega) = -e \delta n(\bar{k}\omega) = \left[-e^2 \int \frac{\bar{k} \cdot (\partial/\partial \mathbf{p}) f_0}{\omega - \bar{k} \cdot \bar{p}/m} d\bar{p} \right] \varphi^{\text{int}}(\bar{k}\omega) \quad . \quad (16b)$$

Let us now turn to the basic theory of electrostatics in continuous media. We know from this theory¹² that:

- (a) The dielectric polarization in the medium is defined by

$$\rho^{\text{ind}} = \langle \rho \rangle = -\nabla \cdot \bar{P}$$

where $\langle \rho \rangle$ is the average charge in the medium.

- (b) By definition the internal electric field is given by

$$\nabla \cdot \bar{E} = 4\pi \langle \rho \rangle = -4\pi \nabla \cdot \bar{P} \quad .$$

- (c) By definition

$$\bar{E} + 4\pi \bar{P} = \bar{D} \quad .$$

- (d) The constitutive equation between $\bar{D}$ and $\bar{E}$ is given by

$$\bar{D} = \epsilon \bar{E} \quad .$$

From the above it follows readily that in $\bar{k}\omega$ -space we have

$$\rho^{\text{ind}}(\bar{k}\omega) = -i\bar{k} \cdot \bar{P}(\bar{k}\omega) = -\frac{i\bar{k}}{4\pi} [\bar{D}(\bar{k}\omega) - \bar{E}(\bar{k}\omega)] \quad .$$

On the other hand we know that, in a frequency-dispersive medium, $\bar{D}(\omega) = \epsilon(\omega) \bar{E}(\omega)$. Generalizing this relationship to a space-dispersive medium we write $\bar{D}(\bar{k}\omega) = \epsilon(\bar{k}\omega) \bar{E}(\bar{k}\omega)$, therefore,

$$\begin{aligned} \rho^{\text{ind}}(\bar{k}\omega) &= -\frac{i\bar{k}}{4\pi} \cdot [\epsilon(\bar{k}\omega) - 1] \bar{E}(\bar{k}\omega) \\ &= -\frac{k^2}{4\pi} [\epsilon(\bar{k}\omega) - 1] \varphi(\bar{k}\omega) \quad . \end{aligned} \quad (17)$$

Comparing Eqs. (17) and (16b), we then have, according to Lindhard, the definition of the "longitudinal dielectric constant." It is called longitudinal because it describes electrostatic effects inside the medium and is given by the expression

$$\epsilon^L(\bar{k}\omega) = 1 + \frac{4\pi e^2}{k^2} \int \frac{\bar{k} \cdot (\partial/\partial \mathbf{p}) f_0}{\omega - \bar{k} \cdot \bar{p}/m} d\bar{p} \quad . \quad (18)$$

Returning now to our equation for the induced charge due to an external potential [Eq. (13)], we obtain

$$\rho(\bar{k}\omega) = -\frac{k^2}{4\pi} \left[\frac{\epsilon^L(\bar{k}\omega) - 1}{\epsilon^L(\bar{k}\omega)} \right] \varphi^{\text{ext}}(\bar{k}\omega) \quad (19a)$$

$$= -\frac{k^2}{4\pi} \left[1 - \frac{1}{\epsilon^L(\bar{k}\omega)} \right] \varphi^{\text{ext}}(\bar{k}\omega) \quad (19b)$$

and

$$C(\bar{k}\omega) = -\frac{k^2}{4\pi} \left[1 - \frac{1}{\epsilon^L(\bar{k}\omega)} \right] \quad (20)$$

For the explicit determination of $\epsilon^L(\bar{k}\omega)$ we need only note that, in the integral of Eq. (18), we have to make the following interpretation of the singularity:

$$\frac{1}{\omega - \bar{k} \cdot \bar{p}/m} \rightarrow \mathcal{P} \frac{1}{\omega - \bar{k} \cdot \bar{p}/m} + \frac{\pi}{i} \delta(\omega - \bar{k} \cdot \bar{p}/m) \quad ,$$

where $\mathcal{P}$ denotes the principal part. Simple integrations then give

$$\epsilon^L(\bar{k}\omega) = 1 + \left(\frac{1}{k^2 \lambda_D^2} \right) [\text{R}(k\omega) + i\text{I}(k\omega)] \quad (21)$$

where λ_D is the Debye length

$$\lambda_D^2 = \frac{\kappa T}{4\pi e^2 n_0}$$

and

$$\text{R}(k\omega) = \text{R}(x) = \text{R}\left(\frac{\omega}{kv_m}\right) = 1 - 2xe^{-x^2} \int_0^x e^{z^2} dz \quad (22)$$

$$\text{I}(k\omega) = \text{I}(x) = \text{I}\left(\frac{\omega}{kv_m}\right) = \sqrt{\pi}xe^{-x^2} \quad (23)$$

where $v_m = 2\kappa T/m$ is the mean thermal velocity of the electrons at temperature T . It can be easily shown that

$$\epsilon^L(\bar{k}\omega) \rightarrow 1 - \left(\frac{\omega_p}{\omega}\right)^2 \quad \text{for} \quad \frac{\omega}{kv_m} \gg 1 \quad (24)$$

$$\epsilon^L(\bar{k}\omega) \rightarrow 1 + \left(\frac{1}{k\lambda_D}\right)^2 \quad \text{for} \quad \frac{\omega}{kv_m} \ll 1 \quad (25)$$

It is important that at the limit of small wave numbers the longitudinal dielectric constant goes to the classical dielectric constant $\epsilon(\omega)$ since we find, in both constants, the plasma frequency ω_p . However, the two constants have a completely different meaning and cannot be used indiscriminately. The longitudinal constant implies density waves similar to acoustic waves with a frequency ω_p and a wave number k along the direction of propagation. On the other hand, the classical dielectric constant implies a propagation of a transverse electromagnetic wave. A space-dispersive, transverse dielectric constant can be developed for transverse propagation in order to generalize the classical dielectric constant, but it is outside the scope of this report.

Let us use now the two limiting forms of $\epsilon^L(\bar{k}\omega)$ in Eq. (19). For $\omega/\bar{k}v_m \gg 1$, we have

$$\rho(\bar{k}\omega) \rightarrow \frac{k^2}{4\pi} \left[\frac{\omega_p^2}{\omega^2 - \omega_p^2} \right] = -\frac{k^2}{4\pi} D(\omega) \varphi^{\text{ext}}(\bar{k}\omega) \quad (26)$$

or, using the continuity equation,

$$\bar{j}(\bar{k}\omega) \rightarrow \frac{i\omega}{4\pi} \left[\frac{\omega_p^2}{\omega^2 - \omega_p^2} \right] \bar{E}^{\text{ext}}(\bar{k}\omega) \quad .$$

For $\omega/\bar{k}v_m \ll 1$, we have

$$\rho(\bar{k}\omega) \rightarrow -\frac{k^2}{4\pi} \left[\frac{1}{1 + k^2 \lambda_D^2} \right] \varphi^{\text{ext}}(\bar{k}\omega) \quad . \quad (27)$$

The first limit clearly indicates the singularity at the plasma frequency, the second limit indicates the independence of the response function from the frequency. To obtain more insight we will plot the function $(4\pi/k^2) C(\bar{k}\omega) = D(\omega)$. The imaginary part of $\bar{D}(\omega)$ that was lost while the limit for $\omega/\bar{k}v_m \gg 1$ was taken can be reintroduced by the substitution

$$\frac{1}{\omega \pm \omega_p} \rightarrow \mathcal{P} \frac{1}{\omega \pm \omega_p} + \frac{\pi}{i} \delta(\omega \pm \omega_p) \quad .$$

Thus

$$\begin{aligned} D(\omega) &= \frac{\omega_p^2}{\omega^2 - \omega_p^2} = \frac{\omega_p}{2} \left[\frac{1}{\omega - \omega_p} - \frac{1}{\omega + \omega_p} \right] \\ &= \frac{\omega_p}{2} \left[\mathcal{P} \frac{1}{\omega - \omega_p} - \mathcal{P} \frac{1}{\omega + \omega_p} + \frac{\pi}{i} \delta(\omega - \omega_p) - \frac{\pi}{i} \delta(\omega + \omega_p) \right] \quad . \end{aligned}$$

The functions $D'(\omega)$ and $D''(\omega)$ are plotted in Fig. 1. In a real situation the singularities will be considerably smoother. This is indicated by the dotted curves. Before ending this section, we shall derive some more useful formulas.

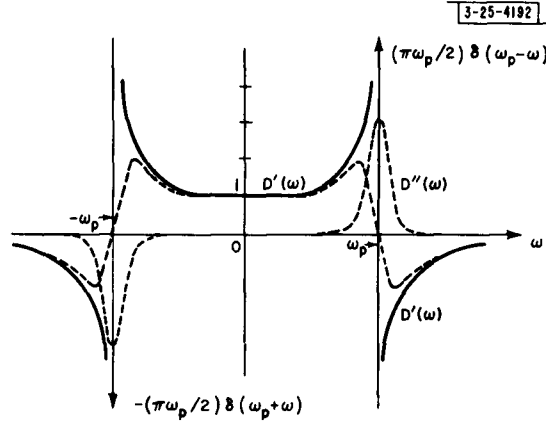


Fig. 1. The real and imaginary parts of the response function are similar to the dispersion relations for an oscillator.

The use of Poisson's equation was inherent in the previous treatment. In fact, it is contained in the equation

$$\mathbb{F}^{\text{int}}(\bar{q}t) = -\frac{\partial}{\partial \bar{q}} \int \frac{e^2}{|\bar{q} - \bar{q}'|} f_1(\bar{p}'\bar{q}'t) d\bar{p}'d\bar{q}'$$

which, in $\bar{k}\omega$ -space, takes the form

$$\varphi(\bar{k}\omega) = -\frac{4\pi e}{k^2} \int f_1(\bar{p}\bar{k}\omega) d\bar{p} \quad (28)$$

or

$$k^2 \varphi^{\text{int}}(\bar{k}\omega) = -4\pi e \delta n(\bar{k}\omega) = 4\pi \rho^{\text{int}}(\bar{k}\omega) ,$$

which is the Fourier transform of

$$\nabla^2 \varphi^{\text{int}}(\bar{q}t) = -4\pi \rho^{\text{int}}(\bar{q}t) . \quad (29)$$

If, however, external charges are introduced in the medium, Poisson's equation would have to be written as

$$\nabla^2 \varphi(\bar{q}t) = -4\pi \rho^{\text{int}}(\bar{q}t) - 4\pi \rho^{\text{ext}}(\bar{q}t)$$

or

$$k^2 \varphi(\bar{k}\omega) = 4\pi \rho^{\text{int}}(\bar{k}\omega) + 4\pi \rho^{\text{ext}}(\bar{k}\omega) . \quad (30)$$

Introducing Eq. (17) for the internal charge, we then have¹¹

$$k^2 \epsilon^L(\bar{k}\omega) \varphi(\bar{k}\omega) = 4\pi \rho^{\text{ext}}(\bar{k}\omega) . \quad (31)$$

Equation (31) gives the response of the internal potential to an external charge. Equation (19) can then be considered the dual of Eq. (31), since the first corresponds to a circuit driven by a voltage source and the second corresponds to a circuit driven by a current source.

Combining Eqs. (17) and (19) we obtain

$$\varphi^{\text{int}}(\bar{k}\omega) = \frac{\varphi^{\text{ext}}(\bar{k}\omega)}{\epsilon^L(\bar{k}\omega)} . \quad (32)$$

Further, since $\varphi^{\text{tot}} = \varphi^{\text{int}} + \varphi^{\text{ext}}$, we have

$$\varphi^{\text{tot}}(\bar{k}\omega) = \frac{\epsilon^L(\bar{k}\omega) + 1}{\epsilon^L(\bar{k}\omega)} \varphi^{\text{ext}}(\bar{k}\omega) . \quad (33)$$

To complete this section we include the Kramers-Kronig relationships. These relations result from the principle of causality and should be applicable to any well-behaved response function. Although these relations have to be reworked for a space-dispersive medium, it is believed that, for low thermal velocities of the electrons, as compared with the velocity of light, no serious modifications need be expected. Separating the response function $C(\bar{k}\omega)$ into its real and imaginary parts

$$C(k\omega) = C'(k\omega) + iC''(k\omega) ,$$

the Kramers-Kronig relations are

$$C'(k\omega) = \frac{1}{\pi} \mathcal{P} \int \frac{C''(k\omega)}{\omega - \omega'} d\omega' \quad (34)$$

and

$$C''(k\omega) = -\frac{1}{\pi} \mathcal{P} \int \frac{C'(k\omega')}{\omega - \omega'} d\omega' \quad (35)$$

The following relations can also be derived:

$$C(k\omega) e^{\pm i\omega t} = \frac{1}{\pi i} \mathcal{P} \int \frac{C(k\omega') e^{\pm i\omega' t}}{\omega - \omega'} d\omega' \quad (36)$$

where the plus implies $t > 0$ and the minus implies $t < 0$. Using Eqs. (34) and (20), for $\omega = 0$, we find the useful

$$C'(k0) = \frac{1}{\pi} \mathcal{P} \int \frac{C''(k\omega)}{\omega} d\omega \quad .$$

However, from the explicit expression for C , we also find

$$-\text{Re} \left\{ \frac{\epsilon^L(k0) - 1}{\epsilon^L(k0)} \right\} = \frac{1}{\pi} \mathcal{P} \int \frac{1}{\omega} \text{Im} \left\{ \frac{1}{\epsilon^L(k\omega)} \right\} d\omega = \frac{1}{1 + k^2 \lambda_D^2} \quad (37)$$

D. ENERGY INPUT INTO THE SYSTEM

With the available results we can now proceed to determine the energy input to a plasma from an external force. In the usual theory of plasma, this energy input can occur only because of the collisions of electrons with neutral particles in a gas that is not fully ionized. In the simplified form of this theory the collision frequency is the important parameter. This, as we have noted in Sec. I-B, implies that the electrons are moving in the system on straight trajectories until they encounter a neutral particle. In a fully ionized gas the collisions have to be interpreted on a continuous basis, and we may expect that the continuous interaction of a test particle with the field of all the others may lead to more pronounced "deflections" than a single short-range collision. In any case, these continuous collisions should lead to the loss mechanism inside the system. We now place our plasma thermodynamically in an insulated container and electrically between the plates of a condenser. Then we go back to Eq. (11) which we multiply by $(1/2) m v^2 = (1/2m) p^2$ and integrate with respect to $\bar{v}$ or $\bar{p}$. We expand f into the equilibrium f_0 and f_1 , but this time retain the nonlinear terms. We note that terms containing $(\partial/\partial \bar{p}) f_0$ are odd, and therefore yield zero on integration. What remains is

$$\int \frac{1}{2m} p^2 \frac{\partial f_1}{\partial t} d\bar{p} + \int \frac{1}{2m} p^2 \frac{\partial f_1}{\partial \bar{q}} \cdot \frac{\bar{p}}{m} d\bar{p} + \bar{F}^{\text{in}} \cdot \int \frac{1}{2m} p^2 \frac{\partial f_1}{\partial \bar{p}} d\bar{p} = -\bar{F}^{\text{ex}} \cdot \int \frac{1}{2m} p^2 \frac{\partial f_1}{\partial \bar{p}} d\bar{p} \quad .$$

The first term gives the rate of change of the kinetic energy density $\mathcal{K}$, the second term gives the divergence of kinetic energy current vector $\bar{S}_{\mathcal{K}}$, while the other two terms give $-\bar{F} \cdot \int (\bar{p}/m) f_1 (\bar{p}\bar{q}) d\bar{p}$. However, $\bar{F} = -e\bar{E}$ and, since by definition the average current in the system is $-(e/m) \int \bar{p} f_1 (\bar{p}\bar{q}) d\bar{p} = \bar{j}(\bar{q})$, we finally obtain

$$\frac{\partial \mathcal{K}}{\partial t} + \nabla_{\bar{q}} \cdot \bar{S}_{\mathcal{K}} + \bar{E}^{\text{int}} \cdot \bar{j}^{\text{ind}} = -\bar{E}^{\text{ext}} \cdot \bar{j}^{\text{ind}} \quad (38)$$

The interaction energy, therefore, is given by the right-hand side of Eq. (38),

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= -\bar{E}^{\text{ext}}(\bar{q}t) \cdot \bar{j}^{\text{ind}}(\bar{q}t) \\ &= -\iint \bar{E}^{\text{ext}}(\bar{k}\omega) \cdot \bar{j}^{\text{ind}}(\bar{k}'\omega') \exp[i(\bar{k} + \bar{k}')\bar{q} - i(\omega + \omega')t] \frac{d\bar{k}d\bar{k}'}{(2\pi)^4} \frac{d\omega d\omega'}{(2\pi)^4} . \end{aligned}$$

Integrating over time t and space $\bar{q}$, we then obtain

$$\iint \frac{\partial \mathcal{E}}{\partial t} dtd\bar{q} = -\iint \bar{E}^{\text{ext}}(\bar{k}\omega) \cdot \bar{j}^{\text{ind}}(-\bar{k}, -\omega) \frac{d\bar{k}d\omega}{(2\pi)^4} .$$

The continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{j} = 0$$

becomes $\rho(\bar{k}\omega) = \bar{k} \cdot \bar{j}(\bar{k}\omega)/\omega$ in $\bar{k}\omega$ -space. Since $\bar{E}(\bar{k}\omega) = -i\bar{k}\varphi(\bar{k}\omega)$, we can write

$$\iint \frac{\partial \mathcal{E}}{\partial t} dtd\bar{q} = \iint i\omega \varphi^{\text{ext}}(\bar{k}\omega) \rho^{\text{ind}}(-\bar{k}, -\omega) \frac{d\bar{k}d\omega}{(2\pi)^4}$$

and, remembering Eq. (19) for the response to an external potential,

$$\begin{aligned} \iint \frac{\partial \mathcal{E}}{\partial t} dtd\bar{q} &= \iint i\omega \varphi^{\text{ext}}(\bar{k}\omega) C(-\bar{k}, -\omega) \varphi^{\text{ext}}(-\bar{k}, -\omega) \frac{d\bar{k}d\omega}{(2\pi)^4} \\ &= \iint i\omega |\varphi^{\text{ext}}(\bar{k}\omega)|^2 C^*(\bar{k}\omega) \frac{d\bar{k}d\omega}{(2\pi)^4} \\ &= \frac{1}{2} \iint i\omega |\varphi^{\text{ext}}(\bar{k}\omega)|^2 [C^*(\bar{k}\omega) - C(\bar{k}\omega)] \frac{d\bar{k}d\omega}{(2\pi)^4} , \end{aligned}$$

or, finally,

$$\iint \frac{\partial \mathcal{E}}{\partial t} dtd\bar{q} = \iint |\varphi^{\text{ext}}(\bar{k}\omega)|^2 \omega \text{Im}[C(\bar{k}\omega)] \frac{d\bar{k}d\omega}{(2\pi)^4} . \quad (39)$$

From Eq. (20) it follows that

$$\text{Im}[C(k\omega)] = \frac{k^2}{4\pi} \text{Im} \left[\frac{1}{\epsilon L(\bar{k}\omega)} \right] . \quad (40)$$

The results we have just obtained are very important. In the first place, we have shown that, as expected, the continuous interaction of the electrons through the electrostatic forces leads to dissipation. Then we have indicated a specific method for evaluating this dissipation. Finally, we have put all this in a very appealing form through the longitudinal dielectric constant. However, the most significant property that can be deduced from the above discussion comes from a consideration of the theory of Callen and Welton.¹³ According to this theory, whenever a dissipation process exists in a system, a fluctuation is associated with it. This theory can be considered as the generalization of the older theory of Nyquist. We shall not reproduce here the derivations of Callen and Welton. Nevertheless, a very short account of some of their results may be helpful. They show, for example, that dissipation involved in viscous drag on a moving

airborne particle leads to a fluctuation in position and velocity of the particle. They show that the acoustic radiation resistance of a small oscillating sphere leads to a formula for the pressure and density fluctuation in a gas. They show that radiation resistance to acceleration of an electrical charge leads to a fluctuation of the electric field that is equivalent to black-body radiation.

In the next section we shall proceed with the calculation of the density fluctuation in a plasma.

E. DENSITY FLUCTUATIONS

To derive the fluctuation spectrum, Callen and Greene¹⁴ argue as follows. They first suppose that the system is under the influence of an external force from $t = -\infty$ to the time $t = 0$ when the force is removed. The system is then left on its own. Let us compute the transient response of the system, using the results obtained in Sec. I-C.

From Eq. (19) we have

$$\rho(\vec{q}t) = \int C(k\omega) \varphi^{\text{ext}}(k\omega) \exp[i\vec{k} \cdot \vec{q} - i\omega t] \frac{d\vec{k}d\omega}{(2\pi)^4} \quad (41)$$

If the externally applied potential was constant to $t = 0$ and, acting at the position of the "particle" at $q = 0$,

$$\varphi^{\text{ext}}(\vec{q}t) = \begin{cases} -\varphi_0 \delta(\vec{q}) & t \leq 0 \\ 0 & t > 0 \end{cases} ,$$

then

$$\varphi^{\text{ext}}(k\omega) = -\varphi_0 \pi \left\{ \delta(\omega) + \frac{1}{\pi i} \mathcal{P} \frac{1}{\omega} \right\} .$$

Introducing this into Eq. (37), we obtain

$$\rho(\vec{q}t) = - \int \frac{d\vec{k}e^{i\vec{k} \cdot \vec{q}}}{(2\pi)^4} \cdot \pi \left\{ C(k0) + \frac{1}{\pi i} \mathcal{P} \int \frac{C(k\omega) e^{-i\omega t}}{\omega} d\omega \right\} \varphi_0 \quad (42)$$

From Eq. (36) and for $t > 0$, we have

$$C(k0) = \frac{1}{\pi i} \mathcal{P} \int \frac{C(k\omega) e^{+i\omega t}}{\omega} d\omega ,$$

and therefore Eq. (41) can also be written as

$$\rho(\vec{q}t) = - \int \frac{d\vec{k}e^{i\vec{k} \cdot \vec{q}}}{(2\pi)^3} \left[\frac{1}{\pi i} \mathcal{P} \int \frac{C(k\omega)}{\omega} \cos \omega t d\omega \right] \varphi_0 \quad (43)$$

Equation (43) describes the evolution (decay) of the system after the driving force was removed.

Callen and Greene further argue that, since the autocorrelation of the density in time as given by

$$R(\tau) = \langle \rho(t) \rho(t + \tau) \rangle = \int d\rho' W(\rho') \rho' \langle \tau, \rho' | \rho \rangle \quad (44a)$$

(where $W(\rho') d\rho'$ is the probability of finding ρ in the range $\rho' < \rho < \rho' + d\rho'$, and $\langle \tau, \rho' | \rho \rangle$ is the expectation value of ρ at a given time if it is known that ρ had the value ρ' at τ seconds earlier) and since the expectation value of ρ is, in fact, given by Eq. (43), there is a clear connection between the autocorrelation and the response function. The above results should be

generalized to include spatial variations that are applicable to our problem. Instead of Eq. (44a) we must consider

$$R(\bar{Q}, \tau) = \langle \rho(\bar{q}, t) \rho(\bar{q} + \bar{Q}, t + \tau) \rangle = \int d\rho' W(\rho') \rho' \langle \bar{Q}, \tau; \rho' | \rho \rangle \quad , \quad (44b)$$

where $\langle Q\tau, \rho' | \rho \rangle$ is the expectation value of ρ at a given time and space point if it is known that ρ had the value ρ' at τ seconds earlier and at Q centimeters farther up. By using the above arguments we then find, according to Callen and Greene, that

$$R(\bar{Q}, \tau) = -\frac{\kappa T}{\pi i} \int \frac{d\bar{k} e^{i\bar{k} \cdot \bar{Q}}}{(2\pi)^3} \left[\rho \int \frac{C(\bar{k}\omega)}{\omega} \cos \omega \tau d\omega \right] \quad , \quad (45)$$

where κ is the Boltzmann constant and T the temperature. Equation (45) is the result we wanted. Now we shall seek the spectral form of Eq. (45). We first note that

$$R(\bar{Q}, \tau) = \lim_{\substack{T, V \\ \rightarrow \infty}} \frac{1}{TV} \iint_{-\infty}^{\infty} \rho(\bar{q}, t; TV) \rho(\bar{q} + \bar{Q}, t + \tau; TV) d\bar{q} dt$$

where, as usual,

$$\lim_{\substack{T, V \\ \rightarrow \infty}} \frac{1}{TV} \iint_{-\infty}^{\infty} \rho(\bar{q}, t; TV) d\bar{q} dt = \lim_{\substack{T, V \\ \rightarrow \infty}} \frac{1}{TV} \int_{-T}^T \int_V \rho(\bar{q}, t) d\bar{q} dt \quad .$$

Then we denote by $S(\bar{k}\omega)$ the following expression

$$\begin{aligned} S(\bar{k}\omega) &= \lim_{\substack{T, V \\ \rightarrow \infty}} \frac{1}{TV} \rho^*(k\omega; VT) \rho(\bar{k}\omega; VT) \\ &= \iint d\bar{Q} d\tau \exp[-i\bar{k} \cdot \bar{Q} + i\omega\tau] \lim_{\substack{T, V \\ \rightarrow \infty}} \frac{1}{TV} \iint_{-\infty}^{\infty} \rho(\bar{q}, t; VT) \rho(\bar{q} + \bar{Q}, t + \tau; VT) d\bar{q} dt \quad . \end{aligned}$$

Thus

$$R(\bar{Q}, \tau) = \iint S(\bar{k}\omega) \exp[i\bar{k} \cdot \bar{Q} - i\omega\tau] \frac{d\bar{k}}{(2\pi)^3} \frac{d\omega}{2\pi} \quad . \quad (46)$$

Comparing Eq. (45) with Eq. (46) and observing that

$$C'(k\omega) = C'(k, -\omega) \quad (\text{even})$$

$$C''(k\omega) = -C''(k, -\omega) \quad (\text{odd})$$

we find

$$S'(\bar{k}\omega) = -(2) \frac{\kappa T C''(k\omega)}{\pi \omega} \quad , \quad (47)$$

where (2) is to be included only if $\omega > 0$ is considered. Equation (47) expresses the connection of the spectral properties of the fluctuations in a system in terms of the imaginary part of the response function. More explicitly for the plasma

$$S'(\bar{k}\omega) = (2) \frac{\kappa T}{\pi} \frac{1}{\omega} \text{Im} \left\{ -\frac{1}{\epsilon^L(\bar{k}\omega)} \right\} \frac{k^2}{4\pi} \quad (48)$$

We can now use the Kramers-Kronig relation [Eq. (34)] to integrate Eq. (47) and find the "integrated spectrum"

$$\begin{aligned} \int S'(\bar{k}\omega) d\omega &= -\frac{\kappa T}{\pi} \int \frac{C''(k\omega)}{\omega} d\omega = -\kappa T C'(k0) \\ &= \kappa T \frac{k^2}{4\pi} \text{Re} \left\{ \frac{\epsilon^L(k0) - 1}{\epsilon^L(k0)} \right\} \\ &= \kappa T \frac{k^2}{4\pi} \frac{1/k^2 \lambda_D^2}{1 + 1/k^2 \lambda_D^2} \\ &= e^2 n_0 \frac{k^2 \kappa T}{4\pi e^2 n_0} \frac{1}{1 + k^2 \lambda_D^2} \\ &= e^2 n_0 \frac{k^2 \lambda_D^2}{1 + k^2 \lambda_D^2} \quad (50) \end{aligned}$$

We note that we have derived the fluctuation spectrum of the charge density. Since $\rho^*(\bar{k}) \rho(\bar{k}) = e^2 \delta n^*(\bar{k}) \delta n(\bar{k})$,

$$|\delta n(\bar{k})|^2 = n_0 \frac{k^2 \lambda_D^2}{1 + k^2 \lambda_D^2} \quad (51)$$

By taking limits for wavelengths ($k = 2\pi/\lambda$) very much larger than the Debye length and very much smaller than the Debye length, we obtain

$$|\delta n(\bar{k})|^2 = \begin{cases} n_0 & , \quad k\lambda_D \gg 1 \\ n_0 (k\lambda_D)^2 & , \quad k\lambda_D \ll 1 \end{cases} \quad (52)$$

$$(53)$$

The first result, giving a fluctuation equal to the mean number of electrons per unit volume, is not surprising since this is, in fact, the fluctuation of any system in equilibrium composed of noninteracting particles. What is intriguing, but hardly unexpected, is that the fluctuation for the long wavelength limit is considerably reduced. The intuitive interpretation of this result is that the plasma (in a phenomenological way) is like a very tight membrane which can only execute oscillations. To understand this model better, we have to look at the spectral quality of the density fluctuations. The spectrum is given by Eq. (48) (divided by e^2). We have, more explicitly,

$$|\delta n(\bar{k}\omega)|^2/n_0 = \frac{\kappa T}{4\pi e^2 n} \frac{k^2}{\pi \omega} \frac{\text{Im} \{ \epsilon^L(\bar{k}\omega) \}}{|\epsilon^L(\bar{k}\omega)|^2} \quad (54)$$

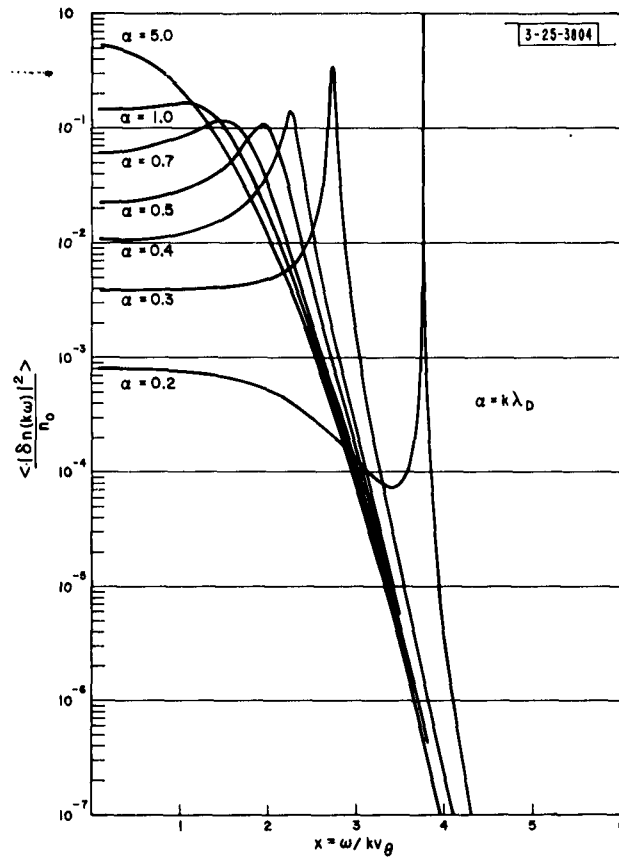


Fig. 2. Mean-square fluctuation of electron density in a positive ion sea.

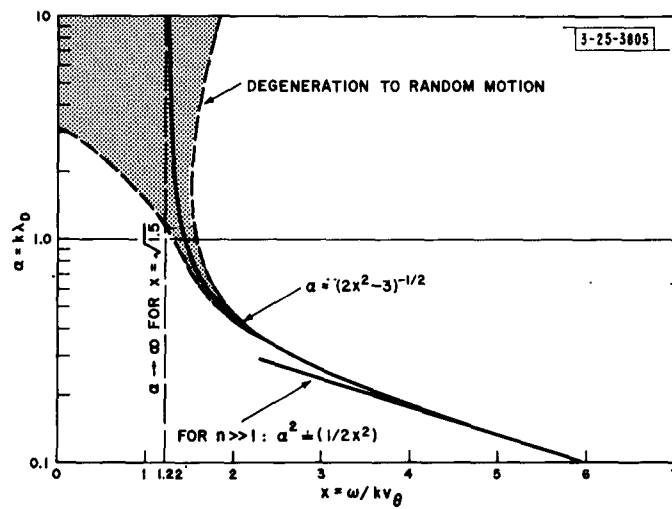


Fig. 3. Gabor's conception of the dispersion relation.

since $-\text{Im}\{\epsilon^{L*}(\bar{k}\omega)\} = \text{Im}\{\epsilon^L(\bar{k}\omega)\}$. With the help of Eqs. (22) and (23) we find that, in the limit of $\omega/\bar{k}v_m \gg 1$, Eq. (54) becomes sharply peaked, indicating that the fluctuation spectrum loses the quality of a random phenomenon. In fact, it does not represent a fluctuation, but an oscillation at a frequency very close to the plasma frequency ω_p .

Figure 2 plots the mean-square fluctuation of the electron density (in the absence of ions). We note that, as an observation limits itself to smaller and smaller k 's or larger wavelengths, the spectrum of $|\delta n|^2$ becomes concentrated more and more near one frequency. In the other limit (of small wavelengths), the spectrum is Gaussian, as it should be for a neutral gas. Since the sharp peak near $\omega \doteq \omega_p$ corresponds to a longitudinal oscillation, we have obtained the natural motion of the plasma in equilibrium.

To complete this picture, we shall now refer to the analysis of Gabor¹⁵ for the density fluctuations. First we note that the dispersion relation $\omega = \omega(\bar{k})$ for the plasma can be obtained from the numerical solution of

$$\epsilon^L(\bar{k}\omega) = 0 \quad (55)$$

which gives the well-known relation

$$\omega^2(k) \doteq \omega_p^2 + \frac{3\kappa T}{m} k^2 \quad (56)$$

The exact solution of Eq. (55) would give a curve as shown by the solid line in Fig. 3.[†] Now this curve can be considered as the projection of the relative maxima of Fig. 2 on the plane $(\omega/\bar{k}v_\Theta, k\lambda_D)$. Referring to the analysis of Gabor, who found that at large k 's the dispersion curve effectively becomes a diffuse line describing the degeneration of organized oscillations to a random motion of the particles, we may plot a similar curve but with "equipotentials" around the classical dispersion relation (Fig. 4). Thus the dispersion relation is a diffuse line for all $\bar{k}$'s but becomes sharper as $\bar{k}$ is decreased. Gabor succinctly expressed this behavior of the plasma as follows:

"For waves much shorter than the Debye length there will be no dispersion law, but merely a [statistical] correlation between frequency and wave number."

It is the same picture that we get from the work of Pines and Bohm⁴ who find that the density fluctuations can only have wave numbers smaller than k_D . For $\bar{k} > k_D$ the plasma behaves statistically like a set of neutral particles interacting with short-range forces.

From the above discussion we conclude that a very useful model of scattering from a plasma can be constructed if the plasma is viewed as a continuum whose density varies periodically in space and time. We have to keep in mind, however, that these macroscopic density variations, or fluctuations, reflect in our measuring scale the granular nature of matter in the microscopic scale. In general, this granular structure is not accessible. What we see and measure are its effects on our measuring apparatus. In the particular case of a plasma, we can conceive of three distinct measuring systems. We can place the plasma between the plates of a capacitor and measure the fluctuating voltage or current. We can send an electromagnetic wave through the plasma and measure scattering due to fluctuations in the density (a continuum with uniform

[†] In terms of $x = (\omega/\bar{k}v_m)$ and $a = k\lambda_D$, Eq. (56) becomes $a^2 = (2x^2 - 3)^{-1}$.

density does not scatter radiation). We can, finally, send a beam of electrons through the system and measure the interaction of the beam with the plasma.

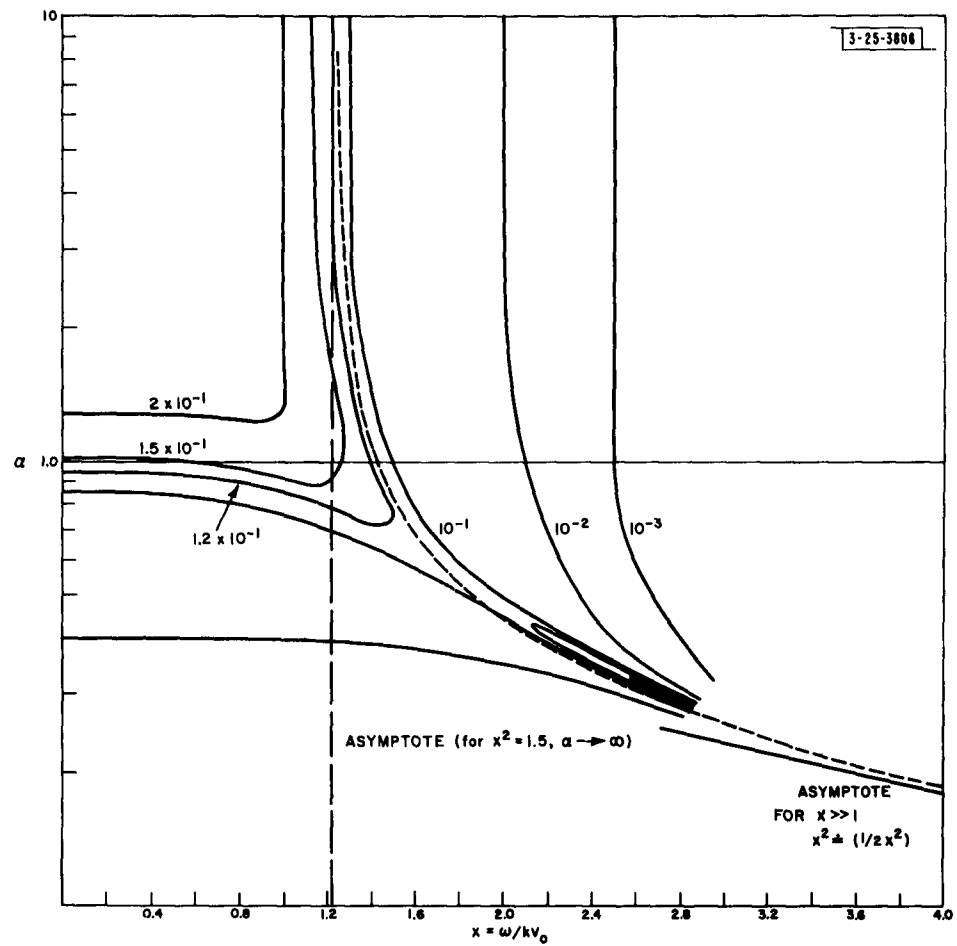


Fig. 4. Dispersion relation in a plasma.

II. THE ION-CARRIED ELECTRONIC SPECTRUM

A. INTRODUCTION

In Part I, we gave a general picture of the models, concepts, and techniques used to establish the fluctuation spectrum of a plasma composed of electrons in a neutralizing sea of positive charge. We reached the conclusion that, on a macroscopic scale, the fluctuation spectrum of the electrons could be interpreted as a density wave propagating in the plasma with a given frequency ω and a wave number k , provided the measuring scale is larger than the Debye length. It would be reasonable to expect that similar results could be obtained for a model plasma in which the role of ions and electrons is reversed (where the ions are the particles moving in a neutralizing negative charge). The only difference between the two models would come from the fact that ions are heavier. Since the characteristic parameter $x_e = \omega/kv_{m,e}$ in the fluctuation spectrum of the electrons is inversely proportional to the mean thermal velocity $v_{m,e} = \sqrt{2kT/m_e}$, where m_e is the mass of the electrons, it is clear that the results obtained in Part I could be scaled and used for the ions in a negative continuum if a new parameter were to be defined as

$$x_i = x_e \sqrt{M/m} = (\omega/kv_{m,e}) \sqrt{M/m} \quad ,$$

where M is the mass of the ions.

In reality, however, a plasma can be composed of electrons, negative ions, positive ions, and neutrals; therefore, the models with positive or negative neutralizing seas can only be useful under restricted circumstances. In general, it is expected that the fluctuation spectrum of any of the constituents of the plasma will be seriously affected by the presence of the other constituents. On an intuitive basis, which rests on the assumption of perfect neutrality in a plasma, we should expect that ion fluctuations (or ion density waves) will be closely followed by electron density waves or fluctuations. On these grounds alone and because the ionic mass M is much larger than the electronic mass m , giving an ion plasma frequency $\omega_{p,i} = \omega_{p,e} \sqrt{m/M}$, we can make a guess that the portion of the electronic spectrum between $\omega = 0$ and $\omega = \omega_p$ will definitely contain an important contribution due to ionic waves. The problem is to compute the exact amount of this contribution. As stated in Part I, all the effort in the theoretical description of the incoherent scattering from the ionospheric plasma has centered in the evaluation of the effect of the ions on the spectrum of the electron fluctuations. The purpose of Part II is to introduce a simple technique by which the response function of a plasma with an arbitrary number of charged constituents can be evaluated with little effort. Nevertheless, we shall also indicate the method by which the equation of motion for electrons [Eq. (4) of Part I] can be modified to include the effect of the ions.

B. THE EQUATION OF MOTION FOR THE ION-ELECTRON PLASMA

The generalization of Eq. (4) of Part I to derive the equation of motion for the ion-electron plasma can be obtained by considering the Hamiltonian of the system with electrons and ions. This Hamiltonian contains pair Coulomb interactions between electrons and electrons, ions and ions, ions and electrons. If $N_e(\vec{p}qt)$ is the density of electrons and $N_i(pqt)$ the density of ions, from the procedure described in Sec. I-B, we can easily obtain the following exact pair of coupled equations.

$$\begin{aligned}
& \frac{\partial N_e(\bar{p}\bar{q}t)}{\partial t} + \frac{\bar{p}}{m} \frac{\partial N_e(\bar{p}\bar{q}t)}{\partial \bar{q}} - \frac{e}{m} \bar{E}^{\text{ext}}(\bar{q}t) \frac{\partial N_e(\bar{p}\bar{q}t)}{\partial \bar{p}} + e^2 \frac{\partial}{\partial \bar{q}} \int \frac{1}{|\bar{q} - \bar{q}'|} \\
& \times \{N_i(\bar{p}'\bar{q}'t) - N_e(\bar{p}'\bar{q}'t)\} \cdot \frac{2N_e(\bar{p}\bar{q}t)}{\partial \bar{p}} = 0
\end{aligned} \tag{57}$$

$$\begin{aligned}
& \frac{\partial N_i(\bar{p}\bar{q}t)}{\partial t} + \frac{\bar{p}}{M} \frac{\partial N_i(\bar{p}\bar{q}t)}{\partial \bar{q}} + \left(\frac{e}{M} \bar{E}^{\text{ext}}(\bar{q}t) \frac{\partial N_i(\bar{p}\bar{q}t)}{\partial \bar{p}} \right) - e^2 \frac{\partial}{\partial \bar{q}} \int \frac{1}{|\bar{q} - \bar{q}'|} \\
& \times \{N_i(\bar{p}'\bar{q}'t) - N_e(\bar{p}'\bar{q}'t)\} \cdot \frac{\partial N_i(\bar{p}\bar{q}t)}{\partial \bar{p}} = 0
\end{aligned} \tag{58}$$

where $\bar{E}^{\text{ext}}$ is the external field. The parenthesis in Eq. (58) indicates that the external field acting on the ions can be disregarded if we are interested in the response of the electrons. Since an externally applied field acts on both electrons and ions, it is obvious that disregarding the term in parenthesis implies that the external field acting on electrons readjusts itself to a different value.

The pair of equations for N_e and N_i can now be homogenized in order to obtain the coarse-grained properties of the ion-electron plasma. Writing

$$\langle N_e(\bar{p}\bar{q}t) \rangle_{\text{av}} = f_e(\bar{p}\bar{q}t)$$

$$\langle N_i(\bar{p}\bar{q}t) \rangle_{\text{av}} = f_i(\bar{p}\bar{q}t)$$

$$\begin{aligned}
\langle N_e(\bar{p}'\bar{q}'t) N_e(\bar{p}\bar{q}t) \rangle_{\text{av}} &= \langle N_e(\bar{p}'\bar{q}'t) \rangle \langle N_e(\bar{p}\bar{q}t) \rangle + \varphi_{ee}(\bar{p}'\bar{q}'\bar{p}\bar{q}t) \\
&= f_e(\bar{p}'\bar{q}'t) f_e(\bar{p}\bar{q}t) + \varphi_{ee}
\end{aligned}$$

$$\begin{aligned}
\langle N_i(\bar{p}'\bar{q}'t) N_e(\bar{p}\bar{q}t) \rangle_{\text{av}} &= \langle N_i(\bar{p}'\bar{q}'t) \rangle \langle N_e(\bar{p}\bar{q}t) \rangle + \varphi_{ie}(\bar{p}'\bar{q}'\bar{p}\bar{q}t) \\
&= f_i(\bar{p}'\bar{q}'t) f_e(\bar{p}\bar{q}t) + \varphi_{ie}
\end{aligned}$$

and corresponding equations for

$$\langle N_i(\bar{p}'\bar{q}'t) N_i(\bar{p}\bar{q}t) \rangle_{\text{av}} \quad , \quad \langle N_e(\bar{p}'\bar{q}'t) N_i(\bar{p}\bar{q}t) \rangle_{\text{av}}$$

we obtain a set of coupled equations by expanding the single particle distribution functions in the usual manner ($f_{e,i} = f_{0e,i}(\bar{p}) + f_{1e,i}(\bar{p}\bar{q}t)$) and neglecting all the pair correlations φ_{ee} , φ_{ii} , φ_{ei} , φ_{ie}

$$\left\{ \frac{\partial f_{1e}}{\partial t} + \frac{\bar{p}}{m} \frac{\partial f_{1e}}{\partial \bar{q}} + e^2 \frac{\partial}{\partial \bar{q}} \int \frac{1}{|\bar{q} - \bar{q}'|} \{f_{1i}(\bar{p}'\bar{q}'t) - f_{1e}(\bar{p}'\bar{q}'t)\} \cdot \frac{\partial f_{0e}}{\partial \bar{p}} = \frac{e}{m} \bar{E}^{\text{ext}}(\bar{q}t) \frac{\partial f_{0e}}{\partial \bar{p}} \right. \tag{59}$$

$$\left. \frac{\partial f_{1e}}{\partial t} + \frac{\bar{p}}{M} \frac{\partial f_{1i}}{\partial \bar{q}} - e^2 \frac{\partial}{\partial \bar{q}} \int \frac{1}{|\bar{q} - \bar{q}'|} \{f_{1i}(\bar{p}'\bar{q}'t) - f_{1e}(\bar{p}'\bar{q}'t)\} \cdot \frac{\partial f_{0i}}{\partial \bar{p}} = \left(-\frac{e}{M} \bar{E}^{\text{ext}}(\bar{q}t) \frac{\partial f_{0i}}{\partial \bar{p}} \right) \right. \tag{60}$$

Following the same technique as in Sec. I-B, we can determine

$$\delta n_{e,i} = \int f_{e,i}(\bar{p}\bar{q}t) d\bar{p}$$

and also any desired response function. From this point on, the problem becomes a problem of algebra. It is obvious that, if more components are introduced in the plasma, the equations increase in number. The calculations required to compute the induced charges or the response functions, also increase. To avoid these complexities, we have developed a method which reduces the problem to one in network theory. This method is discussed in the next section.

C. RESPONSE FUNCTION OF A COMPOSITE PLASMA

In Sec. I-C we argued that the plasma can be described in terms of the classical theory of dielectrics. We shall now show that this model of a plasma can lead effortlessly to the calculation of the response function of any constituent of the plasma in the presence of the remaining constituents. Consider first a dielectric on which we have placed an external charge ρ^{ext} . The presence of the external charge polarizes the dielectric, or in other words, induces a charge inside the dielectric. The potential inside the dielectric is given by Poisson's equation

$$\nabla^2 \phi(\bar{q}t) = -4\pi\rho^{\text{ind}}(\bar{q}t) - 4\pi\rho^{\text{ext}}(\bar{q}t) \quad (61a)$$

or in $\bar{k}\omega$ -space

$$k^2 \phi(\bar{k}\omega) = 4\pi\rho^{\text{ind}}(\bar{k}\omega) + 4\pi\rho^{\text{ext}}(\bar{k}\omega) \quad (61b)$$

From Eq. (17) of Part I,

$$\rho^{\text{ind}}(\bar{k}\omega) = -\frac{k^2}{4\pi} [\epsilon^L(\bar{k}\omega) - 1] \phi(\bar{k}\omega) \quad (62)$$

relating the induced charge to the induced potential and the Poisson equation, we can then obtain

$$k^2 \epsilon^L(\bar{k}\omega) \phi(\bar{k}\omega) = 4\pi\rho^{\text{ext}}(\bar{k}\omega) \quad (63)$$

the generalization of Poisson's equation in matter and for a space-dispersive medium. Suppose now that the medium consists of a number of different charged particles. It is evident that the external charge brought to the dielectric from outside will induce different charges on each of the constituents. If each of the constituents were to be described by a dielectric constant of its own, then we could write

$$\rho_j^{\text{ind}}(\bar{k}\omega) = -\frac{k^2}{4\pi} [\epsilon_j^L(\bar{k}\omega) - 1] \phi_j(\bar{k}\omega) \quad (64)$$

where the subscript j denotes the j^{th} constituent. It is clear, however, that the potential in the system has to adjust itself so that it will be common to all constituents; therefore, $\phi_j(\bar{k}\omega) = \phi(\bar{k}\omega)$. Since the total induced charge has to be the sum of the individual induced charges, we can write, for the Poisson equation in $\bar{k}\omega$ -space,

$$k^2 \phi(\bar{k}\omega) = -k^2 \sum_{j=1}^r [\epsilon_j^L(\bar{k}\omega) - 1] \phi(\bar{k}\omega) + 4\pi\rho^{\text{ext}}(\bar{k}\omega) \quad (65)$$

In an electrostatic notation we could further write

$$\left(- \sum_{j=0}^r C_j(\bar{k}\omega) \right) \varphi(\bar{k}\omega) = \rho^{\text{ext}}(\bar{k}\omega) \quad , \quad (66a)$$

where

$$C_0(k\omega) = -\frac{k^2}{4\pi}$$

$$C_j(k\omega) = -\frac{k^2}{4\pi} [\epsilon_j^L(\bar{k}\omega) - 1] \quad , \quad j \neq 0 \quad . \quad (66b)$$

The above discussion suggests that the composite plasma can be described as a bank of "capacitors" driven by a charge source. The equivalent network of the composite plasma is shown in Fig. 5. This corresponds to the physical situation where, for example, an electron is shot at a volume containing a plasma. It can also correspond to a situation in which neutral molecules in the plasma are photo-ionized; in this case ρ^{ext} corresponds to one electron and one positive ion beginning their existence at some point in the plasma and moving with relatively high velocities.

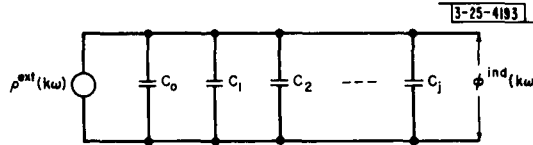


Fig. 5. The equivalent circuit of a multicomponent plasma driven by a charge source.

The above description, however, is not adequate for the study of fluctuations in any of the components of the plasma. We have seen in Sec. I-C that for this problem we need to know the induced charge on the component in question when an external potential is acting on this component. Had the network of Fig. 5 been a real electric network, then we could compute the desired response by making a "plier's entry" at the j^{th} branch. Introducing a potential at this entry, we could calculate the charge induced on the j^{th} capacitor, and therefore the response function. This is equivalent to the calculations that would have been required to solve the system of equations of Sec. II-B. The network for the calculation of the response to an applied potential is shown in Fig. 6. It is obvious that we can use this network for the calculation of any desired transfer function, that is, the response at the i^{th} capacitor which is the consequence of an applied potential through a plier's entry at the j^{th} branch.

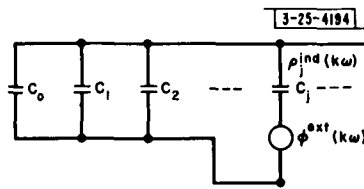


Fig. 6. The equivalent circuit of a multicomponent plasma driven by an external potential.

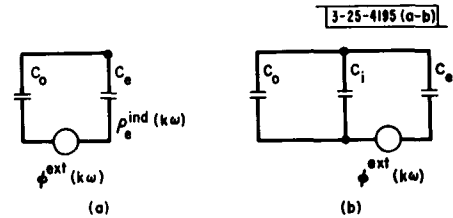


Fig. 7. The equivalent circuit for: (a) electrons in a positive sea and (b) electrons and ions.

As a first application, consider a plasma with electrons in a positive sea. The equivalent network is shown in Fig. 7(a-b). Since the capacitors are in series, we have

$$\rho_e^{\text{ind}}(\bar{k}\omega) = \frac{C_0 C_e}{C_0 + C_e} \varphi^{\text{ext}}(\bar{k}\omega) = C(\bar{k}\omega) \varphi^{\text{ext}}(\bar{k}\omega) \quad .$$

Introducing the values of the C's from Eq. (66b), we have

$$C_e(\bar{k}\omega) = -\frac{k^2}{4\pi} \frac{\epsilon_e^{L(\bar{k}\omega)} - 1}{\epsilon_e^{L(\bar{k}\omega)}} \quad , \quad (67)$$

which is the same as the value obtained in Part I, Eq. (20). The second application concerns the electron-ion plasma. From the circuit of Fig. 7(b) we obtain

$$C_{ee}(\bar{k}\omega) = \frac{(C_0 + C_i) C_e}{C_0 + C_i + C_e} \quad (68)$$

$$= -\frac{k^2}{4\pi} \frac{\epsilon_i^{L(\bar{k}\omega)} [\epsilon_e^{L(\bar{k}\omega)} - 1]}{\epsilon_i^{L(\bar{k}\omega)} + \epsilon_e^{L(\bar{k}\omega)} - 1} \quad . \quad (69)$$

This is the response function required to compute the fluctuation spectrum of the electrons in the presence of ions and is pertinent to the incoherent scattering from the ionospheric plasma. Proceeding with the same technique, we find that the response of a plasma with different types of ions is

$$C_{ee}(k\omega) = -\frac{k^2}{4\pi} \frac{1 - r + \left(\sum_{j=1}^r \epsilon_j^{L(\bar{k}\omega)} \right) \cdot (\epsilon_e^{L(\bar{k}\omega)} - 1)}{\epsilon_e^{L(\bar{k}\omega)} + \left(\sum_{j=1}^r \epsilon_j^{L(\bar{k}\omega)} \right) - r} \quad , \quad (70)$$

where r is the number of types of ions. In a similar way, we can determine the transfer response function for a two-component plasma when the potential is applied at the C_e branch and the response is measured at the C_i branch. The result is

$$C_{e,i}^t(k\omega) = -\frac{k^2}{4\pi} \frac{[\epsilon_e^{L(\bar{k}\omega)} - 1] [\epsilon_i^{L(\bar{k}\omega)} - 1]}{\epsilon_e^{L(\bar{k}\omega)} + \epsilon_i^{L(\bar{k}\omega)} - 1} \quad . \quad (71)$$

We can also find, by inspection, the total response, that is, the total charge induced when the applied potential acts on C_e and C_i . The result is

$$C_{e+i}(k\omega) = \frac{k^2}{4\pi} \frac{\epsilon_e^{L(\bar{k}\omega)} + \epsilon_i^{L(\bar{k}\omega)} - 2}{\epsilon_e^{L(\bar{k}\omega)} + \epsilon_i^{L(\bar{k}\omega)} - 1} \quad . \quad (72)$$

This completes the calculations for the derivation of a few of the response functions required to describe a plasma. These functions can be put to immediate use for the calculation of the mean-square fluctuations in a plasma. Referring to Sec. I-E, we find that the spectral quality of the fluctuations is given by

$$S(\bar{k}\omega) = \frac{\kappa T}{\pi} \frac{1}{\omega} \text{Im} \{-C(k\omega)\} \quad , \quad (73)$$

where $C(k\omega)$ can be any of the functions we have derived in this section. Since, as a result of the Kramer-Kronig relations, the integrated spectrum is given by the real part of $C(k\omega)$ evaluated at $\omega = 0$,

$$\int S(\bar{k}\omega) d\omega = \kappa T \operatorname{Re} \{-C(k0)\} \quad , \quad (74)$$

we can compute the integrated density fluctuation of any component with elementary algebra. In fact, referring again to Part I, Eq. (25), we have

$$\epsilon_j^L(k0) = 1 + (1/k\lambda_{Dj})^2 \quad , \quad (75)$$

where now

$$\lambda_{Dj}^2 = \frac{\kappa T_j}{4\pi e^2 n_j} \quad . \quad (76)$$

In a multicomponent plasma we must have

$$\sum_{j=1}^r n_j = n_0$$

so that the total number of the different kinds of ions is equal to the number of electrons. If negative ions are to be included, we must impose the condition,

$$\sum_{j=1}^r n_{j(+)} = n_0 + \sum_{j=1}^s n_{j(-)} \quad .$$

Using the above results in Eq. (69) we readily obtain

$$|\delta n_{ee}(\bar{k})|_{T_e=T_i}^2 = n_0 \frac{1 + k^2 \lambda_D^2}{2 + k^2 \lambda_D^2} \quad , \quad (77)$$

which is the familiar result of the density fluctuations in an electron-ion plasma. With equal ease we find the fluctuation correlation between electrons and ions from the transfer response function. For equal temperatures it has the value

$$|\delta n_e \delta n_i^*| = n_0 \frac{1}{2 + k^2 \lambda_D^2} \quad . \quad (78)$$

Finally, the total density fluctuation is

$$|\delta n_{tot}|^2 = n_0 \frac{2k^2 \lambda_D^2}{2 + k^2 \lambda_D^2} \quad . \quad (79)$$

Many more results can be computed (for example, $|\delta n_i(\bar{k})|^2$) but these are left as exercises for the reader. The spectral shape of the fluctuations can be computed from the response functions, but the calculations are involved because of the many variables. In Fig. 8 we present a three-dimensional plot of the fluctuation spectrum of the ion-electron plasma when the temperatures

of the two constituents are the same. This plot represents the value of $S(\bar{k}\omega)$ in Eq. (73) when $C(k\omega)$ is given by Eq. (69).

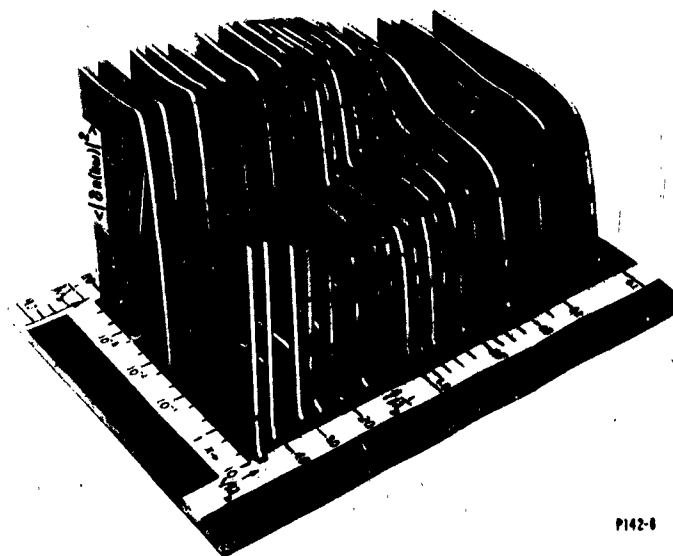


Fig. 8. The fluctuation spectrum of an ion-electron plasma as a function of $x = \omega/kv_D$. The parameter is $1/k\lambda_D$, where k is the wave number and λ_D is the Debye length.

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