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RM-3413-PR
JANUARY 1963

DATA TRANSMISSION ERRORS: COMPARISON OF LF RADIO WITH TELEPHONE FACILITIES

P. Mertz

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PREFACE

This Memorandum compares estimates of probable error rates in an IF radio link with similar figures for data transmission over telephone facilities. It should result in better understanding of the comparisons between radio and wireline data circuit performance.

SUMMARY

In this Memorandum the expected error distribution in the data system, coming from the LF atmospheric noise, illustrates a double type of distribution which has been encountered before. This approaches a purely random or Poisson type of distribution for shorter intervals of time, but approaches a "bunched" or hyperbolic type of distribution over the long run.

The specifications on error performance appear at first to be generous and lenient, but they really describe a data transmission system which is of quite high grade. This is primarily because the tolerances allow only a small interval of time during which the performance can be relatively poor.

The effect is enhanced also by the longer-time distribution approaching the hyperbolic law. This gives a greater ratio than a purely random distribution would, between peak and average incidence of errors. The long-time curve fits in pretty well, with curves obtained from data on a variety of telephone facilities. It differs from them, first in following a much more systematic seasonal and hour-of-the-day trend, and second in not conforming so well to a strict hyperbolic law.

CONTENTS

LIST OF FIGURES

I. INTRODUCTION

A preliminary report by Crichlow, Disney and Davis[1] outlines the general engineering of a special LF radio data transmission link, particularly from the standpoint of atmospheric noise.

The report examines the error performance which is specified for this link, and on the basis of data on LF atmospheric noise collected in the past years, computes the transmitter radio power emission required to ensure this specified performance.

In the course of the analysis the authors examine in some detail the effect of the atmospheric noise in causing errors in the system. It is the objective of the present Memorandum to compare the radio statistical methods of analyzing error occurrences with those which have been discussed for general data transmission over telephone facilities, and which have led to the various "hyperbolic" distribution laws.[2]

The radio methods of specifying noise and error occurrences concentrate essentially on very short periods of nearly worst performance. The methods used in telephone facilities, on the other hand, lean towards characterizations of average performance. Thus to compare the two it is necessary to study the statistical distributions of the noise and the errors over substantial periods of time.

From this it is possible to translate figures given in the one art into figures in the other.

Such a translation does not, of course, answer the question of how much noise one finds in a radio facility as compared with a general telephone facility. There is so much variation in each of these that every case is to be considered individually, and only the broad method of doing this is treated. Hence specific parameters of the radio data system or of the telephone facility data systems that are mentioned are used only illustratively for the comparisons, and are not meant as fixed specifications for the systems.

II. SYSTEM PARAMETERS

The data transmission parameters in the system considered are:

Radio carrier frequency	150 kc (LF)
Bits per sec	1.52
Bandwidth	8 cps
Modulation	FM
Five-minute period	8 messages
One message	57 bits

The system is very slow, as data systems go. The bandwidth allowance is obviously quite generous.

The error tolerance is stated in this way:

- At least one error-free message must be received within 5 min with a probability of 99 per cent.
- The performance must be at least this good during 99 per cent of the noisiest season and the time of day (worst 3 months of year, and worst 8 hr of each day in these 3 months).

The error tolerance has been translated into a tolerance on bit errors (with the assumption that the bit errors are independent) in Ref. 1. The details of this translation are reproduced in Appendix A.

The result is that, during 1 per cent of the noisiest season (and time of day) the bit error rate averages 1460 errors per 10^5 bits, to yield a failed 5 min period with a probability of 1 per cent.

On casual inspection, as has been said, this seems a very lenient performance requirement. When it is analyzed further, however, it will appear much stricter.

III. HOURLY AND SEASONAL DISTRIBUTIONS OF LF ATMOSPHERIC NOISE

The atmospheric noise in the LF band has been studied in some detail over the years. It has been found to show a strong systematic trend in intensity over the hours of the day, and also over the months of the year.

An illustration of the general variation in intensity over the day is shown in Fig. 1. This is plotted from recordings[3] made at Boulder, Colorado of mean atmospheric noise power at 113 kc, over the hours of the day during January and July, respectively, in the years 1957 to 1959.

It is convenient to consider the hours between 8 PM and 4 AM as the night-time "noisy hours" during the daily period. Engineering requirements met during this period are, therefore, safe during the less noisy remainder of the day.

Similarly, an illustration of the general variation in intensity of the atmospheric noise over the year[3] appears in Fig. 2. Here the median of the noise power during the night-time hours is plotted separately from the corresponding median during the day-time hours.

It is noted in Fig. 2 that the atmospheric noise is most intense during the summer, and therefore the months of June, July and August are again separated out for engineering purposes. The net "noisiest season" is then taken as the night-time period during those three months. Since one-third of the day is involved, the total period is, therefore, one cumulated month, or one-twelfth (or 8.33 per cent) of the year.

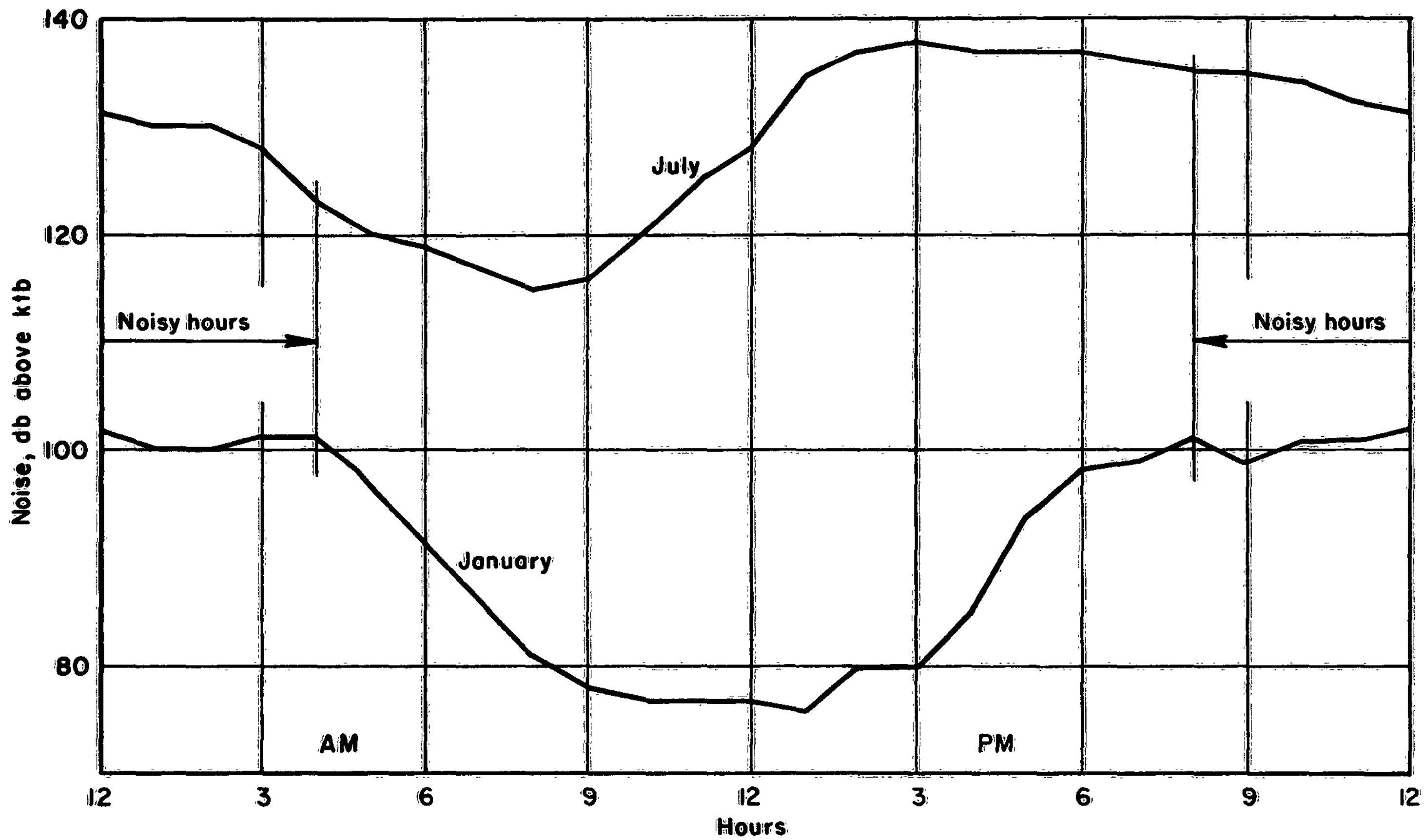

Fig.1 — Diurnal variation in noise intensity
113 kc, Boulder, Colorado

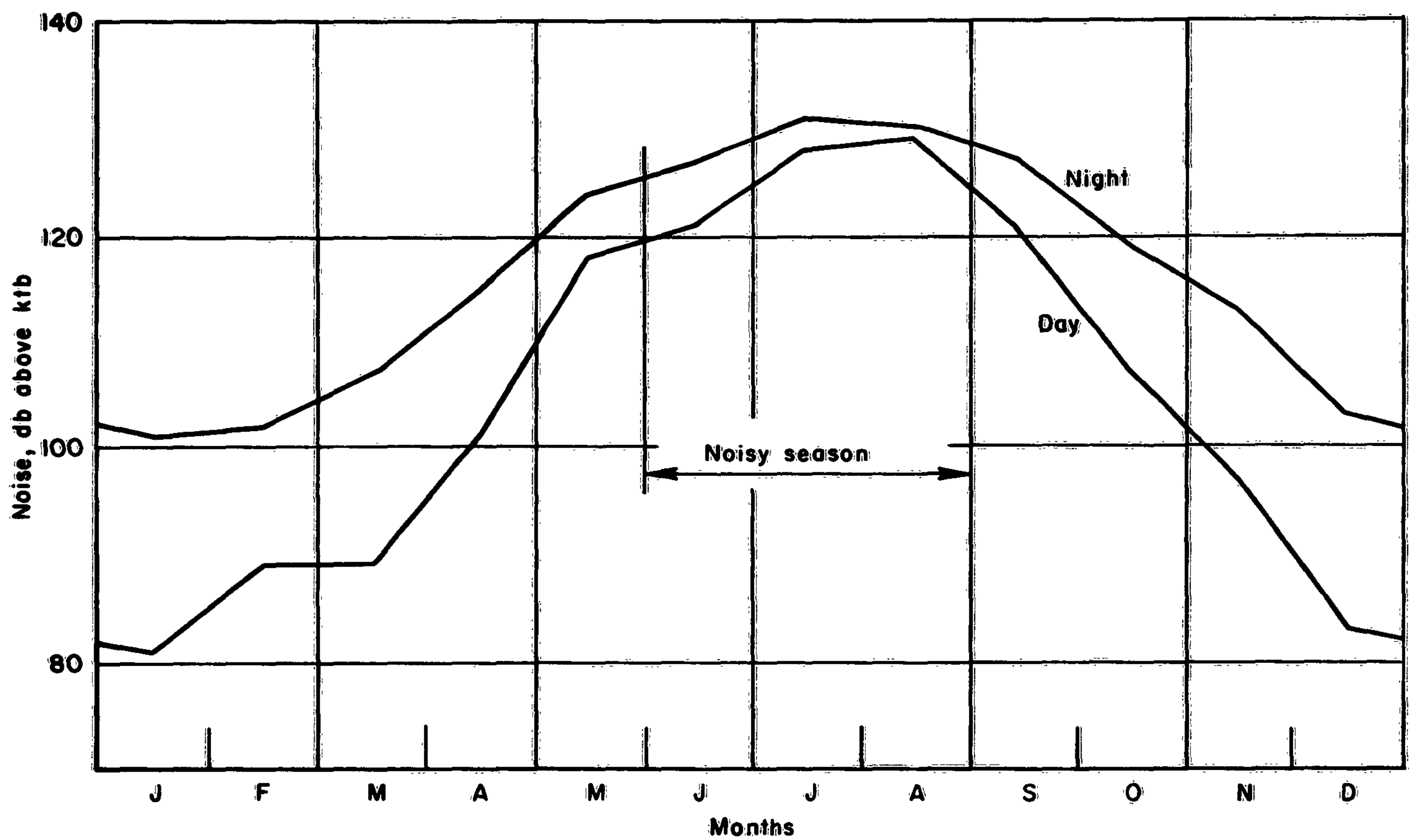

Fig.2—Yearly variation in intensity of atmospheric noise
113 kc, Boulder, Colorado

The ordinates plotted in Figs. 1 and 2 are the mean power averaged over a period of several minutes, expressed in db above kTb. This last is the thermal noise power obtainable from a noiseless (of itself) and loss-less antenna circuit, and is computed from k, as Boltzmann's constant, T as absolute temperature in ^{o}K, and b as the frequency bandwidth (150 cycles used in the actual meter).

The frequency (113 kc) is somewhat different from the 150 kc for the link under consideration. There are procedures for converting the data, but since the figures are only used here illustratively this has not been done. Also, for the purposes of Ref. 1, data were averaged from five stations around the world, including the Canal Zone and Singapore. For simplicity, Figs. 1 and 2 cover only the station at Boulder.

The reader could infer from Figs. 1 and 2 that it might be more useful to choose different boundaries from those illustrated for the noisy hours and the noisy season. Those boundaries have actually been chosen, however, on the basis of a much wider experience, and are fairly well established at the present time.

IV. SHORT-TIME NOISE DISTRIBUTION

The data of Figs. 1 and 2 give only a median value for the average noise power. At any given time the current atmospheric noise may be greater or less than this. The percentages of time that the noise level is higher were presented in Fig. 3 of Ref. 1, which has been reproduced as Fig. 3 herewith. These data were given for 150 kc, the frequency used for this study. They cover specifically the "noisiest season" of the summer night-time hours.

In Ref. 1 there was then a further figure, to indicate the short-time (down to bit-by-bit) distribution of the atmospheric noise intensity. This was Fig. 1 in Ref. 1, and here it is reproduced as Fig. 4. Again it is specifically directed to the summer night-time hours, at 150 kc, and for an 8-cycle band.

The median level (50 per cent point) comes at 5 db below the average power level. This is merely a characteristic of the specific noise distribution.

At the left-hand side of the plot the percentages in the abscissae, as determined by the curve, have been brought over against the decibel scale. Therefore, for any given number of decibels above the average power, the percentage scale against it indicates how much of the time that given level is exceeded. This and the particular point singled out will be used in estimating error occurrences.

It will be noted that an unconventional probability scale is used for the abscissae of Fig. 4. This is a scale proposed by Gumbel[4] particularly in connection with extremal statistics. The

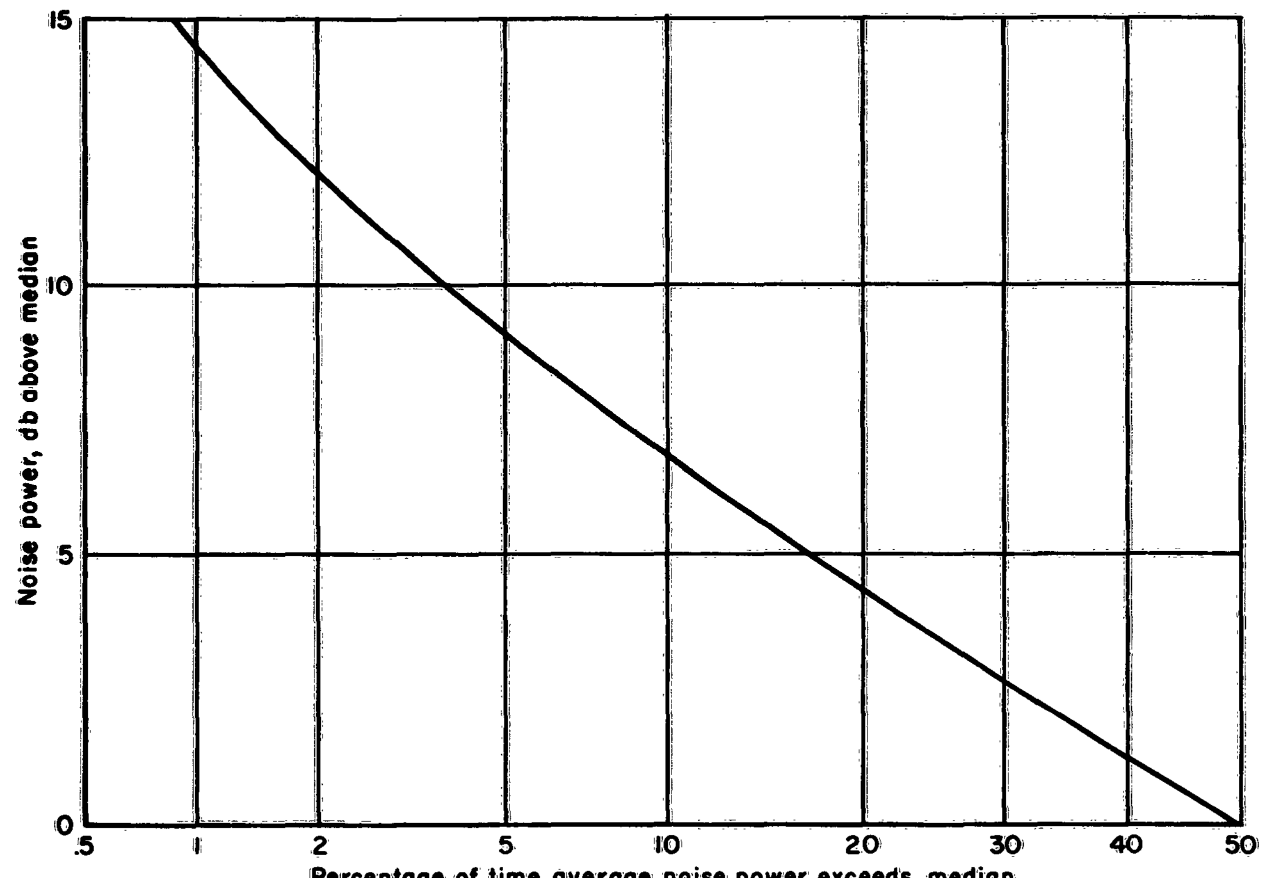

Fig. 3—Noise intensity distribution
150 kc, summer, nighttime

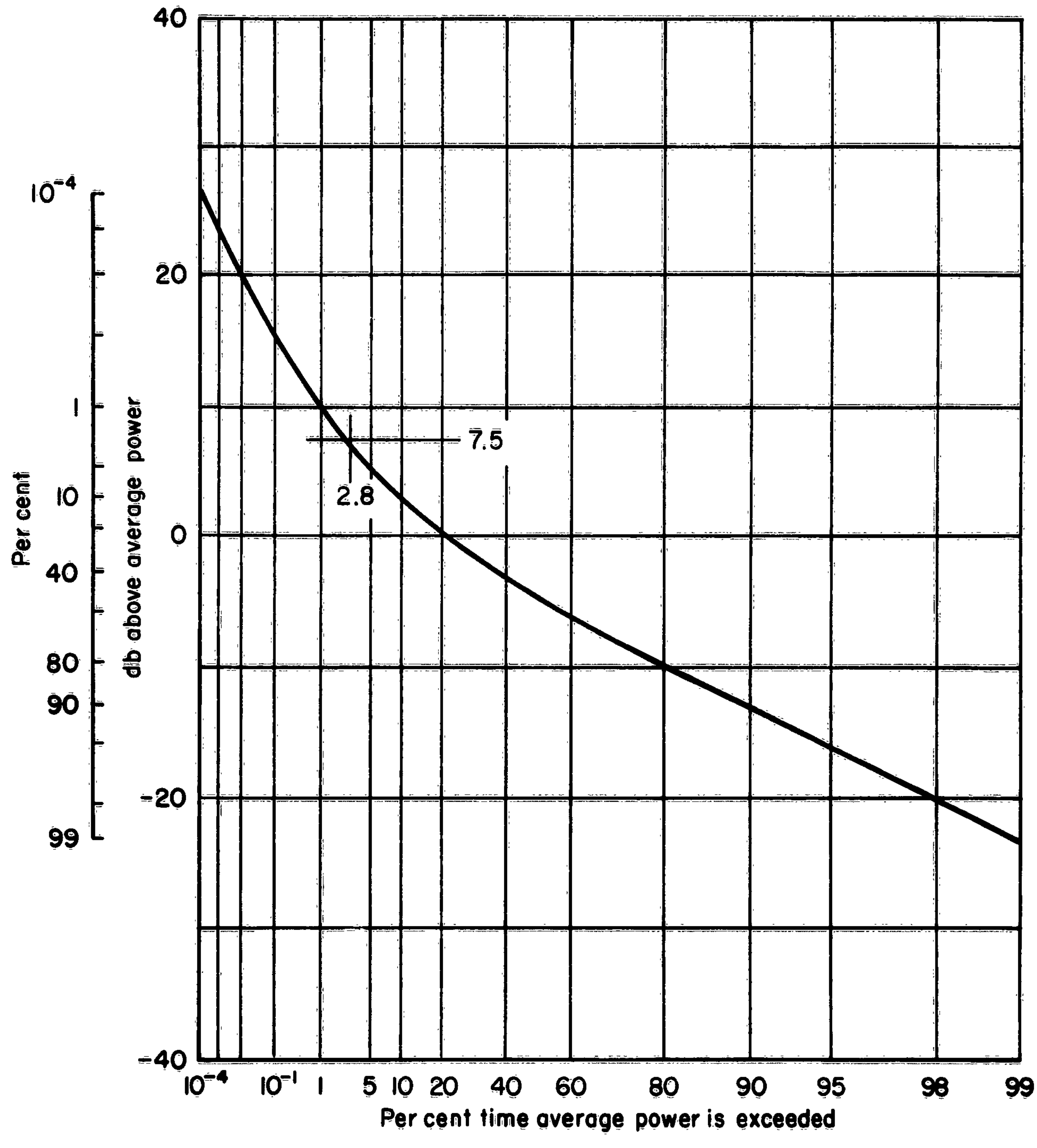

Fig.4— Short-term noise distribution

150 kc, summer, nighttime, 8 c/s bandwidth

scale is such that a Rayleigh distribution plots as a straight line. The right-hand portion of the curve in the figure (toward the higher percentages) is governed by a Rayleigh distribution, and hence approaches a straight line.

V. OVER-ALL NOISE DISTRIBUTION

It is possible to plot the distribution of Fig. 2 in terms of the fraction of the year during which a given atmospheric noise level is exceeded. This leads to curve A (noise) of Fig. 5, using the extreme left-hand ordinate Scale II. The abscissa scale is the conventional probability scale. Such a plot is of interest, but it is really desirable to examine the course of the curve at much shorter intervals of time than it covers.

The curve has been extended to these shorter intervals by the use of Fig. 3. The method is given in Appendix B. The extension is given by curve B of Fig. 5.

By a repetition of the application of this procedure it is possible to extend the distribution curve still further. This uses Fig. 4, and leads to the extended curve C of Fig. 5.

The noise amplitude distribution A-B-C in Fig. 5 then gives an approximate trend all the way from a period shorter than one bit (0.66 sec) to the whole of the year. Some various periods of interest involved are indicated in the scale of abscissae at the bottom of the plot.

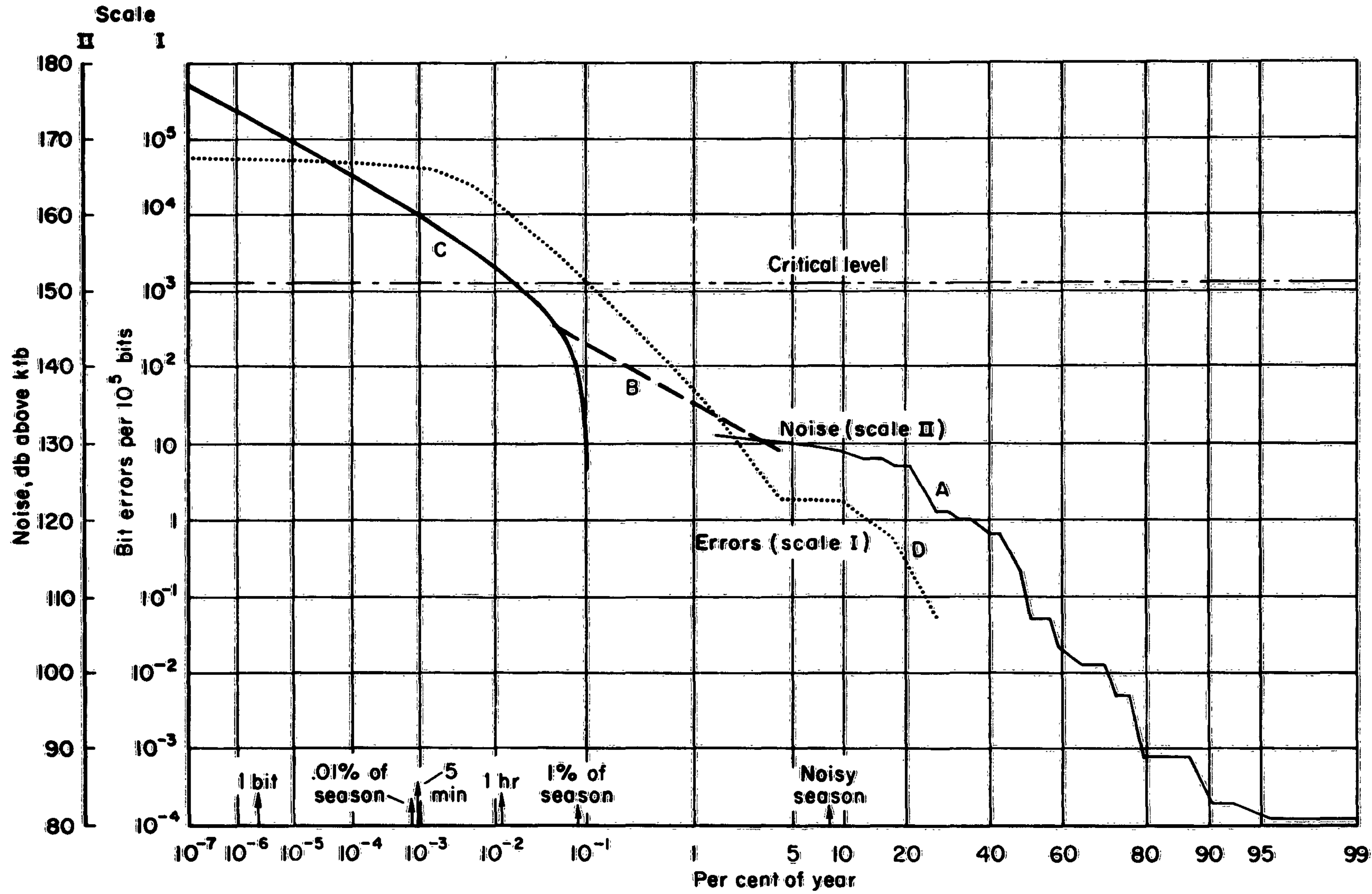

Fig. 5 — Overall noise and error incidence distributions

VI. TRANSLATION TO ERROR INCIDENCE

Such a distribution of noise amplitudes is, of course, of great interest but the data transmission engineer is more fundamentally interested in what these noise amplitudes do to his expected error rates. A translation from one to the other can be based on a secondary reference[5] quoted in Ref. 1, to the effect that the "errors in a binary narrow-band frequency modulation system can be considered as being one-half the probability of the noise envelope exceeding the carrier envelope."

The details of this translation are given in Appendix C. The results are plotted as the dotted curve D (errors) in Fig. 5, using the left-hand ordinate Scale I. This scale gives the error frequency in terms of bit errors per 10^5 bits. In the determination the "critical level" assumed is that of the carrier envelope, and is given as db above kTb in the receiving antenna circuit. Because of the various adjustments which have been ignored, as already mentioned, this comes out at 151 db; compared with 144 db obtained in Ref. 1. The final adjustment which is made in the critical level, however, is to secure the specified error rate (of not more than 1400 bit errors per 10^5 bits transmitted) for the worst 1 per cent of the noisy season. Thus the <u>relative</u> position of the critical level with respect to the noise distribution curve of Fig. 5 is the same, at this point, as in Ref. 1. Also, therefore, the bit error rate at this point is the same.

VII. ERROR INCIDENCE IN DATA COMMUNICATION OVER TELEPHONE FACILITIES

In the use of general telephone facilities for data transmission the experience has been that there is some trace of systematic tendency for error occurrences to vary through the day and through the week. It is, however, very much less pronounced than for the atmospheric noise as illustrated in Figs. 1 and 2; and it has generally been ignored in studying the distributions quantitatively.

The distribution of errors over long periods of time has been found to differ from what would be expected if the occurrences were purely random. The distribution is described in terms of the probability of finding at least c errors in a short time interval for which the long-time average occurrences is a. It has been found that in actual experience where the distribution covers a long enough time this can be fairly closely approximated by the equation of a hyperbola, i.e.,

$$P(a, c) = a/(Ac + a) \qquad (1)$$

where $P(a, c)$ = probability of at least c errors in time interval for which long-time average is a errors

A = a constant which depends upon the overall duration of the test, in terms of the time intervals above

(Where the distribution covers a short time--say under 1/2 hr or so--and where the incidence of errors is high, it seems to revert toward the Poisson form. This will be discussed later.)

Figure 6 shows some plots of actual error distributions observed over a number of experimental tests, each of several weeks to a month duration. These have been published in Lincoln Laboratory reports,[6-9] and are:

Test A. Kingston-Canaveral tests (1960) ($a = 23.2/10^5$)

Test B. Kingston-Canaveral tests (1958) - Circuit A ($a = 11.6/10^5$)

Test C. Kingston-Canaveral tests (1958) - Circuit B ($a = 9.5/10^5$)

Test D. Lexington-South Truro test (1959) - Circuit A ($a = 0.61/10^5$)

Test E. Lexington-South Truro test (1959) - Circuit B ($a = 0.86/10^5$)

Test G. Hawaiian Cable tests (1960) ($a = 0.355/10^5$)

Tests A, B, and C were on long-haul telephone facilities (1500 mi) and were carried out with equipment which had been designed for shorter-haul work, and over what were considered fair, but not excellent, circuits. Tests D and E were over comparatively short-haul telephone facilities (125 mi) which were considered quite good. Test G was over what was considered an excellent facility. While long (5000-mi loop), the submarine cable is well isolated from most noise sources and shows great freedom from errors. For these tests the percentage in the abscissae is that of the total test, and not that of the whole year.

The plot of Fig. 6 is on "hyperbolic" probability paper, which has been designed so that the hyperbolic distribution of Eq. (1) plots as a straight line[2] with a slope of -1. While this slope is not always exactly met, particularly in case G, a fair approximation is obtained for the plots.

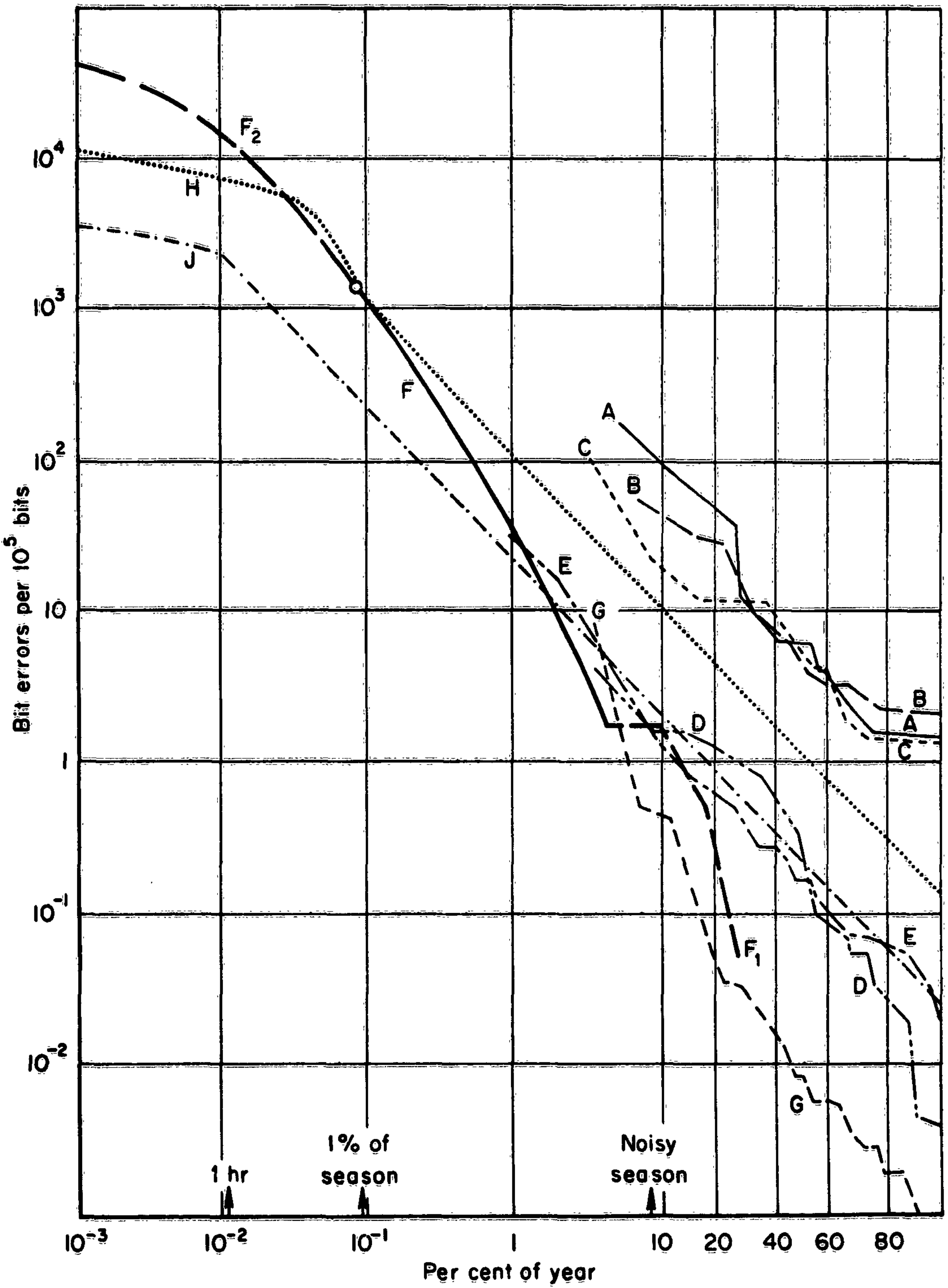

Fig. 6 — Experimental results — measured performance characteristics of various communications circuits — comparison with estimates from atmospheric noise for LF radio link

VIII. COMPARISON OF LF WITH TELEPHONE FACILITIES

Curve D of Fig. 5 has then been transferred as curve F onto Fig. 6 (and here the abscissa percentage is that of the whole year). The portion of curve F that is most certain (see Appendix C) is shown as a solid line. Less certain portions (marked F_1 and F_2) are shown as dashed lines.

The first remark that occurs to one in the comparison of curve F with the others in Fig. 6 is that it describes a facility which ranks well with these others. It is roughly comparable with the Lexington-South Truro and Hawaiian cable circuits, and distinctly better than the Kingston-Canaveral circuits.

It is true that for the time during which the specifications are really critical the error performance allowed is quite poor. But the specifications have been placed on a very small fraction of the time when the performance is nearly at its worst. This is a smaller proportion of the time than was even plotted for the other facilities. When the average over the year is taken, the times of better performance improve the figure considerably, according to the estimated trend of the distribution, and show the specified facility to be of really quite good quality.

This points to a moral which can well be taken to heart in the engineering of data transmission systems. It is that the user is much more concerned with the performance of the system when it is close to failure than when it is operating properly. If he places what seem to be quite generous tolerances on the performance during such times, these can nevertheless by their infrequency specify a system that is really excellent during the greater portion of its operating period. Thus the engineer must concentrate his own attention on the design for these infrequent and nearly worst periods of operation, rather than on a long-term average of performance. This, of course, was done in Ref. 1.

In the plot of Fig. 6 the situation is exaggerated by the fact that the estimated error distribution curve F has a slope that is somewhat steeper than the -1 which corresponds to a strictly hyperbolic law. Thus in going toward the infrequent occurrences it rises rather fast to high error incidence levels.

IX. ANALYTIC APPROXIMATIONS

The plots H and J of Fig. 6 refer to approximations by formulas to the error distributions coming during the short and very poor periods of operation which are described by the specifications.

It has been found that, over telephone facilities, although a given error distribution may for the most part follow a hyperbolic law, there is a tendency during shorter periods of high error incidence for the distribution to veer toward the completely random or Poisson type.[10] This seems to occur also in the case of the LF atmospheric noise. It is indicated in Fig. 6 by the rapidly diminishing slope of curve F toward its upper left-hand end. Of course, the diminishing slope here results from the fact that even with the noise level 100 per cent of the time above the critical level, the error probability is only 0.5, or $5 \times 10^4/10^5$ bits. Crichlow[11] notes, "During periods of several minutes to about an hour, the statistical properties of atmospheric noise remain essentially constant," and beyond these periods, that the statistics change.

Curve H is a plot of the bit error probability obtained through Eq. (5) of Appendix A. In Ref. 1 the specifications were used on the assumption that the errors, during the period involved, were independent. Combinations of bit errors and also of message errors could then be computed using simple probability formulas (as reproduced in Appendix A). These combinations, in the form described by the specifications, were then themselves combined with actual short-time noise amplitude probabilities as plotted in Fig. 4, to bridge the gap between the 5-min interval and the 1 per cent of the noisy season.

The bridging is illustrated analytically by curve H of Fig. 6. The analysis is given in the first part of Appendix A. Curve H is then continued on toward larger fractions of the yearly cycle on the assumption of a hyperbolic law--namely, it is a straight line with a slope of -1.

It is also possible to analyze the marginal 5-min period coming 1 per cent of 1 per cent of the noisy season. (This is actually 4.4 min, but conceptually can be extended to 5 min at the same error rate.) The analysis[10] is on the basis of the longest expected error-free period in the 5 min, assuming a purely Poisson distribution. Again following the Poisson law, this can be extended to 1 hr. It gives curve J of Fig. 6, and the details are explained in Appendix A. The curve is extended to periods beyond 1 hr on the assumption of a hyperbolic law, namely a straight line with slope -1 (as for curve H).

In comparing curves H and J with F one notes that they are both below curve F_2. This suggests that the atmospheric noise pulses during this period do not follow an exact Poisson distribution, but follow some compromise between the Poisson and the hyperbolic distribution.

One further notes that neither curves H nor J, in their hyperbolic extensions, approximate the estimated experimental curve F as well as would be desired. Curve H is not too bad around 0.1 per cent of the year, but it is too high around 10 to 30 per cent of the year. Curve J is reasonably good around 1 to 20 per cent of the year, but too low around 0.01 to 0.1 per cent. Also, even though lower than H, it is still too high around 30 per cent of the year.

The conclusion is reached that the estimated error frequency curve F can be only roughly approximated by combinations of Poisson and hyperbolic laws, over the great time range that extends between the short period during which error performance is specified, and the full year. Of course, it is to be remembered that curve F itself is only a rough approximation, particularly at both its extreme ends.

APPENDIX A

In the critical 5-min period, which is just failing, Ref. 1 computes the relation between the requirements imposed and the bit error probability p_b. It is assumed that the errors are independent. Below is essentially a copy of the derivation:

The probability p_m of a 57-bit message in error is

$$p_m = 1 - (1 - p_b)^{57} \tag{2}$$

The probability p_u of a 5-min period (having 8 messages) with no correct messages (i.e., a failed 5-min period) is

$$p_u = p_m^8 \tag{3}$$

The probability p_s of a 5-min period with at least one correct message is

$$p_s = 1 - p_u = 1 - p_m^8 \tag{4}$$

The condition placed in Section II is that p_s at least equal 0.99, or p_u be no greater than 0.01. Thus

$$0.01 \geq p_u = (1 - (1 - p_b)^{57})^8 \tag{5}$$

Equation (3) is plotted at A in Fig. 7, and Eq. (5) as B in the same figure.

Equation (5) is expected to hold during 1 per cent of the one cumulated month noisy season. This gives an implicit solution for p_b, namely

$$p_b = 0.01460 \tag{6}$$

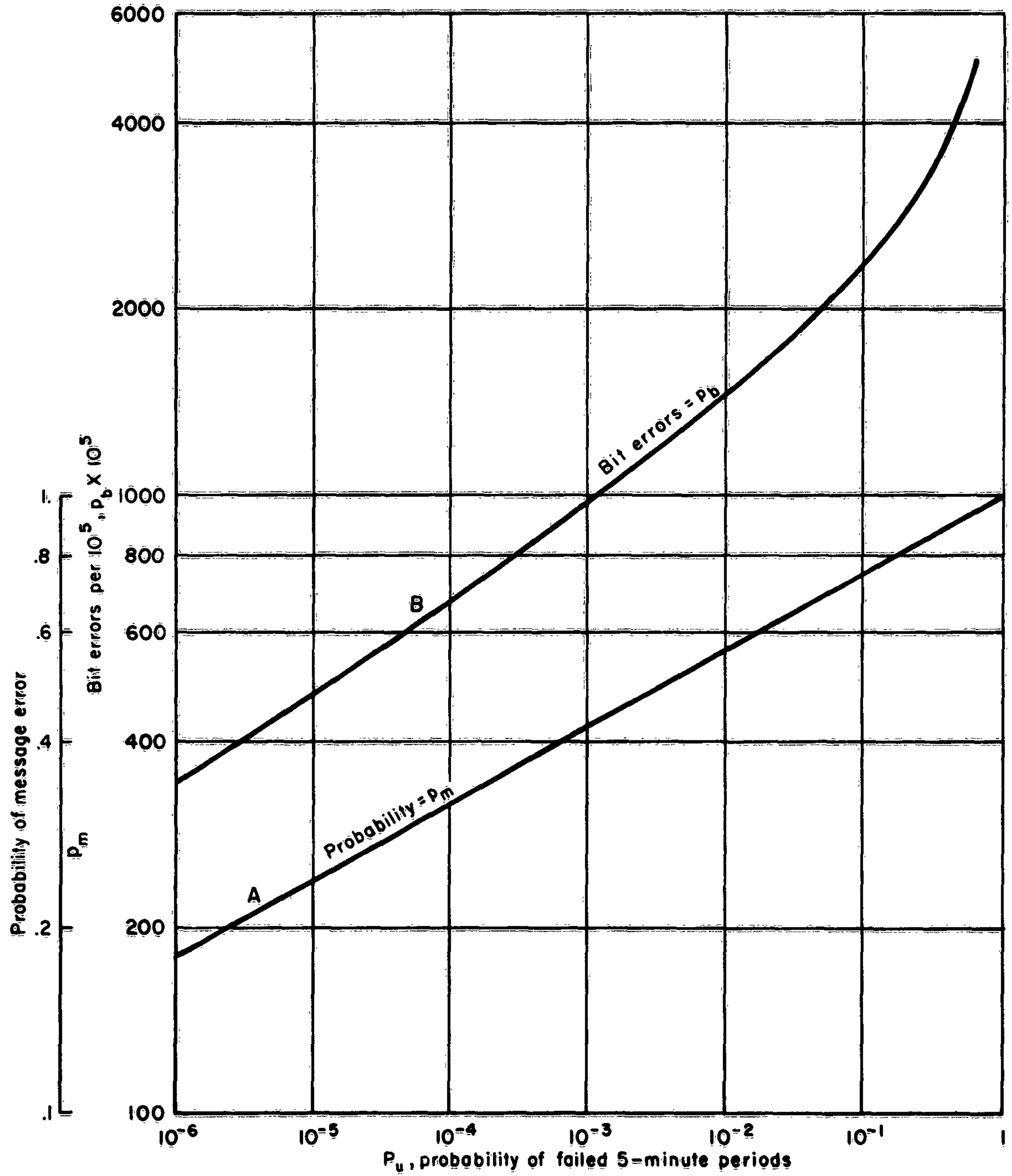

Fig. 7—Relationship between requirements and bit-error probability

$$p_b = 1460 \text{ bit errors}/10^5 \text{ bits} \qquad (7)$$

In Ref. 1 p_b is rounded to 0.014; and doubled to give the fraction of time the noise is to be permitted to exceed the critical level, or 2.8 per cent. At the single marked point in Fig. 4, it indicates that the critical level needs to be 7.5 db above the average noise power.

It is to be noticed that if p_s is taken as equal to 0.01 (or 0.00084 per cent of the entire year), then the implicit solution for p_b comes out

$$p_b = 11{,}100 \text{ bit errors}/10^5 \text{ bits} \qquad (8)$$

This is used for the left-hand extremity of curve H in Fig. 6 (though it plots just a little off scale).

It is also possible to compute the expected bit error rate during the worst 5 min by a technique described in Ref. 10. It is there noted that the longest error-free interval u_o in a Poisson distribution of duration T and with average error rate a (a, T, and u_o being measured in the same time units) is given by the implicit relation

$$1 = - aT \, Ei \, (- au_o) \qquad (9)$$

where Ei(x) is the exponential integral function tabulated by Jahnke and Emde.[11]

The message length is 57 bits. If u_o be taken as 56 bits, then during an interval of 5 min no correct message will be received. The time can be measured in bits, to give T = 8 x 57 = 456 bits.

The solution to Eq. (9) is then*

$$a = 0.0311 \text{ bit errors/bit} \tag{10}$$

$$= 3110 \text{ bit errors}/10^5 \text{ bits} \tag{11}$$

This is used for the left-hand extremity of curve J in Fig. 6. It is extended to 1 hr by the use of the Campbell[13] curves. For this purpose the 5 min is used as a unit of time, and the number of errors during it is taken as 456a = 14.2 errors. The average number of errors expected in the 1-hr period comes out 9.35 per 5-min period, or 2,050 bit errors per 10^5 bits.

*Revisions being considered for Ref. 10 would lead to a = .0585 bit errors/bit. This would raise curve J somewhat in Fig. 6.

APPENDIX B

The distribution as given is plotted as the bar diagram in Fig. 8. The left-most bar shown is that from 0 to 0.1, and the point characterizing it is plotted at its mid-value, namely at 0.05.

If a finer structure distribution is then available, showing how the values are arrayed in the region between 0 and 0.1, it is possible to subdivide the lowest bar. This is indicated in Fig. 9. Here the subdivision into 5 sub-bars, each covering 1/50th of the main distribution, is shown. The lowest point can then be placed at 0.01.

This procedure is exact if the region of validity of the sub-distribution is exactly delineated as in the interval 0 to 0.1, and not encroaching in the next higher interval, and if averages, or medians, are used consistently throughout. In the application used in this Memorandum these conditions are not completely met and, therefore, the results are only approximate.

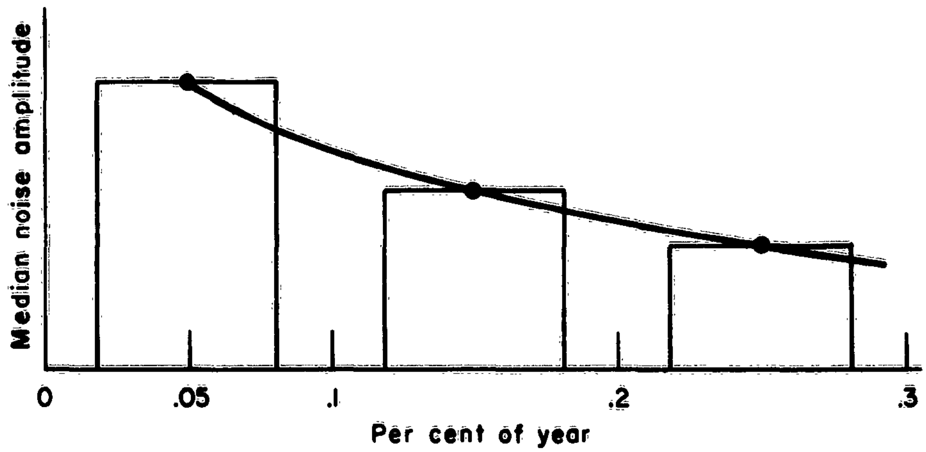

Fig. 8— Longer-term noise distribution (Appendix B)

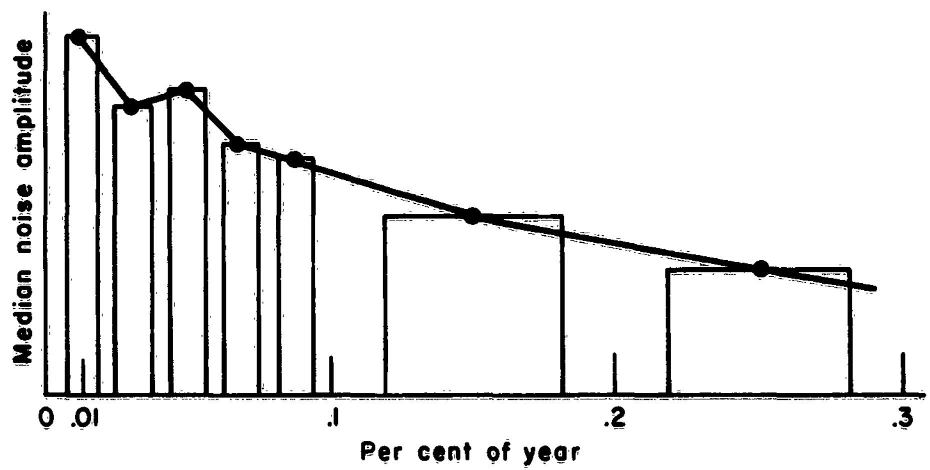

Fig. 9— Short—term noise distribution, obtained from distribution of variations above and below longer-term median level

APPENDIX C

In Fig. 10 the smoothed noise level is plotted as a distribution. By "smoothed" it is meant that it holds for long periods of time compared with one bit.

The instantaneous noise level fluctuates above or below this smoothed value, at intervals of the order of one bit duration. The proportion of the time that the smoothed level, plus the fluctuations, cross to above the critical level is a measure of the error occurrence. According to Ref. 5, with an FM signal errors occur half this much of the time when the critical level is taken as the carrier level.

The vertical scale showing the short-time noise level fluctuations, plotted at the left-hand side of Fig. 4, is transferred to Fig. 10. For each abscissa of Fig. 10, the percentage on this vertical scale corresponding to "0 db above average power" in Fig. 4 is adjusted to the ordinate of the "smoothed noise level" of Fig. 10. In the illustration it is at about 21 per cent.

The percentage at which the critical level crosses the vertical scale indicates how often the short-time noise level reaches the critical level. Half this figure indicates the expected error frequency.

The smoothed noise level is the curve A-B-C of Fig. 5. The critical level is obtained from the point 2.8 per cent, 7.5 db of Fig. 4. This is added to the 1 per cent, 14.5 db of Fig. 3, which is further added to the mean noisy season noise level in Fig. 2 of

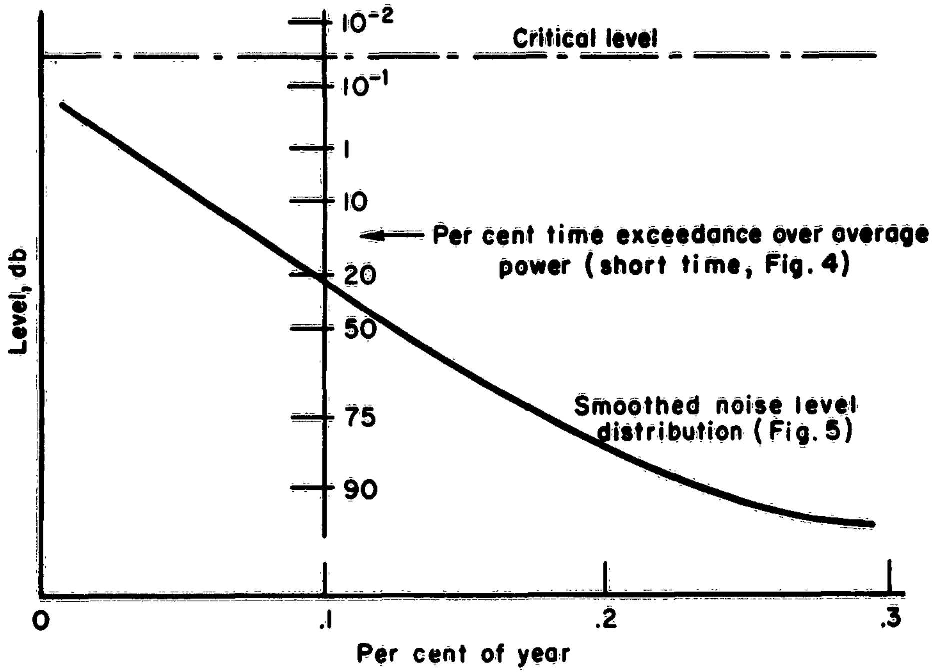

Fig. 10 — Conversion from noise level to error incidence, for Fig. 5

129 db, giving 129 + 14.5 + 7.5 = 151 db. (In Ref. 1 the 129 db is 122 db, giving a critical level of 144 db.)

The procedure is most accurate when essentially using the curves of Fig. 4 and Fig. 3 (namely, curve B of Fig. 5). This is the portion drawn as a solid line in Fig. 6.

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