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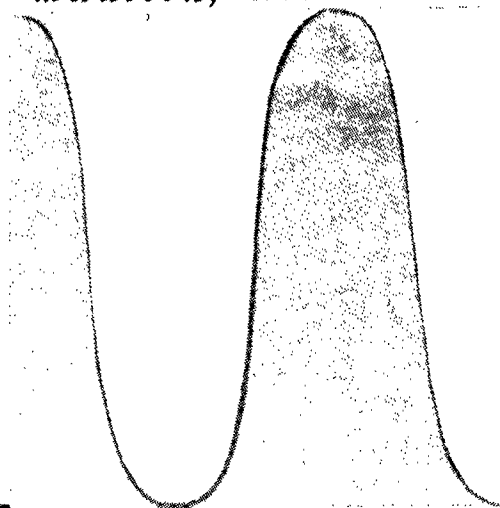
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CONJUGATE FUNCTIONS IN ORLICZ SPACES

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ABSTRACT

The purpose of this report is to show that the mapping of a function on the unit circle into its conjugate is a bounded operation in an Orlicz space if and only if the Orlicz space is reflexive.

CONJUGATE FUNCTIONS IN ORLICZ SPACES

Robert Ryan

1. The purpose of this paper is to prove the following results:

Theorem 1. Let $\widetilde{f}(x) = -\frac{1}{\pi} \int_0^{\pi} \frac{f(x+t) - f(x-t)}{2 \tan \frac{1}{2} t} dt = \lim_{\epsilon \rightarrow +0} \left\{ \frac{1}{\pi} \int_{\epsilon}^{\pi} \right\}.$

The mapping $f \rightarrow \widetilde{f}$ is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.

Beginning with the classical result by M. Riesz for the L_p spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund's book [6; vol. II, pp. 116-118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of $f \rightarrow \widetilde{f}$. Our results are more general than Lozinski's results since we use the definitions of an Orlicz space given by A. C. Zaanen [5] which includes, for example, the space L_1 .

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Section 2 contains preliminary material about Orlicz spaces.

In Section 3 we prove that boundedness implies reflexivity and in

Section 4 we prove the converse.

2. Let $v = \varphi(u)$ be a non-decreasing real valued function defined for $u \geq 0$. Assume that $\varphi(0) = 0$, that φ is left continuous and that that φ does not vanish identically. Let $u = \psi(v)$ be the left continuous inverse of φ . If $\lim_{u \rightarrow \infty} \varphi(u) = l$ is finite then $\psi(v) = \infty$ for $v > l$; otherwise $\psi(v)$ is finite for all $v \geq 0$. The complementary Young's functions Φ and Ψ are defined by

$$\Phi(u) = \int_0^u \varphi(t) dt, \quad \Psi(v) = \int_0^v \psi(s) ds.$$

Φ is an absolutely continuous convex function for $0 \leq u < \infty$ and Ψ is absolutely continuous and convex in the interval where it is finite. If $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ this interval is $0 \leq v < \infty$. If $\lim_{u \rightarrow \infty} \varphi(u) = l$ is finite we say that Ψ jumps to infinity at $v = l$.

Φ is said to satisfy the Δ_2 -condition if there is a constant $k > 0$ and a $u_0 \geq 0$ such that $\Phi(2u) \leq k \Phi(u)$ for $u \geq u_0$. This is equivalent to satisfying the inequality $\Phi(lu) \leq kl \Phi(u)$ for all sufficiently large u , where l is any number greater than one (for a proof and further details see [1; p. 23]).

The Orlicz space $L_\Phi = L_\Phi(0, 2\pi)$ consists, by definition, of all measurable complex functions f defined on the unit circle for which $\|f\|_\Phi = \sup \int_0^{2\pi} |f(t)g(t)| dt < \infty$, where the supremum is taken over all functions g with $\int_0^{2\pi} \Psi|g(t)| dt \leq 1$. The space L_Ψ is defined by interchanging Φ and Ψ . The Orlicz space $L_{M\Phi}$ is defined to be the set of all measurable complex functions f for which $\|f\|_{M\Phi} = \sup \int_0^{2\pi} |f(t)g(t)| dt < \infty$, where the supremum is taken over all g with $\|g\|_\Psi \leq 1$. $L_{M\Psi}$ is similarly defined. The spaces L_Φ , L_Ψ , $L_{M\Phi}$ and $L_{M\Psi}$ are all Banach spaces with their respective norms when functions equal almost everywhere are identified.

The spaces L_Φ and $L_{M\Phi}$ consist of the same functions and

$$\|f\|_{M\Phi} \leq \|f\|_\Phi \leq 2\|f\|_{M\Phi}.$$

The same is true replacing Φ by Ψ .

The space L_Φ is reflexive with dual space $L_{M\Psi}$ if and only if both Φ and Ψ satisfy the Δ_2 -condition.

Two Young's functions Φ_1 and Φ_2 are said to be equivalent ($\Phi_1 \sim \Phi_2$) if and only if there exist positive constants k_1, k_2 , and u_0 such that $\Phi_1(k_1 u) \leq \Phi_2(u) \leq \Phi_1(k_2 u)$ for $u \geq u_0$. It is clear that $\sim$ is an equivalence relation and that the Δ_2 -condition is an equivalence

class property. If $\Phi_1 \sim \Phi_2$ then L_{Φ_1} and L_{Φ_2} consist of the same functions and the norms $\| \cdot \|_{\Phi_1}$ and $\| \cdot \|_{\Phi_2}$ are equivalent. Conversely, if L_{Φ_1} and L_{Φ_2} have the same elements then $\Phi_1 \sim \Phi_2$ [1; p. 112].

3. In this section we will show that if $f \rightarrow \tilde{f}$ is bounded then L_{Φ} is reflexive. Let $S_n(f)$ denote the n^{th} partial sum of the Fourier series of f and write $D_n(t) = \sin(n + \frac{1}{2})t / 2 \sin \frac{1}{2}t$. If $\|\tilde{f}\|_{\Phi} \leq C \|f\|_{\Phi}$ for all $f \in L_{\Phi}$ then it follows [6; vol. I, p. 266] that $\|S_n(f)\|_{\Phi} \leq A \|f\|_{\Phi}$ for all $f \in L_{\Phi}$ and all n , where A is a positive constant independent of n and f . Thus, the following result is ostensibly more general than the corresponding part of Theorem 1.

Theorem 2. If $\|S_n(f)\|_{\Phi} \leq A \|f\|_{\Phi}$ for all $f \in L_{\Phi}$ and all n then L_{Φ} is reflexive.

The proof of Theorem 2 uses the following two lemmas given by S. Lozinski in [3]. Lozinski proved these lemmas under more restrictive conditions on φ than we have assumed. Nevertheless, Lozinski's proofs remain valid for the functions as we have defined them.

Lemma 1. $\frac{\varphi(u)}{250} \log \frac{n}{u\varphi(u)} \leq \|D_n\|_{\Phi}$ for $u\varphi(u) \geq 1$.

Lemma 2. If $\|S_n(f)\|_{\Phi} \leq A \|f\|_{\Phi}$ for all $f \in L_{\Phi}$ and all n then $\|D_n\|_{\Phi} \leq 2\pi A \frac{n + \Phi(u)}{u}$ for $0 < u < \infty$.

Proof of Theorem 2. Our proof is a variation of the one given by Lozinski in [4]. From Lemmas 1 and 2 we have

$$(1) \quad \varphi(v) \log \frac{n}{v\varphi(v)} \leq k \frac{n + \Phi(u)}{u}$$

for $v\varphi(v) \geq 1$ and $0 < u < \infty$. $k = \frac{2\pi A}{250}$. Our immediate aim is to show that for all sufficiently large $\lambda > 1$.

$$(2) \quad \log \left(\frac{\lambda}{2} \right) \leq 2k \frac{\varphi(v)}{\varphi\left(\frac{v}{\lambda}\right)}$$

for $v \geq v_0$, where v_0 depends upon λ .

$$\text{For any } \lambda > 1, \Phi(u) = \int_0^u \varphi(t) dt > \int_{u/\lambda}^u \varphi(t) dt$$

$$\text{and hence } \Phi(u) > \left(u - \frac{u}{\lambda}\right) \varphi\left(\frac{u}{\lambda}\right) = (\lambda - 1) \frac{u}{\lambda} \varphi\left(\frac{u}{\lambda}\right).$$

Thus

$$(3) \quad \log \frac{(\lambda - 1)n}{\Phi(v)} < \log \frac{n}{\frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right)}.$$

By combining (3) and (1) we see that

$$(4) \quad \varphi\left(\frac{v}{\lambda}\right) \log \frac{(\lambda-1)n}{\Phi(v)} \leq k \frac{n + \Phi(v)}{v}$$

whenever $\frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right) \geq 1$. Let $n = [\Phi(v)] =$ greatest integer in $\Phi(v)$.

Then (4) becomes

$$(5) \quad \varphi\left(\frac{v}{\lambda}\right) \log \left\{ (\lambda-1) \frac{[\Phi(v)]}{\Phi(v)} \right\} \leq k \frac{[\Phi(v)] + \Phi(v)}{v} \leq 2k \frac{\Phi(v)}{v}.$$

For every sufficiently large λ there exist a $v_0 \geq 0$ such that for $v \geq v_0$

$$(6) \quad 1 < \frac{\lambda}{2} \leq (\lambda-1) \frac{[\Phi(v)]}{\Phi(v)} \quad \text{and}$$

$$(7) \quad \frac{v}{\lambda} \varphi\left(\frac{v}{\lambda}\right) \geq 1.$$

Using (5), (6) and the fact that $\Phi(v) \leq v\varphi(v)$ we get inequality (2)

for $v \geq v_0$. Since λ can be arbitrarily large (2) implies that

$\lim_{u \rightarrow \infty} \varphi(u) = \infty$ and hence that Ψ does not jump to infinity. We

next show that Ψ satisfies the Δ_2 -condition.

Let λ be large but fixed and write $l = \frac{1}{2k} \log \left(\frac{\lambda}{2}\right)$. Then

(2) states that

$$(8) \quad l\varphi\left(\frac{t}{\lambda}\right) \leq \varphi(t)$$

for $t \geq v_0$.

This implies, on taking inverses, that there is a number s_0 such that for $s \geq s_0$

$$(9) \quad \psi(s) \leq \lambda \psi\left(\frac{s}{\ell}\right).$$

$$\text{Thus } \int_{s_0}^v \psi(s) ds \leq \lambda \int_{s_0}^v \psi\left(\frac{s}{\ell}\right) ds = \lambda \ell \int_{\frac{s_0}{\ell}}^{\frac{v}{\ell}} \psi(s) ds \quad \text{or}$$

$$(10) \quad \Psi(v) - \Psi(s_0) \leq \lambda \ell \left[\Psi\left(\frac{v}{\ell}\right) - \Psi\left(\frac{s_0}{\ell}\right) \right].$$

This shows that for sufficiently large v

$$(11) \quad \Psi(\ell v) \leq 2\lambda \ell \Psi(v)$$

and hence proves that Ψ satisfies the Δ_2 -condition.

If $\|s_n(f)\|_{\Phi} \leq A\|f\|_{\Phi}$ for all $f \in L_{\Phi}$ then it follows that $\|s_n(g)\|_{M\Psi} \leq A\|g\|_{M\Psi}$ for all $g \in L_{M\Psi}$ or, equivalently, that $\|s_n(g)\|_{\Psi} \leq 2A\|g\|_{\Psi}$ for all $g \in L_{\Psi}$. Since we have shown that Ψ does not jump to ∞ we can interchange the rôle of Φ and Ψ in the above argument to show that Φ satisfies the Δ_2 -condition. This proves that L_{Φ} is reflexive and completes the proof of Theorem 2.

4. In this section we prove a general result about reflexive Orlicz spaces which combined with the classical results of M. Riesz [6; vol. I, p. 256 and p. 266] yields the unproved half of Theorem 1 as well as the converse of Theorem 2.

Theorem 3. Suppose that T is a bounded linear operator on L_p into L_p for $1 < p < \infty$. Then if L_Φ is reflexive T is defined and bounded on L_Φ into L_Φ .

Proof. The proof consists of showing that Φ can be replaced by an equivalent function Φ_1 ($\Phi \sim \Phi_1$) such that Φ_1 satisfies the conditions of the Marcinkiewicz theorem on the interpolation of operations i.e. such that

$$(12) \quad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt = O\left\{\frac{\Phi(u)}{u^\beta}\right\} \quad \text{and}$$

$$(13) \quad \int_1^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt = O\left\{\frac{\Phi(u)}{u^\alpha}\right\}$$

for $u \rightarrow \infty$, where $1 < \alpha < \beta < \infty$.

The assumption that L_Φ is reflexive implies that $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ and hence that $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$. By [1; p. 16] Φ is equal for sufficiently

large values of u to a function M of the form $M(u) = \int_0^u p(t) dt$

where p is a non-decreasing right continuous function with

$\lim_{u \rightarrow 0} p(u) = 0$ and $\lim_{u \rightarrow \infty} p(u) = \infty$. Clearly $\Phi \sim M$.

By [1; p. 46] the function M_1 defined by $M_1(u) = \int_0^u \frac{M(t)}{t} dt$

is equivalent to M and hence to Φ . The derivative of M_1 is continuous and strictly increasing.

Since L_Φ is reflexive both Φ and Ψ satisfy the Δ_2 -condition.

Thus both M_1 and its conjugate Young's function N_1 satisfy the

Δ_2 -condition [1; p. 23]. According to [1; pp. 26-27] this implies

the existence of numbers a, b , and $u_0 \geq 0$ with $1 < a < b < \infty$ such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b$$

for all $u \geq u_0$. If we define Φ_1 by

$$\Phi_1(u) = \begin{cases} \frac{M_1(u_0)}{u_0^a} u^a & \text{for } u \leq u_0 \\ M_1(u) & \text{for } u \geq u_0 \end{cases}$$

we obtain a function $\Phi_1 \sim \Phi$ such that

$$(14) \quad 1 < a \leq \frac{u\varphi_1(u)}{\Phi_1(u)} \leq b$$

for all $u \geq 0$.

We next show that Φ_1 satisfies (12) and (13) for suitably chosen a and β . In particular choose a and β such that $1 < a < a \leq b < \beta < \infty$. This is clearly possible. In what follows all of the integrals will exist as finite numbers because of (14).

Integration by parts shows that

$$(15) \quad \int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt = \beta \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt - \frac{\Phi_1(u)}{u^\beta} \quad \text{and}$$

$$(16) \quad \int_0^u \frac{\varphi_1(t)}{t^a} dt = a \int_0^u \frac{\Phi_1(t)}{t^{a+1}} dt + \frac{\Phi_1(u)}{u^a}.$$

From (14) we obtain

$$(17) \quad \int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt \leq b \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt \quad \text{and}$$

$$(18) \quad \int_0^u \frac{\varphi_1(t)}{t^a} dt \geq a \int_0^u \frac{\Phi_1(t)}{t^{a+1}} dt.$$

Combining (15) with (17) and (16) with (18) shows that

$$(19) \quad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt \leq \frac{1}{\beta - b} \left\{ \frac{\Phi_1(u)}{u^\beta} \right\} \quad \text{and}$$

$$(20) \quad \int_0^u \frac{\Phi_1(t)}{t^{a+1}} dt \leq \frac{1}{a - a} \left\{ \frac{\Phi_1(u)}{u^a} \right\}.$$

This shows that Φ_1 satisfies (12) and (13). Thus by the Marcinkiewicz theorem and Theorem 10.14 of [6; vol I, p.174] there exists a constant K_1 such that $\|Tf\|_{\Phi_1} \leq K_1 \|f\|_{\Phi_1}$ for all $f \in L_{\Phi_1}$. Since $\Phi \sim \Phi_1$ there is a constant K such that $\|Tf\|_{\Phi} \leq K \|f\|_{\Phi}$ for all $f \in L_{\Phi}$. This completes the proof of Theorem 3.

Statements of the standard corollaries of Theorem 1 can be found in [2].

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