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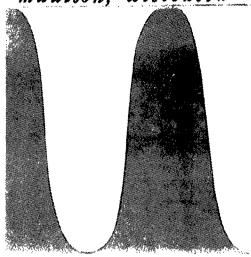*by the*

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SOME THEOREMS CONCERNING

SLOWLY VARYING FUNCTIONS

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ABSTRACT

Conditions are given for $f(t)$ and the regularly varying function $\varphi(x) = x^\gamma L(x)$, so that the asymptotic relation

$$\int_0^\infty f(t)\, \varphi(xt)\, dt \sim \varphi(x) \int_0^\infty f(t)\, t^\gamma\, dt \ .$$

holds true as $x \to \infty$, or as $x \to +0$.

As a special case one obtains extensions of some results concerning the asymptotic behavior ot the Laplace transform of regularly varying functions.

SOME THEOREMS CONCERNING SLOWLY VARYING FUNCTIONS

J. Karamata[1]

1. Let $L(x)$ be a slowly varying function in the neighborhood of ∞, or of $+0$, i.e. such that $L(x) > 0$ for $x > 0$, and that, for every $t > 0$,

$$\frac{L(xt)}{L(x)} \to 1 \ , \tag{1}$$

as $x \to \infty$, or as $x \to +0$, respectively. We assume that $L(x)$ is continuous and every $x > 0$ (which is not absolutely necessary; see [1] and [2]).

It is well-known (see [3]) that for every $\eta > 0$,

$$x^{-\eta} L(x) \to 0, \quad \text{and} \quad x^{\eta} L(x) \to \infty \ . \tag{2}$$

and if we put

$$P(x) = \sup_{0 < t \leq x} \{ t^{\eta} L(t) \} \quad \text{and} \quad Q(x) = \sup_{t \geq x} \{ t^{-\eta} L(t) \} \ , \tag{3}$$

then we have

$$P(x) \sim x^{\eta} L(x) \quad \text{and} \quad Q(x) \sim x^{-\eta} L(x) \ , \tag{4}$$

as $x \to \infty$, or as $x \to +0$, respectively.

Both of these relations follow easily from the following fundamental property of slowly varying functions:

If the relation (1) holds true for each fixed $t > 0$, then it holds true

uniformly for $t \in [a,b]$, where $0 < a < b < \infty$, (see [1], [2] and [3]).

Let finally $f(t)$ be integrable in every finite interval and let

$$T(x) = \int_0^\infty f(t)\, L(xt)\, dt \ .$$

The study of the asymptotic behavior of $T(x)$, or more precisely of the

relation

$$T(x) \sim L(x) \int_0^\infty f(t)\, dt \ , \tag{5}$$

as $x \to \infty$, or as $x \to +0$, was the object of numerous papers; without

mentioning those dealing with special forms of the function $L(x)$, we remind

the reader that K. Knopp [4] considered Césàro, and Hölders transforms of order

k, which correspond to the case that $f(t) = k(1-t)^{k-1}\, t^\eta$, respectively

$f(t) = \dfrac{1}{\Gamma(k)} (\log 1/t)^{k-1}\, t^\eta$, $0 < t \leq 1$, $\eta > -1$, as $x \to \infty$. He also studied [4],

[5] Abel-Laplace transforms, that is the case that

$$f(t) = e^{-t}\, t^\eta, \ t > 0, \ \eta > -1 \ ,$$

as $x \to +\infty$, while G. Doetsch [6] considered the same case for $x \to +0$.

However, the main argument of the proof is not clearly elaborated by the latter.

Finally the general case of the relation (5) was stated by S. Aljančić, R. Bojanić

and M. Tomić [7], [8]. We shall give here simpler proofs of their results, as

well as some additional remarks.

<u>2.</u> To this end we observe first that the asymptotic behavior of $T(x)$, as

$x \rightarrow \infty$, <u>or</u> as $x \rightarrow +0$, solely depends on the behavior of $f(t)$ in the neighborhood

of both $t = +0$ <u>and</u> $t = \infty$, since the assumption (1) implies the uniform

convergence of (1) in every finite interval $[a,b]$, $a > 0$. However, it

follows from the assumption (1) only, in view of Fatou's lemma, that

$$\liminf_{x \rightarrow \infty, \text{ or } x \rightarrow +0} \frac{1}{L(x)} \int_0^\infty f(t) \, L(xt) \, dt \geq \int_0^\infty f(t) \, dt \; , \tag{6}$$

if $f(t) \geq 0$ for every $t \geq 0$. From this observance one easily derives the

following theorem.

$\underline{\text{Theorem 1.}}$ Let $L(x)$ be a slowly varying function in the neighborhood of

∞ , or $+0$, and let $F(t)$ be non-negative for $t \geq 0$, integrable in $[0,\infty]$ and

such that

$$\int_0^\infty F(t) \, L(xt) \, dt \sim L(x) \int_0^\infty F(t) \, dt \; , \tag{7}$$

as $x \rightarrow \infty$, or as $x \rightarrow +0$; if

$$|f(t)| \leq F(t) \quad \text{for } t \geq 0 \; ,$$

then

$$\int_0^\infty f(t) \, L(xt) \, dt \sim L(x) \int_0^\infty f(t) \, dt \; , \tag{8}$$

as $x \rightarrow \infty$, or as $x \rightarrow +0$, respectively.

<u>Proof.</u>   From the assumption $|f| \leq F$ follows that

$$F(t) \pm f(t) \geq 0 \quad \text{for} \quad t \geq 0 \quad ;$$

therefore, by (6),

$$\int_0^\infty F(t)\, L(xt)\, dt \pm \int_0^\infty f(t)\, L(x)\, dt \geq$$

$$\geq L(x) \int_0^\infty F(t)\, dt \pm L(x) \int_0^\infty f(t) + o(L(x)) \quad ;$$

this implies, in view of the assumption (7), that

$$\pm \int_0^\infty f(t)\, L(xt)\, dt \geq \pm L(x) \int_0^\infty f(t)\, dt + o(L(x)) \quad ,$$

as $x \to \infty$, or as $x \to +0$, respectively, which completes the proof of (8) .

<u>Theorem 2.</u>   Let $L(x)$ be a slowly varying function in the neighborhood of ∞, or $+0$, and let $f(t)$ be nonequative for $t \geq 0$, integrable in $[0, \infty]$ and such that for some $\eta_0 > 0$ we have both

$$\int_0^a f(t)\, t^{-\eta_0}\, dt < \infty \quad \text{and} \quad \int_a^\infty f(t)\, t^{\eta_0}\, dt < \infty \quad . \tag{9}$$

Then

$$\int_0^\infty f(t)\, L(xt)\, dt \sim L(x) \int_0^\infty f(t)\, dt \quad , \tag{10}$$

holds as $x \to \infty$, or as $x \to +0$, respectively.

Proof. Let $0 < \eta \leq \eta_0$ and $a > 0$, then (3) and (4) imply that

$$\int_0^\infty f(t)\, L(xt)\, dt = x^{-\eta} \int_0^a f(t)\, t^{-\eta} \{(xt)^\eta L(xt)\}\, dt + x^\eta \int_a^\infty f(t)\, t^\eta \{(xt)^{-\eta} L(xt)\}\, dt$$

$$\leq x^{-\eta} P(ax) \int_0^a f(t)\, t^{-\eta}\, dt + x^\eta Q(ax) \int_a^\infty f(t)\, t^\eta\, dt =$$

$$\leq \{a^\eta \int_0^a f(t)\, t^{-\eta}\, dt + a^{-\eta} \int_a^\infty f(t)\, t^\eta\, dt\}\, L(x) + o(L(x)) \ .$$

Finally, taking $\eta \to 0$, one gets

$$T(x) \leq L(x) \int_0^\infty f(t)\, dt + o(L(x))$$

as $x \to \infty$, or as $x \to +0$, respectively, which together with (6), completes the proof of (10) .

It should be stressed that both assumptions (9) are indispensable in each of the cases $x \to \infty$ and $x \to +0$.

In fact, if e.g. $x \to \infty$, and if

$$f(t) = \ell(t)/t \ ,$$

where $\ell(t)$ is a slowly varying function as $t \to +0$, such that the integral

$$\int_{+0} \frac{\ell(t)}{t}\, dt \quad \text{converges} \ ,$$

then the function

$$\ell^*(t) = \int_0^{1/x} \frac{\ell(t)}{t}\, dt$$

tends to 0 and is a slowly varying function as $x \to \infty$, (see [3]). In this case it suffices to let

$$L(x) = \begin{cases} l^*(1) & \text{for } 0 \le x < 1 \ , \\[2mm] \dfrac{l^*(x)}{1 + \log x} & \text{for } \quad x \ge 1 \ . \end{cases}$$

Then we have

$$\frac{1}{L(x)} \int_0^\infty f(t)\, L(xt)\, dt \to \infty, \quad \text{as } x \to \infty \ ,$$

since

$$\frac{1}{L(x)} \int_0^\infty f(t)\, L(xt)\, dt \ge \frac{l^*(1)}{L(x)} \int_0^{1/x} f(t)\, dt = l^*(1)(1 + \log x) \ .$$

An immediate consequence of the theorems 1 and 2 is the following theorem.

__Theorem 3.__   Let $\alpha < \beta$ and $t^\eta f(t)$ be integrable in $[0, \infty]$ for $\alpha < \eta < \beta$, i.e.

$$\int_0^\infty t^\eta \, |f(t)|\, dt < \infty \quad \text{for } \alpha < \eta < \beta \ ; \tag{11}$$

let $\varphi(x)$ be a regularly varying function in the neighborhood of ∞, or of $+0$, i.e. such that,

$$\varphi(x) \sim x^\gamma L(x) \ ,$$

as $x \to \infty$, or as $x \to +0$. Then we have for every $\alpha < \gamma < \beta$,

$$\int_0^\infty f(t)\, \varphi(xt)\, dt \sim \varphi(x) \int_0^\infty f(t)\, t^\gamma dt \;,$$

as $x \to \infty$, or as $x \to +0$, respectively. In general the theorem is not true if $\gamma = \alpha$, or $\gamma = \beta$, even if the integral (11) converges for $\eta = \alpha$, or $\eta = \beta$.

3. Let us return to the above mentioned theorems of G. Doetsch and K. Knopp mentioned above concerning the Laplace transform

$$\Phi(s) = \int_0^\infty e^{-st}\, \varphi(t)\, dt$$

of the regularly varying function

$$\varphi(x) = x^\gamma L(x) \;.$$

For $s = 1/x$, x positive and real, one obtains

$$\Phi(s) = x \int_0^\infty e^{-t}\, \varphi(xt)\, dt = x^{\gamma+1} \int_0^\infty e^{-t}\, t^\gamma\, L(xt)\, dt \;,$$

Hence, the theorems of Doetsch and Knopp follow from theorem 3, with

$$f(t) = e^{-t}\, t^\gamma \;.$$

More precisely, if $\varphi(x) = x^\gamma L(x)$ is regularly varying in the neighborhood of ∞, or $+0$, we have

$$\Phi(s) = \int_0^\infty e^{-st}\, \varphi(t)\, dt \sim \frac{\Gamma(\gamma+1)}{s}\, \varphi(1/s) \;,$$

for every $\gamma > -1$, as $s \to +0$ (theorem of Knopp) or as $s \to \infty$ (theorem of Doetsch).

One can complete these results by extending the concept of regularly varying functions to the complex domain as follows.

Let $\Phi(s)$ be holomorphic and $\neq 0$ in the domain D: $|\text{arc } s| < \frac{\pi}{2}$, $s \neq 0$. Call $\Phi(s)$ regularly varying in the Stolz's angle if, for every positive real number $\lambda > 0$, the limit

$$(\text{St}) \lim \Phi(\lambda s)/\Phi(s) = A_\lambda$$

exists as $s \to 0$, or as $s \to \infty$, in the Stolz's angle, i.e. if for every $\epsilon > 0$ and $0 < \alpha < \frac{\pi}{2}$, there is $\eta_{\epsilon,\alpha} > 0$ such that

$$|s| < \eta_{\epsilon,\alpha} \text{ and } |\text{arc } s| < \alpha \implies |\frac{\Phi(\lambda s)}{\Phi(s)} - A_\lambda| < \epsilon ,$$

or

$$|s| > \eta_{\epsilon,\alpha} \text{ and } |\text{arc } s| < \alpha \implies |\frac{\Phi(\lambda s)}{\Phi(s)} - A_\lambda| < \epsilon ,$$

respectively.

The study of properties of regularly varying functions of a complex variable will be the object of another report. Here we shall only mention certain results concerning the Laplace transform of a regularly varying function defined on the real axis.

If $\Phi(s)$ is regularly varying in the Stolz's angle, then there exists a complex constant γ such that

$$\Phi(s) = s^\gamma L(s), \quad s \in D ,$$

where the function L(s) is slowly varying in the Stolz's angle, i.e. such

that for every positive real λ

$$(St) \lim \; L(\lambda s)/L(s) = 1 \; .$$

Moreover, for every S(s) such that

$$(St) \lim S(s)/s = \zeta, \quad |arc \; \zeta| < \frac{\pi}{2} \; ,$$

we have

$$(St) \lim \; \Phi(S)/\Phi(s) = \zeta^{\gamma} \; .$$

Using these results , one can strengthen the theorems of Doetsch and

Knopp as follows.

Let the function

$$\varphi(t) = t^{\gamma} \ell(t)$$

be defined on the real positive axis and be regularly varying as $t \to \infty$, or as

$t \to +0$. Then its Laplace transform

$$\Phi(s) = \int_0^{\infty} e^{-st} \varphi(t) \, dt \; ,$$

where $R\gamma > -1$, is regularly varying in the Stolz's angle in the neighborhood

of 0, or ∞ , respectively; moreover,

$$s^{\gamma+1} \; \Phi(s) \sim \Gamma(\gamma+1) \; \ell(1/ |s|) \; ,$$

as $s \to 0$, or as $s \to \infty$, in the Stolz's angle.

4. Theorem 3 can be extended to functions $f(t)$ which are not absolutely integrable in $[0,\infty]$ by suitably restricting the class of regularly varying functions. To this end we shall first prove the following lemmas.

Lemma 1. Let $L(x)$ be defined for $x > 0$ and such that

$$x^\eta L(x) \to 0 \quad \text{as} \quad x \to +0 \tag{12}$$

for some $\eta > 0$; assume also that

$$L(x) = L_1(x) L_2(x) \quad , \tag{13}$$

and that there exists a $\delta > 0$, such that

$$0 \le \varphi_1(x) \stackrel{\text{def}}{=\!=} x^\delta L_1(x) \quad \text{increases,} \tag{14}$$

and

$$0 \le \varphi_2(x) \stackrel{\text{def}}{=\!=} x^{-\delta} L_2(x) \quad \text{decreases,}$$

and that for every $\eta' > 0$,

$$x^{\eta'} L_1(x) \to 0 \quad \text{as} \quad x \to +0 \quad . \tag{15}$$

Then

$$\int_0^a \left| d\{t^\eta L(t)\} \right| \le a^\eta \{ 2\, \frac{\eta + \delta - \epsilon}{\eta - 2\epsilon}\, G_1(a)\, G_2(a) - L(a) \} \quad , \tag{16}$$

for every $a > 0$ and $0 < \epsilon < \eta/2$, where

$$x^\epsilon G_i(x) = \sup_{0 < t \le x} \{ t^\epsilon L_i(t) \}, \quad i = 1, 2 \quad . \tag{17}$$

Similarly, if (12) and (15) are replaced by

$$x^{-\eta} L(x) \to 0, \quad \text{as } x \to \infty \ , \tag{18}$$

for one $\eta > 0$, and by

$$x^{-\eta'} L_2(x) \to 0, \quad \text{as } x \to \infty \ , \tag{19}$$

for every $\eta' > 0$, then

$$\int_b^\infty |d\{t^{-\eta} L(t)\}| \le b^{-\eta} \{2 \frac{\eta + \delta - \epsilon}{\eta - 2\epsilon} H_1(b) H_2(b) - L(b) \}, \tag{20}$$

for every $b > 0$, where

$$x^{-\epsilon} H_i(x) = \sup_{t \ge x} \{t^{-\epsilon} L_i(t)\}, \qquad i = 1, 2 \ . \tag{21}$$

Proof.   By (13) and (14) ,

$$|d\{t^\eta L(t)\}| = |d\{t^\eta \varphi_1(t) \varphi_2(t)\}| \le \varphi_2 d\{t^\eta \varphi_1\} - t^\eta \varphi_1 d\varphi_2 =$$

$$\le 2 \varphi_2 d\{t^\eta \varphi_1\} - d\{t^\eta L(t)\} \ ,$$

therefore, by (12)

$$\int_0^a |d\{t^\eta L(t)\}| \le 2 \int_0^a \varphi_2(t) d\{t^\eta \varphi_1(t)\} - a^\eta L(a) \ . \tag{22}$$

On the other hand, by (15) and (17) ,

$$\int_0^a \varphi_2(t)\, d\{t^\eta\, \varphi_1(t)\} = \int_0^a t^{-\delta-\epsilon}\, \{t^\epsilon\, L_2(t)\}\, d\{t^{\eta+\delta}\, L_1(t)\} \leq$$

$$\leq a^\epsilon\, G_2(a) \int_0^a t^{-\delta-\epsilon}\, d\{t^{\eta+\delta}\, L_1(t)\} =$$

$$\leq a^\epsilon\, G_2(a)\{a^{\eta-\epsilon}\, L_1(a) + (\delta+\epsilon) \int_0^a t^{\eta-1-\epsilon}\, L_1(t)\, dt\} =$$

$$\leq G_2(a)\, \{a^\eta\, L_1(a) + (\delta+\epsilon)\, a^\epsilon \int_0^a t^{\eta-1-2\epsilon}\, \{t^\epsilon\, L_1(t)\}\, dt \leq$$

$$\leq G_2(a)\, \{a^\eta L_1(a) + \frac{\delta+\epsilon}{\eta-2\epsilon}\, a^\eta\, G_1(a)\}\ ,$$

and, since $L_1(a) \leq G_1(a)$ by (17), we obtain

$$\int_0^a \varphi_2(t)\, d\{t^\eta\, \varphi_1(t)\} \leq \frac{\eta+\delta-\epsilon}{\eta-2\epsilon}\, a^\eta\, G_1(a)\, G_2(a)\ .$$

Statement (16) results by introducing this inequality in (22) .

The inequality (20) is obtained similarly: by (13), (14) and (18) we have

$$\int_b^\infty |d\{t^{-\eta}\, L(t)\}| \leq -2 \int_b^\infty \varphi_1(t)\, d\{t^{-\eta}\, \varphi_2(t)\} - b^{-\eta}\, L(b)\ ,$$

and by (19) and (21) ,

$$-\int_b^\infty \varphi_1(t)\, d\{t^{-\eta}\varphi_2(t)\} \leq \frac{\eta+\delta-\epsilon}{\eta-2\epsilon}\, b^{-\eta}\, H_1(b)\, H_2(b)\ .$$

From this follows the next lemma.

Lemma 2.   Let $L(x)$ be a function slowly varying in the neighborhhod

of infinity; if

$$L(x) = \varphi_1(x) \, \varphi_2(x)$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are regularly varying and

$$\varphi_1(x) = x^\delta \, L_1(x) \quad \text{increases} \ ,$$

$$\varphi_2(x) = x^{-\delta} \, L_2(x) \quad \text{decreases} \ ,$$

for some $\delta > 0$,   and where $L_1(x)$ and $L_2(x)$ are slowly varying in the

neighborhood of infinity, such that

$$x^\eta \, L_i(x) \to 0, \quad i = 1, 2, \quad \text{as} \ \ x \to +0 \ ,$$

for every $\eta > 0$. Then, for every $a > 0$, $b > 0$ and $\eta > 0$,   and for

$M > 1 + 2\delta/\eta$ there is a x_M such that

$$x > x_M \Longrightarrow \int_0^a |d\{t^\eta \, L(xt)\}| \leq M a^\eta \, L(x) \ , \tag{23}$$

and

$$x > x_M \Longrightarrow \int_b^\infty |d\{t^{-\eta} \, L(xt)\}| \leq M b^{-\eta} \, L(x) \ . \tag{24}$$

An analogous statement holds for the neighborhood of $+0$ instead of ∞ ,

if

$$x^{-\eta} L_i(x) \to 0, \quad i = 1, 2, \quad \text{as} \quad x \to \infty ,$$

for every $\eta > 0$.

Proof. Let

$$K(a) = 2 \frac{\eta + \delta - \epsilon}{\eta - 2\epsilon} G_1(a) G_2(a) - L(a) ,$$

then the inequality (16) implies

$$\int_0^a |d \{t^{\eta} L(xt)\}| \leq a^{\eta} K(ax) ;$$

on the other hand, since $L_1(x)$ and $L_2(x)$ are slowly varying, (17) implies that

$$G_i(ax) \sim L_i(x), \quad i = 1, 2, \quad \text{as} \quad x \to \infty ,$$

for every $\epsilon > 0$. Therefore

$$K(ax) \sim (2 \frac{\eta + \delta - \epsilon}{\eta - 2\epsilon} - 1) L(x), \quad \text{as} \quad x \to \infty,$$

and ϵ being arbitrary the statement (23) obtains.

Analogously, the statement (24) follows from the inequality (20).

Now we can complete theorem 3 by the following theorem.

Theorem 4. Let $\alpha < \beta$ and let $f(t)$ be integrable in every finite interval $[a,b]$, $a > 0$, such that for every η between α and β, $\alpha < \eta < \beta$, the integral

$$\int_{+0}^{\infty} t^{\eta} f(t) dt \quad \text{converges.}$$

Let $\varphi(x) = x^{\gamma} L(x)$ be a product of two monotone functions $\varphi_1(x)$ and $\varphi_2(x)$,

regularly varying either in the neighborhood of infinity and such that

$$x^{\eta} L_i(x) \to 0, \quad i = 1,2, \quad \text{as} \quad x \to +0 \quad,$$

or in the neighborhood of zero and such that $x^{-\eta} L_i(x) \to 0, \quad i = 1,2, \quad \text{as} \quad x \to \infty$,

for every $\eta > 0$. Then, for every $\alpha < \gamma < \beta$,

$$\int_0^{\infty} f(t) \, \varphi(xt) \, dt \sim \varphi(x) \int_0^{\infty} f(t) \, t^{\gamma} dt \quad, \tag{25}$$

as $x \to \infty$, or as $x \to +0$, respectively.

Proof. From the fact that (see [1], or [3])

$$\varphi(xt)/\varphi(x) \to t^{\gamma} \quad,$$

uniformly in every finite interval $[a,b]$, $a > 0$, as $x \to \infty$, or $x \to +0$

respectively,   we first conclude

$$\int_a^b f(t) \, \varphi(xt) \, dt = \varphi(x) \int_a^b f(t) \, t^{\gamma} dt + o(\varphi(x)) =$$

$$= \varphi(x) \int_0^{\infty} f(t) \, t^{\gamma} dt + r(a,b) \, \varphi(x) + o(\varphi(x)) \quad, \tag{26}$$

where

$$r(a,b) = -\int_0^a f(t) \, t^{\gamma} dt - \int_b^{\infty} f(t) \, t^{\gamma} dt$$

can be made arbitrarily small for sufficiently small a and 1/b .

Then by choosing $\gamma - \alpha < \eta < \beta - \gamma$ and writing

$$F_1(x) = \int_0^x f(t)\, t^{\gamma - \eta}\, dt, \quad F_2(x) = \int_x^\infty f(t)\, t^{\gamma + \eta}\, dt \ ,$$

we obtain

$$\int_0^a f(t)\, \varphi(xt)\, dt = x^\gamma \int_0^a f(t)\, t^{\gamma - \eta} \{t^\eta L(xt)\}\, dt =$$

$$= x^\gamma \{a^\eta L(ax)\, F_1(ax) - \int_0^a F_1(t)\, d\, \{t^\eta L(xt)\}\ .$$

Furthermore, since

$$L(ax) \leq M'\, L(x) \quad \text{for } x > x_{M'} \quad ,$$

we have by (23)

$$\left| \int_0^a f(t)\, \varphi(xt)\, dt \right| \leq x^\gamma \{a^\eta\, |F_1(a)|\, M'\, L(x) + M\, a^\eta L(x) \sup_{a < t \leq a} |F_1(t)| \} =$$

$$\leq a^\eta \{M'\, |F_1(a)| + M \sup_{0 < t \leq a} |F_1(a)| \}\, \varphi(x) \leq$$

$$\leq M_1\, a^\eta\, \varphi(x) \ . \tag{27}$$

On the other hand, since

$$L(bx) \leq M''\, L(x) \quad \text{for } x > x_{M''} \quad ,$$

using (24) we obtain analogously

$$\left| \int_b^{\infty} f(t)\, \varphi(xt)\, dt \right| \le b^{-\eta} \{ M'' \, |F_2(b)| + M \sup_{t \ge b} |F_2(t)| \} \, \varphi(x) \le$$

$$\le M_2 \, b^{-\eta} \, \varphi(x) \ . \tag{28}$$

By combining (26), (27) and (28) one gets

$$\left| \int_0^{\infty} f(t)\, \varphi(xt)\, dt - \varphi(x) \int_0^{\infty} f(t)\, t^{\gamma}\, dt \right| \le$$

$$\le \{ M_1 \, a^{\eta} + r(a,b) + M_2 \, b^{-\eta} \} \, \varphi(x) + o(\varphi(x)) \ ,$$

which yields (25) by letting $a \to 0$ and $b \to \infty$ after the limiting passage $x \to \infty$, or $x \to +0$, respectively.

Theorem 4 généralizes a result of Aljanšić, Bojanić and Tomić [7, Théorème 4]. In fact, these authors assume that $L(x)$ is a product of two monotone **slowly** varying functions, while Theorem 4 assumes the factors to be monotone and **regularly** varying only. A slightly stronger case is obtained if e.a. $L(x)$ is assumed to be of bounded variation in every finite interval and if for every $\delta < 0$ there exists a x_{δ} such that (13) and (14) hold for $x > x\delta$. Then the function $L(x)$ is slowly varying in the sense of A. Zygmund [9], i.e. for every $\delta > 0$ there exists x_{δ} such that for $x > x_{\delta}$

$$x^{\delta} L(x) \text{ increases and } x^{-\delta} L(x) \text{ decreases.} \tag{29}$$

However, R. Bojanić has remarked that (29) holds if and only if

$$x \overline{L}'_+ (x) = o(L(x)), \text{ as } x \to \infty \ ,$$

where $\overline{L}'_+(x)$ and $\underline{L}'_+(x)$ denote the upper and lower right derivate of

$L(x)$, respectively. Therefore, one obtains as a corollary of Theorem 4,

that (25) holds if $\varphi(x)$ is differentiable and if

$$x\varphi'(x)/\varphi(x) \to \gamma \quad \text{as} \quad x \to \infty \quad .$$

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