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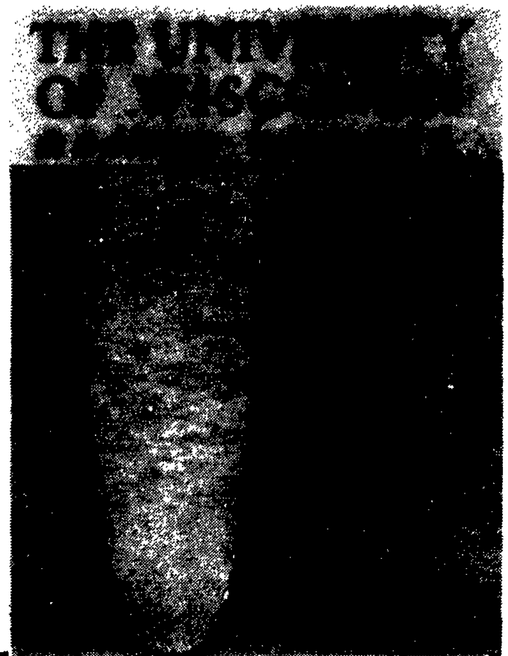

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SHIFT INVARIANT LINEAR MANIFOLDS

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ABSTRACT

This paper gives a characterization of solution manifolds of ordinary differential equations with constant coefficients as finite-dimensional shift invariant linear manifolds of continuous functions.

SHIFT INVARIANT LINEAR MANIFOLDS

P. M. Anselone

This paper concerns a relatively little known characterization of solution manifolds of ordinary differential equations with constant coefficients. Such a manifold consists, of course, of polynomial-exponential functions. In the main theorem, which follows, R is the real line and $C(R)$ is the set of complex continuous functions defined on R.

Theorem 1. Let M be a finite dimensional linear manifold in $C(R)$ which is shift invariant, i.e.,

$$f(t) \in M, \; x \in R \Longrightarrow f(t+x) \in M. \tag{1}$$

Then there exist non-negative integers m_p and complex numbers z_p, $p = 1, \dots, q$, such that M is spanned by the functions

$$t^m e^{z_p t}, \qquad \begin{cases} m = 0, 1, \dots, m_p, \\ p = 1, \dots, q. \end{cases} \tag{2}$$

Equivalently, M is the solution manifold of an ordinary differential equation with constant coefficients.

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This theorem is not really new. A generalization of it to functions defined on a topological group was given in 1948 by Laurent Schwartz [1]. His proof depends on the theory of group representations. The primary purpose of this paper is to present a new, more elementary, proof of Theorem 1. Both the theorem and this proof have features which ought to be of substantial mathematical and pedagogical interest. In particular, the theorem was essential in a paper [2] by D. Greenspan and the author on linear difference-integral equations. It was used in order to determine certain "fundamental" solutions, which are eigenfunctions of a related operator.

Proof of Theorem 1. Let $\{f_i(t): i = 1, \dots, m\}$ be a basis for M. It follows from (1) that, for each $i = 1, \dots, m$ and each $x \in R$, there exist unique complex numbers $\alpha_{ij}(x)$, $j = 1, \dots, m$, such that

$$f_i(t+x) = \sum_{j=1}^{m} \alpha_{ij}(x) f_j(t), \qquad \begin{cases} i = 1, \dots, m, \\ t, x \in R. \end{cases} \tag{3}$$

If we could differentiate (3) with respect to x and then let $x = o$, we should obtain a system of ordinary differential equations with constant coefficients, from which the theorem would follow immediately. The rest of the proof justifies this procedure by showing that the functions α_{ij} are differentiable.

Let $f(t)$ denote the (column) vector with components $f_i(t)$, $i = 1, \dots, m$. Let $A(x)$ denote the matrix with elements $\alpha_{ij}(x)$, $i, j = 1, \dots, m$.

Then (3) is equivqlent to

$$\vec{f}(t+x) = A(x)\vec{f}(t), \qquad t, x \in R. \tag{4}$$

Moreover, there is a unique matrix function $A = \{a_{ij}\}$ which satisfies (4). One implication of the uniqueness is that

$$A(o) = I. \tag{5}$$

Equation (4) yields

$$A(y)A(x)\vec{f}(t) = A(y)\vec{f}(t+x) = \vec{f}(t+x+y) = A(x+y)\vec{f}(t)$$

for all $x, y, t, \in R$. By the uniqueness of A,

$$A(y)A(x) = A(x+y) = A(y+x) = A(x)A(y), \quad x, y \in R. \tag{6}$$

Thus, A is a commutative semigroup of matrices. This fact motivates the steps which follow, but is not actually used to obatin the results.

We prove next that A is continuous, i.e., that each a_{ij}, is a continuous function. For this purpose an auxiliary result is needed.

Lemma. Let g_i, $i = 1, \dots, m$, be m linearly independent real or complex functions defined on an arbitrary (abstract) set P. Then there exists a subset of P,

$$P_m = \{ p_j : j = 1, \ldots, m \},$$

consisting of exactly m points, such that the restrictions of the functions g_i to P_m are linearly independent. Hence, the matrix $\{ g_i(p_j) \}$, $i, j = 1, \ldots, m$, is non-singular.

This lemma is a direct generalization of the familiar theorem in linear algebra that the row and column ranks of a matrix are equal. To see this, consider a "matrix" with rows $i = 1, \ldots, m$ and with the columns indexed by the elements of P. Let $g_i(p)$ be the element in row i and column p. Since the functions g_i are linearly independent, the row rank of this matrix is m. By the same reasoning as in the classical case, the column rank of the matrix also is m, so that there exists a non-singular square submatrix $\{g_i(p_j)\}$, $i, j, = 1, \ldots, m$. Incidentally, the same lemma was used by R. C. Buck [3] in a different connection.

Since the functions $f_i(t)$, $i = 1, \ldots, m$, are linearly independent, the lemma implies that there exists $t_k \in R$, $k = 1, \ldots, m$, such that the matrix $\{ f_i(t_k) \}$ is non-singular. For each $x \in R$, define a matrix function B by

$$B(x) = \{f_i(t_k + x)\}, \quad i, k = 1, \ldots, m, . \tag{7}$$

Then, by (3),

$$B(x) = A(x)B(o), \tag{8}$$

where $B(o)$ is non-singular. It follows that

$$A(x) = B(x)B^{-1}(o). \tag{9}$$

Since B is clearly continuous, A is continuous.

We have proved that A is a continuous semigroup of matrices. One of the earlier results in semigroup theory (cf. [4], pp. 282-284, or [5]) asserts that A is necessarily differentiable. In order to keep this proof elementary and self-contained, we prove here in a few lines that A' exists. By (6),

$$A(x)\int_o^t A(y)\,dy = \int_o^t A(x+y)\,dy = \int_o^{x+t} A(s)\,ds. \tag{10}$$

The continuity of A and (5) imply that

$$\frac{1}{t}\int_o^t A(y)\,dy \to I \quad \text{as} \quad t \to o. \tag{11}$$

It follows that both $\frac{1}{t}\int_o^t A(y)\,dy$ and $\int_o^t A(y)\,dy$ are non-singular for t

sufficiently small and positive. By (10),

$$A(x) = \int_x^{x+t} A(s)\,ds\,[\int_o^t A(y)\,dy]^{-1} \tag{12}$$

for some $t > o$. Since the right member of (12) is differentiable with respect to x, A' exists. Therefore, each a_{ij} is differentiable.

Finally, differentiate (3) with respect to x and then let $x = o$ to obtain

$$f_i'(t) = \sum_{j=1}^{m} a_{ij}'(o) f_j(t), \quad i = 1, \ldots, m, \quad t \in R. \tag{13}$$

This system of ordinary differential equations with constant coefficients yields the desired results.

The conditions in Theorem 1 can be relaxed. For example, it suffices to require (1) only for x sufficiently small and positive. The functions in M need not be defined on all of R; they could be defined on $[o, \infty]$ or on $[o, c]$ for some $c > o$.

A generalization of Theorem 1 to real Euclidean n-space R^n is given below. A typical element of R^n is devoted by $\vec{t} = (t_1, \ldots, t_n)$.

<u>Theorem 2.</u> Let M be a finite dimensional linear manifold in $C(R^n)$ such that

$$f(\vec{t}) \in M, \ \vec{x} \in R^n \implies f(\vec{t} + \vec{x}) \in M.$$

Then there exist integers m_{kp} and complex numbers z_{kp}, $p = 1, \ldots, q$ and $k = 1, \ldots, n$, such that M is spanned by the functions

$$t_1^{m_1} \ldots t_n^{m_n} e^{z_{1p}} \ldots e^{z_{np}}, \left\{ \begin{matrix} m_k = 0, 1, \ldots m_{kp}, \\ p = 1, \ldots q_k, \end{matrix} \right\} k = 1, \ldots, n.$$

This theorem can be proved in much the same way as Theorem 1. Details are omitted.

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