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APPROXIMATE METHODS FOR ANALYZING NON-LINEAR AUTOMATIC SYSTEMS

By

Ye. P. Popov and N. P. Pal'tov

UNEDITED ROUGH DRAFT TRANSLATION

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FOREWORD

Engineers and scientific workers are increasingly often obliged to face the necessity of taking into account essential nonlinearities in systems for automatic control, stabilization, and regulation, and in follow-up systems. This is necessary not only to allow for the detrimental influence of the nonlinearities inevitably occuring in the system, but also for the application of specific nonlinearities with the goal of improving the dynamic qualities of the system. Therefore, the expansion of the practical application of automatic devices in all areas of technology imposes ever-greater demands upon the development of the nonlinear theory of automatic control.

In the present book, the authors have striven as far as possible to give an account of the approximate methods which are simplest and easiest in practical application for the investigation and design of a definite class of nonlinear automatic systems, and are suitable for the solution of certain synthesis problems for such systems. The initial theoretical basis of these methods is formed by the ideas of harmonic balance and equivalent linearization propounded in the familiar works of N.M. Krylov and N.N. Bogolyubov, and also the special form of the small-parameter method developed in the works of B.V. Bulgakov.

After the first applications of the method of harmonic balance to nonlinear control systems, which are due to L.S. Gol'dfarb, V.A.

Kotel'nikov and R. Dzh. Kokhenburger, the methods indicated above were developed successfully for the solution of new problems in the investigation and design of nonlinear automatic systems. The different methods which are proposed in the present book for the solution of the diverse nonlinear problems that arise from practical requirements, which are reduced to concrete applications, are unified by the general idea of harmonic linearization of nonlinearities in its different forms, invoking in the last chapter the statistical linearization proposed by I.Ye. Kazakov.

The book is oriented toward broad classes of engineers and scientific workers in the field of the technology and theory of systems for automatic control, regulation, measurement, and other applications of automatic devices, and also toward students and post-graduate students.

The work was alotted among the authors of the book as follows. Chapters I, II, V (§§ 1-5), VII (§§ 1-5), VIII, IX (§§ 1-4), and X were written by Ye.P. Popov. Basically, they reflect the works of the author in this area, with their further development toward the improvement of engineering design methods for the planning of closed nonlinear automatic systems (from the point of view of the analysis and synthesis of these systems).

Chapters III, IV, V (§§ 6-9), VI, VII (§§ 6-10), and IX (§§ 5-7) were written by I.P. Pal'tov. In them an account is given not only of the results obtained by the author himself, but also his treatment of the results obtained by other authors in studies of specific closed-loop nonlinear automatic systems on the basis of the methods specified.

Chapter I contains a description of the basic forms of the nonlinearities encountered in various automatic systems, and the

classification of nonlinear systems into three large classes in accordance with the number of variables appearing in the nonlinear functions, and the form of the relationships among these variables. In the subsequent chapters of the book, various problems are solved in parallel for all three classes of nonlinear systems in cases of various forms of nonlinearities. Therefore, the division of the material of the book into chapters is not based on the types of nonlinearities, nor on the classes of nonlinear systems, but on the types of the problems to be solved.

An account is also given in Chapter I of the analytical investigation of various processes in the simplest nonlinear system of the third order by the method of harmonic linearization, with the purpose of demonstrating the simplicity of application of the method and its possibilities, which are easily understandable in this simple case even without the study of the foundations of the method, as presented below.

Chapter II presents in general form the concept of harmonic linearization of nonlinearities and the theoretical foundations for the application of the method of harmonic linearization to the determination of the symmetric self-oscillations of nonlinear automatic systems. Conditions applicable to the type of differential equations (or transfer functions) of the system and enabling us to apply the given approximate method of analysis are given here. These conditions are almost always satisfied in various systems for automatic control, stabilization and regulation, and in follow-up systems.

Further on in Chapter II, an account is given, in general form, of various practical methods for the determination of self-oscillations (and their stability), which permits us in the majority of cases to dispense with the mapping of locus diagrams in the complex

plane. A comparison of these methods with the various types of known methods is presented using a small parameter in one form or another. The identity of the final result of the first approximation in all the methods compared is demonstrated, but it is the diversity of approaches to solution of the problem and of the processes of computation which is essential for practical work.

Chapter II also gives an approximate method for the delineation of regions of equilibrium stability for nonlinear systems and its comparison with the results of the direct method of Lyapunov for various particular types of nonlinear systems. The results of this comparison speak for the simplicity and practical effectiveness of the method, and also for the possibility and expediency of presenting the problem to mathematicians for the development of a rigorous foundation (applying the material presented in § 2.2). Apparently, the facts which are stated permit us to hope that the development of some rigorous theory of stability for the case of the first approximation is possible, not with the help of the Taylor expansion, but with the help of the Fourier expansion, i.e., on the basis not of ordinary, but of harmonic linearization.

In a supplement to this investigation of the absolute stability of a nonlinear system (for arbitrary initial conditions), methods are given at the end of the chapter for approximate evaluation of the stability of nonlinear automatic systems over a limited range of initial conditions, and also the delineation of the region of practical stability of a self-oscillatory nonlinear system, which is definable by the condition that the amplitude of self-oscillation must not exceed a given admissible limit, whereupon the limitations of the admissible self-oscillation frequencies may be taken into account.

Chapter III contains a derivation and summarization of the final expressions for the coefficients of harmonic linearization for various types of nonlinearities which are necessary for the analysis of specific nonlinear automatic systems, as they apply to all the problems of Chapter II.

In Chapter IV, the solution of many practical problems by the methods of Chapter II is assembled, using as examples specific automatic systems and taking various nonlinearities into account. These examples not only pursue the goal of illustrating the application of the method, but in many respects they have an independent significance because only in standard examples is it possible to clarify the mass of different fundamental peculiarities of the dynamic behavior of nonlinear systems.

In addition, practical methods for obtaining the frequency and amplitude of self-oscillations as functions of the various system parameters, which are of interest in the synthesis of automatic systems are indicated in these examples. Finally, these examples may have immediate practical value for the specific fields of engineering to which each of them pertains.

Chapter V develops a method for the analysis of nonsymmetric self-oscillations in a general form in the presence of a constant or slowly-varying external disturbance. The separation of the solution into periodic and constant (or slowly varying) components is nonlinear, i.e., the two parts of the solution are mutually interdependent. They pass the nonlinearity with different interdependent gain constants. On the basis of the method of harmonic linearization, this circumstance permits us to investigate the specific nonlinear characteristics of slowly-varying dynamic processes in self-oscillatory systems and the phenomenon of vibrational smoothing of

nonlinearities with the help of self-oscillations. The stability and quality of the responses in a nonlinear system are ascertained as functions of the magnitude of a constant external disturbance or, in astatic systems of the rate of variation of an external disturbance.

The method of which an account is given in Chapter V permits use of approximations to investigate double-frequency oscillations with a large difference between the frequencies. Oscillations with these frequencies pass the nonlinearity with different gain constants corresponding to the harmonic linearization of different nonlinear functions. Here a dual harmonic linearization is carried out for the lower frequency; specifically, the so-called displacement function is determined initially, and thereupon again undergoes harmonic linearization. Similarly, it is possible to determine triple-frequency oscillations. Usually this occurs in complex automatic control systems. The above division of the motion of the system into double-frequency or triple-frequency motion, which may be performed on the basis of harmonic linearization, permits us to simplify significantly the dynamic analysis of complex multiloop automatic-control systems and to build up successive loops, starting the investigation with the simplest interior loop.

Also in Chapter V, final expressions for coefficients of harmonic linearization and the displacement magnitudes are given for various types of nonlinearities, which are often encountered in automatic systems.

In Chapter VI, the investigation of a series of specific automatic systems is carried out by the methods developed in Chapter V. Self-oscillatory measuring devices and acceleration integrators are investigated with determination of their errors, together with auto-

matic-control systems with several nonlinearities, some of which are nonsymmetric. The displacement of the stability boundary of a nonlinear system on a change in the magnitude of the constant external disturbance is illustrated.

Chapter VII is devoted to the development of approximate methods of investigating the quality of the transient responses in nonlinear automatic systems. First, the asymptotic method of N.M. Krylov and N.N. Bogolyubov, relating to slow settling processes is generalized for the investigation of fast-damping transient responses by introduction of a final term in the equation for the derivative of the amplitude with respect to time. It is demonstrated that in this case, it is possible under certain conditions to reduce the solution to the use of harmonic linearization, but with the addition of certain specific terms. A clear picture of transient-response quality as a function of the system parameters is presented in the form of quality diagrams. Three different general methods are given for the construction of these diagrams, together with two methods specifically intended for more complex nonlinear systems of the second and third classes with two or more nonlinearities.

Here, not only symmetrical oscillatory processes are considered, but also more complex forms of transient responses that can be separated nonlinearly into an aperiodic component and an oscillatory component with a variable amplitude. In particular, we also consider transient responses under conditions of vibrational smoothing of nonlinearities and discuss sliding processes.

Also in Chapter VII, examples of specific investigations by methods described in this chapter are cited for several nonlinear automatic systems. Good agreement is demonstrated between the evaluations obtained for the quality of the transient responses

and the results of numerical-graphical solution of the initial nonlinear equations of the system.

In Chapter VIII, an approximate method for calculation of the higher self-oscillation harmonics and refinement of the frequency and amplitude of the first harmonic, which was found earlier in the first approximation, is presented. Finding the higher harmonics directly with the help of the ordinary expansion of all the oscillatory variables into Fourier series would lead to a large number of unsolvable equations. The problem can be solved quite easily by introducing certain special simplifications which are fully justified by analysis of the small order of magnitude of the various terms of the expansion, which is performed on the basis of the materiel in § 2.2. The method is further developed for the determina-;ion of the self-oscillation frequency as a function of the form of the nonlinearity and of the magnitude of a constant external disturbance, while the first approximation which has hitherto been applied did not furnish the possibility of doing this for single-valued, nonevenly-symmetrical nonlinearities.

It is interesting to note that the refinement of the first approximation being considered here leads to the determination of small higher harmonics for a variable appearing in the nonlinearity, and to the appearance of finite higher harmonics for the nonlinear function itself. This is consistent with the nature of the phenomena being studied. Such a refined harmonic linearization most completely approximates a nonlinear oscillation process (the oscillation of the nonlinear function itself). The examples considered in this same Chapter VIII confirm this and show good agreement with the exact solution of the problem.

Chapter IX contains methods for the investigation of forced

oscillations of nonlinear systems both in general form and for examples of the design of specific automatic systems. Two methods exist for the solution of the problem for single-frequency forced oscillations: a graphic complex-plane method and an analytical method which, in the case of unsolvability of the equations obtained, leads to an uncomplicated graphical construction using real variables, both for symmetric and nonsymmetric oscillations.

Subsequently in Chapter IX, the passage of slowly-varying signals through a nonlinear system in the presence of forced oscillations is considered. In the general case, it is assumed that the forced oscillations pass along every closed loop of the system together with the slowly-varying signals. Their equations separate nonlinearly, that is the interdependence among them is maintained even with different gain constants for each of them. The solution of the problem is considerably simplified for the case where the forced oscillations do not pass through the entire loop. As a particular case, we obtain the well-known problem of vibrational smoothing of nonlinearities with the aid of forced oscillations.

If the slowly-varying signal is oscillatory, then we are in fact investigating double-frequency oscillations with a large difference in frequencies, as was done in Chapter V for self-oscillations. Thus the frequency characteristics of a closed-loop system for a slowly-varying signal may be obtained with secondary high-frequency vibrations present in the system.

The essential departure from self-oscillations here is that the gain constant with which the slowly-varying signal passes the nonlinearity may depend heavily upon the given amplitude and frequency of the external periodic disturbance. Consequently, this external disturbance may strongly affect the dynamic properties with

respect to the slowly-varying signal, and even affect the stability of the system. Therefore, in the case of premeditated introduction of such a disturbance, it may be used for improving the qualities of the nonlinear control system. If, on the other hand, the disturbance is vibrational noise, the latter may detrimentally affect the quality of the system, and, for a sufficiently large amplitude, may even render the system unstable. This emphasizes the importance of developing a suitable variety of approximate methods of calculation.

Subsequently in Chapter IX, we consider computation of the harmonics of single-frequency forced oscillations with refinement of the first harmonic; this provides a possibility of investigating the influence of nonsinusoidal external disturbances upon nonlinear automatic systems. All the general methods of investigation developed in Chapter IX are illustrated for examples of specific nonlinear automatic systems, including some with composite action (with perturbation control).

The final Chapter X is devoted to random processes in nonlinear systems. Here, for the synthesis of approximate methods which are convenient in practice, two forms of the linearization of nonlinearities, the harmonic form and the statistical form, are used both separately and together. Initially, using harmonic linearization only, we investigate slow random processes in nonlinear systems operating under conditions of self-oscillatory and forced vibrations which smooth the nonlinearity and thereupon permit us to use a purely linear theory of random processes. Here, the determination of the smoothed, nonlinear characteristic, and the breakdown of the initial nonlinear equation is carried out in full by the methods of Chapters V and IX. If, however, the ordinary linearization of the smoothed characteristic is not feasible for some reason or other (for example,

because of its sharp curvature), the statistical linearization is also used after the harmonic linearization. In both cases, the amplitude of the vibrations is a random quantity, of which the mathematical expectation and dispersion are determined.

In addition to these two types of problems, Chapter X examines methods for the solution of five more diverse types of problems in which the resolution of the solution sought is performed not with respect to the speed of the processes in time, as above, but into regular (mathematical expectation) and random (centered) components. Initially, the familiar problem of the determination of random processes in the steady-state operating mode of closed-loop, nonlinear automatic systems, which was solved by I.E. Kazakov, is investigated. Subsequently, an analogous method is developed for the investigation of nonsteady-state, slowly-varying regular processes, in the presence of high-frequency random fluctuations. This permits us to investigate new phenomena – specific for nonlinear systems – of the dependence of the stability and quality of the dynamic responses as functions of the spectral-density characteristics of the external fluctuations. Here, as in Chapters V and IX, the approximate analysis is based upon the nonlinear separation of the differential equations, but with another, statistical content. The matter is also reduced to the determination of the displacement function, which is a smoothed, nonlinear characteristic, whereupon the nonsteady-state, slowly-varying regular process is analyzed with the help of either ordinary or harmonic linearization of this displacement function.

The other problems considered in Chapter X relate to the investigation of self-oscillations and the conditions for their existence (or cessation) in the presence of random external disturbances,

both high-frequency and low-frequency, and also in the intermediate case, but with the exclusion of the resonance frequency. All the methods described are illustrated by examples of the analysis of specific automatic systems.

At the end of the book, we list a bibliography of literature, not only that used by the authors, but also of other literature devoted to approximate methods for the investigation and design of nonlinear automatic systems. In the literature indicated, other methods of investigation which are not considered in this book are described. Acquaintance with these methods may be useful to the reader who is generally interested in the nonlinear methods which are used in the investigation of automatic systems.

References devoted to exact methods for the solution of the nonlinear problems of control theory (apart from those which are cited in the test of the book) are not listed in the bibliography, since this is an independent and rich field for investigations.

In order to make inspection of the literature easier, it is classified according to separate groups of chapters, although its numeration is unified. It is arranged in chronological order of publication, with the foreign-language literature in a separate list. Foreign works published in Russian translation are listed with the Russian-language literature. In those cases where the book which is cited has run through two or more editions, only the last edition is listed, with its number indicated. In the text, references to the literature are designated by the appropriate number in square brackets.

In conclusion, it is necessary to say that, like any other book, the present book makes no pretense to exhaustive completeness of its exposition of the development and elucidation of the approximate

methods for the investigation of nonlinear automatic systems. The authors have given an account of those methods for the solution of the nonlinear problems considered in the book which are, in their opinion, firstly, the simplest and most convenient for practical use, and secondly, suitable for extension to the investigation of the most complex, nonlinear automatic systems encountered in practice.

In addition, the authors have attempted to give not simply a collection of good methods, but to unify methods for the solution of diverse nonlinear problems by means of a single idea based upon a single foundation – in this case, upon the harmonic linearization of the nonlinearities in a broad sense, encompassing its different forms. This gave us the possibility of considering a broad range of problems from a single, general point of view, with the reduction of the results to examples of the investigation and design of specific automatic systems.

Apart from the above objective considerations, it is probable that the natural scientific interests of the authors played a role in the choice of the methods of investigation. In particular, they took the point of view that by using the apparatus of differential equations (although the application of frequency characteristics is also valid) as a point of departure, it is possible to obtain the most general and widely applicable methods. Therefore the frequency methods of solution are reflected in the book to a lesser extent. However, it is evident that having a theory of design starting from the differential equations, it is possible to transpose these calculations into the language of frequency characteristics, as well as is already being done presently in the literature for solution of the simplest (as yet) nonlinear problems, for example with the aid of logarithmic frequency characteristics. This may be

extremely useful in many practical problems.

The first attempt to give an integrated account of the approximate methods for the investigation of nonlinear automatic systems based upon the harmonic linearization of nonlinearities, as has been undertaken here, may not be free of shortcomings. The authors will be extremely grateful to all readers who send their preferences and critical comments, as well as reports regarding successful and unsuccessful instances of the application of the methods of which an account has been given to their addresses or to the publisher.

The authors express their gratitude to Academician N.N. Bogolyubov for valuable advice, and also extend thanks to all their colleagues who participated in the solution of the individual problems. The authors are thankful to Editor O.K. Sobolyev and Artist A.I. Klimanov for their monumental work in preparing the manuscript for press.

Chapter 1

INTRODUCTION

§ 1.1. EXAMPLES OF NONLINEARITIES FREQUENTLY ENCOUNTERED

The present volume considers closed-loop automatic systems of all designs and with various purposes (automatic-control systems, follow-up systems, stabilization systems, and closed-loop computing and measuring systems). The limitation of the class of systems being considered, which will be elucidated in what follows, is determined not by their structure or function but by the form of the equations (or transfer functions) describing dynamic processes in the system.

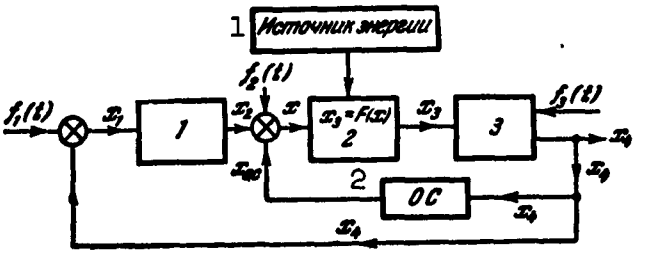


Fig. 1.1. 1) Power source; 2) OS.

A nonlinear automatic system is usually a complex of some arbitrary finite number of links, with the dynamics of the majority of them described by linear equations while one or several (an extremely limited number) of the links is described by nonlinear equations.

In Fig. 1.1, for example, a certain automatic system is represented in the form of a complex of four links which are interconnected in a certain manner. Here, for example, link 1 may corre-

spond to a filter or amplifier, link 2 to a polarized electromagnetic relay and link 3 to a driving motor, while the fourth link OS may correspond to a "rigid" feedback path. The circles with crosses designate the points of addition and subtraction of various disturbances.

Other physical significances are possible for all links with the same general scheme of the system retained.

In examples of such a general type, we will designate by letters $\underline{x}$ with various subscripts all intrinsic variables, which express the interaction of the links of the system among themselves, while all extraneous variables, which express disturbances applied to the system from without, will be designated by letters $\underline{f}$, also with various subscripts. Usually the latter are either given functions of time in explicit form f(t), or vary arbitrarily in a limited range, or are random functions given by their probability characteristics.

Of the extraneous variables, for example in Fig 1.1, $f_1(t)$ designates the setting or controlling disturbance, which the given system must reproduce in the form x_4 at the output (sometimes noise which should not be reproduced may also enter into the composition of $f_1(t)$); $f_2(t)$ in Fig. 1.1 designates a secondary extraneous disturbance, for example, for the formation of forced vibrations (for greater detail, see § 1.9); $f_3(t)$ is the disturbance originating from the object being controlled, to which the variable quantity x_4 reproduced by the system is transmitted [for example, $f_3(t)$ may be the mechanical load on the output shaft of the system].

We may write its own dynamic differential equation for each link of the system. As a matter of fact, the very decomposition of the system into links is undertaken basically for convenience in

spond to a filter or amplifier, link 2 to a polarized electromagnetic relay and link 3 to a driving motor, while the fourth link OS may correspond to a "rigid" feedback path. The circles with crosses designate the points of addition and subtraction of various disturbances.

Other physical significances are possible for all links with the same general scheme of the system retained.

In examples of such a general type, we will designate by letters $\underline{x}$ with various subscripts all intrinsic variables, which express the interaction of the links of the system among themselves, while all extraneous variables, which express disturbances applied to the system from without, will be designated by letters $\underline{f}$, also with various subscripts. Usually the latter are either given functions of time in explicit form f(t), or vary arbitrarily in a limited range, or are random functions given by their probability characteristics.

Of the extraneous variables, for example in Fig 1.1, $f_1(t)$ designates the setting or controlling disturbance, which the given system must reproduce in the form x_4 at the output (sometimes noise which should not be reproduced may also enter into the composition of $f_1(t)$); $f_2(t)$ in Fig. 1.1 designates a secondary extraneous disturbance, for example, for the formation of forced vibrations (for greater detail, see § 1.9); $f_3(t)$ is the disturbance originating from the object being controlled, to which the variable quantity x_4 reproduced by the system is transmitted [for example, $f_3(t)$ may be the mechanical load on the output shaft of the system].

We may write its own dynamic differential equation for each link of the system. As a matter of fact, the very decomposition of the system into links is undertaken basically for convenience in

forming the dynamic equations of the system (apart from the obvious goal of clearly representing the functioning of the system). In all the equations, we shall conventionally designate the derivatives with respect to time briefly by the symbol p, as is the custom in control theory:

$$p=\frac{d}{dt}.$$

In the present example (Fig. 1.1), we write the equation of link 1 (a filter or amplifier) in linear form:

$$(T_1p+1)x_2=k_1x_1, \quad x_1=f_1(t)-x_4, \tag{1.1}$$

where T_1 is the time constant and k_1 is the gain constant. The filter equation may also include the introduction of the derivative into the control equation:

$$(T_1p+1)x_2=k_1(1+kp)x_1, \tag{1.2}$$

where k is the "intensity coefficient" with which the derivative is introduced.

For link 3 (the drive motor) we also write the linear dynamic equation:

$$(T_2p+1)px_4=k_2x_3+f_3(t), \tag{1.3}$$

where T_2 and k_2 are constants and $f_3(t)$ is the disturbance.

The equation of the "rigid" feedback will be

$$x_{o.c}=k_{o.c}x_4, \tag{1.4}$$

where $k_{o.s}$ is a constant. If the feedback is "flexible,"

$$x_{o.c}=k_{o.c}px_4. \tag{1.5}$$

there may also be more complex feedback equations.

Link 2 (polarized relay) is an essentially nonlinear link, which does not submit to the ordinary method of linearization. Its equation may often be written in the form

$$x_3=F(x), \tag{1.6}$$

where F(x) is the characteristic of the relay. The relay switches

on a direct-current voltage x_3, which is equal to +c or – c depending on the change in current direction. Here, according to Fig. 1.1, the relay input $\underline{x}$ is composed of three variables:

$$x = x_3 - x_{0.c} + f_3(t). \tag{1.7}$$

Generally speaking, the relay characteristic may have any one of the forms indicated in Fig. 1.2. In the general case (Fig. 1.2a) there is a dead zone and a hysteresis loop, i.e., a loop expressing a nonlinear (coordinate) delay in operation of the relay, because of the fact that the dropout current x = mb is less than the operate current x = b ($\underline{m}$ is the recovery constant). In a particular case, there is only a dead zone without the loop (Fig. 1.2b) or only the hysteresis loop without the dead zone (Fig. 1.2c), in which case the relay does not have a center position.

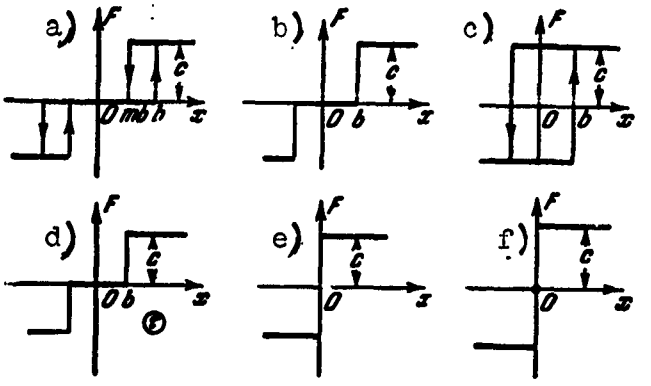


Fig. 1.2.

Sometimes it is important to account for the time delay τ in operation of the relay (Fig. 1.2d). The latter is not represented graphically, since there is no time coordinate in the static characteristic, but we write it symbolically in the form

$$x_3 = F_3(x) = e^{-\tau p} F(x), \tag{1.8}$$

where F(x) is any one of the characteristics of Fig. 1.2. In the case in point (Fig. 1.2d), the time delay τ is introduced simultaneously with the presence of the dead zone when the nonlinear

function F(x) in Formula (1.8) has the form in Fig. 1.2b.

Finally, let us also describe an ideal relay characteristic without a center position (Fig. 1.2e), when

$$F(x)=c\,\operatorname{sign} x,$$

and with the center position (Fig. 1.2e)

$$F(x)=c\,\operatorname{sign} x \quad \text{for } x\neq 0,$$
$$F(x)=0 \quad \text{for } x=0.$$

The latter case is an idealization of the case of Fig. 1.2b for b = 0, while the former is a case of Fig. 1.2c for b = 0. Similarly, the remaining characteristics of Fig. 1.2 [73], may be written analytically, but such a description is not required in what follows.

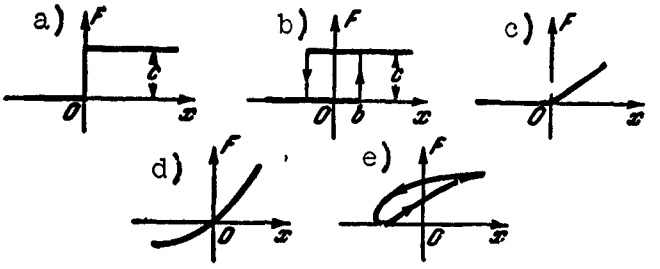


Fig. 1.3.

Relay-type characteristics may also be nonsymmetrical, for example, if a relay or simply a pair of contacts operates under circumstances where a voltage of one polarity is switched on and off (Fig. 1.3a and b). The elastic force of a spring may also be nonsymmetrical, if the latter acts only on one segment of the movement of some device (Fig. 1.3c). The curvilinear characteristics may also be nonsymmetric (Fig. 1.3d and e).

We always write the dynamic equations for every real system with some degree of idealization, with factors of secondary importance disregarded as having little influence upon the solution of the given specific problem. It is evident that the simple equations (1.1)-(1.8) written above do not take into account many circumstances that may be detected and considered important for the detailed study

of each real link of the system in isolation, but which, within certain limits, do not play an essential role in the establishment of the general pattern of the dynamic processes in the system as a whole.

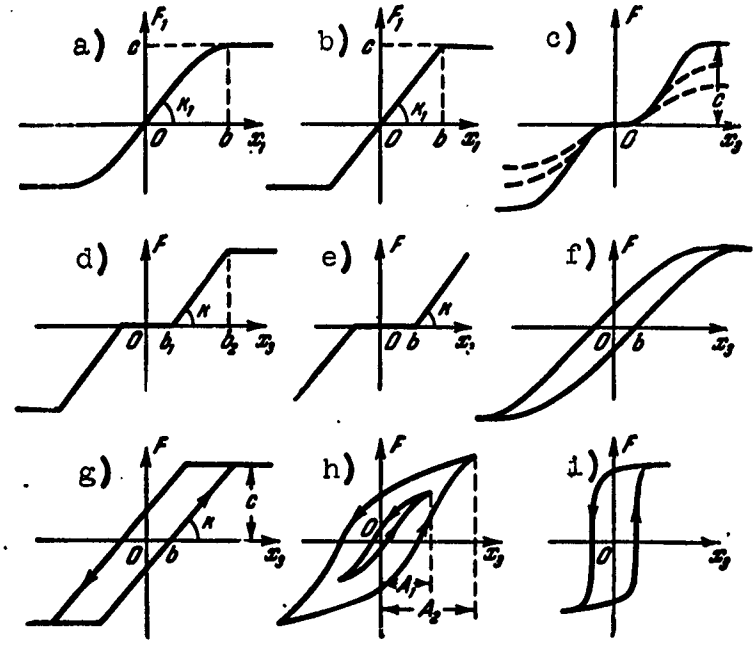


Fig. 1.4.

The linearity hypothesis for all links of the system, other than the relay, is also such an idealization in the given example. In actual fact, the real characteristics of all the links differ from an ideal straight line to a greater or lesser degree. In first approximation, therefore, we may consider them linear only in the fixed range of variation of the variables in which the equations that we have written are valid.

Let us assume that in contrast to the above, in the range of variation of the variables being investigated, we observe saturation or clipping of the output quantity in link 1 (Fig. 1.1), whose influence upon the dynamics of the system must be taken into account.

Then, in addition to (1.6), still another nonlinearity appears in our system, since instead of (1.1) the equation of link 1 takes

the form

$$(T_1 p + 1) x_2 = F_1(x_1), \tag{1.9}$$

where the nonlinear function $F_1(x_1)$ has the form of Fig. 1.4a or b (on the graph of the nonlinear characteristics we shall denote the inclination or steepness of the rectilinear segment not by the angle but by the tangent of the angle, i.e., the gain constant, as, for example, k_1 in Fig. 1.4b).

We may approximate many nonlinearities by analytical functions. For example, the nonlinearity $F_1(x_1)$ given in the form of Fig. 1.4a may be described by the function

$$\left.\begin{aligned} F_1 &= k_1(1 - k x_1^2) x_1 \quad \text{for } |x_1| \leqslant b, \\ F_1 &= c \operatorname{sign} x_1 \quad\quad\quad \text{for } |x_1| \geqslant b, \end{aligned}\right\} \tag{1.10}$$

where from the conditions

$$F_1 = c \text{ and } \frac{dF_1}{dx_1} = 0 \text{ for } x_1 = b$$

we have

$$k_1 = \frac{3c}{2b}, \quad k = \frac{1}{3b^2}. \tag{1.11}$$

In the case of a nonlinearity of the form of Fig. 1.4b, on the other hand, we obtain

$$F_1 = k_1 x_1 \text{ for } |x_1| \leqslant b \text{ and } F_1 = c \operatorname{sign} x \text{ for } x \geqslant b,$$

where $k_1 = c/b$.

For nonlinear introduction of the derivative, we obtain in place of (1.2) the equation

$$(T_1 p + 1) x_2 = F_1(x_1, p x_1). \tag{1.12}$$

We may also visualize a case where the time constant T_1 of the filter changes with variation of the input or output variable, i.e., where in Eq. (1.1)

$$T_1 = F_2(x_1) \text{ or } T_1 = F_2(x_2). \tag{1.13}$$

Then the equation of link 1 takes the form

$$F_2(x_1) p x_2 + x_2 = k_1 x_1 \tag{1.14}$$

or

$$F_2(x_2)\,px_2 + x_2 = k_1 x_1. \tag{1.15}$$

A nonlinearity of the type of Fig. 1.4b – limiting of a linear relationship – is also encountered in mechanical apparatus with arresting devices, for example in the form of restriction of the movement of aircraft control surfaces (the control device of the autopilot).

As we know, in many follow-up systems and automatic regulation and control systems, including the servomechanisms of autopilots, the various types of driving apparatus (electrical, pneumatic, hydraulic) have nonlinear characteristics of the form

$$px_4 = F(x_3), \tag{1.16}$$

where x_4 is the turn angle of the output driving shaft, while x_3 is the electrical or mechanical input quantity (a regulated supply voltage, movement of a slidevalve, and so forth). Here $F(x_3)$ has the form of a characteristic with dead zone and saturation (Fig. 1.4c or d) or even the form of a characteristic with only the dead zone (Fig. 1.4e), when the values of the input variable x_3 are small.

The lag time τ may be introduced in Expression (1.16), as in (1.8)

$$px_4 = F_\tau(x_3) = e^{-\tau p}F(x_3). \tag{1.17}$$

Finally, in the presence of a hysteresis loop of the nonlinear function $F(x_3)$ in Formula (1.16) takes the form of Fig. 1.4f, g, or i. Such loop characteristics are complex in that the width of the loop and even its entire shape may change with variation of the oscillation amplitude A of the input variable x_3 and shifting of the center of oscillation (Fig. 1.4h).

We also encounter descending nonlinearities with drops which have segments with negative slopes (Fig. 1.5a, b, c).

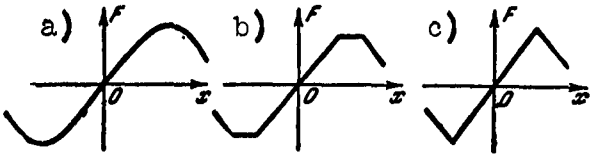


Fig. 1.5.

Let us note that the level c as well as the slope of the nonlinear characteristic, may change with the variation of the drive load, as indicated in Fig. 1.4c by the broken line.

If allowance is made for the inertia of the driven masses, then the equation for the nonlinear drive, like (1.3) which was written earlier, will have instead of (1.16) the form

$$(Tp+1)px_4 = F(x_3). \tag{1.18}$$

Having considered the individual elements of nonlinear systems, let us write as an example the equations of an automatic-control system which holds an aircraft on its course by means of an autopilot (taking slip angle but not roll into consideration). The equations for the aircraft as an object of control [49] will be:

$$\left.\begin{aligned}(T_1p+1)p\psi + k_2\beta &= -k_1\delta + f_1(t),\\ (T_2p+1)\beta &= T_3p\psi + f_2(t),\end{aligned}\right\} \tag{1.19}$$

where ψ is the angle of deflection of the aircraft from the given course, β is the slip angle, and δ is the rudder deflection,

$$T_1 = \frac{J}{M_c^{p\psi}}, \quad T_2 = \frac{mV}{Z^\beta}, \quad k_1 = \frac{M_p^\delta}{M_c^{p\psi}}, \quad k_2 = \frac{M_c^\beta}{M_c^{p\psi}},$$
$$f_1(t) = \frac{M_a(t)}{M_c^{p\psi}}, \quad f_2(t) = \frac{Z_a(t)}{Z^\beta}.$$

In these formulas J, m, and V are, respectively, the moment of inertia, mass, and velocity of the aircraft; $M_s^{p\psi}$, M_s^β, M_s^δ, and Z^β are the slopes (tangents of the angles of inclination) of the aerodynamic characteristics, i.e., of the moment M_s of air resistance to rotation of the aircraft with respect to the angular velo-

city $p\psi$ and the angle β, and of the moments M_r and Z, respectively, of the control surface with respect to the angle δ and of the lateral force with respect to the angle β; M_v and Z_v are the disturbing moment and force leading the aircraft away from the assigned course.

Let us write the equation for the autopilot drive in the form

$$(T_3 p + 1) p\delta = F(x), \tag{1.20}$$

where T_3 is the time constant; F(x) may be given in several of the forms represented on Fig. 1.4. There may also be other forms (Fig. 1.2), corresponding to the case of a constant-velocity drive in contrast to the variable-velocity drive (Fig. 1.4). We write the equation for the variable $\underline{x}$ taking into consideration the signals from the sensors (gyroscopes) and the "rigid" feedback in the form

$$x = (k_1 + k_2 p + k_3 p^2)\psi - k_{o.c}\delta. \tag{1.21}$$

It is determined by the control law which has been selected.

Further, wishing to account for the limitation of the movement of the control surface for the case of linear driving of the control surface, we denote by δ_1 that value of the control-surface deflection angle for which the surface goes as far as the arresting device and stops. Then instead of (1.20) we obtain

$$\left.\begin{aligned} (T_3 p + 1) p\delta = F(x) \text{ for } \delta < \delta_1, \\ p\delta = 0 \text{ for } \delta = \delta_1, \end{aligned}\right\} \tag{1.22}$$

if the elastic properties of the arresting device are not taken into account.

Sometimes it is also necessary to take into account the nonlinearity of the aerodynamic characteristics, which changes Eqs. (1.19), imparting nonlinear form to them.

Quite often the nonlinearity which we encounter is a backlash in some type of mechanical transmission. In the presence of backlash, the displacement $\underline{x}$ of the driving element of the mechanical

transmission will not initially cause any movement y of the driven element as long as the entire backlash – the line segment OH in Fig. 1.6a – is not "chosen" (we shall denote half the total width of the backlash by b). After this the driven element starts its motion, which is represented by the straight line HB. On reversal of the direction of motion of the driving element, the driven element will be stationary at an arbitrary point of the straight line HB for the entire time during which the backlash is "chosen" (transition along any horizontal line segment F = const from the straight line HB to the straight line CD), after which the motion of the driven element starts in the opposite direction along the straight line CD. We describe this nonlinear characteristic of the backlash y = F(x) by the formula (Fig. 1.6a);

$$\left.\begin{array}{rl} & y=x-b \text{ for } \dot{y}>0, \\ & y=x+b \text{ for } \dot{y}<0, \\ \dot{y}=0, & y=\text{const} \text{ for } |x-y|<b. \end{array}\right\} \tag{1.23}$$

It is evident that the smaller the displacement x the more important will be allowance for backlash. For large (in comparison with the quantity b) displacements x, the backlash will not play any essential role.

The influence of the backlash may not be limited to the phenomenon just described. The fact is that in the "time of selection" of the backlash (on the horizontal segments of the graph of Fig. 1.6a) the driven element of the mechanical transmission is separated from the driving element and, consequently, the static and dynamic loads on the driving element are reduced. For example, in the automatic (follow-up) system (Fig. 1.1) this corresponds to disengagement of the controlled object from the driving shaft B (Fig. 1.6b). As a result, the equation of motion of the driving shaft will be

$$\left.\begin{aligned}(T_3p+1)px &= k_3x_3+f_3(t) \text{ for } px_4 \neq 0,\\ [(T_3-T)p+1]px &= k_3x_3 \text{ for } px_4 = 0,\end{aligned}\right\} \quad (1.24)$$

where T is proportional to the moment of inertia of the controlled object, $f_3(t)$ is the load moment from the controlled object, while $x_4 = F(x)$ in accordance with Fig. 1.6a. Simultaneously both the primary and secondary feedback signals change (Fig. 1.6b). During the backlash "selection," the "rigid" feedback signal remains constant, while the "flexible" feedback signal dies out. Consequently, it is as if the transfer function of the whole system were changed during this time.

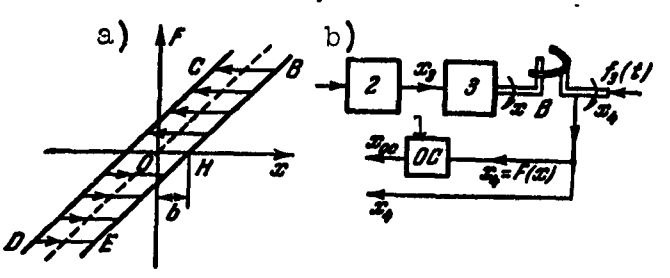


Fig. 1.6. 1) OS.

This type of system nonlinearity, which is associated with a structural change in the transfer functions or differential equations of the system, is also encountered in many other cases (see the example § 3.9). Let us introduce one example of a system in which the nonlinearity consists in a deliberate change of transfer functions (a nonlinear correcting device). We add to the basic linear part of the system $W_1(p)$ (Fig. 1.7) one of the two feedbacks $W_1(p)$ or $W_2(p)$, depending upon the value of the variable $\underline{x}$. For example, for sufficiently small values $|x| < b$, the feedback $W_1(p)$ is cut in, while for larger values $|x| > b$ the feedback $W_2(p)$ is cut in. Then the equation for the nonlinear link of the system will be

$$\left.\begin{aligned}x_1 &= W_1(p)x \text{ for } |x|<b,\\ x_1 &= W_2(p)x \text{ for } |x|>b.\end{aligned}\right\} \quad (1.25)$$

Although each of the feedbacks $W_1(p)$ and $W_2(p)$ here is in itself linear, on the whole a nonlinear correcting device is obtained, since the switch from one feedback to the other is determined by the value of the variable $\underline{x}$ itself.

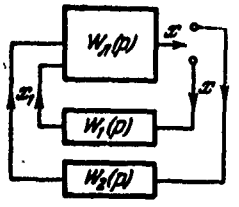


Fig. 1.7.

We illustrate the next type of nonlinearity by reference to the example of the characteristics of a two-phase induction motor (Fig. 1.8a) for various values of the control voltage $\underline{u}$ and the angular velocity ω_{dv}, which in this case is denoted by $\underline{x}$. For linearization of characteristics it is usually assumed that

$$M = c_1 u - c_2 x.$$

But in first approximation this is valid only for the left segment

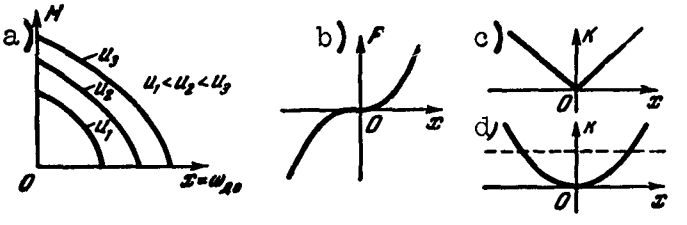


Fig. 1.8.

of the characteristic. If, however, the major part of the characteristic is used, we must take its nonlinearity into account. Keeping in mind that the coefficient c_1 in Fig. 1.8a decreases with increasing $\underline{x}$, while the coefficient c_2 increases, we adopt the following nonlinear expression for description of this characteristic:

$$M = \frac{c_1}{1 + c_3 |x|} u - (c_2 + c_4 |x|) x \qquad (1.26)$$

(the absolute values of $\underline{x}$ are placed in the coefficients, because $\underline{x}$ changes its sign but the coefficients themselves must remain positive numbers). There exist other methods for description of

the nonlinear characteristics of the motor (see Chapter IV).

Then the differential equation of the motor Jpx = M, where J is the moment of inertia of all the masses which are being turned by the motor referred to the motor shaft, may be written in the form

$$Jpx + Jc_3|x|px + c_2x + (c_2c_3 + c_4)|x|x + c_3c_4x^3 = c_1u. \tag{1.27}$$

Here there are three nonlinear functions:

$$F_1 = |x|px, \quad F_2 = |x|x = x^2 \operatorname{sign} x, \quad F_3 = x^3. \tag{1.28}$$

The second and third of these have the form of Fig. 1.8b. The first, however, takes the form

$$F_1 = k(x)px \tag{1.29}$$

with a variable gearing ratio k at a velocity px which [ratio] is a function of the coordinate x taking the form k(x) = |x| (Fig. 1.8c).

As another example of a nonlinear function of the type (1.29) we may introduce the function

$$F = x^2px, \tag{1.30}$$

where $k(x) = x^2$ (Fig. 1.8d). For linear dependence, we would have k = const, as is indicated by the broken line in Fig. 1.8d.

Let us now consider an example of nonlinearity with nonseparating variables. The equation for the two-phase induction motor is sometimes written in the form

$$Jrx + (c_2r + \psi_1^2 + \psi_2^2)x = \psi_1\dot{\psi}_2 - \psi_2\dot{\psi}_1, \tag{1.31}$$

where

$$\psi_1 = \int u_1 dt, \quad \psi_2 = \int u_2 dt,$$

where u_1 and u_2 are the voltages designated in Fig. 1.9, r is the ohmic resistance of the rotor curcuits, c_2 is the coefficient of linear friction, and x is the angular velocity of the shaft.

Let $u_1 = U_0 \cos \omega_0 t$ be the power voltage from the line, while $u_2 = u(t) \sin \omega_0 t$ is the control voltage, which is shifted 90° in phase, and ω_0 is the frequency of the alternating supply current.

We assume that the amplitude of the control voltage u(t) changes relatively slowly.

Substituting the expressions for u_1 and u_2 in Eq. (1.31) and performing simplifications in the process of which we drop dual-frequency oscillatory terms and the terms containing ω_0^3 and higher powers in the denominator, we may finally arrive at the following equation for the motor:

$$(T_1 p + 1)x + bu^2 x = k_1 u, \tag{1.32}$$

where

$$T_1 = \frac{2Jr\omega_0^2}{2c_2 r\omega_0^2 + U_0^2}, \quad b = \frac{1}{2c_2 r\omega_0^2 + U_0^2}, \quad k_1 = \frac{2U_0\omega_0}{2c_2 r\omega_0^2 + U_0^2}.$$

In this case we have the nonlinearity u^2x with nonseparable input and output variables $\underline{u}$ and $\underline{x}$ respectively.

In mechanical oscillatory links, which are described in linear theory by the equation

$$(mp^2 + kp + k_2)x_2 = k_1 x_1,$$

nonlinear friction and a nonlinear restoring force (a nonlinear spring) may occur. Then the equation of the link takes the form

$$mp^2 x_2 + F_1(px_2) + F(x_2) = k_1 x_1. \tag{1.33}$$

The characteristic of the restoring force $F(x_2)$ may deviate from linearity in either direction (Fig. 1.10a and b).

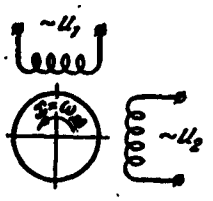


Fig. 1.9.

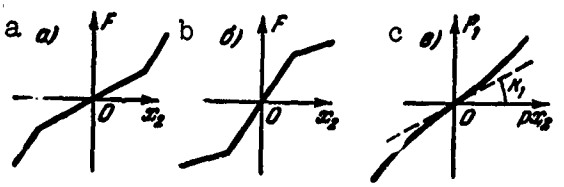


Fig. 1.10.

The friction may be square-law (Fig. 1.10c):

$$F_1(px_2) = k_1 px_2 + c(px_2)^2 \operatorname{sign} px_2 \tag{1.34}$$

or dry (Fig. 1.11a or b); here an extremely important characteristic of dry friction is that for $px_2 = 0$ the frictional force F_1

may assume any value in the range

$$-c \leqslant F_1 \leqslant +c, \tag{1.35}$$

equal at each given moment of time to the sum of all other forces acting (including inertial force). If, therefore, at that moment of time when $px_2 = 0$ the modulus of the sum of all the other forces is less than c, the system will stop. The standstill will continue until the variation of the forces results in a value $|F_1| = c$, after which motion of the system starts again. Consequently, it must be remembered that in this case the characteristic of dry friction (Fig. 1.11a) is different in principle from that of the relay characteristic (Fig. 1.2e), which has a superficially similar form. If, however, it is always found in the motion process of the system that for $px_2 = 0$ the force $|F_1| \geq c$, then there will not be any standstills and the friction characteristic (Fig. 1.11a) will not differ from that of the relay characteristic (Fig. 1.2e). In Fig. 1.11b we show another possible real dry-friction law, while in Fig. 1.11c we show the sum of dry and linear friction.

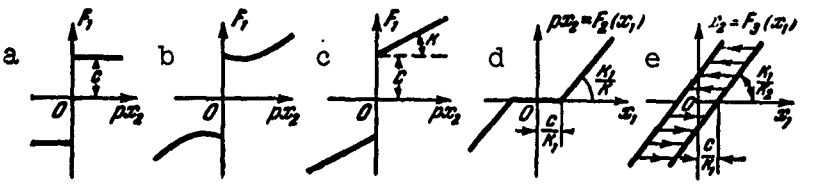


Fig. 1.11.

In the presence of linear and dry friction (Fig. 1.11c) and a linear restoring force the equation of the oscillatory link (1.33) is written in the form

$$mp^2x_2 + kpx_2 + c\,\mathrm{sign}\,px_2 + k_2x_2 = k_1x_1, \tag{1.36}$$

provided that

$$|k_1x_1 - k_2x_2 - mp^2x_2| \geqslant c \quad \text{for} \quad px_2 = 0. \tag{1.37}$$

If, however, at the moment that the value $px_2 = 0$ is reached

in the process of motion, it is found that

$$|k_1x_1 - k_2x_2 - mp^2x_2| < c, \tag{1.38}$$

Eq. (1.36) will be valid only for $px_2 \neq 0$, while for $px_2 = 0$ standstill begins according to (1.35) and will continue while the right member k_1x_1 varies over the range

$$(k_2x_m - c) < k_1x_1 < (k_2x_m + c) \text{ for } px_2 = 0, \tag{1.39}$$

where we denote by x_m the value of x_2 at the moment of stopping. Here, in addition to standstills, the motion may have discontinuities and jolts.

In the case of purely dry friction (Fig. 1.11a) without linear friction, we must set k = 0 in Eq. (1.36).

Thus the nonlinear oscillatory link with dry friction appears quite complex in the presence of mass and a restoring force. The matter is significantly simplified, first in the general case (1.36) for observance of the no-stoppage condition (1.37) (then the dry-friction characteristic reduces to the ideal-relay characteristic) and, secondly, for the presence of standstills in the two particular cases which we shall presently consider.

In cases where the mass may be neglected because of its small size and there is no restoring force, the equation (1.36) of the link in the presence of linear and dry friction (Fig. 1.11c), with (1.35) taken into account, assumes the form

$$\left.\begin{array}{ll} kpx_2 + c \operatorname{sign} px_2 = k_1x_1 & \text{for } px_2 \neq 0, \\ -c < k_1x_1 < +c & \text{for } px_2 = 0. \end{array}\right\} \tag{1.40}$$

It is evident that this is equivalent to the nonlinear function

$$px_2 = F_2(x_1), \tag{1.41}$$

represented in Fig. 1.11d. In this case, therefore, the influence of dry friction reduces to the formation of a dead zone for the velocity as a function of input quantity at the link's output. This

is what usually takes place in drive units with equations of the type (1.16) (see Fig. 1.4e).

Finally, in the case where the mass is neglected, in which case we have purely dry friction (Fig. 1.11a) and a linear restoring force, the link equation (1.36), allowing for (1.39), assumes the form

$$\left.\begin{array}{ll} c\,\operatorname{sign} px_2 + k_2 x_2 = k_1 x_1 & \text{for } px_2 \neq 0, \\ (k_2 x_m - c) < k_1 x_1 < (k_2 x_m + c) & \text{for } px_2 = 0. \end{array}\right\} \tag{1.42}$$

It is easy to see that this is equivalent to the nonlinear function

$$x_2 = F_2(x_1), \tag{1.43}$$

which is represented in Fig. 1.11e, where the horizontal lines correspond to different values of x_m. In the present case, the influence of the dry friction is found equivalent to the backlash in the mechanical transmission (see Fig. 1.6). This takes place in sensors and control elements with light moving parts.

Thus, generally speaking, the complex phenomenon of dry friction, which in the general case results in complex motions that are subjected to analysis only with difficulty, may be described in three particular cases by simpler nonlinear characteristics (the ideal-relay type, the dead-zone type, and the backlash type), the analysis of which does not present any difficulty in what follows.

It would be possible to cite a multitude of other examples of nonlinearities that are encountered in automatic systems. They will be given later in examples and specific problems. Here, however, we shall limit ourselves for the present to the examples which have been indicated, which are sufficient for illustration of the general problems of which an account is given below.

§ 1.2. ON THE INVESTIGATION OF NONLINEAR AUTOMATIC SYSTEMS

We will differentiate between the concept of a nonlinear link of a system and the concept of nonlinearity.

We will call a real link of a system which is described by a nonlinear equation nonlinear. The equation of a nonlinear link may take on any of a rather large number of forms: (1.6), (1.8), (1.9), (1.12), (1.14), (1.15), (1.16), (1.18), (1.22), (1.24), (1.25), (1.27), (1.32), (1.33), (1.36), (1.42), etc. Here both the input and output variables of the link may be found in the nonlinearities.

In what follows, the structure of the individual linear links and their connections in the system will not in themselves be important to us; it is only the resultant equation of the entire ensemble of these links which will be important. In the design of nonlinear automatic systems, it is customary to separate the nonlinear link (for example, link 2 in Fig. 1.1), while all the remaining linear members are conditionally united in one block, which is called the linear part of the system (Fig. 1.12a). Therefore, the diagrams in Fig. 1.12a do not at all imply consideration of simple systems, because the linear part may have an arbitrary structure, including a multiloop structure with linear correcting devices of any type, may be described by linear differential equations of high order. If we have not one but two or more nonlinear links in the system, we accordingly obtain several individual linear parts on separation of the nonlinear links.

However, we may find certain linear terms even in the equation of the nonlinear link itself, as, for example, $(T_1p + 1)x_2$ in Eq. (1.9), px_4 in Eq. (1.16), $Jpx + c_2x$ and c_1u in Eq. (1.27), mp^2x_2 and k_1x_1 in Eq. (1.33), $mp^2x_2 + kpx_2 + k_2x_2$ and k_1x_1 in Eq. (1.36),

etc. Therefore, extending further the idea of unification of linear elements of the system we separate the nonlinearity, that is, the nonlinear function entering into this equation, from the equation of the nonlinear link itself. In the equation of the nonlinear link (1.9), for example, we separate the nonlinearity in the form

$$y = F_1(x_1), \tag{1.44}$$

$$(T_1 p + 1) x_2 = y \tag{1.45}$$

and relate the latter linear equation to the general linear part of the system, calling it the reduced linear part (Fig. 1.12b). Thus, if the real linear part (Fig. 1.12a) was described by some equation

$$Q_2(p) x_1 = R_2(p) x_2 + S_1(p) f_1(t) + S_2(p) f_2(t), \tag{1.46}$$

while the real nonlinear link is described by Eq. (1.9), then according to (1.45) and (1.46) the equation of the reduced linear part in this case (Fig. 1.12b) will be

$$(T_1 p + 1) Q_2(p) x_1 = R_2(p) y + (T_1 p + 1) S_1(p) f_1(t) + (T_1 p + 1) S_2(p) f_2(t), \tag{1.47}$$

while the nonlinearity will take the form (1.44). In the case of (1.16), we obtain a similar expression for x_4 and $x = F(x_3)$ if we replace $(T_1 p + 1)$ by p in (1.47).

In the presence of a lag τ in the system, the latter will also be referred to the linear part. For example, in the case where the linear part is described by Eq. (1.46), while the nonlinear member of type (1.17) has the equation

$$p x_2 = e^{-\tau p} F(x_1), \tag{1.48}$$

we may write for the reduced linear part:

$$p Q_2(p) x_1 = R_2(p) e^{-\tau p} y + p S_1(p) f_1(t) + p S_2(p) f_2(t), \tag{1.49}$$

while for the nonlinearity

$$y = F(x_1). \tag{1.50}$$

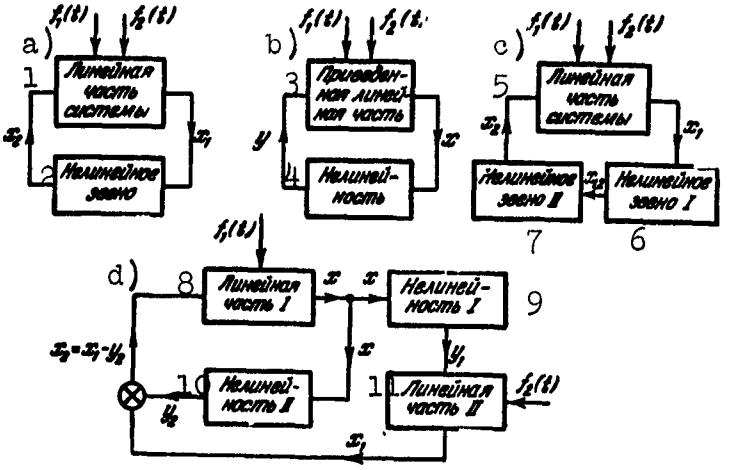


Fig. 1.12. 1) Linear part of system; 2) nonlinear link; 3) reduced linear part; 4) nonlinearity; 5) linear part of system; 6) nonlinear link I; 7) nonlinear link II; 8) linear part I; 9) nonlinearity I; 10) nonlinearity II; 11) linear part II.

Further, when the equation of the nonlinear member has the more complex form (1.27) denoting by y the nonlinearity

$$y = [c_3|x|px + (c_2c_3 + c_4)|x|x + c_3c_4x^3, \tag{1.51}$$

we rewrite Eq. (1.27) in the form

$$[px + y + c_2x = c_1u. \tag{1.52}$$

Let the equation of the linear part (Fig. 1.2a) have the form (1.46), with $x_1 = u$ and $x_2 = x$. Then having excluded the quantity $x_1 = u$ from Eqs. (1.46) and (1.52), we obtain the equation of the reduced linear part (Fig. 1.12b):

$$\left[\left(\frac{J}{c_1}p + \frac{c_2}{c_1}\right)Q_3(p) + R_3(p)\right]x = -\frac{1}{c_1}Q_3(p)y + S_1(p)f_1(t) + S_2(p)f_2(t). \tag{1.53}$$

Here the nonlinearity will take the form (1.51).

The situation is similar in the case of the nonlinear link (1.33) and in many other cases a nonlinearity of the form y = = F(x, px) or in particular, y = F(x) is separated in a similar manner. This is the case for systems with dry friction in three cases: (1.36)-(1.37), (1.40)-(1.41), and (1.42)-(1.43). The more

general case of dry friction, which is characterized by Formulas (1.36), (1.38), and (1.39) together, does not reduce to such a simple form.

Let us refer to those systems in whose equations only one variable quantity, possibly with its derivative, appears in the nonlinear functions as nonlinear systems of the first class [46]. The most widely used nonlinear systems of the first class are those in which the expression of the nonlinearity in general notation takes the form

$$y = F(x, px) \quad \text{or} \quad y = F(x), \tag{1.54}$$

here the equation of the reduced linear part may be represented in the form*

$$Q(p)x = -R(p)y + S_1(p)f_1(t) + S_2(p)f_2(t). \tag{1.55}$$

Here the operator polynomials Q(p) and R(p) may be connected in the most diverse ways with the operator polynomials $Q_L(p)$ and $R_L(p)$ of the real linear part of the system. For example, in the case (1.47)

$$Q(p) = (T_1 p + 1)Q_L(p), \quad R(p) = R_L(p);$$

in the case (1.49)

$$Q(p) = pQ_L(p), \quad R(p) = R_L(p)e^{\tau p},$$

while in the case (1.53)

$$Q(p) = \left(\frac{J}{c_1}p + \frac{c_2}{c_1}\right)Q_L(p) + R_L(p), \quad R(p) = \frac{Q_L(p)}{c_1}.$$

Systems with two nonlinear links connected in series (Fig. 1.12c) may be reduced in the majority of cases to such nonlinear systems of the first class. We may consider two such links as a single more complex nonlinear link, since it is possible to obtain as a result of two nonlinear operations a nonlinear equation directly connecting the variables x_2 and x_1.

The backlash in a follow-up system with allowance for the dif-

ference of the moments of inertia is an example of such a complex nonlinearity; this is described by the equations (1.24), which, in the general case, take the form

$$F_1(p^3x,\ p^2x) = k_3x_3 + f_3(t), \quad x_4 = F_2(x).$$

Here, as they apply to the system of Fig. 1.12c, the variables x_3, x_4, and $\underline{x}$ play the roles of the quantities x_1, x_2, and $x_{1,2}$, respectively.

However, a scheme of the type of Fig. 1.12c may sometimes also lead to a nonlinear system of the third class.

As we see, the type of nonlinear system of the first class (1.54)-(1.55) which is being considered embraces an extremely wide range of nonlinear automatic systems. The overwhelming majority of nonlinear automatic systems which have been considered in the literature up to this time belong to it. The restrictions imposed upon the equation of the linear part (1.55) will be indicated below (§ 2.2).

But nonlinear systems of the first class may also be of another type. For example, if in the scheme in Fig. 1.12d the nonlinear links I and II are described by the relationships

$$y_1 = F_1(x, px), \quad y_2 = F_2(x, px), \tag{1.56}$$

then the equation of the linear part of the system will take the form

$$Q(p)x = -R_1(p)y_1 - R_2(p)y_2 + S_1(p)f_1 + S_2(p)f_2, \tag{1.57}$$

i.e., in contrast to (1.55), the linear part has two inputs (y_1 and y_2) instead of the one input $\underline{y}$. Similarly, we may also conceive of a linear part with $\underline{n}$ inputs, which correspond to nonlinearities in one and the same variable.

Finally, the system shown in Fig. 1.7 may also belong to the nonlinear systems of the first class if the situation reduces to

a nonlinear function of one variable $\underline{x}$ (see § 2.1 concerning this).

Let us refer to those systems whose equations have two (or more) variables connected with each other by linear differential equations as nonlinear systems of the second class.

For example, if there is a nonlinear link (1.15) in the system and no other nonlinearities, then it belongs to the first class, since the product $F_3(x_2)px_2$ is a nonlinear expression of the type (1.54). But if the nonlinear link is described by Eq. (1.14), then, even without the presence of other nonlinearities, the system will belong to the second class of nonlinear systems, since the product $F_2(x_1)px_2$ is a nonlinear expression of the type

$$y = F(x_1, px_2), \tag{1.58}$$

where x_1 and px_2 are connected with each other by a linear differential equation, namely by the equation of the linear part of the system (1.46).

A system with a nonlinear link (1.32) belongs in exactly the same way to the nonlinear systems of the second class.

In both of these cases, the diagram of the system has the form of Fig. 1.12a with one nonlinear link.

However, nonlinear systems with two (or several) nonlinear links may also belong to the nonlinear systems of the second class, i.e., the scheme of Fig. 1.13a in the case where the equation of the nonlinear link I contains a nonlinear function of the output variable x_2 (and, possibly, of its derivative) as, for example, (1.33), while the equation of the nonlinear link II contains a nonlinear function of the input variable x_3 (and its derivative), as for example (1.16). The linear differential equation connecting the variables x_2 and x_3 will be the equation of the linear part II (Fig. 1.13a).

If, however, the linear part II is described by one of the simple relationships

$$x_3 = kx_2, \quad x_3 = kpx_2, \quad x_3 = \frac{k}{p} x_2,$$

then in the majority of cases a system with the scheme of Fig. 1.13a may be included among systems of the first class.

Similarly, we also obtain in the diagram of Fig. 1.13b a system of the second class, when the input variables (x_1 and x_5) are found in the nonlinear functions in both nonlinear links. These variables are connected by the linear part I.

A system of the type 1.7 may be assigned to the systems of the second class in the case where a nonlinear function of two variables is obtained (see § 2.1).

We shall apply the term nonlinear systems of the third class to those systems in whose equations two (or more) variables interconnected by nonlinear differential equations are found in the nonlinearities.

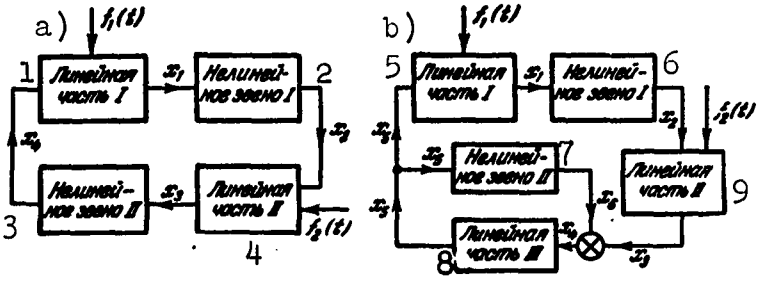


Fig. 1.13. 1) Linear part I; 2) nonlinear link I; 3) nonlinear link II; 4) linear part II; 5) linear part I; 6) nonlinear link I; 7) nonlinear link II; 8) linear part III; 9) linear part II.

This class includes, for example, systems with the scheme of Fig. 1.13a when we have in the equations of both nonlinear links either the input variables (x_1 and x_3) or the output variables (x_2 and x_4) appearing in the nonlinearities. These variables may not be related to each other by linear differential equations.

A system with the scheme of Fig. 1.13b will be a system of the third class, for example, if the input variable x_1 is in the nonlinearity in nonlinear link I, while the output variable x_6 appears similarly in the nonlinear link II.

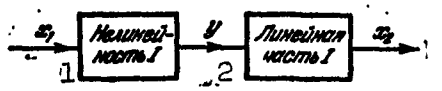


Fig. 1.14. 1) Nonlinearity I; 2) linear part I.

We obtain a system of the third class, for example, in the case where, in investigation of the automatic system (Fig. 1.1) there exist a saturation in the form (1.9) in link 1 and a relay characteristic (1.6) in link 2. Indeed, adopting in Eq. (1.9) the notation

$$y = F_1(x_1), \quad (T_1 p + 1) x_3 = y,$$

we break down link 1 into two parts (Fig. 1.14) and obtain a nonlinear system with two nonlinearities $y = F_1(x_1)$ and $x_3 = F(x)$; here the variables x_1 and $\underline{x}$, which are in the nonlinear functions, may not be connected with each other by linear equations.

A system with a scheme of the type of Fig. 1.2c also belongs to the systems of the third class in the cases where a sequence of two nonlinear links is described by Eq. (1.23) and (1.32) with $\underline{x}$ replaced by px. The equations of two such nonlinear links may be broken down into three parts:

$$y = F(x), \quad (T_1 p + 1) px = x_2, \quad x_2 = k_1 u + b u^2 px,$$

which in fact leads to the scheme of Fig. 1.13a, which has already been discussed; here, $x_1 = u$, $x_3 = x$, and $x_4 = y$. Thus, the linear part II appears between the two nonlinearities, which corresponds to a system of the third class.

If, however, we may neglect the time constant T_1 in Eq. (1.32), then we may reduce such a system with two nonlinear links to a nonlinear system of the second class.

The operation with structural diagrams was used above exclusively with the object of achieving clarity for the classification of nonlinear systems. For the solution of the nonlinear problems by the method studied below, however, there is no need at all to use structural diagrams if the differential equations of the automatic system are given. In what follows, the solution of specific problems does not require additional transformations, either structural or analytical. The equations will be solved in that form in which they are given. Usually they are given in the form of a system of several equations for each link, as for example, (1.1)-(1.7) or (1.19)-(1.21) etc. The methods set forth below also permit us to solve the problem in the case where not the differential equations but rather the frequency characteristics of the links, some of which are experimental, are given.

In theoretical investigations devoted to the theory of stability and the theory of oscillations, we most frequently proceed from a system of equations which are solved for their first derivatives;

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n). \tag{1.59}$$

but, as is evident from the previously written equations (1.1)-(1.7) and (1.19)-(1.21) certain transformations must be performed at this point in order to write the equations for the automatic system in the form (1.59) (or in any special canonical form). Inasmuch as such transformations are not necessary from the point of view of the methods of solution considered below, we will not, as a rule, utilize (1.59) as the form of the equations.

The most general notation for the equations of the linear automatic system, written by links, is the following ([45], page 217):

$$D_{i1}(p)x_1 + D_{i2}(p)x_2 + \ldots + D_{im}(p)x_m = f_i(t) \quad (i = 1, 2, \ldots, m), \tag{1.60}$$

where $D_{il}(p)$ are the operator polynomials and $\underline{m}$ is the number of links. In specific problems, many of the operator polynomials $D_{il}(p)$ will be zeroes, since by far not all of the $\underline{m}$ variables enter into the equation of each link. For example, if we adopt the notation $\psi = x_1$, $\beta = x_2$, $\delta = x_3$, $x = x_4$, we will then have for the three equations (1.19) and (1.21):

$$\begin{aligned}
&D_{11}(p) = (T_1 p + 1)p, && D_{12}(p) = k_2, && D_{13}(p) = k_1, && D_{14}(p) = 0,\\
&D_{21}(p) = -T_2 p, && D_{22}(p) = T_2 p + 1, && D_{23}(p) = 0, && D_{24}(p) = 0,\\
&D_{31}(p) = -(k_4 + k_5 p + k_6 p^2), && D_{32}(p) = 0, && D_{33}(p) = k_{o.c}, && D_{34}(p) = 1.
\end{aligned}$$

It is this notation that we shall henceforth use for the equations of type (1.60), since it corresponds best of all to the equations of automatic systems in the form in which they are directly obtained in synthesis of the equations for each link.

Taking this into account, let us write the equations of a closed-loop system for the basic type of nonlinear systems of the first class in the following form:

$$\begin{aligned}
&D_{11}(p)x_1 + \ldots + D_{1l}(p)x_l + \ldots + D_{1m}(p)x_m = f_1(t),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&D_{k1}(p)x_1 + \ldots + D_{kl}(p)x_l + F(x_l, px_l) + \ldots + D_{km}(p)x_m = f_k(t),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&D_{m1}(p)x_1 + \ldots + D_{ml}(p)x_l + \ldots + D_{mm}(p)x_m = f_m(t).
\end{aligned} \tag{1.61}$$

For example, in an aircraft course-control system, a fourth nonlinear equation (1.20) is added to the three linear equations written above (consequently, in this case $k = \underline{l} = m = 4$), where for the same notation as above we have:

$$D_{41}(p) = 0,\ D_{42}(p) = 0,\ D_{43}(p) = (T_3 p + 1)p,\ D_{44}(p) = 0,$$

where $f_4(t) = 0$ here in the same way that $f_3(t) = 0$.

The system of Eqs. (1.61) corresponds exactly to the system (1.54)-(1.55) if we define

$$x_l = x, \quad F(x_l, px_l) = y. \tag{1.62}$$

Then in Eq. (1.55) we will have

$$Q(p)=\begin{vmatrix} D_{11}(p) \dots D_{1l}(p) \dots D_{1m}(p) \\ \dots\dots\dots\dots\dots \\ D_{k1}(p) \dots D_{kl}(p) \dots D_{km}(p) \\ \dots\dots\dots\dots\dots \\ D_{m1}(p) \dots D_{ml}(p) \dots D_{mm}(p) \end{vmatrix}, \tag{1.63}$$

and R(p) is the minor of the (k, 1)th element of this determinant; $S_1(p)$ and $S_2(p)$ are the minors of the (1, 1)th and (2, 1)th elements, if $f_1(t)$ and $f_2(t)$ are respectively in the first and second rows.

For another type of system of the first class (with two nonlinear functions of one variable in different links), which are described by Eqs. (1.56)-(1.57), we have in general form:

$$\left.\begin{aligned} D_{11}(p)x_1+\dots+D_{1l}(p)x_l+\dots+D_{1m}(p)x_m&=f_1(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ D_{k1}(p)x_1+\dots+D_{kl}(p)x_l+F_1(x_l, px_l)+\dots+D_{km}(p)x_m&=f_k(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ D_{j1}(p)x_1+\dots+D_{jl}(p)x_l+F_2(x_l, px_l)+\dots+D_{jm}(p)x_m&=f_j(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ D_{m1}(p)x_1+\dots+D_{ml}(p)x_l+\dots+D_{mm}(p)x_m&=f_m(t), \end{aligned}\right\} \tag{1.64}$$

here in Eq. (1.57) Q(p) will have the same expression (1.63) as before, $R_1(p)$ and $R_2(p)$ will be the minors of the (k, 1)th and (j, 1)th elements of the determinant (1.63), while

$$y_1=F_1(x_l, px_l), \quad y_2=F_2(x_l, px_l). \tag{1.65}$$

The equations of the nonlinear system of the first class with several nonlinearities in the same variable in different links will also have a similar form.

The nonlinear systems of the second and third classes will contain nonlinearities in different variables in their equations, for example:

$$\left.\begin{aligned} D_{11}(p)x_1+\dots+D_{1l}(p)x_l+\dots+D_{1s}(p)x_s+\dots+ \\ +D_{1m}(p)x_m&=f_1(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ D_{k1}(p)x_1+\dots+D_{kl}(p)x_l+F_1(x_l, px_l)+\dots+ \\ +D_{ks}(p)x_s+\dots+D_{km}(p)x_m&=f_k(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{1.66}$$

$$\begin{aligned} &D_{j1}(p)\,x_1 + \ldots + D_{jl}(p)\,x_l + \ldots + D_{js}(p)\,x_s + F_s(x_s,\ px_s) + \\ &\qquad + \ldots + D_{jm}(p)\,x_m = f_j(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &D_{m1}(p)\,x_1 + \ldots + D_{ml}(p)\,x_l + \ldots + D_{ms}(p)\,x_s + \ldots + \\ &\qquad + D_{mm}(p)\,x_m = f_m(t), \end{aligned} \qquad \text{(1.66 Cont'd.)}$$

or for the presence of a nonlinearity of the type (1.58):

$$\left.\begin{aligned} &D_{11}(p)\,x_1 + \ldots + D_{1m}(p)\,x_m = f_1(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &D_{k1}(p)\,x_1 + \ldots + D_{km}(p)\,x_m + F(x_l,\ px_l,\ x_s,\ px_s) = f_k(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &D_{m1}(p)\,x_1 + \ldots + D_{mm}(p)\,x_m = f_m(t). \end{aligned}\right\} \qquad (1.67)$$

As has already been said, we will not have any further need to perform structural or analytical transformations of the equations into any special canonical forms for the solution of specific problems. They may be solved in the form in which they are given in the conditions of the problem.

Generally speaking, the mathematical investigation of nonlinear dynamic systems presents major difficulties. At the present time, however, engineers and scientific workers in the field of automation in all its diverse applications are increasingly often obliged to deal with nonlinear dynamic phenomena. In closed-loop automatic-control, stabilization, and regulation systems in particular, the presence of nonlinearities in the characteristics of some of the links lead in practice to the most unexpected phenomena, which change the real dynamics of the object fundamentally in comparison with the results of the calculations carried out for the system according to the linear theory of control. In modern, complex automatic control systems, it is frequently the case that an apparently high-quality automatic system designed according to linear control theory is found in practice to be unsuitable with respect to its dynamic characteristics because of the influence of nonlinearities that actually occur in the system and were not allowed for in the design and sometimes even loses stability in contradiction to the calculated

results.

In other cases, the nonlinearities unavoidably occurring in the individual links of the automatic systems may even exert a favorable influence on the dynamic properties of this system. More and more frequently, therefore, various nonlinear correcting devices are being introduced specifically into automatic systems, either to compensate for the detrimental influence of nonlinearities already present in the system (which in themselves are not removable), or independently of the latter, to impart desirable dynamic properties to the system as a whole.

To ascertain the influence exerted on the dynamic properties of the system by the nonlinearities actually present in it, and also for the specific introduction of nonlinear correcting devices into it (independently of whether a problem of analysis or synthesis is being solved) it is therefore necessary to have convenient engineering methods for the investigation and design of nonlinear automatic systems. The problem of the development of such methods is extremely difficult if we take into account, firstly, that even the linear theory of control has still not, on the whole, reached perfection from the point of view of engineering design requirements; secondly, that there do not exist such universal mathematical methods for the solution of nonlinear differential equations, as there are in linear theory, but only various methods for certain particular forms of nonlinear equations; thirdly and lastly, that even in relatively simple nonlinear problems, a large number of qualitatively new phenomena which are little studied and therefore unexpected by the engineer who has gotten used to thinking in terms of linear dynamic phenomena frequently come to light. With this is connected the fact that now the solution of many extremely important

and extremely interesting problems of automation is somehow or other based upon the development of nonlinear methods of the theory of automatic control.

Only a small number of nonlinear problems of the theory of automatic control submit to exact mathematical solution. Moreover, if we take into account that the majority of practical problems require the solution of nonlinear differential equations of higher than the second order, an exact solution, even if it is performed, is often found to be too complex for application in engineering design, although in a number of cases it leads to fundamentally important results. In connection with this, the development of approximate methods for investigation of the dynamic properties of nonlinear automatic systems acquires paramount importance for the theory and practice of systems of automatic control and regulation.

Major importance in this area has devolved upon the approximate methods based upon the ideas of harmonic balance and equivalent linearization proposed in the familiar works of N.M. Krylov and N.N. Bogolyubov, and upon the special form of the small-parameter method developed in the work of B.V. Bulgakov. The various methods for analytical investigation and design of nonlinear automatic systems that are considered in the present book and combined here under the general term method of harmonic linearization (in the last chapter in the book, the method of statistical linearization is adjoined to it in addition) are based fundamentally upon these works.

These approximate methods permit us to solve many problems connected with the investigation of the dynamic properties of nonlinear automatic systems extremely effectively and with an order of accuracy which is completely sufficient for engineers. However, the presence and improvement of effective approximate methods does

not, of course, exclude the need for the further development of exact solutions, which was started in the USSR basically in two scientific schools: namely those of A.A. Andronov and A.I. Lur'ye. The development of the latter solutions is important not only from the point of view of direct acquisition of results, but also especially for the investigation of various particular refined forms of nonlinear-system dynamic processes which cannot be done approximately, with the object of establishing the limits of applicability of the approximate methods.

In all cases it is important to correctly estimate the relative value of the exact and approximate solutions. In striving toward application of the rigorous methods where possible, it should not be forgotten that we must often introduce a number of simplifications into the given equation of the nonlinear system to obtain an exact solution. In this case the approximate solution of the initial nonlinear equation may be found more valuable in practice than the exact solution indicated, since the qualitative picture of the dynamic phenomena being investigated may be lost in the simplifications of the given equation. This applies in particular to cases of lowering of the order of the differential equation from the third to the second by neglecting the influence of one of the system parameters (for example, one of the time constants).

Previous experience and experimental data are of major importance for reliable practical application of approximate methods. However, the problem of the rigorous mathematical justification of the approximate solutions used in practice (in particular, those described in the present book) and the estimation of the degree of their exactness for various classes of nonlinear differential equations of automatic-system dynamics, are extremely important. It would be

expedient to attract the attention of mathematicians to this difficult problem, having in mind the effectiveness of the indicated approximate methods which has already been proven in practice.

Many theoretical and practical problems which present major difficulties for analytical solution may at the present time be solved relatively easily by means of electrical analog and digital computers both completely mathematically, and by means of partial simulation with certain units of the real apparatus or the controlled object(with all of their real nonlinear dynamic properties) connected. However, notwithstanding the colossal role of computers and simulation in engineering applications, including the design of complex automatic systems, not one of the above problems in the development of theoretical investigations and the development of practical design methods has been removed from the agenda.

On the other hand, the development of just the simplest design methods, which permit us in first approximation, even if only roughly, to evaluate the fundamental dynamic properties of the automatic system being designed with allowance for nonlinearities, acquires particular importance in connection with the possibility of solution of complex problems on machines. Such evaluation permits us to produce an elementary first outline of the structure of the automatic system and to determine at least the region of most advantageous values of the parameters of this system. After this we may make the final choice of the structure and the parameters of the system on the computers and by simulation. Without the indicated preliminary computational stage, the use of machines and simulation would lead blindly to the trial-and-error method, since each solution on the machine only gives the result for the given numerical values of the system parameters, and it is not known beforehand which values of

them should be taken. Thus the preliminary, even coarsely approximate and simple calculation (or theoretical investigation) illuminates paths for the subsequent use of the computers. It also retains this role for the development of automatically programmed computers and computers which perform design synthesis automatically. In addition, we must keep in mind that for many automatic systems which are not as complex, the approximate nonlinear calculation may give an acceptable final result immediately. As another valuable property of the analytical investigation, we have the possibility of obtaining from it important practical recommendations of a more general character which are valid not only for a specific system under consideration, but also for the entire class of similar systems.

The tools for analytical investigation of dynamic processes in nonlinear automatic systems, by which it is expedient to develop practical methods for the design of these systems are essentially different for monotonic and oscillatory processes.

It is much simpler to use an exact solution and ordinary numerical or graphical methods of solution in investigation of monotonic processes in nonlinear systems than in the case of oscillatory processes, even in systems described by equations of high order. The fact is that in many cases the dynamics of the nonlinear system may be described by means of a number of linear differential equations with different coefficients for different parts of the process, which are limited by certain dimensions of the unknown variable (in the simplest case this is expressed in the form of a nonlinear static characteristic formed from segments of straight lines). Then to obtain an exact solution it is sufficient to solve these linear equations separately for each segment and match the values of the variable and its derivatives at the end of one segment and the beginning of

the following segment. Such a "method of alignment" which gives an exact solution to the problem, may prove suitable for investigation of a monotonic process, since the latter more often than not breaks down into two or three parts and therefore the relationships obtained for the quantities from the adjustment of segments may not be too cumbersome for analysis. A monotonic process is suitable for numerical and even graphical solution of nonlinear equations in that here we may obtain good accuracy even for a subinterval of approximate integration which is not very small.

On the other hand, for the solution of oscillatory processes directly from the given nonlinear differential equation, even in cases where the problem may be reduced to the solution of linear equations by segments, we obtain such a large number of segments that the relationships obtained by the method of alignment prove to be impossible to analyze and therefore useless in practice (especially for equations of high order). This method may be helpful in searching for periodic solutions, and is in fact used for this purpose, since there it is sufficient to consider a small number of segments within one period. In the synthesis of oscillatory processes by numerical or graphical methods, we may easily obtain a large error and even an invalid result because of the rapid variation of the variable over time, not to speak of the impossibility of obtaining any general relationships. For systems of the second order, the method of representation of the processes in the phase plane appears superior in this case.

The above circumstances oblige us to seek special methods for approximate analysis (and calculation) of the oscillatory processes in nonlinear automatic systems; these methods will differ from those for investigation of monotonic processes, to which we may more easily

apply the ordinary tools for the solution of differential equations. These special methods consist in choosing quantities to characterize the form of the oscillatory process which vary monotonically or, in any case, slowly. For not very complex forms of nonlinear oscillations we may choose such variables as, first, the coordinate of some central line (the shift of the center of oscillation), the maximal deflection from it (envelope) and the time of one complete oscillation (or the frequency). In the majority of cases, knowledge of these quantities is quite sufficient. In a periodic process these will accordingly be: the constant component, the amplitude, and the period (or frequency). In the general case of an oscillatory process, all three quantities change in the course of time. The definition of very complex forms of oscillatory processes does not reduce to knowledge of the three quantities. However, such processes are encountered extremely rarely in the design of automatic systems. In the present book we consider certain complex oscillatory-process forms which are separated nonlinearly into sums of simple processes.

When we speak of the application of various methods of investigation to the determination of oscillatory and monotonic processes, it is necessary to keep in mind that in the design of nonlinear automatic systems, we know beforehand the character of the dynamic process being investigated in far from all cases. Therefore, in addition to knowing how to perform calculations for the oscillatory process itself, we must know how to determine initially whether the process is oscillatory or monotonic for any specified structure, specified system parameters, specified extraneous disturbance and specified initial conditions. In other words, we must know how to find the boundaries for the existence of oscillatory processes as functions of the system parameters and extraneous disturbances (and

sometimes also of the initial conditions), as well as the boundaries for the existence of damping and divergent processes (this is connected with determination of the stability of a nonlinear system). This is important for solution of the problem of synthesis of an automatic system with allowance for nonlinearity both in those cases when we strive to obtain definite oscillatory processes and in those cases where it is necessary to avoid oscillatory character in the processes.

In the present book we develop the simplest approximate methods for the solution of various problems in the field of the investigation and design of nonlinear automatic systems, starting from the above-indicated methods of considering oscillatory processes with the determination of their boundaries of existence and with the delineation of the stability regions of nonlinear systems. Here, we consider not only oscillatory processes as such, but also monotonic processes accompanied by vibrational phenomena, as well as random processes. We also consider dual-frequency processes with a large frequency difference.

We investigate systems with several nonlinearities, some of which are nonsymmetrical, for various extraneous disturbances to the system. For almost all cases we develop methods that enable us to get by without loci in the complex plane. They are also applied to systems with nonlinearities that are not uniquely defined, and to those systems in which the equivalent transfer function of the nonlinearity is a function not only of amplitude, but also of the frequency and displacement of the center of the oscillations, in both steady-state and in transient processes. Here we concieve of the nonlinearity not only in the form of a nonlinear static characteristic, but also in the form of differential equations (or transfer functions) of an

individual part of a system which change as functions of the input variable.

All the investigations in the present book are carried out with the purpose of developing methods for the design of nonlinear systems which are more suitable for engineers and are illustrated by a large number of solutions of specific problems in this area.

§ 1.3. AN EXAMPLE OF FINDING SYMMETRICAL SELF-OSCILLATIONS

In the subsequent chapters we shall give a detailed account of the method of harmonic linearization in general form and its diverse applications to various specific types of nonlinear problems. We shall also elucidate certain questions pertaining to the basis for the method and its comparison with other existing methods for the solution of similar nonlinear problems.

However, before starting a general exposition of the problem, it is expedient to demonstrate the basic ideas of the design method being considered with an example of an extremely simple nonlinear (relay) automatic system; this will be done here and in all the subsequent sections of the first chapter [109].

The illustration of the solution of various nonlinear problems in this extremely simple example permits even the reader who is still not acquainted with the methods of harmonic linearization to satisfy himself of the simplicity of its general conception and the entire procedure of its application to practical designs. In addition, we may best of all acquire an initial clear conception of the character of the problems solved and the phenomena studied in the process; these will help the reader to realize more clearly the meaning of the more general calculations figuring in the following chapters.

As an example, we select the automatic system whose diagram is represented in Fig. 1.1; this is described by Eqs. (1.1)-(1.7).

In the present section we shall consider symmetrical selfoscillations of this system, i.e., stable natural periodic oscillations of the nonlinear system in the absence of extraneous disturbances* to the system ($f_1 = f_2 = f_3 = 0$). Therefore the system diagram takes the form of Fig. 1.15, and the equations will be

$$(T_1 p + 1) x_2 = -k_1 x_0 \tag{1.68}$$

$$x_3 = F(x), \quad x = x_2 - k_{o.s} x_0 \tag{1.69}$$

$$(T_2 p + 1) p x_4 = k_2 x_3 \tag{1.70}$$

where F(x) is the simplest of relay characteristics, i.e., the ideal relay characteristic (Fig. 1.2e), as shown in Fig. 1.15.

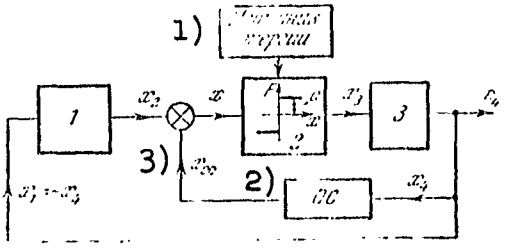


Fig. 1.15. 1) Power sources; 2) OS; 3) $\underline{x}_{o.s}$.

Strictly speaking, the self-oscillations of the nonlinear system will always have a nonsinusoidal form, but it will often be close to a sine curve. In the system considered (Fig. 1.15), the output x_3 of the relay will have a rectangular form for any law of variation of the relay input x (Fig. 1.6). Consequently, x_3 is always far from sinusoidal. In the case of periodic oscillations with the fundamental frequency Ω we may expand it in Fourier series

$$x_3 = \sum_{r=1}^{\infty} B_r \sin(r\Omega t + \beta_r), \tag{1.71}$$

where B_r and β_r are the amplitude and phase of the r^{th} harmonic, with only odd harmonics (r = 1, 3, 5, ...) present here in view of the odd symmetry and uniqueness of the nonlinear function F(x).

Let us find what form the output x_4 of the linear link 3 then assumes:

$$x_4 = \sum_{r=1}^{\infty} C_r B_r \sin(r\Omega t + \beta_r + \varphi_r), \tag{1.72}$$

where, according to the equation of the link being considered (1.70), we have:

$$\left.\begin{aligned} C_r &= \left| \frac{k_2}{p(T_2 p + 1)} \right|_{p = jr\Omega} = \frac{k_2}{r\Omega \sqrt{T_2^2 r^2 \Omega^2 + 1}}, \\ \varphi_r &= \arg \left(\frac{k_2}{p(T_2 p + 1)} \right)_{p = jr\Omega} = -\frac{\pi}{2} - \operatorname{arctg} T_2 r\Omega \\ & (r = 1, 3, 5, \ldots). \end{aligned}\right\} \qquad (1.73)$$

From the formula for C_r it is evident that the amplitudes of the higher harmonics (third, fifth, etc.) pass to the output of the linear link 3 with significantly less "amplification" than the first harmonic ($r = 1$); this applies with greater force the higher the number

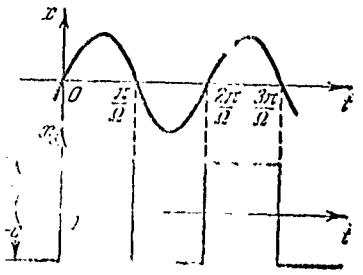


Fig. 1.16.

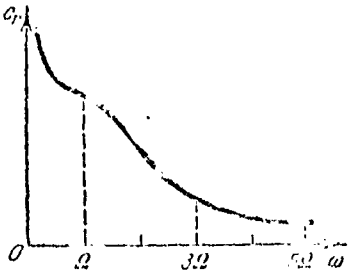


Fig. 1.17.

of the harmonic (Fig. 1.17). From a practical point of view, the high-frequency oscillations do not pass through the link in question (the link acts as if it were a low-pass filter). Electromechanical and mechanical drives always have this property. In addition, it must be taken into account that even the amplitudes B_r of the higher harmonics of the rectangular input quantity x_3 are also smaller the higher the harmonic number.

As a result, we may, under these conditions, regard the quantity x_4 as close from a practical point of view to the sine curve defined by the first harmonic:

$$x_4 \approx C_1 B_1 \sin(\Omega t + \beta_1 + \varphi_1). \qquad (1.74)$$

All this reasoning has been carried out for any single-frequency periodic form of the curve of x (Fig. 1.16). But if for this case x_4

is found to be close to the sine curve (1.74), then according to Eqs. (1.68) and (1.69) the quantities x_2 and $\underline{x}$ will also be close to sinusoidal; here

$$x_2 \approx D_1 C_1 B_1 \sin(\Omega t + \beta_1 + \varphi_1 + \psi_1),$$

where

$$D_1 = \frac{k_1}{\sqrt{T_1^2\Omega^2 + 1}}, \quad \psi_1 = -\pi - \operatorname{arctg} T_1\Omega, \tag{1.75}$$

and

$$x = x_2 - k_{o.c} x_4 \approx$$
$$\approx C_1 B_1 [D_1 \sin(\Omega t + \beta_1 + \varphi_1 + \psi_1) - k_{o.c} \sin(\Omega t + \beta_1 + \varphi_1)].$$

Consequently, determination of the symmetrical self-oscillations of the system under the conditions indicated above may be based upon finding the sinusoidal periodic solution of the given equations for the variables x_4, x_2, and $\underline{x}$. However, it is impossible for us to find the output variable x_3 of the relay in sinusoidal form. Nor is there any necessity for this, since, having determined only the oscillation frequency Ω of the variable $\underline{x}$, we may obtain the entire curve of x_3 (Fig. 1.16).

Thus, we shall find the periodic solution for the variable $\underline{x}$ approximately in the form

$$x = A \sin \Omega t, \tag{1.76}$$

where A and Ω are the unknown amplitude and frequency.* Inasmuch as all the variables ($\underline{x}$, x_2, x_4) are approximately determined according to the above formulas by the first harmonic of the quantity x_3, we do not take the complete expression of the nonlinear function (1.71) to find the solution (1.76), but only its first harmonic in the form**

$$[x_3]_1 = [F(x)]_1 = B_1 \sin \Omega t, \tag{1.77}$$

where B_1 is the Fourier coefficient:

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} F(A \sin \psi) \sin \psi \, d\psi, \quad \psi = \Omega t.$$

Taking Expression (1.76) for x into account, we rewrite Formula (1.77) in the form

$$[F(x)]_1 = qx, \tag{1.78}$$

where (see Fig. 1.16)

$$q = \frac{B_1}{A} = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi)\sin\psi\, d\psi =$$

$$= \frac{1}{\pi A}\int_0^{\pi} c\sin\psi\, d\psi + \frac{1}{\pi A}\int_{\pi}^{2\pi}(-c)\sin\psi\, d\psi,$$

i.e.,

$$q = \frac{4c}{\pi A}. \tag{1.79}$$

The operation (1.78) is known as harmonic linearization of a nonlinearity, while the quantity q is called the harmonic gain constant of the nonlinear link in question; this constant indicates the amplification of the first oscillation harmonic in the link considered. Here it is inversely proportional to the amplitude A of the input quantity x (Fig. 1.18), since the amplitude of the first harmonic at the output of such a relay is constant and equal to $4c/\pi$ (independently of the frequency and amplitude of x). Consequently, it is as if the nonlinear link $x_3 = F(x)$ were replaced in this investigation by a linear link with a definite gain constant q, which, however, takes different values for different amplitudes.

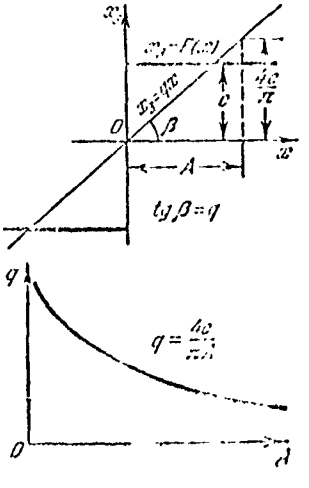


Fig. 1.18.

After the substitution (1.78), the equations of our system (1.68)-(1.70) are rewritten in the form

$$\left.\begin{aligned} (T_1 p + 1)x_2 &= -k_1 x_1, \\ x_3 &= qx, \quad x = x_2 - k_{o.c}x_1, \\ (T_2 p + 1)px_1 &= k_2 x_3. \end{aligned}\right\} \tag{1.80}$$

Inasmuch as we are looking for the solution in the form (1.76) with constant amplitude A, then q = const and (1.80) is a system of

linear equations with constant coefficients. But the peculiarity of this system consists in the fact that the magnitude of the constant coefficient q is unknown here. It is determined from (1.79) when A is found.

We write the characteristic equation of the system (1.80), retaining this same letter p for the variable of this equation:

$$T_1T_2p^3+(T_1+T_2)p^2+(1+T_1k_2k_{o.c}q)p+(k_1+k_{o.c})k_2q=0, \tag{1.81}$$

where q is expressed in terms of A according to Formula (1.79).

Assuming a sinusoidal form of the solution (1.76), the presence of a pair of purely imaginary roots $p = \pm j\Omega$ is required in this equation. Therefore, we make the substitution $p = j\Omega$ in the equation (1.81) and determine for which conditions it satisfies the given equation. With this substitution, Eq. (1.81) assumes the form

$$X(q, \Omega)+jY(q, \Omega)=0, \tag{1.82}$$

or

$$\left.\begin{aligned} X&=(k_1+k_{o.c})k_2q-(T_1+T_2)\Omega^2=0,\\ Y&=(1+T_1k_2k_{o.c}q)\Omega-T_1T_2\Omega^3=0. \end{aligned}\right\} \tag{1.83}$$

From the first equation of (1.83), substituting (1.79), we find

$$\Omega^2=\frac{4c(k_1+k_{o.c})k_2}{\pi A(T_1+T_2)}, \tag{1.84}$$

while from the second equation of (1.83), taking account of (1.84) and (1.79), we obtain:

$$1+T_1k_2k_{o.c}\frac{4c}{\pi A}-\frac{4cT_1T_2}{\pi A(T_1+T_2)}(k_1+k_{o.c})k_2=0,$$

from which

$$A=\frac{4ck_2T(T_2k_1+T_1k_{o.c})}{\pi(T_1+T_2)}, \tag{1.85}$$

whereupon from (1.84)

$$\Omega^2=\frac{k_1+k_{o.c}}{T_1(T_2k_1+T_1k_{o.c})}. \tag{1.86}$$

Inasmuch as A and Ω are positive quantities in a physical sense,

we will have, according to (1.85) and (1.86), the relationship

$$T_2k_1 - T_1k_{0.c} > 0. \tag{1.87}$$

as the existence condition of the periodic solution.

Thus it is evident that we may easily determine by purely algebraic means the magnitudes of the amplitude A (1.85) and frequency Ω (1.86) of the periodic solution which is being sought (1.76). Here A and Ω are expressed in general form in terms of the parameters of the system being investigated. Therefore, Formulas (1.85) and (1.86) may be used for investigation of the influence of any one of the parameters on the oscillation of the system over a wide range of variation of each parameter. In changing the system parameters, we must remember that the property of nonpassage of higher harmonics throughout the link 3 must be preserved. To verify this according to (1.73) we must calculate

$$C_r = \frac{k_3}{r\Omega\sqrt{r^2T_2^2\Omega^2+1}},$$

where Ω is determined by Formula (1.86), and require that $C_r \ll C_1$, i.e., that

$$\frac{k_3}{r\Omega\sqrt{r^2T_2^2\Omega^2+1}} \ll \frac{k_3}{\Omega\sqrt{T_2^2\Omega^2+1}},$$

from which, after simplification and the substitution of (1.86), we obtain the condition

$$\frac{(r^2+1)\,T_2^2(k_1+k_{0.c})}{T_1(T_2k_1-T_1k_{0.c})}+1 > 0. \tag{1.88}$$

This condition does not impose any limitations upon the system parameters if they are all positive, since the solution (1.86) has meaning only for $T_2k_1 > T_1k_{0.s}$, and here Condition (1.88) is always fulfilled. In this problem therefore, the approximate method of solution described is completely justified for any positive values of the system parameters over the whole region (1.87) of existence of the periodic solution.

§ 1.4. SELF-OSCILLATIONS AND EQUILIBRIUM STABILITY AS FUNCTIONS OF SYSTEM PARAMETERS

Above we found the amplitudes (1.85) and the frequency (1.86) of the periodic solution. This periodic solution with the amplitude A will correspond to real self-oscillations in the system if it is stable.

For investigation of the stability of the periodic solution, we shall assume that in its neighborhood a small change in the form of the solution corresponds to small changes in the initial conditions, i.e., that the solution retains the form $x = a \sin \psi$, but with the amplitude a(t) (Fig. 1.19) slowly varying over time; for a periodic solution it takes the constant value a = A. The system (1.80) and the characteristic equation (1.81) also remain valid for a transient process in the presence of small deviations from the periodic solution, where, by analogy with Formula (1.79), we have:

$$q = \frac{4c}{\pi a}. \qquad (1.89)$$

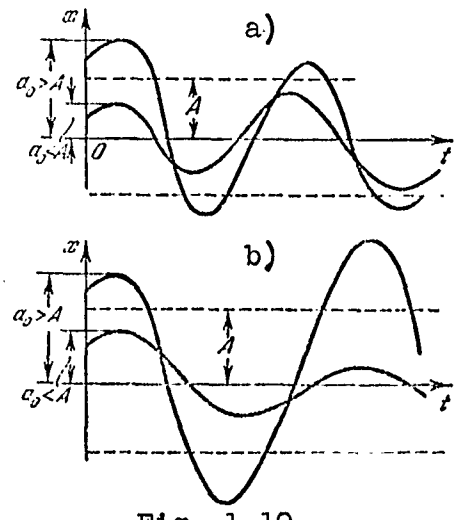


Fig. 1.19.

The periodic solution will be stable if for the initial value $a_0 > A$ (Fig. 1.19a) the amplitude a decreases in the transient process, tending toward the steady-state value A (here we are considering small deviations of a_0 from A). In this case, our linear characteristic equation (1.81) must satisfy the Hurwitz criterion for $a_0 > A$ (the oscillations are damped), while for $a_0 < A$ the criterion is not satisfied (the oscillations diverge).

If the periodic solution with amplitude A that has been found is unstable (Fig. 1.19b), there will be no self-oscillations with

the given amplitude A in the system.

Thus we must now establish whether we have in the given problem the case represented in Fig. 1.19a or the case represented in Fig. 1.19b.

For this, as indicated, we apply the Hurwitz criterion to Eq. (1.81). The first condition – all coefficients positive – is always satisfied here. The second condition will be

$$(T_1 + T_2)(1 + T_1 k_2 k_{o.c} q) - T_1 T_2 (k_1 + k_{o.c}) k_2 q > 0,$$

or, taking (1.89) into account,

$$\pi a (T_1 + T_2) + 4cT_1^2 k_2 k_{o.c} > 4cT_1 T_2 k_1 k_2. \qquad (1.90)$$

Inasmuch as we obtain an equality on substitution of the value a = A from (1.85), it is evident that for $a_0 > A$ the condition (1.90) is satisfied, while for $a_0 < A$ it is not satisfied.

Hence we draw the conclusion that in the present problem we have the case of the stable periodic solution represented in Fig. 1.19a, i.e., the periodic solution (1.76), (1.85), (1.86) that has been found actually determines the self-oscillations of the system.

Here Formulas (1.85) and (1.86) give the amplitudes A and frequencies Ω of the self-oscillations as functions of the various system parameters, as represented in Fig. 1.20.

In addition to the graphs in Fig. 1.20, which give A and Ω as functions of some one of the system parameters, we may also construct the direct dependences of A and Ω on any two parameters in the form of lines of equal values of A and Ω in the plane of these two parameters. In the plane of the parameters K_1 and $K_{o.s}$, for example, the equation of the lines A = const will, according to (1.85), be

$$k_{o.c} = \frac{T_2}{T_1} k_1 - \frac{\pi A (T_1 + T_2)}{4ck_2 T_1^2};$$

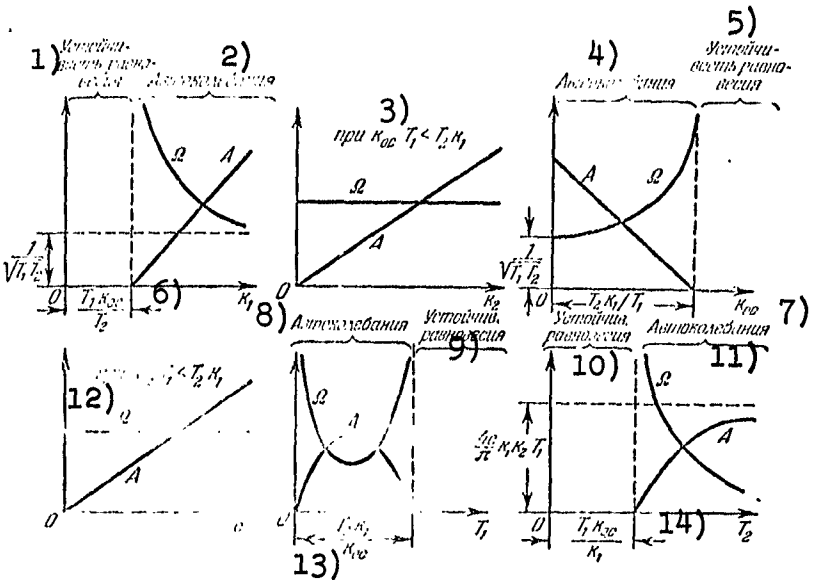


Fig. 1.20. 1) Equilibrium stability; 2) self-oscillations; 3) for $k_{o.s}$; 4) self-oscillations; 5) equilibrium stability; 6) $k_{o.s}$; 7) $k_{o.s}$; 8) self-oscillations; 9) equilibrium stability; 10) equilibrium stability; 11) self-oscillations; 12) for $k_{o.s}$; 13) $k_{o.s}$; 14) $k_{o.s}$.

these are parallel oblique straight lines (Fig. 1.21a). The equation for the lines Ω = const, on the other hand, will be according to (1.86)

$$k_{o.c} = \frac{T_1 T_2 \Omega^2 - 1}{T_1^2 \Omega^2 + 1} k_1;$$

these are straight lines coming from the origin (Fig. 1.21b). Using these same formulas, we may construct the lines A = const and Ω = = const in the plane of the parameters k_2 and $k_{o.s}$ (Fig. 1.22) and so forth.

Thus, we may obtain the region of existence of self-oscillations in the plane of any two system parameters; here, their amplitude and frequency will be known at each point.

The remaining part of the plane of the two parameters, which is not filled by the lines A = const and Ω = const, signifies the absence of self-oscillation in the system. Also in Fig. 1.20, we may see the region of no self-oscillations for each of the parameters

individually. The values of the parameters must lie in this region in cases where it is necessary that the system operate without self-oscillations; here we must satisfy outselves that the equilibrium state of the system in this region will be stable.

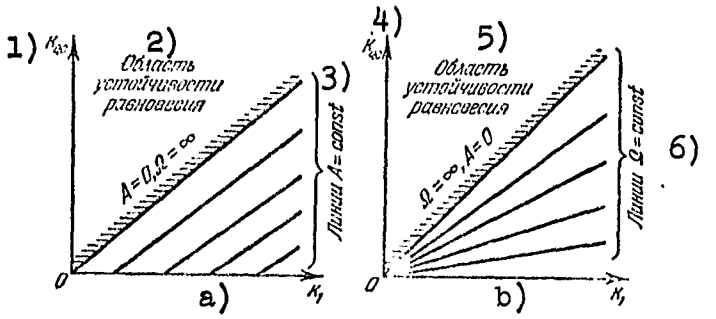


Fig. 1.21. 1) $k_{o.s}$; 2) region of equilibrium stability; 3) lines A = const; 4) $k_{o.s}$; 5) region of equilibrium stability; 6) lines Ω = const.

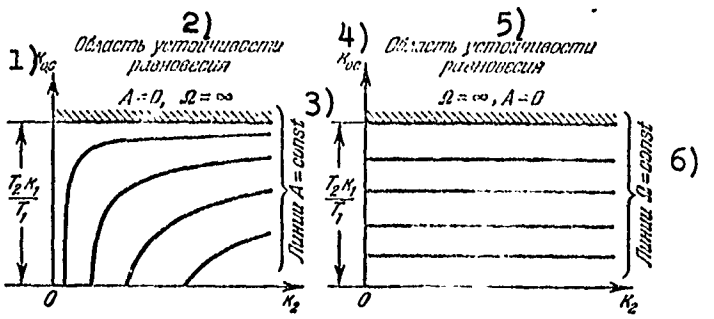


Fig. 1.22. 1) $k_{o.s}$; 2) region of equilibrium stability; 3) lines A = const; 4) $k_{o.s}$; 5) region of equilibrium stability; 6) lines Ω = const.

Inasmuch as the properties of the system vary gradually on variation of its parameters, we shall have beyond the boundaries of the region of self-oscillations* (above the lines A = 0, $\Omega = \infty$ in Figs. 1.21 and 1.22) and for arbitrary initial conditions a non-periodic oscillatory process which is either damped or diverging. In the former case the characteristic equation (1.81) will satisfy the Hurwitz criterion for all values of the amplitude a, i.e., Con-

dition (1.90) will be satisfied. In the second case it will not be satisfied.

Let us find whether Condition (1.90) as represented in Fig. 1.22 is satisfied in the region $k_{o.s} > T_2k_1/T_1$. Dividing both sides of (1.90) by $4cT_1^2k_2$, we obtain:

$$\frac{\pi a (T_1 + T_2)}{4cT_1^2k_2} + k_{o.c} > \frac{T_2k_1}{T_1},$$

from which it is evident that this condition is satisfied for arbitrary a in the region indicated. In Fig. 1.22, therefore, in the region above the line A = 0, $\Omega = \infty$, the oscillations are damped for any values of their amplitude, i.e., the equilibrium state of the system is stable. We obtain similar results for the regions of equilibrium stability in Figs. 1.21 and 1.20.

This result agrees fully with the physical pattern of the phenomena. Indeed, it is evident from Figs. 1.20, 1.21, and 1.22 that the amplitude of the self-oscillations decreases with increasing feedback constant $k_{o.s}$, and for a sufficiently large value of $k_{o.s}$, the self-oscillations in the system are suppressed and the equilibrium state of the system becomes stable. This corresponds to the well-known general property of all negative-feedback loops in control systems. In the absence of feedback ($k_{o.s} = 0$), however, self-oscillations take place for any relationship of the system parameters, and their amplitude increases with the gain constants and time constants, which is also consistent with what happens in practice.

Let us note that self-oscillations are not always desirable phenomena. At small amplitudes and a frequency in the safe range, they are often useful for eliminating stoppages due to dry friction and backlash, for increasing the sensitivity of the controller, etc.

(smoothing or "linearization" of nonlinearities by means of self-oscillations). We also encounter automatic systems in which it is necessary to produce specific stable self-oscillatory operating modes with amplitude and frequency completely determined.

Formulas (1.85) and (1.86) and the graphs of Figs. 1.20-1.22 give the magnitude of the self-oscillation amplitude A for the variable $\underline{x}$. For the variables x_4 and x_2 (see Fig. 1.15), we may calculate the amplitudes A_4 and A_2 for the transfer functions resulting from the given equations (1.80):

$$A_4 = \frac{k_4 q(A)}{\Omega \sqrt{T_4^2\Omega^2 + 1}} A, \quad A_2 = \frac{k_1}{\sqrt{T_1^2\Omega^2 + 1}} A_4. \tag{1.91}$$

As for the variable x_3, however, it has the form of a periodic rectangular curve (Fig. 1.16) with the period $2\pi/\Omega$.

§ 1.5. AN EXAMPLE OF QUALITY EVALUATION OF SYMMETRICAL OSCILLATORY TRANSIENT PROCESSES

Above we considered self-oscillations and delineated the stability region of the system's equilibrium state in the absence of self-oscillations. Now in the same system without any extraneous disturbance (Fig. 1.15), let us investigate the transient process for any point in the parameter plane (Fig. 1.20-1.22), both in the region of existence of self-oscillations and in the region of equilibrium stability of the system, but only where an oscillatory transient process takes place. In some cases it may be damped slowly, in others rapidly, up to the case where the damping practically takes place in one period and only one overshoot is observed.

Let us assume that for the variable $\underline{x}$ the process has the shape shown in Fig. 1.23a, where envelopes 1 and 2 are approximately symmetrical with respect to the axis of $\underline{t}$. The damped oscillations of the input variable x have variable frequency here in contrast to the linear system. Ordinarily, the frequency increases with de-

creasing amplitude (as also occurs for self-oscillations, Fig. 1.20), although there may also be other cases. As a result, the switching frequency of the relay will also be variable (Fig. 1.23b). We shall assume that this frequency changes smoothly with the course of time. It is obvious that here the filter property (Fig. 1.17) of the link 3 (Fig. 1.15) will hold as before. This provides the possibility of finding a solution with a variable $\underline{x}$ in the transient process in the form of the smooth curve indicated in Fig. 1.23.

If a linear system of the third order were considered, then the solution for the oscillatory transient process would have the form

$$x = a_0 e^{\xi_1 t} \sin(\omega_1 t + \psi_0) + C e^{\eta_1 t},$$

which would correspond to the following three roots of the characteristic equation:

$$p_{1,2} = \xi_1 \pm j\omega_1, \quad p_3 = \eta_1.$$

If $|\eta_1| \gg |\xi_1|$, then for the initial condition $\dot{x}_0 = 0$ or for small $\dot{x}_0$ (as, for example, in Fig. 1.23) the last term of the equation is small, as a consequence of which

$$x \approx a_0 e^{\xi_1 t} \sin(\omega_1 t + \psi_0). \tag{1.92}$$

We shall assume that in the nonlinear system, in view of its smooth oscillatory character, we may also approximately describe the sought curve of x(t) (Fig. 1.23) by a damped sine surve, but with a variable frequency ω and a variable damping exponent ξ which will vary smoothly and quite slowly with respect to time. Here the oscillation amplitude $\underline{a}$ may very rapidly, that is, the process may be fast-damping, as it also is for constant $\xi = \xi_1$.

By such means we may describe a broad class of nonlinear oscillatory processes without imposing restrictions upon the speed of variation of the amplitude, i.e., retaining the possibility of investigating fast-damping (for $\xi < 0$) or diverging ($\xi > 0$) oscilla-

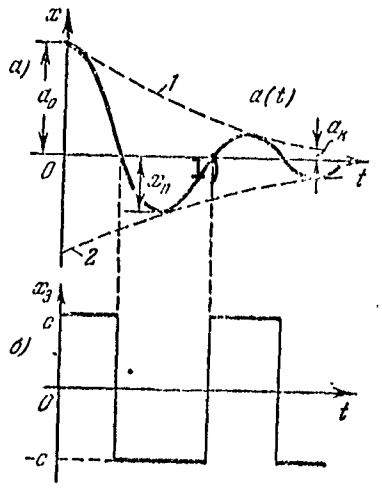


Fig. 1.23.

tory transient processes.

Let us note that here the solution x(t) will be considered over a finite time interval (one, two, or several oscillation periods), after which the process may be considered damped from a practical point of view.

Having set out to consider the frequency ω and the damping exponent ξ as variable, it is impossible to write the expressions for the amplitude and phase of the oscillations in the form (1.92), as for a linear system. We write the sought solution in the form

$$x = a \sin \psi, \tag{1.93}$$

where

$$\frac{da}{dt} = a\xi(a), \quad \frac{d\psi}{dt} = \omega(a). \tag{1.94}$$

We may easily verify that for $\xi = \text{const} = \xi_1$ and for $\omega = \text{const} = \omega_1$, Formulas (1.93) and (1.94) are reduced to the form (1.92). For the same variables ξ and ω we have

$$x = a_0 e^{\int \xi dt} \sin\left(\int \omega dt + \psi_0\right).$$

In the case being considered, therefore, the quality of a nonlinear transient process is determined in first approximation by two quantities: the damping exponent ξ(a) and the frequency ω(a). Having determined them, we may thereupon evaluate the shape of the transient process (i.e., estimate the damping time and "overshoot") over any interval of variation of the amplitude a: from the initial value a_0 to the final value a_k (Fig. 1.23), at which we can consider the process damped from a practical point of view. Hence the

problem is reduced to finding the quantities $\xi(a)$ and $\omega(a)$.

We defined the quantities ξ_1 and ω_1 in a linear system as the real and imaginary parts of a pair of complex roots of the characteristic equation. Here we shall proceed similarly. For this purpose, let us perform the harmonic linearization of the nonlinearity.

In the example being considered (Fig. 1.15), Formulas (1.78) and (1.89) retain their previous form, since the earlier process of the calculation of the harmonic gain constant q(a) corresponds to the form of the solution (1.93), which is adopted here. Therefore the equations of the harmonically linearized system will also be the same here:

$$\left.\begin{array}{c}(T_1p+1)x_2=-k_1x_3,\\ x_3=qx,\quad x=x_2-k_{o.c}x_4,\\ (T_2p+1)px_4=k_2x_3.\end{array}\right\} \tag{1.95}$$

Hence the characteristic equation also preserves its previous form:

$$T_1T_2p^3+(T_1+T_2)p^2+(1+T_1k_2k_{o.c}q)p+(k_1+k_{o.c})k_2q=0, \tag{1.96}$$

where, according to (1.89):

$$q=\frac{4c}{\pi a}. \tag{1.97}$$

This equation contains in its coefficients the quantity q, which is a function of a, and permits us to find the variables $\xi(a)$ and $\omega(a)$ by finding pairs of complex roots of this equation, in contrast to the linear system with constant coefficients. Such a process of computation is completely admissible in view of the sufficiently slow change of the quantities $\xi(a)$ and $\omega(a)$ which we assumed earlier.

To find the pairs of complex roots, we substitute $p = \xi + j\omega$ in the left-hand side of the characteristic equation (1.96). As a result we obtain an expression consisting of real and imaginary parts:

$$X(q, \xi, \omega) + jY(q, \xi, \omega) = 0,$$

where

$$\left.\begin{aligned} X &= T_1T_2\xi^3 + (T_1 + T_2)\xi^2 + (1 + T_1k_2k_{o.c}q)\xi + \\ &\quad + (k_1 + k_{o.c})k_2q - (3T_1T_2\xi + T_1 + T_2)\omega^2 = 0, \\ Y &= [3T_1T_2\xi^2 + 2(T_1 + T_2)\xi + 1 + T_1k_2k_{o.c}q]\omega - T_1T_2\omega^3 = 0. \end{aligned}\right\} \qquad (1.98)$$

These two equations also permit us to find the two quantities $\xi(a)$ and $\omega(a)$ for various combinations of the system parameters and thereby evaluate the quality of the transient processes with the object of choosing optimum system parameters.

From the second equation of (1.98), taking (1.97) into account, we find:

$$\omega^2 = 3\xi^2 + 2\frac{T_1 + T_2}{T_1T_2}\xi + \frac{1}{T_1T_2} + \frac{4ck_2k_{o.c}}{\pi T_2 a}, \qquad (1.99)$$

while from the first equation

$$a = \frac{4ck_2}{\pi f(\xi)}\left[k_1 - T_1k_{o.c}\left(\frac{1}{T_2} + 2\xi\right)\right], \qquad (1.100)$$

where

$$f(\xi) = \frac{T_1 + T_2}{T_1T_2} + 2\left[1 + \frac{(T_1 + T_2)^2}{T_1T_2}\right]\xi + 8(T_1 + T_2)\xi^2 + 8T_1T_2\xi^3 \qquad (1.101)$$

Let us note that as a particular case for $\xi = 0$, we obtain the amplitude (1.85) and the frequency (1.86) of the periodic solution (self oscillations) from the above.

In order to clearly represent the quantities $\xi(a)$ and $\omega(a)$ obtained here, which characterize the quality of the transient process, we proceed as follows. Let us construct the lines ξ = const and the lines ω = const in the plane with the coordinates k_1 and $\underline{a}$ (Fig. 1.24). According to Formula (1.100), the lines ξ = const in this plane will be straight lines whose slopes and initial abscissa are functions of the quantity ξ. According to (1.99), however, the quantity ω will decrease as $\underline{a}$ increases along each line ξ = const. Therefore the lines ω = const will rise less steeply than the lines

ξ = const, and are generally speaking, curvilinear.

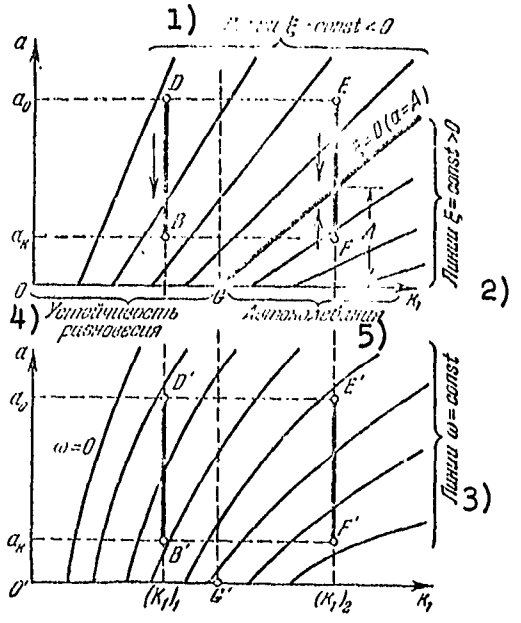


Fig. 1.24. 1) Lines; 2) lines; 3) lines; 4) equilibrium stability; 5) self oscillations.

The lines $\xi = 0$ in Fig. 1.24 gives the amplitude A of the self-oscillations as a function of the coefficient k_1, as indicated in the first graph of Fig. 1.20. To the left of the point G (Fig. 1.24) we have $\xi < 0$ for all values of a; this corresponds to stability of the equilibrium state of the system. To the right of the point G we have $\xi < 0$ only for $a \geqslant A$, i.e., for an amplitude greater than the self-oscillation amplitude (above the lines $\xi = 0$), and $\xi > 0$ for smaller amplitudes ($a < A$). This corresponds to the presence of a stable self-oscillatory process and instability of the equilibrium state of the system in the range of values of the parameter k_1 which is being considered.

The line $\omega = 0$ (Fig. 1.24) is a boundary line to the left of which our analysis has no meaning (we obtain imaginary frequency values). We may assume that we have a monotonic aperiodic transient process to the left of this line (we confine ourselves to consideration of oscillatory processes only). Even the boundary of the region of oscillatory processes ($\omega = 0$) should be considered a coarse approximation.

Let us call the diagram represented in Fig. 1.24 the quality diagram for damping of nonlinear processes. It may also be constructed similarly for any other system parameters figuring in Fig. 1.20. If we construct such a diagram for a linear system, then all the lines

ξ = const and ω = const in it will be vertical.

On the basis of the diagram being considered, we may appraise the quality of the transient process for various values of the parameter k_1. Let us choose two different values: $(k_1)_1$ to the left of point G' and $(k_1)_2$ to the right of point G'. Let us consider the behavior of the transient process for the interval of amplitude variation from a_0 to a_k. We draw the vertical lines DB, D'B', and EF E'F' (Fig. 1.24) accordingly, and on a separate graph (Fig. 1.25) we represent the variation of the quantities ξ and ω, which are taken from Fig. 1.24, along these lines as functions of amplitude. The direction of time movement of the transient process is indicated by arrows (for negative ξ, the amplitude decreases, while for positive ξ it increases). The amplitude and frequency of the self-oscillations are designated by A and Ω. We may judge from the magnitude of ξ the damping speed of the transient process, and from ω we may judge the number of oscillations during the transient process and the magnitude of the overshoot x_p (Fig. 1.23). This will be described in detail in Chapter VII. There we shall also consider the problem of conversion of these quantities, which have been found for the variable $\underline{x}$, into other variables, in particular into the output quantity x_4.

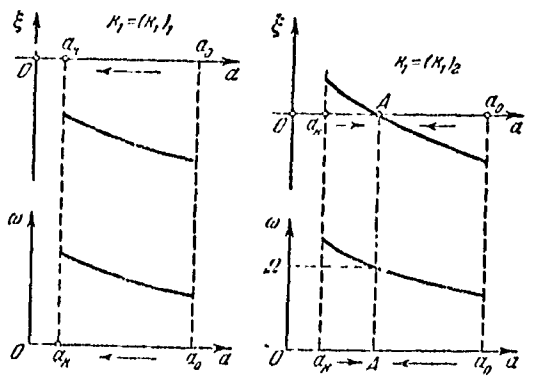


Fig. 1.25.

Thus, with the help of damping diagrams of the nonlinear processes made up for various system parameters, we may choose the optimum range of values of these parameters, starting from a desired property of the oscillatory transient process* or requiring that the transient process be monotonic.

Let us note that close to the self-oscillatory mode of operation (i.e., close to the line $\xi = 0$, Fig. 1.24), the process of generation of self-oscillations is characterized by relatively small values of ξ. In this case, according to (1.101), we have approximately:

$$f(\xi) \approx \frac{T_1 + T_2}{T_1 T_2} + 2\left[1 + \left(\frac{T_1 + T_2}{T_1 T_2}\right)^2\right]\xi.$$

Substituting this in Formula (1.100), we find from it in explicit form

$$\xi = -\frac{T_1 + T_2}{2T_1T_2}\frac{a - A}{Ha + D}, \tag{1.102}$$

where A is the self-oscillation amplitude as determined by Formula (1.85), i.e.,

$$A = \frac{4ck_2T_1(T_2k_1 - T_1k_{0.c})}{\pi(T_1 + T_2)};$$

$$H = 1 + \frac{(T_1 + T_2)^2}{T_1T_2}, \quad D = \frac{4c}{\pi} k_2k_{0.c}T_1.$$

This permits us to carry out integration of Eq. (1.94) and find a(t), the oscillation envelope [208].

§ 1.6. EXAMPLE OF NONSYMMETRICAL SELF-OSCILLATIONS AND STATIC ERRORS OF THE SELF-OSCILLATORY SYSTEM

Let us first consider the same automatic system (Fig. 1.15) without extraneous disturbances, but with a nonsymmetrical nonlinearity in the simplest form (Fig. 1.26a). In this case, it is evident that a constant component must develop in the steady-state oscillatory mode of operation. Hence the approximate solution for the self-oscillations, in contrast to (1.76), must be sought here in the form

$$x = x^0 + x^*, \text{ where } x^* = A \sin \Omega t, \tag{1.103}$$

where x^0 is the constant component and x* is the periodic (oscillatory) component.

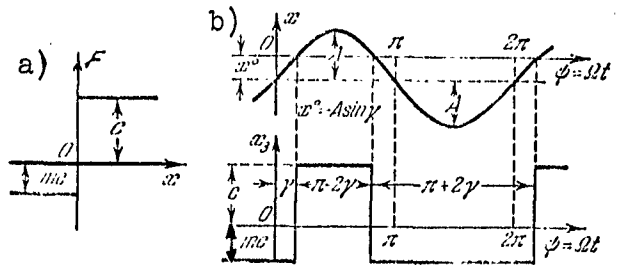


Fig. 1.26.

In correspondence with this, we must also add the constant component to the expansion of the nonlinear function F(x) in Fourier series (1.71). Keeping in mind as previously that the higher harmonics do not pass through the link 3 (see § 1.3), let us also restrict ourselves here to the constant component of the first harmonic. Then, in much the same way as with (1.78), we shall have

$$[x_3]_{0,1} = [F(x)]_{0,1} = x_3^0 + qx^*, \tag{1.104}$$

where, according to the familiar formulas for the coefficients of Fourier series, we have (see Fig. 1.26b):

$$x_3^0 = \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A\sin\psi)\,d\psi = \frac{(\pi-2\gamma)c-(\pi+2\gamma)mc}{2\pi} =$$

$$= \frac{(1-m)c}{2} - \frac{(1+m)\gamma c}{\pi},$$

$$q = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi)\sin\psi\,d\psi =$$

$$= \frac{c}{\pi A}\int_{+\gamma}^{\pi-\gamma}\sin\psi\,d\psi - \frac{mc}{\pi A}\int_{\pi-\gamma}^{2\pi+\gamma}\sin\psi\,d\psi = \frac{2(1+m)c}{\pi A}\cos\gamma.$$

Hence, taking into account that according to Fig. 1.26b, $x^0 = -A\sin\gamma$ we obtain

$$x_3^0 = \frac{(1-m)c}{2} + \frac{(1+m)c}{\pi}\arcsin\frac{x^0}{A}, \tag{1.105}$$

$$q = \frac{2(1+m)c}{\pi A}\sqrt{1-\left(\frac{x^0}{A}\right)^2}. \tag{1.106}$$

After the harmonic linearization (1.104) with (1.103) taken into account, the equations of the system (1.68)-(1.70) take the form

$$\left.\begin{aligned} (T_1 p+1)x_3 &= -k_1 x_4, \quad x_3 = x_3^0 + qx^*, \\ x^0 + x^* &= x_2 - k_{o.c} x_4, \quad (T_2 p+1) p x_4 = k_2 x_2. \end{aligned}\right\} \tag{1.107}$$

It is evident that the remaining variables x_2 and x_4 will also be composed of constant and periodic components:

$$x_2 = x_2^0 + x_2^*, \quad x_4 = x_4^0 + x_4^*.$$

From (1.107), applying (1.105), we obtain the following equations here for the constant components

$$\left.\begin{aligned} x_3^0 &= -k_1 k_4^0, \quad x_3^0 = \frac{(1-m)c}{2} + \frac{(1+m)c}{\pi} \arcsin \frac{x^0}{A}, \\ x^0 &= x_2^0 - k_{o.c} x_4^0, \quad 0 = k_2 k_2^0. \end{aligned}\right\} \tag{1.108}$$

For the periodic components of (1.107), however, we find the equations

$$\left.\begin{aligned} (T_1 p+1)x_3^* &= -k_1 x_4^*, \quad x_3^* = qx^*, \\ x^* &= x_2^* - k_{o.c} x_4^*, \quad (T_2 p+1) p x_4^* = k_2 x_2^*. \end{aligned}\right\} \tag{1.109}$$

where the quantity q is determined by Formula (1.106).

From Eq. (1.108) we find:

$$\left.\begin{aligned} x_2^0 &= 0, \quad x^0 = -A \sin \frac{\pi(1-m)}{2(1+m)} = A \cos \frac{\pi}{1+m}, \\ x_3^0 &= \frac{k_1 x^0}{k_1 + k_{o.c}}, \quad x_4^0 = -\frac{x^0}{k_1 + k_{o.c}}. \end{aligned}\right\} \tag{1.110}$$

Equations (1.109) for the periodic components, however, coincide with the earlier (1.80), but here there is a new value for q. Substituting the expression x^0 from (1.110) into Formula (1.106), we find:

$$q = \frac{2(1+m)c}{\pi A} \sin \frac{\pi}{1+m}. \tag{1.111}$$

Therefore, making use of the earlier characteristic equation (1.81) and Eqs. (1.83), we obtain here instead of (1.85) a solution for the self-oscillation amplitude in the form

$$A = \frac{A_c}{2}(1+m)\sin\frac{\pi}{1+m}, \tag{1.112}$$

where

$$A_c = \frac{4ck_2T_1(T_2k_1 - T_1k_{o.c})}{\pi(T_1+T_2)}$$

represents the self-oscillation amplitude for a symmetric nonlinearity (for m = 1).

The solution for the self-oscillation frequency leads to the same expression as for symmetric oscillations:

$$\Omega^0 = \frac{k_1 + k_{o.c}}{T_1(T_2k_1 - T_1k_{o.c})}. \tag{1.113}$$

The magnitudes of the constant components for all the variables are then determined according to Formulas (1.110).

The resulting graphs of the amplitude A and displacement x^0 as functions of the quantity m, which characterizes the nonsymmetry of the nonlinearity (Fig. 1.26a), are presented in Fig. 1.27.

Let us pass now to consideration of the static errors of a nonlinear automatic system operating under self-oscillatory conditions in the presence of an extraneous disturbance.

Let us assume that only one extraneous disturbance $f_1(t)$, which we shall call the setting or controlling disturbance (Fig. 1.28), acts upon the automatic system whose diagram is represented in Fig. 1.1. The equations of the system (1.1)-(1.7) take the form

$$(T_1p+1)x_1 = k_1x_1, \quad x_1 = f_1(t) - x_0, \tag{1.114}$$

$$x_2 = F(x), \quad x = x_2 - x_{o.c}, \tag{1.115}$$

$$(T_2p+1)px_0 = k_2x_2, \quad x_{o.c} = k_{o.c}x_0. \tag{1.116}$$

Let F(x) have the extremely simple symmetric form (Fig. 1.2e) as shown in Fig. 1.28.

Let us first consider the case of application of a constant-magnitude extraneous disturbance $f_1(t) = \text{const} = f_1^0$. Even for a symmetrical nonlinearity, this may cause the appearance of constant

components which, according to the equations of the system being considered (1.114)-(1.116), may be determined as follows:

$$\left.\begin{aligned} x_3^0 &= k_1(f_1^0 - x_4^0), \\ x_2^0 &= F(x^0), \quad x^0 = x_3^0 - x_{0.c}^0, \\ 0 &= k_2 x_4^0, \quad x_{0.c}^0 = k_{0.c} x_4^0. \end{aligned}\right\} \tag{1.117}$$

From this we obtain

$$\left.\begin{aligned} x_3^0 &= 0, \quad x^0 = 0, \quad x_3^0 = x_{0.c}^0 = k_{0.c} x_4^0, \\ x_4^0 &= \frac{k_1 f_1}{k_1 + k_{0.c}}. \end{aligned}\right\} \tag{1.118}$$

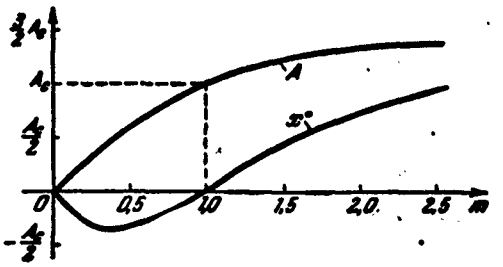


Fig. 1.27

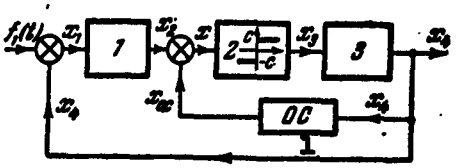


Fig. 1.28. 1) OS.

In the absence of a secondary feedback or on connection of a "flexible" [proportional-plus-derivative] feedback, we would have $x_2^0 = 0$, as in an astatic control system or in an ordinary follow-up system, with $x_4^0 = f_1^0$, i.e., we would have exact reproduction of the input quantity if we disregard the magnitude of the self-oscillation impressed upon the output, since the self-oscillations may be made as small as we wish.

Inasmuch as $x^0 = 0$, then the solution for the self oscillations of the variable $\underline{x}$ has the symmetric form (1.76), as a consequence of which all the computations of §§ 1.3 and 1.4 also remain valid here. Hence, the self-oscillation amplitude and frequency in the system being considered (Fig. 1.28) are not functions of the magnitude of the constant extraneous disturbance f_1^0, and here the constant components (1.118) are added only for certain variables. This was the result of the presence of the integrating link 3 in the

system and the application of the external disturbance outside of this link.

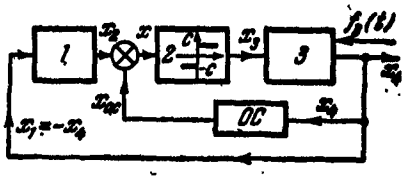


Fig. 1.29.

Let us also consider the case of application of an extraneous disturbance $f_3(t)$, which we shall call a perturbing disturbance (load), to the integrating link 3 (Fig. 1.29). Here the equations of the system (1.1)-(1.7) are

$$(T_1 p + 1) x_2 = -k_1 x_4, \tag{1.119}$$

$$x_3 = F(x), \quad x = x_2 - x_{o.c}, \tag{1.120}$$

$$(T_3 p + 1) p x_4 = k_3 x_3 + f_3(t), \quad x_{o.c} = k_{o.c} x_4. \tag{1.121}$$

Let there also be in this system a constant external disturbance $f_3(t) = \text{const} = f_3^0$ (Fig. 1.29).

In this case the solution for the self-oscillations of the variable x must be sought in the form (1.103), using the harmonic linearization in the form (1.104). Inasmuch as the nonlinear characteristic is symmetric in the case being considered, we must assume that $m = 1$. Here we find from Formulas (1.105) and (1.106)

$$x_3^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A}, \tag{1.122}$$

$$q = \frac{4c}{\pi A} \sqrt{1 - \left(\frac{x^0}{A}\right)^2}. \tag{1.123}$$

According to (1.119)-(1.121), the system equations after the harmonic linearization (1.104) will be

$$\left.\begin{aligned} &(T_1 p + 1) x_2 = -k_1 x_4, \quad x_3 = x_3^0 + q x^0, \\ &x = x_2 - k_{o.c} x_4, \quad (T_3 p + 1) p x_4 = k_3 x_3 + f_3^0. \end{aligned}\right\} \tag{1.124}$$

From this, taking (1.122) into account, we obtain the system of equations for the constant components:

$$\left.\begin{aligned} &x_2^0 = -k_1 x_4^0, \quad x_3^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A}, \\ &x^0 = x_2^0 - k_{o.c} x_4^0, \quad 0 = k_3 x_3^0 + f_3^0. \end{aligned}\right\} \tag{1.125}$$

For the periodic components we obtain from (1.124) the following system of equations:

$$\left.\begin{aligned}(T_1p+1)\,x_3^0 &= -k_1x_4^0, \quad x_3^0 = qx^0,\\ x^0 &= x_3^0 - k_{o.c}x_4^0, \quad (T_2p+1)\,px_4^0 = k_2x_3^0.\end{aligned}\right\} \tag{1.126}$$

The second and last of Eqs. (1.125) give

$$\frac{x^0}{A} = -\sin\frac{\pi f_3^0}{2ck_2}. \tag{1.127}$$

The system of equations (1.126) agrees with the previous system (1.109), with the new value of q, which, according to (1.127) and (1.123), will have the expression

$$q = \frac{4c}{\pi A}\cos\frac{\pi f_3^0}{2ck_2} \tag{1.128}$$

in place of the previous one (1.111). Here, therefore, the solution for the self-oscillation amplitudes will be instead of (1.112)

$$A = \frac{4ck_2T_1(T_2k_1 - T_1k_{o.c})}{\pi(T_1+T_2)}\cos\frac{\pi f_3^0}{2ck_2}. \tag{1.129}$$

The self-oscillation frequency is the same as before (1.113), i.e., it is not a function of the magnitude of the extraneous disturbance.*

Then the constant component x is determined from Formula (1.127), i.e.,

$$x^0 = -\frac{2ck_2T_1(T_2k_1 - T_1k_{o.c})}{\pi(T_1+T_2)}\sin\frac{\pi f_3^0}{ck_2}. \tag{1.130}$$

As we see, the magnitude of the self-oscillation amplitude A (like the displacement x^0) is a function of the magnitude of the constant extraneous disturbance f_3^0 (Fig. 1.30a). Here let us note that the computations performed are valid for the condition

$$-ck_2 < f_3^0 < +ck_2 \tag{1.131}$$

as follows from (1.127) on the basis of the requirement $x^0 \leq A$. Otherwise ($x^0 > A$) there will not be any oscillations in the system at all, since the quantity x will retain the same sign and for the whole time the relay will be connected in one direction.

The amplitudes and the displacements for all the remaining

variables may then be determined according to Formulas (1.91) and (1.125). To obtain the curve of $x_3(t)$ for the output of the relay link, however, it is sufficient, according to Fig. 1.26b, to know only the quantity Ω and

$$\gamma = -\arcsin\frac{x^0}{A} = \frac{\pi f_3^0}{2ck_3}. \qquad (1.132)$$

The case which has been described, when $f_3(t) = \text{const} = f_3^0$ is applied to the integrating link 3 (Fig. 1.29), corresponds to the case of determination of the static errors x^0 and x_4^0 due to the constant load f_3^0 at the output of the self-oscillatory relay automatic system.

Let us now turn to the system represented in Fig. 1.28 and consider <u>variation of the setting (controlling) extraneous disturbance $f_1(t)$ at constant speed</u>:

$$f_1(t) = c_1 t. \qquad (1.133)$$

Let us write the form of the solution for the quantity x_4 (Fig. 1.28), allowing for self-oscillations, in the form

$$x_4 = x_4^0 + c_4 t + x_4^*, \qquad (1.134)$$

where $x_4^0 = \text{const} = c_4 = \text{const}$ and x_4^* is the periodic component. Hence we have the derivative

$$px_4 = c_4 + px_4^*.$$

According to (1.116), therefore, the solution for the variable x_3 will have the form*

$$x_3 = x_3^0 + x_3^*, \text{ where } x_3^0 = \frac{c_4}{k_3}. \qquad (1.135)$$

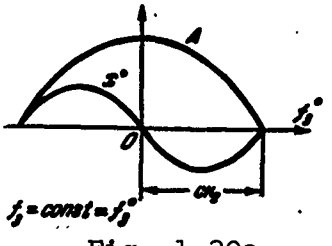


Fig. 1.30a.

It follows from this according to (1.115) that we also have a similar form of the solution for the variable <u>x</u> (without a component proportional to time):

$$x = x^0 + x^*, \text{ where } x^* = A\sin\Omega t. \qquad (1.136)$$

Further, according to (1.115), (1.116), and (1.134):

$$x_2 = x^0 + x^* + k_{0.c}(x_4^0 + c_4 t + x_4^*),$$

i.e., we must assume that x_2 has a solution of the form

$$x_2 = x_2^0 + c_2 t + x_2^*, \tag{1.137}$$

where

$$x_2^0 = x^0 + k_{0.c}x_4^0, \quad c_2 = k_{0.c}c_4, \quad x_2^* = x^* + k_{0.c}x_4^*.$$

However, for the absence of a "rigid" [proportional] secondary feedback ($k_{0.s} = 0$) the variable $x_2 = x = x^0 + x^*$ will not contain the component $c_2 t$ (astatic system).

Formulas (1.135) and (1.136) show that the pattern of variation of the variables $\underline{x}$ and x_3 in time has the same form here as in Fig. 1.26b, but now for the asymmetric ($m = 1$) relay characteristic $x_3 = F(x)$ shown in Fig. 1.28. Hence, we retain here the harmonic-linearization formulas (1.104) and (1.123).

Allowing for the manner of variation of the variables in (1.133)-(1.137), and also Formulas (1.122), we obtain the following equations for the constant components from the given system of equations (1.114)-(1.116):

$$\left.\begin{aligned} T_1 c_3 + x_3^0 &= -k_1 x_4^0, \quad x_3^0 = \frac{2\varepsilon}{\pi}\arcsin\frac{x^0}{A}, \\ x^0 &= x_2^0 - k_{0.c}x_4^0, \quad c_3 = k_3 x_3^0. \end{aligned}\right\} \tag{1.138}$$

For the components which are proportional to time, we obtain on the basis of these same equations:

$$c_3 = k_1(c_1 - c_4), \quad 0 = c_2 - k_{0.c}c_4. \tag{1.139}$$

We obtain the previous system (1.126) for the periodic components. If we take into account here that the second and last of Eqs. (1.138) give the expression

$$\frac{x^0}{A} = \sin\frac{\pi c_4}{2\varepsilon k_3}, \tag{1.140}$$

which agrees with (1.127) on the substitution $c_4 = -f_3^0$, it becomes clear that the previous solution (1.129) and (1.130) is valid here

with f_3^0 replaced by $-c_4$. However, the quantity c_4 is not given. It must first be determined from Eqs. (1.139); namely

$$c_4 = \frac{k_1 c_1}{k_1 + k_{o.c}} \tag{1.141}$$

(for $k_{o.s} = 0$, we would have $c_4 = c_1$, as in an ordinary astatic system).

Thus, we obtain here the solution

$$A = \frac{4ck_2T_1(T_2k_1 - T_1k_{o.c})}{\pi(T_1 + T_2)} \cos \frac{\pi k_1 c_1}{2ck_2(k_1 + k_{o.c})},$$

$$\Omega^2 = \frac{k_1 + k_{o.c}}{T_1(T_2k_1 - T_1k_{o.c})},$$

$$x^0 = \frac{2ck_2T_1(T_2k_1 - T_1k_{o.c})}{\pi(T_1 + T_2)} \sin \frac{\pi k_1 c_1}{ck_2(k_1 + k_{o.c})}.$$

The amplitude A of the self-oscillations of the variable $\underline{x}$ and the displacement x^0 are shown in Fig. 1.30b as functions, on the basis of the above, of the rate of change c_1 of the extraneous disturbance $f_1(t)$. Then the constant components of all the remaining variables and the coefficient c_2 of the time-proportional component for the variable x_2 are determined from the simple algebraic relationship (1.138) and (1.139), while the amplitudes of the self-oscillation of the variables x_4 and x_2 are determined from Formulas (1.91).

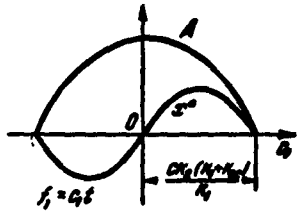


Fig. 1.30b.

The case which has been described, $f_1 = c_1 t$, corresponds to determination of the constant magnitude of the error x^0 (or c_4 for the variable x_4) of the automatic system being considered in the constant-speed following mode.

§ 1.7. EXAMPLE OF SLOWLY VARYING SIGNALS IN SELF-OSCILLATORY SYSTEMS

In the automatic system considered above (Fig. 1.28), let the extraneous input disturbance $f_1(t)$ change according to an arbitrary

law with respect to time, but relatively slowly, so that for the interaction among the links of the system the quantities x_1^0 may then be considered approximately constant within one period of the self-oscillation (Fig. 1.31). In this case, all the intrinsic variables in the system will have the form

$$x = x^0 + x^*, \quad x_1 = x_1^0 + x_1^*, \ldots, \tag{1.142}$$

where $x^0(t)$ and $x_2^0(t)$ are slowly varying components and x^*, x_2^* are oscillatory components, with

$$x^* = A \sin \psi, \quad \Omega = \frac{d\psi}{dt}, \tag{1.143}$$

where, generally speaking, A and Ω also change slowly with time together with the variation of the component x^0.

Assuming that all the slowly-changing components are constant within a self-oscillation period, we take the earlier form of the harmonic linearization (1.104), namely

$$[x_2]_{0;1} = [F(x)]_{0;1} = x_2^0 + qx^*, \tag{1.144}$$

where, in agreement with the symmetry of the nonlinear characteristic (Fig. 1.28) for m = 1, we have from (1.105) and (1.106):

$$x_2^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A}, \tag{1.145}$$

$$q = \frac{4c}{\pi A} \sqrt{1 - \left(\frac{x^0}{A}\right)^2}. \tag{1.146}$$

Substituting (1.142), (1.144), and (1.145) in the given system of equations (1.114)-(1.116), we write the equations for the slowly changing components separately:

$$\left.\begin{gathered} (T_1 p + 1) x_1^0 = k_1 [f_1(t) - x_{01}^0], \\ x_2^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A}, \quad x^0 = x_2^0 - k_{o.c} x_{01}^0, \\ (T_2 p + 1) p x_1^0 = k_2 x_2^0, \end{gathered}\right\} \tag{1.147}$$

and the equations for the oscillatory components:

$$\left.\begin{gathered} (T_1 p + 1) x_1^* = -k_1 x_{01}^*, \\ x_2^* = q x^*, \quad x^* = x_2^* - k_{o.c} x_{01}^*, \\ (T_2 p + 1) p x_1^* = k_2 x_2^*, \end{gathered}\right\} \tag{1.148}$$

where, according to (1.146) $\underline{q}$ is a function not only of the amplitude A, but also of the magnitude of the slowly varying component x^0. In turn, the unknown A also appears in Eqs. (1.147). Therefore the systems of equations (1.147) and (1.148) may not be solved independently of each other. This is a consequence of the nonvalidity of the principle of superposition for nonlinear systems.

Let us proceed as follows. From Eqs. (1.148), we find the self-oscillation amplitude A and frequency Ω as functions of the slowly varying component x^0 (and the system parameters), and thereupon we substitute the resulting function $A(x^0)$ in Eqs. (1.147), from which we then determine all the slowly varying components as functions of time for the given function $f_1(t)$.

Inasmuch as Eqs. (1.148) coincide with (1.80), Eqs. (1.83) are also valid here and from them we find the self-oscillation frequency

$$\Omega^2 = \frac{k_1 + k_{0.c}}{T_1(T_2 k_1 - T_1 k_{0.c})} \tag{1.149}$$

and the expression for $\underline{q}$ in the form

$$q = \frac{T_1 + T_2}{T_1 k_2 (T_2 k_1 - T_1 k_{0.c})}. \tag{1.150}$$

Equating the right-hand parts of Eqs. (1.146) and (1.50), we obtain a biquadratic equation for determination of the amplitude A as a function of x^0:

$$\left(\frac{A}{A_c}\right)^4 - \left(\frac{A}{A_c}\right)^2 + \left(\frac{x^0}{A_c}\right)^2 = 0, \tag{1.151}$$

where we have adopted the notation

$$A_c = \frac{4ck_2 T_1 (T_2 k_1 - T_1 k_{0.c})}{\pi (T_1 + T_2)}. \tag{1.152}$$

The quantity A_c represents the value of the amplitude A for $x^0 = 0$, as in (1.85).

From (1.151) we find

$$\left(\frac{A}{A_c}\right)^2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{x^0}{A_c}\right)^2} \tag{1.153}$$

(we write only the plus sign in front of the radical since it alone satisfies the condition $A = A_c$ for $x^0 = 0$). In Fig. 1.32a we present the resulting dependence of the self-oscillation amplitude A on the magnitude of the slowly varying component x^0.

Substituting the function $A(x^0)$ from (1.153) into Formula (1.145), we find*:

$$x_0^0 = \frac{2e}{\pi} \arcsin \frac{\frac{x^0}{A_c}}{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{x^0}{A_c}\right)^2}}} = \frac{e}{\pi} \arcsin \frac{2x^0}{A_c} \quad \left(x^0 < \frac{A_c}{2}\right), \tag{1.154}$$

which is presented graphically in Fig. 1.32b. For a comparatively narrow interval of variation $|x^0| < A_c/3$, Formula (1.154) may be written approximately in the linear form

$$x_0^0 = k_n x^0, \text{ where } k_n = \frac{2e}{\pi A_c}; \tag{1.155}$$

here the quantity A_c is determined from the system parameters according to Formula (1.152).

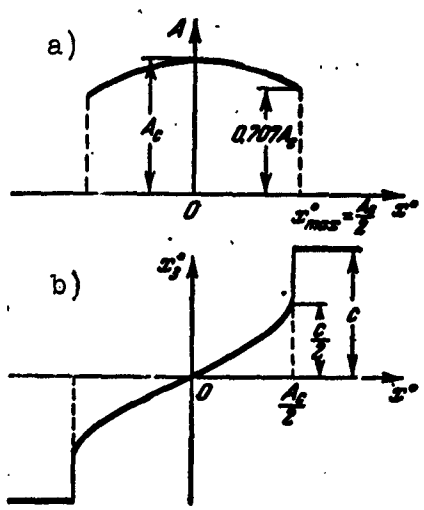


Fig. 1.32.

According to Fig. 1.32b, the curve of (1.154) is situated between the lines $x_0^0 = \frac{2e}{\pi A_c} x^0$ and $x_0^0 = \frac{e}{A_c} x^0$. Therefore, on variation of x^0 in the complete interval $0 \leq |x^0| \leq A_c/2$, we may for the first approximation replace the entire curve of (1.154) by the central ray

$$x_0^0 = \left(\frac{1}{\pi} + \frac{1}{2}\right) \frac{e}{A_c} x^0,$$

i.e., we can take the following value of k_n:

$$k_n = \left(\frac{1}{\pi} + \frac{1}{2}\right) \frac{e}{A_c}. \tag{1.156}$$

Thus, we have drawn the very interesting and important conclusion that in the presence of self-oscillatory vibrations, slowly varying signals pass through the automatic system as if the discon-

tinuous relay characteristic (Fig. 1.28) were replaced by a continuous characteristic (Fig. 1.32b) in the interval $-A_c/2 \leq x^0 \leq A_c/2$. For $|x^0| > A_c/2$, there are no self-oscillations, the relay is cut in in one direction and $|\dot{x}_3^0| = c$.

This phenomenon is called vibrational smoothing of nonlinear characteristics by means of self-oscillations. For comparatively small values of the slowly varying component x^0, we may consider this characteristic linear: (1.155) or (1.156). We may fully expect a sufficiently small order of magnitude of x^0 in good automatic systems, since x^0 represents a certain mismatch (Fig. 1.28). As a result, all the slowly functioning processes in the relay self-oscillatory system being considered may be calculated with respect to ordinary purely linear equations in place of (1.147):

$$\left.\begin{array}{c}(T_1 p+1)x_1^0 = k_1[f_1(t)-x_4^0],\\ x_2^0 = k_2 x^0, \quad x^0 = x_3^0 - k_{o.c}x_4^0,\\ (T_3 p+1)p x_4^0 = k_3 x_2^0,\end{array}\right\} \tag{1.157}$$

where the equivalent gain constant of the nonlinear link k_n for a slowly varying component is defined by Formulas (1.155) and (1.152) in terms of all the remaining system parameters. With a change in x^0 in the complete interval (to $A_c/2$) we may take the value of k_n in the form (1.156). For $|x^0| > A_c/2$ the phenomenon of vibrational smoothing ceases to exist.

When we find this solution $x^0(t)$ from Eqs. (1.157), we may, making use of the functional relationship (1.153) (Fig. 1.32a), also trace the time variation of the self-oscillation amplitude superimposed upon this process (Fig. 1.31). Here we must remember that Formula (1.153) and Fig. 1.32a give the oscillation amplitude A only for the variable $\underline{x}$. The oscillation amplitudes of the remaining variables are determined by finding A with the aid of Relation-

ships (1.91).

It is important to note that the self-oscillatory component x_4^*, which is superimposed upon the slowly varying component x_4^0 (Fig. 1.31a), represents the periodic error of the system. Therefore the system parameters must be chosen so that the quantity A_4 obtained from (1.91) will be as small as possible (in order that in practice it will not be perceived in the output signal x_4 of the system). At the same time, the oscillation amplitude A of the relay output $\underline{x}$ (Fig. 1.31b) must be sufficiently large – in any case larger than the quantity x^0 – for relay switching to take place at all times.

As concerns the self-oscillation frequency Ω, however, according to (1.149) it does not depend upon x^0 in the system being considered and will remain constant at all times.

It is evident that for slowly elapsing processes we may apply all methods of linear automatic-control theory to Eqs. (1.157) for the determination of all forms of the dynamic errors of the automatic system, for both analysis and synthesis of the system. We may synthesize ordinary linear frequency characteristics for slowly elapsing processes, even including logarithmic characteristics, which are valid for a low-frequency interval which is larger the higher the self-oscillation frequency bounding this interval above.

§ 1.8. AN EXAMPLE OF THE DETERMINATION OF THE QUALITY OF NONSYMMETRICAL OSCILLATORY TRANSIENT PROCESSES

In the preceding paragraph we obtained a means of investigating slowly (in comparison with the self-oscillation period) elapsing dynamic processes arising in a nonlinear system under the influence of extraneous slowly varying disturbances (controlling or perturbing). It is evident that the very same method may also be used for

approximate analysis of transient processes in a nonlinear automatic system in the absence of extraneous disturbances in cases where the transient may be regarded as consisting of oscillatory and aperiodic components x^* and x^0, respectively (Fig. 1.33):

$$x = x^0 + x^*, \ldots, \quad x_i = x_i^0 + x_i^*, \tag{1.158}$$

$$x^* = a \sin \phi, \quad \omega = \frac{d\phi}{dt}, \tag{1.159}$$

where $x^0(t)$, $a(t)$, and $\omega(t)$ are slowly varying functions of time that characterize the transient process in the nonlinear system in question. Below we shall consider separately the case where $a(t)$ varies rapidly.

The form of the solution (1.158) and (1.159) which is adopted here agrees with (1.142) and (1.143). Therefore we take advantage of the same results of harmonic linearization (1.144)-(1.146). Considering the case of the absence of extraneous disturbances, we substitute (1.158) in the appropriate equations for the system dynamics (1.68)-(1.70). Let us separate from the latter the equations for the aperiodic components

$$\left.\begin{gathered} (T_1 p + 1) x_1^0 = -k_1 x_4^0, \\ x_3^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{a}, \quad x^0 = x_2^0 - k_{o.c} x_4^0, \\ (T_2 p + 1) p x_4^0 = k_2 x_3^0, \end{gathered}\right\} \tag{1.160}$$

and the equations for the oscillatory components

$$\left.\begin{gathered} (T_1 p + 1) x_1^* = -k_1 x_4^*, \\ x_3^* = q x^*, \quad x^* = x_2^* - k_{o.c} x_4^*, \\ (T_2 p + 1) p x_4^* = k_2 x_3^*, \end{gathered}\right\} \tag{1.161}$$

where

$$q = \frac{4c}{\pi a} \sqrt{1 - \left(\frac{x^0}{a}\right)^2}. \tag{1.162}$$

These last agree fully with the previous equations (1.148), while the former equations (1.160) differ from the earlier ones (1.147) by the absence of the extraneous disturbance $f_1(t)$. There-

fore complete validity is retained here for all the previous results (1.149)-(1.156) and Fig. 1.32; here, the aperiodic components of the transient process may bc determined in first approximation from the linear equations

$$\begin{gathered}(T_1p+1)x_1^0=-k_1x_2^0\\ x_2^0=k_nx^0,\quad x^0=x_3^0-k_{0.c}x_2^0\\ (T_2p+1)\,px_3^0=k_2x_2^0\end{gathered}\tag{1.163}$$

where k_n is the equivalent gain constant of the nonlinear link for the aperiodic component:

$$k_n=\frac{2c}{\pi A_c},\quad A_c=\frac{4ck_2T_1(T_2k_1-T_1k_{0.c})}{\pi(T_1+T_2)}\tag{1.164}$$

for the interval of variation $|x^0| \leq A_c/3$ or

$$k_n=\left(\frac{1}{\pi}+\frac{1}{2}\right)\frac{c}{A_c}\tag{1.165}$$

for the full interval of variation $|x^0| \leq A_c/2$. If, however, $|x^0| > A_c/2$, the formulas lose their significance, since the relay will be cut in in one direction at all times.

Thus, on solving the linear equations (1.163), we find the aperiodic component $x^0(t)$ of the transient process. Having determined the latter, we find from Formula (1.153) or from the graph of Fig. 1.32a the amplitude variation a(t) of the transient-process oscillatory component:

$$\left(\frac{a}{A_c}\right)^2=\frac{1}{2}+\sqrt{\frac{1}{4}-\left(\frac{x^0}{A_c}\right)^2}.$$

As regards the frequency of the oscillatory component, however, it remains constant in the transient process for the example under consideration, according to (1.149). It depends only upon the system parameters and has the same value as for the self-oscillatory operating mode.

Hence it is evident that the calculation which has been carried out for the transient process has meaning only for that region of

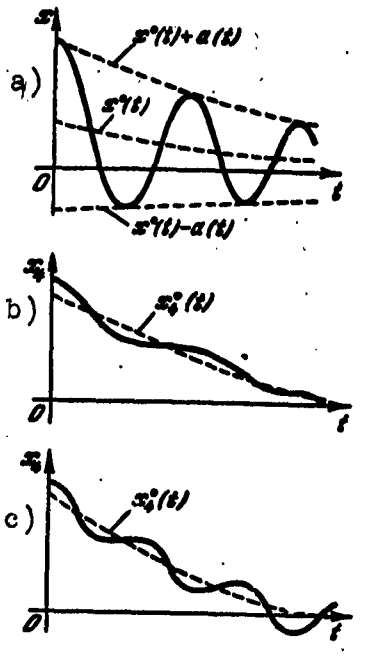


Fig. 1.33.

the system parameters in which the self-oscillatory operating mode exists. Hence, this investigation is applicable to transient processes as a result of which self-oscillations are stabilized (stabilization process of self-oscillations). Here, to guarantee the switching of the relay, the variable x must necessarily change its sign in the process of the oscillations (Fig. 1.33a). The other variables may, however, vary during this time in a sliding-process mode.

We will apply the term sliding process to a transient process whose basic component varies monotonically, as for example, $x_4^0(t)$ in Fig. 1.33, but with small-amplitude vibrations resulting from the operation of the relay in the vibrational mode superimposed upon it.* In the case being considered, such a sliding process may take place for the output variable x_4 of the system being considered, both outside of self-oscillatory region (Fig. 1.33b) and in the region of existence of self-oscillations (Fig. 1.33c), when small-amplitude self oscillations are generated for the variable x_4 as a result of the sliding process. Hence, the analysis which was carried out earlier in this paragraph is valid only for the second of these cases.

Subsequently (Chapter VII) it will be shown that such a method is also used successfully in other systems even where self-oscillations are absent.

For investigation of nonsymmetric oscillatory transient processes in the absence of self-oscillations we must use the method of § 1.5 in the example under consideration. This is also used for the region of existence of self-oscillations in investigation of transient processes of rapidly-varying amplitude, when it is impossible in Formula (1.159) to regard a(t) as a slowly varying function of time, but necessary to have recourse to Expression (1.94):

$$\frac{da}{dt} = a\xi(a), \tag{1.166}$$

in which the damping index ξ is a slowly changing function of time. The magnitude of the amplitude a will vary rapidly for large values of ξ and for small ξ it will vary slowly. Let us consider both variants for the cases of the presence and absence of self-oscillations.

In introducing a slowly varying aperiodic component of the transient process, we shall make use of Expression (1.166) together with (1.158) and (1.159) in contrast to the previous case (1.93). Here the equations for the oscillatory components (1.161) agree with the previous ones (1.95), which have already been solved above, but now we must introduce the new expression (1.162) for q. Therefore, applying the characteristic equation (1.96) and substituting $p = \xi + j\omega$ in it according to the method of § 1.5, we arrive at the same expressions (1.98) for X and Y.

Taking (1.162) into account, we find from the second equation of (1.98):

$$\omega^2 = 3\xi^2 + 2\frac{T_1+T_2}{T_1T_2}\xi + \frac{1}{T_1T_2} + \frac{4ck_2k_{o.c}}{\pi T_2 a}\sqrt{1-\left(\frac{x^0}{a}\right)^2}, \tag{1.167}$$

while for the first equation of (1.98)

$$\frac{a}{\sqrt{1-\left(\frac{x^0}{a}\right)^2}} = \frac{4ck_2}{\pi f(\xi)}\left[k_1 - T_1k_{o.c}\left(\frac{1}{T_2}+2\xi\right)\right], \tag{1.168}$$

where

(1.169)

As will be evident from comparison of these formulas with Formulas (1.99)-(1.101), the earlier damping diagrams of the transient processes (Fig. 1.24) will also be valid here if we plot $a/\sqrt{1-(x^0/a)^2}$ instead of $\underline{a}$ on the ordinate axis of these diagrams (Fig. 1.34). We may conclude from this that in the presence of an aperiodic component in the oscillatory transient process, the point corresponding to the oscillation amplitude determined will appear lower. According to Fig. 1.34, this means that the quantity ξ will be smaller in the region of the damped processes ($\xi < 0$), i.e., the damping time will be greater (the damping of the oscillations is slowed down). In the region of the diverging processes ($\xi > 0$) the quantity ξ will be larger, i.e., the increase in the amplitude will be faster (the process of stabilization of self-oscillation is accelerated).

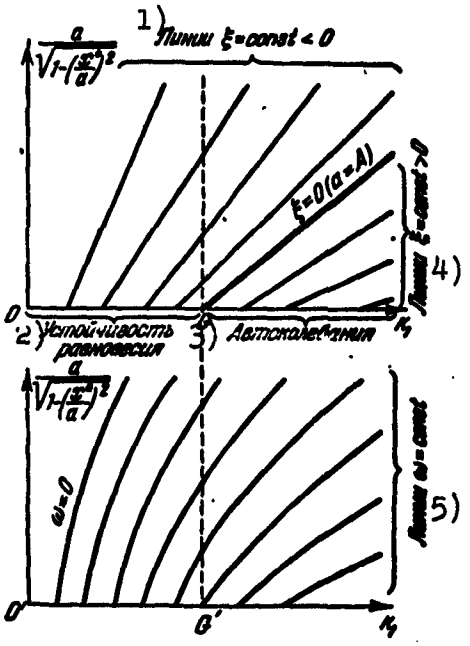


Fig. 1.34. 1) Lines ξ = const < < 0; 2) equilibrium stability; 3) self oscillations; 4) lines ξ = const > 0; 5) lines ω = = const.

With the approach to the self-oscillatory mode (i.e., to the line $\xi = 0$), however, the magnitude of the quotient x^0/a will decrease, tending toward zero, since the self-oscillations themselves are symmetrical in the system under consideration. Close to them [sic], therefore, the process of establishment of self-oscillation remains as previously and will be

described by Expression (1.102).

For the investigation of transient processes outside the region of self-oscillation for small values of the damping index ξ, we have from (1.169):

$$f(\xi)=\frac{T_1+T_2}{T_1T_2}+2\left[1+\frac{(T_1+T_2)^2}{T_1T_2}\right]\xi, \tag{1.170}$$

as a consequence of which for the presence of an aperiodic component x^0 in the transient process, we obtain from (1.168) the expression for the damping exponent ξ in explicit form

$$\xi=-\frac{T_1+T_2}{2T_1T_2}\frac{a+E\sqrt{1-\left(\frac{x^0}{a}\right)^2}}{Ha+D\sqrt{1-\left(\frac{x^0}{a}\right)^2}}, \tag{1.171}$$

where

$$E=\frac{4ck_2T_1(T_1k_{0.c}-T_2k_1)}{\pi(T_1+T_2)},$$

$$H=1+\frac{(T_1+T_2)^2}{T_1T_2},\quad D=\frac{4c}{\pi}k_2k_{0.c}T_1.$$

Thus we may approximately evaluate the influence of the aperiodic component in the transient process on the rate of damping of the oscillations, both in the case of predominance of the oscillatory component and in the sliding mode, as well as its influence upon the rate of generation of the self-oscillations. The calculation being considered does not pursue the goal of constructing the transient-process curve for given system parameters; other methods exist for this. Here the goal consists in investigation of the influence of various system parameters over a wide range of their variation (Fig. 1.34 indicates in particular the variation of the parameter k_1). This is necessary for preliminary choice of desirable values for the parameters, after which we may find exact time curves for the processes by numerical-graphical or machine methods for selected numerical values of the parameters (or for a selected narrow range of their variation).

The width of the ranges of variation of the system parameters

for various structural configurations, which may be examined from the point of view of transient-process quality, is, of course, determined by the validity of the assumptions concerning the character of the solutions and lying at the foundation of the present method of investigation.

§ 1.9. AN EXAMPLE OF THE DETERMINATION OF SINGLE-FREQUENCY FORCED OSCILLATIONS

Up until this time, all the extraneous disturbances in the automatic system have been regarded as slowly varying. Now let us assume that the frequency of the variation of the extraneous disturbance is not so small that it may be considered slowly varying, i.e., let us assume that the frequency of the extraneous disturbance is of the same order as the natural self-oscillation frequency of the system, or higher than it.

Let us assume that the extraneous disturbance $f_1(t)$ in the automatic system (Fig. 1.28) varies according to the sine law

$$f_1 = B \sin \Omega_v t. \tag{1.172}$$

Generally speaking, the forced oscillations of the nonlinear system may have an extremely complex form here. However, cases of simple forms of forced oscillations are also possible.

As a first approximation, let us find the forced oscillations for the variable $\underline{x}$ (Fig. 1.28) in the form of a sine curve

$$x = A_v \sin(\Omega_v t + \varphi) \tag{1.173}$$

with a given frequency Ω_v of the extraneous disturbance. The amplituve A_v and the phase shift φ are the unknowns. This is equivalent to construction of the frequency characteristics of a closed-loop nonlinear system.

The variable x_3 will have a rectangular shape (Fig. 1.16), while, in first approximation, the variables x_4, x_2, and $\underline{x}$ will

have the form of a sine curve, if Condition (1.88) [nonpassage of higher harmonics through link 3 (Fig. 1.28)] is fulfilled. Here the harmonic-linearization formulas (1.78) and (1.79) remain valid, where, in calculating $\underline{q}$, we must assume according to (1.173) that $\psi = \Omega_v t + \varphi$.

The assumption (1.173) concerning the oscillations of a system with the frequency Ω_v of the extraneous disturbance is not always valid for the case of self-oscillatory systems, but only for definite conditions, when the self-oscillations "break down" and the system passes completely to oscillations with the extraneous frequency Ω_v (the seizure condition, otherwise called forced synchronization). These conditions are also subject to determination.

In order to solve the problem by the same methods as those for finding the self-oscillations (§ 1.3), let us express the quantity $f_1(t)$ in terms of $\underline{x}$, using Formulas (1.172) and (1.173). For this we write

$$f_1 = B\sin\Omega_0 t = B\sin[(\Omega_0 t+\varphi)-\varphi] = \\ = B\cos\varphi\sin(\Omega_0 t+\varphi) - B\sin\varphi\cos(\Omega_0 t+\varphi).$$

Remembering that, according to (1.173),

$$\sin(\Omega_0 t+\varphi) = \frac{x}{A_0}, \quad \cos(\Omega_0 t+\varphi) = \frac{px}{A_0\Omega_0},$$

we finally obtain an expression for $f_1(t)$ in the form

$$f_1 = \left(\frac{B\cos\varphi}{A_0} - \frac{B\sin\varphi}{A_0\Omega_0}p\right)x. \tag{1.174}$$

Then, taking (1.78) and (1.174) into account, the given equations of our system (1.114)-(1.116) take the form

$$\left.\begin{gathered}(T_1p+1)x_2 = k_1x_1, \quad x_1 = \left(\frac{B\cos\varphi}{A_0} - \frac{B\sin\varphi}{A_0\Omega_0}p\right)x - x_4, \\ x_3 = qx, \quad x = x_3 - k_{0.c}x_4, \\ (T_2p+1)px_4 = k_2x_3\end{gathered}\right\} \tag{1.175}$$

where

$$q = \frac{4c}{\pi A_0}.$$

Thus the problem of determining forced oscillations under seizure conditions reduces to the solution of a homogeneous system, as in the case of self-oscillations, but with the introduction into it of an artificial secondary link ($\underline{x}$ in the second equation). The method of solution will be completely the same as in § 1.3, but because of this additional factor the result which is obtained will, of course, be different. An additional difference consists in that in § 1.3, A and Ω were unknowns, while here A_V and Ω are unknown (the frequency Ω_V is given).

The characteristic equation for system (1.175) has the form

$$T_1T_2p^3+(T_1+T_2)p^2+(1+T_1k_2k_{o.c}q)p+(k_1+k_{o.c})k_2q= \\ =k_1\left(\frac{B\cos\varphi}{A_3}-\frac{B\sin\varphi}{A_3\Omega_3}p\right)(T_2p+1)p \qquad (1.176)$$

[it differs from the previous (1.81) in that there is an additional expression on the right instead of zero].

To find the sine-curve periodic solution with the given frequency Ω_V, we substitute $p = j\Omega_V$ in the characteristic equation (1.176) and separate the real and imaginary parts:

$$X=(k_1+k_{o.c})k_2q-(T_1+T_2)\Omega_3^2=-\frac{k_1B\Omega_3}{A_3}(T_2\Omega_3\cos\varphi-\sin\varphi),$$
$$Y=(1+T_1k_2k_{o.c}q)\Omega_3-T_1T_2\Omega_3=\frac{k_1B\Omega_3}{A_3}(T_2\Omega_3\sin\varphi+\cos\varphi).$$

Solving these two equations for $\cos\varphi$ and $\sin\varphi$ and applying (1.179), we obtain two equations for determining A_V and φ:

$$-\pi k_2B\cos\varphi=E-\pi A_3,\quad \pi k_1B\sin\varphi=H-D\pi A_3, \qquad (1.177)$$

where

$$E=\frac{4ck_2(T_2k_1+T_2k_{o.c}-T_1k_{o.c})}{T_2^2\Omega_3^2+1},\quad H=\frac{4ck_2(k_1+k_{o.c}+T_1T_2k_{o.c}\Omega_3^2)}{\Omega_3(T_2^2\Omega_3^2+1)},$$
$$D=T_1\Omega_3.$$

Squaring both equations of (1.177) and adding term to term, we obtain the quadratic equation

$$(1+D^2)\pi^2A_3^2-2(E+HD)\pi A_3+E^2+H^2-\pi^2k_1^2B^2=0,$$

from which we find the amplitude of forced oscillations

$$A_3=\frac{E+HD\pm\sqrt{(E+HD)^2-(1+D^2)(E^2+H^2-\pi^2k_1^2B^2)}}{\pi(1+D^2)}. \qquad (1.178)$$

After this, dividing the second of the equations (1.177) by the first, we find the phase shift

$$\varphi=-\operatorname{arctg}\frac{H-D\pi A_3}{E-\pi A_3}. \qquad (1.179)$$

The amplitude A_V is a real positive quantity by its physical

meaning. Therefore we shall assume that the sought solution for forced oscillations exists in the form (1.173) (i.e., in other words, that the seizure phenomenon takes place) in that case where Formula (1.178) gives us at least one real positive result for A_v. On the basis of this, taking into account the fact that $E + HD > 0$ for positive system parameters, we obtain the seizure condition

$$(E+HD)^2 \geqslant (1+D^2)(E^2+H^2-\pi^2 k_1^2 B^2). \tag{1.180}$$

On the basis of Formula (1.178), we may construct a curve of the forced-oscillation amplitude A_v as a function of the frequency Ω_v for different amplitudes B of the extraneous disturbance and also ascertain the influence of the various system parameters upon the amplitude of the forced oscillations for the case where the seizure condition (1.180) is satisfied. This condition may be written in the form

$$\pi^2 k_1^2 B^2 \geqslant \frac{(H-ED)^2}{1+D^2}. \tag{1.181}$$

Hence it is evident that seizure takes place (i.e., the nonlinear system in question develops single-frequency forced oscillations with a frequency Ω_v imposed from without) only in the case where the amplitude B of the extraneous periodic disturbance exceeds a certain threshold value determined by Formula (1.181). This seizure threshold depends upon the relationship of the system parameters and upon the magnitude of the externally imposed frequency Ω_v, since they are used in calculating the quantities E, H, and D which figure here.

Similarly, we may also calculate the forced oscillations of the nonlinear system in question for the case where the load $f_3(t)$ varies according to a sine curve (Fig. 1.29).

Let us pass now to another problem. In § 1.7, it was shown that with the help of self-oscillations we may produce vibrational smoothing of discontinuous linear characteristics, and relay-type characteristics in particular. This may also be done with the help of forced oscillations, by feeding an extraneous disturbance $f_2(t)$ of sufficiently high frequency Ω_v to the input of the relay (Fig. 1.35) and guaranteeing the seizure conditions for it. Hence we shall now assume that

$$f_2 = B \sin \Omega_v t. \tag{1.182}$$

Let the input control disturbance $f_1(t)$ be slowly varying, as in § 1.7. Then a slowly varying component arises in all links. Therefore the solution for the variable x, for example, will have the form

$$x = x^0 + qx^*, \quad x^* = A_0 \sin(\Omega_v t + \varphi), \tag{1.183}$$

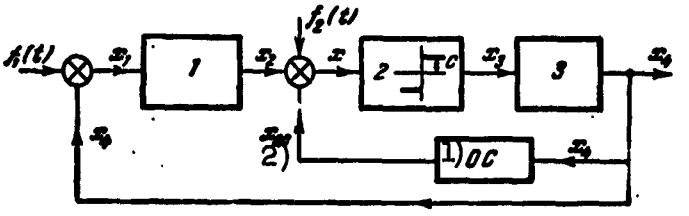


Fig. 1.35. 1) OS; 2) $x_{o.s}$.

The harmonic linearization of the nonlinearity leads to the previous formulas (1.144)-(1.146). Therefore, here as with (1.147) and (1.148), we may write equations for the slowly varying components:

$$\left.\begin{gathered}(T_1 p + 1) x_1^0 = k_1 [f_1(t) - x_{\text{ос}}^0], \\ x_{\text{ос}}^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A_0}, \quad x^0 = x_1^0 - k_{\text{о.с}} x_{\text{ос}}^0, \\ (T_2 p + 1) p x_1^0 = k_2 x_2^0,\end{gathered}\right\} \tag{1.184}$$

and for the oscillatory components (with allowance for the new disturbance ($f_2(t)$):

$$\left.\begin{array}{c}(T_1p+1)x_1^*=-k_1x_1^*,\\ x_3^*=qx^*,\quad x^*=x_3^*-k_{o.c}x_4^*+f_2(t),\\ (T_2p+1)px_4^*=k_2x_3^*,\end{array}\right\} \tag{1.185}$$

where q is determined by Formula (1.146), i.e.:

$$q=\frac{4c}{\pi A_3}\sqrt{1-\left(\frac{x^*}{A_3}\right)^2}. \tag{1.186}$$

According to Formula (1.174) we must substitute in Eqs. (1.185)

$$f_2=\left(\frac{B\cos\varphi}{A_3}-\frac{B\sin\varphi}{A_3\Omega_3}p\right)x^*. \tag{1.187}$$

It would then be possible to solve the equation for the oscillatory components (1.185) by the same method used to solve Eqs. (1.175). It is true that the result would be different, since here the external periodic disturbance plays a different part and there is a different expression for q. However, a simplified solution will be presented here.

We shall keep in mind that in the scheme of Fig. 1.35, the disturbance $f_2(t)$ is introduced with the sole purpose of creating a vibrational mode of operation of the relay (Fig. 1.31b) and that the parameters of the linear link 3 (Fig. 1.35) are such that for a given frequency Ω_v the amplitude of the forced vibrations x_4 at the output of the system is very small (Fig 1.31a). If now we also consider the amplitudes of the periodic components of the variables x_2 and $x_{o.s}$ (Fig. 1.35) to be very small, then, taking (1.182) into account, we may rewrite Eqs. (1.185) in the form

$$\left.\begin{array}{c}(T_1p+1)x_1^*=-k_1x_1^*,\\ x_3^*=qx^*,\quad x^*=f_2(t)=B\sin\Omega_3t,\\ (T_2p+1)px_4^*=k_2x_3^*,\end{array}\right\} \tag{1.188}$$

whence, according to (1.183), we find directly

$$A_3=B,\quad \varphi=0 \tag{1.189}$$

and, according to (1.186):

$$q=\frac{4c}{\pi B}\sqrt{1-\left(\frac{x^*}{B}\right)^2}. \tag{1.190}$$

where the quantity B is given as the amplitude of the extraneous periodic disturbance $f_2(t)$. Consequently, the coefficient $\underline{q}$ is a function only of x^0, and for each specific value of x^0 the forced oscillations of any variable are calculated from purely linear equations with constant coefficients (1.188). In particular, the amplitude of the forced oscillations x_4 at the output of the system will be

$$A_4 = \frac{k_2 q B}{\Omega_v \sqrt{T_1^2\Omega_v^2+1}} = \frac{4ck_2}{\pi\Omega_v}\sqrt{\frac{1-\left(\frac{x^0}{B}\right)^2}{T_1^2\Omega_v^2+1}}. \qquad (1.191)$$

We must ensure a small order of magnitude of A_4 by the choice of the frequency Ω_v and the other parameters entering here, while at the same time B must be larger than the maximum magnitude that is possible in the system being considered for the slowly varying mismatch x^0.

The small size of the amplitudes of the variables x_2 and $x_{o.s}$ is verified according to the formulas:

$$A_2 = \frac{k_1 A_4}{\sqrt{T_1^2\Omega_v^2+1}}, \quad A_{o.c} = k_{o.c} A_4. \qquad (1.192)$$

It is always possible to guarantee small order of magnitude in A_4, A_2, and $A_{o.s}$ by the choice of a sufficiently high frequency Ω_v. If they are found not to be small in comparison with B, it is necessary to solve the complete system of equations (1.184)-(1.187).

Further, according to the second of Formulas (1.184), taking (1.189) into account, we have

$$x_3^* = \frac{2c}{\pi}\arcsin\frac{x^0}{B}. \qquad (1.193)$$

This gives us the smoothed relay characteristic shown in Fig. 1.36a. In a certain interval of variation of the mismatch x^0 (even in the interval $0 < |x^0| < 0.7B$) the relationship (1.193) may be

replaced by a linear relationship:

$$x_3^0 = k_n x^0, \text{ where } k_n = \frac{2c}{\pi B}. \qquad (1.194)$$

However, for the variation of x^0 in the full range as far as $|x^0| = B$ we might use the value k_n, in analogy to (1.156).

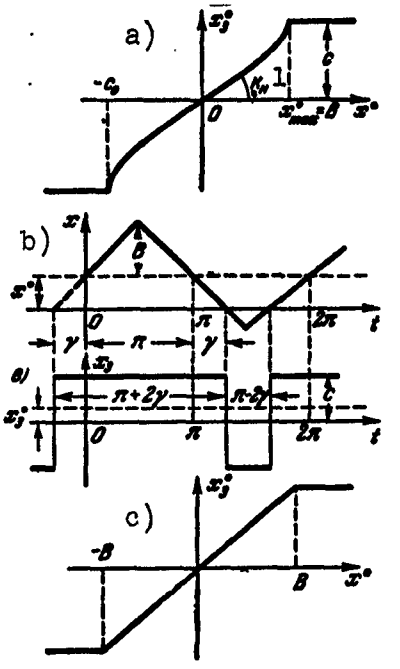


Fig. 1.36. 1) k_n.

In view of the sufficient freedom in assigning the amplitude B in the case in question (Fig. 1.36a), the smoothing zone may be made wider than in the case of self-oscillations (Fig. 1.32b), where it is completely determined by the relationship of the system parameters alone.

Thus, all the slowly elapsing processes in the system that are connected with the reproduction of $f_1(t)$, as well as the transient processes for the slowly varying components, may now be calculated by Eqs. (1.184), which, according to (1.194), are linear for a certain interval of x^0.

For the whole smoothing zone to be linear, saw-tooth oscillations (Fig. 1.36b) are used as the disturbance $f_2(t)$. In the presence of a displacement x^0 we find from similarity of the triangles (Fig. 1.36b):

$$\gamma = \frac{\pi x^0}{2B}.$$

According to Fig. 1.36c, therefore, the average component at the output of the relay, x_3^0, will be

$$x_3^0 = \frac{c(\pi + 2\gamma) - c(\pi - 2\gamma)}{2\pi} = \frac{2\gamma c}{\pi},$$

or

$$x_3^0 = k_n x^0, \text{ where } k_n = c/B. \qquad (1.195)$$

The corresponding smoothed characteristic is shown in Fig. 1.36d. But it should not be forgotten that such a linear smoothed characteristic is obtained only when the amplitudes of the forced vibrations of the variables x_2 and $x_{o.s}$ are very small in comparison with B, since the forced vibrations of these variables, which are generated with the frequency of the saw tooth oscillations, add to them according to Fig. 1.35 and will distort the saw tooth shape of the oscillations at the relay input and change their amplitude, hence distorting the linear dependence of Fig. 1.36d.

In concluding the chapter, it must be noted that only elementary examples are considered in it and certainly far from all of the types of problems to be solved in what follows by the method of harmonic linearization. But it may be hoped that the reader will obtain here an initial conception concerning the ideas and the possibilities of the method, which permits us to approach this method intelligently from the point of view of more complex problems as well.

[Footnotes]

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36	*The number of external disturbances is not a factor.
54	*The external loop remains only in the form of a direct-current power supply to the relay contacts.
56	*The initial phase is assumed zero for the variable x. The appropriate phase shift will be computed for the other variables.
56	**Here there is no phase shift because of the odd symmetry and single-valuedness of the nonlinear function in question.
63	*At least near this region.

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72 *See also § 1.8.

78 *This conclusion is valid only in first approximation (see § 8.6).

79 *The proportional component is absent.

84 *The second expression in Formula (1.154) is obtained from the first substitution $2x^0/A_t = \sin \xi$.

89 *Normally in solution of the sliding-process problem by methods other than that developed here, the frequency of vibration is regarded as infinite and the vibrations themselves are not investigated.

[List of Transliterated Symbols]

15 o.c = o.s = obratnaya svyaz' = feedback
15 OC = OS = obratnaya svyaz' = feedback
23 c = s = soprotivleniye = resistance, drag
24 p = r = rul' = control surface
24 в = v = vozmushchayushchiy = perturbing, disturbing
26 л = l = lineynyy = linear
27 дв = dv = dvigatel' = motor
67 к = k = konechnyy = final
71 п = p = pereregulirovaniye = overshoot
84 н = n = nelineynyy = nonlinear
93 в = v = vneshniy vozdeystviye = external disturbance (letter symbol Ω)

Chapter II

THE FOUNDATIONS OF THE METHOD OF HARMONIC LINEARIZATION AND COMPARISON WITH OTHER METHODS

§ 2.1. HARMONIC LINEARIZATION OF NONLINEARITIES FOR SYMMETRICAL OSCILLATIONS

In Chapter I, we demonstrated, using the simplest examples, the fundamental idea of the method of harmonic linearization and illustrated the possibilities of the method for the investigation of nonlinear automatic systems and for the choice of the basic parameters of such systems. The present chapter gives a systematic account of the problem in general form, and various special points in practical application of the method to numerous specific problems and examples will be pointed out subsequently.

Strictly speaking, no real engineering devices possess ideal straight-line static characteristics for all input values. The majority of the real static characteristics are curvilinear for sufficiently large input values (saturation, clipping, etc.). A number of static characteristics are essentially nonlinear for small input values (dead zone, backlash, hysteresis, etc.), but may be considered linear for some middle range of input values (when dead zone, backlash, or hysteresis have only weak influence). We have already discussed typical nonlinearities in the automatic systems at the beginning of Chapter I.

In view of the impossibility of taking these phenomena exactly into account, some idealized mathematical description by means of

equations is always formed in engineering calculations for the system being investigated, so as to obtain the simplest computational method while retaining its principal essential features. Linear equations are the simplest and have been studied most thoroughly. Therefore there is a completely natural tendency to linearize all real nonlinear characteristics by these or other means, i.e., to replace them in one way or another by a straight line, which usually passes along a tangent to the curve at its origin (line OD in Fig. 2.1).

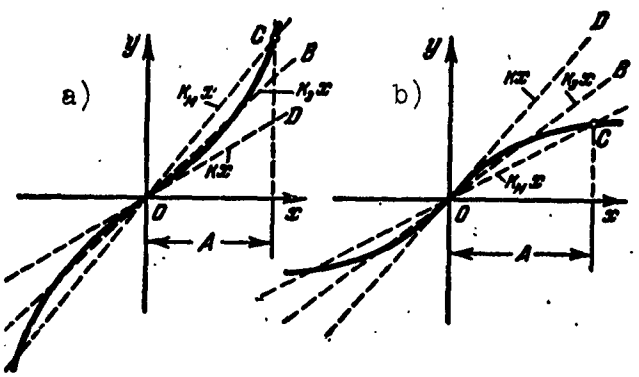


Fig. 2.1.

This method of linearization corresponds analytically to the expansion of the nonlinear function $y = F(x)$ in Taylor series, discarding all terms of the series higher than the first, i.e., here we perform the substitution

$$y = kx, \quad \text{where} \quad k = \left(\frac{dF}{dx}\right)_{x=0}. \tag{2.1}$$

Geometrically, k is the slope of the curve at the point O (the tangent of the angle of inclination). Hence, for sufficiently small departures x from the point O, the link being considered in the system will behave as if it were a linear link with a gain constant k. This method of linearization is used in the entire linear theory of stability (the so-called analysis of stability with respect to the first linear approximation for small deviations)

and in the linear theory of automatic control.

Sometimes, instead of the tangent [line] (2.1), it is necessary to use a secant [line] for linearization, imparting to it a definite constant slope k (Fig. 2.2).

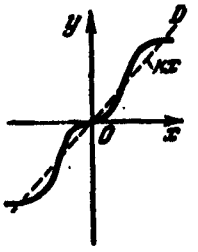


Fig. 2.2.

We shall use the term ordinary linearization for this generally used method of linearization (2.1) by means of a Taylor series with reference to the tangent kx (Fig. 2.1) or a secant kx (Fig. 2.2), where the curve is replaced by a straight line with a constant slope which is not a function of the shape and the magnitude of the input variable x (in a certain sufficiently small interval).

Not let the variable x vary so that it corresponds to some point C on the real characteristic y = F(x) (Fig. 2.1), where the deviation from the linear dependence is quite large.

If here the variable x has changed jumpwise from 0 to the value x = A in question (Fig 2.3a), then the form of the curve y = F(x) is not a factor. The equivalent linear dependence, which for the instantaneous jump x = A under consideration would give the same value y = F(A), will be

$$y = k_m x, \text{ where } k_m = \frac{F(A)}{A}, \qquad (2.2)$$

i.e., k_m is equal to the slope of the straight line OC (Fig. 2.1).

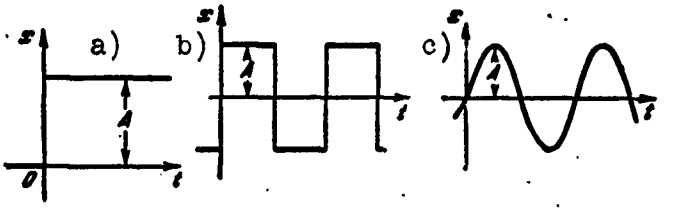


Fig. 2.3.

If, however, the variable x varies from zero to the point C

along some given time curve x(t), then the process of variation of the variable y = F(x) over time will evidently be closer to the process in a linear link having a gain constant

$$y = k_e x, \tag{2.3}$$

which is intermediate between $\underline{k}$ and k_m; here k_e is the slope of some intermediate secant OB (Fig. 2.1), with

$$k < k_e < k_m \quad \text{or} \quad k_m < k_e < k \tag{2.4}$$

for characteristics with increasing (Fig 2.1a) and decreasing (Fig. 2.1b) slopes, respectively.

The numerical value of the equivalent gain constant k_e may be determined for each position of the point C for the given law x(t), if we are given the equivalence condition – the equality of the areas under the curve of y(t) for nonlinear and linear dependences of $\underline{y}$ on $\underline{x}$, or equalities of energies, or other conditions.

Let us cite two examples of equivalent linearization of a nonlinearity for periodic variation of the variable $\underline{x}$.

For variation of $\underline{x}$ in the form of a rectangular periodic curve with an amplitude A (Fig. 2.3b) corresponding to the point C (Fig. 2.1), the variable y = F(x) also takes the form of rectangular periodic oscillations of amplitude F(A). Exactly the same oscillations are also obtained for linear dependence of $\underline{y}$ on $\underline{x}$ with the gain constant k_m determined by Formula (2.2). For example, if the nonlinear characteristic y = F(x) in Fig. 2.1 is described by the equation

$$y = k_1 x + k_3 x^3, \tag{2.5}$$

then the equivalent gain constant for the rectangular oscillations of the input variable $\underline{x}$ will be

$$k_e = \frac{F(A)}{A} = k_1 + k_3 A^2 \tag{2.6}$$

and the equivalent linearization of the nonlinearity takes the form

$$y = k_m x = (k_1 + k_3 A^2)\, x.$$

The equivalent gain constant k_m is a function of the amplitude of the input oscillations, i.e., it assumes different constant values for rectangular periodic disturbances with different amplitudes.

We shall call this method of linearizing (2.2) nonlinear characteristics F(x) rectangular linearization.

Now let the variable x vary according to the sinusoidal oscillation law (Fig. 2.3c)

$$x = A \sin \Omega t. \tag{2.7}$$

According to Fig. 2.1, we obtain periodic oscillations of a complex form for the variable y = F(x). If we assume linear dependence of y upon x, then we would obtain the sinusoidal oscillations

$$y = A_1 \sin \Omega t. \tag{2.8}$$

Let us adopt the following equivalent-linearization condition. We choose the equivalent gain constant

$$k_3 = \frac{A_1}{A} \tag{2.9}$$

in such a way that for the linear dependence (2.8) the oscillations of the variable y correspond exactly to the first harmonic of the complex nonlinear oscillations of the variable y = F(x) for sinusoidal variation of x (it is assumed that the first harmonic plays the fundamental role).

The shape of the periodic nonlinear oscillations y = F(x) for x = A sin Ωt is determined by the expression F(A sin Ωt). Its first harmonic is determined by the expansion of this periodic function in Fourier series. Because of the oddness of the characteristic (Fig. 2.1) there will not be any cosine terms here, and the first harmonic will have the form

$$y_1 = A_1 \sin \Omega, \tag{2.10}$$

where

$$A_F=\frac{1}{\pi}\int_0^{2\pi}F(A\sin\psi)\sin\psi\,d\psi \qquad (\psi=\Omega t).$$

According to (2.9), the condition of equivalent linearization $A_1 = A_F$ that has been adopted leads to the following equivalent gain constant:

$$k_0=\frac{1}{\pi A}\int_0^{2\pi}F(A\sin\psi)\sin\psi\,d\psi. \tag{2.11}$$

For example, if the nonlinear characteristic (Fig. 2.1a) has the form (2.5), then Formula (2.11) gives us

$$k_0=\frac{k_1}{\pi A}\int_0^{2\pi}A\sin^2\psi\,d\psi+\frac{k_3}{\pi A}\int_0^{2\pi}A^3\sin^4\psi\,d\psi= \\ =k_1+\frac{3}{4}k_3A^2. \tag{2.12}$$

Hence, the result of linearization will be

$$y=k_0x=\left(k_1+\frac{3}{4}k_3A^2\right)x.$$

This means that the first harmonic of the complex nonlinear oscillations $y = F(x)$ for $x = A \sin \Omega t$ will be

$$\overline{y_1}=\left(k_1+\frac{3}{4}k_3A^2\right)A\sin\Omega t.$$

Given equality of the oscillation amplitude obtained for the linear dependence of y on x, to the first-harmonic amplitude of the nonlinear oscillations $y = F(x)$ for $x = A \sin \Omega t$, we shall call such a method of linearization as (2.11) harmonic linearization.

Here, as in the preceding case (2.6), the equivalent gain constant is a function of the oscillation amplitude of the variable x, i.e., it assumes different constant values for sinusoidal oscillations of the variable x with different amplitudes A.

This fundamental departure of harmonic linearization (like rectangular linearization) from the ordinary linearization method (2.1) makes it an extremely valuable weapon for the investigation of dynamic processes in nonlinear automatic systems, since, unlike ordinary linearization, it permits us to describe certain specific

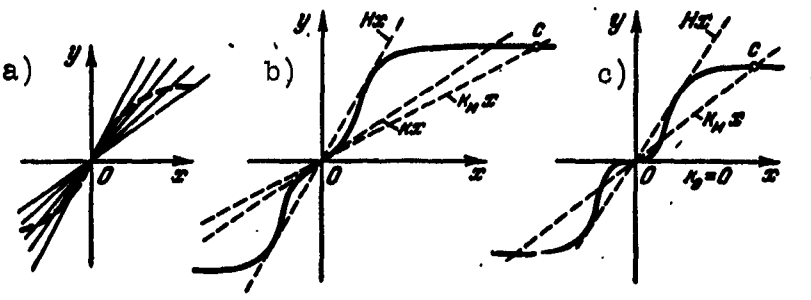


Fig. 2.4.

nonlinear phenomena.

The fact of the matter is that we do not replace the nonlinear characteristic by one straight line in this case, but by a bundle of straight lines (Fig. 2.4a), whose slopes are functions of the oscillation amplitude of the variable $\underline{x}$, i.e., of the magnitude of the "operating" segment of the curve of F(x) which is encompassed in the oscillation process. For each periodic process being considered (i.e., for each A) the equivalent characteristic is linear (k_e = const), but from process to process (for different A) the slope of the straight line varies, i.e., after harmonic linearization (as after rectangular linearization) all the nonlinear properties are nevertheless retained in a certain sense.

Let us note that for the nonlinear characteristic (2.5), ordinary linearization according to Formula (2.1) gives

$$k=(k_1+3k_3x^2)_{x=0}=k_1. \tag{2.13}$$

Comparing this with (2.6) and (2.12), we see that the value k_e (2.12) which is found for harmonic linearization actually satisfies the inequality (2.4). For more complex forms of the nonlinear characteristics (Fig. 2.4b and c) the inequality (2.4) is replaced by the following:

$$0<k_0<H, \tag{2.14}$$

where the relationships between k_e, k_m, and $\underline{k}$ may be of widely

diverse types.

In what follows we shall make extensive use of just this harmonic linearization with various supplements and refinements. It is best of all suited to the essence of the problem of investigating nonlinear automatic systems, although, generally speaking, depending upon the adoption of one or another equivalent linearization condition and upon the shape of the variation of $\underline{x}$ in time (for example, for exponential variation and other types), we may also introduce other forms of linearization with other values of the equivalent gain constant k_e for the same nonlinear characteristic. For harmonic linearization we stipulate the use of a special symbol for the equivalent gain constant: $k_e = q$, which we shall call the harmonic gain constant for the nonlinearity in question.

As has already been shown, for single-valued odd nonlinearities F(x) (see Fig. 1.4a-e, Fig. 1.5; also see relay and other single-valued characteristics; see § 1.1) in the case of symmetric oscillations ($x = A \sin \Omega t$), the harmonic gain constant according to (2.11) will be

$$q = \frac{1}{\pi A} \int_0^{\pi} F(A \sin \psi) \sin \psi \, d\psi, \tag{2.15}$$

where, because of the uniqueness and symmetry, the computation may be carried out according to the formula

$$q = \frac{4}{\pi A} \int_0^{\frac{\pi}{2}} F(A \sin \psi) \sin \psi \, d\psi. \tag{2.16}$$

The result of the harmonic linearization of a single-valued nonlinearity $y = F(x)$ will be

$$y = qx, \quad q = q(A), \tag{2.17}$$

which corresponds to the first harmonic of the nonlinear oscillations $y_1 = q(A)A \sin \Omega t$.

In the case of a loop-type, oddly symmetrical nonlinearity

$y = F(x)$ (Fig. 2.5; see also Fig. 1.2a and c and other loop-type characteristics) with symmetrical oscillations $x = A \sin \Omega t$, the expansion of the periodic nonlinear function in Fourier series gives, in contrast to (2.10), a first harmonic with two components:

$$y_1 = A_F \sin \Omega t + B_F \cos \Omega t, \tag{2.18}$$

where

$$A_F = \frac{1}{\pi}\int_0^{2\pi} F(A \sin \psi) \sin \psi \, d\psi, \quad B_F = \frac{1}{\pi}\int_0^{2\pi} F(A \sin \psi) \cos \psi \, d\psi.$$

For $x = A \sin \Omega t$ we have

$$px = A\Omega \cos \Omega t \quad \left(p = \frac{d}{dt}\right).$$

Therefore the adopted harmonic-linearization condition, for which the equivalent linear dependence of $\underline{y}$ on $\underline{x}$ should give us a sine curve equal to the first harmonic of the nonlinear oscillations

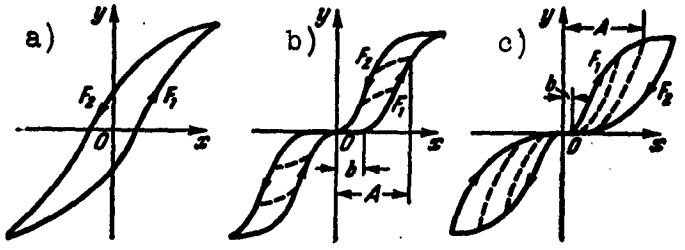


Fig. 2.5.

$y = F(x)$ for $x = A \sin \Omega t$, leads, according to (2.18), to the following result:

$$y = qx + \frac{q'}{\Omega} px, \quad q = q(A), \quad q' = q'(A), \tag{2.19}$$

where

$$\left.\begin{aligned} q &= \frac{1}{\pi A}\int_0^{2\pi} F(A \sin \psi) \sin \psi \, d\psi, \\ q' &= \frac{1}{\pi A}\int_0^{2\pi} F(A \sin \psi) \cos \psi \, d\psi, \end{aligned}\right\} \tag{2.20}$$

where, in view of the specific character of the symmetry, the computations may be carried out in this case by the formulas:

$$\left.\begin{aligned} q &= \frac{2}{\pi A}\int_0^{\pi} F(A\sin\psi)\sin\psi\, d\psi, \\ q' &= \frac{2}{\pi A}\int_0^{\pi} F(A\sin\psi)\cos\psi\, d\psi, \end{aligned}\right\} \qquad (2.21)$$

while it is now impossible to make use of Formula (2.16).

The linearized function (2.19) does not now reduce, as it did earlier, to one harmonic gain constant q(A); now it includes an additional term which is a function of the rate of variation of the variable x. This is a consequence of the fact that the loop-type nonlinearity F(x) (Fig. 2.5) is strictly speaking, itself dependent upon the rate px, or more exactly, upon the sign of the rate, since for a rate $px > 0$ the process passes along the curve F_1, while for $px < 0$ it passes along the curve F_2 (Fig. 2.5).

Let us note that in Fig. 2.5a and b (as in Fig. 1.2a) we have depicted hysteresis-type loops (nonlinear lag), while in Fig. 2.5c we have depicted a forcing-type loop (nonlinear lead). On harmonic linearization in the former case we accordingly obtain $q' < 0$, while in the second case we obtain $q' > 0$, i.e., the nonlinear lag or lead is transformed into a linear lag or lead consisting in the introduction of a negative or positive derivative (see (2.19)).

The linearized expression (2.19) indicates that in the case in question, the first harmonic of the complex nonlinear oscillation $y = F(x)$ for $x = A \sin \Omega t$ will be

$$y_1 = C_1 A \sin(\Omega t + D_1), \qquad (2.22)$$

where the amplitude gain C_I and the phase shift D_I are

$$C_1 = \sqrt{q^2 + (q')^2}, \quad D_1 = \operatorname{arctg}\frac{q'}{q}, \qquad (2.23)$$

here the phase shift D_I has the same sign as q' (the phase lag or phase lead of the first oscillation harmonic in the presence of a hysteresis loop or forcing loop respectively).

In complex notation, with the harmonic oscillations represented

by uniformly rotating vectors (Fig. 2.6a)

$$x = Ae^{j\Omega t}, \quad y_1 = C_1 Ae^{j(\Omega t + D_1)},$$

we have:

$$y_1 = W_n x, \quad \text{where} \quad W_n = q(A) + jq'(A). \tag{2.24}$$

Here W_n is the gain-phase characteristic of the nonlinearity in question for the first oscillation harmonic (see Fig. 2.6b, where two curves are shown: one for a forcing loop of the type in Fig. 2.5c, the other for the hysteresis loop of Fig. 2.5b).

For the general case of an arbitrary nonlinear function

$$y = F(x, px), \tag{2.25}$$

which may be expanded in Fourier series for $x = A \sin \Omega t$, the first harmonic will also have the form (2.18). As a result of harmonic linearization we obtain (in the absence of a constant component):

$$y = qx + \frac{q'}{\Omega} px. \tag{2.26}$$

In this general case, the constants q and q' will be functions not only of the amplitude, as before, but also of the frequency Ω:

$$q = q(A, \Omega), \; q' = q'(A, \Omega),$$

since the formulas for calculating them (which correspond to the coefficients of the Fourier series) assume the form

$$\left.\begin{aligned} q &= \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi, A\Omega\cos\psi)\sin\psi\, d\psi, \\ q' &= \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi, A\Omega\cos\psi)\cos\psi\, d\psi. \end{aligned}\right\} \tag{2.27}$$

For the first oscillation harmonic, the gain-phase characteristic of the nonlinearity in question will be

$$W_n = q(A, \Omega) + jq'(A, \Omega) \tag{2.28}$$

with the preceding expressions (2.23) for the amplitude gain and the phase shift.

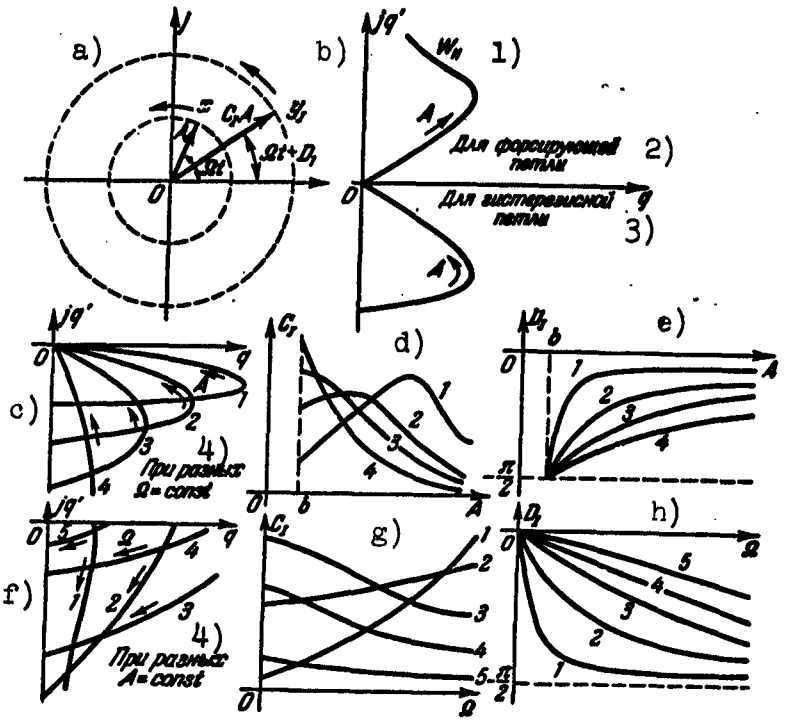


Fig. 2.6. 1) W_n; 2) for forcing loop; 3) for hysteresis loop; 4) for different A = const.

It is evident that Formulas (2.25)-(2.28) are general ones from which the earlier formulas (2.16)-(2.19), (2.21) and (2.24) follow as particular cases. Therefore these general formulas will be used henceforth, since the results obtained may always be applied to the simpler preceding cases.

Let us note that for a nonlinearity of the form (2.25), ordinary linearization of the type (2.1) gives the linear expression

$$y = k_1 x + k_2 px$$

with constant coefficients (Taylor-series coefficients), that are independent of the shape of the time variation of x:

$$k_1 = \left(\frac{\partial F}{\partial x}\right)^0, \quad k_2 = \left(\frac{\partial F}{\partial px}\right)^0,$$

where the superscript zero denotes the substitution x = 0, px = 0.

In contrast to (2.28), the gain-phase characteristic for such

a linear link has the simpler form:

$$W = k_1 + jk_2\Omega,$$

which is independent of the oscillation amplitude of the input variable x. For nonlinearities of the type of Fig. 2.5, on the other hand, as we showed in Eqs. (2.24) and in Fig. 2.6b, the amplitude-phase characteristic for the first harmonic on harmonic linearization is a function of the amplitude A of the oscillations of the input quantity x and is not a function of the frequency Ω.

In the general case of harmonic linearization of nonlinearities, the gain-phase characteristic (2.28) for the first harmonic is a function of both quantities A and Ω. It may be represented on the complex plane (q, jq') in the form of a family of curves with a variable parameter A for different constant values of Ω (Fig. 2.6c) or, conversely, in the form of a family of curves with a variable parameter Ω for different constant values of A (Fig. 2.6f). From the gain-phase characteristic of the nonlinearity, we may construct curves of the gain characteristic C_1 and the phase characteristic D_1 individually, as is shown in Fig. 2.6d and e (first method) and in Fig. 2.6g and h (second method). These characteristics correspond to the effect of a nonlinear lag (negative phase D_I of the first harmonic). Characteristics with nonlinear lead are similarly represented.

It is useful to keep the following in mind. If the nonlinearity takes the form of a sum (or difference) in a nonlinear system of the first class (see § 1.2), for example:

$$y = F_1(x) + F_2(x) + F_3(px) + F_4(x, px), \qquad (2.29)$$

then we may linearize each individual term harmonically and add the results, since the integral (2.27) of the sum of the functions is equal to the sum of the integrals of each of the functions added,

which fact has already been made use of in (2.12). However, from the point of view of multiplication and other operations, for example:

$$y = F_1(x) F_2(x) \quad \text{or} \quad y = [F(x)]^2,$$

individual harmonic linearization is, of course, impossible, and we must apply Formulas (2.27) to the nonlinear function as a whole, using any general rules of integration.

In the presence of nonlinearities in not only the variable x itself and its rate of change px, but also in the acceleration p^2x, for example

$$y = F_1(p^2x) + F(x, px),$$

for $x = A \sin \Omega t$ we must make the substitution

$$p^2x = -A\Omega^2 \sin \Omega t.$$

Then as a result of harmonic linearization we obtain

$$y = \frac{q''}{\Omega^2} p^2x + \frac{q'}{\Omega} px + qx,$$

where q and q' are as before (2.27) and

$$q'' = \frac{4}{\pi A} \int_0^{\frac{\pi}{2}} F_1(-A\Omega^2 \sin \psi) \sin \psi \, d\psi$$

for the single-valued oddly symmetrical function $F_1(p^2x)$.

Let us now turn to harmonic linearization of a nonlinearity which is expressed in the form of a variation of the structure of differential equations or transfer functions (see, for example, Eqs. (1.25) and Fig. 1.7).

Let us assume that the periodic solution for the variable x (Fig. 1.7) is close to the sine curve $x = A \sin \Omega t$. Let us construct the solution for the variable x_1, finding all segments 1 (Fig. 2.7) from the first equation of (1.25) and all segments 2 from the second equation. Then the curve x_1 will have the exact shape of the output of the nonlinear link being considered (Fig. 1.7)

for $x = A \sin \Omega t$. Hence we determine the coefficients of harmonic linearization $\underline{q}$ and $\underline{q}'$ by expanding this curve in Fourier series, i.e.,

$$q = \frac{1}{\pi A} \int_0^{2\pi} x_1(\psi) \sin \psi \, d\psi, \quad q' = \frac{1}{\pi A} \int_0^{2\pi} x_1(\psi) \cos \psi \, d\psi. \tag{2.30}$$

In particular, in view of the half-period symmetry we obtain for Eqs. (1.25):

$$q = \frac{2}{\pi A} \int_{-\psi_1}^{+\psi_1} x_1^{(1)}(\psi) \sin \psi \, d\psi + \frac{2}{\pi A} \int_{\psi_1}^{\pi - \psi_1} x_1^{(2)}(\psi) \sin \psi \, d\psi, \tag{2.31}$$

where, according to Fig. 2.7,

$$\psi_1 = \arcsin \frac{b}{A}.$$

A similar formula may also be written for $\underline{q}'$.

For all the segments 1 (Fig. 2.7) we have

$$x_1^{(1)}(\psi) = U_1(\Omega) A \sin \psi + V_1(\Omega) A \cos \psi + x_n^{(1)}(\psi), \quad \psi = \Omega t, \tag{2.32}$$

while for all segments 2

$$x_1^{(2)}(\psi) = U_2(\Omega) A \sin \psi + V_2(\Omega) A \cos \psi + x_n^{(2)}(\psi), \tag{2.33}$$

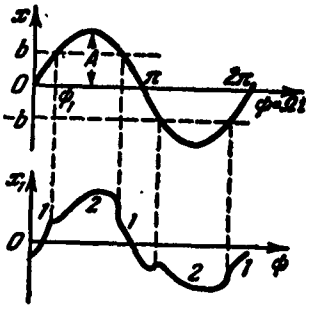


Fig. 2.7.

where $U_1(\Omega)$ and $V_1(\Omega)$ are the real and imaginary parts of the gain-phase characteristic of the first feedback loop with the transfer function $W_1(j\Omega)$, while $U_2(\Omega)$ and $V_2(\Omega)$ are the similar components for the second feedback loop with the transfer function $W_2(j\Omega)$. In addition, if we have in the denominators of the transfer functions $W_1(p)$ and $W_2(p)$ time constants. which smooth the stepwise transition from one interval to the other, we add the transitional components $x_n^{(1)}$ and $x_n^{(2)}$. Otherwise there are no transitional components $x_n^{(1)}$ and $x_n^{(2)}$.

If it is difficult to evaluate the integrals (2.30), then we must turn to the graphical method presented in Chapter III. It

is also used for graphical assignment of nonlinearities.

We must keep the following practical recommendation in mind. Before starting to carry out the harmonic linearization of a nonlinearity of the type of Fig. 1.7 according to Formulas (2.30), it is always necessary to consider the possibility of reducing it to a simpler form and choosing the method of calculation which is optimal from the point of view of simplicity.

Indeed, in many particular cases of the expressions $W_1(p)$ and $W_2(p)$, harmonic linearization of this type of nonlinearity is significantly simplified. For example, if we are concerned with "rigid" [proportional] feedback with various constants, then $W_1(p) = k_1$, $W_2(p) = k_2$, and the nonlinearity in question reduces to a simple nonlinear characteristic of the form of Fig. 1.10a or b. For continuous variation of $k_{o.s}$ we obtain a nonlinear characteristic of the form Fig. 1.8b. The same considerations also apply to simple "flexible" [proportional-plus-derivative] feedback, when $W_1(p) = k_1 p$, $W_2(p) = k_2 p$. In this case for continuous variation of $k_{o.s}$ we will have $x_1 = k_{o.s}(x)px$, where $k_{o.s}(x)$ is an odd function*, for example of the form of Fig. 1.8c or d. In all these cases, harmonic linearization of the nonlinearity is carried out more simply and Formula (2.15) may be used.

In exactly the same way, harmonic linearization becomes simple if we change any one term in a transfer function having a more complex form. For example, if

$$W_1(p) = \frac{k}{T_1 p + 1}, \quad W_2(p) = \frac{k}{T_2 p + 1},$$

then this is equivalent to a nonlinearity of the type (1.14) or (1.15). If the quantity $\underline{k}$ is also changed, then the feedback equation for the scheme of Fig. 1.7 assumes the form

$$T(x) p x_1 + x_1 = k(x) x.$$

Here we are obliged to deal with a nonlinear system of the second class, for which a method of harmonic linearization is given below [47].

In nonlinear systems of the second and third classes (see § 1.2) we encounter either separate nonlinearities with respect to different variables (see, for example, Fig. 1.13):

$$y=F(x,\ px),\quad y_2=F_2(x_2,\ px_2), \tag{2.34}$$

or a mixed nonlinearity including both variables, as for example (1.32) or in more general form

$$y=F_1(x,\ px;\ x_2), \tag{2.35}$$

if it does not break down into a sum of nonlinearities of the type (2.34)

In the case of separate nonlinearities (2.34), performing harmonic linearization separately on each nonlinearity for $x = A \sin \Omega t$ and $x_2 = A_2 \sin (\Omega t + \varphi)$, we obtain

$$y=qx+\frac{q'}{\Omega}px,\quad y_2=q_2x_2+\frac{q_2'}{\Omega}px_2, \tag{2.36}$$

where $\underline{q}$ and $\underline{q}'$ and q_2 and q_2' are calculated from Formulas (2.27), with $\psi = \Omega t$ in the first case and $\psi = \Omega t + \varphi$ in the second case, which is not indicated in the results. It is evident that in the general case

$$q=q(A,\ \Omega),\quad q'=q'(A,\ \Omega),\quad q_2=q_2(A_2,\ \Omega),\quad q_2'=q_2'(A_2,\ \Omega), \tag{2.37}$$

while in the most frequently encountered cases

$$q=q(A),\quad q_2=q_2(A_2). \tag{2.38}$$

Here, in systems of the second class, we determine the relationship between the amplitudes A and A_2 by means of linear differential equations or by means of the transfer function defining the relationship between the variables $\underline{x}$ and x_2 in the automatic system being investigated, i.e., if

$$x_2=W_2(p)x, \tag{2.39}$$

then for $x = A \sin \Omega t$ and $x_2 = A_2 \sin (\Omega t + \varphi)$ we have

$$A_2 = |\bar{W}_2(j\Omega)|\bar{A}, \quad \varphi = \widetilde{\arg} W_2(j\Omega). \tag{2.40}$$

A system of the third class differs from one of the second in that the variables $\underline{x}$ and x_2 are related to each other by nonlinear differential equations. Therefore a relationship of the type (2.40) may be obtained for them only when it is possible to linearize harmonically the nonlinearity connecting x_1 and x_2 and to include the gain-phase characteristic of this nonlinearity for the first harmonic in $W_2(j\Omega)$. The existence condition for such a possibility will be ascertained below in § 2.2.

But in the case of the mixed nonlinearity (2.35), which does not break down into a sum of nonlinearities (2.34), it is necessary before performing harmonic linearization, setting $x = A \sin \Omega t$ and $x_2 = A_2 \sin (\Omega t + \varphi)$, first to reduce the whole expression (2.35) to the form (2.25) by means of Formulas (2.40). For this the variable x_2 is expressed in terms of $\underline{x}$:

$$x_2 = A_2 \cos \varphi \sin \Omega t + A_2 \sin \varphi \cos \Omega t = \frac{A_2 \cos \varphi}{A} x + \frac{A_2 \sin \varphi}{A\Omega} px,$$

or, taking Expressions (2.40) into account:

$$x_2 = U_2(\Omega) x + \frac{V_2(\Omega)}{\Omega} px, \tag{2.41}$$

where $U_2(\Omega)$ and $V_2(\Omega)$ are the real and imaginary parts of the gain-phase characteristic $W_2(\Omega)$ connecting the variables x_2 and $\underline{x}$ in the automatic system in question. In Formula (2.41) the quantities $U_2(\Omega)$ and $U_2(\Omega)$ and $V_2(\Omega)/\Omega$ play the role of constant coefficients whose values are functions of the frequency of the oscillations being investigated. The substitution (2.41) permits us to reduce the mixed nonlinearity (2.35) to the form $y = F(x, px)$ and to perform harmonic linearization on it according to the general formulas (2.26) and (2.27), as in a system of the first class.

The formulas obtained in this section for harmonic lineariza-

tion of nonlinearities will be used subsequently in their present form, and will also be modified in subsequent chapters for non-symmetrical oscillations, for damped oscillations, in allowance for higher harmonics, and also for random processes.

§ 2.2. THE FOUNDATIONS OF THE METHOD OF HARMONIC LINEARIZATION

Those methods of solution of nonlinear problems in the design of various automatic systems that are based on harmonic linearization will be justified in the present chapter, as may be seen from § 2.1, on the proposition that the variable x appearing in the nonlinear function varies sinusoidally: $x = A \sin \Omega t$. If this condition is fulfilled, the form of the variation of the remaining variables will not play any role. This last remark has very important practical significance for the following reason.

Automatic systems are, as a rule, systems with many degrees of freedom, consisting of several links that are interconnected in different ways. Therefore we are always obliged to deal with several variable physical quantities characterizing the disturbances imposed by the links upon each other (see Chapter I).

In a dynamic control process or following process, etc., the variation of different variables in one and the same system will, generally speaking, be represented by several time curves that differ from each other. This is the case to an even greater degree in nonlinear systems (see, for example, Fig. 1.16). This is a well-known fact from linear theory.

From the very beginning, therefore, we must direct our attention to the fact that for the use of approximate methods of analysis which are based upon definite assumptions concerning the form of the solution (the method of harmonic linearization also belongs to this class of melhods), we must, firstly, be certain that the be-

havior of the respective variable in the system will be close to the form of the solution which has been assumed in the method under consideration, and secondly, we must apply the solution obtained to analysis of the behavior of this same variable, without extending it (without additional operations) to other variables; this rule will be observed in what follows.

Thus, as we have already said, we assume in the method of harmonic linearization developed in the present chapter that for variables in the nonlinear function, the periodic solution is sufficiently close to the sinusoidal ($x = A \sin \Omega t$). For other variables in the same system, however, no limitations are imposed upon the form of the solution. It may differ from the sine curve as greatly as we please (as, for example, in Fig. 1.16). Here it is only assumed that the fundamental oscillation frequency is retained for all the variables.

Let us show in general form which properties of the nonlinear system may be used to justify the use of the method of harmonic linearization for finding a periodic solution (self-oscillations) in the presence of a strongly expressed nonlinearity and the consequent possibility that the form of the solution will be markedly different from the sinusoidal form for a number of variables in the system other than the one in the nonlinearity.

We may describe free motion (transient processes and self-oscillations) for an extremely broad range of nonlinear automatic systems of the first class (see § 1.2) by means of a homogeneous differential equation in the form

$$Q(p)x + R(p)F(x, px) = 0 \quad \left(p \equiv \frac{d}{dt}\right), \tag{2.42}$$

where Q(p) and R(p) are polynomials of arbitrary degree with real

constant coefficients, with the degree of R(p) lower than the degree of Q(p). Let us denote the given nonlinear function, which differs essentially from the linear, by F(x, px). As a rule, it is precisely in this case that there is practical meaning in taking account of a nonlinearity in an approximate calculation for the system. As a rule, the generation of self-oscillations in real automatic systems is itself also connected with the presence of some essential nonlinearity in the system.

However, the purpose of the method of harmonic linearlization is to use linear methods for extensive investigation of such a system with an essential nonlinearity.

Thus, for the determination of self-oscillations (for specific methods of solution of the problem, see below in § 2.3), we assume that the solution x(t) of the nonlinear differential equation (2.42) is sufficiently close to the solution $x = A \sin \Omega t$ of some linear differential equation obtained according to (2.26) through replacement of the given nonlinear function F(x, px) by the expression

$$F(x, px) = qx + \frac{q'}{\Omega} px, \tag{2.43}$$

where $\underline{q}$ and $\underline{q}'$ are determined by Formulas (2.27). On this substitution, the differential equation of the system (2.42) assumes the form

$$\left[Q(p) + R(p)\left(q + \frac{q'}{\Omega} p\right)\right] x = 0. \tag{2.44}$$

This equation is a linear differential equation with constant coefficients, since although according to (2.27) the coefficients $\underline{q}$ and $\underline{q}'$ are also functions of the unknown amplitude A and frequency Ω of the solution which is being sought, these quantities A and Ω

are nevertheless constant insofar as a periodic solution is being sought.

Thus we require that the periodic solution $x = A \sin \Omega t$ of the artificially introduced linear equation (2.44) be adequately close to the solution of the given nonlinear equation (2.42). This is the case if the linear differential equation (2.44) differs from the given differential equation (2.42) to a sufficiently small degree. It is evident that we may expect this in far from all cases. Certain conditions restricting the class of equations of the type (2.42), which satisfies this requirement, must exist.

Let us set ourselves the objective of finding the conditions which the expressions Q(p), R(p), and F(x, px) appearing in the given nonlinear equation (2.42) must satisfy in order that it will differ from the linear equation (2.44) to a sufficiently small degree, notwithstanding the presence of the strong nonlinearity F(x, px), when we seek a periodic solution x(t) close to the sinusoidal [77].

To start the analysis of this problem, let us write the solution of the given nonlinear differential equation (2.42) in the form

$$x = x_1 + \varepsilon x_v(t), \tag{2.45}$$

where

$$x_1 = A_1 \sin \Omega_1 t, \tag{2.46}$$

$x_v(t)$ is an arbitrary bounded time function, while ε is a small parameter.

Let us note that for the present, the solution (2.45) is not compared to the solution $x = A \sin \Omega t$ of the linear equation (2.44). This will be done below. In contrast to the latter, therefore, the true first harmonic of the periodic solution of the nonlinear equa-

tion (2.42) is denoted otherwise here – namely, by $x_1 = A_1 \sin \Omega_1 t$.

In the case of the presence of the periodic solution (2.45) of Eq. (2.42) we may write on the assumption that x_1 is an exact expression of its first harmonic

$$\varepsilon x_2(t) = \varepsilon \sum_{k=2}^{\infty} A_k \sin(k\Omega_1 t + \varphi_k). \tag{2.47}$$

Let us represent the given nonlinear function F(x, px) in the form

$$F(x, px) = F(x_1, px_1) + [F(x_1 + \varepsilon x_2, px_1 + \varepsilon px_2) - F(x_1, px_1)], \tag{2.48}$$

where, using a Taylor series, we may represent the expression in the square brackets in the form

$$\begin{aligned}[F(x_1 + \varepsilon x_2, px_1 + \varepsilon px_2) - F(x_1 px_1)] = \varepsilon \left[\frac{\partial}{\partial x} F(x_1, px_1) x_2 + \right. \\ \left. + \frac{\partial}{\partial px} F(x_1, px_1) px_2\right] + \varepsilon^2[\ldots] + \varepsilon^3[\ldots] + \ldots\end{aligned} \tag{2.49}$$

Hence the second term in Formula (2.48), which is given there in square brackets, will be small if the partial derivatives of F with respect to $\underline{x}$ and with respect to px are finite. We may count on their being small even for the discontinuous nonlinear functions encountered in practice (for example, relay-type functions), where the indicated derivatives will be delta-functions.

Expanding each of the two terms in Formula (2.48) individually in Fourier series, we obtain for the first one (allowing for the relationship $p \sin \Omega_1 t = \Omega_1 \cos \Omega_1 t$

$$F(x_1, px_1) = F_0 + \left(C + \frac{B}{\Omega_1} p\right) \sin \Omega_1 t + \sum_{k=2}^{\infty} F_k, \tag{2.50}$$

where, according to the formulas for the Fourier-series coefficients, we have

$$\left.\begin{aligned} F_0 &= \frac{1}{2\pi} \int_0^{2\pi} F(A_1 \sin \psi, A_1\Omega_1 \cos \psi)\, d\psi, \\ C &= \frac{1}{\pi} \int_0^{2\pi} F(A_1 \sin \psi, A_1\Omega_1 \cos \psi) \sin \psi\, d\psi, \\ B &= \frac{1}{\pi} \int_0^{2\pi} F(A_1 \sin \psi, A_1\Omega_1 \cos \psi) \cos \psi\, d\psi, \end{aligned}\right\} \tag{2.51}$$

F_k are higher harmonics, which we write in the form

$$F_k = N_k \sin(k\Omega_1 t + \eta_k) \qquad (k = 2, 3, \ldots). \tag{2.52}$$

Their amplitudes N_k may not be considered small in comparison with the amplitude of the first harmonic $\sqrt{C^2 + B^2}$, since the given nonlinear function F(x, px) is strongly nonlinear. These amplitudes will not be small for at least one or several lower values of k (possibly only for odd values) but for $k \to \infty$ we must have $N_k \to 0$.

For the second term of (2.48), which is small, we write the expansion in Fourier series in the form

$$[F(x_1 + \varepsilon x_2, px_1 + \varepsilon p x_2) - F(x_1, px_1)] = \varepsilon \sum_{k=0}^{\infty} \Phi_k, \tag{2.53}$$

where

$$\varepsilon\Phi_k = \varepsilon G_k \sin(k\Omega_1 t + \vartheta_k) \qquad (k = 0, 1, 2, \ldots). \tag{2.54}$$

Substituting Expressions (2.45), (2.50), and (2.53) in the given nonlinear equation (2.42), we obtain

$$Q(p)x_1 + Q(p)\varepsilon x_2 + R(p)\left(C + \frac{B}{\Omega_1}p\right)\sin\Omega_1 t + \\ + R(p)\sum_{k=2}^{\infty} F_k + R(p)\varepsilon\sum_{k=0}^{\infty}\Phi_k + R(p)F_0 = 0. \tag{2.55}$$

In order to obtain identical equality to zero for the expression in question, we must set all harmonics equal to zero individually. For the zero harmonics we obtain

$$F_0 = \varepsilon\Phi_0$$

or taking (2.51) into account, with a degree of accuracy to within ε

$$\int_0^{2\pi} F(A_1 \sin\psi, A_1\Omega_1 \cos\psi)\, d\psi = 0. \tag{2.56}$$

This is the first requirement imposed upon the given nonlinear function F(x, px) – the absence of a constant component*.

For the first harmonics of Eq. (2.55), we obtain the expression

$$Q(p)x_1 + R(p)\left(C + \frac{B}{\Omega_1}p\right)\sin\Omega_1 t + R(p)\varepsilon\Phi_1 = 0. \tag{2.57}$$

Taking (2.46) and (2.54) into account and applying the complex method for determining the amplitudes and phases of the sinusoidal oscillations, we obtain

$$A_1 \sin \Omega_1 t = -\left|\frac{R(j\Omega_1)}{Q(j\Omega_1)}\right| \sqrt{C^2+B^2} \sin(\Omega_1 t + \gamma + \beta) - \\ -\left|\frac{R(j\Omega_1)}{Q(j\Omega_1)}\right| \varepsilon G_1 \sin(\Omega_1 t + \vartheta_1 + \beta),$$

if Q(p) does not have purely imaginary roots; here,

$$\gamma = \operatorname{arctg}\frac{B}{C}, \quad \beta = \arg\frac{R(j\Omega_1)}{Q(j\Omega_1)}. \tag{2.58}$$

Upon the basis of this we may write with an accuracy to within ε:

$$A = \left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \sqrt{C^2+B^2}, \quad \gamma + \beta(\Omega) = \pi. \tag{2.59}$$

By means of these relationships we determine the first approximation of A and Ω for the amplitude A_1 and the frequency Ω_1 of the first harmonic of the periodic solution. It may be verified easily that the first approximation (2.59) is a solution of the approximate linear equation (2.44) for purely imaginary roots, if we take into account that according to (2.27) and (2.51) we have C = Aq, B = = Aq' on the substitution $A = A_1$ and $\Omega = \Omega_1$.

It is evident from this that for the above condition of small order of magnitude for the terms in square brackets in Expression (2.48) (at least small order of magnitude of the first harmonic of the Fourier-series expansion of the expression in these brackets), the linear equation (2.44) actually determines to an order of accuracy within ε the <u>first harmonic</u> of a periodic solution of the nonlinear equation (2.42) which is close to the sinusoidal solution (if this periodic solution exists).

Let us now turn to the terms of Eq. (2.55) which contain higher harmonics. For each of them, taking (2.47) into account, we have the equation

$$Q(p)\varepsilon A_k \sin(k\Omega_1 t + \varphi_k) + R(p) F_k + R(p)\varepsilon\Phi_k = 0 \quad (k = 2, 3, \ldots), \tag{2.60}$$

from which, according to (2.52) and (2.54), we obtain:

$$\varepsilon A_k \sin(k\Omega_1 t + \varphi_k) = -\left|\frac{R(jk\Omega_1)}{Q(jk\Omega_1)}\right| N_k \sin(k\Omega_1 t + \eta_k + \beta_k) - \\ -\left|\frac{R(jk\Omega_1)}{Q(jk\Omega_1)}\right| \varepsilon\Phi_k \sin(k\Omega_1 t + \vartheta_k + \beta_k) \quad (k = 2, 3, \ldots), \tag{2.61}$$

where we define

$$\beta_k = \arg\frac{R(jk\Omega_1)}{Q(jk\Omega_1)}.$$

In order that Equality (2.61) may be observed, i.e., in order that all the left members $[\varepsilon A_k \sin(k\Omega_1 t + \varphi_k)]$ or, what is the same thing, the quantity $\varepsilon x_v(t)$ in the sought solution (2.45) may be small in reality, we must require small order of magnitude for all the quantities

$$\left|\frac{R(jk\Omega_1)}{Q(jk\Omega_1)}\right| N_k \quad (k = 2, 3, \ldots). \tag{2.62}$$

Each of them must be a small quantity, at least of order ε, i.e., in comparison with A_1, the quantities (2.62) must be at least of the same small order of magnitude as the quantity $\varepsilon x_v(t)$ in comparison with x_1.

Taking (2.59) into account, this results in an important requirement being imposed upon the expressions Q(p) and R(p) and F(x, px), which figure in the given differential equation (2.42), i.e.,

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| N_k < \left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \sqrt{C^2 + B^2}. \tag{2.63}$$

Inasmuch as we are considering the essential nonlinearity F(x, px), then as we have already remarked in writing Formulas (2.52), the quantities N_k may not be considered small in comparison with $\sqrt{C^2 + B^2}$ (at least for lower values of $\underline{k}$). Therefore we must require that

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| < \left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \quad (k = 2, 3, \ldots). \tag{2.64}$$

In addition, requiring that the degree of the polynomial R(p) be lower than the degree of the polynomial Q(p), we obtain

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right|_{k\to\infty} \to 0. \tag{2.65}$$

Let us note that Conditions (2.63) and (2.64) are written here for each higher harmonic separately, while $\varepsilon x_v(t)$ are written here for all the higher harmomics. However, fulfillment of Conditions (2.64) individually (without addition) will be sufficient in real problems for attaining small order of magnitude for $\varepsilon x_v(t)$, since for the quantity $\sum_{k=2}^{\infty}\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| N_k$ which, according to (2.61), enters into Expression (2.47) for εx_v, only some (a small number) of the quantities N_k may have essential significance. Here small order of magnitude of magnitude of the entire sum of higher harmonics $\varepsilon x_v(t)$ is ensured by the rapid approach to zero of the practically occurring quantities N_k and Expression (2.65) as $k \to \infty$.

As we see from § 1.2, the expression

$$W_л(p) = \frac{R(p)}{Q(p)}$$

is the transfer function of the reduced linear part of the system (Fig. 1.12b). Let us construct the gain-frequency characteristic of the reduced linear part

$$|W_л(j\omega)| = \left|\frac{R(j\omega)}{Q(j\omega)}\right|, \tag{2.66}$$

as shown, for example, in Fig. 2.8. Then Conditions (2.64) and (2.65) obtain a graphical interpretation. In the simplest systems, where the reduced linear part corresponds to the real linear part of the system (as was the case in the example of a relay type system in § 1.3), this graphical interpretation of Conditions (2.64) and (2.65) acquires a physical meaning – nonpassage of the higher har-

monics through the linear part of the system. This last property is known as the filter property.

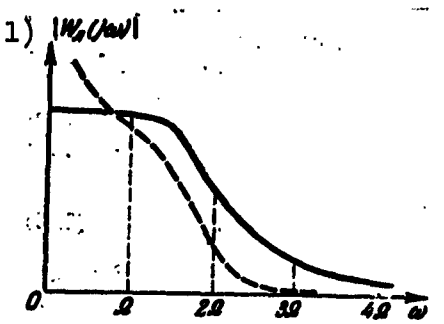


Fig. 2.8. 1) $|W_1(j\omega)|$.

Hence the mathematical conditions (2.64) and (2.65) derived here, whose fulfillment is necessary for application of the harmonic linearization method to systems with strongly expressed nonlinearities, are a generalization of the real filter property for all systems whose equations may be reduced to the form (2.42).

Let us therefore call Conditions (2.64) and (2.65) the generalized filter property. The generalization consists in the fact that in the general case, this property is required not of the actual part of the system, but of the reduced linear part (see § 1.2). This generalization is extremely important for all nonlinear automatic systems that do not have a clearly expressed structural layout of the form of Fig. 1.12b, when the nonlinearity enters in arbitrary fashion into the equation of any link of the system; here the variable x in the nonlinear function may be not an input variable, but rather an output variable or even some intermediate variable, which is introduced artificially in writing the equations of the complex nonlinear link.

In order to strengthen the requirement of the generalized filter property, let us add to the rule stipulated above [absence of the purely imaginary roots of the polynomial $Q(p)$] still another auxiliary condition for the absence of roots with a positive real part. This guarantees stable passage of the oscillations through the reduced linear part. However, we allow the presence of zero roots

in the polynomial Q(p); they only ameliorate nonpassage of the higher harmonics of the reduced linear part, since they change the curve of $|W_1(j\omega|$ as indicated by the broken line in Fig. 2.8. Hence the generalized filter property is reinforced by the requirement that the reduced linear part of the system be stable or neutral.

Thus we lay the foundation for the application of the method of harmonic linearization to nonlinear automatic systems of the first class, whose dynamics are described by a differential equation of the type (2.42), and derive conditions which the expressions for Q(p), R(p), and F(x, px) must satisfy in order that the automatic system being considered may be investigated by the method indicated, i.e., that it may have a solution for the variable x which is close to a sinusoidal solution. Consideration of the most diverse specific forms of automatic systems shows subsequently that these conditions are ordinarily well satisfied in practice. Thus, in the example of a relay-type system in § 1.3, the filter property (2.64) leads to Expression (1.88), which is satisfied for any relationship of the positive system parameters when a periodic solution (self-oscillations) is possible in the system.

The condition of small order of magnitude of Expression (2.49), which lies at the basis of the derivation, is always fulfilled in practice no matter how strong the nonlinearity F(x, px). From a theoretical point of view, however, what is important is that the mathematical basis for the requirement of the generalized filter property (2.64) in order to obtain the near-sinusoidal solution (2.45) no matter how strong the nonlinearity F(x, px), must also be based upon a definite requirement imposed upon the nonlinearity itself by ensuring small order of magnitude of Expression (2.49). This latter requirement may, in turn, be transformed into a require-

ment of finite derivatives dq/dA and dq'/dA, i.e., a requirement of adequate smoothness of variation of the harmonic-linearization coefficients q and q' (2.27) on variation of the oscillation amplitude close to the value A for the periodic solution sought.

According to (1.57) and (1.56) the free-oscillation equation will be as follows for another type of system of the first class:

$$Q(p)x + R_1(p)F_1(x, px) + R_2(p)F_2(x, px) = 0, \tag{2.67}$$

where the degree of the polynomials $R_1(p)$ and $R_2(p)$ is lower than the degree of the polynomial $Q(p)$.

By analogy with the preceding calculations, we arrive at the conclusion that to obtain a solution x(t) which is close to the sinusoidal solution, we must require fulfillment of two conditions of the generalized filter property:

$$\left|\frac{R_1(jk\Omega)}{Q(jk\Omega)}\right| < \left|\frac{R_1(j\Omega)}{Q(j\Omega)}\right|, \quad \left|\frac{R_2(jk\Omega)}{Q(jk\Omega)}\right| < \left|\frac{R_2(j\Omega)}{Q(j\Omega)}\right| \quad (k = 2, 3, \ldots), \tag{2.68}$$

where

$$\left|\frac{R_1(jk\Omega)}{Q(jk\Omega)}\right|_{k\to\infty} \to 0 \text{ and } \left|\frac{R_2(jk\Omega)}{Q(jk\Omega)}\right|_{k\to\infty} \to 0; \tag{2.69}$$

the latter condition is ensured by the fact that the degrees of $R_1(p)$ and $R_2(p)$ are lower than the degree of $Q(p)$. Here the nonlinearities $F_1(x, px)$ and $F_2(x, px)$ may be as strong as we please, but we postulate for them small order of magnitude of the expression of type (2.49). This holds true for the nonlinearities encountered in practice. As before, Condition (2.68) is supplemented by requiring the absence of purely imaginary roots and roots with a positive real part in the polynomial $Q(p)$.

Similar considerations are also easily extended to systems of the second and third classes. In them we must satisfy the generalized filter property (2.64) for the reduced linear parts connecting each of the variables in the nonlinear function, for example x and

x_2 in Expressions (2.34) and (2.35) with the variable $\underline{y}$ (or with the variables y_1 and y_2), which introduces a nonlinearity into the given system. For example, if for a nonlinear system including two nonlinearities (2.34) with respect to different variables we are given the equations

$$\left.\begin{aligned} Q_1(p)\,x &= -R_1(p)\,y_2, \\ Q_2(p)\,x_2 &= R_2(p)\,y_1, \end{aligned}\right\} \tag{2.70}$$

which corresponds to the scheme of Fig. 2.9, then we are required to observe conditions of the same form as (2.68) and (2.69), with replacement of the Q(p) by $Q_1(p)$ and $Q_2(p)$, respectively. The situation is also similar for other types of schemes (Fig. 1.13 and others).

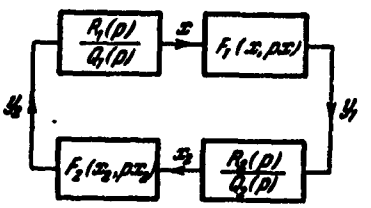


Fig. 2.9.

Let us remark that the fulfillment of all of these filter conditions for reduced linear parts is necessary only in the case where all the nonlinearities may be made as strong as we wish. If, however, one of the nonlinearities does not cause the appearance of strong higher harmonics, then we must apply Condition (2.63) to the appropriate part of the system, this imposes significantly fewer requirements upon the linear part (due to the small order of magnitude of the actual higher-harmonic amplitudes N_k produced by the nonlinearity in comparison with the amplitude of the first harmonic $\sqrt{C^2 + B^2}$).

If, however, for some segment of the system (for example, the segment $y_1x_2y_2$ in Fig. 2.9) the indicated conditions are not satisfied, then we must consider this entire segment as one nonlinear link and make use of Formula (2.30), replacing $x_1(\psi)$ by $y_2(\psi)$ in it and regarding only the variable $\underline{x}$ as sinusoidal. Here the function $y_2(\psi)$ is determined as follows. Assigning the form of the solu-

tion $X = A \sin \psi$, we find (analytically or graphically), in accordance with the scheme of Fig. 2.9, the exact expression of the function

$$y_1(\psi) = F_1(\sin \psi, A\Omega \cos \psi).$$

Thereupon by exact solution of the equation $Q_2(p)x_2 = R_2(p)y_1$, we determine $x_2(\psi)$, where $\psi = \Omega t$. This equation is usually easy to solve, since the nonfulfillment of the filter condition for the transfer function R_2/Q_2 is most often the case when $Q_2(p)$ is of the first or second degree. Finally, we find the exact expression

$$y_2(\psi) = F_2[x_2(\psi), px_2(\psi)].$$

Then the scheme of Fig. 2.9 reduces to a scheme with one nonlinearity unifying three links, for which

$$q = \frac{1}{\pi A}\int_0^{2\pi} y_2(\psi) \sin \psi \, d\psi, \quad q' = \frac{1}{\pi A}\int_0^{2\pi} y_2(\psi) \cos \psi \, d\psi,$$

and with one linear part $W_1(p) = R_1(p)/Q_1(p)$.

§ 2.3. METHODS OF DETERMINING SYMMETRICAL SELF-OSCILLATIONS

Let us first turn to nonlinear systems of the first class, whose equations reduce to the form

$$Q(p)x + R(p)F(x, px) = 0, \tag{2.71}$$

where $Q(p)$ and $R(p)$ are polynomials with constant coefficients; here the degree of $R(p)$ is lower than the degree of $Q(p)$ and the polynomial $Q(p)$ has no purely imaginary roots or roots with positive real parts (we allow the presence of zero roots). For such systems with an essential nonlinearity $F(x, px)$, we require fulfillment of the generalized filter property

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| < \left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \quad (k = 2, 3, \ldots) \tag{2.72}$$

for application of the method of harmonic linearization (Fig. 2.8); this guarantees that the periodic solution for the variable x will be close to the sinusoidal form (2.45) for an arbitrarily

strong nonlinearity. Let us recall only that the derivation of of this property is based upon small order of magnitude of Expression (2.49); this requires adequate smoothness of variation of the harmonic-linearization coefficients q(A) and q'(A) close to the periodic solution sought. We may rely on this fully for all possible nonlinearities which are actually encountered in automatic systems.

Here we seek a periodic solution which will have the approximate form

$$x = A \sin \Omega t \tag{2.73}$$

under the assumption that the given nonlinearity F(x, px) does not give a constant component, i.e.,

$$\int_0^{2\pi} F(A \sin \psi, A\Omega \cos \psi)\, d\psi = 0, \quad \psi = \Omega t. \tag{2.74}$$

In order to find the amplitude A and the frequency Ω of the periodic solution (2.73), we first perform harmonic linearization of the nonlinearity according to (2.26) and (2.27), i.e., the substitution

$$F(x, px) = qx + \frac{q'}{\Omega} px, \tag{2.75}$$

where

$$\left.\begin{aligned} q &= \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \psi, A\Omega \cos \psi) \sin \psi \, d\psi, \\ q' &= \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \psi, A\Omega \cos \psi) \cos \psi \, d\psi. \end{aligned}\right\} \tag{2.76}$$

Computation according to these formulas gives us the expression for the harmonic gain constants as functions of the amplitude and frequency of the periodic solution sought, i.e.,

$$q(A, \Omega), \text{ and } q'(A, \Omega). \tag{2.77}$$

In many particular cases, as in the examples of Chapter 1, q and q' may be functions only of the amplitude A and not functions of the frequency Ω.

Making the substitution (2.75) in Eq. (2.71), we obtain a linear differential equation with constant coefficients:

$$\left[Q(p)+R(p)\left(q+\frac{q'}{\Omega}p\right)\right]x=0, \tag{2.78}$$

where the coefficients $\underline{q}$ and $\underline{q}'$ are, according to (2.77), functions of the sought quantities A and Ω (or only of A), while we seek the solution of the equation in the form (2.73). The latter is possible only in cases where the characteristic equation of the closed system

$$Q(p)+R(p)\left(q+\frac{q'}{\Omega}p\right)=0 \tag{2.79}$$

has the pair of purely imaginary roots $p = \pm j\Omega$. This is also a point of departure for finding the amplitude A and frequency Ω of the periodic solution.* At the same time, as we have already mentioned, the characteristic equation of the open linear part (reduced)

$$Q(p)=0 \tag{2.80}$$

should have no purely imaginary roots or roots with positive real parts. We may have zero roots in Eq. (2.80), but in the equation of the closed system (2.79) we should not have them (the same also applies for roots with positive real parts).

We must keep in mind that the substitution (2.75) approximately reflects only the first harmonic of the function F(x, px), since in the right member of (2.75) we set $x = A \sin \Omega t$, $px = A\Omega \cos \Omega t$. This is also the harmonic-linearization condition adopted in § 2.1. The higher harmonics in Expression (2.75) are discarded even though they are not small (strong linearity), only because the justification for disregarding them in the solution is guaranteed by the fulfillment of the property (2.72), as was shown in § 2.2.

Thus, the equality (2.75) taken alone is in itself not even

approximately valid for strong nonlinearities, inasmuch as the right-hand side is an approximate expression only of the first harmonic of the left-hand side. But in spite of this, Eq. (2.78), which is obtained by using it, differs little from the initial nonlinear equation (2.71), and both of these equations have periodic solutions which are close to each other.

The solution may be expressed briefly as follows: notwithstanding the absence of a small parameter in the substitution (2.75), there is, in contrast, a small parameter both for Eqs. (2.71) and (2.78) themselves and for their solutions; as follows from § 2.2. Thus, we must keep in mind the conditional nature of the equality sign in Expression (2.75), and not impart to it any other sense than that indicated above.

There are six possible methods for determining the amplitude A and frequency Ω of the periodic solution (in application to the different specific problems, however, there may be an even greater detail diversity of computational methods). In synthesis problems for nonlinear automatic systems, we focus our principal attention here on the determination of A and Ω as functions of the system parameters with the object of choosing the latter.

The first method [47]. This method is most suitable for general investigations in the majority of problems. Let us substitute the purely imaginary value $p = j\Omega$ into the characteristic equation (2.79) and, in the resulting complex expression

$$Q(j\Omega)+R(j\Omega)(q+jq')=0 \qquad (2.81)$$

let us separate the real and imaginary parts in the form

$$X(\Omega)+jY(\Omega)=0, \qquad (2.82)$$

where X and Y are polynomials in powers of Ω; here, according to (2.77), the amplitude A which is being sought will enter into their

coefficients in addition to other quantities. Thus we obtain from (2.82) two algebraic (possibly transcendental) equations:

$$\left.\begin{aligned} X(A,\ \Omega)&=0,\\ Y(A,\ \Omega)&=0 \end{aligned}\right\} \tag{2.83}$$

with two unknowns A and Ω. Inasmuch as the amplitude A does not enter directly into Expression (2.82), but only in the form of the coefficients $\underline{q}$ and $\underline{q}'$, which are functions of A, then a more suitable notation for Eqs. (2.83) will be

$$\left.\begin{aligned} X(q,\ q',\ \Omega)&=0,\\ Y(q,\ q',\ \Omega)&=0, \end{aligned}\right\} \tag{2.84}$$

where in the general case

$$q=q(A,\ \Omega),\quad q'=q'(A,\ \Omega) \tag{2.85}$$

and in many particular problems

$$q=q(A),\quad q'=q'(A). \tag{2.86}$$

For simple expressions for $\underline{q}$ and $\underline{q}'$ (as, let us say, in the example of § 1.3), we may pass directly to Eqs. (2.83) and find the formulas for A and Ω in explicit form. In more complex cases it is expedient to use graphs of the functions (2.85) or (2.86) that have been constructed beforehand for different types of nonlinearities (see Chapter IV), for the solution of Eqs. (2.84).

By their physical meaning, the amplitude A and frequency Ω of the periodic solution are real positive numbers. Therefore, if as a result of the solution of the pair of equations (2.83) or (2.84), only one of the two unknowns A and Ω is negative (real) or, on the other hand, imaginary or complex, then we will assume that the periodic solution (2.73) is lacking and hence, self-oscillations close to the form (2.73) do not exist in the system in question.

If, however, the solution of the pair of equations (2.83) or (2.84) provides real positive values for the unknowns A and Ω, then

we will assume that a periodic solution close to (2.73) exists,* although this assumption apparently also still requires additional rigorous mathematical proof.* However, the existence of a periodic solution still does not indicate the presence of self-oscillations in the system in question, since only the stable periodic solution of Eq. (2.71) corresponds to self-oscillations. However, the unstable periodic solution has another physical meaning, as will be seen from what follows. Therefore, in order to determine self-oscillations, we must still investigate the stability of the periodic solution which has been found; this will be discussed below in § 2.4.

The second method of determining A and Ω is based upon the fact that the condition for the presence of a pair of purely imaginary roots in the characteristic equation (2.79) is equality to zero of the next-to-last Hurwitz determinant

$$H_{n-1}=0. \tag{2.87}$$

According to the footnote to Formula (2.79), all the remaining Hurwitz determinants are positive. Further, we must use one of the equations (2.83) as the second equation that must be additionally introduced in order to find the two unknowns (A and Ω).

For example, if we write the characteristic equation (2.79) in the form

$$b_0p^n+b_1p^{n-1}+\ldots+b_{n-1}p+b_n=0,$$

where the coefficients b_0, ..., b_n may be functions of the unknowns A and Ω (or only of A), then for a fourth-order system (n = 4) Eq. (2.87) has the form

$$b_3(b_1b_2-b_0b_3)-b_4b_1^2=0, \tag{2.88}$$

while the second of Eqs. (2.83) gives

$$b_1\Omega^2=b_3.$$

For the system of the third order the two similar equations have the form

$$b_1 b_2 - b_0 b_3 = 0, \quad b_0 \Omega^2 = b_2. \tag{2.89}$$

All the general comments which were indicated under the first method also hold true here.

Let us note that in the first method considered above, we used the first of Eqs. (2.83) in place of (2.87). In particular, we used in place of (2.88)

$$b_0 \Omega^4 - b_2 \Omega^2 + b_4 = 0, \tag{2.90}$$

while in place of the first of Eqs. (2.89) we had

$$b_3 - b_1 \Omega^2 = 0.$$

It is evident that the second method being considered here may have practical significance only if the coefficients b_0, ..., b_n are not functions of Ω, i.e., only for the case of a single-valued nonlinearity F(x). Then Eq. (2.87) contains the one unknown A, which is also determined from this same equation independently of the frequency Ω, The latter is thereupon determined separately from the second equation of (2.83). Generally speaking, however, the first method of solution of the problem is more suitable in the majority of cases. An important development of the method of using the next-to-last Hurwitz determinant (2.87) in the problem being considered has been given by K. Magnus [148], [153].

The third method. Like the first method, this method of determining A and Ω also starts with substitution of a purely imaginary value $p = j\Omega$ into the characteristic equation (2.79). But the complex expression (2.81) which is thereupon obtained is written, taking (2.66) and (2.28) into account, in the form [19]

$$\frac{R(j\Omega)}{Q(j\Omega)} = -\frac{1}{q + jq'} \quad \text{or} \quad W_z(j\Omega) = -\frac{1}{W_x}. \tag{2.91}$$

The left-hand side of this equation contains only one of the unknowns – the frequency Ω. In the general case, the right-hand side contains both the unknowns A and Ω, but in many simpler cases it contains only A.

In the latter case the solution of Eq. (2.91) may be found graphically, as the intersection of two curves in the complex plane, one of which, $W_{\underline{l}}(j\Omega)$, corresponds to the variation $0 < \Omega < \infty$, while the second, $-1/W_n$, corresponds to the variation $0 < A < \infty$. If the curves intersect, we shall assume that a periodic solution close to (2.73) exists (here we must verify fulfillment of all the conditions indicated at the beginning of the present section). In order to determine the self-oscillations we must still analyze the stability of the periodic solution (see § 2.4). If, however, the curves indicated do not intersect, then the periodic solution (2.73) is lacking and there are no self-oscillations close to the sinusiodal in the system under consideration.

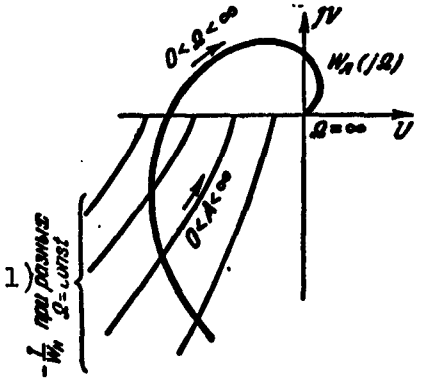


Fig. 2.10. 1) $-1/W_n$ for various Ω = const.

In the general case, however, where $\underline{q}$ and $\underline{q}'$ are functions of A and Ω, we must construct a series of curves of $-1/W_n$ corresponding to the right-hand side of Equality (2.91) for various values of Ω for $0 < A < \infty$ (Fig. 2.10). Here we must find in this series a curve whose value of Ω will coincide with the value of Ω on the curve of $W_{\underline{l}}(j\Omega)$ at their point of intersection.*

The fourth method of determining A and Ω, which may be used in particularly complex cases when other methods are found too unwieldy, consists in the following [43]. Considering the quantities

X and Y in Equalities (2.83) as rectangular coordinates, we must construct curves for a series of specific value of A for $0 < \Omega < \infty$ (Fig. 2.11). The curve which passes through the origin and by which the sought values A and Ω are determined satisfies the equalities (2.83). In practice, therefore, we do not have to construct entire curves, but only the segments close to the origin. Then we must analyze the stability of the resulting periodic solution (see § 2.4).

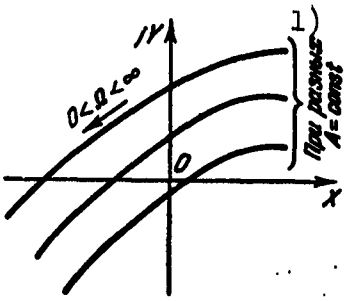


Fig. 2.11. 1) For various A = = const.

If there is no value of A whose curve passes through the origin, there is no periodic solution close to the form (2.73) (and, consequently, no self-oscillation).

The fifth method. For the general case of problems in which each of the harmonic-linearization coefficients q and q' is a complicated function of both unknowns A and Ω, i.e.,

$$q = q(A, \Omega), \quad q' = q'(A, \Omega), \tag{2.92}$$

we may use yet another method of solution, as follows.

Assigning various values of A and Ω, let us construct two series of curves according to Formulas (2.92): $q(\Omega)$ and $q'(\Omega)$ for various A = const (Fig. 2.12, a and b). Then from Eqs. (2.84) we express

$$q = Z_1(\Omega), \quad q' = Z_2(\Omega), \tag{2.93}$$

and plot these two curves on the same graphs. Now it remains to find points C and B on these two curves such that at these points the curves $Z_1(\Omega)$ and $Z_2(\Omega)$ intersect lines with identical values of A at the same value of Ω. The resulting values of A and Ω will be a solution of the problem, i.e., the amplitude and frequency of the periodic solution sought.

In many problems encountered in practice we shall have in place of (2.92)

$$q=q(A) \quad \text{и} \quad q'=q'(A). \tag{2.94}$$

Then the curves of $\underline{q}$ and $\underline{q}'$ in Fig. 2.12 will have the form of horizontal straight lines for different amplitudes.

In the simplest case, when we have in the system a single-valued oddly-symmetrical nonlinearity F(x) for which q = q(A) and q' = 0, we may express from Eqs. (2.84) or even from (2.87)

$$q(A)=Z(\Omega). \tag{2.95}$$

Then, excluding q(A) from Eqs. (2.84), we find the frequency Ω as a function of the system parameters. Next, plotting the curve of the function q(A) (Fig. 2.12c), we construct on it according to (2.95) horizontal lines q = Z(Ω) for various constant values of Ω, i.e., for different relationships of the system parameters. The intersection points of these straight lines with the curve q(A) (for example, the points A_1 and A_2 in Fig. 2.12c) determine the amplitudes of the periodic solutions in each case. If there are no intersections, then there will also be no periodic solutions in the system. In the simplest cases we solve Eq. (2.95) analytically.

The sixth method. The main problem of practical calculations for specific automatic systems is usually the choice of the system parameters, starting from desirable values of the amplitude A and the frequency Ω of the self-oscillations or from the stability requirement of a system without self-oscillations. Thus for specific systems it is important to obtain the quantities A and Ω as functions of one or several system parameters (for example, as functions of the gain constant of the linear part $\underline{k}$, of the feedback constant $k_{o.s}$, of some time constant T, etc.). This was illustrated in the simplest example in § 1.4 (Fig. 1.20).

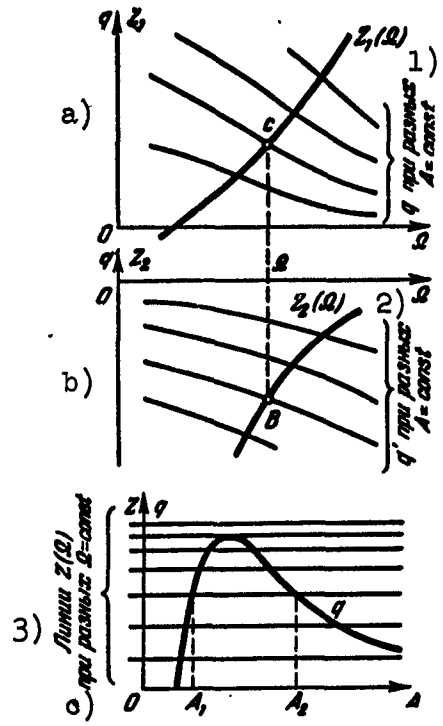


Fig. 2.12. 1) q for various A = const; 2) q' for various A = const; 3) lines Z(Ω) for various Ω = const.

It is obvious that any one of the five methods indicated above may also be used for the solution of this problem if we determine the quantities A and Ω by varying any system parameter. Other methods, which will be simpler in a number of cases [47], may also be used for solution of the problem in question. In fact, if we must find the functions A(k) and Ω(k), we may assume that there are not two independent variables A and Ω in Eqs. (2.83), but three: A, Ω, and k; i.e., they are written in the form

$$\left.\begin{aligned} X(A, \Omega, k)=0, \\ Y(A, \Omega, k)=0 \end{aligned}\right\} \qquad (2.96)$$

or rather in the form of (2.84)

$$\left.\begin{aligned} X(q, q', \Omega, k)=0, \\ Y(q, q', \Omega, k)=0, \end{aligned}\right\} \qquad (2.97)$$

where q and q' are given functions of A and Ω or only of A.

Usually the quantity A enters into Eqs. (2.96) or (2.97) in a complicated manner by way of q and q'. Therefore it is often much simpler to determine not the quantity A from these equations, but the quantity k (or another parameter), assigning values to A. In this we have the essence of the sixth method. This method may have many variants, depending upon the complexity of Eqs. (2.96) or (2.97). The following variants may be regarded as quite universal.

1. Let us exclude the parameter k from Eqs. (2.96) or (2.97),

as a result of which we obtain the equation

$$\Omega = \Omega(A), \tag{2.98}$$

to which we add one of Eqs. (2.96) or (2.97), which is solved in the form

$$k = f(A, \Omega). \tag{2.99}$$

Further, on the basis of (2.98) and (2.99), we may easily construct the sought functions A(k) and Ω(k).

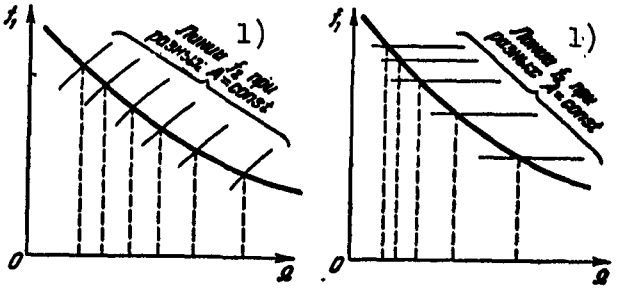


Fig. 2.13. 1) the lines of f_2 for various A = const.

In many practical problems the solution (2.98) is obtained directly in explicit form. In more complex cases, we obtain, after exclusion of k from Expressions (2.97) an equation of the form

$$f_1(\Omega) = f_2(q, q') \quad \text{or} \quad f_1(\Omega) = f_2(q, q', \Omega). \tag{2.100}$$

In this case the problem is solved graphically by plotting the curve of $f_1(\Omega)$ and a series of curves $f_2(\Omega)$ for different values A = const; this is represented in Fig. 2.13, where two variants are shown: 1) f_2 is a function of A and Ω; 2) f_2 is a function only of A. The coordinates of all points of intersection give the sought function (2.98), and thereupon the values of k according to Formula (2.99).

2. If we must choose two system parameters, then we may make use of the above method individually for each parameter. But we may also proceed otherwise. Let us call these two parameters k and T and write (2.96) in the form

$$X(A, \Omega, k, T)=0, \quad Y(A, \Omega, k, T)=0. \tag{2.101}$$

Hence we express

$$k=f_1(A, \Omega), \quad T=f_2(A, \Omega). \tag{2.102}$$

From these formulas, assigning different values A = const, we may construct in the parameter plane (Fig. 2.14) lines T(k) from the parametric equations k = k(Ω) and T = T(Ω), varying Ω. Thus we obtain the lines A = const. Next, connecting points on these lines with the same values of Ω, we obtain the lines Ω = const (Fig. 2.14). The diagram obtained permits us to choose directly the two system parameters T and k, starting from the desired values of the amplitude A and the frequency Ω of the self-oscillations or, on the other hand, starting from the requirement that self-oscillations be absent (the region in Fig. 2.14 which is not filled by curves). An example of such a diagram was given in Figs. 1.21 and 1.22.

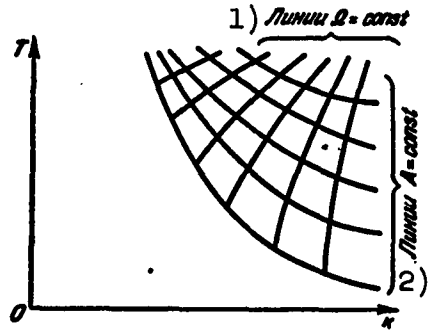


Fig. 2.14. 1) Lines Ω = const; 2) lines A = const.

Let us note that all the methods other than the second (p. 138), may also be applied to systems containing a pure lag element. Then we introduce the multiplier $e^{-\tau p}$ into the characteristic equation at the appropriate place, for example:

$$Q(p)+R(p)e^{-\tau p}\left(q+\frac{q'}{\Omega}p\right)=0, \tag{2.103}$$

here, after the substitution p = jΩ in the third method, Formula (2.91) will automatically contain the factor $e^{-\tau j\Omega}$, while in the first, fourth, fifth, and sixth methods we must separate it into real and imaginary parts in the form

$$e^{-j\tau\Omega}=\cos\tau\Omega-j\sin\tau\Omega. \tag{2.104}$$

All the methods described above relate to nonlinear systems of the first class that satisfy equations of the type (2.71). In

principle they are all equivalent to each other. The choice of one of these six methods for the solution of a specific problem is dictated by computational convenience in each case.

For another type of nonlinear system of the first class which is described by an equation of the form (2.67):

$$Q(p)x + R_1(p)F_1(x, px) + R_2(p)F_2(x, px) = 0, \tag{2.105}$$

harmonic linearization leads to the characteristic equation

$$Q(p) + R_1(p)\left(q_1 + \frac{q_1'}{\Omega}p\right) + R_2(p)\left(q_2 + \frac{q_2'}{\Omega}p\right) = 0 \tag{2.106}$$

with the two unknowns A and Ω entering into the coefficients q_1, q_1', q_2, and q_2', which are calculated from Formulas (2.76). From comparison of (2.106) with (2.79), it is evident that here the first, second, fourth, and sixth of the methods of solution of the problev described above will be valid if we take into account only the presence not of two, but of four coefficients depending on the unknown amplitude A (and in the general case also on Ω). However, we are not able to apply the third and fifth methods directly, since it is impossible in the general case to write expressions of the type (2.91) and (2.93) for the characteristic equation (2.106). We may, however, use modifications of these methods. Thus, for the third method we obtain from (2.106) with $p = j\Omega$ the equation

$$\frac{R_1(j\Omega)}{Q(j\Omega)} = -\frac{1}{q_1 + jq_1'} - \frac{R_2(j\Omega)}{Q(j\Omega)}\frac{q_2 + jq_2'}{q_1 + jq_1'}. \tag{2.107}$$

For its graphical solution, we must construct curves corresponding to the entire right-hand side of Eq. (2.107) in Fig. 2.10 in place of the curve $-1/W_n$. For the fifth method, in exactly the same fashion, in place of the quantities $\underline{q}$ and $\underline{q}'$ in Eqs. (2.93) we will have combinations of them, also including functions of Ω.

In the case where each of the nonlinearities $F_1(x)$ and $F_2(x)$ in the system equations (2.105) is a single-valued oddly-symmetrical

function, we will have from Eq. (2.106) with $p = j\Omega$

$$Q(j\Omega)+R_1(j\Omega)q_1(A)+R_2(j\Omega)q_2(A)=0.$$

Here, separating the real and imaginary parts, we obtain the equations

$$X(q_1, q_2, \Omega)=0, \quad Y(q_1, q_2, \Omega)=0,$$

from which we may express

$$q_1(A)=Z_1(\Omega), \quad q_2(A)=Z_2(\Omega),$$

and the solution reduces to the construction of simple graphs of the type of Fig. 2.12 with horizontal lines of q_1 and q_2 (in place of $\underline{q}'$) for different values A.

As we have already said, the remaining four methods retain their previous form for the system described by Eq. (2.105).

In nonlinear systems of the second class with a mixed nonlinearity of the form (2.35), making use of the substitution (2.41), we arrive at an equation of the type (2.71). For such systems, therefore, all six methods of solution of the problem described above remain without change.

In nonlinear systems of the second and third classes with two individual nonlinearities of the type (2.34), we obtain more complex equations. The characteristic equation may be reduced to the form(2.106), but only with the essential difference that we shall have the coefficients $\underline{q}$, $\underline{q}'$, q_2, q_2', which, according to (2.37), contain not two, but three unknowns: A, A_2, and Ω. Therefore we add to them still another relationship of the type of the first in (2.40). Thus do we investigate systems of the second class. For systems of the third class, however, terms of the type (2.28) are introduced into the relationship (2.40). After substituting the quantities A_2 from the relationship (2.40) into the expressions for q_2 and q_2', the subsequent solution of the problem will be

similar to the solution of the problem for systems of the first class with an equation of the type (2.106). Concerning systems with many single-valued nonlinearities, see [103].

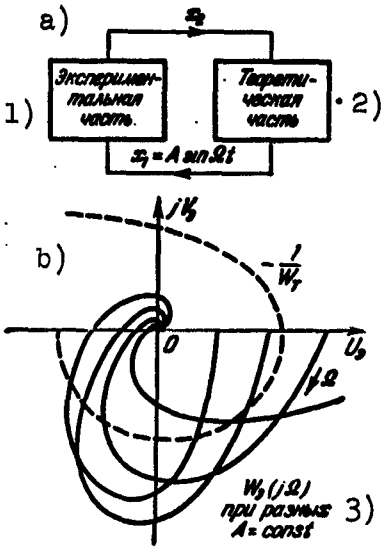


Fig. 2.15. 1) Experimental part; 2) theoretical part; 3) $W_e(j\Omega)$ for various A = = const.

The use of experimental characteristics. For the determination of self-oscillations we may make use not only of the equations for transfer functions, but also the experimental frequency characteristics of separate links or large blocks of the system. For example, let the automatic system being investigated be decomposed into two parts (Fig. 2.15a), the first of which is realized in its natural [prototype] form or in the form of a working model (experimental part), while the second is only designed theoretically, with differential equations or transfer functions formulated for it (theoretical part).

If the experimental part is linear (as is seen from the independence of the experimentally obtained frequency characteristics of the oscillation amplitudes for the input quantity, when calculation of the self-oscillations is carried out by the third method, with the nonlinearity separated from the theoretical part, while the experimental part is connected to the linear links of the theoretical part, i.e., the experimentally obtained gain-phase characteristic $W_e(j\Omega)$ enters into the formulation of $W_1(j\Omega)$ in Fig. 2.10.

If, however, the experimental part contains a nonlinearity (of arbitrary form), then the gain-phase frequency characteristic

which is recorded from it for the first harmonic $W_e(j\Omega)$ will have a different form for various oscillation amplitudes A of the input quantity (Fig. 2.15b). According to the frequency criterion, the passage of the general gain-phase characteristic of the open circuit ($W_e W_t$) through the point −1 or

$$W_e = -\frac{1}{W_t}. \tag{2.108}$$

is a condition for the appearance of a sinusoidal periodic solution in such a system (Fig. 2.15a). In the case where the theoretical part of the system is linear, W_t depends only upon the frequency Ω. In this case, on drawing the curve

$$-\frac{1}{W_t(j\Omega)}$$

(the broken curve in Fig. 2.15b), we obtain a graphic solution of Eq. (2.108); here the amplitude and frequency of the periodic solution are determined at the intersection point of the curve of $-1/W_t$ with that one of the curves W_e which has exactly the same value of Ω at this point as on the curve $-1/W_t$.

However, in the case where the theoretical part of the system also contains a nonlinearity, then we must break down the expression of its gain-phase characteristic W_t into two factors, separating the part which is not a function of amplitude:

$$W_t = W_{t.n}(A, \Omega) W_{t.l}(j\Omega).$$

Then Eq. (2.108) may be written in the form

$$W_e W_{t.n} = -\frac{1}{W_{t.l}}.$$

In order to solve it graphically, we must draw the family of resultant curves $W_e W_{t.n}$ instead of the family W_e in Fig. 2.15b with the present value of the parameter Ω for different A = const, while in pace of $-1/W_t$ we plot the curve $-1/W_{t.l}$.

Proximity of the sought periodic solution to the sinusoidal is

guaranteed, as previously, by the nonpassage of the higher harmonics in the appropriate parts of the system; here, the presence or absence of such a property within the experimental part of the system is directly evident upon registration of its frequency characteristics.

§ 2.4. THE STABILITY OF THE PERIODIC SOLUTION

Let us assume that we have found by one of the methods of § 2.3 the approximate periodic solution in the form $x = A \sin \Omega t$ for the given differential equation, for example (2.71) or (2.105). In order to determine whether it corresponds to self-oscillations or not, we must analyze its stability, especially for the case of the simultaneous presence of two or more periodic solutions.

The rigorous classical method of stability analysis consists in the formation of a linearized differential equation in small deviations from the solution being investigated. Let us introduce the variable $x = x^* + \Delta x$, where $x^* = A \sin \Omega t$. Then, according to (2.71), the above equation in small deviations from the periodic mode (equation in variations) will be

$$Q(p)\Delta x + R(p)\left[\left(\frac{\partial F}{\partial x}\right)^* \Delta x + \left(\frac{\partial F}{\partial px}\right)^* p\Delta x\right] = 0, \qquad (2.109)$$

where the asterisk designates the substitution $x = x^* = A \sin \Omega t$. Hence the partial derivatives which are indicated by asterisks represent periodic coefficients. Thus, stability analysis (for small deviations) of the periodic solution found in § 2.3 for the nonlinear equation (2.71) reduces to determination of whether the solution of the linear equation (2.109) is damped or diverges, i.e., the problem reduces to equilibrium stability analysis ($\Delta x = 0$) with reference to a linear differential equation with periodic coefficients. The situation is also similar for the system described by

Eq. (2.105), and also for nonlinear systems of the second and third classes.

A certain general approach to stability analysis of systems with periodic coefficients exists in the well-known work of A.M. Lyapunov (see also [34], p. 146 and [71], Chapter III). There are also a number of specific investigations due to various authors. In the overwhelming majority of cases, however, the analysis of equations with periodic coefficients is an extremely difficult and often impossible problem. Let us therefore turn to approximate methods.

One of the approximate methods consists in replacement of the periodic coefficients by constants equal to their average value over the period. Here Eq. (2.109) is replaced by an equation with constant coefficients:

$$[Q(p)+R(p)(\varkappa+\varkappa' p)]\Delta x=0, \tag{2.110}$$

where

$$\varkappa=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial F}{\partial x}\right)^*d\psi, \quad \varkappa'=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial F}{\partial px}\right)^*d\psi, \quad \psi=\Omega t. \tag{2.111}$$

Then we apply any one of the well-known stability criteria for linear systems (Hurwitz, Mikhaylov, Nyquist) to Eq. (2.110).

Inasmuch as here one equation (2.109) is replaced by another (2.110), then far from all of the results of the investigation will be valid for Eq. (2.109). We may only assert that such a method is useful for a certain class of problems [51] and that in the specific problems to which it is applied, the results are found to be valid.

Thus, for the example considered in § 1.3, we have, according to (1.68)-(1.70), the following system of equations in small deviations from the periodic mode:

$$\left.\begin{aligned}&(T_1p+1)\Delta x_2=-k_1\Delta x_4,\\&\Delta x_3=\left(\frac{dF}{dx}\right)^*\Delta x,\quad \Delta x=\Delta x_2-k_{o.c}\Delta x_4,\\&(T_2p+1)p\Delta x_4=k_2\Delta x_3.\end{aligned}\right\}\qquad(2.112)$$

Let us replace the periodic coefficient (dF/dx)* by its average value over the period (2.111). Inasmuch as x = A sin ψ, then

$$\frac{dx}{d\psi}=A\cos\psi=\sqrt{A^2-x^2}.$$

Taking this into account and shifting the limits of integration in (2.111) through $-\pi/2$ we obtain the period-averaged coefficient

$$\varkappa=\frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\left(\frac{dF}{dx}\right)^*d\psi=\frac{1}{\pi}\int_{-A}^{+A}\frac{dF}{dx}\frac{dx}{\sqrt{A^2-x^2}}.$$

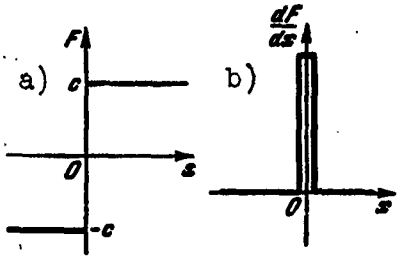


Fig. 2.16.

According to Fig. 2.16, the derivative dF/dx is a momentary pulse of area 2c at x = 0. Hence, the integrand is everywhere equal to zero apart from the point x = 0, where $\sqrt{A^2 - x^2}$ = A, since it is a constant it may be taken out from under the integral sign. The integral of the remaining expression (dF/dx)dx gives the area under the curve dF/dx (Fig. 2.16), which is equal to 2c. Hence, the period-averaged value of the periodic coefficient will be

$$\varkappa=\frac{2c}{\pi A}.\qquad(2.113)$$

Substituting it in the equations of System (2.112) in place of (dF/dx)*, we write the characteristic equation:

$$T_1T_2p^3+(T_1+T_2)p^2+(1+T_1k_2k_{o.c}\varkappa)p+(k_1+k_{o.c})k_2\varkappa=0.\qquad(2.114)$$

According to the Hurwitz criterion, the stability condition will be

$$(T_1+T_2)(1+T_1k_2k_{o.c}\varkappa)-T_1T_2(k_1+k_{o.c})k_2\varkappa>0,$$

or taking (2.113) into account

$$\pi A(T_1+T_2)+2cT_1k_2(T_1k_{o.c}-T_2k_1)>0.$$

Substituting here the value found in § 1.3 for the amplitude A of the periodic solution (1.85), we obtain the stability condition for the periodic solution in the form

$$T_2 k_1 - T_1 k_{0.c} > 0,$$

which agrees with the existence condition for the periodic solution (1.87). Hence, the periodic solution found in the example of § 1.3 is stable (i.e., it represents self-oscillations) over the whole range of its existence.

Another category of approximate methods for stability analysis of a periodic solution [43] and [47], which will also be given prime consideration subsequently, is based upon the following considerations.

The classical method for analyzing the stability of the periodic solution consists in analysis of the transient process for small deviations from this solution with respect to a linear equation with periodically varying coefficients. Whatever the character of such a transient process, determination of the variation of the total coordinate $\underline{x}$ in the process of determining a periodic solution requires adding the small values of the function $\Delta x(t)$ obtained from (2.109) to the values $x = A \sin \Omega t$ of the periodic solution.*

Without separating out the small deviations of the transient process, let us consider all of this nonstationary oscillatory process as a whole, assuming that close to the analyzed periodic solution for the variable $\underline{x}$ in the nonlinear function it has the form of damped or diverging oscillations, as shown in Fig. 2.17a (for a stable periodic solution) and in Fig. 2.17b (for an unstable periodic solution). Here, inasmuch as the nonstationary oscillations x(t) being considered are close to sinusoidal, they are described well enough by the preceding harmonically linearized characteristic

equation (2.79):

$$Q(p)+R(p)\left(q+\frac{q'}{\Omega}p\right)=0, \tag{2.115}$$

or, on the other hand, by (2.106):

$$Q(p)+R_1(p)\left(q_1+\frac{q'_1}{\Omega}p\right)+R_3(p)\left(q_3+\frac{q'_3}{\Omega}p\right)=0. \tag{2.116}$$

Calling the entire left-hand side L(p) for brevity, we reduce the characteristic equation to the form

$$L(p)=b_0p^n+b_1p^{n-1}+\ldots+b_{n-1}p+b_n=0, \tag{2.117}$$

where some of the coefficients contain the quantities $\underline{q}$ and $\underline{q}'$, in whose formulas (2.76) the amplitude is now not the constant A as previously, but the variable a(t), which deviates from A to a small degree, i.e., varies slowly over time close to the value A; thus, $q = q(a, \Omega)$, $q' = q(a, \Omega)$ or, on the other hand, $q = q(a)$, $q' = q'(a)$.

Such an approach is fundamental for the system being considered above, where the reduced linear part is stable or neutral and does not let the higher harmonics pass, if we take into account the condition introduced in § 2.2 that with variation of the amplitude close to its values for the periodic solution in question, the harmonic-linearization coefficients $\underline{q}$ and $\underline{q}'$ and their derivatives with respect to $\underline{a}$ vary sufficiently smoothly. In compliance with § 2.3 we also add the requirement that for the characteristic equation of a harmonically linearized system (2.115) or (2.116) all the Hurwitz determinants other than the next-to-last H_{n-1} be positive (on the other hand, the quantity H_{n-1} varies in the neighborhood of zero). This guarantees closeness of the nonlinear system being considered to a linear system located on the oscillatory stability boundary. Hence, it also guarantees the oscillatory character of the processes in the nonlinear system in question (Fig. 2.17);

for the variable x, these are close to the sinusoidal in the neighborhood of the periodic mode of operation being considered for any small variation of the system parameters and the quantity a (possibly, with some small variation in the frequency Ω). These initial conditions are well justified in practice for the majority of nonlinear automatic systems.

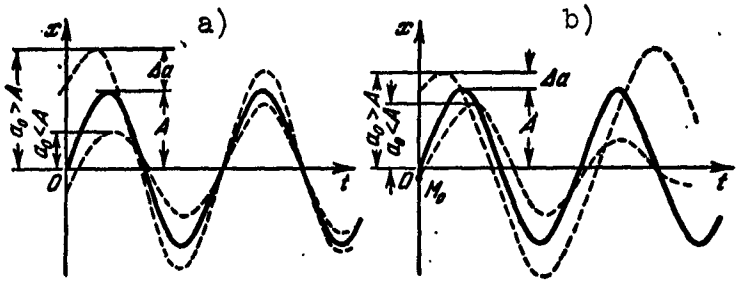


Fig. 2.17.

From the point of view of simplicity of solution, we must keep in mind that the requirement indicated here (that all the Hurwitz determinants other than the next-to-last H_{n-1} be positive for third- and fourth-order systems simply indicates all the coefficients positive in the characteristic equation (2.117) of the harmonically linearized closed system, while for second-order systems it indicates that the first and last coefficients are positive. For systems higher than the fourth order, instead of verifying the positivity of the indicated Hurwitz determinants for Eq. (2.117), we may require satisfaction of any of the stability criteria (Hurwitz, Mikhaylov, Nyquist) for a polynomial having a degree two lower than the degree of Eq. (2.117), i.e.,

$$L_1(p)=\frac{L(p)}{p^2+\Omega^2}. \tag{2.118}$$

All this ensures the presence of negative real parts in all roots other than one pair of purely imaginary roots occurring in the characteristic equation (2.117) of the harmonically linearized

system (by which we also establish the closeness of the behavior of the nonlinear system in question to that of the linear system located on the oscillatory stability boundary).

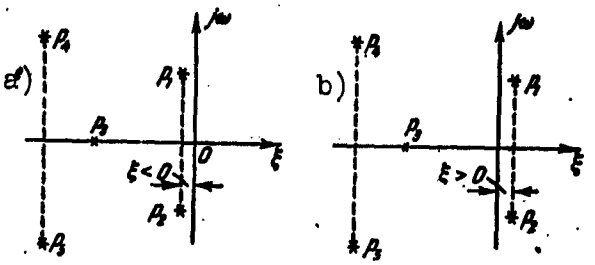


Fig. 2.18.

Thus, for stability analysis of a periodic solution, we shall impart a small deviation Δa to the amplitude, i.e., instead of A in the characteristic equation (2.117) we substitute $a = A + \Delta a$. This causes a certain change in the coefficients of Eq. (2.117), as a consequence of which the above pair of purely imaginary roots $p_{1,2}$, which determine the value of $a = A$, shift slightly to the left or right from the imaginary axis (Fig. 2.18), acquiring a small real part ξ ($\xi < 0$ or $\xi > 0$). It may correspond to slowly damping or slowly diverging oscillations, which are shown by the broken lines in Fig 2.17. All the remaining roots of the characteristic equations (2.117), if they lie to the left of the imaginary axis, cannot influence these processes essentially, at least after some finite time interval has elapsed after the beginning of the process. This is similar to the pattern of variation of the processes in the linear system located on the oscillatory stability boundary on a small change in some of its parameters that determine the coefficients of the characteristic equation (2.117) in the same way as q and q' enter into them, with the variation of the coefficients q and q' due to a change in the oscillation amplitude a for the same parameters of the system in question corresponding here to

variation of the parameters of the linear system.

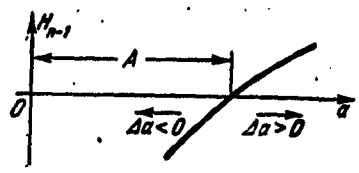


Fig. 2.19.

The first criterion. Now it is evident that under these conditions, stability of a periodic solution, i.e., obtaining a process pattern for the establishment of self-oscillations of the type in Fig. 2.17a, requires that for $\Delta a > 0$ Eq. (2.117) satisfy the Hurwitz criterion, while for $\Delta a < 0$ all the conditions of the Hurwitz criterion be satisfied except one: $H_{n-1} < 0$ (Fig. 2.19). Then for $\Delta a > 0$ the process will be damped, while for $\Delta a < 0$ it will diverge, as is shown in Fig. 2.17a. Let us recall that for $\Delta a = 0$, i.e., for the periodic solution itself, $H_{n-1} = 0$ according to (2.87).

Let us note that here we were only speaking of the amplitude deviations $a = A + \Delta a$. Meanwhile the frequency Ω, which may also take small variations in the process of determining self-oscillations, may in the general case also enter into the coefficients of the characteristic equation. Therefore we must require that the approximate criterion indicated for stability of the periodic solution be fulfilled not only for the given value of Ω (of the periodic solution being investigated), but also for small deviations $\Delta\omega$, if the quantity Ω enters into the expression for the coefficients of the characteristic equation (2.117).

Let us express the criterion described in a compact analytic form. For a small deviation Δa we must have a definite change in sign for the next-to-last Hurwitz determinant H_{n-1}. Hence (Fig. 2.19) we require for stability of the periodic solution firstly that

$$\left(\frac{\partial H_{n-1}}{\partial a}\right)^* > 0 \quad \text{or} \quad \left(\frac{\partial H_{n-1}}{\partial q}\frac{\partial q}{\partial a} + \frac{\partial H_{n-1}}{\partial q'}\frac{\partial q'}{\partial a}\right)^* > 0, \tag{2.119}$$

where the asterisk indicates the substitution of the quantity $a = A$ corresponding to the periodic solution whose stability is being analyzed.* Here the sign of Expression (2.119) must not change for a small deviation ω in either direction of the value Ω corresponding to the periodic solution being investigated if the quantity Ω enters into the coefficients $\underline{q}$ and $\underline{q}'/\Omega$.

Secondly, for values of A and Ω corresponding to the periodic solution being investigated, all remaining Hurwitz determinants must be positive, except for the determinant H_{n-1} already considered. For systems of third and fourth orders, this is equivalent simply to all coefficients of the characteristic equation positive. Therefore the verification of the second condition indicated, which is equivalent to fulfillment of the stability criterion for the polynomial (2.118), need be carried out only in analysis of systems of fifth and higher order.

This criterion is particularly convenient for use in determination of the periodic solution by the second method (§ 2.3), although this does not, of course, exclude its use in other cases as well. Thus, in the example of § 1.3, according to (1.81) and (1.89), we have

$$H_{n-1} = (T_1 + T_2)(1 + T_1 k_2 k_{o.c} q) - T_1 T_2 (k_1 + k_{o.c}) k_2 q = \\ = T_1 + T_2 + \frac{4c}{\pi a} T_1 k_2 (T_1 k_{o.c} - T_2 k_1)$$

and, consequently,

$$\frac{\partial H_{n-1}}{\partial a} = \frac{4c}{\pi a^2} T_1 k_2 (T_2 k_1 - T_1 k_{o.c}).$$

The criterion (2.119) is fulfilled for $T_2 k_1 - T_1 k_{o.s} > 0$, which corresponds to stability of the periodic solution (self-oscillations) found in § 1.3 over the whole range of its existence (1.87).

The second criterion. Let us further consider the following variant of the approximate stability criterion of the same type, in which, instead of the Hurwitz criterion, we use the Mikhaylov criterion.

Let us write the expression of the Mikhaylov curve for the characteristic equation (2.115) or (2.116) by substituting in it $p = j\omega$:

$$L(j\omega) = Q(j\omega) + R(j\omega)\left(q + \frac{q'}{\Omega} j\omega\right) \tag{2.120}$$

or, on the other hand,

$$L(j\omega) = Q(j\omega) + R_1(j\omega)\left(q_1 + \frac{q_1'}{\Omega} j\omega\right) + R_2(j\omega)\left(q_2 + \frac{q_2'}{\Omega} j\omega\right),$$

where ω denotes the current value of the parameter of the Mikhaylov curve, in distinction from the periodic-solution frequency Ω which enters into the coefficients of these expressions. We separate the real and imaginary parts in these expressions

$$L(j\omega) = X(\omega) + jY(\omega), \tag{2.121}$$

whose coefficients are codetermined by the amplitude A and, in the general case, also by the frequency Ω of the periodic solution being investigated.

As we know, for the case of a periodic solution, i.e., for the presence of a pair of purely imaginary roots $p = \pm j\Omega$ in the characteristic equation, the Mikhaylov curve passes through the origin (Fig. 2.20); at the point of the curve coinciding with the origin, the parameter ω is equal to the absolute value of the imaginary root Ω (to the frequency of the periodic solution). Therefore, Expressions (2.83), (2.84), and others considered above are a particular case of (2.121) for a point located at the origin (Fig. 2.20).

Let us impart a small deviation to the amplitude: $a = A + \Delta a$, as a consequence of which the coefficients of Expression (2.121)

are changed and the Mikhaylov curve is deflected from the origin in either direction (in Fig. 2.20 we show only small segments 1 and 2 of the deflected curves). In cases when the Mikhaylov criterion, which is known from the linear theory of automatic control, is satisfied (curve 1 Fig. 2.20), there will be damped oscillations in the system, while if it is not satisfied (curve 2), there will be divergent oscillations.

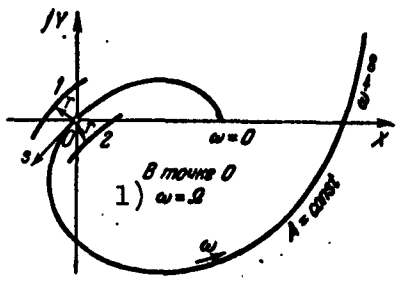


Fig. 2.20. 1) At point 0.

Hence for stability of the periodic solution, i.e., in order to obtain the pattern of the processes shown in Fig. 2.17a, we require that the Mikhaylov criterion be satisfied for $\Delta a > 0$ and not satisfied for $\Delta a < 0$.

It is convenient to use this criterion when the frequency and amplitude of the periodic solution are determined graphically by the fourth method (§ 2.3). Here we must keep in mind that the curves represented in Fig. 2.11 will coincide exactly with the Mikhaylov curves (Fig. 2.20) only if the frequency Ω does not enter into the coefficients of the characteristic equation. Otherwise the curves of Fig. 2.11 will differ somewhat from the Mikhaylov curves used in Fig. 2.20. This is evident on comparison of Equalities (2.81) and (2.120).

The third criterion. Analytically the same criterion for the stability of the periodic solution may be expressed as follows. The shift of the point 0 of the Mikhaylov curve (Fig. 2.20) for a small change in a may be characterized by a vector $\vec{r}$ with the projections

$$X_r = \left(\frac{\partial X}{\partial a}\right)^* \Delta a, \quad Y_r = \left(\frac{\partial Y}{\partial a}\right)^* \Delta a; \tag{2.122}$$

however, the shift of the Mikhaylov-curve coordinates for a small change in ω is determined by a vector $\vec{s}$ (Fig. 2.20) with the projections

$$X_s=\left(\frac{\partial X}{\partial \omega}\right)^* \Delta\omega, \quad Y_s=\left(\frac{\partial Y}{\partial \omega}\right)^* \Delta\omega. \tag{2.123}$$

In both cases, the asterisk corresponds to substitution of the values $a = A$ and $\omega = \Omega$ of the periodic solution being investigated into the partial derivatives of the expressions for X and Y, which enter into (2.121). This substitution corresponds to the initial position of the point). Let us find the angle between the vectors $\vec{r}$ and $\vec{s}$. According to the rules of vector algebra (from the formula for vector multiplication) we have

$$\sin(\widehat{rs})=\frac{X_rY_s-Y_rX_s}{rs}, \tag{2.124}$$

where r and s are the moduli of the corresponding vectors. From Fig. 2.20 it is evident that if we take $\Delta\omega > 0$, satisfaction of the (second) criterion expressed above for stability of the periodic solution requires that the vector $\vec{s}$ be deflected from the vector $\vec{r}$ counterclockwise for $\Delta a > 0$ and clockwise for $\Delta a < 0$. It follows from this according to (2.124) that

$$\frac{X_rY_s-Y_rX_s}{rs}\begin{cases}>0 \text{ for } \Delta a>0 \text{ и } \Delta\omega>0,\\ <0 \text{ for } \Delta a<0 \text{ и } \Delta\omega>0.\end{cases}$$

Inasmuch as the quantities r and s are positive as vector moduli, we find after substitution here of the values of (2.122) and (2.123) that stability of the periodic solution requires, <u>firstly</u>, that the following condition [49] be satisfied:

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^*-\left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^*>0, \tag{2.125}$$

here, the asterisk indicates that in the partial derivatives, which are taken in general form from Expression (2.121), we must sub-

stitute the values a = A and $\omega = \Omega$ for the periodic solution whose stability is being analyzed. Here we must verify further that the inequality sign of (2.115) is preserved on a small deviation of Ω if the latter enters into the coefficients of Expression (2.121). It is convenient to calculate the derivatives of X and Y with respect to a, which enter into (2.125), in the form

$$\frac{\partial X}{\partial a} = \frac{\partial X}{\partial q}\frac{\partial q}{\partial a} + \frac{\partial X}{\partial q'}\frac{\partial q'}{\partial a}, \ldots$$

Secondly, in addition to satisfying Condition (2.125), we require that the entire path of the remaining part of the Mikhaylov curve (with the exclusion of the one point 0 at the origin) as shown in Fig. 2.20, satisfy the Mikhaylov criterion. The latter condition must be verified specifically only for systems of the fifth order and higher. As regards systems of the third and fourth orders, this reduces to the simple requirement of all coefficients of the characteristic equation positive (2.117).

Let us note that Condition (2.125) is equivalent to the preceding condition (2.119), while the auxiliary second condition indicated here is equivalent to the preceding condition of all remaining Hurwitz determinants positive (other than H_{n-1}) or to the fulfillment of the stability criterion for the polynomial (2.118).

It is especially convenient to make use of the criterion (2.125) when the amplitude and frequency of the periodic solution are determined by the first, fifth, or sixth methods (§ 2.3), although we do not exclude the possibility of its use in all other cases as well.

Thus, for the example of § 1.3, we have according to (1.81) and (1.79)

$$\left.\begin{aligned} X &= (k_1 + k_{o.c})\, k_2 q - (T_1 + T_2)\,\omega^2, \\ Y &= (1 + T_1 k_2 k_{o.c} q)\,\omega - T_1 T_2 \omega^3, \end{aligned}\right\} \quad q = \frac{4c}{\pi a},\ \frac{dq}{da} < 0$$

and hence for positive values of the system parameters

$$\left(\frac{\partial X}{\partial a}\right)^* = (k_1 + k_{o.c})\, k_2 \frac{dq}{da} < 0, \quad \left(\frac{\partial X}{\partial \omega}\right)^* = -2(T_1 + T_2)\,\Omega < 0,$$

$$\left(\frac{\partial Y}{\partial a}\right)^* = T_1 k_2 k_{o.c}\,\Omega \frac{dq}{da} < 0, \quad \left(\frac{\partial Y}{\partial \omega}\right)^* = 1 + T_1 k_2 k_{o.c} q - 3T_1 T_2 \Omega^2 = -2T_1 T_2 \Omega^2 < 0;$$

the latter follows from the second equality of (1.83).

Therefore the criterion (2.125) assumes the form

$$-T_2 (k_1 + k_{o.c}) \frac{dq}{da} + (T_1 + T_2) k_{o.c} \frac{dq}{da} > 0.$$

But since dq/da < 0, this inequality reduces to the following:

$$T_2 k_1 - T_1 k_{o.c} > 0.$$

Thus, the periodic solution found in § 1.3 will be stable over the entire region of its existence (1.87).

Let us introduce still another analytical derivation of the third stability criterion (2.125) for the periodic solution [47].

As we know, the periodic solution x = A sin Ωt may be written in complex form (Fig. 2.6a):

$$x = A \operatorname{Im} e^{j\Omega t}. \tag{2.126}$$

Let us write the transient process which is obtained in the system after the formation of small amplitude and frequency deviations Δa and Δω approximately in the form

$$x = (A + \Delta a) e^{-kt} \sin (\Omega + \Delta\omega) t \tag{2.127}$$

or, by analogy with (2.126), in the complex form

$$x = (A + \Delta a) \operatorname{Im} e^{j(\Omega + \Delta\omega + jk)t}. \tag{2.128}$$

Inasmuch as the solution (2.126) is determined, according to (2.82) by the condition

$$X(A, \Omega) + jY(A, \Omega) = 0, \tag{2.129}$$

the solution (2.128) formally corresponds to the condition

$$x(A + \Delta a, \Omega + \Delta\omega + jk) + jY(A + \Delta a, \Omega + \Delta\omega + jk) = 0.$$

Let us expand this expression into a Taylor series, adopting

the notation $\Omega + \Delta\omega + j\xi = \omega$. Them, making use of (2.129), we obtain:

$$\left(\frac{\partial X}{\partial a}\right)^* \Delta a + \left(\frac{\partial X}{\partial \omega}\right)^* (\Delta\omega + j\xi) + j\left(\frac{\partial Y}{\partial a}\right)^* \Delta a + j\left(\frac{\partial Y}{\partial \omega}\right)^* (\Delta\omega + j\xi) = 0,$$

where the asterisk denotes the substitution, into the partial derivatives, of the values a = A and $\omega = \Omega$ that correspond to the periodic solution being investigated.

Separating the real and imaginary parts here, we have two equalities; eliminating $\Delta\omega$ from them, we find

$$\xi = \frac{\left(\frac{\partial X}{\partial a}\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial Y}{\partial a}\frac{\partial X}{\partial \omega}\right)^*}{\left(\frac{\partial X}{\partial \omega}\right)^* + \left(\frac{\partial Y}{\partial \omega}\right)^*} \Delta a.$$

But, as is evident from (2.127), the transient process will converge to self-oscillations with amplitude A from both sides only if Δa and ξ have the same sign. Hence, the stability condition for for a periodic solution will be

$$\left(\frac{\partial X}{\partial a}\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial Y}{\partial a}\frac{\partial X}{\partial \omega}\right)^* > 0,$$

Q.E.D.

The fourth criterion. For the use of the third method of determining the periodic solution (§ 2.3) it is convenient to make use of the approximate frequency criterion of stability [19].

If we conditionally decompose the system into a linear part and a nonlinear part (Fig. 1.12b), then according to Formula (2.66) we may write the expression for the gain-phase characteristics of the reduced linear part*

$$W_л = \frac{R(j\omega)}{Q(j\omega)}$$

and, according to (2.28), the expression for the gain-phase characteristic for the nonlinearity (with respect to the first harmonic)

$$W_н = q + jq'.$$

Here the open-loop gain-phase characteristic for the whole system (with respect to the first harmonic) will be

$$W = W_1 W_2 = \frac{R(j\omega)}{Q(j\omega)}(q + jq'). \qquad (2.130)$$

Therefore Eq. (2.91), which was used in the third method (§ 2.3), may be treated as equating the gain-phase characteristic W to negative unity for $\omega = \Omega$; this corresponds to passage of the curve of open-loop gain-phase characteristic of the whole system through the point C (Fig. 2.21a). As we know, in the Nyquist frequency criterion this is the condition for the appearance of sinusoidal oscillations with the frequency Ω in a closed-loop system.

Let us now supply the amplitude deviation $a = A + \Delta a$ in the coefficients of the expression for W (2.130). Then the gain-phase characteristic is shifted (curve 1 or 2 in Fig. 2.21a). If here the Nyquist frequency criterion of stability is satisfied (for example, curve 1), the oscillations will be damped; if not, (curve 2) they will diverge. Let us recall that we are considering only systems in which the open loop is stable or neutral, since Q(p) does not have purely imaginary roots and roots with positive real parts.

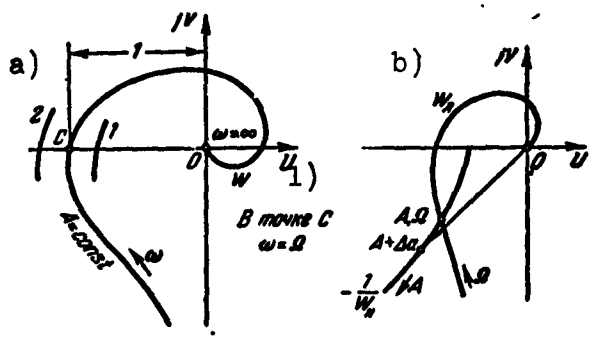


Fig. 2.21. 1) At point.

Therefore, in order that the pattern of the processes shown in Fig. 2.17a may be the case, i.e., in order that the periodic solution may be stable, we require that for $\Delta a > 0$ the frequency criterion be

satisfied (curve 1 Fig. 2.21a), while for $\Delta a < 0$ it is not to be satisfied (curve 2). This must also hold true for small variations of Ω in the case where Ω appears in the expressions for the coefficients $\underline{q}$ and $\underline{q}'$.

Inasmuch as in the third method (§ 2.3) we do not construct the entire characteristic W as a whole, but only in separate parts, in the form of curves of $W_{\underline{l}}$ and $-1/W_n$ (Fig. 2.10), it is also desirable to reformulate the present approximate stability criterion for the periodic solution. The stability of the periodic solution is determined by the overlapping (for $\Delta a < 0$) and the nonoverlapping (for $\Delta a < 0$) of the point C by the characteristic W (Fig. 2.21a). This corresponds to nonoverlapping of the end of the vector $-1/W_n$ by the characteristic $W_{\underline{l}}$ for an increased value of the amplitude $a = A + \Delta a$ (Fig. 2.21b) and to the overlapping for a decreased value of the amplitude.

Therefore, stability of the periodic solution requires that the characteristic $W_{\underline{l}}$ not overlap the point of the characteristic $-1/W_n$ with the increased amplitude $A + \Delta a$; in other words, we require that the direction of the reckoning of the values of A along the characteristic of the nonlinearity $-1/W_n$ emerge from within the gain-phase characteristic of the reduced linear part $W_{\underline{l}}$ at the point of intersection (Fig. 2.21b).

In those cases where the coefficients $\underline{q}$ and $\underline{q}'$ are functions of Ω, we must trace the satisfaction of this criterion for small deviations of the quantity Ω in both directions from the value of the frequency of the periodic solution being investigated. This may be verified by reference to the neighboring curves $-1/W_n$ from the family of these curves shown in Fig. 2.10.

This criterion is derived only for systems of the first class

with equations of the type (2.115), but at the same time all the preceding criteria remain unchanged for systems of the type (2.116) and others.

Let us remark in conclusion that approximate criteria for the stability of a periodic solution, which are associated with the Mikhaylov and Nyquist criteria, i.e., the analytic criterion (2.125) and the graphical criteria (Fig. 2.20 and 2.21), may also be applied to nonlinear systems containing a pure lag element, as, for example, in Eq. (2.103). Here in Fig. 2.21b the factor $e^{-j\tau\omega}$ must be included in the characteristic of the reduced linear part.

The damping exponent. By analogy with linear systems, we shall assign the term damping exponent to the magnitude ξ of the real part, which appears in the pair of imaginary roots on deviation of the system from a periodic solution (Fig. 2.18). Inasmuch as we are considering here small deviations from the periodic solution, the damping exponent ξ is also considered to be a small quantity in the present section. The quantity ξ may be determined by means of the substitution $p = \xi + j\omega$ into the characteristic equation (2.117); here, due to the small order of magnitude of ξ, the result of this substitution may, using a series expansion, be written in the form

$$L(\xi + j\omega) \approx L(j\omega) + \xi L'(j\omega) = 0. \qquad (2.131)$$

Here the prime indicates the first derivative of the left-hand side of the characteristic equation L with respect to p, with the subsequent substitution $p = j\omega$, where ω is the value of the oscillation frequency, which may undergo small deviations from the value Ω in the periodic solution being investigated. The amplitude A, which enters into the coefficients of Eq. (2.117), is considered here to be an independent variable, also deviating by a small amount from its value A in the periodic solution being investigated.

Separating the real and imaginary parts in Eq. (2.131), we obtain:

$$X(a, \omega)+\xi X_1(a, \omega)=0, \quad Y(a, \omega)+\xi Y_1(a, \omega)=0, \tag{2.132}$$

where the real and imaginary parts of the following expressions figure:

$$L(j\omega)=X(a, \omega)+jY(a, \omega), \quad L'(j\omega)=X_1(a, \omega)+jY_1(a, \omega).$$

From the two equations of (2.132), we determine the two unknowns ω and ξ as functions of the amplitude A for transient processes in the neighborhood of the periodic solution. Here the magnitude of the damping exponent

$$\xi=f(q, q')=\varphi(a). \tag{2.133}$$

is of basic interest.

The functions f and φ symbolize the fact that the amplitude a enters into the expressions for L and L' not directly, but through the coefficients q and q', which are functions of A. As we see, the magnitude of the damping exponent ξ in a nonlinear system is, in contrast to the linear system, a function not only of the system parameters, but also of the magnitude of the oscillation amplitude a.

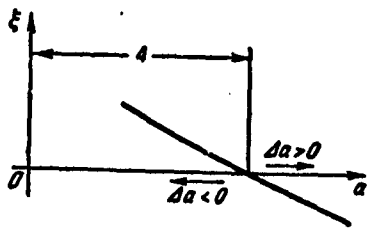


Fig. 2.22.

In order that the periodic solution may be stable, it is required, firstly, that, according to Fig. 2.22, the inequality

$$\left(\frac{d\xi}{da}\right)^*=\frac{\partial f}{\partial q}\left(\frac{\partial q}{\partial a}\right)^*+\frac{\partial f}{\partial q'}\left(\frac{\partial q'}{\partial a}\right)^*<0, \tag{2.134}$$

be satisfied; here, this inequality must also be satisfied for small deviations ω in the neighborhood of the value Ω, if the latter enters into the expressions for q and q'. Inequality (2.134) is equivalent to the inequalities (2.119) and (2.125). Secondly, we must still satisfy the second condition in the form in which it is formulated after Formula (2.119) or (2.125).

Thus, for the example of § 1.3, we have according to (1.81)

$$L(p) = T_1T_2p^3 + (T_1 + T_2)p^2 + (1 + T_1k_2k_{o.c}q)p + (k_1 + k_{o.c})k_2q,$$
$$L'(p) = 3T_1T_2p^2 + 2(T_1 + T_2)p + 1 + T_1k_2k_{o.c}q.$$

Therefore Eqs. (2.132) assume the form

$$(k_1 + k_{o.c})k_2q - (T_1 + T_2)\omega^2 + \xi(1 + T_1k_2k_{o.c}q - 3T_1T_2\omega^2) = 0,$$
$$(1 + T_1k_2k_{o.c}q)\omega - T_1T_2\omega^3 + 2\xi(T_1 + T_2)\omega = 0.$$

From the second equation we obtain

$$\omega^2 = \frac{1 + T_1k_2k_{o.c}q}{T_1T_2} + \frac{2(T_1 + T_2)}{T_1T_2}\xi. \qquad (2.135)$$

Substituting ω^2 in the first equation and making use of (1.89), we find $\xi(a)$ in the form of (1.102):

$$\xi = -\frac{T_1 + T_2}{2T_1T_2}\frac{\Delta a}{Ha + D}, \text{ i.e., } \left(\frac{d\xi}{da}\right)_{a=A} < 0, \qquad (2.136)$$

Q.E.D. [see (2.134)]. As we see, the damping exponent is in first approximation proportional here to the magnitude of the deviation Δa of the oscillation amplitude from its value in the periodic process, since the quantity A in the denominator of (2.136) varies by a small amount. We determine the curve of variation of the frequency $\omega(a)$ from Formula (2.135).

§ 2.5. COMPARISON WITH OTHER APPROXIMATE METHODS FOR SECOND-ORDER SYSTEMS

The method of harmonic linearization developed in the present book serves for investigation of near-sinusoidal solutions of nonlinear differential equations. For this same purpose we also use various versions of the small-parameter method, asymptotic methods, and others. Comparison of the results obtained is therefore of interest.

The idea of the small-parameter method consists in the following. An arbitrarily given system of differential equations

$$\frac{dx_i}{dt} = F_i(t, x_1, \ldots, x_n, \varepsilon) \quad (i = 1, 2, \ldots, n), \qquad (2.137)$$

whose solution is being sought is represented in the form

$$\frac{dx_i}{dt} = F_{i0}(t, x_1, \ldots, x_n) + \varepsilon F_{i1}(t, x_1, \ldots, x_n) + \\ + \varepsilon^2 F_{i2}(t, x_1, \ldots, x_n) + \ldots \quad (i = 1, 2, \ldots, n), \tag{2.138}$$

where ε is a small parameter; here, the solution $x_i^*(t)$ of the "generating" system

$$\frac{dx_i^0}{dt} = F_{i0}(t, x_1^0, \ldots, x_n^0) \quad (i = 1, 2, \ldots, n) \tag{2.139}$$

is known.

For a sufficiently small value of ε, the given system (2.137) will be close to the generating system (2.139). Let us assume that here the unknown solution $x_i(t)$ of the given system (2.137) will be close to the known solution $x_i^*(t)$ of the generating system (2.139). For this last hypothesis to be justified, however, we must fulfill certain conditions imposing limitations on the problem, since, notwithstanding the small difference of the system (2.137) from (2.139), the difference between the solutions $x_i(t)$ and $x_i^*(t)$ may in the general case prove larger not only quantitatively, but even qualitatively.

Therefore when one system of Eqs. (2.137) is replaced by another (2.139) that is even slightly different, we must always ascertain conditions for which a natural solution $x_i(t)$ of the system (2.137) that is close to $x_i^*(t)$ corresponds to the known solution $x_i^*(t)$ of the system (2.139). Here the solution $x_i(t)$ for $\varepsilon = 0$ must tend toward $x_i^*(t)$, since, according to (2.138), the system (2.137) tends towards the generating solution (2.139) for $\varepsilon = 0$. Section 2.2 was devoted to solution of problems of this type for the method of harmonic linearization being considered.

For comparison of the different methods, let us consider the nonlinear second-order equation

$$\frac{d^2x}{dt^2} + F\left(x, \frac{dx}{dt}\right) = 0, \tag{2.140}$$

assuming that it may be reduced to the form*

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right), \tag{2.141}$$

where ε is a small parameter and

$$\varepsilon f\left(x, \frac{dx}{dt}\right) = \omega_0^2 x - F\left(x, \frac{dx}{dt}\right). \tag{2.142}$$

In other words, to find the periodic solution we assume that the nonlinear equation (2.140) is close to the linear equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \tag{2.143}$$

which will play the role of the generating equation in the case being considered.

As an example, let us consider the equation of a relay-type automatic control system in the form (see [49], p. 95)

$$T_1 \frac{d^2x}{dt^2} + \frac{dx}{dt} + k_1 F_1(x) = 0, \tag{2.144}$$

here $F_1(x)$ is given as the loop-type relay function shown in Fig. 2.23a. This equation has the form of (2.140), where

$$F\left(x, \frac{dx}{dt}\right) = \frac{1}{T_1}\frac{dx}{dt} + \frac{k_1}{T_1} F_1(x). \tag{2.145}$$

reducing the equation of the system being considered (2.144) to the form (2.141), we obtain

$$\varepsilon f\left(x, \frac{dx}{dt}\right) = \left[\omega_0^2 x - \frac{1}{T_1}\frac{dx}{dt}\right] - \left[\frac{k_1}{T_1} F_1(x)\right], \tag{2.146}$$

where ω_0^2 is as yet unknown. The sense of the last expression consists in the fact that the loop-type nonlinear function $F_1(x)$ (Fig. 2.23a) contains both the amplification equivalent to the term $\omega_0^2 x$, and the operate lag of the hysteresis-type relay, which causes a phase lag in the oscillations equivalent to that formed by the term $-\frac{1}{T_1}(dx/dt)$. In fact, if there were no loop (Fig. 2.23b), then the oscillations of the variable x (Fig. 2.23c) would be transmitted without a phase shift (Fig. 2.23d). The presence of the loop

(Fig. 2.23a) leads to a lag in the phase oscillations (Fig. 2.23e). The linear expression

$$\left[\omega_0^2 x - \frac{1}{T_1}\frac{dx}{dt}\right],$$

also gives a similar qualitative result, as shown in Fig. 2.23g and h, although the forms of the curves are different. From comparison of Fig. 2.23e and h it is evident that the linear expression which has been written for the sinusoidal oscillations of x may be used as the first (fundamental) oscillation harmonic of the variable F_1 with appropriate choice of the oscillation frequency and amplitude of the variable x. Then the linear and nonlinear terms of the right-hand side of Expression (2.146) will differ only in the magnitude of the higher harmonics, i.e., the residual nonlinearity $\varepsilon f(x, dx/dt)$ corresponds to the higher harmonics of the oscillations.

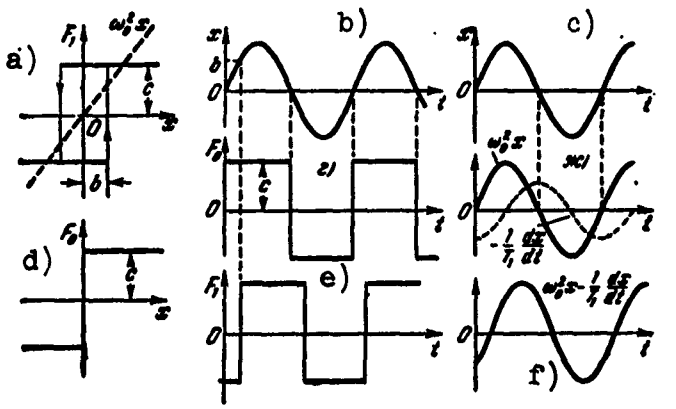


Fig. 2.23.

Here it is extremely important to note that for reduction of the second-order nonlinear equation (2.140) to the form (2.141), the presence of a linear term with a first derivative [see Expression (2.146)] in the presence of a loop-type nonlinearity (Fig. 2.23a) is absolutely obligatory, since one linear term $\omega_0^2 x$ does not conform even qualitatively to the oscillation pattern which is being studied, since it does not reflect the phase shift of the oscilla-

tions. On the other hand, if there is no loop (Fig. 2.23b), the linear term with the first derivative must be lacking for us to assume that the nonlinear equation under consideration is reducible to Form (2.141). Let us also remark that if the loop were not of the hysteresis type, but of the forcing type (lead type), i.e., if the arrows in Fig. 2.23a were reversed, then the first derivative in the given equation (2.144) must be negative in sign. Briefly, for reduction of the nonlinear equation to Form (2.141), the essential phase shift of the oscillations caused by the nonlinearity must be compensated by the presence of the appropriate linear terms. This is the physical explanation for the appearance of the periodic solution, since damping of the oscillations (+dx/dt) is guaranteed in the presence of a hunting nonlinear effect (hysteresis loop) and hunting of the oscillations (–dx/dt) in the presence of a damping nonlinear effect (lead-type loop). From the energy point of view, this corresponds to mutual compensation of the influx and expenditure of energy.

In the small-parameter method, the periodic solution of Eq. (2.141) is sought in the form of the series

$$x = x^*(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots,$$

where the initial approximation, as a solution of the generating equation (2.143), is

$$x^* = A \sin \omega_0 t.$$

Without giving an account of the small-parameter method itself, we indicate only that in it, the first approximation as applied to Eq. (2.141) is obtained in the form

$$x = A \sin \Omega t, \tag{2.147}$$

where the amplitude A is determined from the condition

$$\int_0^{2\pi} \varepsilon f(A \sin \psi,\ A\omega_0 \cos \psi) \cos \psi \, d\psi = 0, \quad \psi = \omega_0 t, \tag{2.148}$$

while the frequency Ω is

$$\Omega = \omega_0 + \varepsilon\nu, \tag{2.149}$$

here the frequency correction $\varepsilon\nu$ is determined by the formula

$$\varepsilon\nu = -\frac{1}{2\pi\omega_0 A}\int_0^{2\pi} \varepsilon f(A\sin\psi,\ A\omega_0\cos\psi)\sin\psi\, d\psi. \tag{2.150}$$

For the example considered above, according to (2.146), Eqs. (2.148) and (2.150) give

$$\left.\begin{aligned} -\frac{\pi A\omega_0}{T_1} + \frac{4cbk_1}{T_1 A} &= 0, \\ \varepsilon\nu = -\frac{\omega_0}{2} + \frac{2ck_1}{\pi T_1\omega_0 A}\sqrt{1-\frac{b^2}{A^2}}. \end{aligned}\right\} \tag{2.151}$$

From these two equations we determine the two unknowns A and $\varepsilon\nu$ if ω_0 is given, i.e., if we are given the slope of the averaging straight line in Fig. 2.23a. In this example, however, the magnitude of ω_0 is not stipulated by any initial conditions of the problem and it remains undetermined in Expression (2.146). Let us therefore select it such that it is equal to the sought frequency of the periodic solution, i.e.,

$$\omega_0 = \Omega. \tag{2.152}$$

Then, according to (2.149), we have $\varepsilon\nu = 0$. Substituting this in (2.151), we obtain equations for the amplitude and frequency of the periodic solution (2.147) in the form

$$\left.\begin{aligned} \left(\frac{A}{b}\right)^2\sqrt{1-\left(\frac{b}{A}\right)^2} &= \frac{4ck_1T_1}{\pi b}, \\ \Omega &= \frac{4ck_1}{\pi b}\left(\frac{b}{A}\right)^2, \end{aligned}\right\} \tag{2.153}$$

the first equation of which is solved graphically (Fig. 2.24).

Let us note that the small-parameter method also permits us to construct higher-order approximations for the periodic solution being sought. However, this involves major difficulties.

The small-parameter method developed by H. Poincare and A.M. Lyapunov [71], was used by L.I. Mandel'shtam and developed in detail

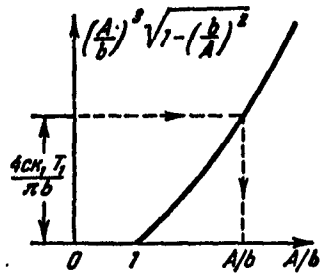


Fig. 2.24.

by A.A. Andronov as it applies to the vacuum-tube oscillator [8]. An approximate method for solution of these same problems in the first approximation, using slowly varying coefficients and based upon simple intuitive considerations, was proposed at the same time by B. van der Pol [5].

The method of slowly varying coefficients, unlike the small-parameter method, does not provide the possibility of forming higher-order approximations, but is limited to only one approximate solution, which, however, does not diminish its practical importance. This method permits us to determine not only the periodic solution itself, but also the time process of its establishment in the neighborhood of this periodic solution [sic]. We seek the solution of the nonlinear equation (2.141) approximately in the form

$$x = u(t)\cos\omega_0 t + v(t)\sin\omega_0 t$$

or in the form

$$x = a(t)\sin[\omega_0 t + \varphi(t)], \tag{2.154}$$

where in the former case u(t) and v(t) are determined from the equations

$$\frac{du}{dt} = -\frac{1}{2\pi\omega_0}\int_0^{2\pi} \varepsilon f(u\cos\psi + v\sin\psi,\ -u\omega_0\sin\psi + v\omega_0\cos\psi)\sin\psi\, d\psi,$$

$$\frac{dv}{dt} = \frac{1}{2\pi\omega_0}\int_0^{2\pi} \varepsilon f(u\cos\psi + v\sin\psi,\ -u\omega_0\sin\psi + v\omega_0\cos\psi)\cos\psi\, d\psi,$$

while in the latter case

$$\frac{da}{dt} = -\frac{1}{2\pi\omega_0}\int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\, d\psi, \tag{2.155}$$

$$\frac{d\varphi}{dt} = -\frac{1}{2\pi\omega_0 a}\int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\, d\psi. \tag{2.156}$$

These expressions are obtained by averaging the nonlinear functions over the period (the so-called "abridged equations").

For the periodic solution we will have a = const = A. Here, therefore, according to (2.155), the condition for the determination of its amplitude coincides with Condition (2.148) for the first approximation of the small-parameter method.

According to (2.154), the oscillation frequency here will be

$$\omega = \omega_0 + \frac{d\varphi}{dt}.$$

Hence according to (2.156), the frequency correction $d\varphi/dt$ obtained here for the periodic solution ($a = A$, $\omega = \Omega$) agrees with the frequency correction (2.150) for the first approximation of the small-parameter method.

N.M. Krylov and N.N. Bogolyubov [3], [7], [102] have proposed and developed an asymptotic method that permits us to form higher approximations not only for the periodic solution, but also for the time process of its establishment in the neighborhood of this periodic solution.

In the asymptotic method of Krylov and Bogolyubov, we seek the solution of the nonlinear equation (2.141) in the form*

$$x = a\sin\psi + \varepsilon u_1(a, \psi) + \varepsilon^2 u_2(a, \psi) + \ldots, \qquad (2.157)$$

here the quantities $a(t)$ and $\psi(t)$ are determined by the equations

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots, \\ \frac{d\psi}{dt} &= \omega_0 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots; \end{aligned}\right\} \qquad (2.158)$$

the functions u_1, u_2, ..., A_1, A_2, ..., B_1, B_2, ..., figuring here for any m-th approximation are determined in such a way that Expression (2.157) will satisfy the given nonlinear equation (2.141) with an accuracy to within the small order of magnitude ε^{m+1}. In particular, the first approximation has the form

$$x = a\sin\psi, \quad \frac{da}{dt} = \varepsilon A_1(a), \quad \omega = \frac{d\psi}{dt} = \omega_0 + \varepsilon B_1(a), \tag{2.159}$$

where

$$\left.\begin{aligned} \varepsilon A_1(a) &= \frac{1}{2\pi\omega_0}\int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\, d\psi, \\ \varepsilon B_1(a) &= -\frac{1}{2\pi\omega_0 a}\int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\, d\psi. \end{aligned}\right\} \tag{2.160}$$

As we see, the method of slowly varying coefficients considered above reduces to the first approximation of the more general asymptotic method of Krylov and Bogolyubov. It is also evident that the first approximation of the latter method agrees with the first approximation of the small-parameter method for the case of the periodic solution.

For the example considered above, let us write Eqs. (2.159) according to (2.146) and (2.152) allowing for (2.160) in the form

$$\frac{da}{dt} = -\frac{a}{2T_1} + \frac{2cbk_1}{\pi T_1 \Omega a}, \tag{2.161}$$

$$\omega = \frac{\Omega}{2} + \frac{2ck_1}{\pi T_1 \Omega a}\sqrt{1 - \frac{b^2}{a^2}}. \tag{2.162}$$

Applying Expression (2.153) for Ω we obtain from (2.161):

$$\frac{da}{dt} = \frac{A^2 - a^2}{2T_1 a},$$

where A is the amplitude of the periodic solution (of the self-oscillations) as determined from Fig. 2.24.

As a result of the integration of this equation, we find the time variation of the amplitude in the process of establishment of the self-oscillations:

$$a^2 = A^2 + (a_0^2 - A^2)e^{-\frac{t}{T_1}}, \tag{2.163}$$

where a_0 is the initial value of $\underline{a}$ for t = 0 (Fig. 2.25a). Hence it is evident that the picture of the oscillatory processes represented in Fig. 2.17a, which corresponds to stable self-oscillations, is the case in the given system.

Equation (2.162), however, may be represented in the form

$$(2\omega - \Omega)\,\Omega = \frac{4ck_1}{\pi T_1 a}\sqrt{1 - \frac{b^2}{a^2}}.$$

But it follows from (2.159) and (2.152) that $\Omega = \omega = \varepsilon B_1$. Therefore the left member of the equation in question will be $\omega^2 - \varepsilon^2 B_1^2$, and, neglecting ε^2, we may write the equation itself in the form

$$\omega^2 = \frac{4ck_1}{\pi T_1 a}\sqrt{1 - \frac{b^2}{a^2}}, \qquad (2.164)$$

by which the oscillation frequency is also determined as a function of amplitude (Fig. 2.25b) in establishment of self-oscillations.

In addition to developing the asymptotic method, Krylov and Bogolyubov [7, 102] showed that the energy balance (developed later by K.F. Teodorchik [36]) led to the same results in first approximation, i.e., reduction of the nonlinear to a linear problem with equivalent energy relationships over the period, as did the harmonic balance (used by Gol'dfarb [19]) and equivalent linearization methods (the latter forms the basis of the harmonic-linearization method developed here). Bogolyubov [32] obtained similar results from a theory of perturbations approaching that of Lagrange, reinforcing them by calculation of higher harmonics as well as the first approximation. Higher approximations of the averaging method are also developed [102]. We must also mention the extremely important studies of K. Magnus [48] in this area.

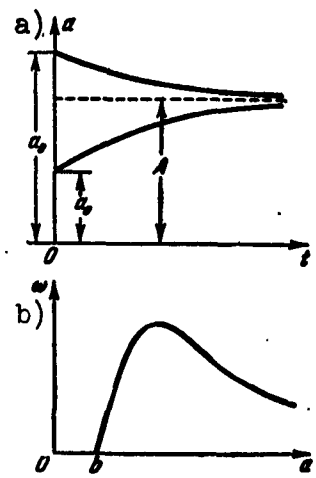


Fig. 2.25.

It is interesting that for an equation like (2.141), extremely diverse approaches give identical first-approximation results. We might cite many more fine variations on the computational method (belonging in principle, however, to one of the above methods) due to

various authors and giving the same results. There are special methods that produce, in certain particular problems, results which are closer to an exact solution (see, for example, [44], [123]). Further we must take special note of the works of Yu.A. Mitropol'skiy [102] and V.O. Kononyenko [93] with reference to non-linear systems with variable coefficients.

Let us dwell on the connection between the methods considered above and the methods of harmonic linearization developed here. Let us first note that N.M. Krylov and N.N. Bogolyubov also proposed the following transformations, which permit us to make use of the first approximation of their asymptotic method directly for Eq. (2.140).

Let us find [102] the square of the frequency ω in the first approximation of the asymptotic method. According to (2.159)

$$\omega^2 = \omega_0^2 + 2\omega_0 \varepsilon B_1(a) + \varepsilon^2 B_1^2(a).$$

Restricting ourselves to the first approximation, we discard the term in ε^2 and substitute the expression for $\varepsilon B_1(a)$ from (2.160). Then

$$\omega^2 = \omega_0^2 - \frac{1}{\pi a}\int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\, d\psi. \tag{2.165}$$

But it follows from (2.142) that

$$F\left(x, \frac{dx}{dt}\right) = \omega_0^2 x - \varepsilon f\left(x, \frac{dx}{dt}\right). \tag{2.166}$$

Making use of this formula, we may easily verify that for $x = a\sin\psi$ we have

$$\int_0^{2\pi} F(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\, d\psi = \omega_0^2 \pi a - \int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\, d\psi.$$

Therefore Formula (2.165) for the square of the frequency may be rewritten in the form

$$\omega^2 = \frac{1}{\pi a}\int_0^{2\pi} F(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\,d\psi. \tag{2.167}$$

Noting further that for x = a sin ψ, we obtain from (2.166)

$$\int_0^{2\pi} F(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\,d\psi = -\int_0^{2\pi} \varepsilon f(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\,d\psi,$$

we may write according to (2.159) and (2.160)

$$\frac{da}{dt} = -\frac{1}{2\pi\omega_0}\int_0^{2\pi} F(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\,d\psi. \tag{2.168}$$

Hence for the periodic solution (a = const = A, ω = const = Ω) we have:

$$\int_0^{2\pi} F(A\sin\psi,\ A\omega_0\cos\psi)\cos\psi\,d\psi = 0, \tag{2.169}$$

while from (2.167)

$$\Omega^2 = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi,\ A\omega_0\cos\psi)\sin\psi\,d\psi. \tag{2.170}$$

On the other hand, if the nonlinear equations (2.140) were solved directly by the method of harmonic linearization, then to find a periodic solution to Eq. (2.140) we would have, according to (2.75) and (2.76), to represent it in the form

$$\frac{d^2x}{dt^2} + \frac{q'(A,\ \Omega)}{\Omega}\frac{dx}{dt} + q(A,\ \Omega)\,x = 0 \tag{2.171}$$

and write the characteristic equation

$$p^2 + \frac{q'(A,\ \Omega)}{\Omega}p + q(A,\ \Omega) = 0. \tag{2.172}$$

The requirement of the presence of a pair of purely imaginary roots p = $\pm$ jΩ in this quadratic equation reduces to the following:

$$q'(A,\ \Omega) = 0, \quad \Omega^2 = q(A,\ \Omega). \tag{2.173}$$

In particular, we would obtain in the example of (2.144)

$$q' = \frac{\Omega}{T_1} - \frac{4cbk_1}{\pi T_1 A^2}, \quad q = \frac{4ck_1}{\pi T_1 A}\sqrt{1 - \frac{b^2}{A^2}}.$$

As we see, in the expressions for $\underline{q}$ and $\underline{q}'$ we have directly expressed not only the nonlinearity F_1 itself, but also the linear term occurring in the initial equation, as, for example, in (2.144), since according to (2.145) it is included in the formation of the function F(x, dx/dt) of Eq. (2.140).

It may be easily seen that on application of the notation of (2.76), these equations of the method of harmonic linearization (2.173) coincide exactly with the equations of the first approximation of the asymptotic method (2.169) and (2.170), if Ω is replaced in ω_0 in them, and also with the equations of the first approximation of the small-parameter method in the particular example of (2.151) and in the general form of (2.148) and (2.150), if we substitute (2.142) in them, assuming that $\omega_0 = \Omega$ and, consequently, $\varepsilon\nu = 0$.

In general in problems of automatic-control theory, where there is no explicitly separated quantity ω_0 beforehand in the specified equations, it is most expedient in using the asymptotic method and the small-parameter method to choose the quantity ω_0 such that it is equal to the sought frequency of the periodic solution ($\omega_0 = \Omega$, $\varepsilon\nu = 0$), as was illustrated above in the example. In short, reducing Eq. (2.140) to the form (2.141), we must write the latter for problems of automatic-control theory in the form

$$\frac{d^2x}{dt^2} + \Omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right),$$

where, like the amplitude A, the quantity Ω is determined in the process of solving this equation by the formulas of the first approximation.

In problems of automatic-control theory, the method of harmonic linearization leads us by the simplest and most direct route to

results identical to those of the small-parameter or asymptotic method, with proper choice of the generating frequency $\omega_0 = \Omega$ in either. This is of great value for many practical engineering problems where the specified nonlinear equation does not contain the generating frequency ω_0 in explicitly separated form.

It is interesting to note here that in expounding the small-parameter (Poincare) method, Bulgakov also notes that if the generating (linearizing) function is not explicitly separated in the given nonlinear equation, which then has the form (2.140), it may be found from equations like (2.169) and (2.170) along with the amplitude (see [45], p. 321); this actually confirms full agreement of the Bulgakov small-parameter method with the harmonic-linearization method as regards results.

According to § 2.4, the harmonically linearized equation (2.171) may also be applied to investigation of nonsteady processes of establishment of self-oscillations for small deviations from the periodic solution. Then it assumes the form

$$\frac{d^2x}{dt^2} + \frac{q'(a, \omega)}{\omega}\frac{dx}{dt} + q(a, \omega)x = 0 \cdot \qquad (2.174)$$

or, in many problems,

$$\frac{d^2x}{dt^2} + \frac{q'(a)}{\omega}\frac{dx}{dt} + q(a)x = 0, \qquad (2.175)$$

which also agrees with the equation of N.M. Krylov and N.N. Bogolyubov (see [102], p. 95). Here we obtain equations similar to linear equations. But while the ordinary linear system has a constant damping exponent ξ and frequency ω, they will be variables here:

$$\xi(a) = -\frac{q'(a)}{2\omega}, \quad \omega^2(a) = q(a), \qquad (2.176)$$

where ξ and q'(a) are small quantities (for the periodic solution itself, however, $\xi = 0$ and $q' = 0$). In particular, we have in the example of (2.144)

$$q' = \frac{\omega}{T_1} - \frac{4cbk_1}{\pi T_1 a^2}, \quad q = \frac{4ck_1}{\pi T_1 a} \sqrt{1 - \frac{b^2}{a^2}}.$$

As we see, we do not, generally speaking, obtain small order of magnitude for q'(a) here at the expense of the nonlinearity $F_1(x)$ itself. It is in itself essential, and the coefficient q' for it is finite. But due to the presence of the linear term giving the summand ω/T_1, the general coefficient q' for the entire function F(x, px) determined by Formula (2.145) may be small on the whole for suitable values of ω and a.

Let us note that while in a linear system we write the solution for the oscillatory transient process in the form $x = a_0 e^{\xi t} \sin \omega t$ it is impossible to write it in such a form here as a consequence of the variability of ξ and ω. Only the differential notation

$$\frac{da}{dt} = a\xi(a), \quad \frac{d\psi}{dt} = \omega(a), \quad x = a \sin \psi, \tag{2.177}$$

which is identical in first approximation to the linear solution indicated, will be valid for the process of establishment of the periodic solution; this also agrees with (2.159). Here ξ and ω are determined by Formulas (2.176) as functions of the amplitude a, which varies over time.

The stability of the periodic solution is determined here by the sign of the quantity $\xi(a)$ (in the work of N.M. Krylov and N.N. Bogolyubov by $\Phi(a)/a$; see [102], p. 79). If $\xi(a)$ changes its sign from plus to minus with increasing a, the periodic solution is stable. This agrees with the approach used in § 2.4.

Here we have dealt with small deviations of the oscillations from the periodic process. Below (Chapter 7) we shall carry out an extension of the asymptotic method of N.M. Krylov and N.N. Bogolyubov, and the harmonic-linearization method to fast-damping oscillatory processes (for large, but slowly varying values of ξ).

We may easily verify that for the example of a nonlinear control system considered above, the use of the equations of the harmonic linearization method (2.173) and (2.176) yields exactly the same results as the first approximation of the small-parameter method (2.153) and the asymptotic method (2.161) and (2.164).

§ 2.6. COMPARISON WITH OTHER APPROXIMATE METHODS FOR HIGH-ORDER SYSTEMS

Let us now turn to nonlinear systems whose dynamics are described by high-order equations. In the works of N.M. Krylov and N.N. Bogolyubov [7], [28], [102] there is an extension of the asymptotic methods to a high-order equation with a small parameter. The book by I.G. Malkin [71] contains a description of the small-parameter method for high-order systems. However, there are also direct applications of the small-parameter method to nonlinear problems of automatic control theory in the works of B.V. Bulgakov [45], A.I. Lur'ye [34], A.M. Letov [18] and others. Therefore let us dwell here on a comparison of the method of harmonic linearization developed in the present book with precisely these applications of the small-parameter method. The statement of the problem in general form was indicated at the beginning of § 2.5.

B.V. Bulgakov (see [45], Chapter 12) first considers a system of equations with, as usual, a small parameter, but in a form suitable for the analysis of automatic systems:

$$D_{i1}(p)\,x_1 + D_{i2}(p)\,x_2 + \ldots + D_{im}(p)\,x_m = \varepsilon f_i(x_1, \ldots, x_m, \varepsilon) \quad (i = 1, 2, \ldots, m), \tag{2.178}$$

where D_{ir} are operator polynomials. This notation conforms better than (2.137) to the notation for the equation of an automatic system for each link. ($\underline{m}$ is the number of links into which the system is broken down); in practical problems many of the $D_{ir}(p)$ will be zeros,

since far from all of the variables $x_1, \ldots, x_m$ enter into each equation of the link. Here ε denotes a small parameter, so that the right-hand sides of Eqs. (2.178) are small nonlinear terms.

In seeking periodic solutions to the characteristic equation of the linear system (which Bulgakov calls a simplified system) obtained from (2.178) for $\varepsilon = 0$, i.e., solutions to the equation

$$L(p) = \begin{vmatrix} D_{11}(p), D_{12}(p), \ldots, D_{1m}(p) \\ \cdots\cdots\cdots\cdots \\ D_{m1}(p), D_{m2}(p), \ldots, D_{mm}(p) \end{vmatrix} = 0, \qquad (2.179)$$

we set forth the requirement that it have one pair of purely imaginary roots $p = \pm j\omega_0$. In other words, the linear system obtained from (2.178) for $\varepsilon = 0$ is a generating system having a periodic solution (in the present case, a sinusoidal solution). The latter serves as a "zeroth" approximation to the solution of the nonlinear system.

From a first approximation to the periodic solution of the nonlinear system (2.178), in terms of the small nonlinear terms, we determine the frequency correction $\varepsilon\nu$ (see (2.149)) on the right-hand sides of (2.178). According to the first approximation of the small parameter method, the formula derived by B.V. Bulgakov for this correction as it applies to the notation of (2.178), has the form ([45], p. 755)

$$-2A\omega_0 L_1(j\omega_0) E_{kr}(j\omega_0)\varepsilon\nu = \frac{1}{\pi}\sum_{i=1}^{m} E_{ir}(j\omega_0)\int_0^{2\pi} \varepsilon f_i(x_1^*, \ldots, x_m^*, 0) e^{-j\psi} d\psi, \qquad (2.180)$$

where

$$L_1(p) = \frac{L(p)}{p^2 + \omega_0^2},$$

$x_1^*, \ldots, x_m^*$ is a periodic solution of the generating linear system and $E_{kr}(j\omega_0)$ is a nonzero minor of any (k, r) element of the determinant (2.179) for $p = j\omega_0$. Here A is the amplitude determined from this same complex equation (it is equivalent to two real equations

and determines two quantities: $\varepsilon\nu$ and A).

Further, B.V. Bulgakov turns to nonlinear equations in which, in contrast to (2.178), small nonlinear terms are not separated in explicit form; this, by the way, is the case in the majority of practical problems. Here he chooses the generating system for cases of one and two nonlinearities, where the nonlinearities are single-valued and oddly-symmetrical, i.e., $\int_0^{2\pi} F(A\sin\psi)\,d\psi = 0$. The choice of the system indicated was made so as to determine a generating frequency ω_0 and corresponding coefficients of the linear generating equation which, even for $\varepsilon = 0$, will give the best approximation to a nonlinear system in the periodic mode.

In the case of one unique nonlinearity $F(x_1)$ (in Bulgakov's work, $g(y_1)$; see [45], p. 785), the system equations have the form

$$\left.\begin{aligned} &D_{11}(p)x_1+\ldots+D_{1l}(p)x_l+\\ &\qquad+\ldots+D_{1m}(p)x_m=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{k1}(p)x_1+\ldots+D_{kl}(p)x_l+F(x_l)+\\ &\qquad+\ldots+D_{km}(p)x_m=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{m1}(p)x_1+\ldots+D_{ml}(p)x_l+\ldots+\\ &\qquad+D_{mm}(p)x_m=0, \end{aligned}\right\} \qquad (2.181)$$

where the small parameter does not figure explicitly. In order to reduce System (2.181) to the form (2.178), we write it as follows:

$$\left.\begin{aligned} &D_{11}(p)x_1+\ldots+D_{1l}(p)x_l+\ldots+D_{1m}(p)x_m=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{k1}(p)x_1+\ldots+D_{kl}(p)x_l+hx_l+\ldots+\\ &\qquad+D_{km}(p)x_m=\varepsilon f(x_l),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{m1}(p)x_1+\ldots+D_{ml}(p)x_l+\ldots+D_{mm}(p)x_m=0, \end{aligned}\right\} \qquad (2.182)$$

where

$$\varepsilon f(x_l)=hx_l-F(x_l).$$

Bulgakov determines the quantity h from the condition that the characteristic equation of the linear generating system, i.e., System (2.182) for $\varepsilon = 0$, have one pair of purely imaginary roots $p = \pm j\omega_0$,

which simultaneously determines the generating frequency ω_0 itself (here Bulgakov uses the Hurwitz criterion $H_{n-1} = 0$, which is equivalent to our second method, § 2.3). After this we apply Eq. (2.180), from which we determine the amplitude A and the frequency correction $\varepsilon\nu$; here the sought frequency of the periodic solution will be $\Omega = \omega_0 + \varepsilon\nu$.

Without giving an account of the solution procedure of [45], we present only its final results. To determine the amplitude we obtain the equation

$$hA = J(A), \tag{2.183}$$

where, if we seek the solution in the form $x_1 = A \sin \psi$,

$$J(A) = \frac{1}{\pi}\int_0^{2\pi} F(A \sin \psi) \sin \psi \, d\psi \tag{2.184}$$

(in the work of Bulgakov, $x_1 = A \cos \psi$; this is not essential), while the frequency correction

$$\varepsilon\nu = 0, \text{ i.e. } \Omega = \omega_0 \tag{2.185}$$

which, for a single-valued nonlinearity, is connected with equality to zero of the integral

$$\int_0^{2\pi} F(A \sin \psi) \cos \psi \, d\psi = 0. \tag{2.186}$$

From the formulas which have been written we may now arrive at a quite definite conclusion. Comparison of (2.183) and (2.184) with the first of Formulas (2.20) suggests the exact equality

$$h = \frac{J(A)}{A} = q, \tag{2.187}$$

while comparison of (2.186) with the second formula of (2.20) corresponds exactly to the already known fact that the coefficient $q' = 0$ for single-valued nonlinearities. The difference consists in the fact that in the method of harmonic linearization, we replace $F(x)$ by $q(A)x$ directly in the given nonlinear-system equation. Not-

withstanding the simpler approach to solution of the problem in the method of harmonic linearization, we have exact agreement of its results with the first approximation of the small-parameter method for single-valued odd nonlinearities. This will also be shown below for the more general case.

In Bulgakov's work, the practical method for finding the periodic solution reduces to the following. We plot the curve of J(A) (Fig. 2.26) and, according to Equality (2.183), the rays hA are plotted on this same graph. The intersection points of each of these rays with the curve J(A) give us the values of the sought amplitude A of the periodic solution for a definite combination of system parameters (and, consequently, for a fixed frequency $\omega_0 = \Omega$), since, as we have indicated, the quantity h is determined from the characteristic equation of the linear generating system (2.182) for $\varepsilon = 0$ together with the frequency $\omega_0 = \Omega$.

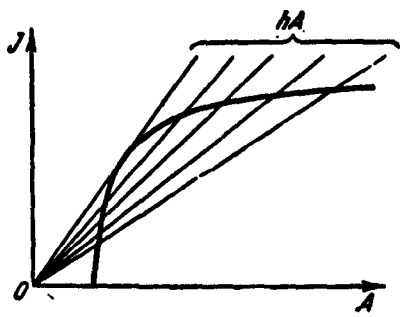


Fig. 2.26.

This practical method due to Bulgakov (Fig. 2.26) is essentially similar to one of our particular methods (Fig. 2.12c), which was described above in § 2.3 for the corresponding simplest case of a nonlinear system of the first class. The agreement of the two results is evident from the fact that in Fig. 2.12c, according to (2.187), $q = J(A)/A$ and the quantity $z(\Omega) = h$, although the methods for finding them are different.

Thus, we may say that for the case of a single-valued nonlinearity, when $q' = 0$, that the harmonically linearized equation (2.78) is thereby in essence itself a linear equation which should be taken as the equation of the generating system in seeking a periodic solu-

tion by the small parameter method. Here it is important that the amplitude and frequency of the periodic solution which are found from it correspond exactly to the first approximation of the small-parameter method, since in the present case (i.e., where $q' = 0$), the frequency correction is, as already noted, equal to zero. It is for this reason that it is not necessary in the method of harmonic analysis, to perform any of the auxiliary operations that play a part in finding the first approximation in the small-parameter method.

Let us note that as long ago as 1951 [34], A.I. Lur'ye obtained full agreement in the results for the small-parameter method and the harmonic-balance (and hence harmonic linearization) method in the case of a single-valued nonlinearity $F(x)$. The calculations cited above only confirm this. Subsequently, however, extending the ideas of B.V. Bulgakov, we shall show that even in the case of a loop-type nonlinearity $F(x)$, and in the general case of a nonlinearity $F(x, px)$, we may, by appropriate synthesis of the generating system in the small-parameter method, also obtain complete agreement for the results of the two methods and, in addition, attain the best results of application of the small-parameter method itself.

Before entering upon this, let us first recall the case considered by B.V. Bulgakov, where we have present in the system two single-valued nonlinearities with respect to different variables (an example of a nonlinear system of the second class):

$$\left.\begin{aligned}
&D_{11}(p)x_1+\ldots+D_{1l}(p)x_l+\ldots+D_{1s}(p)x_s+\ldots+D_{1m}(p)x_m=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&D_{k1}(p)x_1+\ldots+D_{kl}(p)x_l+F_1(x_l)+\ldots+D_{ks}(p)x_s+\ldots+D_{km}(p)x_m=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&D_{j1}(p)x_1+\ldots+D_{jl}(p)x_l+\ldots+D_{js}(p)x_s+F_2(x_s)+\ldots+D_{jm}(p)x_m=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&D_{m1}(p)x_1+\ldots+D_{ml}(p)x_l+\ldots+D_{ms}(p)x_s+\ldots+D_{mm}(p)x_m=0.
\end{aligned}\right\} \qquad (2.188)$$

In this case, B.V. Bulgakov [45] accordingly introduces two coefficients h_1 and h_2, with the result that on reduction of System (2.188)

to the form (2.178) the functions

$$\varepsilon f_1 = h_1 x_1 - F_1(x_1) \text{ and } \varepsilon f_2 = h_2 x_2 - F_2(x_2) \tag{2.189}$$

appear on the right-hand sides of the equations. In the characteristic equation of the generating linear system obtained from (2.188) by the substitution (2.189) for $\varepsilon = 0$, we have two unknown coefficients h_1 and h_2. Therefore, imposing the requirement of the presence of a pair of purely imaginary roots $p = \pm j\omega_0$ in the characteristic equation of the generating system, we obtain from it

$$h_1 = h_1(\omega_0) \text{ and } h_2 = h_2(\omega_0), \tag{2.190}$$

where the magnitude of the generating frequency ω_0 is as yet unknown. It is then determined together with the amplitude by application of Eq. (2.180) and the supplementary relationship between the amplitudes of the two variables of the type (2.40). As a result, Bulgakov also obtains a frequency correction equal to zero in this case (see [45], page 789), i.e., $\Omega = \omega_0$. Thus, we easily establish complete agreement of the first approximation of the small-parameter method with the results of the harmonic-linearization method, even for a nonlinear system of the second class, notwithstanding the incomparably greater simplicity of the approach to the solution of the problem in the latter method.

Let us now turn to the case of a single nonlinearity of the more general form $F(x, px)$, which was not considered by B.V. Bulgakov and, in particular, the loop-type nonlinearity $F(x)$ that played a part in the preceding paragraphs. We shall show that even in this more general case the harmonically linearized equation (2.78) for $q' \neq 0$) is also the linear equation that should be taken as the generating equation best approximating the given nonlinear system in determination of the periodic solution by the small-parameter method, since in this case the generating frequency ω_0 is exactly

equal to the frequency Ω of the first approximation i.e., the frequency correction will, as before, be equal to zero ($\varepsilon\nu = 0$) [64].

Let the nonlinear system be given by the equations:

$$\left.\begin{aligned} &D_{11}(p)x_1+\ldots+D_{1l}(p)x_l+\ldots+D_{1m}(p)x_m=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{k1}(p)x_1+\ldots+D_{kl}(p)x_l+F(x_l, px_l)+\ldots+D_{km}(p)x_m=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{m1}(p)x_1+\ldots+D_{ml}(p)x_l+\ldots+D_{mm}(p)x_m=0. \end{aligned}\right\} \qquad (2.191)$$

Following the same general procedure for the small-parameter method as B.V. Bulgakov, we write in place of (2.191) the following system:

$$\left.\begin{aligned} &D_{11}(p)x_1+\ldots+D_{1l}(p)x_l+\ldots+D_{1m}(p)x_m=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{k1}(p)x_1+\ldots+D_{kl}(p)x_l+h_1x_l+h_2px_l+\ldots+D_{km}(p)x_m=\varepsilon f_k(x_l, px_l),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &D_{m1}(p)x_1+\ldots+D_{ml}(p)x_l+\ldots+D_{mm}(p)x_m=0, \end{aligned}\right\} \qquad (2.192)$$

where

$$\varepsilon f_k(x_l, px_l)=h_1x_l+h_2px_l-F(x_l, px_l). \qquad (2.193)$$

For $\varepsilon = 0$, we obtain a certain linear generating system. The quantities h_1 and h_2 appear linearly in the characteristic equation of this system $L(p) = 0$ of the type (2.179). Therefore, requiring the presence of a pair of purely imaginary roots, we may obtain by means of the Hurwitz determinant or from the equation $L(j\omega_0) = 0$ after separating the real and imaginary parts

$$h_1=h_1(\omega_0) \text{ and } h_2=h_2(\omega_0) \qquad (2.194)$$

in much the same way as in the preceding case (2.190), which was considered by B.V. Bulgakov.

Let us make further use of the equation of the first approximation for the small-parameter method (2.180). In the relationships (2.194), as in Bulgakov's work for the case of two nonlinearities, the generating frequency ω_0 is still not determined. Making use of this, let us define it to best advantage, i.e., in such a way that the frequency correction ($\varepsilon\nu = 0$), according to (2.180), we must satisfy the condition

$$\int_0^{2\pi} \varepsilon f_k(x_1^*, px_1^*) e^{-jv} d\psi = 0.$$

Substituting (2.193) for $x_1^* = A \sin \psi$ ($\psi = \omega_0 t$) here and making use of the familiar formula

$$e^{-j\psi} = \cos \psi - j \sin \psi,$$

we obtain

$$\int_0^{2\pi} (h_1 A \sin \psi + h_2 A \omega_0 \cos \psi) \cos \psi d\psi - j \int_0^{2\pi} (h_1 A \sin \psi + h_2 A \omega_0 \cos \psi) \sin \psi d\psi - \int_0^{2\pi} F(A \sin \psi, A\omega_0 \cos \psi) \cos \psi d\psi + j \int_0^{2\pi} F(A \sin \psi, A\omega_0 \cos \psi) \sin \psi d\psi = 0.$$

Setting the real and imaginary parts equal to zero separately, we find

$$\left.\begin{aligned} \pi A h_1 &= \int_0^{2\pi} F(A \sin \psi, A\omega_0 \cos \psi) \sin \psi d\psi, \\ \pi A \omega_0 h_2 &= \int_0^{2\pi} F(A \sin \psi, A\omega_0 \cos \psi) \cos \psi d\psi. \end{aligned}\right\} \qquad (2.195)$$

The four equations obtained in (2.194) and (2.195) permit us to determine all four unknowns h_1, h_2, A and ω_0. Here, inasmuch as we guarantee observance of the condition $\varepsilon\nu = 0$, then the frequency of the first approximation $\Omega = \omega_0 + \varepsilon\nu$ in the periodic solution that has been found is

$$\Omega = \omega_0. \qquad (2.196)$$

Now let us compare the result obtained by the application of the small-parameter method with the harmonic-linearization method. Comparing (2.195) with Formulas (2.76) and allowing for (2.196), we obtain

$$h_1 = q(A, \Omega), \quad h_2 = \frac{q'(A, \Omega)}{\Omega}. \qquad (2.197)$$

Hence it is evident that the linear generating system (2.192) for $\varepsilon = 0$ in the small-parameter method agrees exactly for the manner of its application described here with the harmonically linearized

equation obtained from (2.191) by the direct substitution (2.75). Hence, notwithstanding the difference in their general approach to the solution of the problem and the difference in the computational processes, the results of application of both methods will be identical.

The difference consists in the fact that by the small-parameter method, we segregate the nonlinear terms with a small parameter ε in the given nonlinear system (2.191) and require the presence of a periodic (sinusoidal) solution in the linear generating system (2.192) for $\varepsilon = 0$. Then from the formula for the first approximation of the small-parameter method (2.180) we determine the amplitude and the correction to the generating frequency of the periodic solution, allowing for the terms containing a small parameter. In the harmonic-linearization method, however, the nonlinearity in the given system (2.191) is replaced at the outset by special harmonically linearized terms without separating the small parameter at all. Then the resulting harmonically linearized system is solved as a linear system with the purpose of determining the sinusoidal periodic solution. The operations in the second method are significantly simpler for engineering calculations.

As was shown above, the choice of the generating system in the small-parameter method from the condition of equality of the frequency correction to zero leads, according to the formulas for the first approximation of this method, allowing for terms containing a small parameter, to the same computational formulas as in the method of harmonic linearization.

Thus we show that in the case of a nonlinearity of the general form $F(x, px)$, the harmonically linearized equation (2.78) is that linear equation which in the small-parameter method corresponds to

the generating system best approximating the given nonlinear system and reducing the frequency correction to zero.

The computations performed may be used as a justification for the method of harmonic linearization on the premises of the small-parameter method. Another justification was given in § 2.2.

All of this also agrees with the fact that as was shown by N.M. Krylov and N.N. Bogolyubov, the equivalent linearization corresponds to the first approximation of the asymptotic method, taking into account the first power of the small parameter (this was described in § 2.5 for a second-order system).

We discussed the nonlinearities of the general form F(x, px) earlier. As regards the loop-type nonlinear characteristics often encountered in automatic systems, these will also be designated by F(x), like the single-valued characteristics. However, inasmuch as they have a different form for increasing x than for decreasing x, the integral (2.186) for them is not equal to zero. As a consequence of this, we obtain on introduction into the generating system of the term hx that linearly approximates the loop-type nonlinearity F(x), on the basis of Formula (2.180), a frequenc· correction that is small if the loop is narrow and by no means small if the loop is wide. Therefore, in the presence of a loop-type nonlinearity F(x), in the small-parameter method, we must also proceed from Expression (2.193) and make use of Formulas (2.195), assuming as in the general case F(x, px) that the loop-type nonlinearity F(x) is, strictly speaking, a particular form of the general case of a nonlinearity F(x, px) with an essential dependence upon the sign of the derivative px. Thus, the loop-type nonlinearity F(x) may be denoted by the the symbols F(x, sign px). In this connection, let us recall that in the harmonic-linearization method for loop-type nonlinearities

we always introduce the second coefficient $q' \neq 0$ (see § 2.1).

The requirement of the small-parameter method concerning the presence of a generating frequency in a linear system for $\varepsilon = 0$ is equivalent to the requirement of the presence of a pair of purely imaginary roots in the characteristic equation of a harmonically linearized system (or the requirement of intersection of frequency characteristics in the method of harmonic balance). From the standpoint of control theory, any of these requirements indicates with equal force the desire to find constant-amplitude sinusoidal oscillations in a closed-loop linear system approximately replacing the closed-loop nonlinear system under consideration. Here it is important to emphasize that the generating system in the small-parameter method is precisely a closed-loop linear system. In addition, in investigating nonlinear systems in the theory of automatic control we introduce the concept "linear part", by which we imply the open linear system (Fig. 1.12b). We exclude the nonlinearity from it in such fashion that the number of equations becomes one less than the number of variables. For example, in System (2.71)

$$Q(p)x + R(p)F(x, px) = 0$$

the closed linear system will be a generating system

$$Q(p)x + R(p)(h_1 x + h_2 px) = 0,$$
$$h_1 = q(A, \Omega), \quad h_2 = \frac{q'(A, \Omega)}{\Omega},$$

while the linear part of the nonlinear system will be the open linear system

$$Q(p)x = -R(p)y, \tag{2.198}$$

obtained by excluding a nonlinearity in the form

$$y = F(x, px). \tag{2.199}$$

for the given equation.

For $\varepsilon = 0$, the closed-loop linear system (2.192) will be a

generating system for the nonlinear system (2.191) in exactly the same fashion, while in the sense of control theory, the linear part (in the general case, the reduced linear part, see § 1.2) will be the open linear system

$$\left.\begin{array}{l} D_{11}(p)x_1+\ldots+D_{1l}(p)x_l+\ldots+D_{1m}(p)x_m=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D_{k1}(p)x_1+\ldots+D_{kl}(p)x_l+\ldots+D_{km}(p)x_m=-y, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D_{m1}(p)x_1+\ldots+D_{ml}(p)x_l+\ldots+D_{mm}(p)x_m=0, \end{array}\right\} \qquad (2.200)$$

which is obtained by excluding a nonlinearity in the form

$$y=F(x_l,\ px_l). \qquad (2.201)$$

from (2.191).

It is evident that such a linear part will have properties altogether different from those of the generating system (in particular, in contrast to the generating system, here we require the absence of purely imaginary roots of the characteristic equation). The properties of the linear part (2.198) are determined as properties of an open system in control theory by study of the variation of the "output" quantity $\underline{x}$ for a given variation of the "input" variable $\underline{y}$, i.e., that quantity by which a nonlinearity is introduced into the system. As we know, the frequency characteristic of the linear part

$$W(j\omega)=\frac{R(j\omega)}{Q(j\omega)}$$

for sinusoidal variation of the input quantity $\underline{y}$ is an example of a characteristic determining these properties. For the nonlinearity (2.199), on the other hand, $\underline{x}$ is the input quantity while $\underline{y}$ is the output quantity.

The expression for W(p), which is determined symbolically from Eq. (2.198) as

$$W(p)=\frac{x}{-y}=\frac{R(p)}{Q(p)},$$

is called the transfer function of the linear part.

Exactly as for the linear part expressed by Eqs. (2.200) we may write the transfer function of the linear part in the form

$$W(p)=\frac{x_l}{y}=\frac{E_{kl}(p)}{Q(p)} \tag{2.202}$$

($\underline{y}$ and $x_{\underline{l}}$ play the roles of the input and output of the linear part respectively) and its frequency characteristic

$$W(j\omega)=\frac{E_{kl}(j\omega)}{Q(j\omega)}, \tag{2.203}$$

where Q(p) is the determinant of the system (2.200) and $E_{k\underline{l}}(p)$ is the minor of the (k, $\underline{l}$)-element of this determinant; here, the determinant Q(p) of the linear part unlike the determinant L(p) of the generating system or of the harmonically linearized system, no longer has purely imaginary roots.

Both in the formulas of the small-parameter method written earlier in this section, and in the formulas of the harmonic-linearization method, it is essential that we actually make use of the closeness of the solution to the sinusoidal only for those variables that appear in the nonlinear function. The forms of the solutions for the remaining variables do not figure in the computations. This suggests that a solution for the former variables may be reliable even in cases where the solution for the latter variables differs appreciably from the sinusoidal. In the method of harmonic linearization, this is substantiated by the filter condition (§ 2.2). Hence, in the small-parameter method the filter condition must also be considered as an auxiliary condition that renders the use of the small-parameter method valid even in cases where the sought periodic solution is close to sinusoidal not for all the system's variables, but only for those within the nonlinear function.

Thus, for example, let us assume that in the nonlinear system

(2.191) or, what is the same thing, (2.200)-(2.201) the periodic solution for the variable x_1 is close to sinusoidal, while the nonlinearity $F(x_1, px_1)$ is such that the variable y, which is here determinable from (2.201), is far from sinusoidal. Then we assume that the linear part of System (2.200) possesses the filter property, i.e., that its frequency characteristic (2.203) satisfied the condition

$$\left|\frac{E_{kl}(jn\Omega)}{Q(jn\Omega)}\right| < \left|\frac{E_{kl}(j\Omega)}{Q(j\Omega)}\right| \text{ and } \left|\frac{E_{kl}(jn\Omega)}{Q(jn\Omega)}\right|_{n\to\infty} \to 0, \qquad (2.204)$$

where n = 2, 3, ..., or n = 3, 5, ..., depending on whether or not the expansion of the function F in Fourier series contains even harmonics. Inasmuch as the frequency characteristic (2.203) determines the variation of the variable x_1 for a given variation of y, y containing higher harmonics, we may say that according to Condition (2.204), the first harmonic (the frequency Ω) will play the major role in the solution for x_1, while the remaining harmonics of the variable y will exert little influence upon the solution for the variable x_1, i.e., the solution for x_1 may actually be close to sinusoidal.

As with Expression (2.202), we may also synthesize a transfer function for any other variable in System (2.200) instead of x_1, and by verifying the filter property for it establish which other variables of this system have a near-sinusoidal solution; we may also find the form of the solution for any variable allowing for the higher harmonics generated by the nonlinear function $y = F(x_1, px_1)$.

Finally, we draw attention to yet another very important circumstance. In speaking of the presence of a pair of purely imaginary roots in the generating system or in the harmonically linearized system, we have had in mind at all times simultaneous realization

of an auxiliary condition: all the remaining roots of the characteristic equation $L(p) = 0$ of this system (other than the pair of purely imaginary roots) have negative real parts of finite magnitude, i.e., we have assumed that the polynomial (2.118)

$$L_1(p) = \frac{L(p)}{p^2 + \omega^2} \tag{2.205}$$

satisfies the Hurwitz criterion (or another linear criterion of stability) with a margin to spare.

Here we have also assumed (see § 2.2) that the open linear part of the system is stable or neutral, i.e., that its characteristic equation $Q(p) = 0$ does not have roots with positive real parts or purely imaginary roots, but only roots with negative real parts and possibly zero roots.

All this has permitted us to speak not simply of closeness of a nonlinear system in a periodic (self-oscillatory) mode of operation to a linear system with a pair of purely imaginary roots, but of its closeness to a linear system located on the oscillatory stability boundary. Thus we have guaranteed that not only that part of the solution which corresponds to the pair of purely imaginary roots, but also that the full solution of the generating system for initial conditions close to the initial conditions of the periodic mode of operation being investigated for the nonlinear system is close to sinusoidal. This corresponds to the position of N.N. Bogolyubov concerning the strong stability of the two-parameter family of particular solutions ([28] and [102], Chapter 4).

The assumptions which have been made relative to all roots of the generating (closed-loop, harmonically linearized) system and the open linear part of the system, together with observance of the filter property, permits us to speak of the applicability of the generating harmonically linearized equation not only for determina-

tion of the periodic solution, but also in first approximation for determination of the process of establishment of the periodic solution, i.e., for slowly damped or slowly diverging oscillations in the neighborhood of the periodic solution, as was also done in § 2.4. In this case, however, the amplitude a and the frequency ω in the solution will not be constant, but will be slowly varying time functions. For this it is, of course, necessary that the coefficients of harmonic linearization $q(a, \omega)$ and $q'(a, \omega)$ [in many practical problems $q(a)$ and $q'(a)$ must also be slowly varying functions of a and ω (or only of a)]. Then the "generating" system will be a linear system – not, however, with constant coefficients, but with coefficients slowly varying over time. Its unique property consists in the fact that the variable coefficients q and q' are not expressed in explicit form as time functions, but depend upon the solution. A similar phenomenon is also the case in determination of a periodic solution where these coefficients although constant, are not, however, given beforehand, since they are functions of the amplitude (and, in the general case, also of the frequency) of the solution being sought. In this sense, the generating harmonically linearized system preserves at least the principal features of the nonlinear system being approximated, which permit us to determine approximately its basic nonlinear features in periodic modes of operation and in the neighborhood of these modes of operation.

In conclusion, we must also note references which are important for the range of phenomena being investigated, i.e., the studies of Yu.A. Mitropol'skiy [102] on nonlinear systems of high order with slowly varying parameters and V.O. Kononyenko [93] on nonlinear systems of high order whose parameters deviate slowly from the periodic.

§ 2.7. DELINEATION OF EQUILIBRIUM-STABILITY REGIONS

Up to the present time we have spoken of the determination of periodic solutions and of their stability, i.e., we have considered a range of nonlinear-system parameters in which there exist periodic solutions, stable (self-oscillations), or unstable. In many cases, however, we must synthesize an automatic system in such a way that self-oscillations do not arise and the equilibrium state of the system will be stable for arbitrary initial conditions. In these cases we must find an equilibrium-stability region of the system outside the region of the periodic solutions.

Generally speaking, the specific properties of nonlinear systems are not limited only to the possibility of the existence of periodic-solution regions in the parameter space. Other complex singularities of the separatrix type and the like may also exist outside the ranges of periodic solutions. In the present section, however, we will be concerned with nonlinear systems for which there are no such singularities in the parameter space outside the range of periodic solutions or at least near them, but there is either an area of equilibrium stability for arbitrary initial conditions or an area of instability. It is these areas of equilibrium stability outside the range of the periodic solutions that will be defined in the present section. Here we assume the presence of a unique equilibrium state of the system or, on the other hand, a unique zone of equilibrium states (for example, within the dead zone of a nonlinear link of the system).

As regards the possibility of stable equilibrium within the range of periodic solutions (for example, in Fig. 2.17b for an unstable periodic solution), § 2.9 will be devoted specifically to this problem.

The absence of a periodic solution is determined by the fact that the characteristic equation of the harmonically linearized system (2.115) or (2.116) has no purely imaginary roots for any possible values of $\underline{q}$ and $\underline{q}'$ for the nonlinearity in question. But as we already know, the harmonically linearized equation is also valid in the nonlinear systems being considered for transient processes in the neighborhood of the periodic solution. Therefore we may say that if Eq. (2.115) or (2.116) satisfies the Hurwitz criterion (or any other linear stability criterion, such as that of Mikhaylov or Nyquist) in the region where the periodic solution is lacking but near it [sic] for all values of $\underline{q}$ and $\underline{q}'$ possible for the nonlinearity in question, then, at least in the neighborhood of the boundary found for the region of no periodic solutions, there will be a region of equilibrium stability for the system. If, however, the harmonically linearized equation (2.115) or (2.116) does not satisfy the Hurwitz (Mikhaylov or Nyquist) stability criterion outside the region of periodic solutions for all values of $\underline{q}$ and $\underline{q}'$ possible for the nonlinearity in question, this will be a region in which the system is unstable.

Thus do we consider determination of the oscillatory boundary* of equilibrium stability for a nonlinear system satisfying the conditions derived in § 2.2 (they are formulated briefly at the beginning of § 2.3). Here we assume that for a harmonically linearized system equation, all the Hurwitz determinants are positive other than the next-to-last H_{n-1}, which may change its sign (or satisfy other conditions equivalent to this).

The determination of the oscillatory equilibrium-stability boundary with respect to the harmonically linearized equation is physically completely natural for nonlinear systems which do not

yield to ordinary linearization, but may have a periodic solution close to the sinusoidal. Below, in § 2.8, we shall show with a number of examples that the results of such a determination of the region of equilibrium stability for a nonlinear system agree with the results of application of the direct method of Lyapunov (for sufficient conditions of stability).

If we make use of Mikhaylov's linear stability criterion, then the equilibrium stability region for nonlinear systems of the classes being considered may be determined as a region of system parameters such that for it the Mikhaylov stability criterion, applied to the harmonically linearized system equation, is realized for any values of q and q' that are possible for the nonlinearity in question. In other words, in this range of system parameters for no possible values of q and q' must a Mikhaylov curve obtained by substituting $p = j\omega$ into the characteristic equation of the harmonically linearized system, for example in (2.115):

$$Q(p)+R(p)\left(q+\frac{q'}{\Omega}p\right)=0, \tag{2.206}$$

or in (2.116):

$$Q(p)+R_1(p)\left(q_1+\frac{q_1'}{\Omega}p\right)+R_2(p)\left(q_2+\frac{q_2'}{\Omega}p\right)=0, \tag{2.207}$$

or other examples, pass through the origin; it must always envelop it (Fig. 2.27). Let us note that the coefficients q and q' in Eq. (2.206) are not independent of each other. For each nonlinearity, a fixed value of q' and a fixed value of Ω for the system in question (on the stability boundary) correspond to each value of q. Just as in Eq. (2.207), all the coefficients q_1, q_1', q_2, q_2', and Ω are interdependent.

In the use of the fourth method of determining the periodic solution (§ 2.3, Fig. 2.11), the equilibrium-stability boundary may be

determined graphically pointwise by choosing suitable boundary values of the parameters, starting with which we satisfy the condition indicated above (see examples in Chapters 4 and 6).

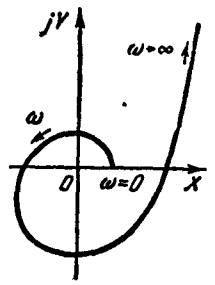


Fig. 2.27.

In the use of the analytical first method (§ 2.3), the equilibrium-stability boundary is determined as the boundary of existence of real positive values of A and Ω in the solution obtained. For example, in § 1.3, according to (1.85) and (1.86), we shall have $T_2k_1 = T_1k_{o.s}$ if we assume that all the parameters are positive. In this case the region of the periodic solution will be $T_2k_1 > T_1k_{o.s}$. In order to be convinced of the fact that the inverse condition $T_2k_1 < T_1k_{o.s}$ determines the region of equilibrium stability of the system, we must verify observance of the Mikhaylov or Hurwitz criterion at any one point in the neighborhood of the resulting boundary which has been found from the characteristic equation of the harmonically linearized system (it was in this way that we solved the problem in § 1.3). This method will be applied to a specific system in Chapter 4.

We must also proceed similarly in application of all the remaining methods of determining a periodic solution, finding the boundary of equilibrium stability as the limiting possible case of the graphical solution there used, after which we must carry out the above verification of the satisfaction of any one of the linear stability criteria for a harmonically linearized system.

Let us now dwell particularly on the case of a nonlinear system with one single-valued oddly symmetrical nonlinearity having an arbitrary configuration. Such systems are encountered often. As is evi-

dent from (2.206), the characteristic equation of the harmonically linearized system in this case has the form

$$Q(p)+R(p)q=0. \tag{2.208}$$

According to the second method (§ 2.3), the condition for the presence of the periodic solution will be equality to zero of the next-to-last Hurwitz determinant:

$$H_{n-1}=0, \tag{2.209}$$

where the magnitude of the determinant H_{n-1} is a function of the coefficient of harmonic linearization q and the system parameters. For variation of any system parameter k, let the magnitude of the determinant

$$H_{n-1}(k, q)$$

be equal to zero for all values of q possible for the nonlinearity in question in some interval $k_1 < k < k_2$ (Fig. 2.28a); however, to the left of the point k_1 we have $H_{n-1} > 0$, while to the right of the point k_2 we have $H_{n-1} < 0$ for all values of q which are possible in the problem being considered. Then the equilibrium-stability region of the system* lies to the left of k_1, while the region of instability lies to the right of the point k_2.

The boundary values k_1 and k_2 may be determined as those values of the parameter k for which the least and greatest values of the determinant H_{n-1} are zero for variation of q in an interval which is fixed for the given nonlinearity. In fact, if the least (for variation of q) value of H_{n-1} (curve 2 in Fig. 2.28b) is zero for some value $k = k_1$, while for $k > k_1$ the equality $H_{n-1} = 0$ is possible for any values of q (curve 3), then for $k < k_1$ we must necessarily have $H_{n-1} > 0$ for all values of q (curve 1). Here we do not consider particular cases that are almost never encountered.

From a practical point of view, as will be shown in what follows,

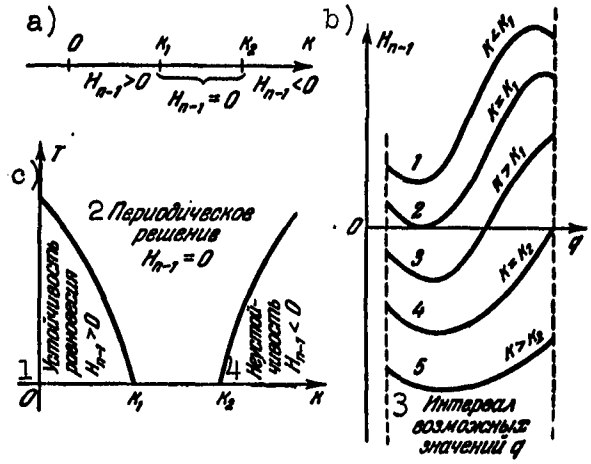


Fig. 2.28. 1) Equilibrium stability; 2) periodic solution; 3) interval of possible values of q; 4) instability.

there is no need to construct these curves.

In exactly the same way, in the plane of any two system parameters (for example, k and T) the case where the least (for variation of q) value of the determinant H_{n-1} is equal to zero, will determine the line which represents the equilibrium-stability boundary of the system (Fig. 2.28c), while the case where the maximum value of the determinant H_{n-1} is equal to zero determines the boundary of the instability region.

These extremal (minimum and maximum) values of the determinant H_{n-1} may be found by setting the derivative

$$\frac{\partial H_{n-1}}{\partial q}=0 \qquad (2.210)$$

equal to zero (excepting the special cases mentioned above) or, on the other hand, by means of direct establishment of values of q, for which the least and greatest values of the determinant H_{n-1} occur, on the boundaries of the interval of variation q, even in the absence of the mathematical extremum.

Eliminating the quantity q from Expressions (2.209) and (2.110), we obtain the equilibrium-stability or -instability boundary for the

system expressed in terms of the system parameters.

However, the quantity $\underline{q}$ obtained here, which corresponds to the mathematical extremum of the determinant H_{n-1}, may lie outside the interval of values of $\underline{q}$ which are possible for the nonlinearity in question, as for example, the magnitude of the maximum H_{n-1} in Fig. 2.28b departs from this interval to the right. Then we take the closest boundary value of it to the right and substitute it into Expression (2.209); this corresponds graphically, for example, to curve 4 in Fig. 2.28b. In addition, we must exclude from consideration those sections of the resulting stability boundary obtained where the condition of all remaining Hurwitz determinants positive is not observed.

Hence, here the equilibrium-stability analysis of the nonlinear system breaks down into two stages [128]:

The 1st stage – elimination of $\underline{q}$ from Eqs. (2.209) and (2.210) – furnishes stability conditions which are sufficient for any form of single-valued oddly symmetrical nonlinearity, since in them the quantity $\underline{q}$ is not bounded ($0 \leq q \leq \infty$), i.e., it is not a function of the form of the nonlinearity;

The 2nd stage – rejection of the superfluous sections of the stability boundary obtained in the first stage and their replacement by others obtained by bounding the interval of possible values of $\underline{q}$ for the nonlinearity in question – leads to the necessary conditions for stability, since here we approach close to the boundary of the region of existence of the periodic solution. (These necessary conditions are approximate in accordance with the approximateness of the method for determining the periodic solution.)

In much the same way, we may also approximate the region of equilibrium stability for a loop-type nonlinearity in the general

case for F(x, px); this will be shown in one of the examples at the end of the present section. But in the general case it is more convenient to make use of the methods indicated at the beginning of the section, employing the Mikhaylov criterion.

Example 1. Let us first present the simplest example of a nonlinear system (Fig. 2.29), whose equation is given in the form

$$\left.\begin{aligned}(T_1p+1)x_2&=-k_1x_1,\\ x_3=F(x),\quad x&=x_3-k_{o.c}x_1,\\ (T_2p+1)px_1&=k_2x_3.\end{aligned}\right\} \tag{2.211}$$

On the substitution F(x) = qx, the characteristic equation of the harmonically linearized system takes the form

$$T_1T_2p^3+(T_1+T_2)p^2+(1+T_1k_2k_{o.c}q)p+(k_1+k_{o.c})k_2q=0. \tag{2.212}$$

The characteristic equation of the open linear part, which, according to (2.80), we obtain here for q = 0, is

$$T_1T_2p^3+(T_1+T_2)p^2+p=0.$$

It has a zero root, but not purely imaginary roots or roots with positive real parts (which is required according to § 2.3), since after removal of p from the parentheses a quadratic trinomial with positive coefficients remains.

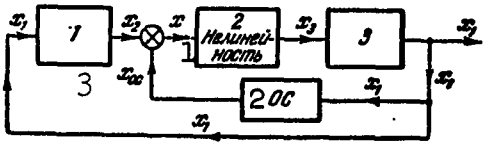


Fig. 2.29. 1) Nonlinearity; 2) OS; 3) x_{os}.

The next-to-last Hurwitz determinant for Eq. (2.212) is

$$H_{n-1}=(T_1+T_2)(1+T_1k_2k_{o.c}q)-T_1T_2(k_1+k_{o.c})k_2q.$$

Removing parentheses, we write Eq. (2.209):

$$H_{n-1}=T_1+T_2+T_1k_2(T_1k_{o.c}-T_2k_1)q=0. \tag{2.213}$$

Here Formula (2.210) assumes the form

$$\frac{\partial H_{n-1}}{\partial q}=T_1k_2(T_1k_{o.c}-T_2k_1)=0. \tag{2.214}$$

The quantity q has not entered into Expression (2.214). Therefore there is no need in the simple example under consideration to

eliminate $\underline{q}$ from (2.213) and (2.214), as was indicated in the general method. Here Expression (2.214) is itself the equation for the boundary of equilibrium stability:

$$k_{o.c} = \frac{T_3 k_1}{T_1}. \tag{2.215}$$

Precisely this will be the boundary of equilibrium stability (but not the stability boundary), because from (2.213) it is evident that where

$$k_{o.c} > \frac{T_3 k_1}{T_1} \tag{2.216}$$

we will have $H_{n-1} > 0$, while the existence conditions for the periodic solution (2.213) may be satisfied only where

$$k_{o.c} < \frac{T_3 k_1}{T_1}, \tag{2.217}$$

since all the parameters and the coefficients $\underline{q}$ are positive (by their physical significance).

The relationships obtained agree with the construction of the areas of equilibrium stability carried out for this example in Chapter 1 (Figs. 1.21 and 1.22).

According to the general method set forth above, we must still verify that all the remaining Hurwitz determinants other than H_{n-1}, which has already been analyzed are positive. In the case in question (a third-order system) this leads to the positive coefficients of the characteristic equation (2.212). From the condition of a constant term positive, we have here the inequalities

$$k_{o.c} > -k_1; \quad k_3 > 0, \tag{2.218}$$

which, combined with (2.216), determine the regions of equilibrium stability shown in Fig. 2.30. This corresponds to the first stage of the analysis, which gives the stability conditions sufficient for any form of nonlinearity, since we have still not yet considered the restriction of possible values of $\underline{q}$. In the second stage of the

investigation, we require for the determination of necessary stability conditions that the value of $\underline{q}$ obtained from (2.213) and (2.214) not transcend the range of values possible for the nonlinearity in question. From (2.213) and (2.214) we have

$$q = \frac{T_1 + T_2}{T_1 k_3 (T_2 k_1 - T_1 k_{0.c})} = \infty. \qquad (2.219)$$

For the nonlinearity considered in § 1.3 (an ideal relay) this is a possible value, since there $0 \leq q \leq \infty$ (Fig. 2.31a). The situation will be the same if in place of relay 2 (Fig. 2.29) we put the nonlinear link $x_3 = F(x)$ with the curvilinear characteristic (Fig. 2.31b), for which $k \leq q \leq \infty$.

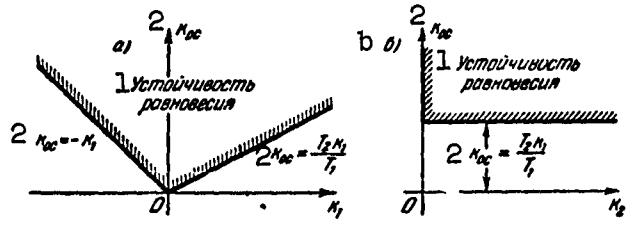


Fig. 2.30. 1) Equilibrium stability; 2) k_{os}.

If, however, we have in this same system (Fig. 2.29) a link with a dead zone (Fig. 2.31c or d) as the nonlinear link 2, or a conditionally linear link with saturation (Fig. 2.31e), or a more complex form of nonlinearity (Fig. 2.31f, g), then for all of them the quantity $\underline{q}$ will have the bounded interval of variation

$$0 \leq q \leq q_{max}; \qquad (2.220)$$

for a nonlinearity of the type of Fig. 2.31h, this interval will be

$$q_{min} \leq q \leq q_{max} \quad (q > 0). \qquad (2.221)$$

In all these cases, therefore, we must take q_{max} for determination of the equilibrium stability boundary in place of the value $q = \infty$ obtained from (2.213) and (2.214). Then, according to (2.213), the stability boundary will be

$$k_{o.c} = \frac{T_3 k_1}{T_1} - \frac{T_1 + T_3}{T_1^2 k_3 q_{max}}; \qquad (2.222)$$

however, Conditions (2.218) remain as before. Figs. 2.32a and b show lines 1 determined by Eq. (2.222). It is evident from comparison

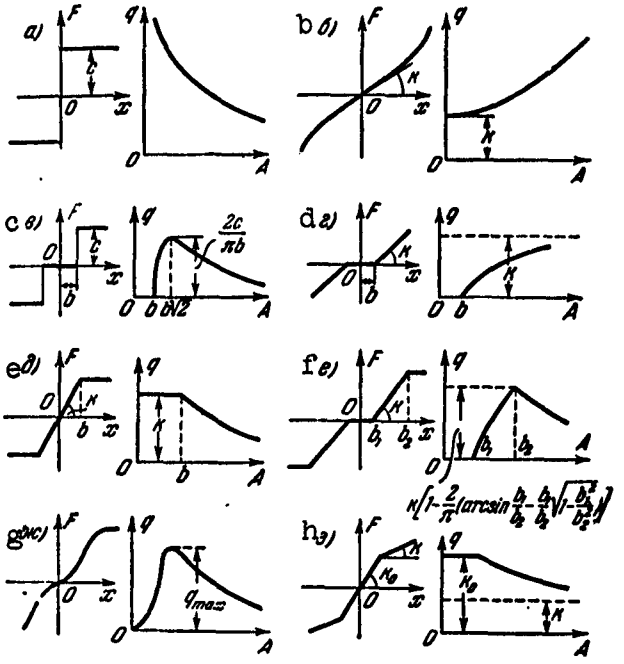


Fig. 2.31.

with Fig. 2.30 that in the problem being considered the bounding of the possible values of q from above enlarges the equilibrium stability region of the system. As a result, we obtain stability conditions which are not only sufficient, but also necessary.

The specific expression for the quantity q_{max} figuring in Figs. 2.32a and b, depends upon the form of the nonlinearity as indicated in Fig. 2.31. Hence, the location of the equilibrium-stability boundary of the system will also depend upon the form of the nonlinearity.

Let us take special note of the case (2.221). There, satisfaction of Equality (2.213) is possible only for the condition

$$\frac{T_3 k_1}{T_1} - \frac{T_1 + T_3}{T_1^2 k_3 q_{min}} < k_{o.c} < \frac{T_3 k_1}{T_1} - \frac{T_1 + T_3}{T_1^2 k_3 q_{max}}, \qquad (2.223)$$

if we are limited to consideration of only positive values of all

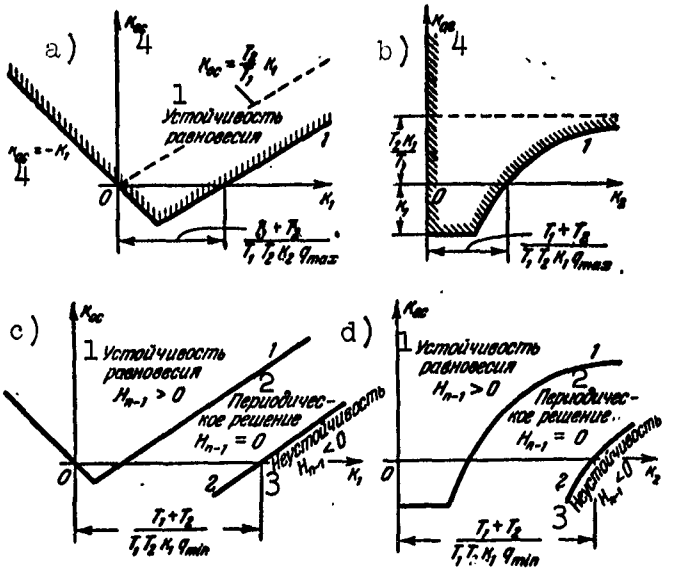


Fig. 2.32. 1) Equilibrium stability; 2) periodic solution; 3) instability; 4) k_{OS}.

the parameters. Therefore, here the region of the periodic solution is again bounded on the other side, where it borders the area of instability of the system (lines 2 in Fig. 2.32c, d). We will also have a similar picture for the nonlinearity of the type Fig. 2.31b, with the only difference that there $q_{max} = \infty$. However, in all the remaining cases except Fig. 2.31b and h, $q_{min} = 0$, as a consequence of which the region of the periodic solution extends indefinitely to the right of line 1 (Fig. 2.32a and b).

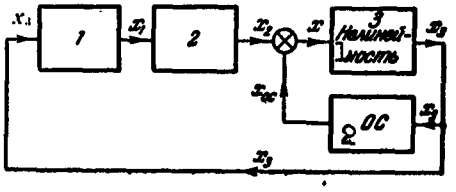


Fig. 2.33. 1) Nonlinearity; 2) OS.

Example 2. Let us introduce an example of another automatic systemtem in which, on the contrary, the quantity q_{min} determines the equilibrium-stability boundary, while q_{max} determines the boundary of instability. Let us consider the system (Fig. 2.33) described by the equations:

$$\left.\begin{aligned}(T_1p+1)\,px_1 &= -k_1x_2,\\ x_2 &= (k_2+k_3p)\,x_1,\\ px_3 &= F(x),\ x = x_3 - k_{o.c}\,x_3\end{aligned}\right\} \qquad (2.224)$$

for positive values of the coefficients.

The characteristic equation of the harmonically linearized system for a single-valued oddly symmetrical linearity F(x) is

$$T_1p^3+(1+k_{o.c}\,T_1q)\,p^2+(k_{o.c}+k_1k_3)\,qp+k_1k_2q=0. \qquad (2.225)$$

For q = 0 we have $T_1p^3 + p^2 = 0$, i.e., the roots of the linear part satisfy the requirements of § 2.3.

The next-to-last Hurwitz determinant for (2.225) is

$$H_{n-1}=(1+k_{o.c}\,T_1q)\,(k_{o.c}+k_1k_3)\,q-T_1k_1k_2q.$$

Therefore Eq. (2.209) assumes the form

$$H_{n-1}=k_{o.c}\,T_1\,(k_{o.c}+k_1k_3)\,q^2+(k_{o.c}+k_1k_3-T_1k_1k_2)\,q=0, \qquad (2.226)$$

while Eq. (2.210) takes the form

$$\frac{\partial H_{n-1}}{\partial q}=2\,k_{o.c}\,T_1\,(k_{o.c}+k_1k_3)\,q+k_{o.c}+k_1k_3-T_1k_1k_2=0. \qquad (2.227)$$

Both of these equations are satisfied for q = 0, if

$$k_{o.c}+k_1k_3-T_1k_1k_2=0$$

or

$$k_2=\frac{k_{o.c}+k_1k_3}{k_1T_1}. \qquad (2.228)$$

This boundary is represented in Fig. 2.34a.

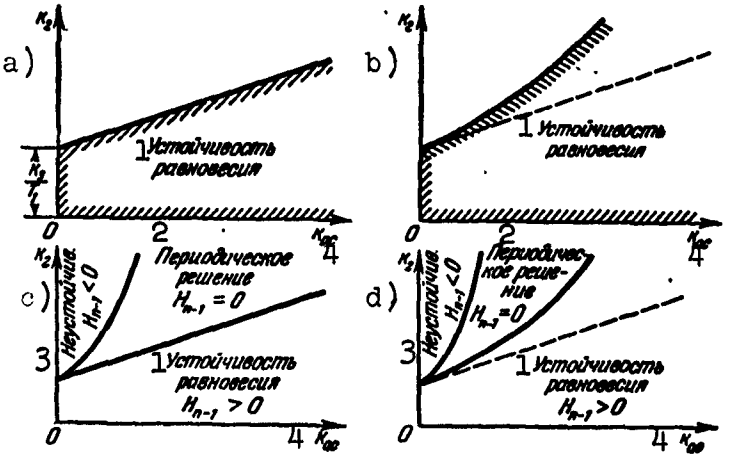


Fig. 2.34. 1) Equilibrium stability; 2) periodic solution; 3) instability; 4) $k_{o.c}$.

If, however, $q \neq 0$, then Eq. (2.226) may be divided by $\underline{q}$ and in place of (2.227) we may write

$$\frac{\partial}{\partial q}\left(\frac{H_{n-1}}{q}\right) = k_{o.c}\, T_1 (k_{o.c} + k_1 k_3) = 0,$$

hence we obtain an additional equilibrium-stability boundary (Fig. 2.34a)

$$k_{o.c} = 0. \tag{2.229}$$

The statement that Equality (2.228) determines just the boundary of equilibrium stability (but not the boundary of instability), follows from the sign of the second derivative

$$\frac{\partial^2 H_{n-1}}{\partial q^2} = 2k_{o.c}\, T_1 (k_{o.c} + k_1 k_3) > 0,$$

i.e., here we actually have a minimum for H_{n-1}.

We may also satisfy ourselves of this by the simple substitution of the value

$$k_3 < \frac{k_{o.c} + k_1 k_3}{k_1 T_1}, \tag{2.230}$$

into Expression (2.226); this gives us $H_{n-1} > 0$ for any positive value of $\underline{q}$ (see the region of equilibrium stability in Fig. 2.34a).

In addition, we have from the positivity condition for the absolute term of the characteristic equation (2.225) $k_2 > 0$.

The region of equilibrium stability which is thus obtained in the first stage of the analysis (Fig. 2.34a) will be the same irrespective of the form of the nonlinearity. This corresponds to the sufficient conditions for stability. They will also be necessary conditions for all cases where the value $q = 0$, which corresponds to Condition (2.228), is possible. This will be the case, in particular, for all the nonlinearities represented in Fig. 2.31 except for two: Fig 2.31b and h. For these,

$$q_{min} = k. \tag{2.231}$$

We must substitute this value into Eq. (2.226) in place of $q = 0$ in

order to obtain the boundary of equilibrium stability (in the second stage of the analysis); this gives us (Fig 2.34b)

$$k_3 = \frac{k_{0.c} + k_1 k_2}{T_1 k_1}(1 + T_1 k_{0.c} q_{min}). \qquad (2.232)$$

In the case in question, the region of equilibrium stability is expanded, in contrast to the preceding example, in the presence of a lower bound for the quantity q (see (2.231)).

However, in the present example the bounding of the quantity q from above results in the appearance of a region of instability for the system. Thus if q varies within the range

$$0 \leqslant q \leqslant q_{max} \text{ or } q_{min} \leqslant q \leqslant q_{max} \qquad (2.233)$$

(all the nonlinearities in Fig. 2.31 except 2.31a and b), then for all positive parameters the condition for the presence of a periodic solution (2.226) may be fulfilled only for

$$k_3 < \frac{k_{0.c} + k_1 k_2}{T_1 k_1}(1 + T_1 k_{0.c} q_{max}). \qquad (2.234)$$

Otherwise $H_{n-1} < 0$ for all possible values of q from the interval (2.233), i.e., there is a region of instability (Fig. 2.34c or d).

Example 3. Let us consider further a third example of a nonlinear system in which the minimum H_{n-1} (the equilibrium stability boundary) is determined not by the values $q = \infty$ and $q = 0$, as in the preceding two examples, but by some intermediate instantaneous value of q. Let us assume that the automatic system (Fig. 2.33) is described by the equations

$$\left.\begin{aligned} (T_1^2 p^2 + T_2 p + 1) x_1 &= -k_1 x_2, \\ x_2 &= (k_3 + k_2 p + k_4 p^2) x_1, \\ p x_3 &= F(x), \quad x = x_2 - k_{0.c} x_3. \end{aligned}\right\} \qquad (2.235)$$

Here for a single-valued oddly symmetrical nonlinearity F(x) the characteristic equation of the harmonically linearized system is

$$T_1^2 p^3 + [T_2 + (k_{0.c} T_1^2 + k_1 k_4) q] p^2 + [1 + (k_{0.c} T_2 + k_1 k_3) q] p + (k_{0.c} + k_1 k_2) q = 0. \qquad (2.236)$$

The next-to-last Hurwitz determinant takes the form

$$H_{n-1}=[T_2+(k_{o.c}\,T_1^2+k_1k_3)\,q][1+(k_{o.c}\,T_2+k_1k_3)\,q]-\\-T_1^2(k_{o.c}+k_1k_2)\,q,$$

as a consequence of which Eq. (2.209) is

$$H_{n-1}=T_2+(\alpha+\beta-\gamma)q+\frac{\alpha\beta}{T_2}q^2=0\ (\text{for } T_2\neq 0), \tag{2.237}$$

where we denote

$$\alpha=T_2(k_{o.c}\,T_2+k_1k_3),\ \beta=k_{o.c}\,T_1^2+k_1k_4,\ \gamma=T_1^2(k_{o.c}+k_1k_2). \tag{2.238}$$

However, Eq. (2.210) assumes the form

$$\frac{\partial H_{n-1}}{\partial q}=\alpha+\beta-\gamma+\frac{2\alpha\beta}{T_2}q=0. \tag{2.239}$$

Determining from this

$$q=-\frac{\alpha+\beta-\gamma}{2\alpha\beta}T_2 \tag{2.240}$$

and substituting it into Eq. (2.237), we obtain the equation for the system's boundary of equilibrium stability in the form

$$(\alpha+\beta-\gamma)^2-4\alpha\beta=0. \tag{2.241}$$

In the plane of the parameters α and β, this equation gives a parabola (Fig. 2.35a), whose axis is the bisector of the coordinate angle (the straight line $\beta = a$). This parabola is tangent to the coordinate axes at the points A($\beta = \gamma$) and B($\alpha = \gamma$). Its vertex C has the coordinates

$$\alpha=\beta=\frac{\gamma}{4}.$$

From the positivity condition for the absolute term (2.236), we have according to (2.238)

$$\gamma>0 \text{ for } q>0. \tag{2.242}$$

Here, the contrast to the preceding examples, the values of $\underline{q}$ along the stability boundary vary according to Formula (2.240), with $\underline{q}$ positive over the section ACB of the curve (Fig. 2.35a); at points A and B we will have $q = \infty$ (for $T_2 \neq 0$), while at point C

$$q=\frac{4T_2}{\gamma}. \tag{2.243}$$

However, according to (2.240), the values of $\underline{q}$ are negative on the sections AE and BD (Fig. 2.35a). The distribution of the values of $\underline{q}$ along the curve EACBD is shown diagramatically in Fig. 2.35b.

By virtue of Condition (2.242) and also keeping in mind that for all the nonlinearities (Fig. 2.31) $q \geq 0$, the sections AE and BD (Fig. 2.35a), where $q < 0$, must be excluded from consideration. On the other hand, it is evident that the condition of the presence of the periodic solution (2.237) for $q > 0$ and $\alpha + \beta > \gamma$ may be satisfied only for $\alpha\beta < 0$, i.e., in the second and fourth quadrants of the plane. Therefore the region of equilibrium stability for the system (in place of Fig. 2.35a) assumes the form of Fig. 2.35c, where the sections $A\beta$ and $B\alpha$ correspond to the values $q = \infty$.

If the quantity $\underline{q}$ has an upper bound in the form of a value q_{max} (all nonlinearities in Fig. 2.31 other than Fig. 2.31a and b), then according to (2.237) the region of equilibrium stability is enlarged on several segments to the hyperbola

$$T_2 + (\alpha + \beta - \gamma) q_{max} + \frac{\alpha\beta}{T_2} q_{max}^2 = 0, \qquad (2.244)$$

where q_{max} has a singular value for each nonlinearity (see Fig. 2.31).

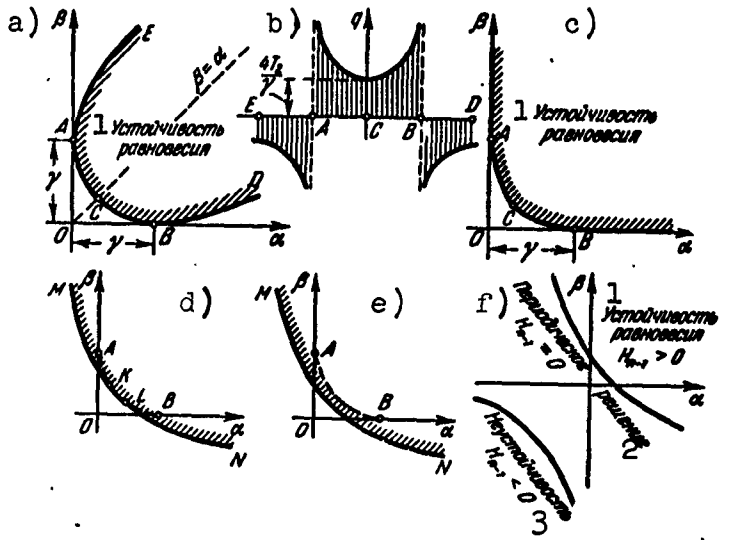


Fig. 2.35. 1) Equilibrium stability; 2) periodic solution; 3) instability.

If here

$$\gamma > \frac{4T_3}{q_{max}},$$

then in accordance with (2.243) and Fig. 2.35b, the boundaries will be the lengths MK and LN of the hyperbola (2.244), while a length of the original parabola remains on the segment KL of the boundary (Fig. 2.35d). If, however,

$$\gamma < \frac{4T_3}{q_{max}},$$

then the stability boundary will be the entire hyperbola (Fig. 2.35e), and for

$$\gamma \leqslant \frac{T_3}{q_{max}}$$

the stability region includes the entire first quadrant of the α, β-plane.

Finally where $\underline{q}$ is bounded below by some value q_{min} (Fig. 2.31b, h) we obtain three regions (Fig. 2.35f); here the boundary of the instability region is the hyperbola

$$T_3 + (\alpha + \beta - \gamma) q_{min} + \frac{\alpha\beta}{T_3} q_{min}^2 = 0.$$

The parameters α, β, and γ, for which all the diagrams in Fig. 2.35 have been drawn, are combinations (2.238) for real system parameters. For example, let us make the choice of the coefficients k_3 and k_4 [the intensity coefficients for introduction of the derivatives into the control law; see (2.235)]. From (2.238) it is evident that here these coefficients enter only into the parameters α and β and do so independently of each other. Therefore the problem is easily solved by means of Fig. 2.35. If, in addition, we must also select the coefficient k_2 with reference to the basic controller signal, then according to (2.238) we must vary the parameter γ, which is also easily done on the basis of Fig. 2.35. Generally speaking, we may also construct a three-dimensional diagram in the coordinates

α, β, and γ.

Example 4. As our last example, let us consider a system with a loop-type nonlinearity. Let us assume that an automatic system is synthesized according to the scheme of Fig. 2.29 and described by Eqs. (2.211); here F(x) is a loop-type nonlinearity having, for example, any one of the forms indicated in Fig. 2.36. After harmonic linearization, Eqs. (2.211) assume the form

$$\left.\begin{aligned} (T_1 p+1)\,x_1 &= -k_1 x_1,\\ x_2 &= \left(q+\frac{q'}{\Omega}p\right)x, \quad x = x_3 - k_{o.c}\,x_1,\\ (T_2 p+1)\,p x_1 &= k_2 x_2, \end{aligned}\right\} \tag{2.245}$$

where q and q' are determined separately for different nonlinearities (Fig. 2.36). The characteristic equation of this system is

$$\begin{aligned} T_1 T_2 p^3 + (T_1 + T_2 - T_1 k_2 k_{o.c}\, q'')\,p^2 + [1 + T_1 k_2 k_{o.c}\, q - \\ - (k_1 + k_{o.c})\,k_2 q'']\,p + (k_1 + k_{o.c})\,k_2 q = 0, \end{aligned} \tag{2.246}$$

where we designate

$$q'' = -\frac{q'}{\Omega} > 0, \tag{2.247}$$

in order to be able to deal with positive numbers (in itself $q' < 0$; see Fig. 2.36).

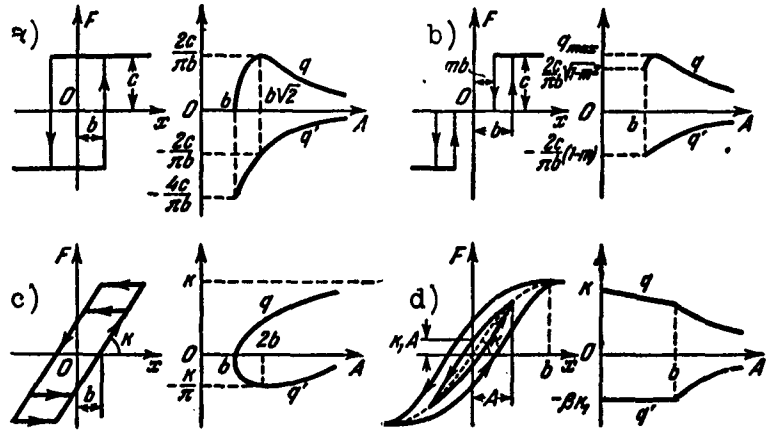


Fig. 2.36.

Here the next-to-last Hurwitz determinant has the form

$$\begin{aligned} H_{n-1} = (T_1 + T_2 - T_1 k_2 k_{o.c}\, q'')[1 + T_1 k_2 k_{o.c}\, q - (k_1 + k_{o.c})\,k_2 q''] - \\ - T_1 T_2 (k_1 + k_{o.c})\,k_2 q. \end{aligned} \tag{2.248}$$

For stability of the system we must have $H_{n-1} > 0$. From Formula (2.248), regarding all the parameters as positive and observing the positivity requirements for all the coefficients of Eq. (2.246), we see that the larger the quantity q", the more the presence of this quantity q" narrows down the region of the system's equilibrium stability. Therefore, for approximate estimation of the maximum narrowing of the equilibrium-stability region, we may take the maximum possible value of q" for the nonlinearity being investigated (also considering a tentative magnitude for the frequency Ω) without, however, violating positivity of all coefficients of Eq. (2.246) for any q, including q = 0.* Thus, denoting

$$q''_{max} = k_3,$$

we shall assume that

$$0 < k_3 < \frac{T_1 + T_2}{T_1 k_2 k_{o.c}}; \qquad k_3 < \frac{1}{(k_1 + k_{o.c}) k_2}. \tag{2.249}$$

By introduction of the constant value $K_3 = q''_{max}$, we replace the nonlinear lag expressed in the form of a hysteresis loop by a linear lag in the form of the introduction of a negative derivative with the maximum possible constant coefficient, i.e., with a margin in the worse direction from the viewpoint of stability. The preceding method may be applied to such a system just as for a system with a unique nonlinearity. According to (2.248), Eqs. (2.209) and (2.210) here assume the form

$$H_{n-1} = (T_1 + T_2 - T_1 k_2 k_3 k_{o.c})[1 - k_2 k_3 (k_1 + k_{o.c})] + \\ + T_1 k_2 (T_1 k_{o.c} - T_2 k_1 - T_1 k_2 k_3 k_{o.c}^2) q = 0, \tag{2.250}$$

$$\frac{\partial H_{n-1}}{\partial q} = T_1 k_2 (T_1 k_{o.c} - T_2 k_1 - T_1 k_2 k_3 k_{o.c}^2) = 0, \tag{2.251}$$

hence we obtain the expression

$$k_1 = \frac{T_1}{T_2} k_{o.c} (1 - k_2 k_3 k_{o.c}), \tag{2.252}$$

for the equilibrium-stability boundary, which in the absence of k_3

agrees with the previous expression (2.215). The new equilibrium-stability boundary takes the form of the parabola OBC (Fig. 2.37), which intersects the previous boundary OM (2.215) at the origin and has an axis parallel to the axis of abscissas k_1. Let us indicate the coordinates of the vertex B of the parabola and the ordinate of the point C (Fig. 2.37):

$$(k_1)_B = \frac{T_1}{4T_2 k_2 k_3}, \quad (k_{o.c})_B = \frac{1}{2k_2 k_3}, \quad (k_{o.c})_C = \frac{1}{k_2 k_3}.$$

The second boundary OD of the stability region, which results from positivity of the constant term of the characteristic equation (2.246), retains the same form $k_{o.s} = -k_1$ as before.

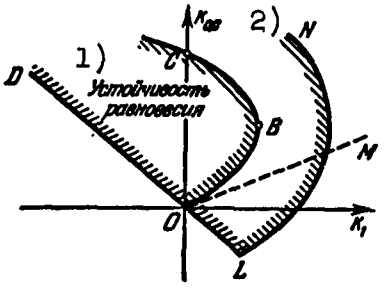


Fig 2.37. 1) Equilibrium stability; 2) $k_{o.s}$.

The stability region DOBC represented in Fig. 2.37 corresponds, as does the previous region (Fig. 2.30a), to the first stage of the analysis, where the quantity q may assume arbitrary values ($0 \leq q \leq \infty$) and $q = \infty$ along the boundary parabola. If, however, the quantity q is bounded by some value q_{max} for the nonlinearity being considered (this obtains for all the nonlinearities in Fig. 2.36), then the stability region is widened and its boundary assumes the form of the curve OLN in Fig. 2.37 (the stability area before expansion was shown in Fig. 2.32a). The equation of this curve is determined by substituting the value q_{max} corresponding to the nonlinearity in question into Eq. (2.250).

Inasmuch as we made use only of the maximum value q" with rough allowance for the quantity Ω, the patterns of the stability region obtained in Fig. 2.37 must be regarded as an approximate estimation of the stability-region narrowing due to the presence of a loop in the nonlinear characteristic (apparently, with a margin). In complex

cases, a more complete analysis may be performed with the aid of the Mikhaylov criterion, as was indicated at the beginning of the present section.

§ 2.8. COMPARISON WITH THE DETERMINATION OF STABILITY REGIONS BY THE DIRECT (SECOND) METHOD OF LYAPUNOV

We have already remarked in the preceding paragraph that the application of harmonic linearization to nonlinearities (in place of the ordinary method of linearization) for determination of the oscillatory equilibrium-stability boundary is completely natural, since in the types of nonlinear systems being considered (see §§ 2.2-2.4) the periodic solution prevailing at the stability boundary is close to sinusoidal (this fact lies at the basis of harmonic linearization). Here, as may be seen from § 2.7, the method of harmonic linearization gives extremely fruitful results.

In the present section we shall show that the equilibrium stability regions obtained in § 2.7 for nonlinear systems agree very well with the results of determination of stability regions by the direct method of Lyapunov, while in the first stage of the analysis (sufficient conditions) and in those cases where all real values of $\underline{q}$ are possible for the nonlinearity being considered (necessary conditions), these results agree exactly with each other. The method of harmonic linearization gives us new necessary stability conditions for a bounded interval of possible values of $\underline{q}$.

We shall make use of the techniques developed by A.I. Lur'ye [34] for applying the method of Lyapunov to analysis of this problem. Let us also note that in § 2.7, the first stage of the analysis gives results close to those of the method used by A.M. Letov [63].

We will apply the Lyapunov method to all the problems for which we obtained stability regions in § 2.7 by the method of harmonic

linearization, and compare the results. In addition, certain problems solved in the book by A.I. Lur'ye [34] using the method of Lyapunov will be solved by the method of harmonic linearization; the results will also be compared.

Example 1. The nonlinear system is described by Eqs. (2.211):

$$\left.\begin{array}{c}(T_1p+1)x_2=-k_1x_1,\\ x_3=F(x),\ x=x_3-k_{o.c}x_1,\\ (T_2p+1)px_1=k_2x_3.\end{array}\right\} \tag{2.253}$$

In the notation of A.I. Lur'ye, this system belongs to the class of systems ([34], p. 25)

$$\dot{\eta}_k=\sum_{\alpha=1}^{n}b_{k\alpha}\eta_\alpha+h_kf(\sigma),\quad \sigma=\sum_{s=1}^{n}j_s\eta_s,$$

and in the problem being considered we have

$$n=3,\quad \eta_1=x_2,\quad \eta_2=x_1,\quad \eta_3=px_1,\quad f(\sigma)=F(x),$$

as a consequence of which System (2.253) assumes the form

$$\left.\begin{array}{l}\dot{\eta}_1=-\frac{1}{T_1}\eta_1-\frac{k_1}{T_1}\eta_2,\quad \dot{\eta}_2=\eta_3,\\ \dot{\eta}_3=-\frac{1}{T_2}\eta_3+\frac{k_2}{T_2}f(\sigma),\ \sigma=\eta_1-k_{o.c}\eta_2.\end{array}\right\} \tag{2.254}$$

The roots of the determinant

$$D(\lambda)=\begin{vmatrix}-\frac{1}{T_1}-\lambda & -\frac{k_1}{T_1} & 0\\ 0 & -\lambda & 1\\ 0 & 0 & -\frac{1}{T_2}-\lambda\end{vmatrix}$$

are

$$\lambda_1=-\frac{1}{T_1},\quad \lambda_2=0,\quad \lambda_3=-\frac{1}{T_2}.$$

Let us note that the determinant $D(\lambda)$ corresponds exactly to the polynomial $Q(p)$ in our general formulas (§§ 2.2-2.4, 2.6). Therefore $D(\lambda) = 0$ is the characteristic equation of the open linear part of the system.

Using the formulas of A.I. Lur'ye ([34], pp. 23 and 26), we calculate further

$$H_1=-\frac{k_1k_2}{T_1T_2},\quad H_2=\frac{k_2}{T_2}\left(\frac{1}{T_1}+\lambda\right),\quad H_3=\frac{k_2}{T_2}\lambda\left(\frac{1}{T_1}+\lambda\right),$$
$$D'(\lambda)=-\lambda\left(\frac{1}{T_2}+\lambda\right)-\left(\frac{1}{T_1}+\lambda\right)\left(\frac{1}{T_2}+\lambda\right)-\lambda\left(\frac{1}{T_1}+\lambda\right),$$

$$\gamma_1=\frac{k_1k_2T_1}{T_1-T_2},\quad \gamma_2=-k_2(k_1+k_{o.c}),\quad \gamma_3=\frac{k_1k_2T_2}{T_2-T_1}+k_{o.c}k_2,$$
$$\beta_1=\frac{k_1k_2}{T_2-T_1},\quad \beta_2=0,\quad \beta_3=\frac{k_1k_2}{T_1-T_2}-\frac{k_{o.c}k_2}{T_2}.$$

Therefore here the quantities Γ^2 and ϑ ([34], p. 78) determining the stability boundaries are:

$$\Gamma^2=\frac{\beta_1}{\lambda_1}+\frac{\beta_2}{\lambda_2}=k_2(k_1+k_{o.c}),$$
$$\vartheta=\frac{(\beta_1-\beta_2)(\lambda_1-\lambda_2)}{4\lambda_1\lambda_2}=-\frac{k_2}{4}\left(2k_1+k_{o.c}-\frac{T_1}{T_2}k_{o.c}\right).$$

According to A.I. Lur'ye ([34], p. 80) the stability conditions have the form

$$\Gamma^2>0 \text{ and } -4\vartheta<\Gamma^2.$$

The first condition gives us

$$k_2(k_1+k_{o.c})>0, \text{ i.e. } k_{o.c}>-k_1 \text{ и } k_2>0, \tag{2.255}$$

while the second gives us

$$k_2(2k_1+k_{o.c})-\frac{T_1}{T_2}k_{o.c}k_2<k_2(k_1+k_{o.c}),$$

from which

$$k_{o.c}>\frac{T_2}{T_1}k_1. \tag{2.256}$$

It is easily seen that both of these stability conditions (2.255) and (2.256) obtained by the Lyapunov method agree completely with the stability conditions (2.218) and (2.216) obtained in § 2.7 by the method of harmonic linearization, which correspond to the graphs in (Fig. 2.30).

As we know, the theorems of the direct Lyapunov method give us sufficient conditions for stability that are not always necessary conditions. Comparison of the results obtained here with the results of Example 1 in § 2.7 suggests that in the present problem, the formulas of Lur'ye corresponding to the method of Lyapunov give us necessary and sufficient stability conditions only for systems with nonlinearities (for example, Fig. 2.31a and b) such that the value of the coefficient q may assume all positive values $0 \leq q \leq \infty$. How-

ever, for the remaining nonlinearities obtained by the Lyapunov method for the Lur'ye formulas the sufficient conditions (2.255) and (2.256) are narrower than the necessary conditions, giving, according to the method of harmonic linearization, a wider region of stability (2.222) (Fig. 2.32) which depends upon the form of the nonlinearity through the quantity q_{max}. However the conditions of A.I. Lur'ye do not depend upon the form of the nonlinearity, i.e., are sufficient for any unique nonlinearity, while at the same time specific definition of the form of the nonlinearity (§ 2.7) leads to a region of stability which is wider for each nonlinearity in question, as is completely natural.

Example 2. The nonlinear system is described by Eqs. (2.224);

$$\left.\begin{aligned} (T_1 p+1)\,p x_1 &= -k_1 x_2, \\ x_2 &= (k_2 + k_3 p)\,x_1, \\ p x_3 = F(x), \quad x &= x_2 - k_{o.c} x_3, \end{aligned}\right\} \tag{2.257}$$

which, in the notation of A.I. Lur'ye, belong to the following class ([34], p. 37):

$$\dot{\eta}_k = \sum_{s=1}^{n} b_{ks}\eta_s + n_k\xi, \quad \dot{\xi} = f(\sigma), \quad \sigma = \sum_{s=1}^{n} j_s\eta_s - r\xi, \tag{2.258}$$

and in the example in question

$$n=2, \quad \eta_1 = x_1, \quad \eta_2 = p x_1, \quad \xi = x_3, \quad f(\sigma) = F(x).$$

The system (2.257) assumes the form

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2, \quad \dot{\eta}_2 = -\frac{1}{T_1}\eta_2 - \frac{k_1}{T_1}\xi, \\ \dot{\xi} &= f(\sigma), \quad \sigma = k_2\eta_1 + k_3\eta_2 - k_{o.c}\xi. \end{aligned}\right\} \tag{2.259}$$

The roots of the determinant

$$D(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\frac{1}{T_1} - \lambda \end{vmatrix}$$

are

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{T_1}.$$

Using the formulas of A.I. Lur'ye ([34], p. 38), we calculate

$$N_1=\frac{k_1}{T_1},\qquad N_2=\frac{k_1}{T_1}\lambda,$$
$$D_1=-\frac{1}{T_1}-\lambda,\qquad D_1'=-1,$$
$$\beta_1=-k_1k_2,\qquad \beta_2=k_1k_2-\frac{k_1k_3}{T_1}.$$

Here the stability question is resolved by the presence of a real solution of the quadratic equation for some variable a_2 ([34], p. 68):

$$2a_2\sqrt{k_{o.c}}+a_2^2T_1+k_1k_2-\frac{k_1k_3}{T_1}=0.$$

Since from this

$$a_2=-\frac{\sqrt{k_{o.c}}}{T_1}\pm\sqrt{\frac{k_{o.c}}{T_1^2}-\frac{k_1k_2}{T_1}+\frac{k_1k_3}{T_1^2}},$$

then the condition for realness of the solution for positive parameters is

$$\frac{k_{o.c}+k_1k_3}{T_1^2}>\frac{k_1k_2}{T_1}$$

or

$$0<k_2<\frac{k_{o.c}+k_1k_3}{k_1T_1}. \tag{2.260}$$

which is also a stability condition for the nonlinear system in question. In this case, also, Condition (2.260), which is obtained by the method of Lyapunov, agrees completely with the stability condition (2.230) obtained by the method of harmonic linearization (see also Fig. 2.34a).

These conditions are sufficient and necessary for all nonlinearities represented in Fig. 2.31 except for two: Fig. 2.31b and h, for which, using the method of harmonic linearization for each specific form of nonlinearity, we obtained the wider stability region (2.234) represented in Fig. 2.34b.

Example 3. The nonlinear system is described by Eqs. (2.235):

$$\left.\begin{aligned}(T_1^2p^2+T_2p+1)x_1&=-k_1x_2,\\ x_2&=(k_2+k_3p+k_4p^2)x_1,\\ px_3&=F(x),\quad x=x_2-k_{o.c}x_3.\end{aligned}\right\} \tag{2.261}$$

A.M. Letov [63] considered such a system in another notation under the name of the second Bulgakov problem. In Chapter 4 of this book, using the direct Lyapunov method with the formulas of A.I. Lur'ye, he obtained the boundary of the system stability region in the form of a parabola for the same type as in Fig. 2.35a. Such a parabola was obtained still earlier by B.V. Bulgakov [45] using the small-parameter method. This parabola corresponds to sufficient stability conditions for any form of single-valued nonlinearity. Figure 2.35 shows the widening of the system's stability region obtained for specific forms of nonlinearities by the method of harmonic linearization. A similar enlargement was obtained by the small-parameter method in the work of B.V. Bulgakov, but in other, less suitable coordinates.

Example 4. In making the substitution (2.247) with the maximum possible value $q''_{max} = k_3$ in Example 4 of § 2.7, we have in fact replaced the specified system with a loop-type nonlinearity (2.211) by the following nonlinear system:

$$\left.\begin{array}{c}(T_1p+1)x_2=-k_1x_1,\\ x_3=F(x)-k_2px,\quad x=x_3-k_{o.c}x_1,\\ (T_2p+1)px_1=k_2x_2,\end{array}\right\} \tag{2.262}$$

where F(x) is a unique oddly-symmetrical nonlinearity. This substitution gives a margin in the stability determination, since the larger the value of k_3, the more will the presence of the quantity k_3 restrict the region of stability.

Adopting, as in Example 1, the notation

$$\eta_1=x_2,\quad \eta_2=x_1,\quad \eta_3=px_1,\quad f(\sigma)=F(x),\quad \sigma=x,$$

we rewrite Eqs. (2.262) in the form

$$\dot\eta_1=-\frac{1}{T_1}\eta_1-\frac{k_1}{T_1}\eta_2,\quad \dot\eta_2=\eta_3,$$
$$\dot\eta_3=-\frac{1}{T_2}\eta_3+\frac{k_2}{T_2}f(\sigma)-\frac{k_2k_3}{T_2}\dot\eta_1+\frac{k_2k_3k_{o.c}}{T_2}\eta_2.$$

Transforming the last equation by substitution of the expression $\dot{\eta}_1$ from the first equation, we arrive finally at the system of equations

$$\left.\begin{aligned} \dot{\eta}_1 &= -\frac{1}{T_1}\eta_1 - \frac{k_1}{T_1}\eta_2, \quad \dot{\eta}_2 = \eta_3, \\ \dot{\eta}_3 &= \frac{k_2 k_3}{T_1 T_2}\eta_1 + \frac{k_1 k_2 k_3}{T_1 T_2}\eta_2 - \frac{1 - k_2 k_3 k_{o.c}}{T_2}\eta_3 + \frac{k_2}{T_2} f(\sigma), \\ \sigma &= \eta_1 - k_{o.c}\eta_3. \end{aligned}\right\} \tag{2.263}$$

The roots of the determinant

$$D(\lambda) = \begin{vmatrix} -\frac{1}{T_1} - \lambda & -\frac{k_1}{T_1} & 0 \\ 0 & -\lambda & 1 \\ \frac{k_2 k_3}{T_1 T_2} & \frac{k_1 k_2 k_3}{T_1 T_2} & -\frac{1 - k_2 k_3 k_{o.c}}{T_2} - \lambda \end{vmatrix}$$

are

$$\lambda_2 = 0, \quad \lambda_1\lambda_3 = \frac{1 - k_2 k_3 (k_1 + k_{o.c})}{T_1 T_2}, \quad \lambda_1 + \lambda_3 = -\frac{T_1 + T_2 - T_1 k_2 k_3 k_{o.c}}{T_1 T_2},$$

here we assume $\lambda_1\lambda_3 > 0$ and $\lambda_1 + \lambda_3 < 0$, which is in agreement with Conditions (2.249).

Then by the formulas of A.I. Lur'ye, as in Example 1, we calculate

$$H_1 = -\frac{k_1 k_2}{T_1 T_2}, \quad H_2 = \frac{k_2}{T_2}\left(\frac{1}{T_1} + \lambda\right), \quad H_3 = \frac{k_2}{T_2}\lambda\left(\frac{1}{T_1} + \lambda\right),$$

$$D'(\lambda) = -3\lambda^2 - 2\frac{T_1 + T_2 - T_1 k_2 k_3 k_{o.c}}{T_1 T_2}\lambda - \frac{1 - k_2 k_3 (k_1 + k_{o.c})}{T_1 T_2},$$

$$\gamma_1 = -\frac{k_2 (k_1 + k_{o.c}) + T_1 k_2 k_{o.c}\lambda_1}{\lambda_1 (2T_1 T_2 + T_1 + T_2 - T_1 k_2 k_3 k_{o.c})},$$

$$\gamma_2 = -\frac{k_2 (k_1 + k_{o.c})}{1 - k_2 k_3 (k_1 + k_{o.c})}, \quad \gamma_3 = -\frac{k_2 (k_1 + k_{o.c}) + T_1 k_2 k_{o.c}\lambda_3}{\lambda_3 (2T_1 T_2 + T_1 + T_2 - T_1 k_2 k_3 k_{o.c})},$$

$$\beta_1 = \gamma_1\lambda_1, \quad \beta_2 = 0, \quad \beta_3 = \gamma_3\lambda_3.$$

After the rather cumbersome manipulations connected with exclusion of the quantities λ_1 and λ_3, we obtain

$$\Gamma^2 = \gamma_1 + \gamma_3 = \frac{k_2 (k_1 + k_{o.c})}{1 - k_2 k_3 (k_1 + k_{o.c})},$$

$$\vartheta = \frac{(\beta_1 - \beta_3)(\lambda_1 - \lambda_3)}{4\lambda_1\lambda_3} = -\frac{k_2 [2T_2 (k_1 + k_{o.c}) - k_{o.c} (T_1 + T_2 - T_1 k_2 k_3 k_{o.c})]}{4T_2 [1 - k_2 k_3 (k_1 + k_{o.c})]}.$$

The stability conditions: $\Gamma^2 > 0$ and $-4\vartheta < \Gamma^2$ assume the form

$$k_2 (k_1 + k_{o.c}) > 0, \text{ i.e. } k_{o.c} > -k_1 \text{ for } k_2 > 0, \tag{2.264}$$

$$k_{o.c} T_1 (1 - k_2 k_3 k_{o.c}) - k_1 T_2 > 0. \tag{2.265}$$

In both cases we assume that starting from the condition $\lambda_1\lambda_3 > 0$, we will have

$$1 - k_3 k_3 (k_1 + k_{o.c}) > 0.$$

As we see, the stability boundaries obtained here by a complex method agree fully with those obtained in Example 4 of § 2.7 by a simpler method – merely by differentiation of the Hurwitz determinant H_{n-1}.

Example 5. Using the method of Lyapunov, A.I. Lur'ye ([34], p. 84) obtains an example of the stability boundary for a nonlinear system described by the following equations:*

$$\left.\begin{aligned} T_a p\varphi &= -\xi, \\ (T_f^2 p^2 + T_k p + \delta)\eta &= \varphi, \\ p\xi = F(x), \quad x &= \eta - \xi. \end{aligned}\right\} \tag{2.266}$$

Let us solve this same problem by the method of harmonic linearization.

For a unique oddly-symmetrical nonlinearity F(x), we obtain after harmonic linearization the characteristic equation of the system (2.266) in the form

$$T_f^2 p^4 + (T_k + T_f^2 q) p^3 + (\delta + T_k q) p^2 + \delta q p + \frac{q}{T_a} = 0.$$

The next-to-last Hurwitz determinant is

$$H_{n-1} = \delta q [(T_k + T_f^2 q)(\delta + T_k q) - T_f^2 \delta q] - \frac{q}{T_a}(T_k + T_f^2 q)^2,$$

as a consequence of which Eq. (2.209) assumes the form

$$H_{n-1} = \left(T_k \delta^2 - \frac{T_k^2}{T_a}\right) q + T_k \left(T_k \delta - 2\frac{T_f^2}{T_a}\right) q^2 + T_f^2 \left(T_k \delta - \frac{T_f^2}{T_a}\right) q^3 = 0. \tag{2.267}$$

Equation (2.210) is

$$\frac{\partial H_{n-1}}{\partial q} = T_k \delta^2 - \frac{T_k^2}{T_a} + 2T_k \left(T_k \delta - 2\frac{T_f^2}{T_a}\right) q + 3T_f^2 \left(T_k \delta - \frac{T_f^2}{T_a}\right) q^2 = 0. \tag{2.268}$$

According to the positivity condition for the constant term of the characteristic equation, one of the stability boundaries will be the value q = 0 (for $T_a > 0$). For q = 0, the two Equations (2.267) and (2.268) are satisfied if

$$T_k\delta^2 = \frac{T_k^2}{T_a}. \tag{2.269}$$

For $q > 0$, Eq. (2.267) may be divided by q and instead of Eq. (2.268) we may write

$$\frac{\partial}{\partial q}\left(\frac{H_{n-1}}{q}\right) = T_k\left(T_k\delta - 2\frac{T_r^2}{T_a}\right) + 2T_r^2\left(T_k\delta - \frac{T_r^2}{T_a}\right)q = 0, \tag{2.270}$$

hence

$$q = \frac{T_k(2T_r^2 - T_aT_k\delta)}{2T_r^2(T_aT_k\delta - T_r^2)}. \tag{2.271}$$

Inasmuch as here $q > 0$, this expression for q may be made use of only given the condition

$$2T_r^2 > T_aT_k\delta > T_r^2. \tag{2.272}$$

Substituting the value of q from (2.271) into Eq. (2.267) divided by q, we obtain

$$4T_r^2(T_aT_k\delta - T_r^2) - T_aT_k^2 = 0. \tag{2.273}$$

We introduce the notation

$$\chi = \frac{T_k}{\delta^2 T_a}, \quad \psi = \frac{\delta T_r^2}{T_k^2}. \tag{2.274}$$

Then Condition (2.269) assumes the form

$$\chi = 1, \tag{2.275}$$

while Conditions (2.273) and (2.272) are

$$\chi = \frac{1}{\psi} - \frac{1}{4\psi^2} \text{ for } \frac{1}{2\psi} < \chi < \frac{1}{\psi}, \tag{2.276}$$

as is shown graphically in Fig. 2.38a. These conditions agree completely with the results of A.I. Lur'ye ([34], p. 84) and represent sufficient conditions for the stability of the system for any form of nonlinearity ($0 \leq q \leq \infty$).

Using the method of harmonic linearization we observe the following in the second stage of the analysis. Along the stability boundary, according to (2.271) and (2.276), the quantity q varies according to the relationship

$$\left.\begin{array}{ll} \frac{T_r^2}{T_k} q = 2\psi - 1 & \text{for } \psi > 0{,}5, \\ q = 0 & \text{for } 0 < \psi < 0{,}5. \end{array}\right\} \qquad (2.277)$$

If the quantity q has the upper bound q_{max} (all the nonlinearities in Fig. 2.31 except for Fig. 2.31a and b), then, adopting the notation

$$\varkappa = \frac{T_r^2}{T_k} q_{max} \qquad (2.278)$$

we obtain a bound for the variation of the quantity (T_r^2/T_k) q along the stability boundary as indicated in Fig. 2.38c, with the result that the previous sufficient boundary (Fig. 2.38a) will be a necessary one only for $0 < \psi < \frac{1+\varkappa}{2}$. For

$$\psi > \frac{1+\varkappa}{2}$$

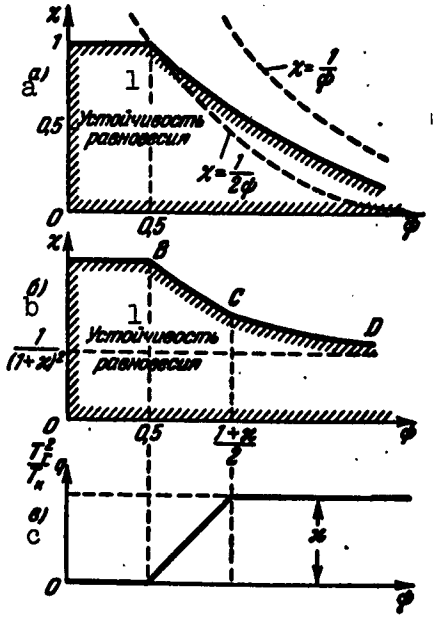


Fig. 2.38. 1) Equilibrium stability.

we must make direct use of substitution of the quantity q_{max} into Eq. (2.267), which, with the notation of (2.274) and (2.278), gives us

$$\chi = \frac{\varkappa}{1+\varkappa} \cdot \frac{1}{\psi} + \frac{1}{(1+\varkappa)^2},$$

where, according to (2.278) and Fig. 2.31, the quantity κ has its own special value for each nonlinearity in question. This new boundary CD is represented in Fig. 2.38b. It enlarges the stability region in accordance with the form of the nonlinearity.

Example 6. Let us introduce still one more example of a nonlinear system considered in the work of A.I. Lur'ye ([34], p. 87) by the Lyapunov method:

$$\left.\begin{aligned}(p^2+2np+\omega^2)\,\eta &= -N\xi,\\ p\xi &= F(x),\quad x=(j_0+j_1p)\,\eta-r\xi.\end{aligned}\right\} \tag{2.279}$$

Let us solve the same problem here by the method of harmonic linearization. After harmonic linearization, the characteristic equation of System (2.279) will be

$$p^3+(2n+rq)\,p^2+(\omega^2+2nrq+j_1Nq)\,p+(\omega^2r+j_0N)\,q=0.$$

The next-to-last Hurwitz determinant will be

$$H_{n-1}=(2n+rq)(\omega^2+2nrq+j_1Nq)-(\omega^2r+j_0N)\,q. \tag{2.280}$$

A.I. Lur'ye considers two cases: a) r = 0; b) $\omega^2 = 0$.

According to (2.280), Eqs. (2.209) and (2.210) are in the case r = 0:

$$H_{n-1}=2n\omega^2+(2nj_1-j_0)\,Nq=0, \tag{2.281}$$

$$\frac{\partial H_{n-1}}{\partial q}=(2nj_1-j_0)\,N=0. \tag{2.282}$$

Hence we obtain the stability boundary in the form

$$nj_1=\frac{1}{2}j_0 \tag{2.283}$$

while from the condition of positive constant term in the characteristic equation (for N > 0 and q > 0)

$$j_0>0. \tag{2.284}$$

These two boundaries are represented in Fig. 2.39 in the form of the straight lines 1 and 2. They agree completely with the results that A.I. Lur'ye obtained by the Lyapunov method.

Let us perform an additional analysis. According to (2.281), we have on the stability boundary (2.283) the value

$$q=\frac{2n\omega^2}{(2nj_1-j_0)\,N}=\infty.$$

If, however, the value of q is bounded from above by the quantity q_{max} (Fig. 2.31), then it is this value that must be substituted in Eq. (2.281) in place of q = ∞ in order to obtain the stability boundary; this gives us

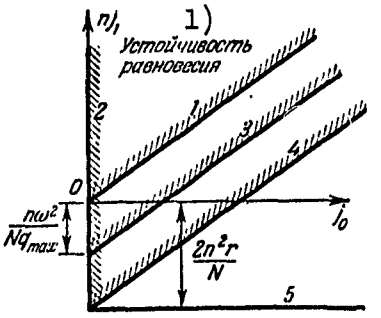


Fig. 2.39. 1) Equilibrium stability.

$$nj_1 = \frac{j_0}{2} - \frac{n\omega^2}{Nq_{max}}. \qquad (2.285)$$

Thus, the stability region is enlarged (Fig. 2.39, line 3) in accordance with the quantity q_{max}, which depends upon the form of the nonlinearity. As a particular case, we may obtain from this the result (cited in the book by A.I. Lur'ye) for a linear system when q = const = $= q_{max} = 1/T_s$.

Finally on the basis of (2.280), Eqs. (2.209) and (2.210) in the case $\omega^2 = 0$ will be

$$H_{n-1} = (4n^2r + 2nj_1N - j_0N)q + r(2nr + j_1N)q^2 = 0, \qquad (2.286)$$

$$\frac{\partial H_{n-1}}{\partial q} = 4n^2r + 2nj_1N - j_0N + 2r(2nr + j_1N)q = 0. \qquad (2.287)$$

Both equations are satisfied for q = 0 if

$$4n^2r + 2nj_1N - j_0N = 0$$

or

$$nj_1 = \frac{j_0}{2} - \frac{2n^2r}{N} \qquad (2.288)$$

(see Fig. 2.39, line 4); this also agrees with the results of A.I. Lur'ye. If, however, $q \neq 0$, then Eq. (2.286) may be divided by q and instead of (2.287) we may write:

$$\frac{\partial}{\partial q}\left(\frac{H_{n-1}}{q}\right) = r(2nr + j_1N) = 0,$$

which, however, is not essential, since the value obtained from this

$$nj_1 = -\frac{2n^2r}{N}$$

determines a straight line lying outside the stability region resulting from (2.288).

CONCLUSION. Thus we see that in all cases, the results of application of the Lyapunov method confirm the validity of determining

the stability region for nonlinear systems (of the types being considered) by the method of harmonic linearization in the first stage. The formulas of A.I. Lur'ye, which are based upon the Lyapunov method, give us sufficient stability conditions that guarantee stability of the system for any form of unique oddly-symmetrical nonlinearity. They also prove to be necessary conditions if all values of q ($0 \leq q \leq \infty$) are possible for the given nonlinearity. Otherwise the necessary conditions for stability are wider. However, for the presence of a loop-type nonlinearity or a nonlinearity of the more general form $F(x, px)$ the region of stability may be narrowed down. The method of harmonic linearization permits us to find these enlargements and reductions of the region of stability for each specific given nonlinearity. In addition, with application of the Mikhaylov criterion, the method of harmonic linearization also permits us to find the region of stability for more complex nonlinear systems of the classes being considered here (see §§ 2.2-2.4), and also the examples in Chapters 4 and 6).

The facts which have been set forth enable us to hope that there is a possibility of the development of some rigorous theory of the stability for the first approximation, not by means of the Taylor-series expansion, but by means of the Fourier-series expansion, i.e., based not upon ordinary, but upon harmonic linearization.

§ 2.9. DETERMINATION OF SYSTEM STABILITY OVER A LIMITED RANGE OF INITIAL CONDITIONS

In our consideration of the delineation of equilibrium-stability regions in § 2.7, we had in mind the stability of an equilibrium state of the system for arbitrary initial conditions. Thus we presupposed the presence of a unique equilibrium state of the system (or of a unique equilibrium zone). Generally speaking, however, a

nonlinear system may have two or more equilibrium states whose regions of stability or instability for all given system parameters are separated by definite relationships of the initial conditions (i.e., initial deviations of the variables and their derivatives or the initial oscillation amplitudes). Examination of such systems may often be reduced to examination of systems with a unique equilibrium state if we determine beforehand the nonlinear characteristics in the neighborhood of each of these states individually, together with the resulting restriction of the limits of variation of these variables. Then we may use the stability-analysis methods of which an account was given in § 2.7 spearately for each of these equilibrium states; here, however, we speak not of the system's stability for arbitrary initial conditions, but of the stability of a given equilibrium state of the system in a restricted region of initial conditions determined by the limits of variation of the variables for which the equations written for oscillations of the system about the equilibrium state being considered are valid.

By this we characterize one concept of system stability over a restricted range of initial conditions. We shall not return to it again; in what follows, we shall assume in all cases that the expression "stability of the system for any initial conditions" (§ 2.7) will always be taken with the implied qualification "for limits within which the investigated dynamic system equations expressing the motion of the system in the neighborhood of the given equilibrium state are valid" (such a restriction may be dictated not only by the above, but also by other causes).

Another concept of system stability over a restricted range of initial conditions requires special analysis and consists in the following.

We have encountered cases where the equilibrium state of a nonlinear system is stable for restricted (sufficiently small) initial deviations, while at the same time for larger initial deviations in the system, the transient process is found to be divergent, in spite of the fact that the same dynamic system equations remain valid with the same coefficients (system parameters) and the same nonlinearity. For a bounded range of initial conditions, such a form of system stability will not be detected directly by the methods of which an account is given in § 2.7 (which determine the stability region in the space of the system parameters for arbitrary initial conditions). The stability region considered here must be delineated either in a space of initial conditions for all the given system parameters, or in the total space of initial conditions and system parameters. In the majority of instances we will be oriented toward the latter case, which is the most general and presents major practical interest for the synthesis problem, i.e., for the choice of the most advantageous parameters for the design of a nonlinear automatic system.

The presence of a single unstable periodic solution is a typical case of bounding of the range of initial conditions for which an equilibrium state of the system is stable. Indeed, if the initial amplitude a_0 is less than the amplitude A found for the unstable periodic solution, then the transient process is damped (Fig. 2.17b), i.e., an equilibrium state of the system is stable "in the small." If, however, the initial amplitude is larger than A, then the process diverges, i.e., the system is unstable "in the large" (if there are no other singularities in addition to the periodic solution indicated, for example if there is no second, stable periodic mode of operation). In particular, for nonlinear systems of the second order this form

of limitation of the stability region in the plane of the initial conditions is represented by an unstable limiting cycle corresponding to an unstable periodic solution (Fig. 2.40a).

In the case where there is a second periodic solution that is stable with a larger amplitude A_2 (Fig. 2.40b), then for large deviations the previous system instability is replaced by stable self-oscillations with the amplitude A_2, represented by a second limiting cycle. However, if the amplitude A_2 is inadmissibly large with respect to the operating conditions of the system in question, then in practice this will be equivalent to instability of the system for initial conditions transcending the limitations of the first limiting cycle (instability "in the large"). In the case where the amplitude A_2 is not large and is not dangerous (or perhaps even desirable) for operation of the system in question, while at the same time the amplitude of the first cycle A_1 (Fig. 2.40b) is very small, in a practical sense the system will be a workable self-oscillatory system. Here an excessively small stability region within the first cycle will not be of practical importance, since the perturbations actually acting upon the system will always take its system out of this range. This is a case of "hard" excitation of self-oscillations taking place as a result of an initial "jolt" of finite magnitude, in contrast to "soft" excitation in the presence of a unique stable periodic solution (Fig. 2.40c), and corresponds to the pattern of the processes in Fig. 2.17a.

There is also the possibility of self-oscillation stability for bounded initial conditions, i.e., within an unstable limiting cycle (Fig. 2.40d) beyond whose limits there is a region of system instability "in the large". Here practical treatment of system behavior depends upon the amplitudes of the two periodic solutions. If the

amplitude A_1 (Fig. 2.40d) is sufficiently large so that actual perturbations are not able to take the system beyond its limits, while the amplitude A_2 is admissible for the operation of the system in question, then the system in question is workable from a practical point of view. If one of these conditions is not satisfied, then we must reject the system, considering it unstable from a practical point of view.

In exactly the same fashion, we may consider a system with one unstable periodic solution (Fig. 2.40a) to be workable from a practical point of view where the amplitude A is sufficiently large. Otherwise the system will be unstable from a practical point of view.

Here we have spoken of system efficiency in the usual practical sense. However, there is the possibility of a special problem in which we may make use of the passage of the system into an unstable state for large deviations in order to bring it into some completely new state. Such a problem may be encountered in the technique of self-

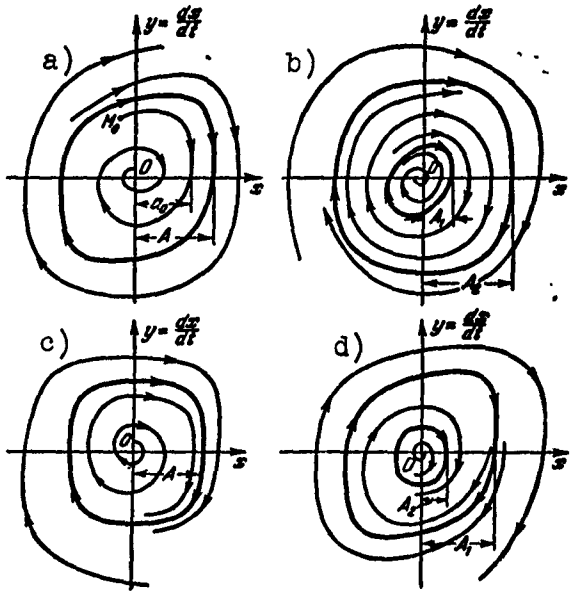


Fig. 2.40.

adaptive automatic systems and in computer engineering. Then by choos-

ing the system parameters we may attain a given value of A for the boundary of the initial conditions beyond which the required system instability appears.

In speaking of the determination of any range of initial conditions, we should always visualize a phase space, the number of whose dimensions is equal to the order of the system. However, in the case of the presence of a periodic solution, this representation may be simplified. Thus, for example, for a second-order system, instead of giving two initial conditions (x_0, y_0) it is sufficient to give one quantity a_0, which we shall call the initial amplitude (Fig. 2.40a and 2.17b), since it is evident that if $a_0 < A$, then the values x_0, y_0 (the point M_0 in Fig. 2.40a and 2.17b) must lie within the range bounded by the limiting cycle, irrespective of the location of the initial point M_0 of the transient process. We shall also make use of this same concept of the initial amplitude for systems of high order in the method of harmonic linearization, although there the pattern of the processes is significantly more complex. This circumstance is based upon the property of high stability described by N.N. Bogolyubov ([102], Chapter IV) and illustrated above on page 155.

In addition, in determining system stability in a bounded region of initial conditions by means of an unstable periodic solution, we must recall that, generally speaking, other restrictions to such a region, for example, in the form of a separatrix, are also possible; however, we shall not be concerned with these restrictions, since they are less essential for the automatic systems being considered.

In § 2.7, we separated the parameter space for a nonlinear system (basically confining ourselves to positive values of the parameters, in accordance with their physical significance) into three regions: the region of system stability for arbitrary initial

conditions, the region of existence of periodic solutions and the region of system instability for arbitrary initial conditions. In the region of existence of the periodic solutions, we shall now determine either self-oscillations or system stability over a restricted range of initial conditions, or the combinations of them described above. Here we shall define the convergence and divergence of oscillatory transient processes for various initial conditions in the region of existence of periodic solutions. In accordance with what has been said above, we will characterize the ensemble of initial conditions by the value of the initial amplitude a_0 alone. Such an approach to solving the problem in first approximation is sufficient for many engineering calculations, particularly in the first design stage of a nonlinear automatic system.

Thus, for example, in the presence of one unstable periodic solution (Fig. 2.17b and 2.40a), the space of the initial conditions (which we may interchange with the axis of the amplitudes a) is divided into two regions: regions of converging and diverging oscillations, as shown in Fig. 2.41a. Here the oscillations may converge not necessarily at the point 0, but at any point within the dead zone in the neighborhood of point 0, the half-width of which we shall denote by the letter b. In the absence of this zone we have b = 0. It is convenient to denote convergence and divergence of the processes by arrows (Fig. 2.41a). In Fig. 2.41 we have also shown the breakdown of the axis of the initial conditions (the amplitude axis) into regions of converging and diverging oscillations for all remaining patterns of the processes represented in Fig. 2.40. Here the arrows indicate, so to speak, the regions of attraction of the nonsteady solutions to steady (equilibrium or self-oscillatory) solutions.

It is evident that the position determined for point A (the amplitude of the periodic solution) in Fig. 2.41a corresponds to the fixing of all the system parameters. Therefore each diagram of the type of Fig. 2.41 corresponds to a definite point of the parameter space of the nonlinear system in question. Consequently, if we add one new dimension to the coordinate axes of the parameter space of the system – the axis of initial conditions (axis of the amplitudes a) – then we obtain the total space of the parameters and the initial conditions, in which it is convenient to represent the regions of attraction (including the regions of system stability over a bounded range of initial conditions) in the presence of periodic solutions. This is how we should approach analysis of the range of system parameters in which periodic solutions exist.

Let us assume that for some nonlinear system we have delineated

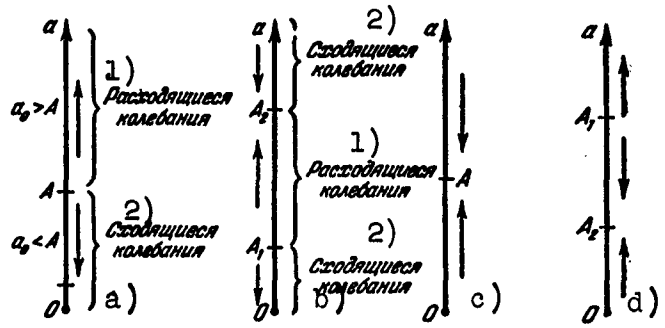


Fig. 2.41. 1) Diverging oscillations; 2) converging oscillations.

(§ 2.7) a region of system stability and a region of existence of periodic solutions (Fig. 2.42a) with respect to any one parameter k (for example, the gain constant), and that these are separated by the "critical" value k_{kr}. Then suppose that we have found a unique stable periodic solution (§§ 2.3 and 2.4) for which the dependence of the amplitude A upon the parameter k is shown in Fig. 2.42b.

Then on the plane (in the general case, in the space) with the coordinate axes k (system parameters) and a (amplitudes of the tran-

sient processes), we may construct (Fig. 2.42c) the boundary dividing it into the two regions indicated above (the dashed line). The line a = A divides the region of existence of periodic solutions into two regions: converging processes (above the line a = A, i.e., for initial amplitudes $a_0 > A$), and diverging processes (below the line a = A, i.e., for $a_0 < A$). Let us denote this by arrows on the diagram (Fig. 2.42c), the sense of which will be apparent without any legends. A similar arrow may also be placed in the region of stability for arbitrary initial conditions ($k < k_{kr}$). The pattern shown in Fig. 2.42c corresponds to Figs. 2.17a, 2.40c, and 2.41c. A similar pattern is shown in Fig. 2.43a for processes of the type of Fig. 2.17b, 2.40a, and 2.41a (for $k_{kr_1} < k < k_{kr_2}$), while in Fig. 2.43b it is shown for Fig. 2.40b and 2.41b (for $k > k_{kr}$). As is evident,

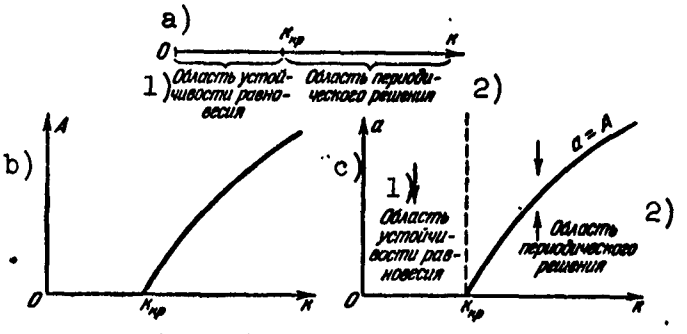


Fig. 2.42. 1) Region of equilibrium stability; 2) region of periodic solution.

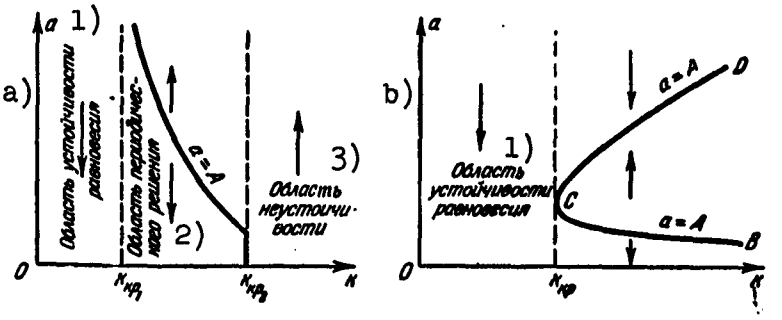


Fig. 2.43. 1) Region of equilibrium stability; 2) region of periodic solution; 3) region of instability.

the unstable periodic solution is the boundary of the regions of attraction.

Inasmuch as the methods of delineating the regions of system stability in the parameter space and the methods of determining periodic solutions and analysis of their stability have already been described in preceding sections, it remains now only to set forth a method for determining the regions of attraction, i.e., the regions of convergence and divergence of the transient processes within the region of existence of periodic solutions for the various initial conditions (as a particular case we also have here the delineation of the range of initial conditions of the type of Fig. 2.43a) for which the equilibrium state of the system is stable. The basis for the method will be the same as in § 2.7.

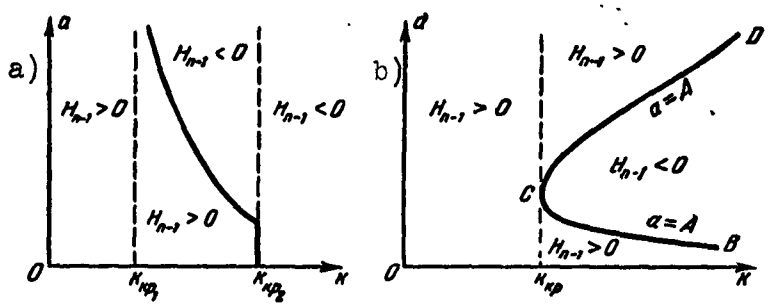


Fig. 2.44.

First of all, let us assume that over the whole region of system parameters and initial amplitudes which is being investigated, we have observed all the conditions for applicability of the method of harmonic linearization, which were established in § 2.2 and briefly formulated at the beginning of § 2.3. In addition, we assume, as in §§ 2.4 and 2.7, that over this entire range all the Hurwitz determinants remain positive for the characteristic equation of the harmonically linearized system, except for the next-to-last H_{n-1}, or, what is the same thing, the Mikhaylov or Hurwitz criterion is

fulfilled for a polynomial of reduced degree (2.118). For systems of order no higher than the fourth, this is equivalent to having the coefficients of the characteristic equation positive over the entire region of system parameters and initial amplitudes which is being investigated. Here we must keep in mind that the amplitude enters into the characteristic equation only through the coefficients of harmonic linearization q and q'. Hence, the expression "over the whole range of initial amplitudes being investigated" in application to the characteristic equation of the harmonically linearized system must necessarily be understood as the expression: "for all values of the coefficients q and q' possible for the nonlinearity in question."

All this guarantees that the real parts of all roots of the characteristic equations of a harmonically linearized system except for one pair of complex roots will be negative. The latter are reduced to a pair of purely imaginary roots on the line a = A (Fig. 2.43) determining the amplitude of the periodic solution, where the next-to-last Hurwitz determinant $H_{n-1} = 0$. Away from this line a = A, either $H_{n-1} > 0$ or $H_{n-1} < 0$. In the former case, the pair of imaginary roots acquires a negative real part and the oscillatory transient process is damped, while in the second case the pair of roots acquires a positive real part, which attests to the presence of diverging oscillations. Thus, for example, the distribution of the signs of the next-to-last Hurwitz determinant H_{n-1} shown in Fig. 2.44 corresponds to the pattern of processes shown in Fig. 2.43. Here it is assumed, of course, that all the remaining Hurwitz determinants are everywhere positive.

Such is a method for evaluating the convergence and divergence of oscillations (determining the regions of attraction) in a non-

linear automatic system subject to calculation by the method of harmonic linearization, for various initial conditions expressed in the form of values of the initial amplitude. This method is easily translated from the language of the Hurwitz criterion to the language of the Mikhaylov criterion; this is essential for systems of high order, particularly in complex problems and in problems with a time lag.

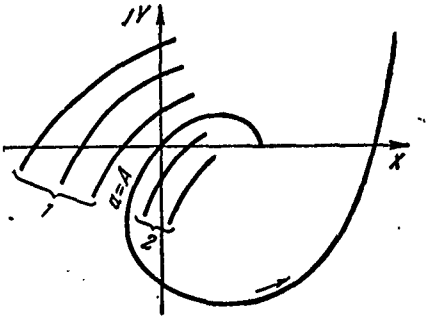


Fig. 2.45.

Passage of the Mikhaylov curve through the origin (Fig. 2.45), corresponds to each point on the line a = A (Fig. 2.44); this is equivalent to the condition $H_{n-1} = 0$. The corresponding contour of the rest of the Mikhaylov curve (it must pass around the origin in a clockwise direction, traversing the necessary number of quadrants) guarantees that the remaining Hurwitz determinants will be positive. If due to the variation of the coefficients q and q', the Mikhaylov curve is shifted into position 1 (Fig. 2.45), which is equivalent to $H_{n-1} > 0$, for all $a > A$, while for all $a < A$ it is shifted into position 2, which is equivalent to $H_{n-1} < 0$, then in the first case the processes are damped, while in the second case they diverge. Thus we obtain here the pattern of processes represented in Fig. 2.42c (for $k > k_{kr}$). In the contrary case we obtain a pattern of the type of Fig. 2.43a ($k_{kr_1} < k < k_{kr_2}$). In the case shown in Fig. 2.43b, however, the Mikhaylov curve should be shifted into position 1 (Fig. 2.45) for all values of a lying under the lower curve a = A and above the upper curve a = A, and should be shifted to position 2 for all values of a lying between the curves a = A.

From consideration of these methods for estimating the convergence and divergence of transient processes for various initial conditions and in various regions of values of the system parameters, we may draw certain conclusions important from the viewpoint of simplification of the engineering calculations.

All the stability criteria set forth in § 2.4 for the periodic solution (other than the method of averaging periodic coefficients), have essentially the same foundation as the method of §§ 2.7 and 2.9, the only difference being that in § 2.4 we investigated the behavior of the system only in a small neighborhood of a periodic solution. Let us assume that by the method of § 2.3 we have found the amplitude of the periodic solution as a function of the system parameters, for example in the form of Fig. 2.42b, and that we have established its stability by the method of § 2.4. We may consider the pattern of convergence and divergence of the processes, including the determination of the stability of an equilibrium state (Fig. 2.42c), that has been obtained in the example under consideration, as a simple extension of the results of the stability analysis of a periodic solution to the entire ($\underline{k}$, $\underline{a}$)-plane. This follows from the fact that the direction indicated in Fig. 2.42c for the arrows near the line $a = A$ is determined directly in the stability analysis of the periodic solution. Now they are only extended without interruption to all initial conditions within the region of existence of the periodic solution, and also beyond the limits of this region (to the left of the point $k = k_{kr}$, Fig. 2.42c), where we obtain the region of equilibrium stability for the system. Also, in exactly the same fashion, all the arrows shown in Fig. 2.43 may be treated as a transfer of the results of stability analysis of the periodic solution onto the entire plane of the parameters and initial conditions

[49]. It goes without saying that for such an extension of the results, we must be certain of the observance of all conditions of which an account is given in the present section.

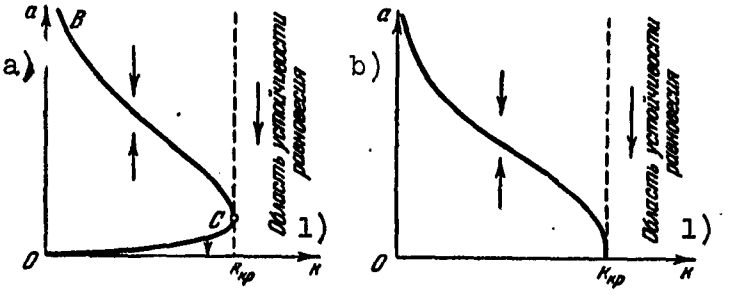


Fig. 2.46. 1) Region of equilibrium stability.

We may also proceed otherwise. After determination of the periodic solutions (§ 2.3), we may pass directly, without investigating their stability, to the investigations of the entire space of the parameters and initial conditions of which an account is given in §§ 2.9 and 2.7. Here the stability and instability of the periodic solutions are ascertained of themselves, without auxiliary analyses.

We will not consider nonlinear systems of another kind rarely encountered in automation, where the dividing lines between the various types of processes do not correspond to the periodic solutions, but are determined otherwise (separatrices, etc.).

All the analyses which have been described are based according to § 2.2, upon the fact that the coefficients of harmonic linearization $\underline{q}$ and $\underline{q}'$ vary smoothly enough with variation of the amplitude, i.e., the derivatives dq/da and dq'/da are bounded. In places of sharp variation of $\underline{q}$ and $\underline{q}'$, the method recounted may yield invalid results. In certain specific problems, for example, instead of a result obtained in the form of Fig. 2.46a, the exact solution gives a result of the type of Fig. 2.46b, i.e., the unstable periodic solution (the line OC) and the region under it are found to be spurious.

However, this does not have essential practical significance, since here the ordinate of the point C is found small, and, consequently, the entire line OC lies so low that in practice it may be neglected. However, the result obtained by the method recounted here for segments of smooth variation of q and q' (which gives the line BC, Fig. 2.46a), is reliable and is the one that will determine the behavior of the nonlinear system in question.

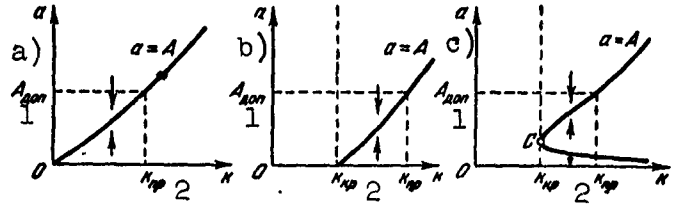


Fig. 2.47. 1) A_{dop}; 2) K_{pr}.

Let us dwell further upon the separation of the stability boundaries of nonlinear systems into those which are dangerous and those which are safe. A dangerous stability boundary is a boundary such that on it an equilibrium state of the system still remains stable over a sufficiently large range of initial conditions (for example, $k = k_{kr_1}$ in Fig. 2.43a), or, on the other hand, one such that in its neighborhood stable self-oscillations with a sufficiently small magnitude appear (for example, Fig. 2.42c). Figures 2.43b and 2.46a may also belong to the latter case if the ordinate of the point C is sufficiently small. If, however, it is so large that the amplitude of the stable self-oscillations becomes inadmissible for the system in question, then the stability boundaries in Figs. 2.43b and 2.46a will be dangerous.

In those cases where the nonlinearity is such that for small deviations the system is described by linear equations, we may speak of safe or dangerous boundaries determined from the linear equations for the system in question. Below in § 4.4 we consider an example of

this type with allowance for the generation of self oscillations.

Finally, let us give a further quantitative expression for the concept of practical stability of self-oscillatory systems [87], of which we have already spoken above in general form. Closed-loop automatic systems which operate at all times in the self-oscillatory mode are often encountered in practice. For example, vibrational voltage controllers, vibrational accelerometers, certain gyro stabilizers with relay control, etc. belong to this class of systems. In general, a stable equilibrium state may be lacking in such systems, and only a stable self-oscillatory mode of operation (Fig. 2.47a) will be possible. Then the stability region of the system must in practice be understood as a parameter region such that in it the self-oscillation amplitude does not exceed the value A_{dop} which is admissible for the system in question (the region $0 < k < k_{pr}$ in Fig. 2.47a). However, such a practical stability boundary k_{pr} may also be introduced into the consideration in those cases where a stability region exists for an equilibrium state (Fig. 2.47b and c), but the system must or is forced to operate in a self-oscillatory mode.

[Footnotes]

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117 *A departure from the case of an even function leads to nonsymmetrical oscillations, which we do not consider in the present chapter.

125 *This corresponds to the content of the present chapter, where we seek symmetrical self-oscillations; in Chapter 5 we shall consider problems in which the requirement (2.56) is not realized.

135 *We also assume that all the remaining roots of Eq. (2.79), other than the one pair of purely imaginary roots indicated, have negative real parts (in order to exclude the possibility of internal resonance, etc.).

138 *If, of course, all the conditions indicated at the beginning of the present section are realized here it is mandatory that they be verified.

138 *This applies with equal force to all methods for solving the problems.

140 *Instead of the gain-phase frequency characteristics $W_{\underline{1}}$, we may make use of logarithmic characteristics; this is not considered in the present book, although it would be useful in a number of cases.

153 *In what follows we shall consider only cases where as a result we obtain the curve of some near-sinusoidal oscillatory process, but with varying amplitude (nonsteady oscillations).

158 *The second of Expressions (2.119) may prove suitable, since $\underline{a}$ appears in H_{n-1} not directly, but in the form of the coefficients $\underline{q}$ and $\underline{q}'$, which are functions of $\underline{a}$ (and sometimes also of Ω).

164 *This is valid only for systems of the first class with an equation of the type (2.115).

171 *For another case, see Chapter 7.

176 *In the work of Krylov and Bogolyubov in place of sin Ψ they write cos Ψ, which is unimportant.

202 *The boundary between the region of equilibrium stability and the region of the periodic solutions, i.e., the region of nondamped oscillations.

205 *We assume that here all the remaining Hurwitz determinants

are positive (for systems as high as the fourth order inclusive, this means simply that the coefficients of the characteristic equation are positive).

220 *By this we ensure the absence of roots with a positive real part in the characteristic equation of the open linear part (2.80), as was stipulated in § 2.3.

229 *The notation $f(\sigma)$ used by A.I. Lur'ye is replaced here by $F(x)$.

[List of Transliterated Symbols]

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105	э = e = ekvivalentnyy = equivalent
105	м = m = mgnovennaya = instantaneous
112	н = n = nelineynost' = nonlinearity
123	в = v = vremya = time
129	л = l = lineynaya = linear
148	э = ė = eksperimental'naya = experimental
149	т = t = teoreticheskaya = theoretical
149	т.н = t.n = teoreticheskaya nelineynaya = theoretical nonlinear
149	т.л = t.l = teoreticheskaya lineynaya = theoretical linear
152	о.с = o.s = obratnaya sviaz' = feedback
242	кр = kr = kriticheskiy = critical
248	доп = dop = dopustimoye = admissible
248	пр = pr = prakticheskiy = practical

Chapter 3

HARMONIC LINEARIZATION OF NONLINEARITIES IN THE PRESENCE OF SYMMETRICAL OSCILLATIONS

In the case of symmetrical oscillations, harmonic linearization of the nonlinear function F(x, px) reduces to its replacement by the expression (see §2.1)

$$F(x, px)=\left[q(A, \Omega)+\frac{q'(A, \Omega)}{\Omega}p\right]x. \tag{3.1}$$

Here we assume that the solution for the input quantity of the nonlinear link is sought in the harmonic form

$$x=A\sin\psi, \quad \psi=\Omega t$$

and that we take into account only the first harmonic in the Fourier expansion of the periodic function of the argument ψ:

$$F(x, px)=F(A\sin\psi, A\Omega\cos\psi).$$

In this case the harmonic-linearization coefficients are determined from the formulas

$$\left.\begin{aligned} q(A, \Omega)&=\frac{1}{\pi A}\int_0^{2\pi}F(A\sin\psi, A\Omega\cos\psi)\sin\psi\, d\psi,\\ q'(A, \Omega)&=\frac{1}{\pi A}\int_0^{2\pi}F(A\sin\psi, A\Omega\cos\psi)\cos\psi\, d\psi. \end{aligned}\right\} \tag{3.2}$$

If the nonlinear function does not depend upon the rate of change of the input, then the harmonic linearization coefficients will be functions only of the oscillation amplitude and Formula (3.2) assumes the form

$$\left.\begin{aligned} q(A) &= \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi)\sin\psi\, d\psi, \\ q'(A) &= \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi)\cos\psi\, d\psi. \end{aligned}\right\} \qquad (3.3)$$

As has already been indicated (§2.1), the same methods for the harmonic linearization of nonlinearities are also applicable where the nonlinear function is also a function of the input acceleration or where both the input and the output of the nonlinear link appear in the nonlinearity (nonlinear system of the second class). This will be shown here in specific examples.

Let us find expressions for the harmonic-linearization coefficients for symmetrical oscillations in nonlinear systems for the nonlinearities which are most frequently encountered. In practice, the presence of complete expressions for the coefficients q and q' significantly facilitates analysis of nonlinear systems by the method of harmonic linearization.

§3.1. RELAY CHARACTERISTICS

By a relay characteristic we mean a variation of the output of a nonlinear link as a function of the input in which at certain fully defined values of the input the output changes stepwise, while for all other values it remains constant.

General form of relay characteristic. The general form of the relay characteristic is shown in Fig. 3.1a. For example, the voltage u_d applied to a motor varies as such a nonlinear function of the control current i_u of the relay (Fig. 3.1c) if the motor is controlled by two neutral electromagnetic relays and is wired as a dynamic-braking* circuit. According to convention, positive and negative values of the current correspond to the connection of one relay or the other for the case of mutually opposite displacements α of

the control element. The different signs of the voltage applied to the motor indicate that the motor is connected for forward or backward rotation. Smoothness of the current variation in the control windings of the relay results from the inductance of the control circuit, while the hysteresis loops are governed by the differing operate and dropout currents of the relay.

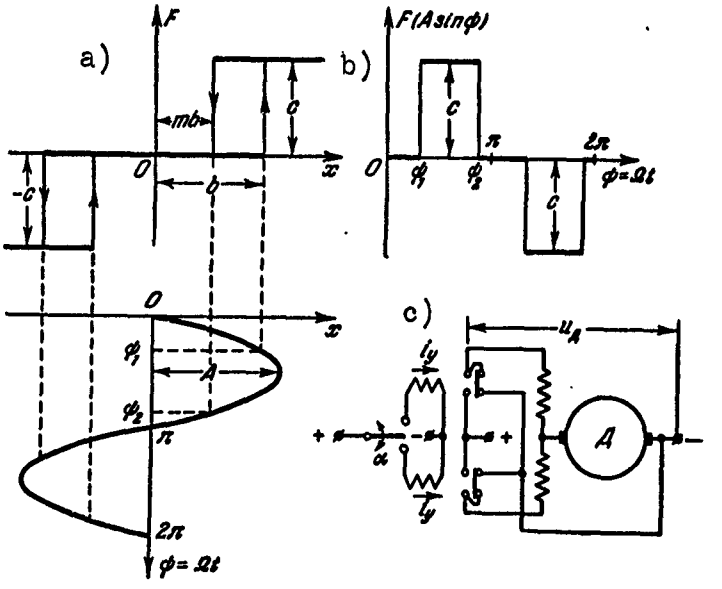


Fig. 3.1.

In order to carry out harmonic linearization of the nonlinear characteristic indicated, we assume that the solution for x is sought in the form

$$x = A\sin\psi, \quad \psi = \Omega t. \tag{3.4}$$

Then in accordance with the curve of F(x) we obtain the periodic function F(A sin ψ) with the argument $\psi = \Omega t$ (Fig. 3.1b), whereby the values of the agrument ψ_1 = arcsin b/A and $\psi_2 = \pi$ – arc sin mb/A will correspond to the switching points x = b and x = mb, respectively of the relay. The return constant of the relay m may be any fraction in the interval

$$-1 \leqslant m \leqslant 1.$$

For an amplitude $A \leq b$ the relays are cut out, i.e., there is no signal transmission in the system. Therefore we will determine the coefficients of harmonic linearization q and q' for the condition $A \geq b$.

In the case being considered the integrals in Formula (3.3) will have equal values for each half-period, so that it is sufficient to consider

$$\int_0^{\pi} = \int_0^{\psi_1} + \int_{\psi_1}^{\psi_2} + \int_{\psi_2}^{\pi}.$$

The first and third integrals are equal to zero, since over these intervals $F(A \sin \psi) = 0$. Then the first coefficient of harmonic linearization will be

$$q(A) = \frac{2}{\pi A} \int_{\psi_1}^{\psi_2} F(A \sin \psi) \sin \psi \, d\psi = \frac{2}{\pi A} \int_{\psi_1}^{\psi_2} c \sin \psi \, d\psi =$$

$$= -\frac{2c}{\pi A} \cos \psi \Big|_{\psi_1}^{\psi_2} = \frac{2c}{\pi A} (\cos \psi_1 - \cos \psi_2).$$

Allowing for the values of ψ_1 and ψ_2, we write the formula for the determination of q(A) in the form

$$q(A) = \frac{2c}{\pi A} \left(\sqrt{1 - \frac{b^2}{A^2}} + \sqrt{1 - \frac{m^2 b^2}{A^2}} \right) \text{ for } A \geq b. \tag{3.5}$$

calculating the coefficient q'(A), we obtain

$$q'(A) = \frac{2}{\pi A} \int_{\psi_1}^{\psi_2} F(A \sin \psi) \cos \psi \, d\psi = \frac{2}{\pi A} \int_{\psi_1}^{\psi_2} c \cos \psi \, d\psi =$$

$$= \frac{2c}{\pi A} \sin \psi \Big|_{\psi_1}^{\psi_2} = -\frac{2c}{\pi A} (\sin \psi_1 - \sin \psi_2),$$

or, allowing for the values of ψ_1 and ψ_2,

$$q'(A) = -\frac{2cb}{\pi A^2} (1 - m) \text{ for } A \geq b. \tag{3.6}$$

The relay characteristics shown in Fig. 3.2 may be considered as particular cases of the general relay-characteristic form. The

dependence shown in Fig. 3.2a may, for example, characterize the variation of the voltage applied to the appliance connected by a three-position polarized relay as a function of the control current. The characteristic of Fig. 3.2b corresponds to a two-position polarized relay. For example, the position of limit switches will vary according to the characteristic of Fig. 3.2c as a function of drive displacement. We obtain the characteristic of Fig. 3.2d if the control relay in the scheme of Fig. 3.1c has a large return constant. We may obtain the ideal relay characteristic of Fig. 3.2e for the case where the consuming element is controlled from two contacts, as, for example, in the case of control of the motor according to the scheme shown in Fig. 3.2f. Here the input is the displacement $\underline{x}$ of the movable contact of K_1. The induction motor has two control windings, which are so connected that the magnetic fluxes which they form are opposed. One of the control windings is permanently connected to the line through an adjusting resistance R_d. The second winding is connected by closing the contacts. We choose the resistance R_d such that the magnetic flux of the first winding will be half the flux of the second. As a result, the torque developed by the motor will in practice vary according to the ideal relay characteristic as a function of the displacement $\underline{x}$ of the movable contact.

We must keep in mind that the relay has time lags governed by the time of motion of the armature; these are not allowed for in the characteristics being considered and are subject to additional consideration.

Let us determine the harmonic-linearization coefficients for the characteristics of Fig. 3.2 as particular cases of the general characteristic form.

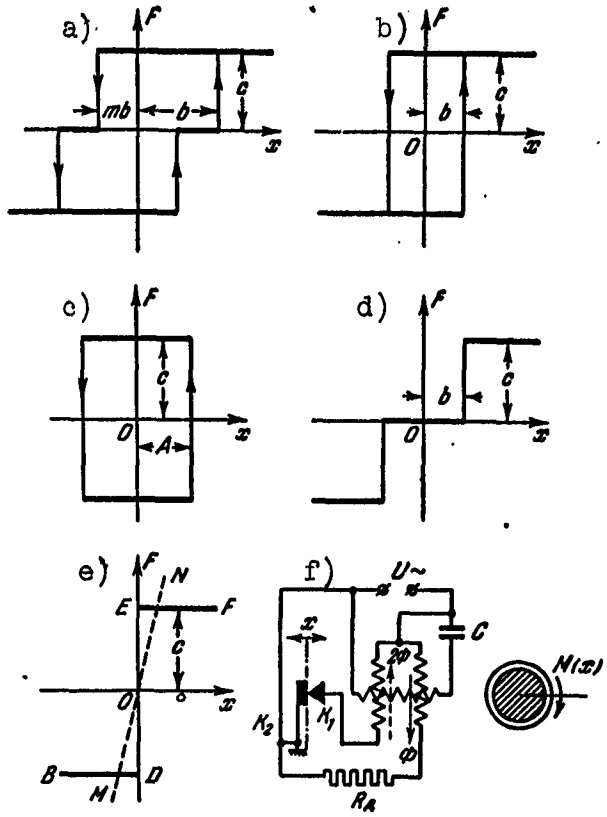


Fig. 3.2.

Relay characteristic with displaced hysteresis loop. Considering the quantity m as negative ($-1 \leq m \leq 0$) in a characteristic of the form of Fig. 3.2a, we obtain in accordance with (3.5) and (3.6):

$$q(A)=\frac{2c}{\pi A}\left(\sqrt{1-\frac{b^2}{A^2}}+\sqrt{1-\frac{m^2b^2}{A^2}}\right) \text{ for } A\geq b, \tag{3.7}$$

$$q'(A)=-\frac{2cb}{\pi A^2}(1+|m|) \text{ for } A\geq b. \tag{3.8}$$

Relay characteristic with hysteresis loop. For the relay characteristic of Fig. 3.2b we have $m = -1$. Therefore in accordance with (3.5) and (3.6) we obtain the values of the harmonic linearization coefficients in the form

$$q(A)=\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}} \quad \text{for } A\geq b, \tag{3.9}$$

$$q'(A)=-\frac{4cb}{\pi A^2} \text{ for } A\geq b. \tag{3.10}$$

Relay-type characteristic with hysteresis loop of variable width. In the relay-type characteristic in Fig. 3.2c, the width of the hysteresis loop is equal to twice the amplitude and varies with the variation of the amplitude. Then, setting b = A in (3.9) and (3.10), we obtain:

$$q(A)=0, \tag{3.11}$$

$$q'(A)=-\frac{4c}{\pi A}. \tag{3.12}$$

Relay-type characteristic with dead zone. We have m = 1 for the relay characteristic in Fig. 3.2d. Hence from (3.5) and (3.6)

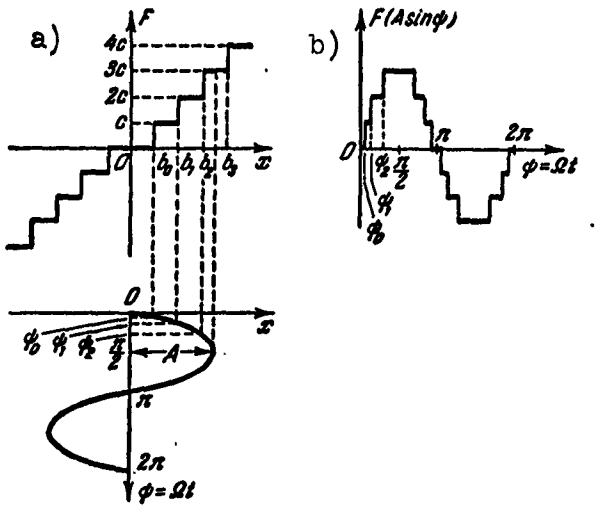


Fig. 3.3.

we obtain the values of the harmonic-linearization coefficients in the form

$$q(A)=\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}},\quad q'(A)=0 \text{ for } A\geq b. \tag{3.13}$$

Ideal relay characteristic. We will have m = 0 and b = 0 for the ideal relay characteristic (Fig. 3.2e). Then in accordance with (3.5) and (3.6) we obtain:

$$q(A)=\frac{4c}{\pi A},\quad q'(A)=0. \tag{3.14}$$

Let us note that the coefficient q'(A) is equal to zero for all single-valued symmetrical characteristics.

Step relay characteristic. The step relay characteristic is shown in Fig. 3.3a. Such a characteristic will, for example, reflect the voltage registered by a laminated potentiometer, as a function of the rotation angle of the slider. Let us calculate the harmonic linearization coefficients for the case of equal steps $b_0 = b_1 - b_0 = b_2 - b_1 = \ldots = b_n - b_{n-1}$, where n steps are swept by the oscillations of the input variable. In this case the coefficient q'(A) is equal to zero. The coefficient q(A) may be determined from Formula (3.3) by integrating over a quarter of a period with allowance for the values of the function $F(A \sin \psi)$ (Fig. 3.3b) and for the corresponding values of the angles. For a value of the argument $0 < \psi < \psi_0$ the function $F(A \sin \psi)$ is equal to zero. Carrying out the computation, we obtain:

$$q(A) = \frac{4}{\pi A}\left[\int_{\psi_0}^{\psi_1} c \sin\psi\, d\psi + \int_{\psi_1}^{\psi_2} 2c \sin\psi\, d\psi + \int_{\psi_2}^{\psi_3} 3c \sin\psi\, d\psi + \ldots\right.$$

$$\left. \ldots + \int_{\psi_{n-1}}^{\psi_n} nc \sin\psi\, d\psi + \int_{\psi_n}^{\frac{\pi}{2}} (n+1)\, c \sin\psi\, d\psi\right] = -\frac{4c}{\pi A}\left[\cos\psi\Big|_{\psi_0}^{\psi_1} + \right.$$

$$\left. + 2\cos\psi\Big|_{\psi_1}^{\psi_2} + 3\cos\psi\Big|_{\psi_2}^{\psi_3} + \ldots + n\cos\psi\Big|_{\psi_{n-1}}^{\psi_n} + (n+1)\cos\psi\Big|_{\psi_n}^{\frac{\pi}{2}}\right] =$$

$$= \frac{4c}{\pi A}\sum_{\nu=0}^{n} \cos\psi_\nu$$

Allowing for the value $\cos\psi_\nu = \sqrt{1 - b_\nu^2/A^2}$, we find

$$q(A) = \frac{4c}{\pi A}\sum_{\nu=0}^{n}\sqrt{1-\frac{b_\nu^2}{A^2}} \quad \text{for } b_n \leqslant A \leqslant b_{n+1}. \tag{3.15}$$

In particular, for the case $b_1 \leq A \leq b_2$ (from 3.15) we obtain:

$$q(A) = \frac{4c}{\pi A}\left(\sqrt{1-\frac{b_0^2}{A^2}} + \sqrt{1-\frac{b_1^2}{A^2}}\right). \tag{3.16}$$

The meaning of harmonic linearization of relay characteristics

is readily understood from the examples cited. Thus, for example, for the ideal relay characteristic (Fig. 3.2e), Expression (3.14) denotes the replacement of the broken-line characteristic BDEF by a straight-line characteristic MN with a slope such that the straight line MN approximately replaces that section of the discontinuous characteristic BDEF covered by the sought amplitude A. Hence the inversely proportional dependence of $\underline{q}$ upon A in Formula (3.41) becomes understandable, since the larger the amplitude A of oscillation of the input variable $\underline{x}$, the shallower will be the rise of the straight line MN which approximately replaces the discontinuous characteristic BDEF.

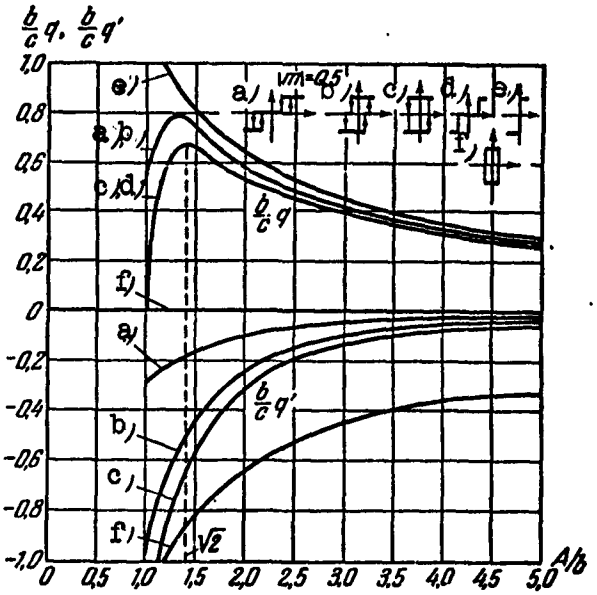


Fig. 3.4.

The situation is similar with the relay characteristic of Fig. 3.2d, for which the slope of the straight line replacing it is determined by Formula (3.13).

Hence, every nonhysteresis relay link in the oscillatory process is equivalent to some linear link with a variable "gearing" ratio (gain constant) q(A) dependent upon the amplitude of the input-

variable oscillations.

As regards the relay link with a hysteresis characteristic, it is replaced according to (3.9) and (3.10) by a linear link with the same coefficient q(A), but in addition to this a negative derivative is introduced into the right-hand side of the equation. The introduction of a negative derivative leads to a lag in the reaction of the link to the input disturbance. This serves as a "linear equivalent", replacing the effect of a nonlinearity in the form of a hysteresis loop. Here, according to (3.10), the coefficient q'(A) before the derivative also decreases with increasing amplitude A of the oscillations of the input variable $\underline{x}$; this is explained by the fact that the influence of the hysteresis loop on the oscillation process in the relay link must be weaker the larger the amplitude in comparison with the width of the hysteresis loop.

For construction of curves of variation of the self-oscillation amplitude and frequency as functions of the system parameters, it is often convenient to make use of the curves of the harmonic-linearization coefficients as functions of amplitude. In addition, determining the stability of the periodic solutions obtained usually requires knowledge not only of the values q(A) and q'(A), but also of the signs of their partial derivatives $\partial q/\partial A$ and $\partial q'/\partial A$, i.e., the direction of the tangent. Therefore it is expedient to construct the curves of variation of the harmonic linearization coefficients as functions of amplitude.

These curves, as constructed from the appropriate formulas, are shown in Fig. 3.4 for relay characteristics. The curves are constructed for the variation of the relative amplitude A/b and relative values of the harmonic-linearization coefficients (b/c)q and (b/c)q'.

§3.2. NONLINEAR CHARACTERISTICS WITH DEAD ZONE, SATURATION AND VARIABLE GAIN CONSTANT

Characteristic with dead zone and saturation. The static characteristic of a link with a dead zone and saturation is shown in Fig. 3.5a. For example the pressure drop Δp in a diaphragm-type output device varies as such a function of the angle of turn α of a control-element valve (Fig. 3.5c). The dead zone of a pneumatic booster is governed by the angle of overlap α_p. Such a characteristic will correspond to an electronic amplifier operating with input variables that bring the amplifier to the saturation point. The dead zone will correspond to the initial gently sloping section of the amplifier characteristic.

Let us calculate the coefficient of harmonic linearization q(A). In this case the coefficient q'(A) is equal to zero, since the characteristic is unique.

Here oscillations are again possible only provided that the amplitude A exceeds half the dead zone, i.e., b_1.

For harmonic oscillations of the input variable $\underline{x}$ of the nonlinear link, the function $F(A \sin \psi)$ will have the form of Fig. 3.5b. The value of the integral in the first formula of (3.3) will be the same for each quarter-period. Then we may write

$$q(A)=\frac{4}{\pi A}\int_0^{\frac{\pi}{2}} F(A\sin\psi)\sin\psi\,d\psi,$$

and since $F(A \sin \psi) = 0$ in the interval $0 \leq \psi \leq \psi_1$,

$$q(A)=\frac{4}{\pi A}\int_{\psi_1}^{\frac{\pi}{2}} F(A\sin\psi)\sin\psi\,d\psi.$$

Substituting the values of $F(A \sin \psi)$ and breaking the integral into two segments, we obtain:

$$q(A)=\frac{4k}{\pi A}\int_{\psi_1}^{\psi_2}(A\sin\psi-b_1)\sin\psi\,d\psi+\frac{4c}{\pi A}\int_{\psi_2}^{\frac{\pi}{2}}\sin\psi\,d\psi=$$

$$=\frac{4k}{\pi}\int_{\psi_1}^{\psi_2}\sin^2\psi\,d\psi-\frac{4kb_1}{\pi A}\int_{\psi_1}^{\psi_2}\sin\psi\,d\psi+\frac{4c}{\pi A}\int_{\psi_2}^{\frac{\pi}{2}}\sin\psi\,d\psi=$$

$$=\frac{4k}{\pi}\left(\frac{\psi}{2}\Big|_{\psi_1}^{\psi_2}-\frac{1}{4}\sin 2\psi\Big|_{\psi_1}^{\psi_2}\right)+\frac{4kb_1}{\pi A}\cos\psi\Big|_{\psi_1}^{\psi_2}-\frac{4c}{\pi A}\cos\psi\Big|_{\psi_2}^{\frac{\pi}{2}}=$$

$$=\frac{2k}{\pi}\left(\psi_2-\frac{1}{2}\sin 2\psi_2-\psi_1+\frac{1}{2}\sin 2\psi_1\right)+\frac{4kb_1}{\pi A}(\cos\psi_2-\cos\psi_1)+$$

$$+\frac{4c}{\pi A}\cos\psi_2.$$

Substituting $c = k(b_2 - b_1)$ into the expression obtained and allowing for the fact that

$$\psi_1=\arcsin\frac{b_1}{A},\quad \psi_2=\arcsin\frac{b_2}{A},$$

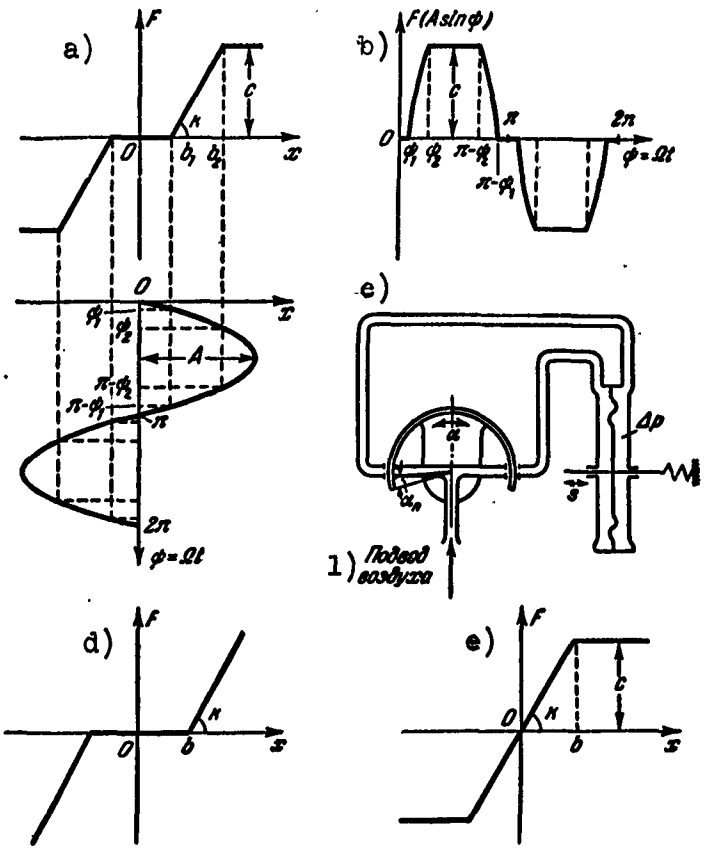


Fig. 3.5. 1) Air supply.

we finally find

$$\left.\begin{aligned} q(A)=\frac{2k}{\pi}\left(\arcsin\frac{b_2}{A}-\arcsin\frac{b_1}{A}+\frac{b_2}{A}\sqrt{1-\frac{b_2^2}{A^2}}-\right.\\ \left.-\frac{b_1}{A}\sqrt{1-\frac{b_1^2}{A^2}}\right),\\ q'(A)=0 \text{ for } A\geqslant b_2. \end{aligned}\right\} \qquad (3.17)$$

If, however, $A < b_2$, then we must make use of Formula (3.18), which is introduced below.

Characteristic with dead zone and without saturation. The characteristic of a link having a dead zone and no saturation (Fig. 3.5d) may be considered as a particular case of the characteristic of Fig. 3.5a for the conditions $A \leq b_2$ and $b_1 = b$. Then $\psi_2 = \pi/2$ and in accordance with (3.17) we obtain

$$\left.\begin{aligned} q(A)=k-\frac{2k}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right),\\ q'(A)=0 \text{ for } A\geqslant b. \end{aligned}\right\} \qquad (3.18)$$

As we see, the link with a dead zone is compared here to a linear link whose gain constant is reduced because of it. This decrease in the gain constant is significant for small amplitudes and is small for large amplitudes; here

$$0\leqslant q(A)\leqslant k \text{ for } b\leqslant A\leqslant\infty.$$

Characteristic with saturation and without dead zone. For the case of a link with saturation and no dead zone (Fig. 3.5e), setting $b_1 = 0$, $b_2 = b$, $\psi_1 = 0$ in (3.17) we obtain

$$\left.\begin{aligned} q(A)=\frac{2k}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right),\\ q'(A)=0 \text{ for } A\geqslant b. \end{aligned}\right\} \qquad (3.19)$$

For values of $A \leq b$, the coefficient $q(A) = k$ (a linear characteristic).

At oscillation amplitudes of the input variable that include the saturation zone, the link in question is replaced, so to speak, by a linear link whose gain constant q(A) is smaller the larger the amplitude (in contrast to the previous case).

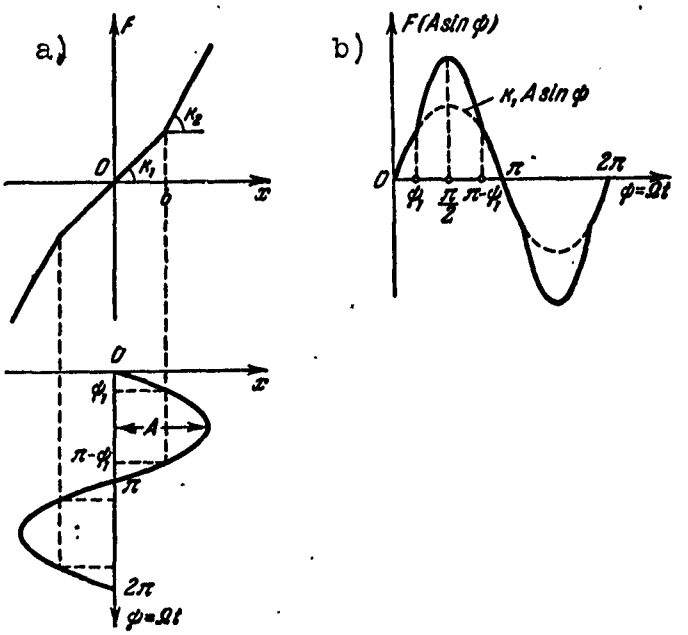


Fig. 3.6.

Characteristic with variable gain constant. The static characteristic of a link with a variable gain constant is shown in Fig. 3.6a. Such a characteristic may be obtained either by approximation of a curvilinear static characteristic or in links with a linear characteristic by the presence of a device for switching the transmission ratio as a function of the input variable.

For a value of $x \leq b$ the link remains linear. Therefore we will determine the value of q(A) for the condition $A \geq b$. For harmonic variation of the input variable x, the function $F(A \sin \psi)$ will be a periodic function of the argument ψ (Fig. 3.6b). Since in this case the value of the integral in (3.3) will be the same for each quarter-period, we calculate q(A) from the formula

$$q(A) = \frac{4}{\pi A} \int_0^{\frac{\pi}{2}} F(A \sin \varphi) \sin \varphi \, d\varphi.$$

Allowing for the appropriate values of the function $F(A \sin \psi)$ and the values of the argument ψ, we obtain

$$q(A)=\frac{4}{\pi A}\int_0^{\psi_1} k_1 A\sin^2\psi\,d\psi+\frac{4}{\pi A}\int_{\psi_1}^{\frac{\pi}{2}}[k_2(A\sin\psi-b)+k_1 b]\sin\psi\,d\psi=$$

$$=\frac{4k_1}{\pi}\int_0^{\psi_1}\sin^2\psi\,d\psi+\frac{4k_2}{\pi}\int_{\psi_1}^{\frac{\pi}{2}}\sin^2\psi\,d\psi-\frac{4k_2 b}{\pi A}\int_{\psi_1}^{\frac{\pi}{2}}\sin\psi\,d\psi+\frac{4k_1 b}{\pi A}\int_{\psi_1}^{\frac{\pi}{2}}\sin\psi\,d\psi=$$

$$=\frac{4k_1}{\pi}\left(\frac{\psi}{2}\Big|_0^{\psi_1}-\frac{1}{4}\sin 2\psi\Big|_0^{\psi_1}\right)+\frac{4k_2}{\pi}\left(\frac{\psi}{2}\Big|_{\psi_1}^{\frac{\pi}{2}}-\frac{1}{4}\sin 2\psi\Big|_{\psi_1}^{\frac{\pi}{2}}\right)+\frac{4k_2 b}{\pi A}\cos\psi\Big|_{\psi_1}^{\frac{\pi}{2}}-$$

$$-\frac{4k_1 b}{\pi A}\cos\psi\Big|_{\psi_1}^{\frac{\pi}{2}}=\frac{4k_1}{\pi}\left(\frac{\psi_1}{2}-\frac{1}{4}\sin 2\psi_1\right)+\frac{4k_2}{\pi}\left(\frac{\pi}{4}-\frac{\psi_1}{2}+\frac{1}{4}\sin 2\psi_1\right)-$$

$$-\frac{4k_2 b}{\pi A}\cos\psi_1+\frac{4k_1 b}{\pi A}\cos\psi_1=k_2-\frac{2}{\pi}(k_2-k_1)\arcsin\frac{b}{A}+$$

$$+\frac{2}{\pi}(k_2-k_1)\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}-\frac{4}{\pi}(k_2-k_1)\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}$$

and hence

$$\left.\begin{aligned} q(A)&=k_2-\frac{2}{\pi}(k_2-k_1)\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right),\\ q'(A)&=0 \text{ for } A\geq b. \end{aligned}\right\} \qquad (3.20)$$

In the case in question, the broken-line characteristic is replaced after linearization by a single straight line with a slope q(A) intermediate between k_1 and k_2; here, this slope varies in the interval $k_1 \leq q(A) \leq k_2$ on variation of the amplitude $b \leq A \leq \infty$. For amplitudes $A \leq b$ we have a linear characteristic with a slope k_1.

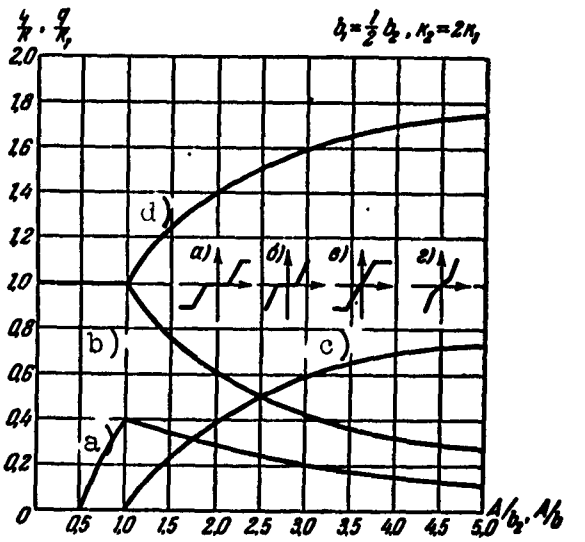


Fig. 3.7.

As in the case of relay characteristics, we may easily construct curves of the variation of the harmonic-linearization coefficients q(A) as functions of the amplitude A for the characteristics under consideration; this is also shown in Fig. 3.7.

§3.3. CHARACTERISTICS WITH HYSTERESIS LOOPS

Characteristic with dead zone, saturation, and hysteresis loop. A nonlinear static characteristic having a dead zone, saturation or

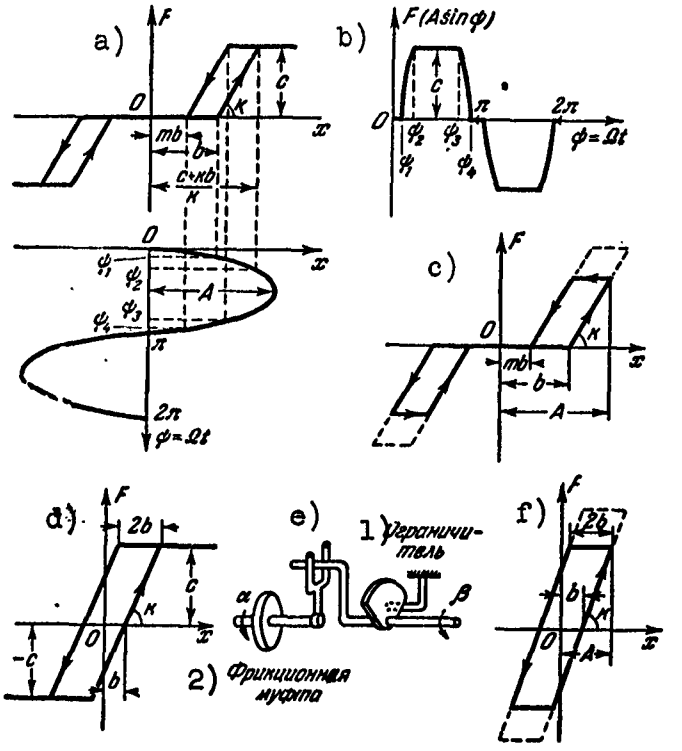


Fig. 3.8. 1) Arresting device; 2) friction clutch.

clipping, and hysteresis loops is shown in Fig. 3.8a. For example, this displacement s of the valve in the diaphragm output mechanism (Fig. 3.5c) will vary as a function or the turning angle α according to such a functional relationship. The dead zone depends upon the angle of valve overlap α_p and the saturation depends upon the limited pressure drop, while the hysteresis loops depend upon the presence of dry friction between the rod and the body of the diaphragm casing.

For harmonic variation of the input $\underline{x}$ of such a nonlinear link, the output will be a periodic function of the argument $\psi = \Omega t$ (Fig. 3.8b). Let us calculate the harmonic linearization coefficients q(A) and q'(A) from Formulas (3.3). Here, as in the case of a general form relay characteristic, m is any interval $-1 \leq m \leq 1$.

The integrals in Formulas (3.3) will be the same for each half-period. The values of the integrals from 0 to ψ_1 and from ψ_4 to π are zero since the functions $F(A \sin \psi)$ is zero on these intervals.

Calculating the coefficient q(A), we obtain

$$q(A) = \frac{2}{\pi A}\int_0^{\pi} F(A\sin\psi)\sin\psi\, d\psi =$$

$$= \frac{2}{\pi A}\left[\int_{\psi_1}^{\psi_2} k(A\sin\psi - b)\sin\psi\, d\psi + \int_{\psi_2}^{\psi_3} c\sin\psi\, d\psi + \right.$$

$$\left. + \int_{\psi_3}^{\psi_4} k(A\sin\psi - mb)\sin\psi\, d\psi\right] = \frac{2k}{\pi}\int_{\psi_1}^{\psi_2}\sin^2\psi\, d\psi - \frac{2kb}{\pi A}\int_{\psi_1}^{\psi_2}\sin\psi\, d\psi +$$

$$+ \frac{2c}{\pi A}\int_{\psi_2}^{\psi_3}\sin\psi\, d\psi + \frac{2k}{\pi}\int_{\psi_3}^{\psi_4}\sin^2\psi\, d\psi - \frac{2mkb}{\pi A}\int_{\psi_3}^{\psi_4}\sin\psi\, d\psi =$$

$$= \frac{2k}{\pi}\left(\frac{\psi}{2}\Big|_{\psi_1}^{\psi_2} - \frac{1}{4}\sin 2\psi\Big|_{\psi_1}^{\psi_2}\right) + \frac{2kb}{\pi A}\cos\psi\Big|_{\psi_1}^{\psi_2} - \frac{2c}{\pi A}\cos\psi\Big|_{\psi_2}^{\psi_3} +$$

$$+ \frac{2k}{\pi}\left(\frac{\psi}{2}\Big|_{\psi_3}^{\psi_4} - \frac{1}{4}\sin 2\psi\Big|_{\psi_3}^{\psi_4}\right) + \frac{2mkb}{\pi A}\cos\psi\Big|_{\psi_3}^{\psi_4} =$$

$$= \frac{2k}{\pi}\left(\frac{\psi_2}{2} - \frac{1}{4}\sin 2\psi_2 - \frac{\psi_1}{2} + \frac{1}{4}\sin 2\psi_1\right) + \frac{2kb}{\pi A}(\cos\psi_2 - \cos\psi_1) -$$

$$- \frac{2c}{\pi A}(\cos\psi_3 - \cos\psi_2) + \frac{2k}{\pi}\left(\frac{\psi_4}{2} - \frac{1}{4}\sin 2\psi_4 - \frac{\psi_3}{2} + \frac{1}{4}\sin 2\psi_3\right) +$$

$$+ \frac{2mkb}{\pi A}(\cos\psi_4 - \cos\psi_3).$$

Taking the values of the appropriate angles into consideration:

$$\psi_1 = \arcsin\frac{b}{A}, \quad \psi_2 = \arcsin\frac{c+kb}{kA},$$

$$\psi_3 = \pi - \arcsin\frac{c+mkb}{kA}, \quad \psi_4 = \pi - \arcsin\frac{mb}{A}$$

and performing transformations, we obtain

$$q(A) = \frac{k}{\pi}\left[\arcsin\frac{c+kb}{kA} + \arcsin\frac{c+mkb}{kA} - \arcsin\frac{b}{A} - \arcsin\frac{mb}{A} + \right.$$

$$+ \frac{c+kb}{kA}\sqrt{1 - \frac{(c+kb)^2}{k^2A^2}} + \frac{c+mkb}{kA}\sqrt{1 - \frac{(c+mkb)^2}{k^2A^2}} -$$

$$\left. - \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}} - \frac{mb}{A}\sqrt{1 - \frac{m^2b^2}{A^2}}\right] \text{for} \quad A \geq \frac{c+kb}{k}. \qquad (3.21)$$

Calculating the coefficient q'(A), we obtain:

$$q'(A)=\frac{2}{\pi A}\int_0^{\pi}F(A\sin\psi)\cos\psi\,d\psi=\frac{2}{\pi A}\Big[\int_{\psi_1}^{\psi_2}k(A\sin\psi-b)\cos\psi\,d\psi+$$

$$+\int_{\psi_2}^{\psi_3}c\cos\psi\,d\psi+\int_{\psi_3}^{\psi_4}k(A\sin\psi-mb)\cos\psi\,d\psi\Big]=\frac{2k}{\pi}\int_{\psi_1}^{\psi_2}\sin\psi\cos\psi\,d\psi-$$

$$-\frac{2kb}{\pi A}\int_{\psi_1}^{\psi_2}\cos\psi\,d\psi+\frac{2c}{\pi A}\int_{\psi_2}^{\psi_3}\cos\psi\,d\psi+\frac{2k}{\pi}\int_{\psi_3}^{\psi_4}\sin\psi\cos\psi\,d\psi-$$

$$-\frac{2mkb}{\pi A}\int_{\psi_3}^{\psi_4}\cos\psi\,d\psi=\frac{k}{\pi}\sin^2\psi\Big|_{\psi_1}^{\psi_2}-\frac{2kb}{\pi A}\sin\psi\Big|_{\psi_1}^{\psi_2}+\frac{2c}{\pi A}\sin\psi\Big|_{\psi_2}^{\psi_3}+$$

$$+\frac{k}{\pi}\sin^2\psi\Big|_{\psi_3}^{\psi_4}-\frac{2mkb}{\pi A}\sin\psi\Big|_{\psi_3}^{\psi_4}=\frac{k}{\pi}(\sin^2\psi_2-\sin^2\psi_1)-$$

$$-\frac{2kb}{\pi A}(\sin\psi_2-\sin\psi_1)+\frac{2c}{\pi A}(\sin\psi_3-\sin\psi_2)+\frac{k}{\pi}(\sin^2\psi_4-\sin^2\psi_3)-$$

$$-\frac{2mkb}{\pi A}(\sin\psi_4-\sin\psi_3).$$

Substituting the values of the angles ψ_1, ψ_2, ψ_3, and ψ_4 and performing transformations, we obtain

$$q'(A)=-\frac{2bc(1-m)}{\pi A^2}\quad\text{for}\quad A\geqslant\frac{c+kb}{k}. \tag{3.22}$$

If, however, $A \leq (c + kb)/k$, then instead of (3.21) and (3.22) we must use the formulas cited below.

Characteristic with dead zone and hysteresis loops but without saturation. A characteristic having a dead zone and hysteresis loops but not saturation is shown in Fig. 3.8c. The needle deflection of such a diaphragm-type output mechanism (Fig. 3.5c) will vary according to such a functional relationship as a function of the position angle of the control-element valve when the pressure drop in the diaphragm casing with oscillatory motion of the diaphragm does not reach the full pressure, i.e., a saturation region is lacking in the characteristic.

For determination of the coefficients q(A) and q'(A) we must in this case set

$$\frac{c+kb}{k}=A, \tag{3.23}$$

in Formulas (3.21) and (3.22), since here c is a variable quantity which is a function of A.

Then we obtain from Formulas (3.21) and (3.22)

$$q(A)=\frac{k}{\pi}\left\{\frac{\pi}{2}+\arcsin\left[1-\frac{b(1-m)}{A}\right]-\arcsin\frac{b}{A}-\arcsin\frac{mb}{A}+\right.$$
$$+\left[1-\frac{b(1-m)}{A}\right]\sqrt{1-\left[1-\frac{b(1-m)}{A}\right]^2}-\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}-$$
$$\left.-\frac{mb}{A}\sqrt{1-\frac{m^2b^2}{A^2}}\right\}\quad\text{for}\quad A\geqslant b, \tag{3.24}$$

$$q'(A)=-\frac{2kb}{\pi A}\left(1-\frac{b}{A}\right)(1-m)\quad\text{for}\quad A\geqslant b. \tag{3.25}$$

Characteristic with saturation and hysteresis loop. A characteristic with saturation and a hysteresis loop is shown in Fig. 3.8d. For example, the turn angle of the output shaft (Fig. 3.8e) will vary according to such a functional relationship as a function of turn angle of the input shaft, in a mechanical transmission in the presence of play and limiting. The characteristics shown may be regarded as a particular case of the characteristic (Fig. 3.8a) for m = -1. Then we obtain from Formulas (3.21) and (3.22)

$$q(A)=\frac{k}{\pi}\left(\arcsin\frac{c+kb}{kA}+\arcsin\frac{c-kb}{kA}+\frac{c+kb}{kA}\sqrt{1-\frac{(c+kb)^2}{k^2A^2}}+\right.$$
$$\left.+\frac{c-kb}{kA}\sqrt{1-\frac{(c-kb)^2}{k^2A^2}}\right)\quad\text{for}\quad A\geqslant\frac{c+kb}{k}, \tag{3.26}$$

$$q'(A)\doteq-\frac{4bc}{\pi A^2}\quad\text{for}\ A\geqslant\frac{c+kb}{k}. \tag{3.27}$$

Characteristic of play or backlash type. A nonlinear characteristic of the play or backlash type is shown in Fig. 3.8f. The turn angle of the output shaft will vary according to such a relationship as a function of the turn angle of the input shaft in a mechanical transmission in the presence of play without limiting. The nonlinear characteristic in question may be regarded as a particular case of a characteristic with dead zone and hysteresis loops (Fig. 3.8c)

for the condition m = – 1. Then in accordance with Formulas (3.24) and (3.25) we obtain

$$q(A)=\frac{k}{\pi}\left[\frac{\pi}{2}+\arcsin\left(1-\frac{2b}{A}\right)+\right.$$
$$\left.+2\left(1-\frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1-\frac{b}{A}\right)}\right] \text{ for } A\geqslant b, \qquad (3.28)$$

$$q'(A)=-\frac{4kb}{\pi A}\left(1-\frac{b}{A}\right) \text{ for } A\geqslant b. \qquad (3.29)$$

The curves of variation of the coefficients q(A) and q'(A) as functions of amplitude for the characteristics considered as shown in Fig. 3.9.

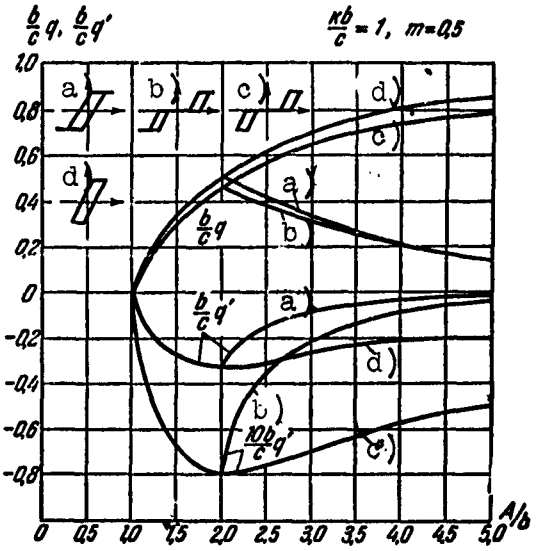


Fig. 3.9.

§3.4. POWER-LAW NONLINEAR CHARACTERISTIC

In synthesizing the equations of links in automatic systems, we may encounter power-law nonlinear characteristics having the form

$F(x)=kx^n$ with n an odd integer;

$F(x)=kx^n \operatorname{sign} x$ with n an even integer.

At high velocities, for example, viscous friction is proportional to the square and to the cube or the velocity. In addition, it is convenient to make use of power-law characteristics for approxi-

mation of nonlinear static characteristics obtained experimentally.

Let us consider the general form of power-law characteristic represented by the curve of Fig. 3.10. Here the coefficient q'(A) is zero due to the uniqueness of the characteristic.

In accordance with (3.3), the coefficient q(A) may be calculated by integrating over a quarter-period.

$$q(A)=\frac{1}{\pi A}\int_0^{2\pi}F(A\sin\psi)\sin\psi\,d\psi=\frac{4k}{\pi A}\int_0^{\frac{\pi}{2}}A^n\sin^n\psi\sin\psi\,d\psi=$$

$$=\frac{4kA^{n-1}}{\pi}\int_0^{\frac{\pi}{2}}\sin^{n+1}\psi\,d\psi=$$

$$=\frac{4nkA^{n-1}}{(n+1)\pi}\int_0^{\frac{\pi}{2}}\sin^{n-1}\psi\,d\psi-\frac{4kA^{n-1}}{(n+1)\pi}\sin^n\psi\cos\psi\Big|_0^{\frac{\pi}{2}},$$

where n is a positive integer.

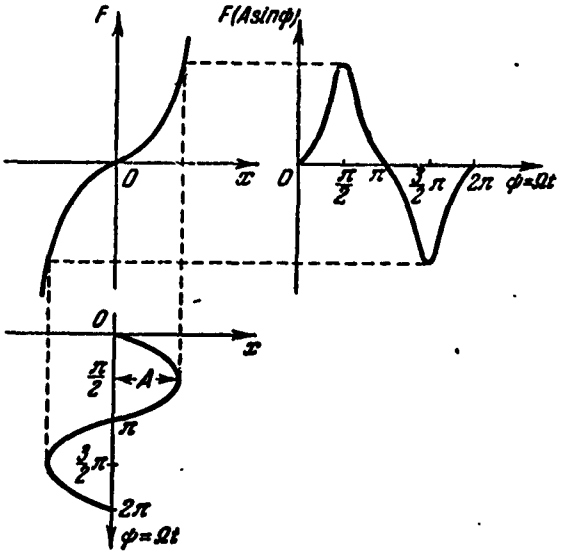


Fig. 3.10.

Since the last term is zero for the given limits of integration, we shall have the formula

$$q(A)=\frac{4nkA^{n-1}}{(n+1)\pi}\int_0^{\frac{\pi}{2}}\sin^{n-1}\psi\,d\psi,$$

for calculation of q(A) where n is a positive integer.

Comparing this with the previous expression, we obtain the recursion formula

$$q_n = \frac{nA^2}{n+1} q_{n-2}, \qquad (3.30)$$

where n is the exponent of the nonlinear function F(x).

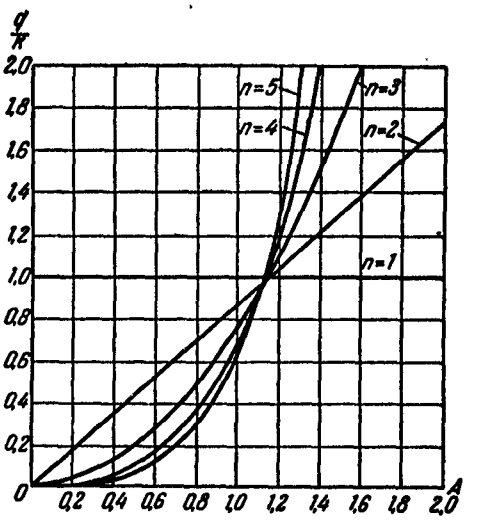


Fig. 3.11.

Therefore to determine the coefficient of harmonic linearization for an oddly symmetric characteristic of any order it is sufficient to know any two values of q (one for an even exponent and one for an odd exponent). Let us choose n = 0 and n = 1 as initial values. For n = 0 we have F(x) = k sign x (the ideal relay characteristic), for which according to (3.14)

$$q(A) = \frac{4k}{\pi A},$$

while for n = 1 we have F(x) = kx (a linear characteristic for which

$$q = k.$$

Starting from this we obtain from Formula (3.30) a general expression for the harmonic-linearization coefficient for any power-law characteristic in the form

$$q(A) = \frac{3 \cdot 5 \ldots n}{4 \cdot 6 \ldots (n+1)} kA^{n-1} \quad \text{for } n \text{ odd},$$

$$q(A) = \frac{4}{\pi} \cdot \frac{2 \cdot 4 \ldots n}{3 \cdot 5 \ldots (n+1)} kA^{n-1} \quad \text{for } n \text{ even}.$$

In particular, we obtain for the characteristic $F(x) = kx^2 \operatorname{sign} x$

$$q(A)=\frac{8kA}{3\pi}=0{,}85kA. \tag{3.31}$$

For the characteristic $F(x) = kx^3$ we have

$$q(A)=\frac{3kA^2}{4}=0{,}75kA^2; \tag{3.32}$$

For the characteristic $F(x) = kx^4 \operatorname{sign} x$

$$q(A)=\frac{32kA^3}{15\pi}=0{,}68kA^3; \tag{3.33}$$

For the characteristic $F(x) = kx^5$

$$q(A)=\frac{5kA^4}{8}=0{,}625kA^4. \tag{3.34}$$

Curves of q/k as a function of A are shown in Fig. 3.11 for power-law nonlinear characteristics.

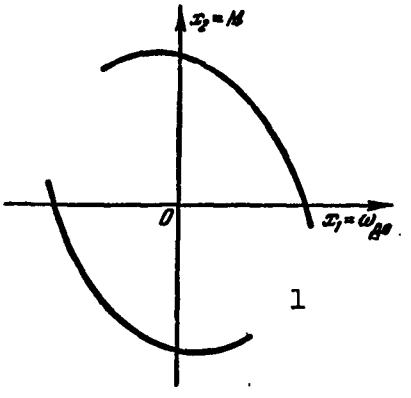


Fig. 3.12. 1) dv.

Power-law nonlinearities of the second class. For approximation of certain experimentally obtained nonlinear characteristics, as, for example, the mechanical characteristics of two-phase asynchronous motors with tubular rotors (Fig. 3.12), we use polynomials in powers of the variables, where both the input and output variables appear in the nonlinearity (nonlinearities of the second class). Let us carry out harmonic linearization for certain nonlinearities of the second class.

The nonlinear function $F(x_1, x_2) = kx_2^2 \operatorname{sign} x_1$. In this case, where the input variable x_1 and the output variable x_2 of the nonlinear link are contained in the nonlinearity, we assume that the solution for the output variable of the nonlinear link is found in

harmonic form

$$x_2 = A_2 \sin \Omega t$$

and that for the input variable also in harmonic form, but with a different amplitude and phase shift relative to the output quantity:

$$x_1 = A_1 \sin (\Omega t - \varphi).$$

The relationships between the amplitudes A_1 and A_2 and the phase shift φ are determined through the frequency characteristics of the linear part separating the variables x_1 and x_2.

Allowing for the relationships which have been introduced, $F(x_1, x_2)$ will be a periodic function of the argument $\psi = \Omega t$ (Fig. 3.13) and is rewritten in the form

$$F(x_1, x_2) = k(A_2 \sin \psi)^* \operatorname{sign} \sin (\psi - \varphi). \quad (3.35)$$

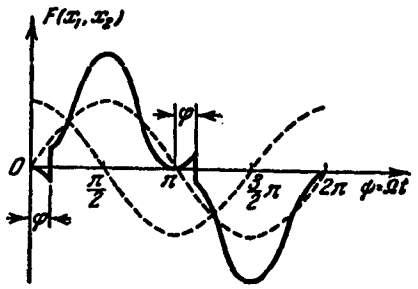


Fig. 3.13.

As may be seen from the curve (Fig. 3.13), the condition of absence of the constant term of the Fourier-series expansion of the function $F(x_1, x_2)$ is realized.

We must keep in mind that the harmonic-linearization coefficients will now be functions not only of the amplitude A_2, but also of the oscillation frequency Ω because of the different phase shifts φ for different frequencies.

Let us determine the harmonic linearization coefficients $q(A_2, \Omega)$ and $q'(A_2, \Omega)$ for the function (3.35). The integrals in Formulas (3.2) will be equal for each half-period. Then calculating the value of $q(A_2, \Omega)$, we obtain

$$q(A_2, \Omega) = \frac{1}{\pi A_2}\int_0^{2\pi} F(x_1, x_2)\sin\psi\, d\psi =$$

$$= \frac{2kA_2}{\pi}\int_0^{\pi} \sin^2\psi \operatorname{sign}\sin(\psi-\varphi)\sin\psi\, d\psi =$$

$$= \frac{2kA_2}{\pi}\left(-\int_0^{\varphi}\sin^3\psi\, d\psi + \int_{\varphi}^{\pi}\sin^3\psi\, d\psi\right) = \frac{2kA_2}{\pi}\left(\cos\psi\Big|_0^{\varphi} - \frac{1}{3}\cos^3\psi\Big|_0^{\varphi}\right) -$$

$$- \cos\psi\Big|_{\varphi}^{\pi} + \frac{1}{3}\cos^3\psi\Big|_{\varphi}^{\pi} = \frac{2kA_2}{\pi}\left(2\cos\varphi - \frac{2}{3}\cos^3\varphi\right)$$

and hence

$$q(A_2, \Omega) = \frac{4kA_2}{\pi}\left(\cos\varphi - \frac{1}{3}\cos^3\varphi\right), \quad \varphi = \varphi(\Omega). \tag{3.36}$$

Calculating the value of $q'(A_2, \Omega)$, we obtain

$$q'(A_2, \Omega) = \frac{1}{\pi A_2}\int_0^{2\pi} F(x_1, x_2)\cos\psi\, d\psi =$$

$$= \frac{2kA_2}{\pi}\int_0^{\pi} \sin^2\psi \operatorname{sign}\sin(\psi-\varphi)\cos\psi\, d\psi =$$

$$= \frac{2kA_2}{\pi}\left(-\int_0^{\varphi}\sin^2\psi\cos\psi\, d\psi + \int_{\varphi}^{\pi}\sin^2\psi\cos\psi\, d\psi\right) =$$

$$= \frac{2kA_2}{3\pi}\left(-\sin^3\psi\Big|_0^{\varphi} + \sin^3\psi\Big|_{\varphi}^{\pi}\right) = \frac{2kA_2}{3\pi}(-\sin^3\varphi - \sin^3\varphi)$$

and hence

$$q'(A_2, \Omega) = -\frac{4kA_2}{3\pi}\sin^3\varphi, \quad \varphi = \varphi(\Omega). \tag{3.37}$$

Let us note that from Formulas (3.36) and (3.37) for $\varphi = 0$, we obtain the values of the coefficients

$$q(A_2) = \frac{8kA_2}{3\pi}, \quad q'(A_2) = 0,$$

corresponding to the power function

$$F(x_2) = kx_2^{\frac{3}{2}}\operatorname{sign} x_2.$$

The nonlinear function $F(x_1, x_2) = kx_2^3 \operatorname{sign} x_2 \operatorname{sign} x_1$. Assuming just as in the previous case that

$$x_2 = A_2 \sin\Omega t,$$
$$x_1 = A_1 \sin(\Omega t - \varphi), \ \Omega t = \psi,$$

we write the nonlinear function $F(x_1, x_2)$ in the form

$$F(x_1, x_2) = k(A_2 \sin\psi)^3 \operatorname{sign}\sin\psi \operatorname{sign}\sin(\psi - \varphi). \tag{3.38}$$

The curve of the function (3.38) has a form similar to the curve of the function (3.35) (Fig. 3.13). Hence the constant term of the function $F(x_1, x_2)$ will also be equal to 0 in the case under consideration.

Calculating $q(A_2, \Omega)$, we obtain

$$\begin{aligned}
q(A_2, \Omega) &= \frac{1}{\pi A_2}\int_0^{2\pi} F(x_1, x_2)\sin\psi\, d\psi = \\
&= \frac{2kA_2^3}{\pi}\int_0^{\pi} \sin^3\psi \operatorname{sign}\sin\psi \operatorname{sign}\sin(\psi-\varphi)\sin\psi\, d\psi = \\
&= \frac{2kA_2^3}{\pi}\left(-\int_0^{\varphi}\sin^4\psi\, d\psi + \int_{\varphi}^{\pi}\sin^4\psi\, d\psi\right) = \\
&= \frac{2kA_2^3}{\pi}\left(-\frac{3}{8}\psi\Big|_0^{\varphi} - \frac{1}{4}\sin 2\psi\Big|_0^{\varphi} + \right. \\
&\qquad \left. + \frac{1}{32}\sin 4\psi\Big|_0^{\varphi} + \frac{3}{8}\psi\Big|_{\varphi}^{\pi} - \frac{1}{4}\sin 2\psi\Big|_{\varphi}^{\pi} + \frac{1}{32}\sin 4\psi\Big|_{\varphi}^{\pi}\right) = \\
&= \frac{2kA_2^3}{\pi}\left(\frac{3}{8}\pi - \frac{3}{4}\varphi + \frac{1}{2}\sin 2\varphi - \frac{1}{16}\sin 4\varphi\right)
\end{aligned}$$

and hence

$$q(A_2, \Omega) = \frac{kA_2^3}{\pi}\left(\frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi\right), \quad \varphi = \varphi(\Omega). \tag{3.39}$$

computing the value of $q'(A_2, \Omega)$, we obtain

$$\begin{aligned}
q'(A_2, \Omega) &= \frac{1}{\pi A_2}\int_0^{2\pi} F(x_1, x_2)\cos\psi\, d\psi = \\
&= \frac{2kA_2^3}{\pi}\int_0^{\pi} \sin^3\psi \operatorname{sign}\sin\psi \operatorname{sign}\sin(\psi-\varphi)\cos\psi\, d\psi = \\
&= \frac{2kA_2^3}{\pi}\left(-\int_0^{\varphi}\sin^3\psi\cos\psi\, d\psi + \int_{\varphi}^{\pi}\sin^3\psi\cos\psi\, d\psi\right) = \\
&= \frac{kA_2^3}{2\pi}\left(-\sin^4\psi\Big|_0^{\varphi} + \sin^4\psi\Big|_{\varphi}^{\pi}\right) = \frac{kA_2^3}{2\pi}(-\sin^4\varphi - \sin^4\varphi)
\end{aligned}$$

and, consequently,

$$q'(A_2, \Omega) = -\frac{kA_2^3}{\pi}\sin^4\varphi, \quad \varphi = \varphi(\Omega). \tag{3.40}$$

In the particular case for the value $\varphi = 0$ we obtain from (3.39) and (3.40) the values of the coefficients

$$q(A_2)=\frac{3kA_2^3}{4},\quad q'(A_2)=0,$$

which correspond to the function

$$F(x_2)=kx_2^3.$$

§3.5. Hysteresis Loops of Electric Circuits Containing Iron

In investigation of automatic systems having electric circuits containing iron, we must allow for nonlinearities in the form of the hysteresis loop (Fig. 3.14a). The magnetic induction of a magnetic circuit will vary according to such a nonlinear relationship as a function of the current flowing in its winding. For dynamoelectric amplifiers, such a characteristic will correspond to the variation of the emf as a function of the control current.

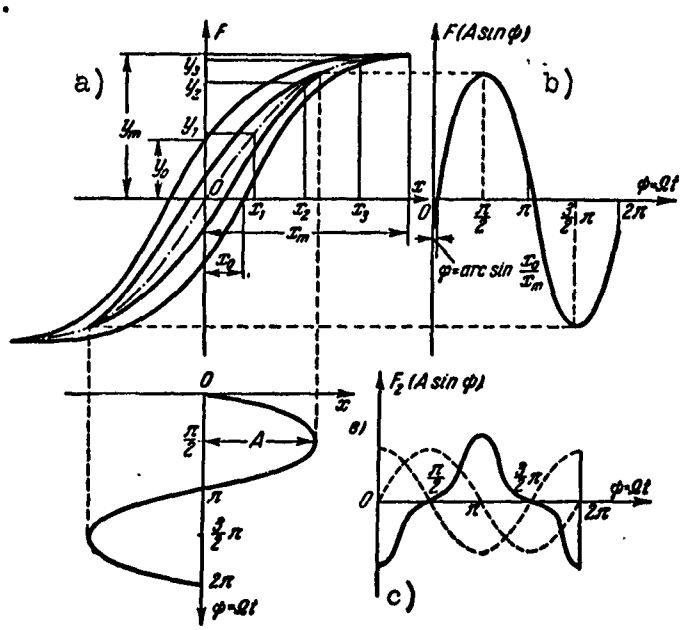


Fig. 3.14.

In the general case, we shall assume that a characteristic of the type of the hysteresis loop F(x) expresses the dependence of the output variable of the nonlinear link upon the input variable in a steady-state mode of operation.

The shape of the hysteresis characteristic is determined by the

material of the magnetic circuit and may be varied by special methods, as for example the use of a secondary winding fed by an alternating current of a frequency higher than the oscillation frequency of the regulated or controlled quantities of the automatic system.

Hysteresis characteristics may not be expressed by an exact analytical dependence, but may be obtained experimentally for the appropriate links of the system. For harmonic linearization of a hysteresis characteristic which has been obtained experimentally, we may first select an approximating function and thereupon perform the linearization according to the ordinary rules.

We must keep in mind that with oscillation of the input variable with a variable amplitude, the hysteresis loop of an electric circuit containing iron will be subject to complex deformations. For approximate calculations we will assume that the hysteresis characteristic remains similar to that shown in Fig. 3.14a with variation of the input-variable oscillation amplitude.

For variation of the input variable $\underline{x}$ according to a harmonic law, the output variable will be a distorted sine curve (Fig. 3.14b). The larger the oscillation amplitude of the variable $\underline{x}$, the larger will be this distortion. In addition, the presence of a hysteresis loop causes a phase lag of the output variable relative to the input variable. The wider the hysteresis loop the larger will be the phase shift. In harmonic linearization, the coefficient $q'(A)$ allows for the phase shift.

We will approximate the hysteresis loop by choosing a function $F_1(x)$ for the basic curve (the broken line in the middle) with ordinates equal to the half-sums of the forward and backward branches of the loop and a secondary function $F_2(x)$ that allows for the ordinates of the branches reckoned from the basic curve. Hence the hysteresis

loop is represented in the form

$$F(x)=F_1(x)+F_2(x).$$

We represent the function $F_1(x)$ analytically in the form of a polynomial in powers of $\underline{x}$. Due to the approximate nature of the harmonic-linearization method itself, we may limit ourselves to three terms of the polynomial and represent the function $F_1(x)$ in the form

$$F_1(x)=Bx+Cx^3+Dx^5, \tag{3.41}$$

where B, C, and D are coefficients determined for three chosen points on the basic curve.

Having chosen three points on a positive section of the basic curve and denoting the abscissas of the points chosen by x_1, x_2, and x_3 and the ordinates by y_1, y_2, and y_3 (Fig. 3.14a), we write three equations for determination of the coefficients B, C, and D:

$$\begin{aligned} y_1&=Bx_1+Cx_1^3+Dx_1^5,\\ y_2&=Bx_2+Cx_2^3+Dx_2^5,\\ y_3&=Bx_3+Cx_3^3+Dx_3^5. \end{aligned}$$

Solving these equations with respect to the unknowns B, C, and D, we obtain

$$\begin{aligned} B&=\frac{y_1x_2^3x_3^3(x_2^2-x_3^2)+y_2x_1^3x_3^3(x_3^2-x_1^2)+y_3x_1^3x_2^3(x_1^2-x_2^2)}{\Delta},\\ C&=\frac{y_1x_2x_3(x_3^5-x_2^5)+y_2x_1x_3(x_1^5-x_3^5)+y_3x_1x_2(x_2^5-x_1^5)}{\Delta},\\ D&=\frac{y_1x_2x_3(x_2^3-x_3^3)+y_2x_1x_3(x_3^3-x_1^3)+y_3x_1x_2(x_1^3-x_2^3)}{\Delta}, \end{aligned}$$

where

$$\Delta=x_1x_2x_3[x_1^3x_2^3(x_2^2-x_1^2)+x_1^3x_3^3(x_1^2-x_3^2)+x_2^3x_3^3(x_3^2-x_2^2)].$$

Thus we determine the basic function $F_1(x)$. We represent the auxiliary function $F_2(x)$ in the form

$$F_2(x)=-\gamma A\left(1-\frac{x^2}{A^2}\right)^{\frac{n}{2}}\operatorname{sign} px, \tag{3.42}$$

where the value n = 2, 3, 4, ... is chosen on the basis of the ex-

perimentally obtained hysteresis characteristic as a function of its shape. The quantity

$$\gamma=\frac{y_0}{x_m} \tag{3.43}$$

is determined from this same experimental characteristic.

Let us note that for determination of the solution for $\underline{x}$ in the form

$$x=A\sin\psi, \quad \psi=\Omega t, \tag{3.44}$$

we will have

$$1-\frac{x^2}{A^2}=\cos^2\psi$$

and hence according to (3.42) the secondary function is written in terms of the argument ψ in the form

$$F_2(A\sin\psi)=-\gamma A\cos^n\psi\,\operatorname{sign}\cos\psi \quad \text{for } n=2,3,4,\ldots \tag{3.45}$$

In accordance with (3.41)-(3.43), the complete approximating function is

$$F(x)=F_1(x)+F_2(x)=Bx+Cx^3+Dx^5-\gamma A\left(1-\frac{x^2}{A^2}\right)^{\frac{n}{2}}\operatorname{sign} px. \tag{3.46}$$

The values of the harmonic-linearization coefficients of the nonlinear terms of the function $F_1(x)$ in x^3 and x^5 were determined in the previous section. Let us now calculate the harmonic-linearization coefficient for the function $F_2(x)$.

The function $F_2(A\sin\psi)$ will be even (Fig. 3.14c) for any whole $\underline{n}$, and hence the coefficient

$$q(A)=\frac{1}{\pi A}\int_0^{2\pi}F_2(A\sin\psi)\sin\psi\,d\psi \tag{3.47}$$

for it is equal to zero.

Let us calculate the values of the coefficient $q'(A)$ for $F_2(x)$ with various values of $\underline{n}$. It is evident from Fig. 3.14c that the integral in Formula (3.3) for the calculation of $q'(A)$ will be the

same for each quarter-period. Then for determination of q'(A) we obtain the formula

$$q'(A)=\frac{4}{\pi A}\int_0^{\frac{\pi}{2}} F_2(A\sin\psi)\cos\psi\,d\psi=-\frac{4\gamma}{\pi}\int_0^{\frac{\pi}{2}}\cos^{n+1}\psi\,d\psi. \qquad (3.48)$$

As is evident here, the coefficient q' is not a function of the oscillation amplitude A.

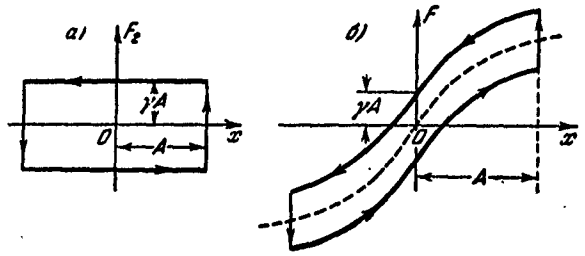


Fig. 3.15.

Formula (3.48) may be transformed to

$$q'=-\frac{4\gamma}{\pi}\left|\frac{\cos^n\psi\,\sin\psi}{n}\right|_0^{\frac{\pi}{2}}-\frac{4\gamma}{\pi}\frac{n}{n+1}\int_0^{\frac{\pi}{2}}\cos^{n-1}\psi\,d\psi.$$

Since the first term is equal to zero, we obtain the following recursion formula on comparing the second term with (3.48):

$$q'_n=\frac{n}{n+1}q'_{n-2};$$

Let us choose n = 0 and n = 1 for the initial values. For n = 0, we obtain from (3.42)

$$F_2(x)=-\gamma A\,\mathrm{sign}\,px,$$

which corresponds to a relay-type hysteresis loop with variable width and height (Fig. 3.15a). In this case, instead of the form shown in Fig. 3.14a, the complete characteristic $F = F_1 + F_2$ assumes the form of Fig. 3.15b.

The value of q' for the case n = 0 under consideration may be obtained either from Formula (3.48) or from the ready expression (3.12)

for $c = \gamma A$, which gives us

$$q' = -\frac{4\gamma}{\pi}.$$

For $n = 1$ we find from Formula (3.48)

$$q' = -\gamma.$$

Therefore, using the recursion formula, we obtain a general expression for the harmonic-linearization coefficient q' of the hysteresis loop for any power n in the form

$$q' = -\frac{3 \cdot 5 \ldots n}{4 \cdot 6 \ldots (n+1)}\gamma \qquad \text{for odd } n;$$

$$q' = -\frac{4}{\pi} \cdot \frac{2 \cdot 4 \ldots n}{3 \cdot 5 \ldots (n+1)}\gamma \qquad \text{for even } n.$$

Hence the coefficient q' will have the values shown in Table 3.1 as a function of the degree n of the approximating function $F_2(x)$.

TABLE 3.1

n	2	3	4	5	6	7	8	9
q'	$-0{,}85\gamma$	$-0{,}75\gamma$	$-0{,}68\gamma$	$-0{,}625\gamma$	$-0{,}582\gamma$	$-0{,}547\gamma$	$-0{,}518\gamma$	$-0{,}492\gamma$

Taking the values of q(A) from (3.32) and (3.34) for the second and third nonlinear terms of the function $F_1(x)$ and remembering that for all n the coefficient q(A) for the function $F_2(x)$ is equal to zero, we obtain a formula for calculation of the harmonic-linearization coefficient q(A) of the general function F(A) in the form

$$q(A) = B + 0{,}75CA^2 + 0{,}625DA^4. \qquad (3.49)$$

Ordinarily the coefficient C in this formula will be negative. The coefficient D may have either sign, depending on the shape of the hysteresis characteristic.

The output-independence of the coefficient q' characterizing the oscillation phase lag of the input is explained for the nonlinearity in question by the adopted condition that the hysteresis characteristic

retains its shape as it shrinks on a decrease in the oscillation amplitude of the input quantity. The values of the coefficient q' will be different for different characteristics in accordance with the shape of the loop.

It is also possible to have a more complex analytical representation of the hysteresis characteristic allowing for its deformation on variations in the input-oscillation amplitude.

§3.6. Nonlinearities of Dry- and Square-Law Friction Types

A nonlinear characteristic of the dry-friction type (Fig. 3.16a) represents the dependence of a dry-friction force or moment upon

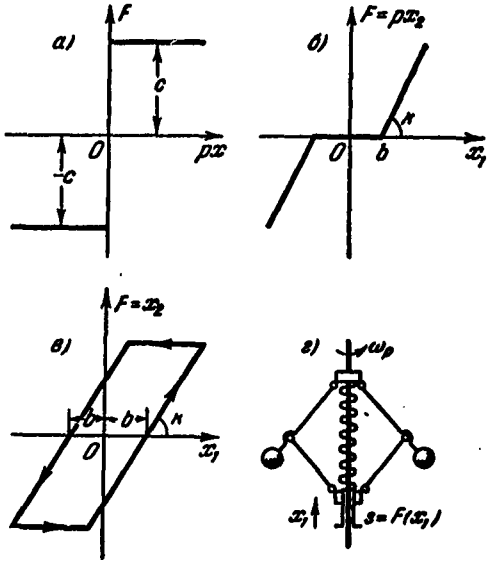


Fig. 3.6.

velocity. Here we must keep in mind that the direction of the frictional force or moment, which is contrary to that of the velocity, is ordinarily taken into account in synthesizing the equations. Such a characteristic is similar in form to the ideal relay characteristic. In contrast to the relay characteristic, we must keep in mind that F(px) may assume not only the values c and – c, but also values $-c \leq F(px) \leq c$, when the rate px = 0 over the course of some period

of time. However, on instantaneous passage of the rate through zero, the sign of the dry frictional force or moment changes instantaneously.

Hence in the case where the input passes instantaneously (without stopping) through the value px = 0 in oscillatory motion, harmonic linearization of the dry-friction characteristic is performed in the same way as for an ideal relay characteristic.

In the case where the rate does not pass instantaneously through the value px = 0, but there are stationary points in the motion of a link with dry friction, the replacement of a dry-friction characteristic by a relay characteristic may prove too crude a method for analysis of the oscillations. Stoppages in the motion of a system with dry friction are possible when, upon approaching the rate px = 0, it is found that the sum of all forces or moments applied to a link in the presence of friction will be less than the dry frictional force or moment.

Let us determine the harmonic-linearization coefficients for a dry friction characteristic which we assume to have the form of the ideal relay characteristic, i.e., for the case where there are no significant stoppages within the oscillation.

Setting

$$px = A_p \sin \Omega t,$$

where A_p is the oscillation amplitude of the rate, we obtain

$$q(A_p) = \frac{4c}{\pi A_p}, \tag{3.50}$$

in accordance with the ideal relay characteristic; here, c is the constant value of the dry-frictional force or moment. The formula for the harmonic linearization of dry friction is

$$\dot{F}(px) = q(A_p)\, px. \tag{3.51}$$

If in analysis of a nonlinear system it is desirable to obtain a solution for the variable $\underline{x}$ itself, then setting

$$x = A \sin \Omega t, \quad px = A\Omega \cos \Omega t,$$

we obtain

$$A_p = A\Omega.$$

Then, denoting

$$q'(A) = \frac{4c}{\pi A}, \tag{3.52}$$

we obtain in accordance with (3.50) and (3.51) the formula for harmonic linearization of the dry-frictional characteristic for this case in the form

$$F(x) = \frac{q'(A)}{\Omega} px. \tag{3.53}$$

In the case where stoppages are present in the motion of a system with dry friction, the analysis is significantly more complex (see §1.1). However, allowance for the stoppages is simple enough in two particular cases.

In the first case we allow for dry and viscous linear friction, while the mass is considered to be equal to zero for the moving elements of a link in the presence of friction (the inertial forces are small). In addition, we assume that there is no restoring force in the link. Then dry friction may be allowed for by means of the nonlinear characteristic with dead zone (Fig. 3.16b), which was introduced earlier and represents the dependence of the velocity px_2 upon the disturbing forces or moments x_1 for the case of a link with dry friction. For variation of the input within the limits $-b \leq x_1 \leq b$, the output, i.e., the rate, is equal to zero. For the case $|x| > |b|$, a rate which is proportional to the difference $|x| - |b|$ develops in the link.

In accordance with (3.18), the harmonic linearization coeffi-

cients in this case have the values

$$q(A)=k-\frac{2k}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right),\quad q'(A)=0\ \text{for}\ A\geqslant b. \tag{3.54}$$

After carrying out harmonic linearization, the equations for a link with dry and linear friction

$$\left.\begin{array}{ll} k_2px_2+c\,\operatorname{sign}px_2=k_1x_1 & \text{for}\quad px_2\neq 0,\\ -c<k_1x_1<c & \text{for}\quad px_2=0 \end{array}\right\} \tag{3.55}$$

should be replaced by the single nonlinear equation

$$px_2=F(x_1).$$

Then in place of the nonlinear function $F(x_1)$ we must substitute the linear expression

$$F(x_1)=q(A)\,x_1,$$

where $\underline{k}$ and $\underline{b}$ in Formula (3.54) for q(A) are determined from the coefficients of Eq. (3.55) from the relationships:

$$k=\frac{k_1}{k_2},\quad b=\frac{c}{k_1}.$$

In the second case, only dry friction and the linear restoring force are allowed for, while the mass of the moving elements (the inertial forces) is considered equal to zero for a link with friction. In such a link we may take a play-type or backlash-type characteristic (Fig. 3.16c) as the nonlinear characteristic. In this case, we take the displacement x_2 as the output of a link with dry friction, while for the input we take x_1 – the forces or moments applied to the movable part of the link. The width of the loop 2b is determined by the magnitude of the dry-frictional force or moment. For example, if we neglect the inertial forces of the movable parts in the centrifugal governor (Fig. 3.16d) and allow for dry friction, then the displacement $\underline{s}$ of the clutch in the centrifugal governor will vary according to the characteristic in Fig. 3.16c as a function of the

forces x_1 applied to the clutch. The clutch displacement starts when the turning rate of the centrifugal governor attains a level for which the force acting upon the clutch overcomes static friction. On a change in the rotational direction of the centrifugal governor, the clutch displacement remains constant as long as the force acting upon the clutch does not vary by an amount equal to 2b.

Assuming that the solution for the input x_1 of a link with friction is sought in harmonic form in accordance with the values of the harmonic-linearization coefficients for a play-type characteristic, we obtain the values of the coefficients q(A) and q'(A):

$$q(A)=\frac{k}{\pi}\left[\frac{\pi}{2}+\arcsin\left(1-\frac{2b}{A}\right)+\right.$$
$$\left.+2\left(1-\frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1-\frac{b}{A}\right)}\right] \text{ for } A\geqslant b, \tag{3.56}$$

$$q'(A)=-\frac{4kb}{\pi A}\left(1-\frac{b}{A}\right) \text{ for } A\geqslant b. \tag{3.57}$$

Here, harmonic linearization of the nonlinear characteristic (Fig. 3.16c) is carried out according to the formula

$$F(x_1)=\left[q(A)+\frac{q'(A)}{\Omega}p\right]x_1. \tag{3.58}$$

After harmonic linearization, the equations for a link with dry friction and linear restoring force

$$\left.\begin{array}{rl} c\,\operatorname{sign} px_2+k_2x_2=k_1x_1 & \text{for } px_2\neq 0, \\ (k_2x_m-c)<k_1x_1<(k_2x_m+c) & \text{for } px_2=0 \end{array}\right\}. \tag{3.59}$$

must be replaced by the single nonlinear equation

$$x_2=F(x_1). \tag{3.60}$$

In (3.59) x_m is the maximum value of x_1 in steady-state oscillatory motion. The nonlinear function $F(x_1)$ in (3.60) must be then replaced by the linear relationship (3.58). The coefficients $\underline{k}$ and $\underline{b}$ in Formulas (3.56) and (3.57) for q(A) and q'(A) are defined through the coefficients of Eq. (3.59) from the relationship:

$$k=\frac{k_1}{k_2}, \quad b=\frac{c}{k_1}.$$

For high velocities we are somtimes obliged to allow for a friction proportional to the square of the velocity. Such a nonlinear characteristic is similar to a power-law characteristic. The only difference from the power-law characteristic will be that here we do not take the quantity $\underline{x}$ for the input, but its rate of change px, i.e., in this case we obtain the nonlinear function

$$F(px)=k(px)^2 \operatorname{sign} px.$$

Assuming that we seek a solution for the velocity in the harmonic form $px = A_p \sin \psi$, $\psi = \Omega t$ in accordance with (3.31) we obtain the value of the harmonic linearization coefficient

$$q(A_p)=\frac{8kA_p}{3\pi}=0{,}85\, kA_p \tag{3.61}$$

and hence

$$F(px)=q(A_p)px=0{,}85\, kA_p px. \tag{3.62}$$

If it is desirable to obtain a solution for the variable $\underline{x}$ itself, then setting

$$x=A \sin \Omega t, \quad px=A\Omega \cos \Omega t,$$

we have

$$A_p=A\Omega.$$

Denoting here

$$q''(A)=\frac{8kA}{3\pi}=0{,}85\, kA, \tag{3.63}$$

we obtain in accordance with (3.61) the harmonic linearization formula in the form

$$F(x)=q''(A)\Omega px. \tag{3.64}$$

§3.7. Nonlinear Characteristics with Leading Loops

Up until this time two-valued loop-type nonlinear characteristics have been considered as lagging characteristics, where on an increase in the input of the nonlinear link the output varies in accordance

with the right-hand branch of the characteristic, while in the case of a decrease, it varies in accordance with the left-hand branch of the characteristic (Fig. 3.17a). Nonlinear characteristics with leading loops are also possible. It is sometimes desirable to create artificially links with nonlinear static characteristics having leading loops, for example, in order to improve the dynamic properties of a nonlinear system.

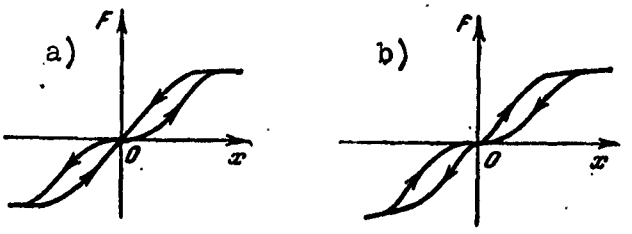


Fig. 3.17.

In a link with a nonlinear characteristic having leading loops (Fig. 3.17b), the output varies in accordance with the left-hand loop on an increase in input, while on a decrease of the input it varies in accordance with the right loop.

Let us determine the harmonic linearization coefficients for certain nonlinear static characteristics with leading loops.

General form of relay characteristic with leading loops. Figure 3.18a shows the general form of relay characteristic with leading loops.

In order to carry out harmonic linearization of the above nonlinear characteristic, we assume that a solution for x is sought in the harmonic form

$$x = A \sin \psi, \quad \psi = \Omega t.$$

Then in accordance with the curve of F(x) we obtain the periodic function F(A sin ψ) with respect to the argument $\psi = \Omega t$ (Fig. 3.18b). The values x = mb and x = b and the values of the argument

$\psi_1 = \arcsin\frac{mb}{A}$, $\psi_2 = \pi - \arcsin\frac{b}{A}$ correspond to the switching points of the relay. Here m is an irrational number in the interval $-1 \leq m \leq 1$.

For the case where the amplitude $A \leq mb$, the relays are cut out and there is no signal transmission in the system. If $mb \leq A \leq b$, then we assume that cutting in and cutting out of the relay take place for the same value $x = mb$. In this case the characteristic with leading loops degenerates into a relay characteristic with a dead zone,

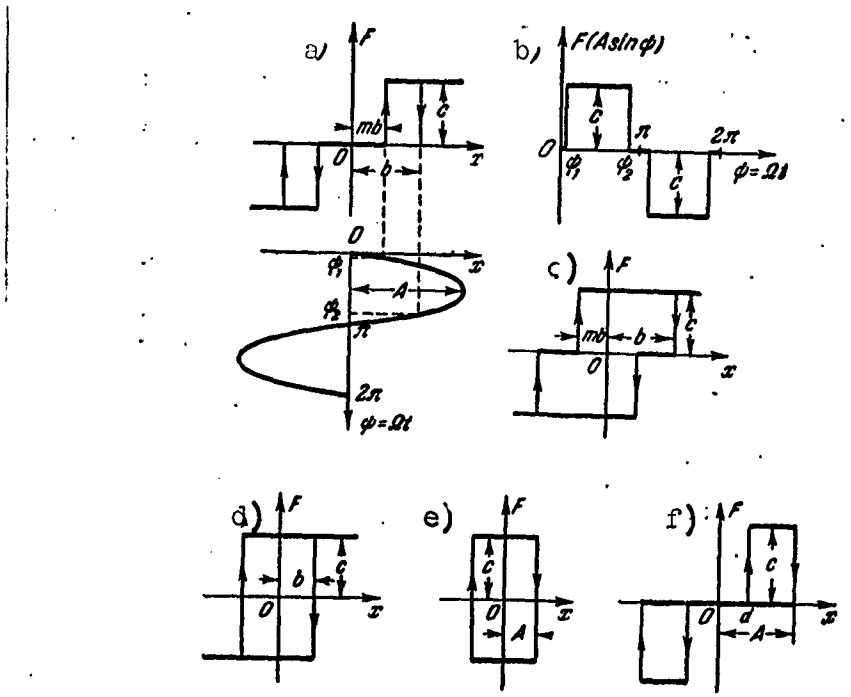


Fig. 3.18.

for which the harmonic-linearization coefficients are determined [see (3.13)]. Here, however, we consider the case for which $A \geq b$ for the case of steady-state oscillation.

Carrying out the calculations according to Formulas (3.3) and allowing for the fact that the values of the integrals in these formulas are the same for each half-period, we obtain

$$q(A) = \frac{2}{\pi A}\int_{\psi_1}^{\psi_2} F(A\sin\psi)\sin\psi\,d\psi = \frac{2}{\pi A}\int_{\psi_1}^{\psi_2} c\sin\psi\,d\psi =$$
$$= -\frac{2c}{\pi A}\cos\psi\Big|_{\psi_1}^{\psi_2} = \frac{2c}{\pi A}(\cos\psi_1 - \cos\psi_2).$$

for the coefficient q(A). Allowing for the values of ψ_1 and ψ_2, we write the formula for the determination of q(A) in the form

$$q(A)=\frac{2c}{\pi A}\left(\sqrt{1-\frac{b^2}{A^2}}+\sqrt{1-\frac{m^2b^2}{A^2}}\right) \text{ for } A\geqslant b. \tag{3.65}$$

Computing the coefficient q'(A), we obtain

$$q'(A)=\frac{2}{\pi A}\int_{\psi_1}^{\psi_2}F(A\sin\psi)\cos\psi\,d\psi=\frac{2}{\pi A}\int_{\psi_1}^{\psi_2}c\cos\psi\,d\psi=$$
$$=\frac{2c}{\pi A}\sin\psi\Big|_{\psi_1}^{\psi_2}=-\frac{2c}{\pi A}(\sin\psi_1-\sin\psi_2),$$

or, allowing for the values of ψ_1 and ψ_2,

$$q'(A)=\frac{2cb}{\pi A^2}(1-m) \text{ for } A\geqslant b. \tag{3.66}$$

Comparing the result obtained with the values of the coefficients for the general-form relay characteristic with lag loops (3.5) and (3.6), we note that the value (3.65) obtained for the coefficient q(A) is the same as (3.5), while the value (3.66) obtained for the coefficient q'(A) is the same in absolute value as (3.6) but has a positive sign. This means that in both cases q(A) expresses in like manner the slope of a straight line replacing the nonlinear characteristic on harmonic linearization. On harmonic linearization, the leading properties of the characteristic in question are expressed by the coefficient q'(A), which determines the fraction of the positive derivative introduced. In characteristics with lagging loops, this derivative was negative. Hence the possibilities for synthesis of nonlinear correcting devices in automatic systems are evident.

Relay characteristic with displaced leading loop. Considering the quantity m as negative ($-1 < m < 0$) in a characteristic having the form of Fig. 3.18c, we obtain the formulas:

$$q(A)=\frac{2c}{\pi A}\left(\sqrt{1-\frac{b^2}{A^2}}+\sqrt{1-\frac{m^2b^2}{A^2}}\right) \text{ for } A\geqslant b, \tag{3.67}$$

$$q'(A)=\frac{2cb}{\pi A^2}(1+|m|) \text{ for } A\geqslant b. \tag{3.68}$$

from (3.65) and (3.66).

Relay characteristic with leading loop. For the relay characteristic of Fig. 3.18d we have m = − 1. Then in accordance with (3.65) and (3.66) we obtain the values of the harmonic-linearization coefficients:

$$q(A)=\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}} \quad \text{for } A\geq b, \tag{3.69}$$

$$q'(A)=\frac{4cb}{\pi A^2} \quad \text{for } A\geq b. \tag{3.70}$$

Relay characteristic with leading loop of variable width. In the relay characteristic of Fig. 3.18e, the width of the loop is equal to twice the amplitude and varies together with the variation of the amplitude. Then setting b = A in (3.69) and (3.70), we obtain

$$q(A)=0, \tag{3.71}$$

$$q'(A)=\frac{4c}{\pi A}. \tag{3.72}$$

It is evident that after harmonic linearization, a link with such a nonlinear characteristic is equivalent to a differentiator. A link with the same form but a lagging characteristic would be equivalent to an integrating link.

Relay characteristic with two lag loops of variable width. In the relay characteristic of Fig. 3.18f, the input value x = d = cons corresponds to operation of the relay. Dropout of the relay takes place on a change in sign of the input velocity, when x = A. Assuming that mb = d and b = A in the general-form relay characteristic, we obtain

$$q(A)=\frac{2c}{\pi A}\sqrt{1-\frac{d^2}{A^2}} \quad \text{for } A\geq d, \tag{3.73}$$

$$q'(A)=\frac{2c}{\pi A}\left(1-\frac{d}{A}\right) \quad \text{for } A\geq d. \tag{3.74}$$

Characteristic with trapezoidal leading loops of variable width.

Let us calculate the harmonic-linearization coefficients for the characteristic shown in Fig. 3.19a.

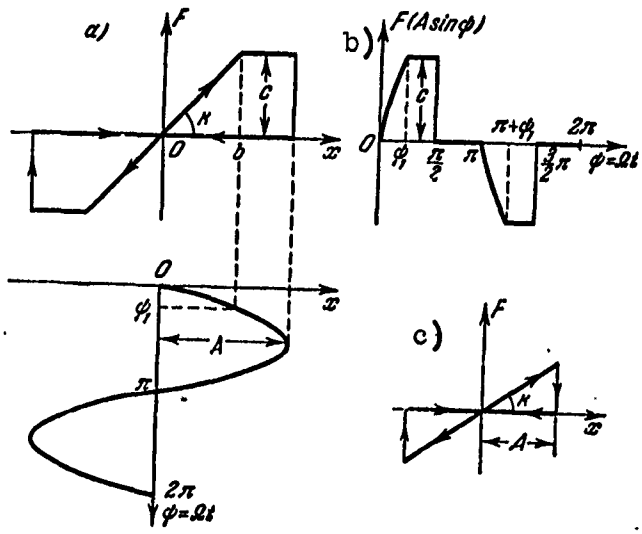


Fig. 3.19.

Assuming that we seek a solution for the input $\underline{x}$ of the nonlinear link in the form

$$x = A \sin \psi, \quad \psi = \Omega t,$$

we obtain the output F(a sin ψ) in the form of a periodic function of the argument $\psi = \Omega t$ (Fig. 3.19b).

Performing the calculations according to Formulas (3.3) and allowing for the fact that the values of the integrals in the formulas are the same for each half-period, we obtain

$$q(A) = \frac{2}{\pi A}\left(\int_0^{\psi_1} kA \sin\psi \sin\psi \, d\psi + \int_{\psi_1}^{\frac{\pi}{2}} c \sin\psi \, d\psi\right) =$$
$$= \frac{2}{\pi A}\left[kA\left(\frac{\psi}{2}\Big|_0^{\psi_1} - \frac{1}{4}\sin 2\psi\Big|_0^{\psi_1}\right) - c\cos\psi\Big|_{\psi_1}^{\frac{\pi}{2}}\right] =$$
$$= \frac{k}{\pi}\left(\psi_1 - \frac{1}{2}\sin 2\psi_1\right) + \frac{2c}{\pi A}\cos\psi_1.$$

for the coefficient q(A). Allowing for the value ψ_1 = arc sin b/A and c = kb, we obtain

$$q(A) = \frac{k}{\pi}\left(\arcsin\frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right) \text{ for } A \geq b. \qquad (3.75)$$

For the coefficient q'(A) we find

$$q'(A)=\frac{2}{\pi A}\left(\int_0^{\psi_1} kA\sin\psi\cos\psi\,d\psi+\int_{\psi_1}^{\frac{\pi}{2}} c\cos\psi\,d\psi\right)=$$
$$=\frac{2}{\pi A}\left(\frac{1}{2}kA\sin^2\psi\Big|_0^{\psi_1}+c\sin\psi\Big|_{\psi_1}^{\frac{\pi}{2}}\right)=$$
$$=\frac{k}{\pi}\sin^2\psi_1+\frac{2c}{\pi A}(1-\sin\psi_1).$$

Allowing for the values ψ_1 = arc sin b/A and c = kb, we obtain

$$q'(A)=\frac{2kb}{\pi A}\left(1-\frac{b}{2A}\right) \text{ for } A\geqslant b. \tag{3.76}$$

In the case where in a steady-state oscillatory process the amplitude values A $\leq$ b, a characteristic with trapezoidal loops degenerates into a characteristic with triangular leading loops of variable width (Fig. 3.19c). For the case of such a characteristic, the quantity b will be a variable and equal to A. Setting b = A in (3.75) and (3.76), we obtain formulas for the calculation of the harmonic-linearization coefficients:

$$q=\frac{k}{2},\quad q'=\frac{k}{\pi}. \tag{3.77}$$

As is evident from (3.77), the harmonic-linearization coefficients for a characteristic with triangular leading loops are not functions of the amplitude A, i.e., a link with such a characteristic will behave like a linear link with the introduction of a derivative. Here the gain constant for the input x is equal to k/2, while the gain constant for the time derivative of the input px is equal to $k/\pi\Omega$.

The harmonic linearization of the nonlinearities is performed simply enough for any piecewise-linear and other nonlinear static characteristics that can be represented analytically. If such representation is difficult, then graphical methods for determining the harmonic-linearization coefficients are also possible. One of the graphical methods is considered in the following section.

§3.8. Graphical Method for Harmonic Linearization of Nonlinearities

In the event that it is difficult to represent the nonlinear static characteristic in analytical form, but it is trustworthy (i.e., obtained experimentally), it is convenient to use a graphical method for determining the harmonic-linearization coefficients. Reference [81] introduces a graphical method based upon the connection of the harmonic linearization coefficients with the shape of the static characteristic. The justification for this is given in that paper. This method may also be extended to both single-valued symmetrical nonlinear static characteristics and two-valued and nonsymmetrical characteristics. Here, without the proof, we introduce a graphical method of determining the harmonic-linearization coefficients of single-valued symmetrical nonlinear static characteristics.

On the basis of the approximate calculations of the integral, the exact formula for calculation of the harmonic-linearization coefficients

$$q(A)=\frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi)\sin\psi\,d\psi$$

may be reduced to the expression

$$q(A)\approx\frac{2}{3A}\left[F(A)+F\left(\frac{A}{2}\right)\right]. \tag{3.78}$$

The approximate dependence (3.78) permits us to determine q(A) by graphical means.

In order to determine q(A) we must plot the right-hand branch of the nonlinear static characteristic F(x) = F(A) (Fig. 3.20) on the curve. Varying the sacle along the axis of abscissas, we obtain the curve F(A/2). Adding the ordinates of the first and second curves, we obtain the curve F(A) + F(A/2). Then we draw a straight line through the point A = – 2/3 parallel to the axis of ordinates. In

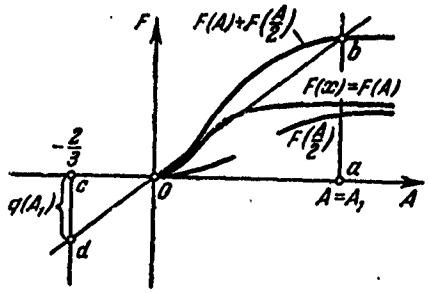


Fig. 3.20.

order to determine the values of $q(A)$ for the case of the value $A = A_1$ in question, we must construct a straight line through the origin and the ordinate $F(A) + F(A/2)$ of the curve. The segment cut off by this line on the line $A = 2/3$ is equal to the value $q(A_1)$ which we seek. This is evident from the similarity of the triangles Ocd and aOb.

For the case of the condition that $Oc = 2/3$, $ab = F(A_1) + F(A_1/2)$, $Oa = A_1$, we have

$$cd = q(A_1) = \frac{2}{3}\frac{F(A_1) + F\left(\frac{A_1}{2}\right)}{A_1}.$$

Performing a similar operation for other values $A = A_2$, $A = A_3$, ..., $A = A_n$, we obtain the curve of $q(A)$.

The accuracy with which the harmonic-linearization coefficients are determined by graphical means is completely sufficient for practical work. In Fig. 3.21 we show for comparison curves of $q(A)$ for the nonlinear characteristics considered earlier as calculated from exact formulas (continuous curves) and those obtained by the graphical method introduced (broken-line curves).

In addition to the graphical method described, we may also make use of the experimental determination of $q(A)$ based upon Reference [113].

In those cases where the harmonic-linearization coefficients are determined by a graphical or experimental method in the form of curves of $q(A)$, the unknowns $\underline{q}$ and Ω for which the equations are solved will occur in the equations for finding the periodic solution (see §2.3). The curves of $q(A)$ are used for determination of the

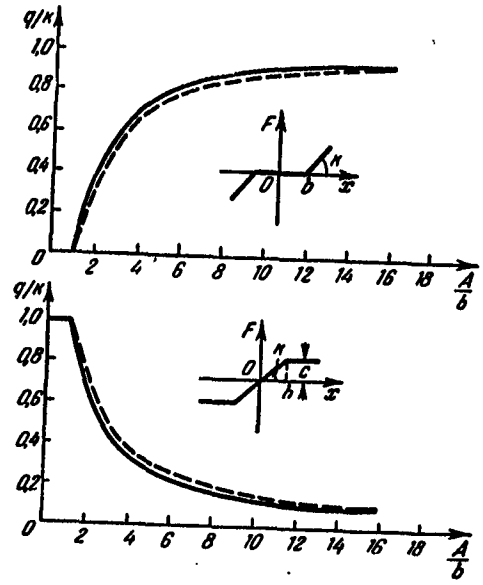


Fig. 3.21.

amplitude A of the periodic solution from the value of q known for this solution. Such a method is also often useful in the case of complex analytical dependence of the harmonic-linearization coefficients upon amplitude. To make use of this method, it is sufficient to construct the curve of q(A) in advance for the nonlinearity being considered.

§3.9. A Relay Follow-Up Device

As shown in [127], a relay output device (Fig. 3.22a) consisting of a separately-excited motor and a relay controlling an armature circuit (Fig. 3.22a) cannot be regarded as a series connection of a relay link and a linear link.

In analysis of processes for a system incorporating a relay-type output device, we may represent the control relay approximately in the form of a link with the static characteristic shown in Fig. 3.22b, for the case of a large relay recovery coefficient.

For the case of a steady-state oscillatory process in the system, the motion of the motor armature over the course of one oscillation

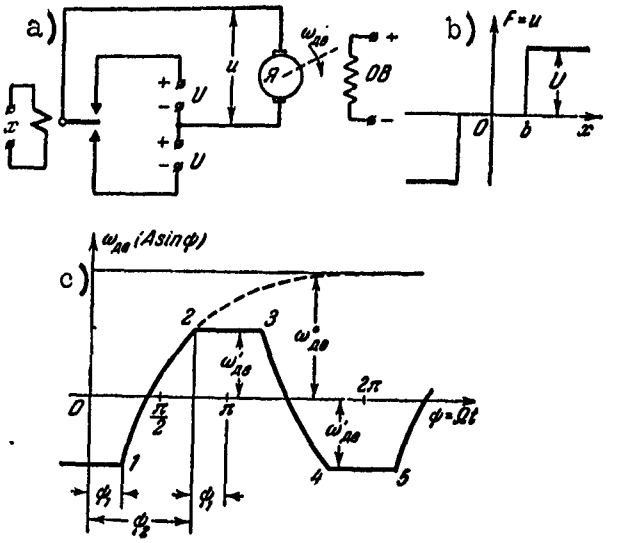


Fig. 3.22.

period is described by different differential equations depending upon whether the relay contacts are closed or opened.

If we do not allow for the load moment and the mechanical characteristics of the motor are assumed linear, then for the case of closed contacts the equation of the motor has the form

$$(Tp+1)\omega_{дв}=kM, \tag{3.79}$$

where ω_{dv} is the angular velocity of the motor shaft, M is the starting torque, T is the electromechanical time constant of the motor and $\underline{k}$ is the transmission ratio.

For the case of the open state of the armature contacts, the motor becomes a flywheel, and the motion of the motor armature is described by the equation

$$Ip\omega_{дв}=0, \tag{3.80}$$

where I is the moment of inertia of the motor armature.

In performing the harmonic linearization in this case, therefore, the relay and the motor must be considered jointly.

Assuming that the input $\underline{x}$ varies according to the sinusoidal relationship

$$x = A \sin \Omega t,$$

determining the periodic function $\omega_{dv} = \omega_{dv}(\psi)$, $\psi = \Omega t$ requires finding a solution for the steady-state mode of operation from Eqs. (3.79) and (3.80) allowing for the relay characteristic (Fig. 3.22b). This solution is shown in Fig. 3.22c. From zero to point 1, the motor has the constant speed $\omega'_{dv} = \text{const}$, whose sign is determined by the previous closed position of the relay contacts. Here the angle ψ_1 is determined by the relationship

$$\psi_1 = \arcsin \frac{b}{A}. \tag{3.81}$$

For the segment 1 – 2, Solution (3.79) gives

$$\omega_{дв} = \omega^0_{дв} - (\omega^0_{дв} + \omega'_{дв}) e^{-\frac{\psi}{2\pi}}, \tag{3.82}$$

where $\omega^0_{dv} = kM$ is the steady-state speed, and ω'_{dv} is the rate at which the motor runs down with contacts open.

The value of the angle ψ_2 is determined by the equality

$$\psi_2 = \pi - \arcsin \frac{b}{a}. \tag{3.83}$$

The absolute value of the motor's rundown rate is determined from (3.82) if in place of ψ we substitute its value for the segment 1 – 2 or the function $\omega_{dv}(\sin \psi)$:

$$\psi_{1-2} = \pi - 2\psi_1. \tag{3.84}$$

Solving (3.82) with respect to ω'_{dv} and allowing for (3.83), we obtain

$$\omega'_{дв} = \frac{1-y}{1+y} \omega^0_{дв}, \tag{3.85}$$

where

$$y = e^{-\frac{\pi - 2\psi_1}{2\pi}}. \tag{3.86}$$

Let us determine the values of the harmonic-linearization coefficients q(A) and q'(A) for a relay output device.

On the basis of Formula (3.3), in accordance with the form of

the periodic function $\omega_{dv}(A \sin \psi)$ (Fig. 3.22c), we obtain

$$q(A)=\frac{1}{\pi A}\int_0^{2\pi}\omega_{\lambda_B}(A\sin\psi)\sin\psi\,d\psi=\frac{2}{\pi A}\left\{\int_0^{\psi_1}(-\omega'_{\lambda_B}\sin\psi)\,d\psi+\right.$$

$$+\int_{\psi_1}^{\psi_2}\left[\omega^0_{\lambda_B}-(\omega^0_{\lambda_B}+\omega'_{\lambda_B})e^{-\frac{\psi}{2\pi}}\right]\sin\psi\,d\psi+\left.\int_{\psi_2}^{\pi}\omega'_{\lambda_B}\sin\psi\,d\psi\right\}=$$

$$=\frac{2}{\pi A}\left\{\omega'_{\lambda_B}(\cos\psi_2+\cos\psi_1)+\omega^0_{\lambda_B}(\cos\psi_1-\cos\psi_2)+\right.$$

$$\left.+\frac{4\pi^2}{1+4\pi^2}(\omega^0_{\lambda_B}+\omega'_{\lambda_B})\left[e^{-\frac{\psi_2}{2\pi}}\left(\frac{1}{2\pi}\sin\psi_2+\cos\psi_2\right)-e^{-\frac{\psi_1}{2\pi}}\times\right.\right.$$

$$\left.\left.\times\left(\frac{1}{2\pi}\sin\psi_1+\cos\psi_1\right)\right]\right\}.$$

for q(A). Since from (3.81), (3.83), and Fig. 3.22 we have

$$\cos\psi_2=-\cos\psi_1,\qquad \sin\psi_2=\sin\psi_1,$$

we finally obtain, allowing for the values of ω^0_{dv} and ω^1_{dv},

$$q(A)=\frac{4kM}{\pi A}\left\{\sqrt{1-\frac{b^2}{A^2}}+\frac{4\pi^2}{(1+4\pi^2)(1+y)}\left[e^{-\frac{\psi_2}{2\pi}}\left(\frac{b}{2\pi A}-\sqrt{1-\frac{b^2}{A^2}}\right)-\right.\right.$$

$$\left.\left.-e^{-\frac{\psi_1}{2\pi}}\left(\frac{b}{2\pi A}+\sqrt{1-\frac{b^2}{A^2}}\right)\right]\right\},\qquad (3.87)$$

where y is determined from Formula (3.86), while ψ_1 and ψ_2 are determined from Formulas (3.81) and (3.83).

For the coefficient q'(A) we obtain

$$q'(A)=\frac{1}{\pi A}\int_0^{2\pi}\omega_{\lambda_B}(A\sin\psi)\cos\psi\,d\psi=\frac{2}{\pi A}\left\{\int_0^{\psi_1}(-\omega'_{\lambda_B}\cos\psi)\,d\psi+\right.$$

$$+\int_{\psi_1}^{\psi_2}\left[\omega^0_{\lambda_B}-(\omega^0_{\lambda_B}+\omega'_{\lambda_B})e^{-\frac{\psi}{2\pi}}\right]\cos\psi\,d\psi+\left.\int_{\psi_2}^{\pi}\omega'_{\lambda_B}\cos\psi\,d\psi\right\}=$$

$$=\frac{2}{\pi A}\left\{-\omega'_{\lambda_B}\sin\psi_1-\omega'_{\lambda_B}\sin\psi_2-\frac{4\pi^2}{1+4\pi^2}(\omega^0_{\lambda_B}+\omega'_{\lambda_B})\times\right.$$

$$\left.\times\left[e^{-\frac{\psi_2}{2\pi}}\left(-\frac{1}{2\pi}\cos\psi_2+\sin\psi_2\right)-e^{-\frac{\psi_1}{2\pi}}\left(-\frac{1}{2\pi}\cos\psi_1+\sin\psi_1\right)\right]\right\}.$$

Substituting the values of ω^0_{dv} and ω^1_{dv}, we finally obtain

$$q'(A)=-\frac{4kM}{\pi A(1+y)}\left\{\frac{b(1-y)}{A}+\frac{4\pi^2}{1+4\pi^2}\left[e^{-\frac{\psi_2}{2\pi}}\left(\frac{1}{2\pi}\sqrt{1-\frac{b^2}{A^2}}+\right.\right.\right.$$

$$\left.\left.\left.+\frac{b}{A}\right)+e^{-\frac{\psi_1}{2\pi}}\left(\frac{1}{2\pi}\sqrt{1-\frac{b^2}{A^2}}-\frac{b}{A}\right)\right]\right\},\qquad (3.88)$$

where y is determined from Formula (3.86), while ψ_1 and ψ_2 are deter-

mined from Formulas (3.81) and (3.83).

Similarly we may also perform linearization of nonlinearities in another similar case, for example, in the case where allowance is made for the hysteresis loops in a relay characteristic where the relay has a small recovery coefficient.

Above in §2.1 a general method was given for performing harmonic linearization in the case where the nonlinearity is expressed in the form of the structural variation of the differential equations or transfer functions for the automatic system.

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[Footnote]

253 In the case of a scheme without dynamic braking, the linearization should be made for the relay and motor considered together (see §3.9).

[List of Transliterated Symbols]

253 д = d = dvigatel' = motor (letter symbol u)
253 у = u = upravleniye = control
254 Д = D = dvigatel' = motor
256 д = d = dobavochnyy = additional, adjusting (letter symbol R)
262 п = p = perekrytiye = overlap
274 дв = dv = dvigatel' = motor
299 Я = Ya = yakor' = armature
299 ОВ = OV = obmotka vozbuzhdeniya = excitation winding

Chapter 4

SYMMETRICAL SELF-OSCILLATIONS AND STABILITY OF AUTOMATIC SYSTEMS

§4.1. Requirements Imposed on Self-Oscillating Systems

Nonlinear closed loop automatic systems frequently oscillate in the steady state. In some systems, the self-oscillating mode is unavoidable for all values of the parameters. For other nonlinear systems, depending on the values of the parameters, it is possible to have either a self-oscillating steady state mode, or else a stationary mode without self-oscillations. In the former case we call the domain of parameter variation the self-oscillation region, and in the latter we call it the region of stability of equilibrium. In addition to the regions indicated, other regions are also possible in the case of nonlinear systems, particularly instability regions. Nonlinear systems, unlike linear ones, can have several stable and unstable equilibrium states, with the stability regions confined not only to certain definite values of the system parameters, but also to definite relationships between the initial conditions.

Depending on the type of automatic system and on its operating conditions, we can ascertain whether self-oscillations can or cannot be tolerated in a given system. Thus, for example, in computers constructed in the form of servomechanisms with small moving masses, a self-oscillating mode is frequently useful, for it reduces the backlash zone due to dry friction. A self-oscillation mode can be useful in vibration smoothing of nonlinear static characteristics (particularly relay characteristics). On the other hand, if the system has

large moving masses, as for example in an aircraft fire control system, self-oscillation cannot be tolerated, since the presence of such a mode leads to a reduction in the firing accuracy, to the occurrence of large overloads in the moving part of the system, and to premature wearing out of the kinematic transmission to the controlled object.

In the analysis of self-oscillating systems, that is, systems for which a self-oscillating operating mode is possible, it is of interest in practice to determine the values of the parameters for which a self-oscillating mode is possible in the system, how each of the parameters of the system influences the self-oscillations, what means can be used to change the amplitude and frequency of the self-oscillations or to suppress the self-oscillations if necessary.

The amplitude of the self-oscillations should be in almost all cases as small as possible, for this determines the accuracy of the system in the steady-state mode under symmetrical oscillations. Only on occasion is it necessary to produce oscillations of specified (not small) amplitude (see §4.16). In the presence of an external signal which is either constant or slowly varying, the amplitudes become deformed, since the center of the oscillations shifts. In this case the errors in the system will be determined by the magnitude of the shift of the oscillation center, and the smaller the oscillation amplitude, the smaller the error will be.

In the present chapter we shall investigate only symmetrical self-oscillations and the stability of nonlinear systems in the absence of an external signal. An investigation of self-oscillating modes under constant and variable external signals will be considered in Chapters 5 and 6 below.

The self-oscillation frequency should in most cases be as high as possible, although it is sometimes specified such as to satisfy other

requirements. The limiting factor in this case may be the increased overload on the mechanical elements with increasing frequency. On the other hand, the requirement of high frequency is necessitated by the fact that in systems that duplicate a master signal, the frequency of self-oscillations should be at least one order of magnitude higher than the frequency of the duplicated signal. Otherwise such a system cannot attain sufficient accuracy. On the other hand, the presence of high self-oscillation frequencies makes it possible to obtain by sufficiently simple means from a closed loop self-oscillating system an output without noticeable oscillation amplitudes, owing to the inertial properties of the individual elements. A change in the self-oscillation frequency is frequently necessary in those self-oscillating systems which operate under vibration conditions, so as to get rid of the system errors resulting from forced vibrations.

Other requirements on the steady state oscillating mode can be stipulated for each specific system, depending on its operating conditions.

In any case it is necessary to know the influence of the system parameters on the form of its steady state mode. If the steady state mode is self-oscillating, it is important to know the dependence of the frequency and of the amplitude of the self-oscillations on each system parameter.

Frequently the accuracy required in the determination of these dependences is not high, and it is sufficient to have merely some indication as to the direction in which a particular parameter must be changed in order to decrease the amplitude and increase the frequency of the self-oscillations. For engineering purposes, it is therefore possible to use approximate methods to obtain answers to these questions in the investigation of self-oscillating modes. In many practical

cases it is most advantageous to investigate the self-oscillations on the basis of the harmonic linearization method. By using this method one can obtain analytic expressions for the amplitude and frequency in terms of the system parameters in explicit form. If the derivation of such explicit relations is difficult, various graphic procedures for solving the resultant equations can be employed (see §2.3).

We shall examine several examples illustrating the derivation of such relationships for self-oscillating systems.

Along with determining the self-oscillation regions, it is necessary to estimate the stability of the periodic solutions. This will be done with the aid of approximate stability criteria for periodic solutions, as described in §2.4.

Finally, we shall also determine in the present chapter the stability regions of the equilibrium state of the system by means of the approximate procedures indicated in §§2.7 and 2.9.

The investigation will be carried out using specific examples of automatic systems of various types and using numerical values in the results, so as to illustrate as fully as possible the application of the methods in engineering practice.

§4 2. System for Stabilization of a Gyro Pendulum

We shall carry out an investigation of self oscillations using the stabilization of a gyroscope as an example [129]. Figure 4.1 shows the schematic diagram of a gyroscope with three degrees of freedom (gyro pendulum) which is unbalanced relative to the axis of the internal gimbal, z. If a force f is applied to the gyroscope in a direction parallel to the axis x of the external gimbal, then the gyroscope will precess about this axis. The rate of precession will be proportional to the magnitude of the force f. Because of the presence of friction in the suspension, a torque arises as the gyroscope moves relative to

the axis of the external gimbal. In addition, a torque may be produced relative to the axis of the external gimbal as a result of inaccurate balancing of the system, which consists of the gyroscope, the internal gimbal, and the external gimbal, relative to the x axis. Under the influence of the torque about the x axis applied to the gyroscope, the latter will start precessing about the axis z of the internal gimbal. Since the unbalanced gyroscope is used as a meter for the force f, it becomes necessary to stabilize the motion of the gyroscope about the x axis in the yO'z plane in order to maintain constant the lever arm l_0 of the measured force. The torque produced about the x axis is a random function of the time, and therefore exact compensation of this torque by applying a torque of opposite sign is practically impossible. Consequently, a closed loop stabilization system is usually employed.

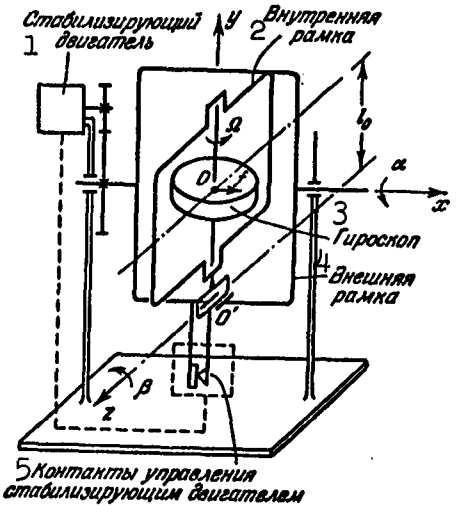


Fig. 4.1. 1) Stabilizing motor; 2) internal gimbal; 3) gyroscope; 4) external gimbal; 5) contacts controlling the stabilizing motor.

For this purpose, a stabilizing motor is geared to the shaft of the outer gimbal; the motor is controlled by relay contacts. One of the contacts controlling the stabilizing motor is secured to the outer

gimbal and the other to the inner gimbal so that whenever the gyroscope axis is perpendicular to the axis of the outer gimbal the contacts touch each other without pressure.

To ensure freedom of motion of the gyroscope about the z axis, one of the contacts is mounted on a spring of low stiffness.

The presence of torque about the x axis causes the gyroscope to rotate about the z axis, and depending on the direction of the torque the contacts will either close or open. The electrical circuit is such as to ensure change of sign of the stabilizing-motor torque with the contacts closed and open. If the motor torque exceeds the friction and unbalance torque applied to the gyroscope relative to the axis of the outer gimbal, then an angular deflection of the gyroscope in the β direction reverses the motor and the gyroscope starts precessing in the opposite direction. This brings about a new reversal of the motor in the opposite direction. Thus, the stabilization system guarantees a self-oscillating operating mode. The torques applied about the x axis cancel out during one period of oscillation. The gyroscope axis, while oscillating in the direction of the angle β, is maintained on the average in a plane perpendicular to the axis of the outer gimbal.

To increase the accuracy with which the gyro pendulum measures an external force, it is necessary to make the self-oscillations have a high frequency and a low amplitude. The gyroscope stabilization system just discussed is an example in which the self-oscillating mode is useful, for such a mode affords a simple solution for the problem of stabilizing the motion of a gyroscope in a plane perpendicular to the axis of the outer gimbal.

Let us investigate the self-oscillations of a gyroscope stabilization system (disregarding the torques due to the drive friction and to the unbalance) by the harmonic linearization method.

The equations of motion of the gyroscope, coordinates directed as shown in Fig. 4.1, can be written in the form

$$\left.\begin{aligned} A_0\ddot{\alpha} + n_\alpha\dot{\alpha} - H\dot{\beta} &= M_{s.d}(\beta), \\ B_0\ddot{\beta} + n_\beta\dot{\beta} + H\dot{\alpha} &= \mu_0, \end{aligned}\right\} \qquad (4.1)$$

where α is the angle of gyroscope rotation about the axis of the outer gimbal; β is the angle of gyroscope rotation about the axis of the inner gimbal; A_0 [g·cm·sec^2] is the moment of inertia of the system comprising the gyroscope and the inner and outer gimbals about the x axis; B_0 [g·cm·sec^2] is the moment of inertia of the gyroscope together with the inner gimbal about the z axis; H [g·cm·sec] is the kinetic moment of the gyroscope; n_α, n_β [g·cm·sec] are the coefficients of viscous friction; $M_{s.d}$ [g·cm] is the torque of the stabilizing motor; μ_0 [g·cm] is the external torque applied to the gyroscope as a result of the measured force f.

Let us reduce Eqs. (4.1) to a single equation with variable β. By determining $\dot{\alpha}$ and $\ddot{\alpha}$ and substituting these values into the first equation of (4.1), we obtain

$$A_0B_0\dddot{\beta} + (A_0n_\beta + B_0n_\alpha)\ddot{\beta} + (n_\alpha n_\beta + H^2)\dot{\beta} = -HM_{s.d}(\beta) + n_\alpha\mu_0.$$

Usually $n_\alpha n_\beta \ll H^2$; then, neglecting $n_\alpha n_\beta$ compared with H^2 and introducing the operator symbol $p = d/dt$, we obtain

$$[A_0B_0p^3 + (A_0n_\beta + B_0n_\alpha)p^2 + H^2p]\beta = -HM_{s.d}(\beta) + n_\alpha\mu_0. \qquad (4.2)$$

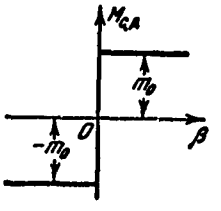


Fig. 4.2.

We shall investigate the gyroscope oscillations with the stabilizing motor connected and without an external torque, that is, $\mu_0 = 0$. Then Eq. (4.2) can be rewritten

$$[A_0B_0p^3 + (A_0n_\beta + B_0n_\alpha)p^2 + H^2p]\beta = -HM_{s.d}(\beta). \qquad (4.3)$$

The torque of the stabilizing motor $M_{s.d}(\beta)$ is a nonlinear function of β. We shall assume that this nonlinear function has the form of an ideal relay characteristic (Fig. 4.2). Such a characteristic is

obtained in practice by using a two-phase induction motor as the stabilizing motor. Since the motor operates in practice with the rotor locked, its torque will be constant. By feeding the motor with alternating current at a frequency on the order of 400 cps, the lag in the occurrence of the torque due to the control voltage is eliminated. The torque of a two-phase induction motor is determined by the power component of the current, which in turn is lag-free relative to the applied voltage.

The presence of a symmetrical nonlinear characteristic in the absence of an external signal makes it possible to carry out harmonic linearization by using the condition that the expected periodic motion in the system is symmetrical.

We seek a periodic solution in a sinusoidal form

$$\beta = A \sin \Omega t.$$

Then, taking into account the values of the coefficients of harmonic linearization (3.14) for an ideal relay characteristic, we replace the torque $M_{s.d}$, which is nonlinear in β, by the relation

$$M_{c.z} = q(A)\beta = \frac{4m_0}{\pi A}\beta.$$

Substituting the value of $M_{s.d}$ in (4.3) we obtain the harmonically linearized equation of the system

$$\left[A_0 B_0 p^3 + (A_0 n_\beta + B_0 n_\alpha) p^2 + H^2 p + \frac{4m_0 H}{\pi A}\right]\beta = 0. \qquad (4.4)$$

In accord with the first method for finding the self-oscillations (§2.3), we substitute into this equation $p = j\Omega$ and, separating the real and imaginary parts, we obtain two equations for the periodic solution (for the variable β):

$$\left.\begin{aligned} \frac{4m_0 H}{\pi A} - (A_0 n_\beta + B_0 n_\alpha)\Omega^2 = 0, \\ H^2 - A_0 B_0 \Omega^2 = 0. \end{aligned}\right\} \qquad (4.5)$$

We estimate the stability of the periodic motion by using the ap-

proximate analytic criterion (2.125), i.e., we determine whether the condition

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0. \qquad (4.6)$$

is satisfied. The asterisks following the derivatives denote that one substitutes in the derivatives the values of the amplitude and frequency of the investigated periodic solution $a = A$ and $\omega = \Omega$.

To determine the corresponding derivatives, we write out, on the basis of Eq. (4.4), the formulas for the real and imaginary parts of the expression of the Mikhaylov curve

$$X(a, \omega) = \frac{4m_0 H}{\pi a} - (A_0 n_\beta + B_0 n_\alpha)\omega^2,$$
$$Y(a, \omega) = H^2\omega - A_0 B_0 \omega^3.$$

We shall denote the amplitudes and the frequencies from now on by $\underline{a}$ and ω, in accordance with the fact that the stability criterion of the periodic equation was obtained in §2.4 from an analysis of a system whose amplitudes were different from those in the periodic solution, i.e., from an analysis of a Mikhaylov curve shifted away from the origin.

After determining the corresponding derivatives, we get

$$\left(\frac{\partial X}{\partial a}\right)^* = -\frac{4m_0 H}{\pi A^2} < 0, \quad \left(\frac{\partial Y}{\partial a}\right)^* = 0,$$
$$\left(\frac{\partial X}{\partial \omega}\right)^* = -2(A_0 n_\beta + B_0 n_\alpha)\Omega < 0, \quad \left(\frac{\partial Y}{\partial \omega}\right)^* = H^2 - 3A_0 B_0 \Omega^2.$$

From the second equation of (4.5) we have

$$\Omega^2 = \frac{H^2}{A_0 B_0},$$

therefore

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = -2H^2 < 0.$$

As can be seen from the signs of all the derivatives, the stability criterion (4.6) is satisfied for the periodic solution. The supposition

that the gyroscope stabilization system is in a self-oscillating mode is thus confirmed.

In the present case we did not have to investigate the stability of the periodic solution, for in our example it is unique and it is obvious from the operating principle of the stabilization system that it corresponds to self-oscillations.

From the second equation of (4.5) we obtain a formula for the self-oscillation frequency expressed in terms of the system parameters:

$$\Omega = \frac{H}{\sqrt{A_0 B_0}}. \tag{4.7}$$

Substituting the value of Ω in the first equation of (4.5), we obtain a formula for the self-oscillation amplitude:

$$A = \frac{4m_0 A_0 B_0}{\pi H (A_0 n_\beta + B_0 n_\alpha)}. \tag{4.8}$$

As can be seen from (4.7), the self-oscillation frequency is determined by the kinetic moment of the gyroscope and by the moments of inertia of the gyroscope and of the suspension gimbals. The frequency is equal to the so-called nutation frequency. The role of the stabilizing motor reduces to maintaining nutation oscillations, which would be damped out by friction in the gyroscope gimbals were the stabilizing motor to be disconnected.

By assigning values to the parameters in (4.7) and (4.8) and varying each of these in succession within the limits of practical interest, we can plot the variations of the frequency and of the amplitude of the self-oscillations with respect to each parameter. Figure 4.3 shows such plots for the case when all other parameters are kept constant: $H = 400$ g·cm·sec, $A_0 = 2$ g·cm·sec^2, $B_0 = 2$ g·cm·sec^2, $n_\alpha = 2$ g·cm·sec, $n_\beta = 1$ g·cm·sec, $m_0 = 30$ g·cm.

For a qualitative evaluation of the results obtained, an experiment was made with a gyroscope unbalanced with respect to the axis of

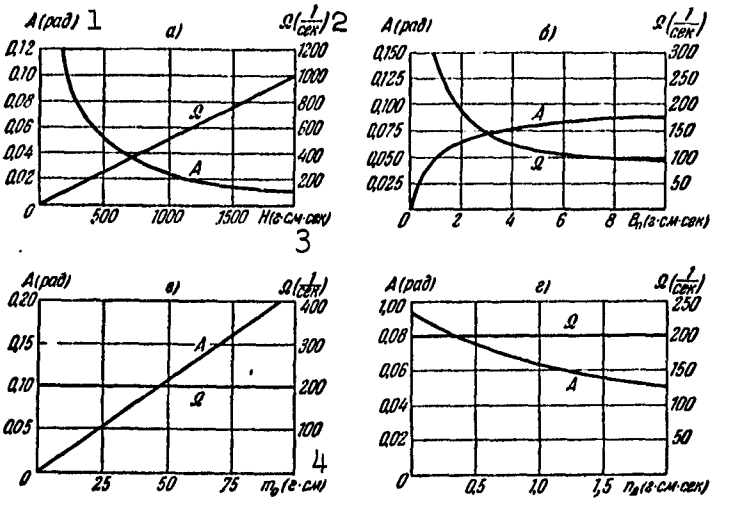


Fig. 4.3. 1) Rad; 2) 1/sec; 3) g·cm·sec; 4) g·cm.

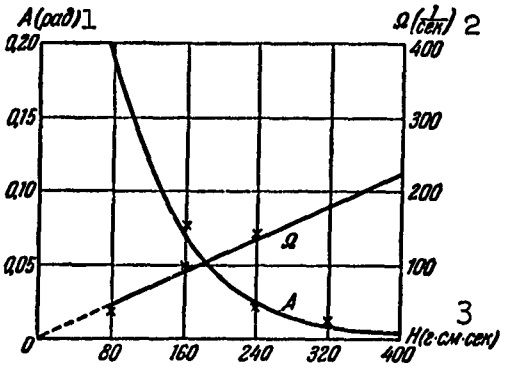


Fig. 4.4. 1) Rad; 2) 1/sec; 3) g·cm·sec.

the internal gimbal, having a polar moment of inertia $J = 0.153$ g·cm·sec^2. Variation of the speed of rotation changed the kinetic moment H of the gyroscope and yielded the experimental dependences A(H) and Ω(H) (Fig. 4.4).

The resultant experimental curves confirmed qualitatively the analytical dependences A(H) and Ω(H).

Plots of the amplitude and frequency of self-oscillations of a gyro pendulum as functions of the system parameters can serve as guidelines for the construction of similar instruments. It follows from the plots that in order to obtain a high frequency and a low amplitude of

gyroscope self-oscillation, it is desirable to increase the kinetic moment of the gyroscope, to decrease the moment of inertia of the gyroscope gimbals, to increase the damping of the gyroscope motion about the suspension axes and decrease as much as possible the torque of the stabilizing motor. Reduction in the torque of the stabilizing motor is limited by the friction torque and by the inaccuracy in the balancing of the gyroscope relative to the axis of the external gimbal. Consequently, it is necessary to strive toward low friction in the suspension shaft and thorough balancing of the gyroscope together with its gimbals relative to the axis of the outer gimbal.

§4.3. Servomechanism with Nonlinear Amplifier

Let us consider a servomechanism intended for remote transmission of the angular positions of a transmitter to a controlled object (Fig. 4.5). The system is fed with direct current. The error transducer is a

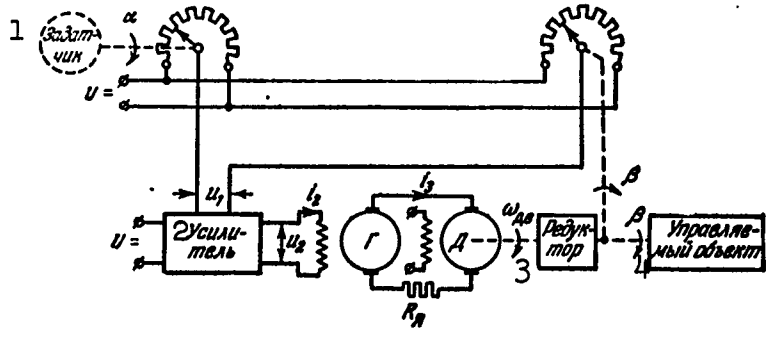


Fig. 4.5. 1) Transmitter; 2) amplifier; 3) reduction gear; 4) controlled object.

rheostat pair. The amplification unit comprises an electronic amplifier and a motor-generator set.

We assume that the output voltage u_1 of the electronic amplifier can attain values such that an upper limit exists for the output voltage u_2 at the output of the electronic amplifier, i.e., we shall consider a nonlinearity of the saturation or limitation type (Fig. 4.6).

We set up the equations for the linear elements of the system.

1. Equation of the error transducer

$$u_1 = k_1\vartheta, \quad \vartheta = \alpha - \beta, \tag{4.9}$$

where ϑ is the error angle; u_1 [v] is the electronic amplifier input voltage; k_1 [v] is the transfer ratio of the error transducer.

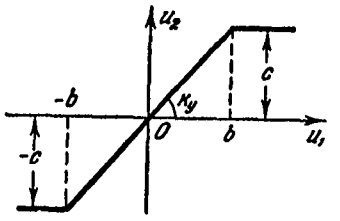


Fig. 4.6.

2. Equation of the generator excitation circuit

$$(T_1 p + 1) i_2 = k_2 u_2, \tag{4.10}$$

where T_1 [sec] is the electric time constant of the excitation circuit; k_2 [a/v] is the gain of the excitation circuit; u_2 [v] is the voltage at the amplifier output with the load connected; i_2 [a] is the current in the excitation circuit.

3. The equation for the armature circuits of the motor generator, neglecting the armature inductances, is

$$R_a i_a = e_r - e_{dv},$$

where $e_g = k_g i_2$ and $e_{dv} = k_{dv} p\beta$. Here k_g and k_{dv} are the gains.

After reducing the equation to standard form, we obtain

$$i_a = k_3 i_2 - k_4 p\beta, \tag{4.11}$$

where

$$k_3 = k_g/R_{ya}, \quad k_4 = k_{dv}/R_{ya} \text{ [v·sec/ohm]}.$$

4. Equation of motion of the motor armature, the reduction gear, and the object with the friction assumed to be proportional to the first power of the velocity, and without account of the external disturbance, will be

$$Jp^2\beta + k_\beta p\beta = M_{dv},$$

where J [g·cm·sec^2] is the moment of inertia of all the elements rotated by the motor, referred to the output shaft of the reduction gear; k_β [g·cm·sec] is the coefficient of viscous friction. If the motor is separately excited, its torque is

$$M_{дв} = k_i i_в.$$

Then the foregoing equation is rewritten

$$(T'_2 p + 1) p\beta = k_5 i_в, \tag{4.12}$$

where $T'_2 = J/k_\beta$ [sec] is the mechanical time constant; $k_5 = k_1/k_\beta$ [a/sec] is the transfer ratio.

Combining (4.11) with (4.12) we get

$$(T'_2 p + 1 + k_4 k_5) p\beta = k_3 k_5 i_2,$$

or

$$(T_2 p + 1) p\beta = \frac{k_3 k_5}{1 + k_4 k_5} i_2, \tag{4.13}$$

where $T_2 = T'_2/(1 + k_4 k_5)$ [sec] is the electromechanical time constant of the motor.

From (4.9), (4.10), and (4.13) we obtain for $\alpha = 0$ an equation for the linear part of the system

$$(T_1 p + 1)(T_2 p + 1) p u_1 = -k_1 u_2, \tag{4.14}$$

where $k_1 = k_1 k_2 k_3 k_5/(1 + k_4 k_5)$ [1/sec] is the gain of the linear part of the servomechanism.

According to the method of harmonic linearization, a characteristic with saturation is replaced in the case of symmetrical oscillations by the relation

$$u_2 = q(A) u_1.$$

In accordance with (3.19) we obtain for a nonlinearity of the form shown in Fig. 4.6

$$q(A) = k_y \quad \text{for } A \leqslant b,$$
$$q(A) = \frac{2k_y}{\pi}\left(\arcsin\frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right) \quad \text{for } A \geqslant b.$$

Substituting the value of u_2 in (4.14) we obtain

$$[(T_1 p + 1)(T_2 p + 1) p + k_1 q(A)] u_1 = 0. \tag{4.15}$$

The characteristic equation of a harmonically linearized system is written in the form

$$T_1T_2p^3+(T_1+T_2)p^2+p+k_\text{л}q(A)=0. \tag{4.16}$$

Substituting $p = j\Omega$ and separating the real and imaginary parts

$$X(A, \Omega) = 0 \text{ and } Y(A, \Omega) = 0, \tag{4.17}$$

we obtain two equations for the determination of the periodic solution

$$\left.\begin{aligned} k_\text{л}q(A)-(T_1+T_2)\Omega^2=0,\\ 1-T_1T_2\Omega^2=0.\end{aligned}\right\} \tag{4.18}$$

From the second equation of (4.18) we obtain a formula for the frequency of the periodic solution in terms of the system parameters:

$$\Omega=\frac{1}{\sqrt{T_1T_2}}. \tag{4.19}$$

Substituting the values of Ω and $q(A)$ in the first equation of (4.18) we obtain a formula relating the amplitude of the periodic solution with the system parameters:

$$\frac{2k_\text{л}k_y}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right)=\frac{T_1+T_2}{T_1T_2}. \tag{4.20}$$

To investigate the stability of the periodic solution, we make use of the approximate criterion (4.6). In accord with (4.16) we obtain

$$\begin{aligned} X(a, \omega)&=k_\text{л}q(a)-(T_1+T_2)\omega^2,\\ Y(a, \omega)&=\omega-T_1T_2\omega^3.\end{aligned}$$

Calculating the partial derivatives with allowance for the sign of $(\partial q/\partial a)^*$, determined from the plot of Fig. 3.7, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^*=k_\text{л}\left(\frac{\partial q}{\partial a}\right)^*<0 \text{ for } A>b, \quad \left(\frac{\partial Y}{\partial a}\right)^*=0,$$
$$\left(\frac{\partial X}{\partial \omega}\right)^*=-2(T_1+T_2)\Omega<0, \quad \left(\frac{\partial Y}{\partial \omega}\right)^*=1-3T_1T_2\Omega^2.$$

On the basis of (4.19) we have

$$\left(\frac{\partial Y}{\partial \omega}\right)^*=-2<0.$$

Applying the criterion (4.6), we obtain

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^*-\left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^*=-2k_\text{л}\left(\frac{\partial q}{\partial a}\right)^*>0 \text{ for } A>b.$$

Consequently, self-oscillations will arise in the system if $A > b$. At values $A < b$ we obtain $q = \text{const} = k_u$ irrespective of the values

of A, and consequently there are no self-oscillations.

We use (4.19) and (4.20) to plot the variation of the amplitude and of the frequency of self-oscillations as functions of each of the system parameters.

It is seen from (4.19) that the self-oscillation frequency depends only on the time constants T_1 and T_2. Each of the time constants exerts a similar influence on the variation of the self-oscillation frequency and amplitude. Consequently we plot the sought curves as a function of the parameters $k = k_1 k_u$, T_2, and b.

Since (4.20) is a transcendental equation with respect to the amplitude, it is advantageous to derive expressions for the parameters in explicit form as functions of the amplitude.

To determine the influence of the servomechanism gain on the self-oscillation amplitude, we obtain from (4.20) the formula

$$k = \frac{\pi (T_1 + T_2)}{2T_1T_2\left(\arcsin \frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right)}. \qquad (4.21)$$

It is clear from (4.21) that when A = b the gain is

$$k_{kp} = \frac{T_1 + T_2}{T_1T_2}, \qquad (4.22)$$

i.e., oscillations arise in a system only in a well-defined region of values of the gain, $k \geq k_{kr}$. At values $A \leq b$, the servomechanism can be regarded as being linear with a gain $k = k_1 k_u$. In accordance with (4.18), putting $q(A) = k_u$, we obtain the condition for the stability limit of the linear system

$$k_{rp} - (T_1 + T_2)\Omega^2 = 0,$$
$$1 - T_1T_2\Omega^2 = 0,$$

hence

$$k_{rp} = \frac{T_1 + T_2}{T_1T_2}. \qquad (4.23)$$

Comparing (4.22) and (4.23) we obtain $k_{gr} = k_{kr}$. Consequently,

when $k < k_{kr} = k_{gr}$ the servomechanism is stable both in the linear part of the characteristic ($u_1 < b$), and in the nonlinear part ($u_1 > b$, Fig. 4.6). When $k \geq k_{kr}$ the presence of saturation in the characteristic of the amplifier causes the servomechanism to operate as a nonlinear system in the self-oscillating mode, with a definite amplitude, whereas in the absence of saturation, the linear system would be subject to a diverging oscillating process in the case of $k > k_{kr}$.

The indicated behavior of servomechanism is observed in practice. Linear servomechanisms frequently have elements with limited linearity sections which go over into a saturation section. Consequently if the gain of a servomechanism is increased to a certain value one obtains not the instability that follows from linear theory, but usually self-oscillations.

To plot the self-oscillation frequency, we specify the following values of the parameters:

$$k = 20 \text{ 1/sec}, \quad T_1 = 0.1 \text{ sec}, \quad T_2 = 1 \text{ sec}, \quad b = 1 \text{ v}.$$

Substituting in (4.19) and (4.20) the values of all the parameters, except the one that is being varied within the limits of interest, and carrying out the calculations for each parameter, we obtain curves showing the dependence of the amplitude and frequency on each parameter.

Formulas (4.19) and (4.20) were obtained during the course of determining the periodic solution for the input of the nonlinear element, i.e., for the input voltage of the amplifier. It is of interest to obtain such a solution for the value of the angle β at the output of the servomechanism.

The frequency of the periodic solution will be the same for any variables of the system. The value of the amplitude of the amplifier output voltage can be readily recalculated in terms of the amplitude of the angle β by using the equation for the error transducer

$$u_1 = k_1\beta.$$

Consequently

$$A = k_1 A_\beta,$$

where A_β is the amplitude of the oscillations of the output shaft of the servomechanism reduction gear.

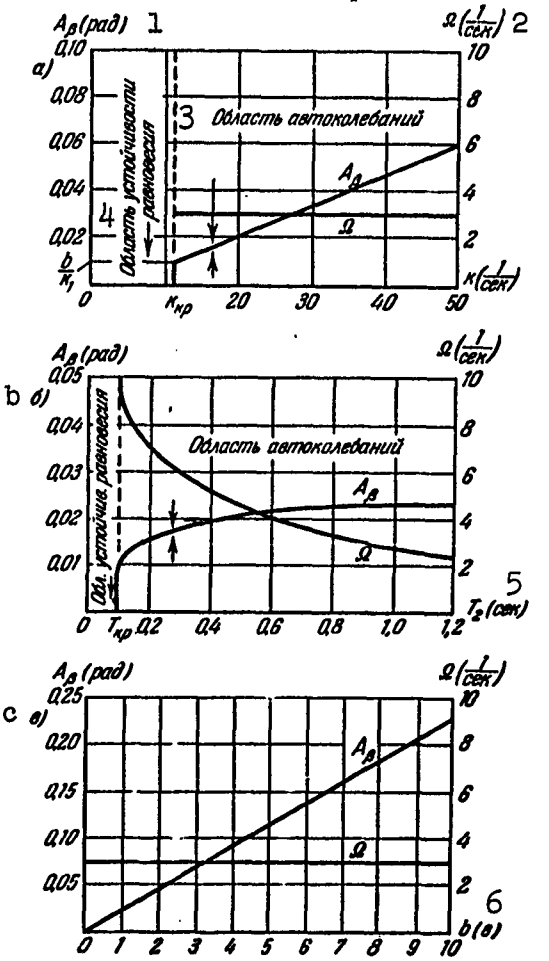


Fig. 4.7. 1) Rad; 2) 1/sec; 3) self-oscillation region; 4) stable equilibrium region; 5) sec; 6) v.

The calculations made for k_1 = 100 v/rad and for the chosen values of the other parameters are plotted in Figs. 4.7a, b, c. The plots illustrate clearly the influence of each parameter on the amplitude and frequency of the self-oscillations.

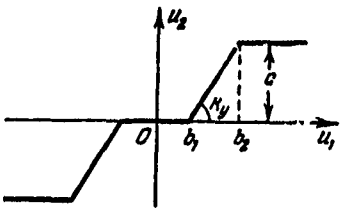


Fig. 4.8.

As can be seen from these drawings, the system can operate either in a stationary self-oscillating mode or in a stable mode without self-oscillations, depending on the values of the parameters.

The amplitude of self-oscillations can be reduced by decreasing the gain of the system k, by decreasing the time constants T_1 and T_2, and by decreasing the linear portion of the amplifier characteristic, i.e., the value of b.

The self-oscillation frequency can be increased only by increasing the time constants T_1 and T_2.

The self-oscillations can be suppressed by reducing the gain and decreasing the time constants.

For the parameter values chosen for the plots, the self-oscillation frequency is $\Omega = 3.16$ 1/sec and the amplitude is $A_\beta = 0.022$ rad.

We now make allowances in the same servomechanism for not only saturation but also for the insensitivity zone of the amplifier (Fig. 4.8). This corresponds to an amplifier in which the tubes have good characteristics with initial nearly horizontal portions. In this case, to find the periodic solution one can use Eqs. (4.18) but with different values of q(A).

For amplitude values $b_1 \leq A \leq b_2$ we have in accordance with formula (3.18) for the harmonic linearization of a characteristic with a dead zone

$$q(A) = k_y - \frac{2k_y}{\pi}\left(\arcsin\frac{b_1}{A} + \frac{b_1}{A}\sqrt{1-\frac{b_1^2}{A^2}}\right).$$

To estimate the stability of the periodic solution we obtain, allowing for the plot of q(A) (Fig. 3.7),

$$\left(\frac{\partial X}{\partial a}\right)^* = k_a\left(\frac{\partial q}{\partial a}\right)^* > 0 \text{ for } b_1 \leqslant A \leqslant b_2$$

and for the former values of

$$\left(\frac{\partial Y}{\partial \omega}\right)^*, \quad \left(\frac{\partial X}{\partial \omega}\right)^*, \quad \left(\frac{\partial Y}{\partial a}\right)^*.$$

Consequently, the periodic solution is unstable for $b_1 \leq A \leq b_2$, and the presence of only a zone of insensitivity in the amplifier cannot bring about self-oscillations in the given system.

For amplitude values $A \geq b_2$ we obtain in accordance with formula (3.17) for the harmonic linearization in the case of the characteristic with an insensitivity zone and saturation:

$$q(A) = \frac{2k_y}{\pi}\left(\arcsin \frac{b_2}{A} - \arcsin \frac{b_1}{A} + \frac{b_2}{A}\sqrt{1-\frac{b_2^2}{A^2}} - \frac{b_1}{A}\sqrt{1-\frac{b_1^2}{A^2}}\right).$$

To determine the stability of the periodic solution we obtain, in accordance with the plot of q(A) (Fig. 3.7),

$$\left(\frac{\partial X}{\partial a}\right)^* = k_s\left(\frac{\partial q}{\partial a}\right)^* < 0 \quad \text{for } A \geq b_2.$$

The remaining derivatives will have the same signs as before.

Consequently, when $A \geq b_2$ the periodic solution is stable.

Let us calculate the variation of the amplitude and frequency of self-oscillations for the previous values of the parameters and for $b_1 = 0.2$ v, $b_2 = b = 1$ v. The frequency Ω of the periodic solution will depend only on the time constants T_1 and T_2 and is given by formula (4.19). The dependence of the amplitude on the parameters of the system is determined, in accord with (4.20), from the following relations:

1) for the case of an unstable periodic solution

$$k - \frac{2k}{\pi}\left(\arcsin \frac{b_1}{A} + \frac{b_1}{A}\sqrt{1-\frac{b_1^2}{A^2}}\right) = \frac{T_1+T_2}{T_1T_2} \qquad (4.24)$$

$$\text{for } b_1 \leq A \leq b_2.$$

2) for the case of a stable periodic solution (self-oscillations)

$$\frac{2k}{\pi}\left(\arcsin \frac{b_2}{A} - \arcsin \frac{b_1}{A} + \frac{b_2}{A}\sqrt{1-\frac{b_2^2}{A^2}} - \frac{b_1}{A}\sqrt{1-\frac{b_1^2}{A^2}}\right) = \frac{T_1+T_2}{T_1T_2} \qquad (4.25)$$

$$\text{for } A \geq b_2.$$

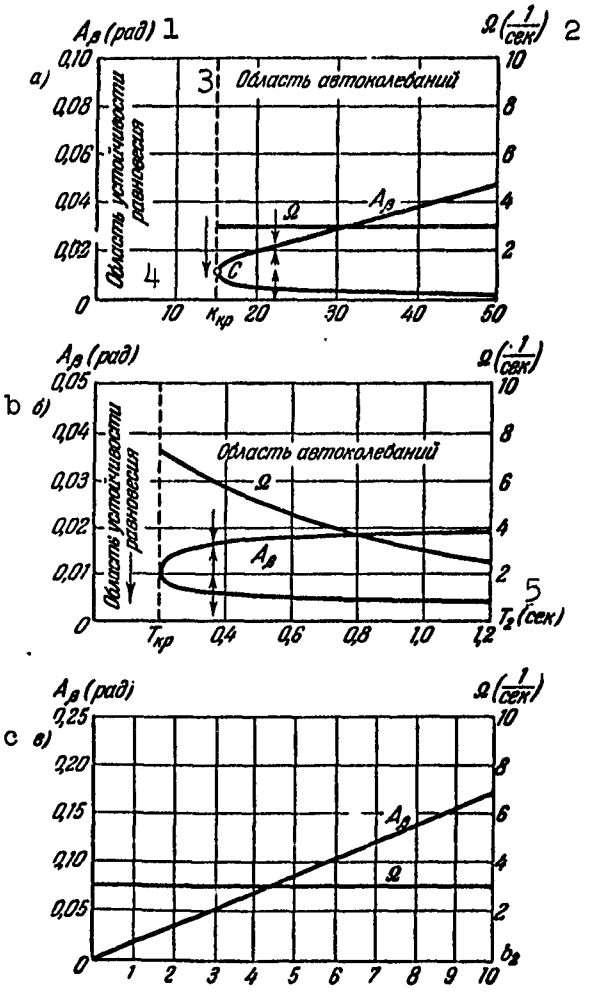


Fig. 4.9. 1) Rad; 2) 1/sec; 3) self-oscillation region; 4) stable equilibrium region; 5) sec.

The dependences of the amplitude $A_\beta = A/k_1$ and of the frequency Ω of the self-oscillations of the system parameters, determined by formulas (4.19), (4.24), and (4.25), are plotted in Fig. 4.9a, b, c.

It can be seen from the plots that in the case when linearity with insensitivity zone and saturation are present, two steady-state modes are likewise possible in the nonlinear system, namely a self-oscillation region and a stability region, with two values for the amplitude of the periodic solutions in the self-oscillation region. The branch with the large amplitudes, as was already shown, corresponds

to a stable periodic solution, namely self-oscillations, as designated by the arrows that converge to the given branch from the top and from the bottom. The branch with the smaller amplitudes corresponds to an unstable periodic solution, as designated by arrows diverging from the given branch. Consequently, the system is stable "in the small" in the self-oscillation region, and has a stable self-oscillating mode "in the large."

At values $k < k_{kr}$ and $T_2 < T_{kr}$ there is no periodic solution. In this case the system is nonlinear. In order to ascertain whether the nonlinear system is stable outside the self-oscillation region, we extrapolate the result obtained by investigating the stability of the self-oscillation at the critical point C (Fig. 4.9) to the region where there is no periodic solution. Actually, at the point C we still have stability "in the small" and an attenuating process "in the large," i.e., the given system is stable when $k < k_{kr}$ and $T_2 < T_{kr}$ for all values of the initial conditions (we bear in mind the fact that $k > 0$ and $T_2 > 0$). This result can be readily confirmed by the method developed in §2.7.

A decrease in the self-oscillation amplitude, an increase in the self-oscillation frequency, and suppression of the self-oscillations are made possible, as we see, by the same means as in the presence of saturation alone. In addition, the presence of an insensitivity zone is seen to narrow down the region of self-oscillations and is therefore one of the means of suppressing self-oscillations.

§4.4. Servomechanism with Nonlinear Drive

In the examples considered in §§4.2 and 4.3, stability regions and self-oscillation regions were singled out for each parameter of the system. If the variations of the amplitude and frequency of self-oscillation are not plotted for each parameter, and if we are inter-

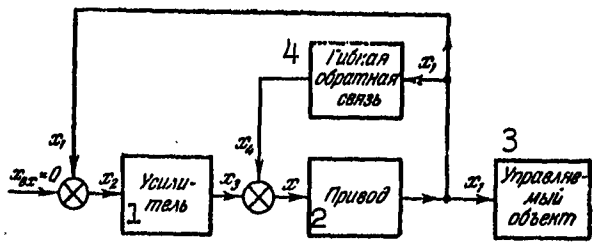


Fig. 4.10. 1) Amplifier; 2) drive; 3) controlled object; 4) derivative feedback.

ested in the behavior of the nonlinear system from the point of view of its stability in the steady state, then it is convenient to plot the possible steady-state modes and to separate the stability regions in the plane of any two (most important) of the parameters [64].

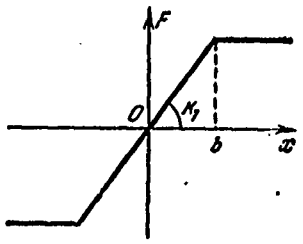


Fig. 4.11

We shall prepare such a plot with the servomechanism whose block diagram is shown in Fig. 4.10 as an example. We assume a nonlinearity of the saturation type without a dead zone (Fig. 4.11) in the drive of the servomechanism object. Such a nonlinearity occurs in hydraulic and pneumatic servo drives controlled by means of slide valves. Upon displacement of the slide valve, the velocity of the servo drive first increases linearly as a function of the slide valve displacement, and when the ports of the valve are completely open, it reaches a maximum value at which it stays constant. Moreover, in the present example we take account of the inertia of the amplifier and of an additional derivative feedback loop in which the drive is included.

The system elements are described by the following equations:

$$\left.\begin{array}{c}(T_1p+1)px_1=F(x),\\ x_2=x_{вх}-x_1,\\ (T_2p+1)x_3=k_2x_2,\\ x=x_3-x_4,\quad x_4=k_{o.c}px_1,\end{array}\right\} \qquad (4.26)$$

the first of which is the equation of the drive with account of the load produced by the object, the second is the equation of the error transducer, the third is the equation of the amplifier, and the fourth is the feedback equation.

Assuming $x_{vkh} = 0$, we seek a solution for $\underline{x}$ in sinusoidal form

$$x = A \sin \psi, \quad \psi = \Omega t.$$

Harmonic linearization of the nonlinearity yields

$$F(x) = q(A)\, x, \tag{4.27}$$

where in accord with (3.19) we have

$$q(A) = \frac{2k_1}{\pi}\left(\arcsin \frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right) \text{ for } A \geqslant b.$$

The characteristic equation corresponding to the differential equation derived from (4.26) and (4.27) is

$$(T_1 p + 1)(T_2 p + 1)p + [(T_2 p + 1)k_{o.c} p + k_2]\, q(A) = 0,$$

or after expanding

$$T_1 T_2 p^3 + [T_1 + T_2 + T_2 k_{o.c}\, q(A)]p^2 + [1 + k_{o.c}\, q(A)]p + k_2 q(A) = 0. \tag{4.28}$$

Let us plot the regions of steady self-oscillation modes and steady equilibrium state on the plane of the parameters k_2 and $k_{o.s}$.

We first determine the stability region for a purely linear system, replacing the linear function F(x) by the linear one $k_1 x$. This makes the characteristic equation, in accordance with (4.28),

$$T_1 T_2 p^3 + (T_1 + T_2 + T_2 k_1 k_{o.c})p^2 + (1 + k_1 k_{o.c})p + k_1 k_2 = 0. \tag{4.29}$$

The stability condition has the form

$$(T_1 + T_2 + T_2 k_1 k_{o.c})(1 + k_1 k_{o.c}) - T_1 T_2 k_1 k_2 > 0. \tag{4.30}$$

The corresponding stability limit is plotted in the plane of the parameters k_2, $k_{o.s}$ in the form of the parabola ABC (Fig. 4.12a). In the entire region above this parabola our linear system is stable.

Let us ascertain how the nonlinearity of F(x) (Fig. 4.11) influences the limit of the stability region of this system. The presence

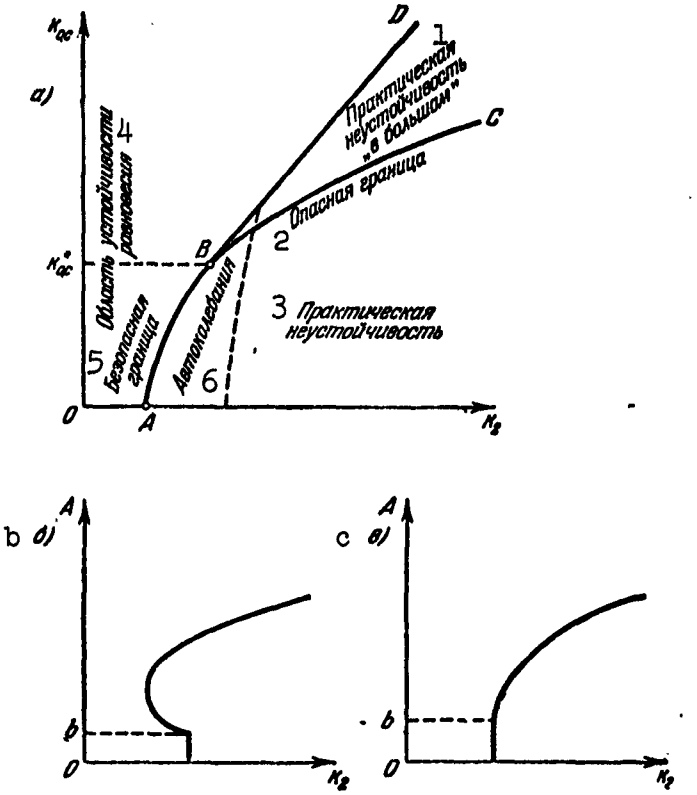


Fig. 4.12. 1) Practical instability "in the large"; 2) danger limit; 3) practical instability; 4) region of stable equilibrium; 5) safety limit; 6) self-oscillations.

of a sinusoidal periodic solution in the harmonically linearized system ($H_{n-1} = 0$) can be determined, in accordance with (4.28), by means of the equation

$$[T_1 + T_2 + T_3 k_{o.c} q(A)][1 + k_{o.c} q(A)] - T_1 T_2 k_3 q(A) = 0,$$

hence

$$k_3 = \left[\frac{T_1 + T_2}{T_1 T_2 q(A)} + \frac{k_{o.c}}{T_1}\right][1 + k_{o.c} q(A)]. \qquad (4.31)$$

This equality determines the condition under which self-oscillations occur and the dependence of the amplitude on the system parameters.

The self-oscillation frequency is determined here by the same method as in the preceding examples. If we replace p by $j\Omega$ in the characteristic equation (4.28) and equate the imaginary parts to zero, then

$$\Omega^2 = \frac{1 + k_{o.c} q(A)}{T_1 T_2}.$$

In formula (4.31) we must put $A \geq b$, for only then does the nonlinear portion of the characteristic enter into operation. When $A \geq b$ the value of q(A) decreases with increasing A (see the plot in Fig. 3.7), i.e.,

$$k_1 \geqslant q(A) \geqslant 0 \text{ for } b \leqslant A \leqslant \infty. \tag{4.32}$$

It is clear that (4.31) is satisfied if k_2 is sufficiently large, with $k_2 = \infty$ when $q = 0$. The lower limit of k_2 need not necessarily correspond to the value $q_{max} = k_1$ (due to the presence of q(A) in the second bracket of formula (4.31)), i.e., the amplitude of the periodic solution can vary with the parameter k_2, which can have a minimum (Fig. 4.12b).

To determine the region of self-oscillations it is necessary to know the minimum of k_2 for which Eq. (4.31) still holds. The minimum of k_2 is determined from the condition that the derivative dk_2/dA must vanish. Since the derivative of k_2 with respect to A can be represented in the form

$$\frac{dk_2}{dA} = \frac{dk_2}{dq} \cdot \frac{dq}{dA},$$

and since dq/dA never vanishes on the nonlinear portion of a static characteristic with saturation (Fig. 3.7), it is sufficient to equate the derivative dk_2/dq to zero in order to determine the region of self-oscillations. As a result we obtain from (4.31) the value of $k_{o.s}$ corresponding to the minimum of k_2:

$$k'_{o.c} = \frac{1}{q(A)} \sqrt{1 + \frac{T_1}{T_2}}.$$

Since

$$0 \leqslant q \leqslant k_1,$$

we have

$$\infty \geq k'_{o.c} \geq \frac{1}{k_1}\sqrt{1+\frac{T_1}{T_2}}.$$

Consequently, if the system parameters are such that

$$k_{o.c} < \frac{1}{k_1}\sqrt{1+\frac{T_1}{T_2}},$$

then there exists no minimum of k_2. Therefore the critical value of $k^*_{o.s}$ will be

$$k^*_{o.c} = \frac{1}{k_1}\sqrt{1+\frac{T_1}{T_2}}. \qquad (4.33)$$

When $k_{o.s} < k^*_{o.s}$ there is no mathematical minimum of k_2, and consequently the self-oscillation region will be determined in accordance with (4.31) and (4.32) by the condition

$$k_2 \geq \left(\frac{T_1+T_2}{T_1T_2k_1}+\frac{k_{o.c}}{T_1}\right)(1+k_{o.c}k_1) \text{ for } k_{o.c} < k^*_{o.c}. \qquad (4.34)$$

When $k_{o.s} \geq k^*_{o.s}$, k_2 does have a minimum

$$k_{2\min} = \frac{k_{o.c}}{T_1}\left(1+\sqrt{1+\frac{T_1}{T_2}}\right)^2,$$

and consequently the corresponding region of self-oscillations will be

$$k_2 \geq \frac{k_{o.c}}{T_1}\left(1+\sqrt{1+\frac{T_1}{T_2}}\right)^2 \text{ for } k_{o.c} \geq k^*_{o.c}. \qquad (4.35)$$

From the obtained relations (4.34) and (4.35) it follows that the limit of the self-oscillation region for $k_{o.s} \leq k^*_{o.s}$, defined by formula (4.34), coincides with the stability limit obtained when the given system is regarded as purely linear (formula 4.30). This is represented by section AB of Fig. 4.12a. On the other hand, the limit of the self-oscillation region for $k_{o.s} > k^*_{o.s}$ does not coincide with the linear stability limit. According to formula (4.35) the latter has the form of the line BD. Consequently, in our example the presence of a nonlinearity of the saturation type narrows down the stability region of the system. A physical explanation of this system can be seen in the cut-off of the feedback signal at large values of x_1, brought about by the character of the saturation type of instability.

Physical considerations (comparison with the corresponding linear system) indicate that in our nonlinear system the region to the left of the line ABD will be a stable one (regardless of the initial conditions). However, even approximate verification of this fact is difficult.

Application of the Hurwitz stability criterion to Eq. (4.28) yields the condition

$$[T_1 + T_2 + T_2 k_{o.c} q(A)][1 + k_{o.c} q(A)] - T_1 T_2 k_2 q(A) > 0.$$

This condition is satisfied for all values of q(A) defined by the inequality (4.32), that is, for all A (for all initial conditions) we have here inequalities that are the inverse of formulas (4.34) and (4.35), thus proving the stability in the region left of ABD.

The behavior of the system inside the self-oscillation region is investigated by the same method as in the preceding examples. It shows that in the region CBD the system will be stable "in the small" and unstable "in the large," namely: when the initial deviations do not go beyond the linear part of the characteristic (Fig. 4.11) or exceed this part only slightly (up to a certain limit), the transient in the system will be damped out and the system will return to the equilibrium state; at large initial deviations the system goes into self-oscillation (hard excitation) with so large an amplitude that in practice it can be regarded as instability.

On the other hand, to the right of the line AB the amplitude of the self-oscillations grows smoothly, starting with a value A = b (soft excitation). It becomes too large only to the right of a certain dashed curve (Fig. 4.12a), beyond which the system must be regarded as practically unstable.

In this connection the stability limit of this system, calculated for the linear case, breaks up into two portions: AB is the safe limit,

since the system does not swing wild even on crossing this limit, and BC is the dangerous limit, for the system may be practically unstable even before it reaches this limit. For this part of the parameter plane, i.e., at sufficiently large feedback coefficients, one cannot employ the linear theory.

§4.5. Temperature Control System with Balanced Relay

Consider the relay system for temperature control shown in Fig. 4.13 [108]. Let us explain its operating principle. Whenever the regulated quantity, namely the temperature ϑ of the regulated object 1, changes the bimetal strip 2 bends. Depending on the direction of the deflection of the end of the bimetal strip, one of two windings of balanced relay 3 is energized. The armature of the relay has three positions. Turning on one of the relay windings causes motor 4 to run in the forward direction, while turning on the second winding and turning off the first reverses the motor. When no current flows in the windings of the relay, the armature occupies a neutral position and the armature of the motor is short circuited. With $\underline{x}$ varying continuously, the voltage $\underline{u}$ on motor 4 changes in discrete steps. The motor actuates, via reduction gear 5, the regulating unit 6 which, by changing its position ξ acts on the regulated object. It is possible to add to the system additional proportional feedback 7, which displaces a panel with contact plates. Then the displacement of the end of the bimetal strip relative to the contact plates will be described by the equations

$$s = x - x_{o.c}, \quad x_{o.c} = k_{o.c}\xi.$$

Let us investigate the system illustrated first with the feedback disconnected, and then in the presence of feedback with and without account of the time delay in the relay. In the investigation we shall take account of the inertia of the regulated object and of the electric drive.

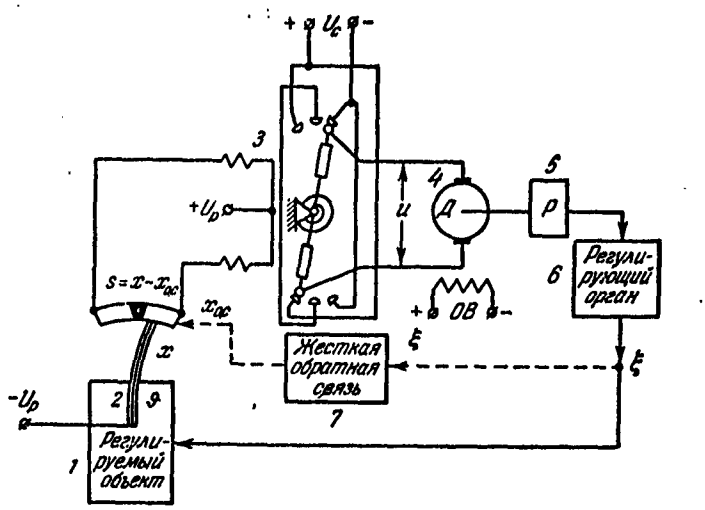


Fig. 4.13. 1) Regulated object; 2) regulating unit; 7) proportional feedback.

Let us write down the equations for the system elements.

1. The equation of the regulated object is

$$(T_1 p + 1)\theta = -k_1 \xi, \tag{4.36}$$

where T_1 [sec] is the time constant of the object and k_1 [deg] is the transfer ratio of the object.

2. The equation of the bimetal strip is

$$x = k_2 \theta, \tag{4.37}$$

where k_2 [deg^{-1}] is the transfer ratio of the bimetal strip.

3. The equation of the drive with regulating unit is

$$(T_2 p + 1) p\xi = k_3 u, \tag{4.38}$$

where T_2 [sec] is the electromechanical time constant of the drive and k_3 [$sec^{-1}v^{-1}$] is the transfer ratio of the drive.

4. The contact unit together with the balanced relay is a nonlinear element.

The voltage u applied to the motor will be a nonlinear function of the displacement x of the end of the bimetal strip relative to the panel with the contact plates. This function is illustrated in Fig. 4.14.

Combining Eqs. (4.36)-(4.38) we obtain the equation of the linear portion of the system:

$$(T_1p+1)(T_2p+1)px=-k_1k_2k_3u. \tag{4.39}$$

For the nonlinear element we write down, in accord with formula (3.13) for the coefficient of harmonic linearization, the following expression:

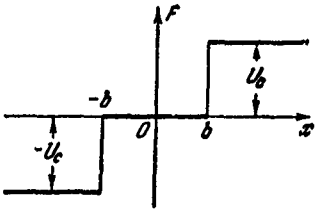


Fig. 4.14

$$u=F(x)=q(A)x, \tag{4.40}$$

where

$$q(A)=\frac{4U_c}{\pi A}\sqrt{1-\frac{b^2}{A^2}} \text{ with } A\geqslant b.$$

Substituting the value of u from (4.40) into (4.39) and denoting by $k_1 = k_1k_2k_3$ [$v^{-1}sec^{-1}$] the gain of the linear portion of the system, we obtain a linearized equation describing the motion of the system proper:

$$[T_1T_2p^3+(T_1+T_2)p^2+p+k_1q(A)]x=0. \tag{4.41}$$

Substituting $p = j\Omega$, we obtain from (4.41) equations for the determination of the amplitude and frequency of the periodic solution

$$k_1q(A)-(T_1+T_2)\Omega^2=0, \tag{4.42}$$

$$1-T_1T_2\Omega^2=0. \tag{4.43}$$

From (4.43) we obtain a formula for the determination of the frequency of the periodic solution:

$$\Omega=\frac{1}{\sqrt{T_1T_2}}. \tag{4.44}$$

Replacing Ω in (4.42) by its value, we obtain a formula for the calculation of the amplitude of the periodic solution as a function of the system parameters

$$k_1q(A)=\frac{T_1+T_2}{T_1T_2}. \tag{4.45}$$

We can see from (4.45), if we take account of the value of q(A), that a periodic solution is possible when $A \geq b$. The time constants of the regulated object, T_1, and of the drive of the regulating unit, T_2,

influence the frequency and amplitude of the periodic solution to an equal degree.

In the investigation of the system it is of interest to obtain a solution for the regulated quantity (in our case, the temperature ϑ). Yet the equations are set up for the output of the linear portion (in our case, the displacement $\underline{x}$ of the end of the bimetallic sensitive element). The frequency of the periodic solution will be the same for any variable in the closed loop. The amplitude of the variable of interest can be obtained by recalculating the amplitude of the output quantity of the linear portion in terms of the transfer function of the linear elements which separate the given variables. For the case considered here, to change over from the amplitude A of the oscillations of the end of the bimetal strip to the amplitude A_ϑ of the temperature fluctuations we must use, in accordance with formula (4.37), the relation $A = k_2 A_\vartheta$. We assume for our calculation that $k_2 = 0.01$ rad/deg, and consequently A_ϑ [deg] = 100 A.

Let us find the dependence of the amplitude and of the frequency of the periodic solution on the parameters $\underline{k_1}$, T_2, and $\underline{b}$. We note that U_s and $\underline{k_1}$ influence the periodic solution to an equal degree, since the value of U_s determines the amplification factor of the nonlinear element. For each parameter that we vary we shall assume the other parameters to have the following values: $\underline{k_1} = 0.01$ sec^{-1}v^{-1}, $T_1 = 10$ sec, $T_2 = 0.1$ sec, $U_s = 25$ v, and $b = 0.01$ rad.

Since formula (4.45) is easier to solve with respect to the parameters than with respect to the amplitude, it is preferable to specify the values of the amplitude and calculate the values of the parameters.

To determine the relation $A(\underline{k_1})$ we obtain from (4.45) the formula

$$k_1 = \frac{T_1 + T_2}{T_1 T_2 q(A)}. \qquad (4.46)$$

Carrying out the calculations in accordance with (4.46) for the assumed values of the parameters, we obtain for the amplitude of the periodic solution two branches (Fig. 4.15a).

A periodic solution is possible in the range $k_{\underline{1}.kr} \leq k_{\underline{1}} \leq \infty$. From the condition that the derivative of $k_{\underline{1}}$ with respect to A must vanish, we obtain

$$k_{\underline{1}.\,кр} = \frac{\pi b (T_1 + T_2)}{2U_c T_1 T_2} \text{ for } A = \sqrt{2}\,b, \tag{4.47}$$

which yields for the assumed values of the parameters $k_{\underline{1}.kr} = 0.0063$ $[v^{-1}sec^{-1}]$. When $A \gg b$, $A(k_{\underline{1}})$ varies in linear fashion and when $k \to \infty$ we have $A \to \infty$. The branch with the large amplitudes belongs to the stable periodic solution, i.e., it corresponds to self-oscillations in the system, while the small-amplitude branch is the stability limit of the system "in the small," designated symbolically by the arrows. This is confirmed by the analytical stability criterion (4.6). In accord with (4.41) we have

$$X(a, \omega) = k_{\underline{1}} q(a) - (T_1 + T_2)\omega^2,$$
$$Y(a, \omega) = \omega - T_1 T_2 \omega^3.$$

After determining the corresponding derivatives, using the plot of Fig. 3.4, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^* = k_{\underline{1}}\left(\frac{\partial q}{\partial a}\right)^* \begin{cases} > 0 \text{ for } A < \sqrt{2}\,b, \\ < 0 \text{ for } A > \sqrt{2}\,b, \end{cases} \left(\frac{\partial Y}{\partial a}\right)^* = 0;$$
$$\left(\frac{\partial X}{\partial \omega}\right)^* = -2(T_1 + T_2)\Omega < 0, \left(\frac{\partial Y}{\partial \omega}\right)^* = 1 - 3T_1 T_2 \Omega^2;$$

since it follows from (4.43) that $\Omega^2 = 1/T_1T_2$, we have $(\partial Y/\partial\omega)^* = -2 < < 0$. Consequently, in accord with criterion (4.6), we have

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* \begin{cases} > 0 \text{ for } A > \sqrt{2}\,b, \\ < 0 \text{ for } A < \sqrt{2}\,b. \end{cases}$$

Transferring the results obtained for the behavior of the system in the self-oscillation region to the region of small values of the gain of the linear portion, $0 \leq k_{\underline{1}} < k_{\underline{1}.kr}$, we see that the motion of

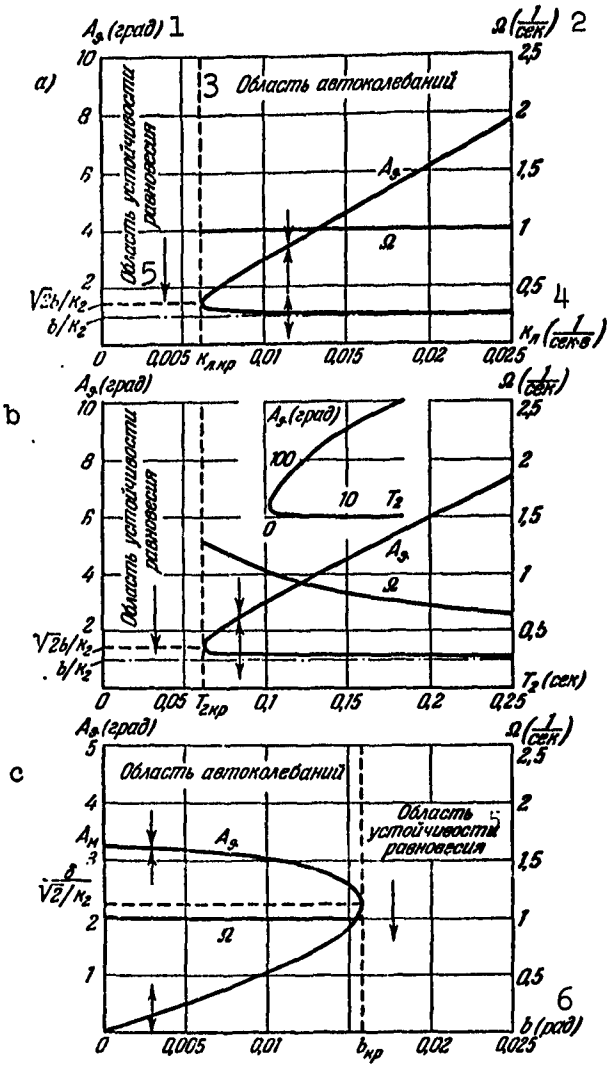


Fig. 4.15. 1) Deg; 2) 1/sec; 3) self-oscillation region; 4) $sec^{-1}v^{-1}$; 5) region of stability of equilibrium; 6) rad.

the system in the region where there is no periodic solution will be stable.

Let us compare the results obtained with the first example in which we investigate the stabilization system of a gyro pendulum, where the role of the amplifying element was assumed by the stabilizing motor (the parameter m_0). We see that the backlash zone exerts a stabilizing influence on the relay system, for in the presence of a characteristic with a backlash zone there appears a stability region without

self-oscillations, with the width of the stability region, $0 \leq k_1 < k_{1.kr}$ proportional, in accordance with (4.47), to the width of the backlash zone 2b. On the other hand, the stability region "in the small" which appears inside the self-oscillation region has no practical significance, since it corresponds to too small deviations.

To determine the relation $A(T_2)$ we obtain from (4.45) the formula

$$T_2 = \frac{T_1}{k_1 T_1 q(A) - 1}. \qquad (4.48)$$

Formulas (4.48) and (4.44) were used to plot curves of $A_\vartheta(T_2)$ and $\Omega(T_2)$ (Fig. 4.15b). Depending on the value of the parameter T_2, we again have two regions: a self-oscillation region when $T_{2kr} \leq T_2 \leq \infty$, and a stability region when $0 \leq T_2 < T_{2kr}$. In the self-oscillation region we have two branches for the values of the amplitudes of the periodic solution.

The branch with the large amplitudes corresponds to self-oscillations, while the branch with the smaller amplitudes corresponds to the stability limit "in the small." As T_2 increases without limit, the self-oscillation amplitude tends in this case to a value

$$A_\infty = \frac{4k_1 U_c T_1}{\pi}.$$

The value of T_{2kr} is determined by equating to zero the derivative of T_2 with respect to A:

$$T_{2xp} = \frac{\pi T_1 b}{2k_1 U T_1 - \pi b} \text{ for } A = \sqrt{2}b$$

and amounts to, for the assumed values of the parameters, $T_{2kr} = 0.063$ sec.

To plot the dependence of the amplitude of the periodic solution on the variation of the backlash zone we obtain from (4.45) the formula

$$b = \frac{A}{\delta}\sqrt{\delta^2 - A^2}, \qquad (4.49)$$

where

$$\delta = \frac{4k_{\text{д}}U_c T_1 T_2}{\pi(T_1 + T_2)}.$$

The curve of $A_{\text{д}}(b)$ is shown in Fig. 4.15c. The branch with the large values of the amplitudes corresponds to self-oscillations, while the branch with the smaller values of amplitudes is the stability limit "in the small." From (4.49) we obtain, under the condition $b = 0$, the maximum value of the self-oscillation amplitude $A_m = \delta$ (for our example $A_m = 3.16^{\circ}C$). From the condition that the derivative of $\underline{b}$ with respect to A must vanish we obtain

$$b_{\text{кр}} = \frac{\delta}{2}.$$

The curves obtained for the variation of the frequency and amplitude of the self-oscillations with the system parameters enable us to choose system parameters that satisfy the conditions imposed by the desired steady-state mode.

From the investigation made it is evident that a decrease in the self-oscillation amplitude and a suppression of the self-oscillations are possible by decreasing the gain of the system, by decreasing the time constants of the object and of the drive of the regulating unit, and also by increasing the backlash zone of the relay element. The frequency of the self-oscillations can be increased only by decreasing the time constants of the object and of the drive of the regulating unit.

We note that in the present case an account of the inertia of the regulator is of fundamental importance. Although the time constant of the object T_1 is larger than the time constant of the drive T_2 by 100 times ($T_1 = 10$ sec, $T_2 = 0.1$ sec), one cannot neglect T_2 compared with T_1. Such a neglect would be tantamount to investigating a system with an assumed second degree equation, which would introduce fundamental qualitative differences compared with an investigation by assuming a

third-degree equation. In fact, when $T_2 = 0$ the condition $Y(A, \Omega) = 0$ is written in the form $\Omega = 0$, i.e., a periodic solution is impossible no matter what the system parameters are. In actuality, however, as can be seen from the investigation made, the self-oscillation region predominates over the stability region without self-oscillations.

For the parameter values assumed in the investigation we find the system will operate in a self-oscillating steady-state mode at a frequency $\Omega = 1$ sec^{-1} and at a temperature fluctuation amplitude $A_\vartheta = 3^\circ$C. Such a mode would be acceptable for many thermal objects. It must be borne in mind, however, that it is the result of a low gain. The reduction in the gain decreases the response speed of the system and increases the steady state error. Such a system is incapable of operating at high rates of variation of the disturbance. To increase the response speed of the system it is necessary to increase the gain, and this increases the amplitude of temperature fluctuations, which in many cases cannot be tolerated.

In order to obtain an acceptable self-oscillating mode with a large gain in such a system, proportional feedback must be used.

Let us analyze the steady-state modes in a system which contains proportional feedback. The equation of the system is written in this case in terms of the variable $\underline{s}$ (displacement of the bimetal strip relative to the contact plates) in the form

$$\{T_1T_2p^3+(T_1+T_2)p^2+\ldots+[1+k_3k_{o.c}T_1q(A)]p+\ldots+(k_4+k_3k_{o.c})q(A)\}s=0, \qquad (4.50)$$

where $k_{o.s}$ is the transfer ratio of the feedback element. The symbols for the other parameters remain as before.

In accord with (4.50) we obtain after making the substitution $p = j\Omega$ two equations for the amplitude and the frequency of the periodic solution:

$$(k_4+k_3k_{o.c})q(A)-(T_1+T_2)\Omega^2=0,$$

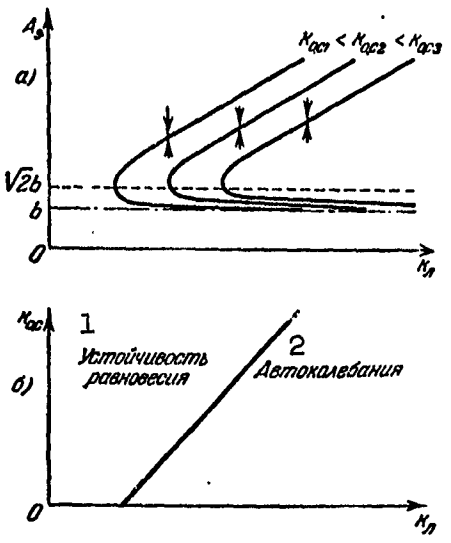


Fig. 4.16. 1) Stability of equilibrium; 2) self-oscillations.

$$1 + k_3 k_{o.c} T_1 q(A) - T_1 T_2 \Omega^2 = 0.$$

Solving the second equation with respect to Ω^2, we obtain

$$\Omega^2 = \frac{1}{T_1 T_2} + \frac{k_3 k_{o.c} q(A)}{T_2}.$$

We see that the additional feedback increases the frequency of the periodic solution (without feedback we have $\Omega^2 = 1/T_1 T_2$). Substituting the value of Ω^2 in the first equation we obtain a relation for the determination of the amplitude of the periodic solution as a function of the system parameters:

$$\left[k_1 + k_3 k_{o.c}\left(1 - \frac{T_1 + T_2}{T_2}\right)\right] q(A) = \frac{T_1 + T_2}{T_1 T_2}.$$

Let us plot the curves of $A_\vartheta = A_\vartheta(k_1)$ and $\Omega = \Omega(k_1)$ in the presence of feedback. Solving the last equation with respect to k_1, we obtain

$$k_1 = \frac{T_1 + T_2}{T_1 T_2 q(A)} + k_3 k_{o.c} \frac{T_1}{T_2}.$$

Comparing the result obtained with (4.46) we see that k_1, unlike the case when there is no feedback, is increased by an amount

$$k_1' = k_3 k_{o.c} \frac{T_1}{T_2}$$

which is independent of the amplitude; consequently, the curve for the amplitudes of the periodic solution $A_\vartheta(k_1)$ will be shifted to the right by an amount k'_1 (Fig. 4.16a). The feedback thus broadens the stability region by shifting the self-oscillation region in proportion to the feedback coefficient $k_{o.s}$ (Fig. 4.16b), and the effect of the feedback, measured as a function of the ratio of the time constants, will be the greater, the larger the time constant of the object is as compared with the time constant of the drive of the regulating unit.

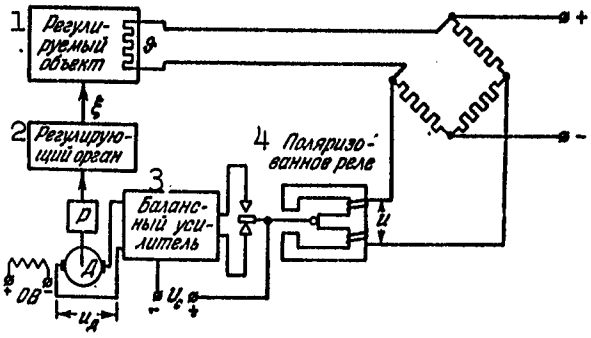


Fig. 4.17. 1) Regulated object; 2) regulating unit; 3) balanced amplifier; 4) polarized relay.

§4.6. Temperature Control System with a Polarized Relay

Let us consider a relay system for temperature control, using a polarized relay. The schematic diagram of the system is shown in Fig. 4.17. Deviations of the regulated quantity ϑ are measured with the aid of a resistance thermometer connected in a bridge circuit. The voltage u from the bridge diagonal is applied to the control winding of a two-position polarized relay. Through a balanced amplifier, the relay controls the motor D, which drives through a reduction gear R the regulating unit, which in turn acts on the object as a result of a change in the value of the regulating action ξ.

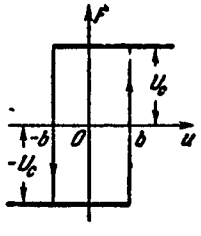


Fig. 4.18

Let us write down the equations for the elements in terms of the deviations.

1. The equation of the regulated object is

$$(T_1 p + 1)\vartheta = -k_1 \xi,$$

where T_1 [sec] is the time constant of the object and k_1 is the transfer ratio of the object.

2. The equation of the sensitive element, which is the bridge with resistance thermometer, is

$$u = k_2 \vartheta,$$

where k_2 [v/deg] is the transfer ratio of the sensitive element.

3. The equation of the drive together with the regulating unit is:

$$(T_2 p + 1) p\xi = k_3 u_d,$$

where T_2 [sec] is the electromechanical time constant of the drive, k_3 [$v^{-1}sec^{-1}$] is the transfer ratio of the drive, and u_d [v] is the voltage applied to the motor armature.

4. The polarized relay together with the amplifier is a nonlinear element with input $\underline{u}$ and output u_d, the latter being dependent in nonlinear fashion on the control voltage of the relay:

$$u_d = F(u).$$

In the case of a two-position polarized relay, the nonlinear function F(u) has the form of a relay hysteresis static characteristic (Fig. 4.18).

Combining the equations of the first three elements, we obtain the equation of the linear portion of the system

$$(T_1 p + 1)(T_2 p + 1) pu = -k_d u_d,$$

where

$$k_d = k_1 k_2 k_3 \left[\frac{1}{\text{sec}}\right].$$

For the nonlinear element we write down the harmonically linearized expression

$$u_d = F(u) = \left[q(A) + \frac{q'(A)}{\Omega} p\right] u,$$

where in accordance with formulas (3.9) and (3.10) the values of the harmonic linearization coefficients are

$$q(a) = \frac{4U_c}{\pi A^2}\sqrt{A^2 - b^2}, \quad q'(A) = -\frac{4U_c b}{\pi A^2} \text{ for } A \geq b.$$

Combining the equation of the linear part of the system and the formulas for the harmonically linearized nonlinear element, we obtain a linearized equation describing the motion of the system proper:

$$\left\{T_1 T_2 p^3 + (T_1 + T_2) p^2 + \left[1 + \frac{k_d q'(A)}{\Omega}\right] p + k_d q(A)\right\} u = 0. \qquad (4.51)$$

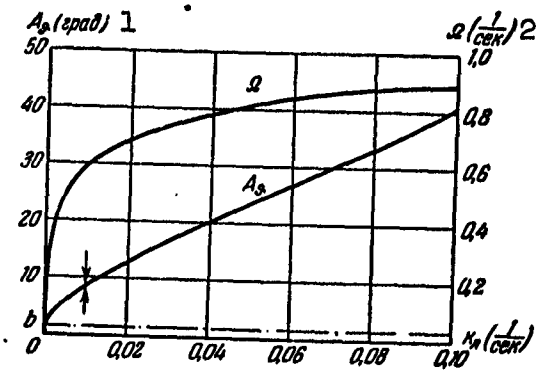


Fig. 4.19. 1) Deg; 2) 1/sec.

Substituting $p = j\Omega$ in (4.51), we write down two equations for the amplitude and frequency of the periodic solution

$$\frac{4k_1U_c}{\pi A^2}\sqrt{A^2-b^2}-(T_1+T_2)\Omega^2=0, \tag{4.52}$$

$$\Omega-\frac{4k_1U_cb}{\pi A^2}-T_1T_2\Omega^3=0. \tag{4.53}$$

Using Eqs. (4.52) and (4.53) we plot the amplitude and the frequency of the temperature fluctuations as functions of the parameter k_1. Eliminating k_1 from (4.52) and (4.53) we obtain a formula for the determination of the amplitude of the periodic solution for specified system parameters and for a specified frequency of the periodic solution:

$$A^2=\frac{(1+T_1^2\Omega^2)(1+T_2^2\Omega^2)}{(1-T_1T_2\Omega^2)^2}b^2. \tag{4.54}$$

Eliminating A^2 from the same equations, we obtain a formula for k_1 in terms of the parameters and the frequency of the periodic solution

$$k_1=\frac{\pi b\Omega(1+T_1^2\Omega^2)(1+T_2^2\Omega^2)}{4U_c(1-T_1T_2\Omega^2)}. \tag{4.55}$$

Plots of $A_\vartheta = A_\vartheta(k_1)$ and $\Omega = \Omega(k_1)$ obtained from formulas (4.54) and (4.55) are shown in Fig. 4.19. The following constant values of the parameters were assumed in the plotting:

$$T_1 = 10 \text{ sec}, \quad T_2 = 0.1 \text{ sec}, \quad U_s = 25 \text{ v}, \quad b = 0.01 \text{ v}.$$

The amplitudes of the relay winding voltage oscillations were recalcu-

lated into amplitudes of temperature fluctuations with $k_2 = 0.01$ v/deg, i.e., $A_\vartheta = 100$ A. It is seen from (4.54) and (4.55) that when $k_1 = 0$ we have $A = b$ and $\Omega = 0$, and when $k_1 = \infty$ we have $A = \infty$ and $\Omega = 1/\sqrt{T_1 T_2}$.

In this example we shall check on the stability of the periodic solution by the method of averaging the periodic coefficients (see §2.4).

In the investigation of stability by the method of averaging the periodic coefficients, the differential equation of the system is expressed in terms of the small deviations from the sought periodic solution and is then analyzed. The equation of the nonlinear element represents in this case a linear dependence of the output on the input, with periodically varying coefficients in the input quantity and in the velocity of the input quantity of the nonlinear element. In the simplest case, when the nonlinear function depends only on the input, as is the case in our example where $u_d = F(u)$, we have, in terms of small displacements,

$$\Delta F(u) = \left(\frac{dF}{du}\right)^* \Delta u,$$

where (dF/du)* is indeed the periodically varying coefficient. This coefficient is averaged over the period by means of the formula

$$\varkappa(A) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{dF}{du}\right)^* d\psi, \quad \psi = \Omega t. \tag{4.56}$$

The asterisk following the brackets denotes that after taking the derivative one substitutes the value of u in periodic form

$$u = A \sin \Omega t.$$

After averaging the periodically varying coefficient of the nonlinear function, the total equation of the system, expressed in deviations from the steady-state oscillating mode, will be linear with constant coefficients. It can be subjected to any customary linear sta-

bility criterion, satisfaction of which will indeed determine the stability of the periodic solution.

To calculate the coefficient $\kappa(A)$ it is necessary, in accord with (4.56), to know the derivative of F(u) with respect to u, a derivative which in our case becomes infinite when $u = b$, in which case $du/dt > 0$, and when $u = -b$, in which case $du/dt < 0$. To circumvent this, we replace the specified characteristic (4.18) by a new characteristic (4.28), from which the specified characteristic is obtained by going to the limit as $h \to 0$.*

For the characteristic of Fig. 4.20a with u varying as $u = A \sin \Omega t$ (Fig. 4.20b), the derivative dF(u)/du assumes the values shown in Fig. 4.20c, where

$$\psi_1 = \arcsin \frac{b}{A}, \quad \psi_2 = \arcsin \frac{b+h}{A}.$$

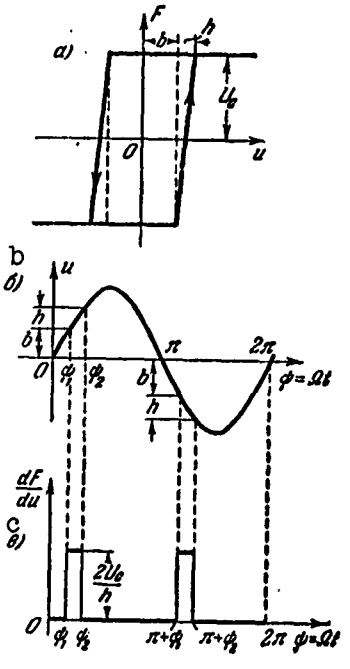


Fig. 4.20

According to Fig. 4.20c, its averaged value, after making the limiting transition to the specified characteristic ($h \to 0$), will be

$$\varkappa(A) = \lim_{h \to 0} \frac{2 \frac{2U_c}{h} (\psi_2 - \psi_1)}{2\pi} = \lim_{\psi_2 \to \psi_1} \frac{2U_c (\psi_2 - \psi_1)}{\pi A (\sin \psi_2 - \sin \psi_1)},$$

since $h = A \sin \psi_2 - A \sin \psi_1$.

Putting $\psi_2 = \psi_1 + \Delta\psi$ and taking the derivatives of the numerator and denominator with respect to $\Delta\psi$ so as to eliminate the indeterminate form, we obtain

$$\varkappa(A) = \lim_{\Delta\psi \to 0} \frac{2U_c}{\pi A \cos(\psi_1 + \Delta\psi)} = \frac{2U_c}{\pi \sqrt{A^2 - b^2}}.$$

Consequently, the nonlinear function of the steady state, expressed in terms of the deviations, can be written upon averaging of the periodic

coefficients in the form

$$\Delta u_1 = \Delta F(u) = \varkappa(A)\,\Delta u = \frac{2U_c}{\pi\sqrt{A^2-b^2}}\,\Delta u. \qquad (4.57)$$

Then, taking (4.57) and the equation of the linear part of (4.39) into account, the characteristic equation for the determination of the stability of the periodic solution will be written in the form

$$T_1T_2p^3+(T_1+T_2)p^2+p+\frac{2U_ck_1}{\pi\sqrt{A^2-b^2}}=0. \qquad (4.58)$$

In accord with the Hurwitz stability criterion it follows from (4.58) that in order for the resultant periodic solution to be stable it is necessary to satisfy the inequality

$$T_1+T_2>\frac{2U_ck_1T_1T_2}{\pi\sqrt{A^2-b^2}}. \qquad (4.59)$$

Substituting into (4.59) the values of A^2 and k_1 from (4.54) and (4.55), we obtain

$$\frac{T_1+T_2}{T_1T_2}>\frac{(1+T_1^2\Omega^2)(1+T_2^2\Omega^2)}{2(T_1+T_2)}. \qquad (4.60)$$

As was shown, the frequency of the periodic solution varies within the range

$$0<\Omega<\frac{1}{\sqrt{T_1T_2}}$$

when $0 < k_1 < \infty$. Substituting in (4.60) the maximum possible value of the frequency, we verify that even in this case the inequality (4.60) is satisfied. Consequently, the periodic solution obtained is stable. The system will operate in a self-oscillating steady mode for all values of k_1.

In the present example we plotted only the curves of $A_y(k_1)$ and $\Omega(k_1)$ (Fig. 4.19). Similar curves can be plotted also for the parameters T_2 and b, in analogy with what was done in the preceding examples.

§4.7. Account of the Time Delay in a Relay System

In the foregoing examples of temperature control systems, we took into account in the nonlinear elements the coordinate lag due to the backlash zone and to the presence of a hysteresis loop in the relay characteristics. In addition to a coordinate lag, systems with relay elements are also subject to time delay. When the end of the bimetal strip (Fig. 4.13) makes contact with the corresponding contact plate, the current in the control circuit of the relay builds up not instantaneously, but exponentially (Fig. 4.21a)

$$i_p = \frac{U_p}{R}\left(1 - e^{-\frac{t}{T}}\right), \quad T = \frac{L}{R},$$

where U_p is the voltage of the relay power source, while L and R are the inductance and resistance of the control winding of the relay. The delay will be determined in this case, first of all, by the time required for the current in the control winding of the relay to build up to the value of the pull-in current of the relay $\tau'_1 = T \ln U_p/(U_p - RI_{sr})$, where I_{sr} is the pull-in current of the relay.

Secondly, it is necessary to add to the delay τ'_1 a delay τ''_1 equal to the time of motion of the relay armature. The total pull-in delay will be

$$\tau_1 = \tau'_1 + \tau''_1.$$

When the end of the bimetal strip moves in the opposite direction, the relay is likewise turned on not instantaneously, but after a time delay τ_2, in which the operating current of the relay decreases to its drop-out value. In this case the current i_p in the control winding decreases exponentially (Fig. 4.21b):

$$i_p = \frac{U_p}{R} e^{-\frac{t}{T}}.$$

From this we get

$$\tau_2 = T \ln \frac{U_p}{I_{отп} R},$$

where I_{otp} is the drop-out current of the relay.

The delays τ_1 and τ_2 in the pull-in and drop-out of the relays are quite fixed and invariant during the course of operation of the given system. On the other hand, the lags due to the backlash zone and to the hysteresis loop are connected with the value of the input quantity, namely the coordinate. These will bring about a time delay which depends on the speed of the process in the system. Consequently the delays τ_1 and τ_2 are frequently called pure time delays, thereby emphasizing their independence of the speed of the process, while the delay due to the backlash and the hysteresis loop is called coordinate lag.

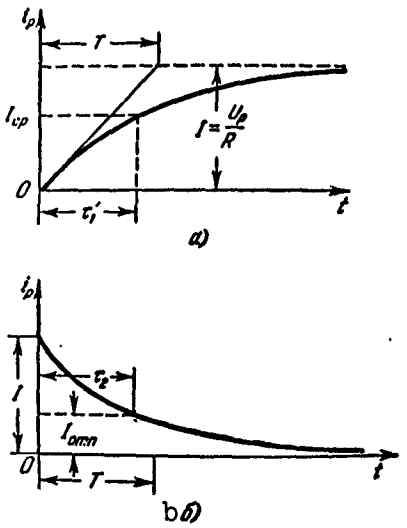


Fig. 4.21

The coordinate lag due to the backlash can be eliminated from a temperature control system with bimetal strip by using the circuit shown in Fig. 4.22. In this case the system is so adjusted as to make the end of the bimetal strip 1 situated at nominal temperature on the boundary between the insulation and the conducting part of plane 2. The smallest displacement of the end of the bimetal strip toward the insulator part of plate 2 causes the relay 3 to become de-energized and the central terminal makes contact under the influence of the spring with the upper terminal thus causing the motor to turn in the required direction. When the bimetal strip goes over onto the

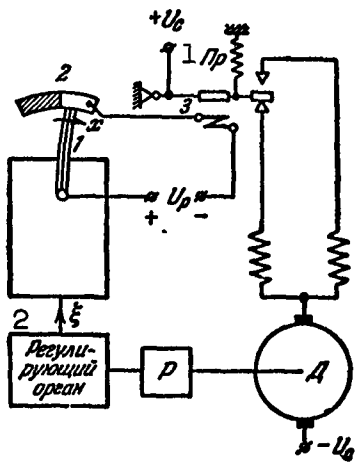


Fig. 4.22. 1) Spring; 2) regulating unit.

conducting part, the central terminal of the relay makes contact with the lower one and the motor starts turning in the opposite direction. It is clear from physical considerations that the system will operate in the self-oscillation mode.

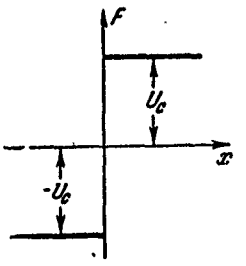


Fig. 4.23

In the present case the static characteristic of the nonlinear element $u_d = F(x)$ will be the same as that of an ideal relay (Fig. 4.23).

If we disregard the pure time delay, then we obtain in accord with (4.52) and (4.53) with $b = 0$ the following equations for the determination of the amplitude and frequency of the self-oscillations:

$$\left.\begin{aligned} \frac{4k_1U_c}{\pi A} - (T_1 + T_2)\Omega^2 = 0, \\ 1 - T_1T_2\Omega^2 = 0. \end{aligned}\right\} \qquad (4.61)$$

From the second equation of (4.61) we obtain a formula for the determination of the self-oscillation frequency

$$\Omega = \frac{1}{\sqrt{T_1T_2}}, \qquad (4.62)$$

while from the first one we obtain, with account of (4.62), a formula for the determination of the amplitude of the self-oscillations in terms of the parameters of the system

$$A = \frac{4k_1U_cT_1T_2}{\pi(T_1 + T_2)}. \qquad (4.63)$$

As can be seen from (4.62) and (4.63), whenever k_1 changes, the self-oscillation frequency remains the same, while the amplitude changes in proportion to the gain. This is illustrated in the curves of Fig. 4.24a for the case of the previously assumed values of the parameters and for $\tau_1 = \tau_2 = 0$. The amplitude of the oscillations of the end of the bimetal strip has been converted into the amplitude of the temperature fluctuation A_ϑ.

Let us take into account the effect of a constant time delay τ in

the given system. The approximate equation of the nonlinear element assumes after harmonic linearization and account of the delay the following form

$$u_A = \frac{4U_c}{\pi A} e^{-\tau p} x. \tag{4.64}$$

The characteristic equation in accordance with (4.51) and with allowance for (4.64) will be

$$T_1 T_2 p^3 + (T_1 + T_2) p^2 + p + \frac{4U_c k_3}{\pi A} e^{-\tau p} = 0. \tag{4.65}$$

Substituting $p = j\Omega$ and recognizing that

$$e^{-j\tau\Omega} = \cos \tau\Omega - j \sin \tau\Omega,$$

we obtain the following equations for the amplitude and frequency of the periodic solution

$$\left.\begin{aligned} \frac{4U_c k_3}{\pi A} \cos \tau\Omega - (T_1 + T_2)\Omega^2 = 0, \\ -\frac{4U_c k_3}{\pi A} \sin \tau\Omega + \Omega - T_1 T_2 \Omega^3 = 0. \end{aligned}\right\} \tag{4.66}$$

From (4.66) we obtain after eliminating $4U_s k_1/\pi A$ the following formula relating the frequency of the periodic solution with the parameters of the system and with the delay

$$(T_1 + T_2)\Omega \operatorname{tg} \tau\Omega = 1 - T_1 T_2 \Omega^2. \tag{4.67}$$

Recognizing that $\sin^2\tau\Omega + \cos^2\tau\Omega = 1$, we obtain from the same equations a formula for the amplitude of the periodic solution

$$A = \frac{4U_c k_3}{\pi\Omega \sqrt{1 + (T_1^2 + T_2^2)\Omega^2 + T_1^2 T_2^2 \Omega^4}}. \tag{4.68}$$

Equation (4.67) is transcendental in Ω. Therefore the value of Ω can be determined only graphically. However, inasmuch as in practice the time delay is small (hundredths of a second), and the expected frequency Ω is likewise small (several units per second), we have $\tan \tau\Omega \approx \approx \tau\Omega$. We then obtain from (4.67) an approximate formula for the determination of the frequency at small values of $\tau\Omega$:

$$\Omega \approx \frac{1}{\sqrt{T_1 T_2 + \tau(T_1 + T_2)}}. \tag{4.69}$$

We see that the presence of a time delay causes a decrease in the frequency of the periodic solution.

Substituting the value of Ω from (4.69) in (4.68) and neglecting after several transformations the value of τ^2 compared with $T_1T_2 + \tau(T_1 + T_2)$, we obtain an approximate formula for the amplitude of the periodic solution:

$$A = \frac{4U_c k_\vartheta}{\pi}\left(\frac{T_1T_2}{T_1+T_2} + \tau\right). \qquad (4.70)$$

From (4.70) we see that the time delay increases the amplitude of the periodic solution. Plots of $A_\vartheta(k_1)$ and $\Omega(k_1)$ for different values of τ, as well as plots of $A_\vartheta(\tau)$ and $\Omega(\tau)$ are shown in Figs. 4.24a and b. In plotting these curves we assumed the previous values of the parameters as well as $k_1 = 0.02$ [$\text{sec}^{-1}\text{v}^{-1}$].

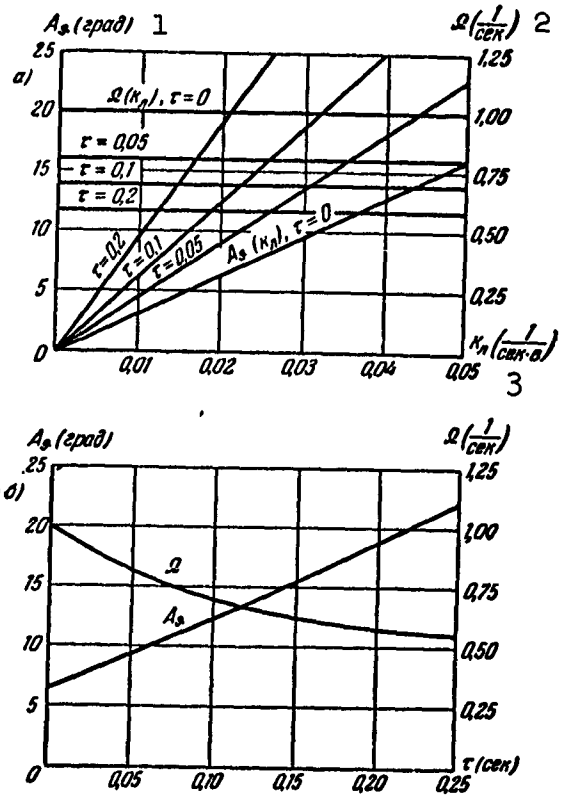


Fig. 4.24. 1) Deg; 2) sec; 3) sec·v.

§4.8. Second Order Servomechanism with Dry Friction and Play

Allowances for dry friction and play are of practical interest, since nonlinearities of this type are frequently encountered in automatic systems with mechanical and electromechanical elements.

If the actuating element of the servomechanism is an electric motor, then dry friction and viscous friction must be taken into account. Dry friction can be regarded as independent of the velocity and its direction changes with that of the velocity. Viscous friction has in general a nonlinear dependence on the velocity. At low velocities the viscous friction is assumed proportional to the first degree of the velocity. With increasing velocity, viscous friction changes from linear into one proportional to the higher powers of the velocity.

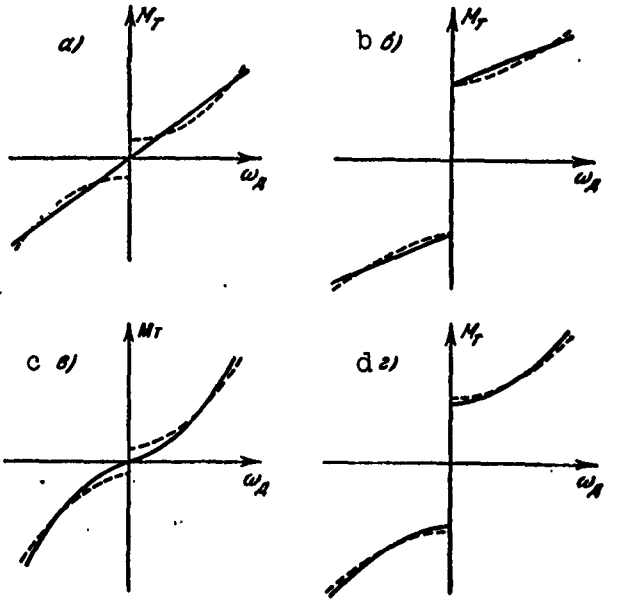


Fig. 4.25

In order to choose the correct velocity dependence of the friction, it is necessary to plot experimentally the dependence of the torque or of the friction forces on the velocity for a given element, varying the velocity over the range expected during the operation of the system. If the dry friction is small and the characteristic of the

torque due to the viscous friction forces differs little from linear, one can assume the total friction to be approximately linear (Fig. 4.25a).

If the dry friction is appreciable, and the viscous friction has a near linear variation, it is necessary to take the dry friction into account in addition to the linear friction (Fig. 4.25b). In this case the system is analyzed as if it were nonlinear. It is also possible to take into consideration quadratic friction alone (Fig. 4.25c) as well as dry and quadratic friction simultaneously (Fig. 4.45d [sic]); other cases are also possible.

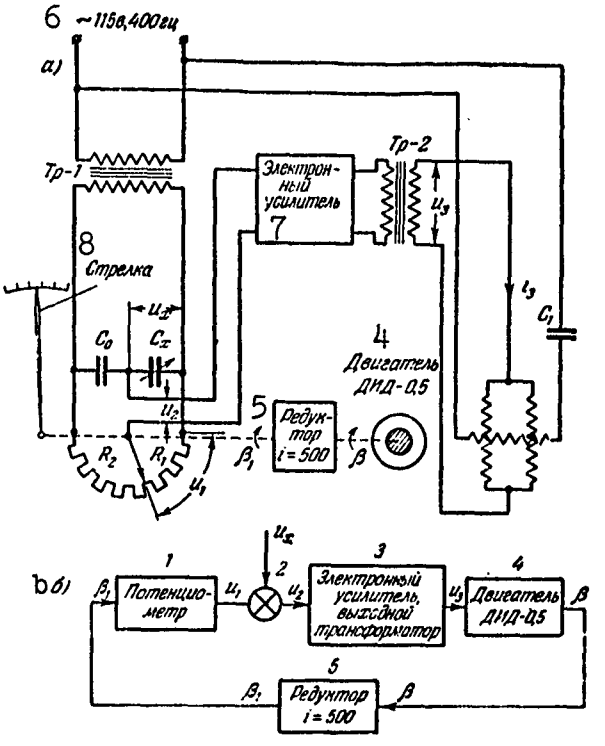


Fig. 4.26. 1) Potentiometer; 3) vacuum tube amplifier and output transformer; 4) motor DID-0.5; 5) reduction gear, i = 500; 6) 115 v, 400 cps; 7) vacuum tube amplifier; 8) pointer.

In angular-position servomechanisms the motion is transferred from the actuating motor to the object and feedback, as a rule, through

a reduction gear. A spur or worm reduction gear is subject to play. The play becomes particularly large in an ordinary spur-gear reduction when the reduction coefficient is large (several pairs of gears). An investigation in which the servomechanism is regarded as nonlinear and account is taken of the play therefore yields a more complete result.

Let us take by way of an example the servomechanism used for a fuel gauge [121].

The fuel gauge is intended for continuous measurement of the volume of liquid fuel in tanks used to supply a thermal engine.

The schematic diagram of the fuel gauge is shown in Fig. 4.26a. The fuel gauge transducer is a capacitor C_x, made in the form of coaxial metallic cylinders, which are placed inside the tank with the fuel. The dielectric constant of the fuel differs from that of air. Therefore as the fuel is used up, the value of the capacitance C_x of the transducer will vary depending on the fuel level.

The capacitance C_x is connected in an AC bridge circuit. The other arms of the bridge are a fixed capacitor C_0 and a potentiometer whose arms have resistances R_1 and R_2, which are varied by displacing the wiper. The wiper is displaced by the actuating motor through a reduction gear. The wiper shaft carries also the pointer of the fuel gauge. The bridge is fed with 115 volt, 400 cycle AC through a transformer Tr-1.

A change in the fuel level will cause a change in the capacitance C_x and upset the equilibrium of the bridge. The voltage u_2 picked off the diagonal of the bridge is fed to a vacuum tube amplifier. The voltage u_3 picked off the output of the vacuum tube amplifier is fed through transformer Tr-2 to the control winding of a two-phase DID-0.5 induction motor. The second winding of the motor is fed from the line through a phase shifting capacitor C_1. Thus, whenever voltage u_3 exists

on the control winding of the motor, the latter rotates and moves, through a reduction gear, the wiper of the potentiometer and the pointer of the fuel gauge indicator. The motor keeps on turning until the wiper is displaced by an angle such that the bridge becomes balanced and the voltage picked off the bridge diagonal, u_2, becomes equal to zero. Thus, the fuel gauge operates on the servomechanism principle with a set-point value u_x and an output β_1.

During the course of operation of fuel gauges it was noted that the pointer of the indicator frequently oscillates with appreciable amplitude (several degrees), making it difficult to read its indications. The presence of self-oscillations in the system is evidence of an appreciable influence of the nonlinearities.

The self-oscillating mode in the operation of such instruments is not harmful. When such devices operate in the self-oscillating mode, the dry friction is appreciably reduced and consequently the operating accuracy of the instrument is increased, since the backlash zone is decreased. However, a self-oscillating mode is acceptable only when the amplitude of the oscillations is small, almost invisible to the eye (fractions of a degree).

Consequently, this system must be investigated with account of the nonlinearities so as to disclose the possibility of obtaining a self-oscillating mode with acceptable amplitude and frequency. If this operating mode is unattainable, then recommendations must be made on how to eliminate the self-oscillations, i.e., how to transfer the operation of the instrument into a stable mode without self-oscillations. Such an analysis is also of interest because other remotely operating instruments are based on a similar principle.

In the present system it is necessary to take into account two nonlinearities: 1) dry friction in the actuating motor and reduction

gear; 2) play due to the presence of gears.

There is practically no nonlinearity connected with the saturation of the amplifier, since the system operates at low amplifier input signals.

Using the approximate method of harmonic linearization of the nonlinearities, we shall determine the range of variation of the parameters, within which self-oscillations and stability without self-oscillation occur, and also the effect of each parameter on the variation of the amplitude and frequency of the self-oscillations.

We shall assume during the course of investigation that the variation of the capacitance C_x and of the voltage u_x occurring as the fuel is consumed, is incomparably slower than the rate of operation of the servomechanism. We shall disregard the variation of the sensitivity of the bridge as it operates over the entire range of variation of resistors R_1 and R_2, and consequently disregard the variation of the gain, but will take these factors into account later on, in the determination of the influence of the gain on the frequency and amplitude of the self-oscillations of the system, i.e., we shall investigate the servomechanism at a constant value of the set-point signal by regarding the servomechanism as being a system that regulates the position of the output shaft of the reduction gear.

A block diagram of the servomechanism is shown in Fig. 4.26b. Elements 1, 2, 3, and 4 are linear. Element 5 is nonlinear, since both the play and the dry friction can be attributed to the reduction gear.

The vacuum tube amplifier together with the output transformer Tr-2 can be regarded as an ideal element. When the alternating voltage u_2 at the input of the vacuum tube amplifier is varied, the output voltage u_3 will have during the transient stage two components, forced and free. The forced component of the voltage u_3 is translated with

practically no inertia upon change in the input voltage and consequently the AC component in the control circuit of the motor will not be delayed; the torque on the rotor of the motor will likewise be produced without delay. The free component of the voltage u_3 and of the current i_3, on the other hand, cannot influence appreciably the torque of the motor.

Let us set up the equations for the linear elements of the system, and let us introduce the nonlinear static relationships in the form of certain nonlinear functions.

1. The equation of the potentiometer is

$$u_1 = k_1 \beta_1, \tag{4.71}$$

where u_1 [v] is the voltage picked off the resistor R_1, β_1 is the angle of rotation of the potentiometer wiper, and k_1 [v] is the transfer ratio of the first element.

2. The equation of the error transducer (bridge) is

$$u_2 = k_2 (u_x - u_1). \tag{4.72}$$

In the investigation of the motion of the system proper (u_x = const), Eq. (4.72) assumes the form

$$u_2 = -k_2 u_1, \tag{4.73}$$

where u_2 [v] is the voltage applied to the input of the vacuum tube amplifier and k_2 is the transfer ratio of the bridge.

3. The equation of the vacuum tube amplifier together with the output transformer is

$$u_3 = k_3 u_2, \tag{4.74}$$

where u_3 [v] is the voltage applied to the control winding of the motor and k_3 is the gain of the vacuum tube amplifier together with the transformer.

4. The equation of motion of the motor rotor, with account of dry friction as well as friction proportional to the first power of the

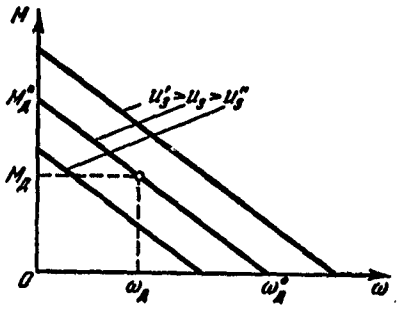


Fig. 4.27

velocity, will be

$$Jp^2\beta = M_д - k_\beta' p\beta - M_т, \qquad (4.75)$$

where β is the displacement angle of the motor rotor, J [g·cm·sec^2] is the moment of inertia of all the elements driven by the motor, referred to the motor shaft, M_d [g·cm] is the torque developed by the motor, M_t [g·cm] is the torque due to dry friction and k_β' [g·cm·sec] is the damping coefficient.

In two-phase induction motors, the torque is directly proportional to the voltage applied to the control winding. During the course of operation of the motor, the torque varies with the speed. The mechanical characteristics of the two-phase induction motor can be approximately represented by straight lines (Fig. 4.27). In accordance with Fig. 4.27, we express the torque of the motor in terms of the control voltage u_3, the nominal locked-rotor torque M_d^0, and the nominal no-load speed ω_d^0:

$$M_д = k_д' u_3 - k_д'' \omega_д \qquad (4.76)$$

where M_d [g·cm] is the torque developed by the motor for a given control voltage u_3 and angular velocity ω_d, k'_d [g·cm/v] is the coefficient representing the variation of the motor torque with the control voltage, and $k''_d = M_d^0/\omega_d^0$ [g·cm·sec] is the coefficient representing the variation of the motor torque with angular velocity.

The torque due to dry friction will be taken into consideration in the form of a nonlinear function of the motor speed:

$$M_т = F_1(p\beta). \qquad (4.77)$$

With allowance for (4.76) and (4.77), Eq. (4.75) is rewritten

$$Jp^2\beta + k_д'' p\beta + k_\beta' p\beta = k_д' u_3 - F_1(p\beta). \qquad (4.78)$$

Reduced to standard form, Eq. (4.78) becomes

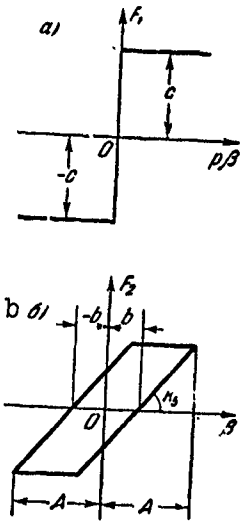


Fig. 4.28

$$(Tp+1)p\beta = k_4 u_3 - \frac{F_1(p\beta)}{n}, \qquad (4.79)$$

where $T = J/n$ [sec] is the electromechanical time constant of the motor, $k_4 = k'_d/n$ [$v^{-1}sec^{-1}$] is the transfer ratio of the motor, and $n = k''_d + k'_\beta$ [g·cm·sec] is the total damping coefficient.

5. The equation of the reduction gear without allowance for play is written in the form

$$\beta_1 = k_5\beta,$$

where β_1 is the angle of rotation of the output shaft of the reduction gear and k_5 is the transfer ratio of the reduction gear.

Inclusion of the play in the reduction gear makes β_1 a nonlinear function of β

$$\beta_1 = F_2(\beta). \qquad (4.80)$$

Reducing (4.71), (4.73), (4.74), (4.79), and (4.80) to a single equation in the variable β, we obtain

$$(Tp+1)p\beta = -k_5 F_2(\beta) - \frac{F_1(p\beta)}{n}, \quad (4.81)$$

where $k_1 = k_1k_2k_3k_4$ [sec^{-1}] is the transfer ratio of the linear part of the system.

The nonlinear static characteristic $F_1(p\beta)$ is plotted in Fig. 4.28a. Analytically, the function $F_1(p\beta)$ can be expressed as

$$F_1(p\beta) = c\,\operatorname{sign} p\beta \begin{cases} p\beta \neq 0, \\ p\beta = 0 \text{ for } Jp^2\beta + np\beta - k'_4 u_3 \geq C \end{cases} \qquad (4.82)$$

in the case when the motor is not being stopped, or by

$$-c < F_1(p\beta) < c, \ \beta = \text{const for } k'_4 u_3 < C$$

when the motor is being stopped.

In determining the self-oscillations, we shall confine ourselves to the condition (4.82).

The static characteristic of the function $F_2(\beta)$ is plotted in Fig. 4.28b.

We shall seek a solution for the motor shaft angle β in the form

$$\beta = A \sin \Omega t, \quad \Omega t = \psi,$$

and for the velocity in the form

$$p\beta = A\Omega \cos \Omega t.$$

Upon harmonic linearization of the nonlinearities in accordance with formulas (3.53), (3.28), (3.29), and (3.52) we obtain

$$F_1(p\beta) = \frac{q_1'(A)}{\Omega} p\beta, \tag{4.83}$$

$$F_2(\beta) = \left[q_2(A) + \frac{q_2'(A)}{\Omega} p\right]\beta, \tag{4.84}$$

where

$$q_1'(A) = \frac{4c}{\pi A}, \tag{4.85}$$

$$q_2(A) = \frac{k_5}{\pi}\left[\frac{\pi}{2} + \arcsin\left(1 - \frac{2b}{A}\right) + 2\left(1 - \frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1 - \frac{b}{A}\right)}\right] \text{for} A \geq b,$$

$$q_2'(A) = -\frac{4k_5 b}{\pi A}\left(1 - \frac{b}{A}\right) \text{ for } A \geq b. \tag{4.86}$$

Taking (4.81), (4.83), and (4.84) into account we obtain a linearized equation that describes the motion of the servomechanism proper

$$\left\{Tp^2 + \left[1 + \frac{k_3}{\Omega} q_2'(A) + \frac{q_1'(A)}{n\Omega}\right]p + k_3 q_2(A)\right\}\beta = 0. \tag{4.87}$$

In accordance with the differential equation (4.87), we write down the characteristic equations for the following cases:

a) when only dry friction is considered and play in the reduction gear is disregarded [$q_2(A) = k_5$, $q'_2(A) = 0$]:

$$Tp^2 + \left[1 + \frac{q_1'(A)}{n\Omega}\right]p + k_3 k_5 = 0; \tag{4.88}$$

b) when only play in the reduction gear is taken into account [$q'_1(A) = 0$]:

$$Tp^2 + \left[1 + \frac{k_3 q_2'(A)}{\Omega}\right]p + k_3 q_2(A) = 0; \tag{4.89}$$

c) when dry friction and play are considered simultaneously:

$$Tp^2+\left[1+\frac{k_2q_2'(A)}{\Omega}+\frac{q_1'(A)}{n\Omega}\right]p+k_2q_2(A)=0. \qquad (4.90)$$

Let us analyze the resultant equations (4.88)-(4.90).

To obtain a periodic solution in a system describable by a second order equation of the form

$$(a_0p^2+a_1p+a_2)x=0$$

we must stipulate satisfaction of the condition

$$a_1=0, \text{ i.e., } a_0p^2+a_2=0. \qquad (4.91)$$

In analogy with the investigations of nonlinear systems described by equations of arbitrary order, we shall employ here, too, the substitution $p = j\Omega$, from which we obtain the condition $X(A, \Omega) = 0$ and $Y(A, \Omega) = 0$, which is equivalent to (4.91).

It is seen from (4.88) and (4.85) that the condition

$$Y(A, \Omega)=\Omega+\frac{4c}{\pi nA}=0$$

cannot be satisfied for any positive values of c, n, Ω, and A. Consequently, dry friction without play cannot bring about self-oscillations in the present system.

On the other hand, Eq. (4.89) can be satisfied by the conditions $X(A, \Omega) = 0$, $Y(A, \Omega) = 0$ with positive values of the amplitude, frequency, and the parameters contained in the equation. Consequently, play causes self-oscillations in the present system.

Let us analyze Eq. (4.89). Substituting $p = j\Omega$, we obtain two equations for the amplitude and frequency of the periodic solution:

$$k_2q_2(A)-T\Omega^2=0, \qquad (4.92)$$

$$\Omega+k_2q_2'(A)=0. \qquad (4.93)$$

We plot the variation of the amplitude and of the frequency of the periodic motion as functions of the system parameters. From (4.93) with allowance for the value of $q'_2(A)$ we obtain

$$\Omega = \frac{4k_2k_5b}{\pi A}\left(1-\frac{b}{A}\right). \qquad (4.94)$$

It is seen from (4.94) that a periodic solution is possible only when $A \geq b$, i.e., when the amplitude exceeds half the width of the play. Substituting (4.94) in (4.92) and taking the value of $q_2(A)$ into account, we obtain

$$\frac{16k_2k_5Tb^2}{\pi A^2}\left(1-\frac{b}{A}\right)^2 = \frac{\pi}{2} + \arcsin\left(1-\frac{2b}{A}\right) + 2\left(1-\frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1-\frac{b}{A}\right)}. \qquad (4.95)$$

By assigning values to the amplitude and to all parameters but one we obtain from (4.95) the dependence of the amplitude on each parameter. From relation (4.94), for the same values of the parameters and for the corresponding values of the amplitude, we determine the frequency of self-oscillation.

The values of the parameters determined experimentally for one fuel-gauge assembly are

1) electromechanical time constant of the motor T = 0.115 sec; 2) over-all gain of the system $k = k_1k_5 = 35.2$ sec^{-1}; 3) transfer ratio of the reduction gear $k_5 = \beta_1/\beta = 1/500$; 4) play in reduction gear 2b = 16 rad.

Formulas (4.94) and (4.95) were used to calculate and plot the variation of the frequency and of the amplitude of the periodic solution with the parameters $k = k_1k_5$, T, and b (Fig. 4.29). The amplitudes of the oscillations of the output shaft of the reduction gear were determined from the amplitudes of the oscillations of the motor shaft using the transfer ratio of the reduction gear:

$$A_{\beta_1} = 0{,}002A.$$

As can be seen from the resultant curves, the system can operate either in a stable mode without self-oscillations or else in a self-oscillating steady-state mode, depending on the values of the parameters k and T.

Within the self-oscillation region there are two branches of the amplitude and frequency values satisfying the periodic solution. The upper branches correspond to self-oscillations. The lower branches are the limits of stability "in the small." This is symbolically designated by arrows.

To check on the stability of the periodic solution corresponding to the branches of the amplitudes with the larger values, we shall employ the approximate analytical criterion (4.6).

It follows from the characteristic equation (4.89) that

$$X(a,\ \omega)=k_s q_2(a)-T\omega^2,$$
$$Y(a,\ \omega)=\left(1+\frac{k_s q_2'(a)}{\Omega}\right)\omega.$$

Calculating the corresponding derivatives with account of the plots of $q_2(a)$ and $q'_2(a)$ (Fig. 3.9) we obtain

$$\left(\frac{\partial X}{\partial a}\right)^*=k_s\left(\frac{\partial q_2}{\partial a}\right)^*>0 \qquad \text{for all values of the amplitude a = A,}$$

$$\left(\frac{\partial Y}{\partial a}\right)^*=k_s\left(\frac{\partial q_2'}{\partial a}\right)^*\begin{cases}\leqslant 0 & \text{for } A\leqslant 2b,\\ >0 & \text{for } A>2b,\end{cases}$$

$$\left(\frac{\partial X}{\partial \omega}\right)^*=-2T\Omega<0 \qquad \left(\frac{\partial Y}{\partial \omega}\right)^*=1+\frac{k_s q_2'(A)}{\Omega}.$$

Since it follows from (4.93) that

$$q_2'(A)=-\frac{\Omega}{k_s},$$

we have

$$\left(\frac{\partial Y}{\partial \omega}\right)^*=0.$$

Taking into account the signs obtained for the partial derivatives, we verify that the stability criterion is fulfilled when A > 2b and is not fulfilled when A < 2b. Consequently, the branch with the large amplitudes belongs to the stable periodic solution.

The assertion that the periodic solution is stable over the entire large-amplitude branch follows from the continuity of the self-oscillating steady-state modes under variation of the system parameters.

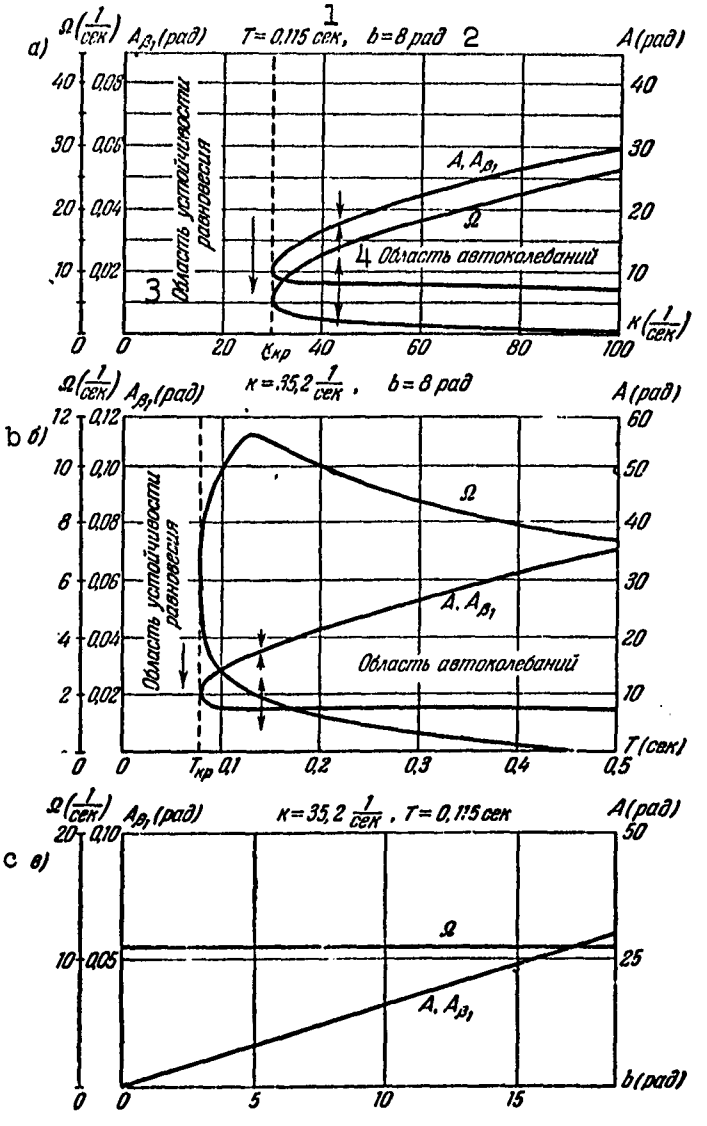


Fig. 4.29. 1) Sec; 2) rad; 3) region of stable equilibrium; 4) region of self-oscillations.

The disparity in the present case between the value of the amplitude separating the solution into two branches and the value of the amplitude obtained from the stability investigation can be attributed to the approximate nature of the stability criterion (failure to take account of the variation of the frequency near the periodic solution). The fact that the branch with small amplitudes belongs to the unstable periodic solution is evident from the fact that this branch lies im-

mediately above the play zone, within which there is no periodic solution.

Extrapolation of the results obtained for the self-oscillation region into the region where there is no periodic solution enables us to conclude that the system is stable for values of the parameter below critical.

For the real values of the parameters measured on the model we obtain from the curves plotted: A = 13.8 rad, $A_{\beta_1} = 0.002\ A = 0.0275$ rad, $\Omega = 10.8\ \text{sec}^{-1}$, and f = 1.72 cps.

The curves plotted enable us to choose parameters such as to reduce the amplitude and increase the frequency of the self-oscillations or else such as to suppress the self-oscillations for systems of similar type, but where dry friction can be neglected. If dry friction cannot be neglected, it must be taken into consideration in the form of a second nonlinearity in the system, on top of the play.

Let us investigate the servomechanism with simultaneous account of the play in the reduction gear and of the friction. From (4.90) we obtain two equations for the amplitude and the frequency of the periodic solution:

$$k_3 q_3(A) - T\Omega^2 = 0, \tag{4.96}$$

$$\Omega + k_3 q_2'(A) + \frac{q_1'(A)}{n} = 0. \tag{4.97}$$

From (4.96) we obtain, taking into account the values of the harmonic-linearization coefficients

$$\Omega = \sqrt{\frac{k_1 k_3}{\pi T}\left[\frac{\pi}{2} + \arcsin\left(1 - \frac{2b}{A}\right) + 2\left(1 - \frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1 - \frac{b}{A}\right)}\right]}, \tag{4.98}$$

while (4.97) yields

$$c = nkb\left(1 - \frac{b}{A}\right) - \frac{\pi n A \Omega}{4}. \tag{4.99}$$

To determine the influence of the dry friction on the self-oscillations, let us determine the dependences A(c) and Ω(c) with all other

parameters constant. For this purpose we specify the values of A in (4.98) and we calculate Ω; we then use the same values of A and the resultant values of Ω and calculate $\underline{c}$ by means of (4.99). Inasmuch as under the assumed values of the parameters the system is near the boundary separating the stability region from the self-oscillation region, we have plotted, for the sake of clarity, the curves A(c) and Ω(c) (Fig. 4.30a) with the time constant T increased by a factor of four. It is evident from the plot that the dry friction causes the self-oscillation region to become narrower.

A similar construction was carried out for the parameter $\underline{n}$, which is the total damping coefficient (Fig. 4.30b). We see that damping likewise narrows down the region of self-oscillations.

No curves were plotted in this case for the variation of the amplitude and frequency of the self-oscillations for the parameters T, $\underline{k}$, and $\underline{b}$, since they remained qualitatively the same as for the case when dry friction is disregarded. The dry friction merely reduces slightly the width of the self-oscillation region.

A plotting of the transient by the Bashkirov grapho-analytic method has confirmed the result obtained by the approximate harmonic linearization method. Experimental measurements made on one fuel gauge assembly are also in good agreement with the theoretical results. The observed self-oscillations with the fuel gauge operating on different portions of the scale had a frequency f = 1.5-2 cps and an amplitude $A_{\beta_1} = 1\text{-}2^{\circ}$. In addition, disappearance of self-oscillation was observed on some portions. This was caused by the fact that at the real values of the parameters, the system operates near the boundary of the stability and self-oscillation regions.

The results of the investigation can be used to select the parameters of the existing fuel gauge design or to improve the latter.

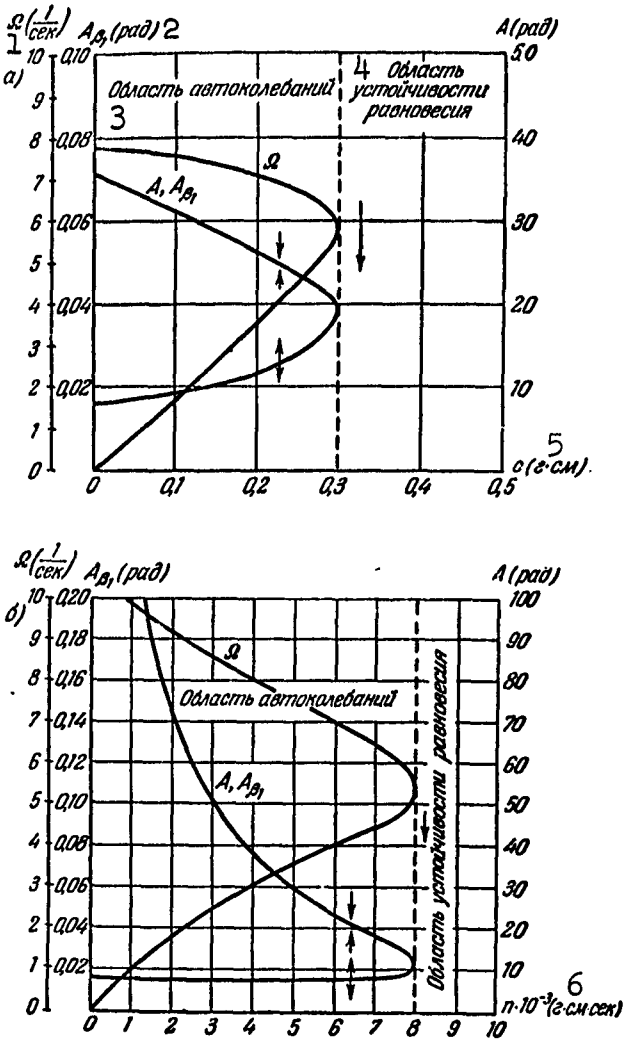


Fig. 4.30. 1) Sec; 2) rad; 3) region of self-oscillations; 4) region of stable equilibrium; 5) g·cm; 6) g·cm·sec.

As can be seen from the analysis, a stable mode without self-oscillation or suppression of self-oscillation can be obtained by reducing the time constant, by reducing the gain of the system, by increasing the dry friction, and by increasing the total damping coefficient. However, only the reduction in the gain and the increase in the damping can be regarded as rational means, since an increase in the dry friction increases the backlash zone of the instrument, and the change in the time constant resulting from changing the moment of inertia can be very insignificant in practice. It is preferable to in-

crease the total damping coefficient than to decrease the gain, for a decrease in the gain brings about a decrease in the operating speed of the system and increases the instrument errors. An increase in the total damping coefficient is structurally feasible either by installing an additional magnetic or air damper with coefficient k_β, or by using a motor with stiffer mechanical characteristics k''_d. Another recommended means of suppressing self-oscillations is to add a supplementary negative feedback loop. For this purpose, the motor can be provided with a tachometer winding, whose voltage is applied to the input of the amplifier, i.e., one can use negative feedback proportional to the motor speed. The use of such feedback decreases the time constant and the gain. The latter can be again increased if necessary.

The instrument can operate in practice in a self-oscillating mode. This greatly reduces the backlash zone of the instrument. On the other hand, obtaining an acceptable self-oscillation amplitude (several tenths of a degree) calls for an appreciable decrease in the play in the reduction gear. This can be done by replacing an ordinary spur gear reduction system by a differential-planetary reduction gear.

§4.9. Third Order Servomechanism with Nonlinear Friction and Play

We shall investigate the system considered in §4.3 (Fig. 4.5). We shall assume that the vacuum tube amplifier has a linear static characteristic for the operating values of the input voltage. We shall take account of dry or quadratic friction in the actuating motor and play in the reduction gear.

The servomechanism is represented by the block diagram shown in Fig. 4.31, where the previous notation is retained for the variables.

Let us write down the equations for the system elements in the case where we investigate the motion of the system proper with allowance for the indicated nonlinearities.

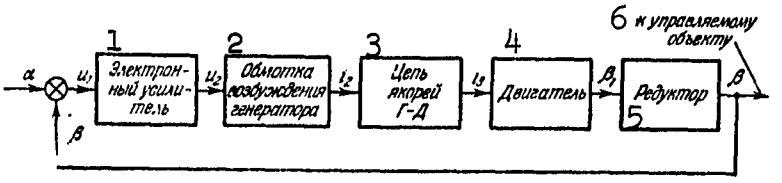


Fig. 4.31. 1) Vacuum tube amplifier; 2) generator exciting winding; 3) generator and motor armature circuit; 4) motor; 5) reduction gear; 6) to controlled object.

1. The equation of the error transducer is

$$u_1 = k_1(\alpha - \beta),\ u_1 = -k_1\beta \text{ for } \alpha = 0. \tag{4.100}$$

2. The equation of the vacuum tube amplifier is

$$u_2 = k_y u_1. \tag{4.101}$$

3. The equation of the generator excitation winding is

$$(T_1 p + 1) i_2 = k_2 u_2. \tag{4.102}$$

4. The equation for the generator and motor armature circuit, neglecting inductance, is

$$i_3 = k_3 i_2 - k_4 p\beta_1. \tag{4.103}$$

5. The equation of motion of the motor armature together with the controlled object will be written in terms of the variable β_1, which is the motor shaft angle. The nonlinear friction will be accounted for by the nonlinear function

$$M_T = F_1(p\beta_1).$$

In accordance with (4.12), the equation of motion of the motor rotor with allowance for nonlinear friction is

$$(T_2' p)\, p\beta_1 = k_5 i_3 - \frac{F_1(p\beta_1)}{k_5'}. \tag{4.104}$$

Combining (4.104) with (4.103), we obtain

$$(T_2 p + 1)\, p\beta_1 = \frac{k_3 k_5}{1 + k_4 k_5} i_2 - \frac{F_1(p\beta_1)}{k_5'(1 + k_4 k_5)}, \tag{4.105}$$

where

$$T_2 = \frac{T_2'}{1 + k_4 k_5}.$$

6. The equation of the reduction gear with allowance for play is written in the form of the nonlinear function

$$\beta = F_2(\beta_1). \tag{4.106}$$

We shall seek a solution for the variable β_1 in harmonic form

$$\beta_1 = A \sin \psi, \quad \psi = \Omega t.$$

We then obtain from the formulas for harmonic linearization of dry and quadratic friction, (3.53) and (3.64), respectively, the following linear relations:

$$F_1(\beta_1) = \frac{q_1'(A)}{\Omega} p\beta_1, \quad F_1(\beta_1) = q_{1*}'(A)\,\Omega p\beta_1, \tag{4.107}$$

where $q'_1 = 4c/\pi A$ if dry friction is considered and $q'_{1*} = 8k_*A/3\pi$ if quadratic friction is considered; c is the constant value of the torque due to the dry friction forces.

The first relation in (4.107) corresponds to a dry friction characteristic in which the values of the friction torque are disregarded for zero speed. Such an account of dry friction can be used in practice to separate the regions of periodic solution.

For the nonlinear characteristic due to play in the reduction gear, we have in accord with the formulas (3.28) and (3.29) for the coefficients of harmonic linearization the following relation

$$\beta = F_2(\beta_1) = \left[q_2(A) + \frac{q_2'(A)}{\Omega} p\right]\beta_1, \tag{4.108}$$

where

$$q_2(A) = \frac{k_6}{\pi}\left[\frac{\pi}{2} + \arcsin\left(1-\frac{2b}{A}\right) + 2\left(1-\frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1-\frac{b}{A}\right)}\right] \text{ for } A \geqslant b,$$

$$q_2'(A) = -\frac{4k_6 b}{\pi A}\left(1-\frac{b}{A}\right) \text{ for } A \geqslant b,$$

b is half the width of the play and k_6 is the transfer ratio of the reduction gear.

Substituting (4.107) in (4.105) we obtain the equations for the cases when dry friction and quadratic friction are taken into account,

respectively:

$$\left.\begin{aligned}\left[T_2p+1+\frac{q_1'(A)}{k_\beta^*(1+k_4k_5)\Omega}\right]p\beta_1&=\frac{k_3k_5}{1+k_4k_5}l_2,\\ \left[T_2p+1+\frac{q_{1*}'(A)\Omega}{k_\beta^*(1+k_4k_5)}\right]p\beta_1&=\frac{k_3k_5}{1+k_4k_5}l_2.\end{aligned}\right\} \tag{4.109}$$

Combining (4.100), (4.101), (4.102) with (4.109), we obtain

$$\left.\begin{aligned}(T_1p+1)\left[T_2p+1+\frac{q_1'(A)}{k_\beta^*(1+k_4k_5)\Omega}\right]p\beta_1&=-\frac{k_1k_yk_2k_3k_5}{1+k_4k_5}\beta,\\ (T_1p+1)\left[T_2p+1+\frac{q_{1*}'(A)\Omega}{k_\beta^*(1+k_4k_5)}\right]p\beta_1&=-\frac{k_1k_yk_2k_3k_5}{1+k_4k_5}\beta.\end{aligned}\right\} \tag{4.110}$$

Replacing β in (4.110), by its value (4.108) and introducing the symbols: $k_1 = k_1k_uk_2k_3k_5/(1 + k_4k_5)$ [sec^{-1}] is the gain of the linear portion of the system and $k' = 1/k_\beta^*(1 + k_4k_5$ [$g^{-1}cm^{-1}sec^{-1}$], we obtain the following general equations of the system

$$\left.\begin{aligned}&\left\{T_1T_2p^3+(T_1+T_2)p^2+\right.\\ &\qquad\left.+\left[1+\frac{k'q_1'(A)}{\Omega}+\frac{k_3q_2'(A)}{\Omega}\right]p+k_3q_2(A)\right\}\beta_1=0,\\ &\left\{T_1T_2p^3+(T_1+T_2)p^2+\right.\\ &\qquad\left.+\left[1+k'q_{1*}'(A)\Omega+\frac{k_3q_2'(A)}{\Omega}\right]p+k_3q_2(A)\right\}\beta_1=0.\end{aligned}\right\} \tag{4.111}$$

Using Eqs. (4.111), we write down the characteristic equations for various cases:

a) when the system is considered to be linear and nonlinear friction and play are disregarded:

$$T_1T_2p^3+(T_1+T_2)p^2+p+k=0, \tag{4.112}$$

where $k = k_1k_6$ [sec^{-1}] is the over-all gain of the system;

b) when dry friction is considered to be the nonlinearity

$$T_1T_2p^3+(T_1+T_2)p^2+\left[1+\frac{k'q_1'(A)}{\Omega}\right]p+k=0; \tag{4.113}$$

c) when quadratic friction is regarded as the nonlinearity

$$T_1T_2p^3+(T_1+T_2)p^2+[1+k'q_{1*}(A)\Omega]p+k=0; \tag{4.114}$$

d) when play in the reduction gear is regarded as the nonlinearity

$$T_1T_2p^3+(T_1+T_2)p^2+\left[1+\frac{k_3q_2'(A)}{\Omega}\right]p+k_3q_2(A)=0. \tag{4.115}$$

Equation (4.112) enables us to estimate the stability of the linear system in the case when the nonlinearities are insignificant. We see that in order for the linear system to be stable it is necessary to satisfy the following condition for positive values of the parameters:

$$T_1 + T_2 > kT_1T_2.$$

The value of the gain at which the system is on the limit of stability, i.e.,

$$k_{gr} = \frac{T_1 + T_2}{T_1T_2},$$

will be called the limiting value.

When $k < k_{gr}$ the system will be stable, and when $k > k_{gr}$ the system will be unstable.

For the time constants assumed above, $T_1 = 0.1$ sec and $T_2 = 1$ sec, we find that the linear system will be stable when the over-all gain is $k < 11$ sec^{-1}, i.e., $k_{gr} = 11$ sec^{-1}.

Let us examine Eq. (4.113). Substituting of $p = j\Omega$ into this equation yields two equations for the determination of the amplitude and frequency of the periodic solution

$$k - (T_1 + T_2)\Omega^2 = 0, \tag{4.116}$$

$$\Omega + k'q_1'(A) - T_1T_2\Omega^3 = 0. \tag{4.117}$$

Let us investigate the stability of the periodic solution, determined by Eqs. (4.116) and (4.117) by means of the analytical criterion

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0.$$

In accordance with (4.113), we obtain

$$X(a, \omega) = k - (T_1 + T_2)\omega^2,$$

$$Y(a, \omega) = \omega + k'\frac{q_1'(a)}{\omega}\omega - T_1T_2\omega^3.$$

Calculating the corresponding derivatives, we obtain

$$\left.\begin{aligned}&\left(\frac{\partial X}{\partial a}\right)^{*}=0, \quad \left(\frac{\partial Y}{\partial a}\right)^{*}=k'\left(\frac{\partial q_1'}{\partial a}\right)=-\frac{4k'c}{\pi A^2}<0,\\&\left(\frac{\partial X}{\partial \omega}\right)^{*}=-2(T_1+T_2)\Omega<0, \quad \left(\frac{\partial Y}{\partial \omega}\right)^{*}=1+k'\frac{q_1'(A)}{\Omega}-3T_1T_2\Omega^2.\end{aligned}\right\} \tag{4.118}$$

Replacing $q'_1(A)$ in (4.118) by its value from (4.117) we obtain

$$\left(\frac{\partial Y}{\partial \omega}\right)^{*}=-2T_1T_2\Omega^2<0. \tag{4.119}$$

From the values obtained for the derivatives we see that the stability criterion is not fulfilled. The periodic solution corresponding to Eqs. (4.116) and (4.117) determines only the limit of stability "in the small."

The frequency of the periodic solution is determined from Eq. (4.116):

$$\Omega=\sqrt{\frac{k}{T_1+T_2}}.$$

Substituting the value of Ω and the value of $q'_1(A)$ into (4.117) we obtain a formula for the amplitude

$$A=\frac{4k'c}{\pi\sqrt{\frac{k}{T_1+T_2}\left(\frac{T_1T_2k}{T_1+T_2}-1\right)}}. \tag{4.120}$$

As can be seen from (4.120), a periodic solution is possible only if

$$T_1T_2k>T_1+T_2, \tag{4.121}$$

and the value of the gain separating the region in which a periodic solution is present from the region where there is no periodic solution is determined from (4.121) by replacing the inequality sign by an equality sign

$$k_{kp}=\frac{T_1+T_2}{T_1T_2}.$$

The value obtained for the coefficient k_{kr} coincides with the value of k_{gr} of the linear system.

It follows from (4.120) that $A \to \infty$ when $k = k_{kr}$ and $A \to 0$ when $k \to \infty$. Therefore the variation of the amplitude of the periodic solu-

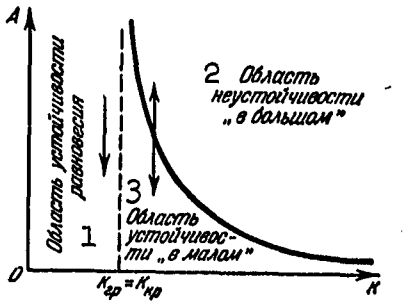


Fig. 4.32. 1) Region of stable equilibrium; 2) region of instability "in the large"; 3) region of stability "in the small."

tion as a function of k can be plotted as shown in Fig. 4.32. The arrows that point away from the curve show symbolically the instability of the periodic solution.

The stability of the nonlinear system in the region where there is no periodic solution can be determined by transferring the results of the investigation of the stability from the region of periodic solution, something also designated symbolically by an arrow. This result is readily confirmed by the method of §2.7.

Consequently, as in the analysis of a servomechanism describable by a second order equation, dry friction does not cause self-oscillations in the present system. To the contrary, it damps the motion in the system. When account is taken of the dry friction, a region of stability "in the small" is added to the stability region obtained when the system is considered to be linear.

Let us examine the behavior of the system in the case when quadratic friction is added to the linear friction. From (4.114) we obtain two equations for the determination of the amplitude and frequency of the periodic solution:

$$\left.\begin{aligned} k-(T_1+T_2)\Omega^2=0,\\ 1+k'q'_1*(A)\Omega-T_1T_2\Omega^2=0. \end{aligned}\right\} \qquad (4.122)$$

The expressions X(a, ω) and Y(a, ω) needed for the determination of the stability of the periodic solution will be:

$$X(a,\omega)=k-(T_1+T_2)\omega^2,$$
$$Y(a,\omega)=\omega+[1+k'q'_1*(A)\Omega]\omega-T_1T_2\omega^3.$$

Now, unlike the case of dry friction, the value of the derivative will

be

$$\left(\frac{\partial Y}{\partial a}\right)^* = k'\left[\frac{\partial q'_{1*}(a)}{\partial a}\right]^* \Omega^2 = \frac{8k'k_*}{3\pi}\Omega^2 > 0$$

and consequently the periodic solution is stable.

The frequency of the periodic solution is determined from the first equation of (4.22):

$$\Omega = \sqrt{\frac{k}{T_1 + T_2}}.$$

The amplitude of the periodic solution is determined from the second equation of (4.122) upon substitution of the value of Ω:

$$A = \frac{3\pi}{8k_0}\sqrt{\frac{T_1+T_2}{k}}\left(\frac{kT_1T_2}{T_1+T_2} - 1\right), \qquad (4.123)$$

where $k_0 = k'k_*$.

In accordance with (4.123), positive values of the amplitude are possible only when

$$k > \frac{T_1+T_2}{T_1T_2}$$

and consequently the critical gain value separating the region of stable equilibrium from the region of self-oscillations will be

$$k_{кр} = \frac{T_1+T_2}{T_1T_2}.$$

This coincides with the limiting value k_{gr} of the linear system.

Consequently, quadratic friction leads to self-oscillation in that region of the parameters, where the linear system would be unstable without this additional friction. The reason for it is the reinforced damping action of the quadratic friction resulting from an increase in the amplitude (and consequently also velocity) of the oscillations, which prevents the system from running away. The changeover of the resistance of the object to motion from linear to quadratic, as was shown earlier, is quite real at high velocities.

Figure 4.33 shows the variation of the oscillation amplitude of

the output shaft of the reduction gear and of the frequency with each parameter. As one parameter was varied, the following constant values were assumed for the other parameters: $T_1 = 0.1$ sec, $T_2 = 1$ sec, $k = 20$ sec^{-1}, $k_0 = 1$ sec, and the transfer ratio of the reduction gear $k_6 = 0.01$. The curves obtained make it possible to choose parameters such as to satisfy the requirements imposed on the system. The suppression of self-oscillations in the system, as can be seen from these curves, is possible by reducing the gain and the time constants of the motor and of the generator excitation circuit (T_1 and T_2 exert similar influences). If it is desirable to retain the self-oscillation mode in such a system, then it is possible to choose the parameters in such a way as to attain, on the basis of the relations obtained, a reduction in the amplitude and an increase in the frequency of the self-oscillations.

Simultaneous reduction in the amplitude and increase in the frequency of the self-oscillation is possible by proper selection of the time constants and gains.

For given values of $\underline{k}$, T_1, and T_2, it is possible to decrease the amplitude without changing the frequency of the self-oscillations by increasing the parameter k_0. This is also verified by physical considerations, since k_0 determines the ratio between the quadratic friction and the linear friction

$$k_0 = \frac{k_0}{k_3 (1 + k_1 k_3)}.$$

The more the quadratic friction predominates over linear friction in the system, the smaller the amplitude of the swing of the system in the self-oscillation region.

One can conclude in general from a comparison of the formulas of harmonic linearization for such systems that located beyond the sta-

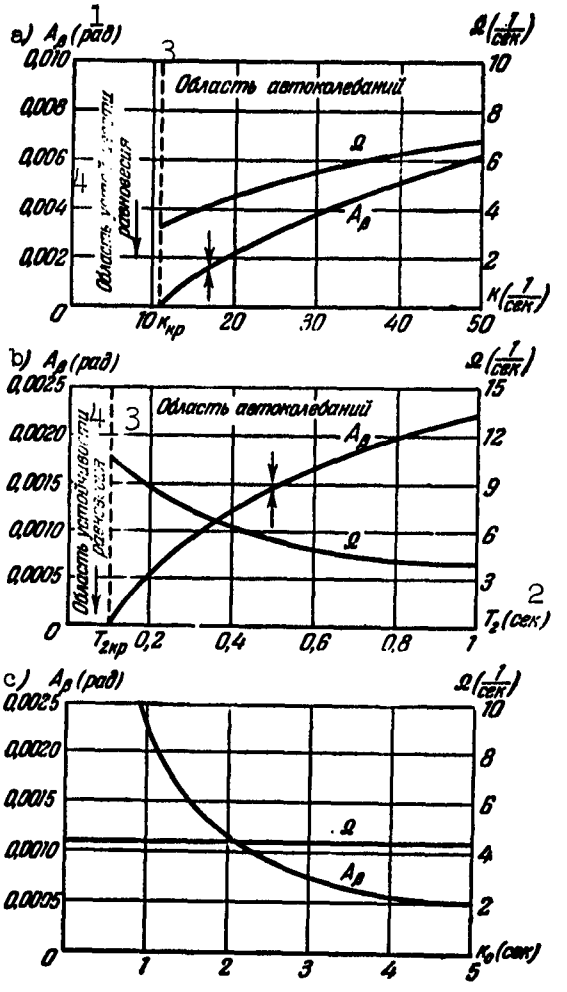


Fig. 4.33. 1) rad; 2) sec; 3) region of self-oscillations; 4) stable equilibrium region.

bility region will be a self-oscillation region if the nonlinear friction increases faster than the first power of the velocity, and an instability region if the nonlinear friction is approximated by a power law with degree smaller than the first (fractional), including dry friction as a limiting case.

Let us analyze the motion of this system with account of a nonlinearity in the form of play in the reduction gear. From Eq. (4.115) we obtain an equation for the determination of the amplitude and frequency of the periodic solution

$$k_1 q_2(A) - (T_1 + T_2)\Omega^2 = 0, \qquad (4.124)$$

$$\Omega + k_1 q_2'(A) - T_1 T_2 \Omega^2 = 0. \qquad (4.125)$$

Equation (4.124) cannot be satisfied by real values of the amplitude A < b, meaning that periodic solutions are possible only when A $\geq$ b. This follows also from physical considerations, since whenever the amplitude of the motor shaft oscillations decreases to less than half the value of the play, the transmission of motion to the subsequent elements is stopped.

To plot the variation of the amplitude and frequency of the periodic solution as a function of the system parameters it is necessary to solve Eqs. (4.124) and (4.125) with respect to A and Ω and to obtain formulas that relate the amplitude and the frequency with each parameter, while the other parameters are kept constant.

Equations (4.124) and (4.125) are transcendental in A and Ω, but can be solved with respect to the parameters $k = k_1 k_6$, T_1, and T_2. Therefore, by expressing for example the parameter k with the aid of the first and second equations with account of the values of $q_2(A)$ and $q'_2(A)$, we obtain

$$k = \frac{\pi (T_1 + T_2)\Omega^2}{\frac{\pi}{2} + \arcsin\left(1 - \frac{2b}{A}\right) + 2\left(1 - \frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1 - \frac{b}{A}\right)}}, \qquad (4.126)$$

$$k = \frac{\pi (\Omega - T_1 T_2 \Omega^3)}{4\frac{b}{A}\left(1 - \frac{b}{A}\right)}. \qquad (4.127)$$

By specifying the same values of Ω in (4.126) and (4.127) and by varying the amplitude, we obtain two families of curves k = k(A). The intersection points of curves (Fig. 4.34a) with like values of Ω will be the graphic solutions of the equations A = A(k) and $\Omega = \Omega(k)$.

Figure 4.34b shows the result of the solution for A_β with the parameters T_1 = 0.1 sec, T_2 = 0.1 sec, and b = 1 rad.

It is seen from the solution obtained that there are three regions

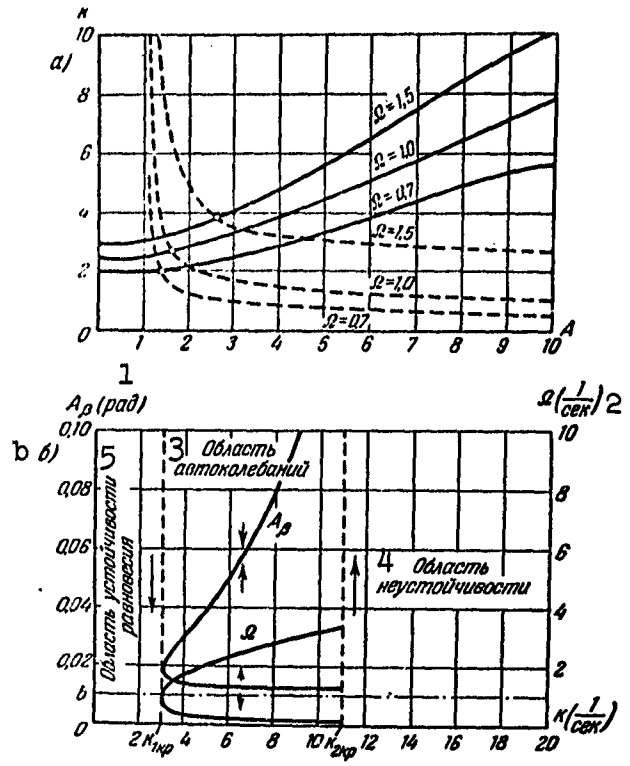


Fig. 4.34. 1) rad; 2) sec; 3) self-oscillation region; 4) instability region; 5) stable equilibrium region.

for the given system, depending on the value of the over-all gain. At low gains, i.e., when $0 < k < k_{1\ kr}$, a stability region is obtained. The value of $k_{1\ kr}$ can be calculated by eliminating Ω from Eqs. (4.126) and (4.127) so as to obtain an expression for $k = k(A)$, and set the derivative dk/dA equal to zero. If such an investigation is too cumbersome, it can be avoided, for an approximate value of $k_{1\ kr}$ is obtained during the course of the graphic solution.

The self-oscillation region corresponds to medium values of the over-all gain, $k_{1\ kr} \leq k \leq k_{2\ kr}$. The value of k_{2kr} is equal to the value of the gain on the stability limit of the same system, but regarded as linear without account of the play in the reduction gear:

$$k_{2\,кр} = k_{гр} = \frac{T_1 + T_2}{T_1 T_2}.$$

This can be seen from an examination of (4.126) and (4.127). The numerator in (4.127) vanishes when $\Omega^2 = 1/T_1T_2$, and the amplitude increases without limit with $\underline{k}$ finite. The substitution $A = \infty$ and $\Omega^2 = 1/T_1T_2$ in (4.126) does indeed yield the value of k_{2kr} given above. Consequently, at the value k_{2kr} we have

$$A=\infty, \quad \Omega_*=\frac{1}{\sqrt{T_1T_2}}. \tag{4.128}$$

Lying beyond the self-oscillation region is the instability region, at values $k > k_{2kr}$. Consequently, the self-oscillation region occupies part of the stability region of the linear system without the play.

From the foregoing example we see that an investigation of this system regarded as linear falls short of representing the real picture of the processes occurring there, and gives an erroneous idea concerning the stability region, since steady-state self-oscillating modes with amplitudes that are not acceptable in practice can occur within the stability region.

In the self-oscillation region there are two branches for the amplitude and frequency of the periodic solution. The upper branches belong to the stable periodic solution, namely self-oscillations, and the lower ones to the unstable periodic solution, i.e., they are the limit of system stability "in the small." This is confirmed on the basis of the stability criterion for periodic solutions

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^*-\left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^*>0.$$

In accordance with (4.115) we obtain

$$X(a, \omega)=k_s q_2(a)-(T_1+T_2)\omega^2,$$

$$Y(a, \omega)=\omega+\frac{k_s q_2'(a)}{a}\omega-T_1T_2\omega^3.$$

Calculating the corresponding derivatives and taking into account

the signs of the derivatives of the coefficients of harmonic linearization, which are obvious from the plots in Fig. 3.9, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^* = k_2\left(\frac{\partial q_2}{\partial a}\right)^* > 0,$$
$$\left(\frac{\partial Y}{\partial a}\right)^* = k_2\left(\frac{\partial q_2'}{\partial a}\right)^* \quad \begin{cases} <0 \text{ for } b<A<2b, \\ >0 \text{ for } A>2b, \end{cases}$$
$$\left(\frac{\partial X}{\partial \omega}\right)^* = -2(T_1+T_2)\Omega < 0, \quad \left(\frac{\partial Y}{\partial \omega}\right)^* = 1 + \frac{k_2 q_2'(A)}{\Omega} - 3T_1T_2\Omega. \tag{4.129}$$

From (4.125) we have

$$1 + \frac{k_2 q_2'(A)}{\Omega} = T_1T_2\Omega^2.$$

We therefore obtain from (4.129) for the derivative

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = -2T_1T_2\Omega^2 < 0.$$

Substituting the obtained values of the derivatives into the inequality serving as the stability criterion, we obtain

$$(T_1+T_2)\left(\frac{\partial q_2'}{\partial a}\right)^* > T_1T_2\Omega\left(\frac{\partial q_2}{\partial a}\right)^*. \tag{4.130}$$

From the plots of $q_2(A)$ and $q'_2(A)$ (Fig. 3.9) we see that for the branch with the larger amplitudes the coefficients $q'_2(A)$ and $q_2(A)$ tend to constant values as $A \to \infty$, and consequently one can assume their derivatives with respect to $\underline{a}$ to be approximately equal to each other in the case of large amplitudes. Then the inequality (4.130) can be rewritten by introducing the maximum value Ω_m, in the form

$$\frac{T_1+T_2}{T_1T_2} > \Omega_m.$$

Taking into account the value $k_{2kr} = (T_1 + T_2)/T_1T_2$, we write the condition for the stability of the periodic solution in the form

$$k_{2кр} > \Omega_м \tag{4.131}$$

or, taking into consideration the value $\Omega_m = 1/\sqrt{T_1T_2}$, in the form

$$T_1+T_2 > \sqrt{T_1T_2}. \tag{4.132}$$

Consequently, the stability criterion is satisfied for the upper branch (Fig. 4.34).

For the lower branch the stability criterion is not satisfied for $A < 2b$, in accordance with the signs of the derivative, and consequently the lower branch belongs to the unstable periodic solution, i.e., it is the limit of the stability of the system "in the small."

The stability or instability of the periodic solution are indicated for the corresponding branches by means of converging and diverging arrows. The arrows converging toward the upper branch of the frequencies indicate symbolically that this branch belongs to the branch of the amplitudes of the stable periodic solution.

By transferring the results obtained in the self-oscillation region to the regions $0 \leq k \leq k_{1kr}$ and $k \geq k_{2kr}$, we verify that the former is a stability region and the latter an instability region for the nonlinear system.

We note that a similar solution can be obtained also by the frequency method (the third method of §2.3). By opening the system and setting up the equation for the linear part we obtain in this case an expression for the frequency characteristic of the linear part of the system in the form

$$W_л(j\omega) = \frac{k}{[1 - T_1 T_2 \omega^2 + (T_1 + T_2) j\omega] j\omega} \cdot \qquad (4.133)$$

and a corresponding characteristic of the nonlinear element

$$-\frac{1}{W_н(a)} = -\frac{1}{\frac{1}{\pi}\left[\frac{\pi}{2} + \arcsin\left(1 - \frac{2b}{a}\right) + 2\left(1 - \frac{2b}{a}\right)\sqrt{\frac{b}{a}\left(1 - \frac{b}{a}\right)} - j\frac{4b}{\pi a}\left(1 - \frac{b}{a}\right)\right]} \cdot \qquad (4.134)$$

Since the characteristic $-1/W_н(a)$ is independent of the parameter k, it is sufficient to construct one characteristic in accordance with (4.134) and a family of characteristics in accordance with (4.133) for different values of k. The points where the curve $-1/W_н(a)$ crosses the family of curves $W_л(j\omega)$ will be the sought solution, making it possible to plot $A = A(k)$ and $\Omega = \Omega(k)$. However, the advisability of using any

particular criterion is determined by the accuracy of the graphic solution. Thus, for example, in the present case the curves (4.133) cross the curve (4.134) at excessively small angles, making it difficult to determine the points of the solution.

§4.10. System for the Control of the Course of an Airplane

We have investigated above nonlinear systems described by equations of order not higher than the third. The investigation methods remain the same in the case of nonlinear systems described by equations of higher order. Let us take, for example, a system for automatic control of the course of an airplane. The block diagram of the system is shown in Fig. 4.35.

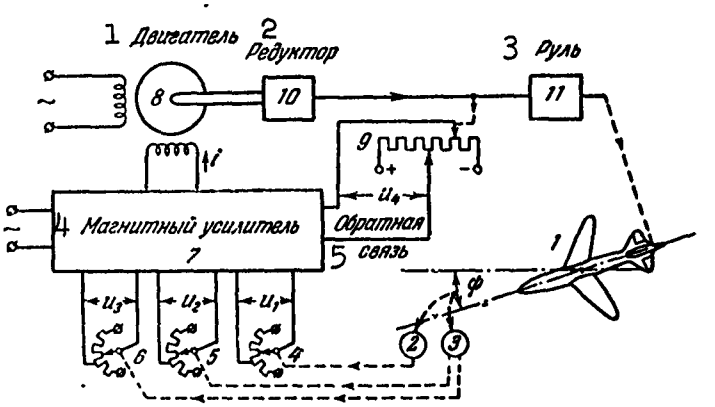


Fig. 4.35. 1) Motor; 2) reduction gear; 3) rudder; 4) magnetic amplifier; 5) feedback.

The regulated object is the airplane, and the regulator is the heading autopilot, which is arbitrarily shown in the figure outside the airplane, while the regulated quantity is the angle ψ through which the airplane axis is turned, measured from the specified heading.

The sensitive elements of the heading autopilot are the free gyroscope 2, which measures the angle ψ, and the damping gyroscope 3, which measures the first derivative $\dot{\psi}$ and the second derivative $\ddot{\psi}$ of the angle ψ with respect to the time (i.e., the angular velocity and

the angular acceleration of the rotation of the airplane about the vertical axis); both gyros are mounted on the airplane, 1.

Current from the sensitive elements flows through potentiometers 4, 5, and 6 to the control windings of a push-pull magnetic amplifier, 7. These three signals are summed and amplified in the magnetic amplifier. The resultant alternating current is applied to one of the windings of the reversible AC motor 8, which turns the control rudder 11 (the regulating organ) through reduction gear 10. The rudder acts on the airplane, thereby closing the control loop.

It is possible to have also additional feedback, 9, in which the wiper of the rheostat turns together with the rudder, and the voltage picked off the rheostat is applied to a fourth control winding in the magnetic amplifier.

The equation of the airplane in its motion along the course, with account of slip, will be [49]:

$$[(T_1p+1)(T_2p+1)+k_2T_2]p\psi=-k_1(T_2p+1)\delta, \qquad (4.135)$$

where ψ is the deviation of the airplane from its heading, δ is the deflection of the control rudder, $T_1 = J/M_s^{\dot{\psi}}$ [sec] is the inertial time constant of the airplane, $T_2 = mv/Z^{\beta}$ [sec], $k_1 = M_r^{\delta}/M_s^{\dot{\psi}}$ [sec^{-1}], and $k_2 = M_s^{\beta}/M_s^{\dot{\psi}}$ [sec^{-1}]. In these formulas, J is the moment of inertia of the airplane about its vertical axis, $M_s^{\dot{\psi}}$ is the slope (tangent of angle of inclination) of the aerodynamic characteristic of the airplane, representing the dependence of the moment of air resistance on the angular velocity of rotation of the airplane about the vertical axis, $\underline{m}$ is the mass of the airplane, $\underline{v}$ is its velocity, Z^{β} is the slope of the aerodynamic characteristic of the airplane, representing the dependence of the lateral force on the slip angle β, $M_r^{\delta} = (\partial M_r/\partial\delta)^0$, where M_r is the torque produced by the rudder, and $M_s^{\beta} = (\partial M_s/\partial\beta)^0$, where M_s is the moment of the aerodynamic resistance of the airplane

to rotation.

An autopilot with feedback into which two derivatives are introduced is described by the following equations:

a) the linear equation of the sensitive elements and the magnetic amplifier

$$(T_3 p + 1) i = (k_3 k_4 + k'_3 k_5 p + k''_3 k_6 p^2)\psi - k_7 k_9 \delta, \qquad (4.136)$$

where i [a] is the current in the control winding of the control-servo motor, T_3 [sec] is the time constant of the magnetic amplifier, k_3, k'_3, and k''_3 are the transfer ratios of the corresponding sensitive elements, k_4, k_5, and k_6 are the gains of the magnetic amplifier for the voltage components obtained from the sensitive elements, and k_7 and k_9 are, respectively, the transfer ratios of the feedback potentiometer and of the magnetic amplifier for the feedback signal;

b) the nonlinear equation of the control-surface servo:

$$p\delta = F(i). \qquad (4.137)$$

The nonlinear function F(i) can assume different forms, depending on the construction of the control servo. Let us assume that in some control servo the rate of swing of the rudder is proportional to the current $\underline{i}$ up to a value $-b \leq i \leq b$, and is constant when $i \leq -b$ and $i \geq b$, i.e., we have a nonlinear characteristic of the saturation type (Fig. 3.5e). In this case, in order to find the periodic solution for the variable $\underline{i}$ (the current in the control circuit of the control servo), we carry out harmonic linearization and replace the nonlinear equation (4.137) by the relationship

$$p\delta = q(A) i. \qquad (4.138)$$

The coefficient of harmonic linearization q(A) is equal to, in accordance with (3.19),

$$q(A) = \frac{2k_{p.m}}{\pi}\left(\arcsin\frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right),$$

where $k_{r.m}$ [$a^{-1}sec^{-1}$] is the gain of the control servo on the linear portion of the static characteristic.

The characteristic equation corresponding to the differential equation of the system for the current i, will be in accordance with (4.135), (4.136), and (4.138)

$$[(T_1p+1)(T_2p+1)+k_2T_2][(T_3p+1)p+k_7k_9q(A)]p+ \\ +k_1(T_2p+1)(k_3k_4+k'_3k_5p+k''_3k_6p^2)q(A)=0.$$

Let us consider a somewhat simpler version of the system. If the autopilot has no additional feedback ($k_9 = 0$) and no second derivative is introduced ($k''_3 = 0$), then the characteristic equation has after transformation the form

$$T_1T_2T_3p^5+[T_1T_2+(T_1+T_2)T_3]p^4+[T_1+T_2+(1+k_2T_2)T_3]p^3+ \\ +[1+k_2T_2+k_1k'_3k_5T_2q(A)]p^2+k_1(k_3k_4T_2+k'_3k_5)q(A)p+ \\ +k_1k_3k_4q(A)=0. \qquad (4.139)$$

Substituting $p = j\Omega$ we obtain equations for the determination of the amplitude and frequency of the periodic solution (for the variable i, namely the current in the electric control-surface servomotor):

$$k_1k_3k_4q(A)-[1+k_2T_2+k_1k'_3k_5T_2q(A)]\Omega^2+ \\ +[T_1T_2+(T_1+T_2)T_3]\Omega^4=0, \qquad (4.140)$$

$$k_1(k_3k_4T_2+k'_3k_5)q(A)-[T_1+T_2+(1+k_2T_2)T_3]\Omega^2+ \\ +T_1T_2T_3\Omega^4=0. \qquad (4.141)$$

Equations (4.140) and (4.141) make it possible to determine the dependences of the amplitude and frequency of the periodic solution on each parameter of the system, with the remaining parameters maintained constant.

Assume that it is required to find the dependence of the amplitude or the frequency on the autopilot transfer ratio relative to the angle of deviation from the course, i.e., on the parameter k_3. Solving (4.140) and (4.141) with respect to k_3, we obtain

$$k_3=\frac{[1+k_2T_2+k_1k'_3k_5T_2q(A)]\Omega^2-[T_1T_2+(T_1+T_2)T_3]\Omega^4}{k_1k_4q(A)}, \qquad (4.142)$$

$$k_3 = -\frac{k'_3 k_4 q(A)}{k_1 T_3} + \frac{[T_1 + T_2 + (1 + k_2 T_3) T_3] \Omega^2 - T_1 T_2 T_3 \Omega^4}{k_1 k_4 T_3}. \qquad (4.143)$$

By plotting a family of curves $k_3(A)$ in accordance with (4.142) and (4.143) for equal values of Ω, we obtain the values of $A(k_3)$ and $\Omega(k_3)$ by determining the points of intersections of curves having equal values of Ω. The result of such a solution in the case when the values of the parameters are $T_1 = 1$ sec, $T_2 = 1$ sec, $T_3 = 0.01$ sec, $k_1 = 1$ sec^{-1}, $k_2 = 1$ sec^{-1}, $k'_3 = 1$ v/sec, $k_4 = 0.1$ a/v, $k_5 = 0.1$ a/v, $k_{r.m} = 1$ sec^{-1}, and b = 1 a is shown in Fig. 4.36.

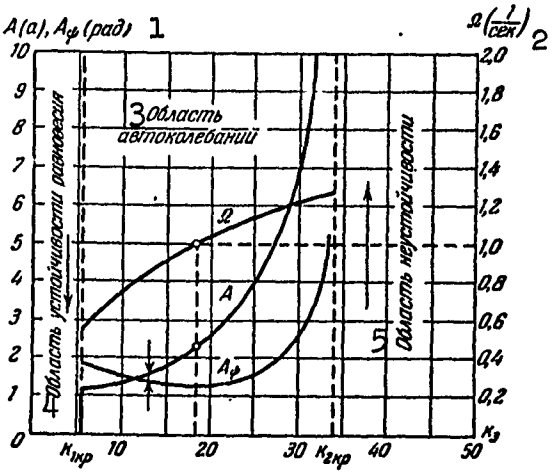


Fig. 4.36. 1) rad; 2) sec; 3) self-oscillation region; 4) stable equilibrium region; 5) instability region.

It follows from the curves in Fig. 4.36 that three regions are possible, depending on the values of the ratio k_3: at small values of k_3 we have a stability region, at medium values of k_3 we have a self-oscillation region, and at large values of k_3 we have an instability region.

In view of the approximate nature of the graphical solution, the instability region is separated here without a rigorous proof that the amplitude A tends to infinity at k_{2kr}. From the solution we find that the amplitude increases to large values, so that for all practical pur-

poses we can assume the state of the system to be unstable.

The foregoing subdivision of the values of k_3 into different regions is valid provided the resulting periodic solution is stable when $k_{1kr} \leq k_3 \leq k_{2kr}$. The stability of the periodic solution will be estimated on the basis of the approximate criterion

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0.$$

From (4.139) it follows that

$$X(a, \omega) = k_1 k_2 k_4 q(A) - [1 + k_2 T_2 + k_1 k_2' k_3 T_2 q(A)]\omega^2 + [T_1 T_2 + (T_1 + T_2) T_3]\omega^4,$$

$$Y(a, \omega) = k_1 (k_3 k_4 T_2 + k_2' k_3) q(A)\omega - [T_1 + T_2 + (1 - k_2 T_2) T_3]\omega^3 + T_1 T_2 T_3 \omega^5.$$

Calculating the corresponding derivative, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^* = k_1 (k_2 k_4 - k_2' k_3 T_2 \Omega^2)\left(\frac{\partial q}{\partial a}\right)^*,$$

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = k_1 (k_3 k_4 T_2 + k_2' k_3) q(A) - 3[T_1 + T_2 + (1 + k_2 T_2) T_3]\Omega^2 + 5 T_1 T_2 T_3 \Omega^4,$$

$$\left(\frac{\partial X}{\partial \omega}\right)^* = -2[1 + k_2 T_2 + k_1 k_2' k_3 T_2 q(A)]\Omega + 4[T_1 T_2 + (T_1 + T_2) T_3]\Omega^3,$$

$$\left(\frac{\partial Y}{\partial a}\right)^* = k_1 (k_3 k_4 T_2 + k_2' k_3)\Omega\left(\frac{\partial q}{\partial a}\right)^*.$$

Substituting into the obtained derivatives the numerical values of the parameters and the corresponding values of q(A) and $(\partial q/\partial a)^*$ for one self-oscillation mode with A = 2.3 a, $\Omega = 1$ [sec^{-1}] and $k_3 = 18$ we obtain a result that satisfies the stability criterion. Consequently, at $k_3 = 18$ (Fig. 4.36) we have self-oscillations. This means that the periodic solution will be stable for all values $k_{1kr} \leq k_3 \leq k_{2kr}$, namely self-oscillations, since the curves of $A(k_3)$ and $\Omega(k_3)$ are continuous and single-valued.

Practical interest attaches to a determination of the amplitude of the self-oscillations of the airplane about the course. In order to convert the amplitudes A of the self-oscillations of the current in

the control circuit of the surface servomotor into the amplitudes A_ψ of the oscillations of the airplane about the course, we make use of Eq. (4.136) without account of the feedback ($k_9 = 0$) or of the second derivative ($k''_3 = 0$). The transfer function converting the angle ψ into the current $\underline{i}$ flowing in the control winding of the rudder motor will be

$$W(p) = \frac{k_3 k_4 + k'_3 k_5 p}{T_3 p + 1}.$$

We obtain the following formula for conversion of the amplitudes

$$A_\psi = \sqrt{\frac{T_3^2 \Omega^2 + 1}{(k'_3)^2 k_5^2 \Omega^2 + k_3^2 k_4^2}}\, A.$$

The values of A_ψ calculated by means of this formula are plotted in Fig. 4.36.

We see from the plot of $A_\psi(k_3)$ that the amplitudes of the oscillations of the airplane about the course are intolerably large in the system assumed for the investigation and for the chosen values of the parameters. Consequently, it is necessary to choose a transfer ratio $k_3 < k_{1kr}$ so as to permit the airplane-autopilot system to operate in the region of stable processes, without self-oscillations. Such a result was obtained without feedback and without a second derivative with respect to the course angle in the control equation. The values of the amplitudes will be lower if feedback and the second derivative are used.

In the presence of nonlinearities of the saturation type, the region of self-oscillations lies beyond the stability region of the linear system (without account of saturation). The self-oscillations have a hard excitation mode at an amplitude equal to half the linearity zone of the nonlinear characteristic, and therefore in practice one cannot as a rule use a self-oscillating mode in this case.

A similar investigation can be made for any other parameter and

for any other airplane-autopilot system.

§4.11. Nonlinear Stabilization System with Time Delay

By way of a second example of an approximate investigation of a nonlinear system using higher-order equations, let us consider a system used to stabilize an aerodynamic object in banking motion. In addition to nonlinearity, we shall take also time delay into account.

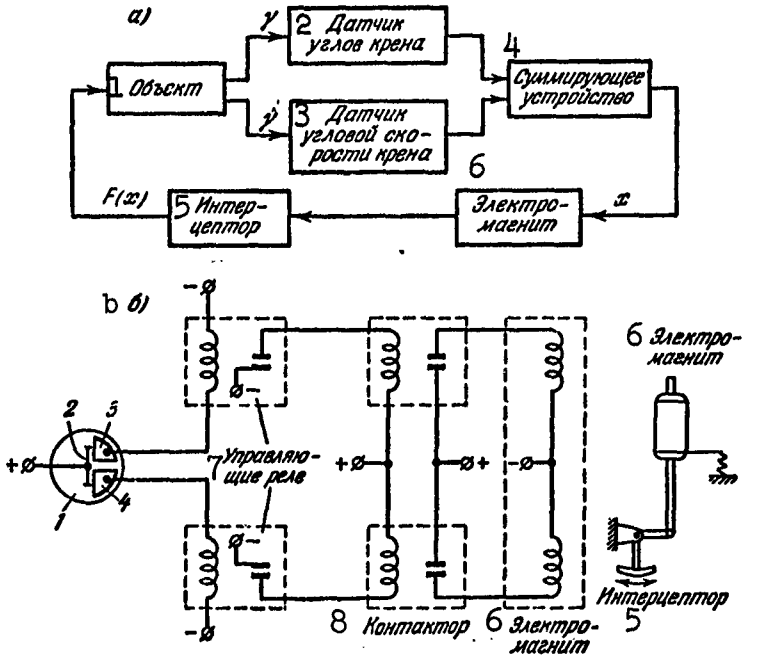


Fig. 4.37. 1) Object; 2) bank angle transmitter; 3) bank angular velocity transmitter; 4) summing unit; 5) interceptor; 6) electromagnet; 7) control relay; 8) contactor.

A block diagram of the system is shown in Fig. 4.37a. The lateral bank angle of the object is measured by the bank angle transmitter, made in the form of a three-degree gyroscope. The angular velocity of the lateral bank is measured by a banking angular velocity transmitter, made in the form of a two-degree gyroscope. The bank angle transmitter gyroscope displaces the brushes, 2, of the contact making unit (Fig. 4.37b), while the angular velocity transmitter gyroscope is connected with disk 1, which carries contact blades 3 and 4. The relative

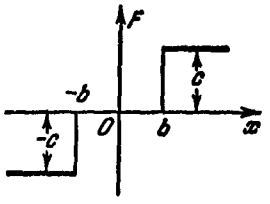


Fig. 4.38

displacement of the brushes and of the disks causes control relays to be energized, and these in turn actuate contactors that control the interceptor electromagnets. The interceptors act on the object and eliminate the undesirable banking angle deviation of the object.

The object, the banking angle transmitter, the banking angular velocity transmitter, and the summing unit are part of the linear portion of the system. The control relay, the contactors, and the interceptor electromagnet comprise the nonlinear portion of the system.

The nonlinear portion of the system can be represented in the form of an element having a static relay characteristic (Fig. 4.38) with a backlash zone (in the contact disk), as well as an element with pure time delay, including the pull-out time and the time of motion of the armatures of the control relays and the contactors.

In accordance with the formula for harmonic linearization of the relay characteristic with backlash zone, taking into account the time delay, we have for the nonlinear portion the following equation

$$F_1(x) = q(A)\,e^{-p\tau}\,x, \qquad x = \gamma + k_\gamma^{\cdot}\dot{\gamma}, \tag{4.144}$$

where γ is the bank angle of the object, $k_\gamma^{\cdot}$ is the transfer ratio of the two-degree gyroscope, $q(A) = \frac{4c}{\pi A}\sqrt{1 - (b^2/A^2)}$ is the coefficient of harmonic linearization, and τ is the time delay.

For the case of horizontal flight, the equation of lateral motion for the variable $\underline{x}$ can be written in the form

$$(p^4 + m_1p^3 + m_2p^2 + m_3p + m_4)\,x + (n_1p^3 + n_2p^2 + n_3p + n_4')\,F_1(x) = 0, \tag{4.145}$$

where the coefficients $m_1, \ldots, m_4, n_1, \ldots, n_3, n'_4$ include the parameters of the system and the aerodynamic coefficients.

Taking (4.144) into account, we obtain from (4.145) the characteristic equation for the harmonically linearized system:

$$p^4 + m_1 p^3 + m_2 p^2 + m_3 p + m_4 + (n_1 p^3 + n_2 p^2 + n_3 p + n_4') q(A) e^{-p\tau} = 0. \qquad (4.146)$$

Substituting $j\omega$ in place of p in the left half of (4.146) and recognizing that

$$e^{-j\omega\tau} = \cos \omega\tau - j \sin \omega\tau,$$

we obtain an analytic expression for the Mikhaylov curve:

$$L(j\omega) = \omega^4 - jm_1\omega^3 - m_2\omega^2 + jm_3\omega + m_4 + (-jn_1\omega^3 - n_2\omega^2 + jn_3\omega + n_4') q(A)(\cos \omega\tau - j \sin \omega\tau). \qquad (4.147)$$

This expression was used subsequently to investigate the stability of the periodic motion in accordance with §2.4.

On the basis of the condition that the Mikhaylov curve must pass through the origin, $X(A, \omega) = 0$ and $Y(A, \omega) = 0$ when $\omega = \Omega$, we write on the basis of (4.147) two equations for the determination of the amplitude and frequency of the periodic solution:

$$\Omega^4 - m_2\Omega^2 + m_4 + q(A)[\cos \Omega\tau(-n_2\Omega^2 + n_4') + \sin \Omega\tau(-n_1\Omega^3 + n_3\Omega)] = 0, \qquad (4.148)$$

$$-m_1\Omega^3 + m_3\Omega + q(A)[\cos \Omega\tau(-n_1\Omega^3 + n_3\Omega) + \sin \Omega\tau(n_2\Omega^2 - n_4')] = 0. \qquad (4.149)$$

Eliminating q(A) from (4.148) and (4.149), we obtain a relation for the determination of the possible frequencies of the periodic solution as functions of the system parameters:

$$\operatorname{tg} \Omega\tau = \frac{\Omega[(m_1\Omega^2 - m_3)(-n_2\Omega^2 + n_4') + (\Omega^4 - m_2\Omega^2 + m_4)(-n_1\Omega^2 + n_3)]}{\Omega^2(m_1\Omega^2 - m_3)(-n_1\Omega^2 + n_3) + (\Omega^4 - m_2\Omega^2 + m_4)(-n_2\Omega^2 + n_4')}. \qquad (4.150)$$

Equation (4.150) is transcendental in Ω. The frequency of the periodic solution can be determined by graphically solving Eq. (4.150). For this purpose it is necessary to plot curves of $\tan \Omega\tau$ and $L_1(\Omega)/L_2(\Omega)$ as Ω is varied from 0 to ∞, where $L_1(\Omega)$ and $L_2(\Omega)$ are the numerator and denominator of the right half of (4.150).

If it is known for certain that the frequencies of the expected

periodic solution are small and the time delay is small, then we can assume approximately

$$\operatorname{tg} \Omega\tau \approx \Omega\tau.$$

Equation (4.150) is then rewritten in the form

$$\tau = \frac{(m_1\Omega^2 - m_3)(-n_2\Omega^2 + n_4') + (\Omega^4 - m_2\Omega^2 + m_4)(-n_1\Omega^2 + n_3)}{\Omega^2(m_1\Omega^2 - m_3)(-n_1\Omega^2 + n_3) + (\Omega^4 - m_2\Omega^2 + m_4)(-n_2\Omega^2 + n_4')}. \qquad (4.151)$$

The solution is now obtained as the intersection of the straight line τ = const and the curve $L_1^*(\Omega)/L_2(\Omega)$, where $L_1^*(\Omega) = L_1(\Omega)/\Omega$, with $0 < \Omega < \infty$.

To determine the dependence of the frequency of the periodic solution on the system parameters, it is necessary to use (4.150) or (4.151) to plot a family of curves for the variation of the parameter of interest. In practice, however, after determining Ω for a definite value of the parameter, it is no longer necessary to construct the frequency variation over a wide range, and it is sufficient to vary the frequency near the values of the expected periodic solution.

Knowing the frequency of the periodic solution we can determine the amplitude by using either of the two equations (4.148) or (4.149). Thus, from (4.148) we obtain a relation for the amplitude:

$$q(A) = \frac{\Omega^4 - m_2\Omega^2 + m_4}{\cos \Omega\tau (n_2\Omega^2 - n_4') + \sin \Omega\tau (n_1\Omega^3 - n_3\Omega)}. \qquad (4.152)$$

Since in final analysis we must know the behavior of the aerodynamic object as it moves relative to the longitudinal axis, it is of interest to determine the amplitude of the oscillations of the object in the angle γ.

Taking into consideration the equation $x = \gamma + k_\gamma^{\cdot}\dot{\gamma}$, which relates x with γ, we obtain the following transfer function for conversion from x to γ:

$$W(p) = \frac{1}{1 + k_\gamma^{\cdot} p}.$$

Consequently, the amplitude A_γ of the object oscillations is determined in terms of the amplitude of the variable $\underline{x}$ by the relation

$$A_\gamma = \frac{A}{\sqrt{1+k_1\Omega^2}}. \qquad (4.153)$$

If the object follows a trajectory other than horizontal, the equation of motion can be written in the form

$$(p^5 + k_1p^4 + k_2p^3 + k_3p^2 + k_4p)\,x + \\ + (n_1p^4 + n_2p^3 + n_3p^2 + n_4p + n_5)\,F(x) = 0,$$

where the coefficients $k_1, \ldots, k_4, n_1, \ldots, n_5$ include the parameters of the system and the aerodynamic coefficients.

An analytic expression for the Mikhaylov curve is written in the form

$$l.(j\omega) = j\omega^5 + k_1\omega^4 - jk_2\omega^3 - k_3\omega^2 + jk_4\omega + \\ + (n_1\omega^4 - jn_2\omega^3 - n_3\omega^2 + jn_4\omega + n_5)(\cos\omega\tau - j\sin\omega\tau)\,q(A). \qquad (4.154)$$

In accordance with (4.154), we obtain equations for the determination of the amplitude and frequency of the periodic solution:

$$k_1\Omega^4 - k_3\Omega^2 + q(A)[\cos\Omega\tau(n_1\Omega^4 - n_3\Omega^2 + n_5) - \\ - \sin\Omega\tau(n_2\Omega^3 - n_4\Omega)] = 0, \qquad (4.155)$$

$$\Omega^5 - k_2\Omega^3 + k_4\Omega + q(A)[\cos\Omega\tau(-n_2\Omega^3 + n_4\Omega) - \\ - \sin\Omega\tau(n_1\Omega^4 - n_3\Omega^2 + n_5)] = 0. \qquad (4.156)$$

Eliminating q(A) from (4.155) and (4.156) we obtain an equation for the determination of the frequency of the periodic solution as a function of the system parameters:

$$\operatorname{tg}\Omega\tau = \frac{(\Omega^5 - k_2\Omega^3 + k_4\Omega)(n_1\Omega^4 - n_3\Omega^2 + n_5) + (k_1\Omega^4 - k_3\Omega^2)(n_2\Omega^3 - n_4\Omega)}{(\Omega^5 - k_2\Omega^3 + k_4\Omega)(n_2\Omega^3 - n_4\Omega) - (k_1\Omega^4 - k_3\Omega^2)(n_1\Omega^4 - n_3\Omega^2 + n_5)}. \qquad (4.157)$$

The amplitude of the periodic solution can be determined from either Equation (4.155) or (4.156). Thus, from (4.155) we obtain a relation for the amplitude:

$$q(A) = \frac{k_3\Omega^2 - k_1\Omega^4}{\cos\Omega\tau(n_1\Omega^4 - n_3\Omega^2 + n_5) - \sin\Omega\tau(n_2\Omega^3 - n_4\Omega)}. \qquad (4.158)$$

The value of q(A) should always be positive, since q(A) is the gain of the nonlinear element. Therefore, denoting the right half of

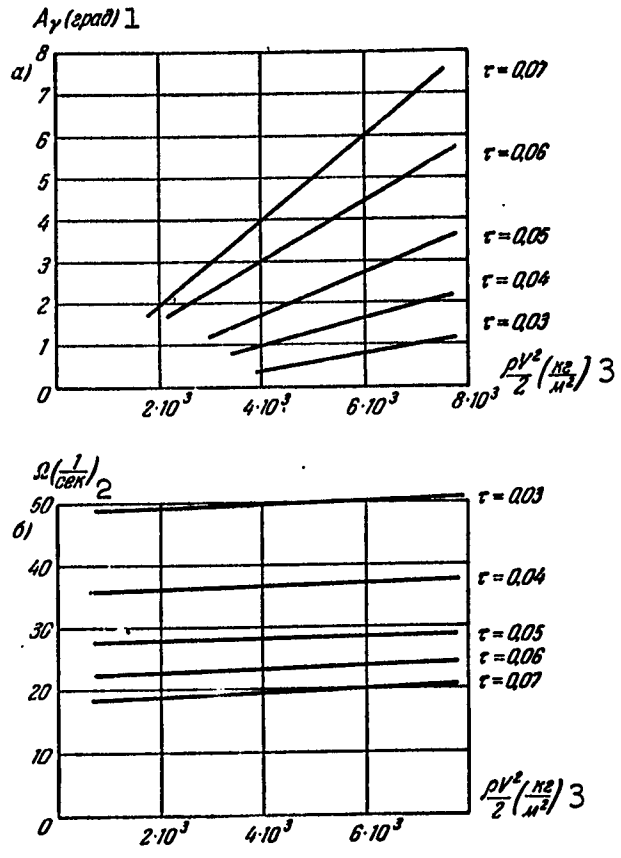


Fig. 4.39. 1) deg; 2) sec; 3) kg/m^2.

(4.158) by $\underline{s}$ and taking the value of q(A) for the nonlinear characteristic under consideration into account, we obtain

$$\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}}=s.$$

This leads to a formula for the amplitude of the periodic solution:

$$A=\sqrt{\frac{8c^2}{(\pi s)^2}\pm\sqrt{\left(\frac{8c^2}{\pi^2 s^2}\right)^2-\frac{16c^2b^2}{\pi^2 s^2}}}. \qquad (4.159)$$

It is seen from (4.159) that for each value of the frequency (for each value of $s = s(\Omega)$), two values of the amplitude are obtained.

The transcendental equation (4.157) makes it possible to calculate (by graphical solution) the values of the frequency, while (4.159) yields the amplitudes of the periodic solution as any of the system parameters of interest is varied.

Conversion of the amplitudes of the periodic solution for the

variable $\underline{x}$ (the relay control voltage) into amplitudes of the banking oscillations A_γ of the object is on the basis of Formula (4.153).

The result of the solution for the sought amplitude and frequency of the periodic solution, as a function of the velocity head $\rho V^2/2$, in the case when the flight trajectory differs from horizontal, is shown for one of the objects in Figs. 4.39$\underline{a}$ and $\underline{b}$, where the branches with larger values, corresponding to the stable periodic solution, i.e., to self-oscillations, are shown for the amplitudes. The stability of the periodic solution was determined in this case on the basis of the approximate criterion.

Depending on the values of the parameters, the object in its banking motion will have either a stable steady state without self-oscillations, or a steady self-oscillating mode, since an investigation of the stability of the obtained periodic solution, using the second graphic criterion (§2.4) and Expressions (4.147) and (4.154) for the Mikhaylov curve, yields a positive result.

§4.12. System of the First Class with Two-Phase Induction Motor

Frequently real nonlinear static characteristics cannot be approximated with sufficient accuracy by a single simple nonlinear dependence or by a piecewise-linear characteristic. It is then necessary to choose in each individual case such analytical relationships as to ensure the desired accuracy of approximation of the real static characteristic of the nonlinear element.* The approximating relation can be reduced here to several elementary nonlinear functions, for which it is necessary to employ the harmonic linearization formulas presented in the preceding chapter, or to carry out harmonic linearization in accordance with the general rules.

By way of an example let us consider an automatic control system in which the regulating unit is driven by a two-phase induction motor

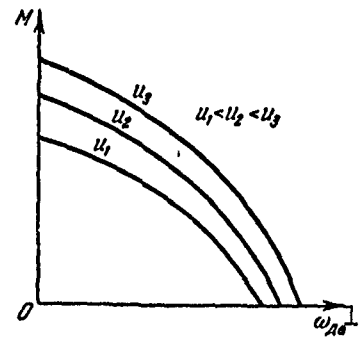


Fig. 4.40. 1) ω_{dv}.

[49]. The mechanical characteristics of such a motor are plotted in Fig. 4.40.

In linearizing the mechanical characteristics, it is usually assumed that

$$M = c_1 u - c_2 \omega_{дв}. \qquad (4.160)$$

But this holds true in first approximation only for the left portion of the characteristic. If a greater portion of the characteristic is employed, it is necessary to take their nonlinearity into account. Bearing in mind that with increasing ω_{dv} the coefficient c_1 decreases while the coefficient c_2 increases, we approximate this characteristic not by (4.160) but by the following nonlinear relation*:

$$M = \frac{c_1}{1 + c_3 |\omega_{дв}|} u - (c_2 + c_4 |\omega_{дв}|)\omega_{дв}. \qquad (4.161)$$

The values of the coefficients c_3 and c_4 must be chosen on the basis of the available experimental static characteristics in such a way as to make the approximating characteristics give better agreement with the real ones. Similarly, it is possible to choose any other more suitable nonlinear law with which to describe the characteristics of the motor.

Neglecting friction we have in accord with (4.161) the following differential equation for the motor

$$Jp\omega_{дв} = \frac{c_1}{1 + c_3 |\omega_{дв}|} u - c_2 \omega_{дв} - c_4 |\omega_{дв}| \omega_{дв}, \qquad (4.162)$$

where J [g·cm·sec^2] is the moment of inertia of all the masses driven by the motor, referred to the motor shaft.

Reducing (4.162) to a common denominator, we obtain

$$Jp\omega_{дв} + Jc_3 |\omega_{дв}| p\omega_{дв} + c_2 \omega_{дв} + (c_2 c_3 + c_4) |\omega_{дв}| \omega_{дв} + c_3 c_4 \omega_{дв}^3 = c_1 u. \qquad (4.163)$$

This equation contains three nonlinear functions

$$F_1(\omega_{дв}) = |\omega_{дв}| p\omega_{дв}, \quad F_2(\omega_{дв}) = |\omega_{дв}| \omega_{дв}, \quad F_3(\omega_{дв}) = \omega_{дв}^3.$$

For $F_2(\omega_{dv})$ and $F_3(\omega_{dv})$ we have already calculated the harmonic linearization coefficients (3.31) and (3.32), and consequently these functions are replaced by the linear relationships

$$F_2(\omega_{дв}) = \frac{8A}{3\pi}\omega_{дв}, \quad F_3(\omega_{дв}) = \frac{3A^2}{4}\omega_{дв}. \tag{4.164}$$

Let us carry out harmonic linearization of the nonlinear function $F_1(\omega'_{dv}) = |\omega_{dv}| p\omega_{dv}$. Since the solution for ω_{dv} is sought in the form

$$\omega_{дв} = A \sin \Omega t,$$

we have

$$p\omega_{дв} = A\Omega \cos \Omega t.$$

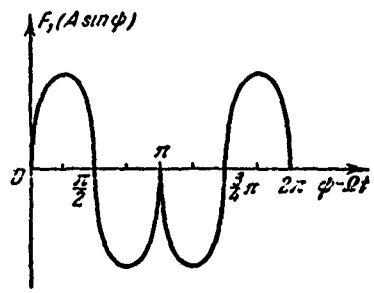


Fig. 4.41

Then the nonlinear function $F_1(\omega_{dv})$ of the argument $\psi = \Omega t$ will be

$$F_1(A \sin \psi) = |A \sin \psi| A\Omega \cos \psi$$

and is represented by the plot of Fig. 4.41.

By virtue of the fact that $F_1(A \sin \psi)$ is an even function, the harmonic linearization coefficient $q_1(A)$ vanishes. The values of the integral in Formula (3.3) for the calculation of $q'_1(A)$ will be the same for one quarter of the period. Then, calculating $q'_1(A)$, we obtain

$$q'_1(A) = \frac{4}{\pi A}\int_0^{\frac{\pi}{2}} F_1(A \sin \psi) \cos \psi \, d\psi =$$

$$= \frac{4A\Omega}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\psi \sin \psi \, d\psi = -\frac{4A\Omega}{3\pi} \cos^3 \psi \Big|_0^{\frac{\pi}{2}} = \frac{4A\Omega}{3\pi}.$$

In accordance with (3.1), we obtain for the nonlinear function $F_1(\omega_{dv})$ the following harmonically linearized expression

$$F_1(\omega_{дв}) = \frac{4A}{3\pi} p\omega_{дв}. \tag{4.165}$$

Substituting (4.164) and (4.165) into (4.163), we obtain the fol-

lowing equation for the two-phase motor (for oscillatory processes):

$$[T_3(1+b_1A)p+(1+b_2A+b_3A^3)]\omega_{дв}=k_3u \qquad (4.166)$$

place of the usual linear equation

$$(T_3p+1)\omega_{дв}=k_3u,$$

where

$$T_3=\frac{J}{c_2},\quad k=\frac{c_1}{c_2},\quad b_1=\frac{4c_3}{3\pi},\quad b_2=2b_1+\frac{8c_4}{3\pi c_2},\quad b_3=\frac{3c_3c_4}{4c_2}. \qquad (4.167)$$

The rate of displacement of the regulating organ, taking into account the transfer ratio of the reduction gear, will be

$$p\xi=k_4\omega_{дв}. \qquad (4.168)$$

The equation of the regulated object and the equation of the sensitive element of the regulator will be chosen in the form

$$\left.\begin{aligned}(T_1p+1)\varphi&=-k_1\xi,\\ u&=k_2\varphi,\end{aligned}\right\} \qquad (4.169)$$

where φ is the deviation of the regulated quantity.

The characteristic equation of the entire closed-loop system will be, in accordance with (4.166), (4.168), and (4.169)

$$[T_3(1+b_1A)p+1+b_2A+b_3A^2](T_1p+1)p+k_1k=0, \qquad (4.170)$$

where $k = k_2k_3k_4$. Substituting here $p = j\Omega$, we obtain two equations to determine the amplitude and frequency of the periodic solution:

$$k_1k-[T_3(1+b_1A)+T_1(1+b_2A+b_3A^3)]\Omega^2=0, \qquad (4.171)$$

$$1+b_2A+b_3A^2-T_1T_3(1+b_1A)\Omega^2=0. \qquad (4.172)$$

Let us examine the influence of the parameter k (the over-all gain of the regulator). From (4.172) we obtain

$$A=\frac{b_1T_1T_3\Omega^2-b_2+\sqrt{(b_1T_1T_3\Omega^2-b_2)^2+4b_3(T_1T_3\Omega^2-1)}}{2b_3}. \qquad (4.173)$$

From (4.171) we obtain, with account of (4.172), the following equation for the parameter k whose influence is of interest to us:

$$k=\frac{T_3}{k_1}(1+b_1A)(1+T_1^2\Omega^2)\Omega^2. \qquad (4.174)$$

Using (4.173) we plot a curve $A = A(\Omega)$ for constant values of the

parameters. On the basis of this curve, by specifying values of Ω and substituting the corresponding values of A, we calculate the values of k by means of (4.174), and consequently plot the functions A(k) and Ω(k).

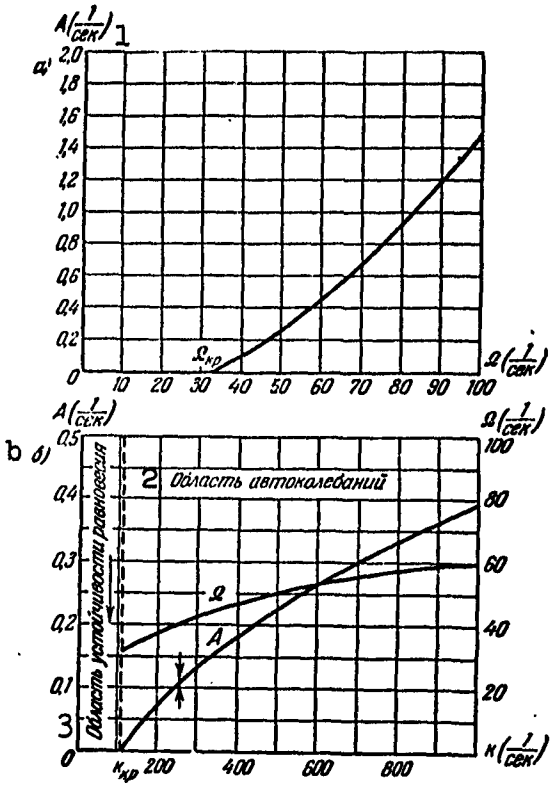


Fig. 4.42. 1) sec; 2) region of self-oscillations; 3) region of stable equilibrium.

The plots made for parameter values $b_1 = 0.05$ sec, $b_2 = 5$ sec, $b_3 = 1$ sec^2, $T_1 = 0.1$ sec, $T_3 = 0.01$ sec, and $k_1 = 1$ are shown in Figs. 4.42a and b.

Let us investigate the stability of the periodic solution obtained by the method of averaging of the periodically varying coefficients. For this purpose we set up the equations of the system in terms of the deviations from the periodic solution.

The deviation equation of the linear portion will be, in accordance with (4.168) and (4.169),

$$(T_1 p + 1) p \Delta u = - k_1 k_2 k_3 \Delta \omega_{дв}.$$

For the equation of the nonlinear element (4.163), which has the form

$$F(\omega_{дв}, \dot{\omega}_{дв}) = c_1 u,$$

we obtain the periodically varying coefficients

$$\frac{\partial F}{\partial \omega_{дв}} = J c_3 p \omega_{дв} \operatorname{sign} \omega_{дв} + c_2 + 2 (c_2 c_3 + c_4) |\omega_{дв}| + 3 c_3 c_4 \omega_{дв}^2,$$
$$\frac{\partial F}{\partial \dot{\omega}_{дв}} = J + J c_3 |\omega_{дв}|.$$

Let us average the coefficients obtained by means of the formulas

$$\varkappa = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial F}{\partial \omega_{дв}} \right)^* d\psi, \quad \varkappa_1 = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial F}{\partial \dot{\omega}_{дв}} \right)^* d\psi.$$

Carrying out the calculations with allowance for the periodic values of ω_{dv} and $p\omega_{dv}$, we obtain

$$\varkappa = c_2 + \frac{1}{2\pi} \left[J c_3 A \Omega \int_0^{2\pi} \cos \psi \operatorname{sign} \sin \psi + 2 (c_2 c_3 + c_4) A \int_0^{2\pi} |\sin \psi| \, d\psi + \right.$$
$$\left. + 3 c_3 c_4 A^2 \int_0^{2\pi} \sin^2 \psi \, d\psi \right].$$

The first integral in the square brackets vanishes. The other two can be calculated by integrating over one quarter of the period. We then have

$$\varkappa = c_2 + \frac{1}{2\pi} \left[8 (c_2 c_3 + c_4) A \int_0^{\frac{\pi}{2}} \sin \psi \, d\psi + 12 c_3 c_4 A_2 \int_0^{\frac{\pi}{2}} \sin^2 \psi \, d\psi \right] =$$
$$= c_2 + \frac{1}{2\pi} \left[8 (c_2 c_3 + c_4) A - \cos \psi \Big|_0^{\frac{\pi}{2}} + 12 c_3 c_4 A^2 \frac{\psi}{2} \Big|_0^{\frac{\pi}{2}} - \frac{1}{4} \sin 2\psi \Big|_0^{\frac{\pi}{2}} \right] =$$
$$= c_2 + \frac{4}{\pi} (c_2 c_3 + c_4) A + \frac{3}{2} c_3 c_4 A^2.$$

Calculating the value of $\varkappa_1$, we obtain

$$\varkappa_1 = J + J \frac{c_3 A}{2\pi} \int_0^{2\pi} |\sin \psi| \, d\psi = J + J \frac{2 c_3 A}{\pi} \int_0^{\frac{\pi}{2}} \sin \psi \, d\psi =$$
$$= J + J \frac{2 c_3 A}{\pi} \Big| - \cos \psi \Big|_0^{\frac{\pi}{2}} = J + J \frac{2 c_3 A}{\pi}.$$

Substituting the averaged values of the periodically varying coefficients into (4.163) and taking into account the notation in (4.167),

we write out the approximate equation for the nonlinear element in terms of the deviations:

$$\left[T_3\left(1+\tfrac{3}{2}b_1A\right)p+\left(1+\tfrac{3}{2}b_3A+2b_3A^2\right)\right]\Delta\omega_{dv}=k_3\Delta u.$$

The characteristic equation for the investigation of the stability of the periodic solution assumes the form

$$\left[T_3\left(1+\tfrac{3}{2}b_1A\right)p+\left(1+\tfrac{3}{2}b_3A+2b_3A^2\right)\right](T_1p+1)p+k_1k=0.$$

From this we get a stability condition (by the Hurwitz criterion)

$$\left[T_3\left(1+\tfrac{3}{2}b_1A\right)+T_1\left(1+\tfrac{3}{2}b_3A+2b_3A^2\right)\right]\times \\ \times\left(1+\tfrac{3}{2}b_3A+2b_3A^2\right)-T_1T_3k_1k\left(1+\tfrac{3}{2}b_1A\right)>0, \qquad (4.175)$$

whereas the presence of a periodic solution, in accordance with the previous characteristic equation (4.170), corresponds to the equation

$$[T_3(1+b_1A)+T_1(1+b_3A+b_3A^2)](1+b_3A+b_3A^2)- \\ -T_1T_3k_1k(1+b_1A)=0. \qquad (4.176)$$

Replacing $T_1T_3k_1k$ in (4.175) by its value from (4.176), we find that in order for the periodic solution to be stable it is necessary to satisfy the inequality

$$\left(T_3+T_1\frac{1+\frac{3}{2}bA+2b_3A^2}{1+\frac{3}{2}b_1A}\right)\left(1+\tfrac{3}{2}b_3A+2b_3A^2\right)> \\ >\left(T_3+T_1\frac{1+b_3A+b_3A^2}{1+b_1A}\right)(1+b_3A+b_3A^2). \qquad (4.177)$$

That the inequality (4.177) is satisfied for large amplitudes A of the periodic solution is obvious. It follows therefore that it is satisfied also for the entire single-valued curve A(k) (Fig. 4.42b). Consequently, the periodic solution obtained is stable; it corresponds to self-oscillations of this system with a frequency and amplitude determined by the plots of Fig. 4.42b.

The self-oscillation amplitude for the regulated quantity φ is determined in terms of the transfer function of the elements that separate the variables φ and ω_{dv}. From (4.168) and (4.169) we obtain

$$(T_1 p + 1) p\varphi = k_1 k_4 \omega_{дв}$$

and consequently, the function for converting ω_{dv} to φ will be

$$W(p) = \frac{\varphi}{\omega_{дв}} = \frac{k_1 k_4}{(T_1 p + 1) p}.$$

After substituting $p = j\omega$ we obtain the complex transfer ratio

$$K = W(j\omega) = \frac{k_1 k_4}{-T_1 \omega^2 + j\omega},$$

the modulus of which is

$$|K| = |W(j\omega)| = \frac{k_1 k_4}{\omega \sqrt{T_1^2 \omega^2 + 1}}.$$

Then the amplitude for the variable φ is determined from the formula

$$A_\varphi = \frac{k_1 k_4}{\Omega \sqrt{T_1^2 \Omega^2 + 1}} A.$$

Extrapolating the result obtained to the self-oscillation region for $k > k_{kr}$, we verify that to the left of the self-oscillation region $k < k_{kr}$ lies a stability region without self-oscillations.

The value of Ω_{kr} is determined in accordance with Formula (4.173) from the condition

$$4b_3 (T_1 T_3 \Omega^2 - 1) = 0,$$

i.e.,

$$\Omega_{кр} = \frac{1}{\sqrt{T_1 T_3}}.$$

Substituting the value of Ω_{kr} and $A = 0$ in (4.174), we obtain a formula for k_{kr}:

$$k_{кр} = \frac{T_1 + T_3}{k_1 T_1 T_3}.$$

We note that the value of k_{kr} coincides with the value of k_{gr} corresponding to the stability limit of the given system, regarded as linear. Indeed, if we disregard the nonlinearities of the mechanical characteristics of the motor, then the characteristic equation assumes the form

$$T_1 T_3 p^3 + (T_1 + T_3) p^2 + p + k_1 k = 0$$

and consequently, in accordance with the Hurwitz stability criterion, the condition for the stability limit will be

$$T_1 = T_3 = k_1 k T_1 T_3,$$

hence

$$k_{cr} = \frac{T_1 + T_3}{k_1 T_1 T_3}.$$

Thus, failure to take into account the nonlinearity of the mechanical characteristics of the drive of the regulating organ gives a result that states that the instability region lies beyond the stability region, whereas in fact the mode there is that of stable self-oscillations with low amplitude near the stability limit (safe limit).

§4.13. System of the Second Class with Two-Phase Induction Motor

The nonlinearities of the mechanical characteristics of two-phase induction motors can also be taken into account by other methods. Thus, for example, in article [94], devoted to an account of the nonlinearity of the characteristics of the ADP series of motors, the authors proposed to approximate the mechanical characteristics of the motors by polynomials in powers of the angular velocity. This approximation covers in the general case both the motor operation and plugging operation (Fig. 3.12). The lower branch of the characteristic corresponds to the opposite polarity of the control voltage of the motor.

The mechanical characteristics are approximated by the following equations:

$$M = M_n - B_1 \omega_{дв} - B_2 \omega_{дв}^n \operatorname{sign} u \quad \text{for even } \underline{n} \qquad (4.178)$$

or

$$M = M_n - B_1 \omega_{дв} - B_2 \omega_{дв}^n \operatorname{sign} \omega_{дв} \operatorname{sign} u \quad \text{for odd } \underline{n}. \qquad (4.179)$$

For the region corresponding to the motor mode, Eqs. (4.178) and (4.179) can be written in simpler form:

$$M = M_n - B_1 \omega_{дв} - B_2 \omega_{дв}^n \operatorname{sign} \omega_{дв} \quad \text{for even } \underline{n} \qquad (4.180)$$

or

$$M = M_n - B_1\omega_{dv} - B_2\omega_{dv}^n \quad \text{for odd } \underline{n}, \tag{4.181}$$

where M_p is the starting torque of the motor, M is the load torque, ω_{dv} is the angular velocity of the motor, $\underline{u}$ is the voltage on the control winding, and B_1 and B_2 are coefficients that are determined during the course of the approximation.

The accuracy of such an approximation can be estimated by comparing the real and approximated characteristics.

Figure 4.43a shows the experimentally plotted characteristics of the ADP-263 motor (solid curves) and the characteristics (dashed curves) plotted in accordance with Eq. (4.181):

$$M = M_n - 12{,}7 \cdot 10^{-3}\omega_{dv} - 1{,}2 \cdot 10^{-6}\omega_{dv}^3,$$

where $M_p = c_1 u$, and $c_1 = 5$ g·cm/v is the coefficient of proportionality between the starting torque of the motor and the control winding voltage. Figure 4.43b shows analogous characteristics of the motor ADP-363, approximated by means of the expression

$$M = M_n - 7{,}24 \cdot 10^{-3}\omega_{dv} - 2{,}76 \cdot 10^{-6}\omega_{dv}^3$$

with $c_1 = 9$ g·cm/v.

The foregoing equations characterize the steady-state operation of the motor. For transient conditions, with only the inertial load taken into account, Eqs. (4.178) and (4.179) assume the form

$$J_2 \frac{d\omega_{dv}}{dt} + B_1\omega_{dv} + B_2\omega_{dv}^n \operatorname{sign} u = c_1 u \quad \text{for even } \underline{n} \tag{4.182}$$

or

$$J_2 \frac{d\omega_{dv}}{dt} + B_1\omega_{dv} + B_2\omega_{dv}^n \operatorname{sign}\omega_{dv} \operatorname{sign} u = c_1 u \quad \text{for odd } \underline{n}, \tag{4.183}$$

where J_2 is the moment of inertia of the motor rotor and of the masses referred to this rotor.

Let us show how to carry out the harmonic linearization of equations of this type. To be specific, we confine ourselves to equations

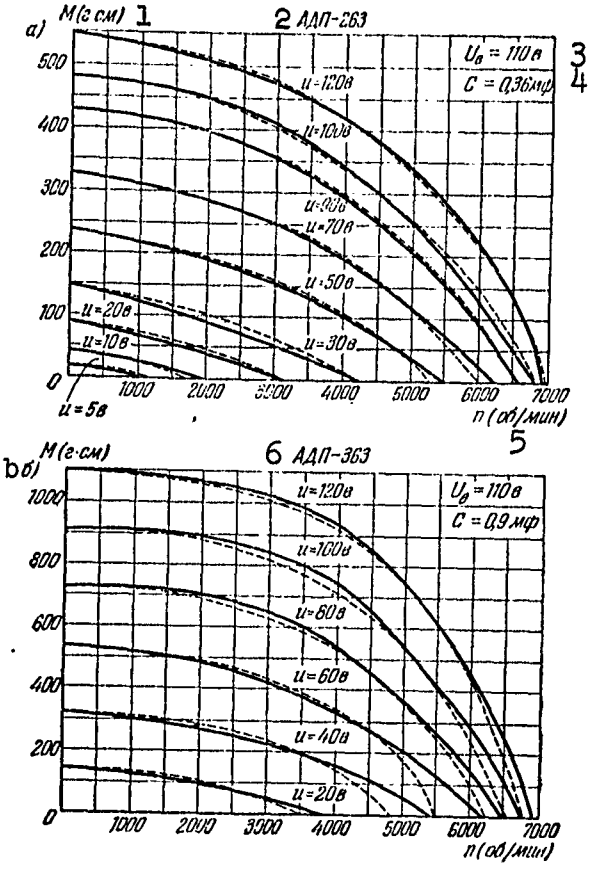


Fig. 4.43. 1) g·cm; 2) ADP-263; 3) v; 4) mF; 5) rpm; 6) ADP-363.

whose nonlinearity is characterized by the second and third powers of the angular velocity. The same solution procedure is used for any other degree.

For the case n = 2 we have in accordance with (4.182)

$$J_2 \frac{d\omega_{дв}}{dt} + B_1\omega_{дв} + B_2\omega_{дв}^2 \operatorname{sign} u = c_1 u. \tag{4.184}$$

Dividing all the terms of (4.184) by B_1, we obtain

$$T_2 \frac{d\omega_{дв}}{dt} + \omega_{дв} + B\omega_{дв}^2 \operatorname{sign} u = k_2 u, \tag{4.185}$$

where

$$T_2 = J_2/B_1 \text{ [sec]}, \quad B = B_2/B_1 \text{ [sec]}, \quad k_2 = c_1/B_1 \text{ [v}^{-1}\cdot\text{sec}^{-1}\text{]}.$$

We introduce the notation

$$F(\omega_{дв}, u) = B\omega_{дв}^2 \operatorname{sign} u. \tag{4.186}$$

In accordance with the previously made classification (see Chapter 1), the nonlinear function (4.186) is of the second class type. Therefore, in the harmonic linearization it is necessary to introduce into the equation of the element, in this case into the equation of the motor, two oscillation amplitudes A_u and A_ω, putting

$$\begin{aligned} \omega_{дв} &= A_\omega \sin \Omega t, \\ u &= A_u \sin (\Omega t + \varphi). \end{aligned} \tag{4.187}$$

The sought periodic solutions for ω_{dv} and $\underline{u}$ are related through the amplitude-phase characteristic $W_{\underline{1}}(j\omega)$ of the linear part of the system by the following relationship:

$$\frac{A_u}{A_\omega} = |W_л(j\omega)| = A, \quad \varphi = \arg W_л(j\omega)$$

and consequently

$$u = AA_\omega \sin (\Omega t + \varphi). \tag{4.188}$$

With (4.187) and (4.188) taken into account, the nonlinear function (4.186) is rewritten

$$F(\omega_{дв}, u) = BA_\omega^3 \sin^3 \Omega t \operatorname{sign} \sin (\Omega t + \varphi). \tag{4.189}$$

In accordance with the method of harmonic linearization, upon satisfaction of the condition

$$\int_0^{2\pi} F(\omega_{дв}, u)\, d\psi = 0, \quad \psi = \Omega t,$$

the nonlinear function (4.189) is replaced by

$$F(\omega_{дв}) = \left[q(A_\omega, \Omega) + \frac{q'(A_\omega, \Omega)}{\Omega} p \right] \omega_{дв}, \tag{4.190}$$

where $q(A_\omega, \Omega)$ and $q'(A_\omega, \Omega)$ are the coefficients of harmonic linearization.

For the nonlinear function (4.189) we obtain, in accordance with Formulas (3.36) and (3.37), the following values for the coefficients of harmonic linearization:

$$\left.\begin{aligned} q(A_\omega, \Omega) &= \frac{4BA_\omega}{\pi}\left(\cos\varphi - \frac{1}{3}\cos^3\varphi\right), \\ q'(A_\omega, \Omega) &= -\frac{4BA_\omega}{3\pi}\sin^3\varphi. \end{aligned}\right\} \tag{4.191}$$

Taking (4.191) into account, we rewrite (4.190) in the form

$$F(\omega_{дв}) = \frac{4BA_\omega}{\pi}\left(\cos\varphi - \frac{1}{3}\cos^3\varphi - \frac{\sin^3\varphi}{3\Omega}p\right)\omega_{дв}. \tag{4.192}$$

We note that when $\varphi = 0$ Eq. (4.192) assumes the form

$$F(\omega_{дв}) = \frac{8BA_\omega}{3\pi}\omega_{дв}, \tag{4.193}$$

corresponding to harmonic linearization of the nonlinearity

$$F(\omega_{дв}) = B\omega_{дв}^2 \operatorname{sign}\omega_{дв}.$$

The harmonically linearized equation of the motor will be, in accordance with (4.185)

$$\left[T_2 p + 1 + q(A_\omega, \Omega) + \frac{q'(A_\omega, \Omega)}{\Omega}p\right]\omega_{дв} = k_2 u, \tag{4.194}$$

where $q(A_\omega, \Omega)$ and $q'(A_\omega, \Omega)$ have the values given in (4.191).

Let us carry out harmonic linearization of the equation

$$T_2\frac{d\omega_{дв}}{dt} + \omega_{дв} + B\omega_{дв}^2 \operatorname{sign}\omega_{дв}\operatorname{sign} u = k_2 u. \tag{4.195}$$

This equation describes, for example, motors of the type ADP-263 and ADP-363.

For periodic processes, we write the nonlinearity

$$F(\omega_{дв}, u) = B\omega_{дв}^2 \operatorname{sign}\omega_{дв}\operatorname{sign} u \tag{4.196}$$

in the form

$$F(\omega_{дв}, u) = B(A_\omega \sin\Omega)^2 \operatorname{sign}\sin\Omega t \operatorname{sign}\sin(\Omega t + \varphi), \tag{4.197}$$

where φ, as before, is determined in terms of the linear portion of the system

$$\varphi = \arg W_л(j\omega).$$

Since the function (4.197) satisfies the condition

$$\int_0^{2\pi} F(\omega_{дв}, u)\,d\psi = 0, \quad \psi = \Omega t,$$

the relation (4.190) remains in force. The coefficients of harmonic linearization assume in this case, in accordance with (3.39) and (3.40), the values

$$\left.\begin{aligned} q(A_\omega, \Omega) &= \frac{BA_\omega^2}{\pi}\left(\frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi\right), \\ q'(A_\omega, \Omega) &= -\frac{BA_\omega^2}{\pi}\sin^4\varphi. \end{aligned}\right\} \qquad (4.198)$$

Taking (4.198) into account, the formula for harmonic linearization for the nonlinearity (4.196) will be

$$F(\omega_{дв}) = \frac{BA_\omega^2}{\pi}\left(\frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi - \frac{\sin^4\varphi}{\Omega}p\right)\omega_{дв}, \qquad (4.199)$$

and when $\varphi = 0$,

$$F(\omega_{дв}) = \frac{3BA_\omega^2}{4}\omega_{дв}. \qquad (4.200)$$

Let us analyze the motion of the closed system proper, with the ADP motor as the nonlinear element. We choose a system of simple structure. We assume that the linear portion of the system is described by the equation

$$(T_1 p + 1)u = -k_1\beta, \qquad (4.201)$$

where T_1 [sec] is the time constant of the linear element, k_1 [v] is the transfer ratio of the linear element, and β is the angular coordinate at the output of the nonlinear element.

Differentiating Eq. (4.201) with respect to the time and recognizing that $p\beta = \omega_{dv}$, we obtain

$$(T_1 p + 1)pu = -k_1\omega_{дв}. \qquad (4.202)$$

Combining the harmonically linearized equation of the motor (4.194) and Eq. (4.202), we obtain the system equation in the variable ω_{dv}:

$$\left[\left(T_1T_2 + \frac{T_1q'}{\Omega}\right)p^3 + \left(T_1 + T_2 + T_1q + \frac{q'}{\Omega}\right)p^2 + (q+1)p + k\right]\omega_{дв} = 0, \qquad (4.203)$$

where $k = k_1k_2$ is the gain of the system. The characteristic equation corresponding to (4.203) will be

$$\left(T_1T_2+\frac{T_1q'}{\Omega}\right)p^3+\left(T_1+T_2+T_1q+\frac{q'}{\Omega}\right)p^2+$$
$$+(q+1)p+k=0. \qquad (4.204)$$

Substituting $p = j\Omega$, we separate the real and imaginary parts $X(A_\omega, \Omega)$ and $Y(A_\omega, \Omega)$, from which, taking into account the values of the coefficients of harmonic linearization (4.191), we obtain two equations for the determination of the amplitude and frequency of the periodic solution (for the variable ω_{dv}):

$$\left[T_1+T_2+\frac{4T_1BA_\omega}{\pi}\left(\cos\varphi-\frac{1}{3}\cos^3\varphi\right)\right]\Omega^2-$$
$$-\frac{4BA_\omega\sin^3\varphi}{3\pi}\Omega-k=0, \qquad (4.205)$$

$$T_1T_2\Omega^2-\frac{4T_1BA_\omega\sin^3\varphi}{3\pi}\Omega-$$
$$-\frac{4BA_\omega}{\pi}\left(\cos\varphi-\frac{1}{3}\cos^3\varphi\right)-1=0, \qquad (4.206)$$

where $\varphi = \arctan T_1\Omega$.

Equations (4.205) and (4.206) enable us to determine the value of the amplitude and frequency of the periodic solution for the angular velocity of the motor with specified system parameters, and also to plot the variation of the amplitude and frequency as functions of the variation of any parameter of the system with the other parameters kept constant. Such plots make it possible to choose the parameters such as to ensure the desired steady-state mode. Let us determine the dependence of the variation of the amplitude and frequency of the periodic solution on the gain. For this purpose we express the amplitude in terms of the frequency using Eq. (4.206) and the gain in terms of the frequency using Eq. (4.205). As a result we obtain after transformation:

$$A_\omega=\frac{\pi(T_1T_2\Omega^2-1)}{4B\left(\frac{T_1\sin^3\varphi}{3}\Omega+\cos\varphi-\frac{1}{3}\cos^3\varphi\right)}, \qquad (4.207)$$

$$k=\left[T_1+T_2+\frac{\left(T_1^2T_2\Omega^2-T_1\right)\left(\cos\varphi-\frac{1}{3}\cos^3\varphi\right)}{\frac{T_1\sin^3\varphi}{3}\Omega+\cos\varphi-\frac{1}{3}\cos^3\varphi}\right]\Omega^2-$$
$$-\frac{(T_1T_2\Omega^2-1)\sin^3\varphi}{T_1\sin^3\varphi\Omega+3\left(\cos\varphi-\frac{1}{3}\cos^3\varphi\right)}. \qquad (4.208)$$

As can be seen from (4.207), positive values of the amplitudes, i.e., a periodic solution, are possible only if

$$\Omega > \frac{1}{\sqrt{T_1 T_2}}$$

and consequently the critical value of the frequency is determined by the relation

$$\Omega_{кр} = \frac{1}{\sqrt{T_1 T_2}} \text{ with } A_\omega = 0. \qquad (4.209)$$

Substituting (4.209) in (4.208) we obtain the value of the critical gain

$$k_{кр} = \frac{T_1 + T_2}{T_1 T_2}.$$

We note that, as in the preceding example, the value of k_{kr} coincides with the value of k_{gr}, determined from the condition for the stability limit of the given system, analyzed on the basis of the linear equation. Consequently, a periodic solution is possible only beyond the stability region of the linear system, when

$$k > k_{кр} = k_{гр}.$$

In the present case there is no need for investigating the stability of the periodic solution, since this was done previously for a similar example. The difference here lies only in the method used to approximate the nonlinear mechanical characteristics of the two-phase induction motor.

Formulas (4.207) and (4.208) enable us to plot the functions $A_\omega(k)$ and $\Omega(k)$. Since it is interesting to know the amplitudes of the angle coordinate at the output of the motor, A_β, the values obtained for A_ω can be recalculated in accordance with the relation

$$A_\beta = \frac{A_\omega}{\Omega}.$$

Let us consider a system with a motor described by Eq. (4.195). In this case the characteristic equation retains the same form (4.204),

but with different values of $\underline{g}$ and q', which are determined from (4.198). Taking this into account, we obtain from the characteristic equation, putting $p = j\Omega$, the following two equations for the amplitude and frequency of the periodic solution:

$$\left[T_1 + T_2 + \frac{T_1 B A_\omega^2}{\pi}\left(\frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi\right)\right]\Omega^2 - \frac{B A_\omega^2 \sin^4\varphi}{\pi}\Omega - k = 0, \tag{4.210}$$

$$T_1 T_2 \Omega^2 - \frac{T_1 B A_\omega^2 \sin^4\varphi}{\pi}\Omega - \frac{B A_\omega^2}{\pi}\left(\frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi\right) - 1 = 0. \tag{4.211}$$

From (4.210) and (4.211) we obtain formulas for the amplitude and the gain in terms of the frequency:

$$A_\omega^2 = \frac{\pi(T_1 T_2 \Omega^2 - 1)}{B\left(T_1 \sin^4\varphi\Omega + \frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi\right)}, \tag{4.212}$$

$$k = \left[T_1 + T_2 + \frac{T_1(T_1 T_2 \Omega^2 - 1)\left(\frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi\right)}{T_1 \sin^4\varphi\Omega + \frac{3}{4}\pi - \frac{3}{2}\sin\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi}\right]\Omega^2 - \frac{(T_1 T_2 \Omega^2 - 1)\sin^4\varphi\Omega}{T_1 \sin^4\varphi\Omega + \frac{3}{4}\pi - \frac{3}{2}\varphi + \sin 2\varphi - \frac{1}{8}\sin 4\varphi}. \tag{4.213}$$

Formulas (4.212) and (4.213) were used to plot $A_\beta(k)$ and $\Omega(k)$ for a system using the ADP-363 motor with the following parameters: $T_1 = 0.02$ sec, $B = B_2/B_1 = 38 \cdot 10^{-6}$ sec, $T_2 = 1.5$ sec, and $J_2 = 0.105$ g·cm·sec^2 (Fig. 4.44, dashed curves 1). The same figure shows the experimentally plotted relationships (solid curves 2) and the relationships obtained by using a simplified approximation such as (4.181) (dashed curves 3).

As can be seen from Fig. 4.44, the approximation of the mechanical characteristics of two-phase induction motor by means of a polynomial in the angular velocity yields a sufficiently accurate result in the investigation.

For our case, even if we take into account the peculiarities of the motor characteristics in the plugging mode, the final result dif-

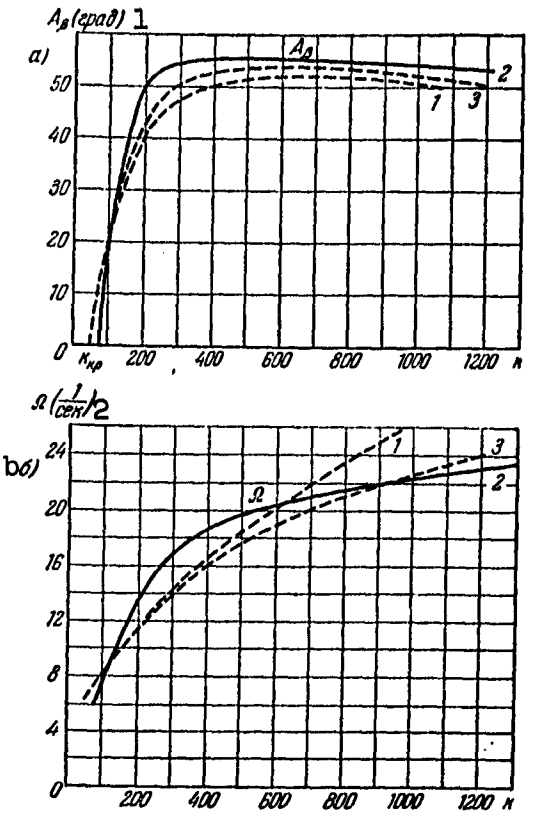


Fig. 4.44. 1) deg; 2) sec.

fers little from the case when the motor-mode equations are used. In the general case, however, allowance for the plugging mode may be advantageous.

In analogy with the foregoing, we can plot the variation of the amplitude and frequency of the self-oscillations as functions of other parameters of the system. This makes it possible to disclose the constructional possibilities for suppressing self-oscillations or for obtaining an acceptable self-oscillating mode.

In the present system it is difficult to obtain a self-oscillating mode with small amplitude, in view of the sharp increase in the amplitude with increasing gain.

§4.14. System with Two-Phase Induction Motor with Account of Dry Friction

We consider a servomechanism with a two-phase induction motor,

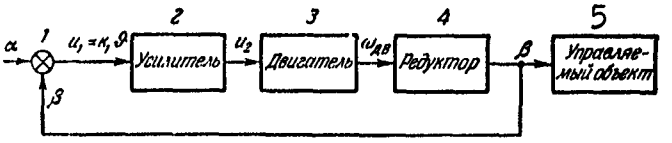


Fig. 4.45. 2) Amplifier; 3) motor; 4) reduction gear; 5) controlled object.

represented by the block diagram of Fig. 4.45. We investigate the system with simultaneous account of two types of nonlinearity – nonlinear mechanical characteristics of the two-phase induction motor, and dry friction in the controlled object. An account of the dry friction (on top of the nonlinear mechanical characteristics) is of practical interest, in view of the widespread use of low-power servomechanisms in which the dry friction torque exerts an appreciable influence on the behavior of the servomechanism.

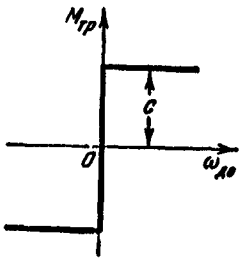


Fig. 4.46

The dry friction on the motor shaft will be represented by the static characteristic of Fig. 4.46. The mechanical characteristics of the two-phase induction motor are shown in Fig. 4.40. If we disregard stoppages of the motor, when the friction torque can assume values $-c < M_{tr} < c$, and if we represent the nonlinearity of the mechanical characteristics by a polynomial in ω_{dv} and confine ourselves to two terms (as was done in §4.13), then the equation of the motor, with allowance for the inertia of the controlled object, will be

$$J\dot{\omega}_{дв} + A_1\omega_{дв} + A_2\omega_{дв}^2 + c\,\operatorname{sign}\omega_{дв} = k_{дв}u_2$$

which after all terms are divided by A_1 assumes the form

$$T_3\dot{\omega}_{дв} + \omega_{дв} + B_1\omega_{дв}^2 + B_3\operatorname{sign}\omega_{дв} = k_3u_2, \qquad (4.214)$$

where

$$T_3 = \frac{J}{A_1},\quad B_1 = \frac{A_2}{A_1},\quad B_3 = \frac{c}{A_1},\quad k_3 = \frac{k_{дв}}{A_1}.$$

The term c sign ω_{dv} determines the dry friction. We separate the nonlinearity here in the form

$$F(\omega_{дв}) = B_1\omega_{дв}^1 + B_2 \operatorname{sign} \omega_{дв}. \qquad (4.215)$$

We seek a solution for the angular velocity ω_{dv} in the form of harmonic oscillations

$$\omega_{дв} = A \sin \psi, \quad \psi = \Omega t.$$

Harmonic linearization of the nonlinear function $F(\omega_{dv})$ yields

$$F(\omega_{дв}) = q(A)\omega_{дв}, \qquad (4.216)$$

where the coefficient of harmonic linearization will be, in accordance with (3.32) and (3.50)

$$q(A) = \frac{3}{4} B_1 A^2 + \frac{4B_2}{\pi A}.$$

Taking the expressions (4.215) and (4.216) into account, we obtain from (4.214) the harmonically linearized equation of the motor

$$[T_2 p + q(A) + 1]\omega_{дв} = k_2 u_2. \qquad (4.217)$$

We assume that a single stage magnetic amplifier is used. Taking into account the inertia of the magnetic amplifier, we write its equation in the form

$$(T_1 p + 1) u_2 = k_2 u_1, \qquad (4.218)$$

where T_1 is the time constant of the control circuit of the magnetic amplifier and k_2 is its gain.

The equation of the error transducer is

$$u_1 = k_1(\alpha - \beta). \qquad (4.219)$$

For the reduction gear, whose output is taken to be the angle of rotation of the controlled object β and the input is the motor speed ω_{dv}, we write

$$p\beta = k_4 \omega_{дв}. \qquad (4.220)$$

Combining Eqs. (4.217)-(4.220) into one equation for the variable ω_{dv} and assuming α = const during the investigation of the motion of

the system proper, we obtain

$$\{T_1T_2p^3+[T_1q(A)+T_1+T_2]p^2+[q(A)+1]p+k\}\omega_{x_0}=0, \qquad (4.221)$$

where $k = k_1k_2k_3k_4$ is the gain of the servomechanism.

Replacing p by $j\Omega$ in the characteristic equation corresponding to the differential equation (4.221), and separating the real and imaginary parts, we obtain two equations for the amplitude and frequency of the periodic solution

$$\left.\begin{aligned} k-[T_1q(A)+T_1+T_2]\Omega^2=0,\\ 1+q(A)-T_1T_2\Omega^2=0. \end{aligned}\right\} \qquad (4.222)$$

To separate the region of periodic solution, we use here the convenient graphoanalytic method which was detailed in §2.3 and called there the fifth method.

A periodic solution is possible only for well-defined values of the amplitude and frequency, satisfying each of the equations in (4.222). It is seen from the second equation that the periodic solution must satisfy the equation

$$q(A)=T_1T_2\Omega^2-1,$$

the left half of which depends only on the values of the amplitude and the right half only on the values of the frequency (for specified parameters).

This leads to the possibility of a graphical determination of the expected periodic solutions. For this purpose it is sufficient to plot the curve q(A), by specifying the values of the amplitude A, and to draw straight lines $T_1T_2\Omega^2 - 1$ for different values of Ω. The points where the plotted curve crosses the straight lines yield the values of the amplitude and frequency of the periodic solution (Fig. 4.47).

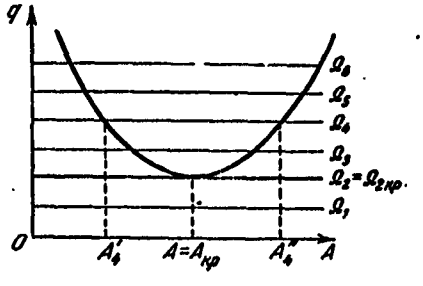


Fig. 4.47

If the straight line for the corresponding value of Ω does not cross the curve q(A), this means that this frequency of periodic solution is impossible.

The case when the straight line corresponding to a given value of Ω is tangent to q(A) if the latter has a maximum or minimum, corresponds to a critical value of the amplitude and frequency of the periodic solution on the boundary between the regions where the periodic solution exists or does not exist. Thus, for the system we are considering, the critical value of the amplitude is determined from the condition that q(A) be a minimum. Taking the derivative of q(A) with respect to A and setting it equal to zero, we obtain, after allowing for the value of q(A),

$$\frac{d\left(\frac{3}{4}B_1A^2+\frac{4B_2}{\pi A}\right)}{dA}=\frac{3}{2}B_1A-\frac{4B_2}{\pi A^2}=0,$$

hence

$$A_{kp}=2\sqrt[3]{\frac{B_2}{3\pi B_1}},\quad \Omega_{kp}^2=\frac{1}{T_1T_2}+\frac{3B_2\sqrt[3]{3\pi B_1}}{\pi T_1T_2\sqrt[3]{B_2}},\quad q_{kp}=\frac{3B_2}{\pi}\sqrt[3]{\frac{3\pi B_1}{B_2}}.$$

If we have a plot (Fig. 4.47), we can readily use the first equation of (4.222) to plot the dependences of the amplitude and frequency of the periodic solution on the total gain of the servomechanism. Replacing Ω^2 in the first equation of (4.222) by its value from the second equation we obtain

$$k=\frac{1+q(A)}{T_1}+\frac{[1+q(A)]^2}{T_2}. \qquad (4.223)$$

Inserting values of q(A) as given by the plot of Fig. 4.47 into Formula (4.223), and knowing the parameters of the system, we obtain the values of k. From the same plot (Fig. 4.47) we determine the values of the amplitude A and of the frequency Ω corresponding to the obtained values of k.

To obtain the value of k_{kr} we substitute in (4.223) the value of $q_{kr}(A)$. The result is an analytic expression for the critical value of the gain of the servomechanism

$$k_{кр}=\frac{D}{T_1}+\frac{D^2}{T_2},\quad D=1+\frac{3B_2}{\pi}\sqrt{\frac{3\pi B_1}{B_2}}. \qquad (4.224)$$

From Formula (4.223) for $\underline{k}$ and from the formula for q(A) we can see that $\underline{k}$ increases with increasing amplitude A and that $k \to \infty$ as $A \to \infty$; in addition, $k \to \infty$ as $A \to 0$. It is obvious from (4.223) that if a minimum exists for q(A), a minimum exists also for k(A). This determines the character of the dependence of the amplitude of the periodic solution on the gain $\underline{k}$ of the servomechanism (Fig. 4.48).

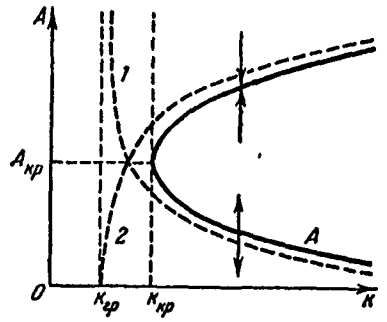


Fig. 4.48

We now show that of the two branches for the amplitude, the rising branch belongs to the stable periodic solution, namely self-oscillations, and the descending branch is the limit of stability "in the small." For this purpose, we employ the approximate criterion (4.6) for the stability of the periodic solution.

It follows from (4.221) that

$$X(a,\ \omega)=k-[T_1q(a)+T_1+T_2]\omega^2,$$
$$Y(a,\ \omega)=[q(a)+1]\omega-T_1T_2\omega^3.$$

Calculating the corresponding derivatives, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^*=-T_1\left(\frac{\partial q}{\partial a}\right)^*\Omega^2\begin{cases}>0 & \text{for } A<A_{кр},\\ <0 & \text{for } A>A_{кр},\end{cases}$$

since we have in accordance with Fig. 4.47

$$\left(\frac{\partial q}{\partial a}\right)^*\begin{cases}>0 & \text{for } A>A_{кр},\\ <0 & \text{for } A<A_{кр}.\end{cases}$$

Furthermore,

$$\left(\frac{\partial Y}{\partial \omega}\right)^*=q(A)+1-3T_1T_2\Omega^2,$$

but since the second equation of (4.222) yields $\Omega^2 = [1 + q(A)]/T_1T_2$, we have

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = -2[1+q(A)] < 0;$$
$$\left(\frac{\partial X}{\partial \omega}\right)^* = -2[T_1 q(A) + T_1 + T_2]\Omega < 0;$$
$$\left(\frac{\partial Y}{\partial a}\right)^* = \left(\frac{\partial q}{\partial a}\right)^* \Omega \begin{cases} > 0 & \text{for } A > A_{kp}, \\ < 0 & \text{for } A < A_{kp}. \end{cases}$$

Substituting the values of the corresponding derivatives and taking their signs into consideration, we find that the criterion

$$\left(\frac{\partial X}{\partial a}\right)^* \left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^* \left(\frac{\partial Y}{\partial a}\right)^* > 0$$

is satisfied when $A > A_{kr}$ for the rising amplitude branch and is not satisfied when $A < A_{kr}$ for the descending amplitude branch. This is symbolically denoted by the arrows (Fig. 4.48) that converge to the branch of the stable periodic solution and diverge away from the branch of the unstable periodic solution.

The behavior of the system now becomes clear for all values of the gain $\underline{k}$. When $k \geq k_{kr}$, if the initial values of the deviation of the motor speed are smaller than the values represented by the descending branch, the system is stable "in the small" (relative to the equilibrium state). When $k \geq k_{kr}$ and the initial deviations are greater than the values of the descending branch, self-oscillations are established in the system, with amplitudes corresponding to the values of the rising branch. Extrapolating the result obtained in the region $k \geq k_{kr}$ (the self-oscillation region) to the region of values $k < k_{kr}$, we verify that in the latter we have stability with respect to the equilibrium state no matter what the initial deviations of the servomechanism motor speed are.

If we regard the investigated servomechanism as linear with $B_1 = 0$ and $B_2 = 0$, and consequently with $q(A) = 0$, we obtain from (4.221) the characteristic equation

$$T_1T_2p^3+(T_1+T_2)p^2+p+k=0.$$

From the conditions for the stability limit

$$T_1+T_2=kT_1T_2$$

we obtain the limiting value of the servomechanism gain

$$k_{gr}=\frac{T_1+T_2}{T_1T_2}. \qquad (4.225)$$

Comparing Formula (4.224) for k_{kr} and (4.225) for k_{gr} we see that the critical value of the gain exceeds the limiting value (corresponding to the stability limit of the linear system) by an amount determined by the second term in the expression for D in the formula for k_{kr}. Consequently, the presence of nonlinearity in the mechanical characteristics of a two-phase induction motor and the presence of dry friction on its shaft lead to a broadening of the stability region with respect to the servomechanism gain compared with the stability region with respect to the structure of the linear servomechanism (with straight-line mechanical characteristics and without dry friction).

This can be explained on the basis of the previously made investigations. Inclusion of only dry friction in the servomechanism (see §4.9 and Fig. 4.32) has led to a broadening of the stability region corresponding to a linear servomechanism, wherein the dry friction has caused the region of stability of the linear system to be augmented by a region of stability "in the small" when $k_{gr} = k_{kr}$. An account of the nonlinearities of the mechanical characteristics of the actuating motor of the servomechanism has led to a result wherein a region of self-oscillations is located beyond the stability region of the linear system when $k_{gr} = k_{kr}$ (see §§4.12 and 4.13, Figs. 4.42b and 4.44a).

Now, when we take dry friction and the nonlinearity of the mechanical characteristics of the actuating motor simultaneously into account, the two foregoing effects are superimposed. If allowance for

dry friction results in a branch of A(k) corresponding to an unstable periodic solution (dashed curve 1 on Fig. 4.48), and allowance for the nonlinearity of the mechanical characteristics yields a branch corresponding to stable periodic solutions (dashed curve 2 on Fig. 4.48), the summary action of the two foregoing nonlinearities should cause the A(k) curve, naturally, to shift toward larger values of k and to be included in the region between the drawn dashed curves, something indeed disclosed by the investigation performed.

§4.15. Relay Type Servomechanism with Linear and Nonlinear Feedback

We shall investigate the relay type servomechanism with supplementary feedback for two cases; in the first case the supplementary feedback will be assumed to be linear in the velocity of the actuating

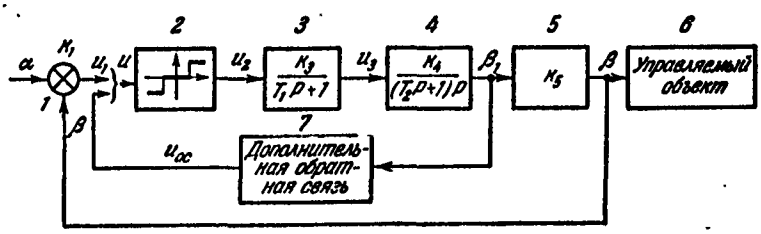


Fig. 4.49. 6) Controlled object; 7) supplementary feedback.

motor. In the second case we introduce nonlinear feedback with a quadratic velocity dependence. This enables us to ascertain which of the two additional feedbacks is advantageous.

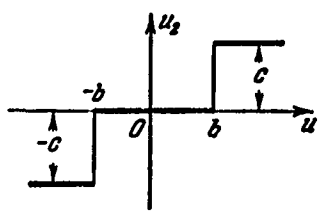


Fig. 4.50

A block diagram of the investigated relay system is shown in Fig. 4.49, where 1 is the error transducer with transfer coefficient k_1, 2 is a relay element with a relay characteristic having a backlash zone, 3 is the second amplification stage, 4 is the motor, 5 is the reduction gear, 6 is the controlled object, and 7 is the supplementary feedback.

The transfer functions of the linear elements are shown on the

block diagram.

For the nonlinear or relay element, which has a static characteristic as shown in Fig. 4.50, we obtain after harmonic linearization

$$u_1 = q(A)\,u, \quad u = u_1 - u_{o.c}. \tag{4.226}$$

where in accordance with (3.13)

$$q(A) = \frac{4c}{\pi A}\sqrt{1 - \frac{b^2}{A^2}}. \tag{4.227}$$

For the first investigated case, the equation for the feedback is

$$u_{o.c} = k_{o.c}\,p\beta_1. \tag{4.228}$$

Taking into account the transfer functions (4.226) and (4.228) of the linear elements, we obtain the characteristic equation corresponding to the harmonically linearized differential equation that describes the proper motion of the system for the variable $\underline{u}$:

$$T_1T_2p^3 + (T_1 + T_2)p^2 + [1 + k_3k_4k_{o.c}q(A)]p + k_1k_2k_4k_5q(A) = 0. \tag{4.229}$$

Substituting $p = j\Omega$ in (4.229) we obtain two equations for the amplitude and frequency of the periodic solution

$$\left.\begin{aligned} k_1k_3k_4k_5q(A) - (T_1 + T_2)\Omega^2 = 0, \\ 1 + k_3k_4k_{o.c}q(A) - T_1T_2\Omega^2 = 0. \end{aligned}\right\} \tag{4.230}$$

Let us delineate the stability regions in the plane of the parameters $k_{o.s}$ and k_1. For this purpose we determine first the critical values of the coefficient of harmonic linearization, $q(A)_{kr}$, of the amplitude, A_{kr}, and of the frequency, Ω_{kr}.

For a relay characteristic with a backlash zone, the critical value of the amplitude, obtained from the condition

$$\frac{dq(A)}{dA} = 0,$$

was determined in §3.1 (Fig. 3.4) and amounts to

$$A_{kp} = \sqrt{2}b. \tag{4.231}$$

The critical value of the coefficient of harmonic linearization will be, in accordance with (4.227) and with allowance for (4.231),

$$q_{kp}(A)=\frac{2c}{\pi b}. \qquad (4.232)$$

From the first equation of (4.230) we get

$$\Omega^2=\frac{k_1k_2k_4k_5q(A)}{T_1+T_2}. \qquad (4.233)$$

Then, taking (4.232) into account, we obtain a formula for the critical value of the oscillation frequency:

$$\Omega^2_{kp}=\frac{2k_1k_2k_4k_5c}{\pi b(T_1+T_2)}. \qquad (4.234)$$

To plot the boundary that separates the region of periodic solutions from the region where there are no periodic solutions, it is necessary to express the parameters $k_{o.s}$ and k_1 in terms of each other for the critical values of the amplitude and frequency.

From the second equation of (4.230) we obtain

$$k_{o.c}=\frac{T_1T_2\Omega^2-1}{k_3k_4q(A)}. \qquad (4.235)$$

Upon substitution of the critical values Ω^2_{kr} and $q_{kr}(A)$ from (4.234) and (4.232) into the last equation, we obtain the equation of the boundary separating the foregoing regions:

$$k_{o.c}=\frac{k_5T_1T_2}{T_1T_2}k_1-\frac{\pi b}{2k_3k_4c}.$$

It can be seen from this formula that the boundary will be a straight line with slope arc tan $k_5T_1T_2/(T_1+T_2)$, shifted downward away from the origin by an amount $\pi b/2k_3k_4c$. In addition to this line, we can draw in the region of the periodic solutions lines of equal amplitudes of the periodic solution. For this purpose we specify in the same formulas values of A other than critical. In the present example these will be lines parallel to the boundary of the periodic solutions.

Within the region of the periodic solution, two values of the amplitude are possible for the same values of the parameters k_1 and $k_{o.s}$, since the curve q(A) (Fig. 3.4) has a maximum.

In order to determine which of the amplitude values, $A > A_{kr}$ or $A < A_{kr}$ belong to the stable periodic solutions (self-oscillations), we can employ the approximate stability criterion (4.6). Extrapolating the result obtained for the region of periodic solutions to the region where there are no periodic solutions, we can ascertain whether the process is stable or not in the latter region.

We shall do this with the investigated system for the following values of the parameters: $k_3 = 25$, $k_4 = 1\ v^{-1}sec^{-1}$, $k_3 = 0.05$, $T_1 = 0.2$ sec, $T_2 = 0.3$ sec, $b = 0.4$ v, and $c = 2$ v.

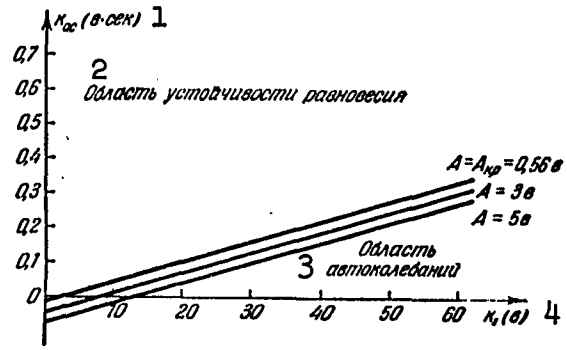


Fig. 4.51. 1) v·sec; 2) stable equilibrium region; 3) self-oscillation region; 4) v.

In accordance with (4.231) and (4.232) we obtain for the assumed values of the parameters $A_{kr} = 0.56$ v and $q_{kr}(A) = 3.19$ v. Carrying out the calculations by means of Formula (4.235) for the critical value and for other possible values of the amplitude within the region of periodic solutions, we obtain the results presented in Fig. 4.51.

We now show that the values $A > A_{kr}$ of the amplitudes of the periodic solution belong to the stable periodic solutions, while the values $A < A_{kr}$ belong to the unstable periodic solution. For this purpose we employ the approximate analytic stability criterion (4.6) for periodic solutions.

It follows from (4.229) that the real and imaginary parts of the

corresponding characteristic equation will be

$$\left.\begin{aligned} X(a,\ \omega) &= k_1k_2k_4k_3q\,(a) - (T_1 + T_2)\,\omega^2, \\ Y(a,\ \omega) &= [1 + k_3k_4k_{o.c}q\,(a)]\,\omega - T_1T_2\omega^3. \end{aligned}\right\} \qquad (4.236)$$

Calculating the derivatives contained in the inequality (4.6), we obtain

$$\left(\frac{\partial X}{\partial a}\right)^* = k_1k_2k_4k_3\left(\frac{dq}{da}\right)^*,$$

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = 1 + k_3k_4k_{o.c}\,q\,(A) - 3T_1T_2\Omega^2,$$

$$\left(\frac{\partial X}{\partial \omega}\right)^* = -2(T_1 + T_2)\,\Omega,$$

$$\left(\frac{\partial Y}{\partial a}\right)^* = k_3k_4k_{o.c}\left(\frac{dq}{da}\right)^*\Omega.$$

From the second equation of (4.230) it follows that

$$1 + k_3k_4k_{o.c}\,q\,(A) = T_1T_2\Omega^2.$$

Therefore

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = -2T_1T_2\Omega^2.$$

Employing the criterion for the stability of the periodic solution

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0$$

with account of the values of the partial derivatives, we obtain a stability condition in the form

$$\left(k_{o.c} - \frac{k_1k_3T_1T_2}{T_1 + T_2}\right)\left(\frac{dq}{da}\right)^* > 0.$$

Since $k_{o.s}$ is smaller than the critical value (4.235) in the region of periodic solutions, i.e.,

$$k_{o.c} < \frac{k_3T_1T_2}{T_1 + T_2}k_1 - \frac{\pi b}{2k_3k_4c},$$

the difference in the first brackets of the inequality will be less than zero. Consequently, in order to satisfy the criterion we must satisfy the inequality

$$\left(\frac{dq}{da}\right)^* < 0.$$

From the plot of q(A) (Fig. 3.4) it follows that this is fulfilled

when $A > A_{kr}$ and is not fulfilled when $A < A_{kr}$. Consequently, the branch of the large amplitudes of the periodic solution belongs to the self-oscillations, and that of the small values belongs to the unstable periodic solution. The results obtained for the region of periodic solutions enable us to conclude that the servomechanism is stable relative to the equilibrium state for values of the parameters k_1 and $k_{o.s}$ corresponding to the region where there is no periodic solution.

Let us examine now the case of nonlinear supplementary feedback. With the same block diagram of the servomechanism (Fig. 4.49), we take the supplementary feedback in the form

$$u_{o.c} = k_{o.c}\, x^2 \operatorname{sign} x, \quad x = p\beta_1.$$

We assume that the periodic solution for the input to the relay element is determined in the form

$$u = A_1 \sin \psi, \quad \psi = \Omega t,$$

and for the output of the nonlinear feedback element in the form

$$x = p\beta_1 = A_2 \sin(\psi + B), \quad \psi = \Omega t,$$

where B is the phase shift of the variable $\underline{x}$ relative to the variable $\underline{u}$.

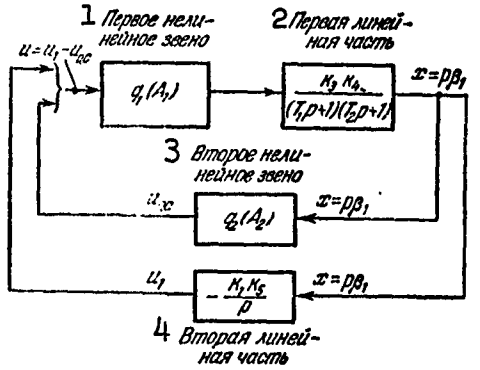


Fig. 4.52. 1) First nonlinear element; 2) first linear part; 3) second nonlinear element; 4) second linear part.

After harmonic linearization of the nonlinearities we obtain: for the relay element

$$u_2 = q_1(A_1)\,u,$$

where in accord with (4.227)

$$q_1(A_1) = \frac{4c}{\pi A_1}\sqrt{1-\frac{b^2}{A_1^2}};$$

for the nonlinear feedback

$$u_{o.c}q_2(A_2)\,x = q_2(A_2)\,p\beta_1,$$

where in accord with (3.31)

$$q_2(A_2) = \frac{8k_{o.c}A_2}{3\pi}.$$

According to the block diagram (Fig. 4.49), we can, for the motion of the system proper with $\alpha = 0$, represent the servomechanism by a simplified block diagram (Fig. 4.52), which comprises two linear parts with corresponding transfer functions and two nonlinear elements with transfer functions $q_1(A_1)$ and $q_2(A_2)$ for the steady-state oscillato· processes.

Following the rules for linear systems and taking thc foregoing transfer functions into account, we obtain a general differential equation for the harmonically linearized servomechanism:

$$\{T_1T_2p^3+(T_1+T_2)p^2+[k_3k_4q_1(A_1)\,q_2(A_2)+1]p+ \\ +k_1k_2k_3k_4k_5q_1(A_1)\}\,u=0.$$

The characteristic equation of the harmonically linearized system will be

$$T_1T_2p^3+(T_1+T_2)p^2+[k_3k_4q_1(A_1)\,q_2(A_2)+1]p+k_1k_2k_3k_4k_5q_1(A)=0.$$

After making the substitution $p = j\Omega$ we obtain two equations for the amplitude and frequency of the periodic solution:

$$\left.\begin{aligned} k_1k_2k_3k_4k_5q_1(A_1)-(T_1+T_2)\,\Omega^2=0,\\ 1+k_3k_4q_1(A_1)\,q_2(A_2)-T_1T_2\Omega^2=0.\end{aligned}\right\} \qquad (4.237)$$

Since the equations in (4.237) contain not only the unknown frequency of the periodic solution but also the unknown amplitudes A_1 and A_2 of the two different variables u and $x = p\beta_1$, one of the amplitudes must be eliminated. We use for this purpose the transfer function of

the elements that separate the foregoing variables. The transfer function for the conversion from u to x (Fig. 4.52) will be

$$W(A_1, p)=\frac{x}{u}=q_1(A_1)\frac{k_3k_4}{T_1T_2p^2+(T_1+T_2)p+1}.$$

Consequently, taking into account the value of $q_1(A_1)$, we obtain

$$A_2(A_1, \Omega)=\frac{4c}{\pi A_1}\sqrt{1-\frac{b^2}{A_1^2}}\frac{k_3k_4}{\sqrt{(1-T_1T_2\Omega^2)^2+(T_1+T_2)^2\Omega^2}}A_1. \quad (4.238)$$

Allowing for the values of $q(A_2)$ and A_2 we obtain from (4.238)

$$q_2(A_1, \Omega)=\frac{3k_{o.c}}{3\pi}A_2=k_{o.c}q_2^*(A_1, \Omega), \quad (4.239)$$

where

$$q_2^*(A_1, \Omega)=\frac{32ck_3k_4\sqrt{A_1^2-b^2}}{3\pi^2A_1\sqrt{(1-T_1T_2\Omega^2)^2+(T_1+T_2)^2\Omega^2}}.$$

In accord with (4.239), the equations in (4.237) assume the form

$$\left.\begin{aligned}k_1k_2k_3k_4q_1(A_1)-(T_1+T_2)\Omega^2=0,\\ 1+k_3k_4k_{o.c}q_1(A_1)q_2^*(A_1, \Omega)-T_1T_2\Omega^2.\end{aligned}\right\} \quad (4.240)$$

Determining the critical values of the variables, we obtain for the amplitude of the oscillations of the variable u, from the condition $dq_1(A_1)/dA_1 = 0$, the critical value

$$A_{1кр}=\sqrt{2}\,b;$$

the critical value of the coefficient of harmonic linearization will in this case be

$$q_1(A_1)_{кр}=\frac{2c}{\pi b}.$$

From the first equation of (4.240) we get

$$\Omega^2=\frac{k_1k_2k_3k_4q_1(A_1)}{T_1+T_2}.$$

Allowing for the value of $q_1(A_1)_{kr}$ we obtain the critical value of the oscillation frequency in the form

$$\Omega_{кр}^2=\frac{2k_1k_2k_3k_4c}{\pi b(T_1+T_2)}. \quad (4.241)$$

The critical value for the coefficient $q^*_2(A_1, \Omega)$ with allowance for the critical values A_{1kr} and Ω_{kr} will be

$$q_1^*(A_1, \Omega)_{кр} = \frac{32ck_3k_4}{3\sqrt{2}\,\pi^2\sqrt{(1-T_1T_2\Omega_{кр}^2)^2+(T_1+T_2)^2\Omega_{кр}^2}}.$$

From the second equation of (4.240), with allowance for the critical values of the variables contained in it, we obtain a formula for the stability limit of the servomechanism with supplementary nonlinear feedback, in the plane of the parameters k_1 and $k_{o.s}$:

$$k_{o.c} = \frac{3\sqrt{2}\,\pi^2 b(T_1T_2\Omega_{кр}^2-1)\sqrt{(1-T_1T_2\Omega_{кр}^2)^2+(T_1+T_2)^2\Omega_{кр}^2}}{64c^2k_3^2k_4^2}, \qquad (4.242)$$

where $\Omega_{kr}^2(k_1)$ is calculated by means of (4.241).

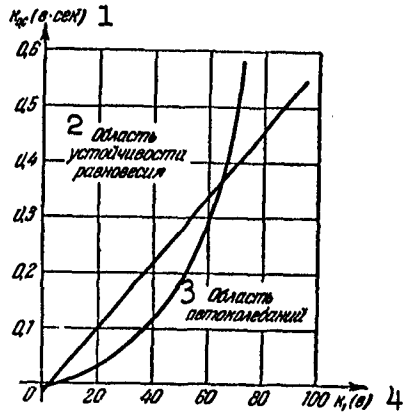


Fig. 4.53. 1) v·sec; 2) stable equilibrium region; 3) self-oscillation region; 4) v.

Carrying out the calculations in accord with Formula (4.242) and including the previously given numerical values of the parameters, we obtain the boundary that separates the stability region from the self-oscillation region, in the form of a certain curve (Fig. 4.53). This curve crosses the line corresponding to the stability limit for the case of linear feedback. No investigations of the stability of the periodic solution are needed here, since the results of such an investigation for linear feedback extend also to include the case of nonlinear feedback (for example, by putting $k_{o.s} = 0$ we obtain the case covered both in the first and in the second investigation).

From the foregoing investigation we can conclude that for certain values of the parameters nonlinear feedback quadratic in the velocity extends the stability region of the servomechanism. The value of $k_{o.s}$ of such a feedback must be adjusted for the specified parameters, for when the values of $k_{o.s}$ increase, the parameters remaining constant, the stability region begins to narrow down.

§4.16. System for Generating Current Oscillations with an Amplidyne

So far we have considered examples where the self-oscillations of nonlinear automatic systems were investigated with an aim toward either determining the possibility of suppressing the self-oscillations, or the possibility of reducing the amplitudes of the self-oscillations.

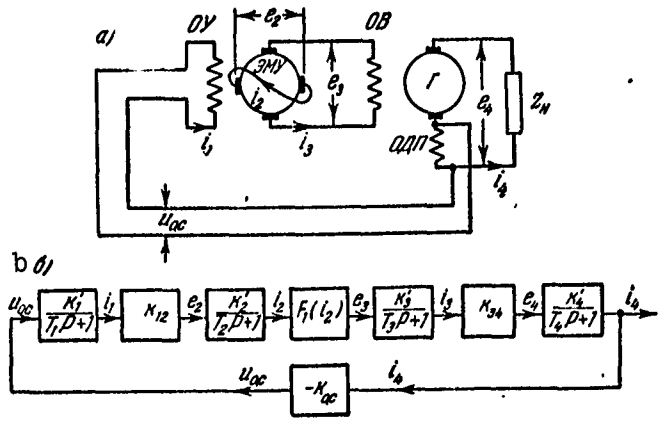


Fig. 4.54

There are closed-loop automatic systems in which the nonlinearities are used to create a steady-state oscillatory mode with definite amplitude and frequency, i.e., to generate oscillations. Such systems include primarily generators in transmitting and receiving units for communication, and relaxation oscillation generators. In the present section we consider the investigation of a high-power system for generating current oscillations at low frequency, on the order of several cycles per second. Such generators are used to demagnetize structures of ferromagnetic materials. Demagnetization is essential, for example, to guarantee normal operation of magnetic compasses installed in these structures. Investigations aimed at creating such a generator were made by I.A. Borodina [132].

We first carry out the investigation in the simplest version, without account of internal feedbacks in the amplidyne that is part of the closed loop. The block diagram of the system for generating cur-

rent oscillations is shown in Fig. 4.54a, where OU is the control winding of the amplidyne, OV is the excitation winding of the generator G, the ODP is the interpole winding, and Z_n is the load impedance. We shall assume that all elements except for the short-circuited winding of the amplidyne are linear. The nonlinearity in the short-circuited winding of the amplidyne will be assumed to be of the saturation type. In this case the system can be represented by the block diagram of Fig. 4.54b. The equations of the system elements are

$$\left.\begin{aligned}(T_1p+1)i_1 = k_1'u_{o.c},\ (T_2p+1)i_2 = k_{12}k_2i_1,\\ (T_3p+1)i_3 = k_3'F_1(i_2),\ (T_4p+1)i_4 = k_{34}k_4'i_3,\ u_{o.c} = -k_{o.c}i_4,\end{aligned}\right\} \quad (4.243)$$

where $F_1(i_2)$ is a nonlinear function that determines the dependence of the voltage induced in the main winding of the amplidyne by the current in the short-circuited quadrature winding.

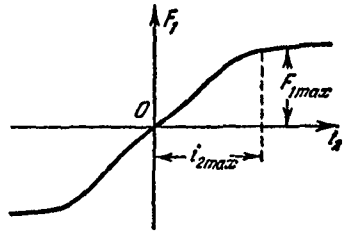


Fig. 4.55

The nonlinear dependence of the amplidyne voltage, which is obtained experimentally, will be represented approximately in the form of a polynomial in powers of i_2, up to a certain value $i_2 < i_{2max}$ and constant when $i_2 > i_{2max}$ (Fig. 4.55), i.e.,

$$\left.\begin{aligned}F_1(i_2) = Bi_2 - Ci_2^3 + Di_2^5\\ \text{for}\quad i_2 \leqslant i_{2max},\\ F_1(i_2) = F_{1max}\operatorname{sign} i_2\\ \text{for}\quad i_2 \geqslant i_{2max},\end{aligned}\right\} \quad (4.244)$$

where the coefficients B, C, and D are determined on the basis of the experimentally plotted curve $F_1(i_2)$ (see §3.5). We disregard here the hysteresis loop, which may be present in similar cases.

Carrying out harmonic linearization of the nonlinear function (4.244) we obtain

$$F_1(i_2) = q_1(A)i_2, \quad (4.245)$$

where

$$q_1(A) = B - 0{,}75cA^2 + 0{,}625DA^4 \quad \text{for} \quad A < i_{2\max},$$

$$q_1(A) = \{B(\arcsin\alpha - \alpha\xi) - cA^2[0{,}75\arcsin\alpha - \alpha\xi(0{,}75 + 0{,}5\alpha^2)] + \frac{DA^4}{6}[3{,}75\arcsin\alpha - \alpha\xi(3{,}75 + 2{,}5\alpha^2 + 2\alpha^4)] + \frac{2}{A}F_{1\max}\xi\} \text{ for } A > i_{2\max};$$

here $\alpha = \frac{i_{2\max}}{A}$, $\xi = \sqrt{1 - \frac{i_{2\max}^2}{A^2}}$.

Combining (4.243) and (4.245) we obtain for the variable i_2 a general differential equation of the harmonically linearized system

$$(a_0p^4 + a_1p^3 + a_2p^2 + a_3p + a_4)i_2 = 0, \tag{4.246}$$

where

$$a_0 = T_1T_2T_3T_4,\ a_1 = T_1T_2T_3 + T_1T_2T_4 + T_1T_3T_4 + T_2T_3T_4,$$
$$a_2 = T_1T_2 + T_1T_3 + T_1T_4 + T_2T_3 + T_2T_4 + T_3T_4,\ a_3 = T_1 + T_2 + T_3 + T_4,$$
$$a_4 = 1 + kk_{o.c}q_1(A),\ k = k_1'k_{12}k_2'k_3'k_{34}k_4'.$$

The characteristic equation corresponding to the differential equation (4.246) will be

$$a_0p^4 + a_1p^3 + a_2p^2 + a_3p + 1 + kk_{o.c}q_1(A) = 0. \tag{4.247}$$

From (4.247) we obtain after substituting $p = j\Omega$ two equations for the amplitude and frequency of the periodic solution

$$\left.\begin{aligned} 1 + kk_{o.c}q_1(A) - a_2\Omega^2 + a_0\Omega^4 = 0, \\ a_3 - a_1\Omega^2 = 0. \end{aligned}\right\} \tag{4.248}$$

From the second equation of (4.248) we obtain a formula for the frequency of the periodic solution

$$\Omega = \sqrt{\frac{a_3}{a_1}} = \sqrt{\frac{T_1 + T_2 + T_3 + T_4}{T_1T_2T_3 + T_1T_2T_4 + T_1T_3T_4 + T_2T_3T_4}}. \tag{4.249}$$

The first equation of (4.248) determines the dependence of the amplitude of the self-oscillations on the system parameters. Thus, for the feedback coefficient $k_{o.s}$, using the value of Ω from (4.249), this dependence is given by the formula

$$k_{o.c} = \frac{a_1a_2a_3 - a_0a_3^2 - a_1^2}{a_1^2kq_1(A)}. \tag{4.250}$$

Let us determine the stability of the obtained periodic solution. For this purpose we calculate the derivatives contained in the expres-

sion (4.6) for the stability criterion of the periodic solution. From (4.247) we obtain

$$X(a,\ \omega) = 1 + kk_{o.c}q_1(A) - a_2\omega^2 + a_0\omega^4,$$
$$Y(a,\ \omega) = a_3\omega - a_1\omega^3.$$

The corresponding partial derivatives will be

$$\left(\frac{\partial X}{\partial a}\right)^* = kk_{o.c}\left(\frac{dq_1}{da}\right)^*,$$
$$\left(\frac{\partial Y}{\partial \omega}\right)^* = a_3 - 3a_1\Omega^2,$$

or, allowing for the value of Ω^2 from (4.249)

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = -2a_3 < 0.$$

Since $(\partial Y/\partial a)^* = 0$, the stability condition reduces to the satisfaction of the inequality

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* > 0,$$

and allowing for the values of the derivatives, to the satisfaction of the condition

$$\left(\frac{dq_1}{da}\right)^* < 0. \qquad (4.251)$$

Figure 4.56 shows a plot of $q_1(A)$ calculated for the amplidyne EMU-50. We see that self-oscillations will be established in the system when $A \geq A_{kr} = 4.2$ a, for the condition (4.251) is satisfied in this case. When $A < A_{kr} = 4.2$ a, the periodic solution is unstable.

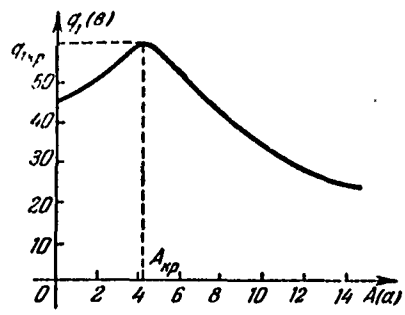


Fig. 4.56

Let us carry out the investigation for the same current generating system with allowance for the internal feedback in the amplidyne. The processes occurring in the system are described by the following equations of the elements [132]:

$$\left.\begin{aligned}
(T_1p+1)i_1 &= k'_1u_{o.c} - \frac{\chi}{k_{12}}e_2 - \frac{T_1(\eta_1+\eta_2)}{k_{12}}pe_4 - \frac{\eta_1}{k_{12}}e_4,\\
e_2 &= k_{12}i_1,\\
(T_2p+1)i_2 &= k'_2e_2,\\
e_3 &= F_1(i_2),\\
(T_3p+1)i_3 &= k'_3e_3,\\
e_4 &= k_{34}i_3,\\
(T_4p+1)i_4 &= k'_4e_4,\\
u_{o.c} &= -k_{o.c}i_4,
\end{aligned}\right\} \qquad (4.252)$$

where χ is a coefficient that takes into account the influence of the demagnetizing flux; η_1 is a coefficient allowing for the undercompensation flux; $\eta_2 = \eta'_2 + \eta_3$; η'_2, η_3 are, respectively, the coefficients that allow for the stray fluxes of the compensation winding and the frontal parts of the armature.

After carrying out the harmonic linearization we obtain from the system (4.252) the following single equation for the variable i_2:

$$\{a_0p^4 + a_1p^3 + [a_2 + T_4k'_3q_1(A)\lambda]p^2 + [a_3 + k_3q_1(A)\gamma]p + 1 + k'_3q_1(A)\xi + kk_{o.c}q_1(A)\}i_2 = 0, \qquad (4.253)$$

where

$$\begin{aligned}
k &= k'_1k_{12}k'_2k'_3k_{34}k'_4,\\
\gamma &= \chi(T_3+T_4) + k_3[T_4\eta_1 + T_1(\eta_1+\eta_2)],\\
\lambda &= \chi T_3 + k_3T_1(\eta_1+\eta_2),\\
\xi &= \chi + k_3\eta_1;
\end{aligned}$$

the coefficients a_0, a_1, a_2, and a_3 have their previous values, and $k_3 = k'_3k_{34}$.

From the characteristic equation corresponding to the differential equation (4.253) we obtain after making the substitution $p = j\Omega$ the following two equations for the determination of the periodic solution

$$\left.\begin{aligned}
1 + k'_3\xi q_1(A) + kk_{o.c}q_1(A) - [a_2 + T_4k'_3\lambda q_1(A)]\Omega^2 + a_0\Omega^4 &= 0,\\
a_3 + k'_3\gamma q_1(A) - a_1\Omega^2 &= 0.
\end{aligned}\right\} \qquad (4.254)$$

From the second equation of (4.254) we get

$$q_1(A) = \frac{a_1\Omega^2 - a_3}{k'_3\gamma}. \qquad (4.255)$$

From the first equation of (4.254) we obtain after allowing for the value of k

$$k_{o.c} = \frac{(a_3 - a_0\Omega^2)\,\Omega^2 - 1}{kq_1(A)} + \frac{T_4\lambda\Omega^2}{k'_1k_{12}k'_3k_{34}k'_4} - \frac{\xi}{k'_1k_{12}k'_2k_{34}k'_4}. \qquad (4.256)$$

Formulas (4.255) and (4.256) show that the amplitude and frequency depend on the system parameters, and in addition, unlike the preceding case, they are interrelated. In this case, two amplitudes are obtained for the same values of the parameters, in accordance with the plot of $q_1(A)$ (Fig. 4.56). We assume here, without proof but on the basis of the preceding arguments, that the branch with the larger values of the amplitudes belongs to the stable periodic solution (self-oscillations), while the branch of the small amplitudes belongs to the unstable periodic solution.

On the basis of Formulas (4.255) and (4.256), calculation yielded the dependence of the amplitude and frequency of the periodic solution on the coefficient $k_{o.s}$ (Fig. 4.57). The following values of the parameters, which were obtained experimentally, were used in the calculations: the time constants of the EMU-50 amplidyne were $T_1 = 0.12$ sec and $T_2 = 0.22$ sec; the time constants of the generator were $T_3 = 0.79$ sec and $T_4 = 0.45$ sec. The parameters $\gamma = 0.020$, $\xi = 0.025$, and $\lambda = -0.0049$ were also obtained experimentally. The gains of the individual elements had the following values: $k'_1 = 1.1 \cdot 10^{-3}$, $k_{12} = 125$ b/a, $k'_2 = 2.1$ a/b, $k'_3 = 0.40$ a/b, $k_{34} = 5.13$ b/a, $k'_4 = 10$ a/b. The coefficients included in the approximating function of the nonlinearity $F_1(i_2)$ were $B = 45.7$, $C = 2.06$, $D = -0.071$.

Figure 4.57 shows also the functions $A(k_{o.s})$ and $\Omega(k_{o.s})$ (dashed curves) for the case when the internal feedback in the amplidyne was disregarded. Failure to take account of the internal feedback leads to a conclusion that the frequency of the generated oscillations cannot be regulated by varying the coefficient $k_{o.s}$.

The experiments performed were in good agreement with the results

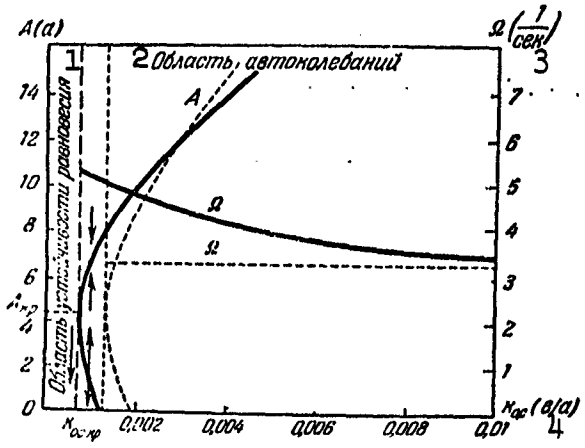


Fig. 4.57. 1) Stable equilibrium region; 2) self-oscillation region; 3) sec; 4) v/a.

obtained when allowance was made for the internal feedback. Consequently, allowance for the internal feedback via the magnetic fluxes of the amplidyne is of essential significance in the representation of the processes occurring in the amplidyne with the aid of differential equations.

The result obtained for values $k_{o.s} > k_{o.s.kr}$ (in the self-oscillation region) makes it possible to conclude that the motion of the system is stable relative to the equilibrium state for values $k_{o.s} < k_{o.s.kr}$ (in the region where there are no periodic solutions).

The self-oscillations were investigated for the variable i_2 (current in the quadrature circuit of the amplidyne). From the practical point of view, it is important to determine the variation of the current i_4 in the load. The obtained amplitudes A of the current i_2 can be converted into amplitudes A_{i_4} of the current i_4 by using the transfer function of the elements separating the variables i_4 and i_2. This transfer function will be, in accord with the block diagram of the system (Fig. 4.54),

$$W(A, p) = \frac{i_4}{i_2} = \frac{k_3 k_4' q_1(A)}{(T_3 p + 1)(T_4 p + 1)}$$

and consequently

$$A_{1_4} = \frac{k_0 k_1' q_1(A)}{\sqrt{(T_1^2\Omega^2+1)(T_2^2\Omega^2+1)}} A. \qquad (4.257)$$

In the system under consideration it is desirable to have a broad range of variation of the amplitude and frequency of the generated oscillations as the system parameters are varied, so as to be able to vary the self-oscillation mode. The range of variation of the amplitude and frequency of the self-oscillations can be broadened by introducing additional feedback proportional to the amplidyne output voltage or the DC generator voltage.

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[Footnotes]

345 Another method was indicated in §2.4.

396 One can also use a graphical method for determining the coefficients of harmonic linearization (see Chapter 3).

397 The absolute values of ω_{dv} are used in the coefficients because ω_{dv} reverses, and the coefficients themselves should remain positive numbers.

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[List of Transliterated Symbols]

309 с.д = s.d = stabiliziruyushchiy dvigatel' = stabilizing motor
314 Д = D = dvigatel' = motor
315 Г = g = generator = generator
315 дв = dv = dvigatel' = motor
315 я = ya = yakor' = armature
316 л = l = lineynyy = linear
317 у = u = usilitel' = amplifier
318 кр = kr = kriticheskiy = critical
318 гр = gr = granitsa = limit

325	вх = vkh = vkhodnoy = input
326	о.с = o.s = obratnaya svyaz' = feedback
332	Р = R = rele = relay
332	ОВ = OV = obmotka vozbuzhdeniya = excitation winding
332	с = s = set' = line
332	р = r = regulirovaniye = control
337	м = m = maksimal'nyy = maximum
342	д = d = dvigatel' = motor
347	ср = sr = srabatyvaniye = operation, pull-in
347	отп = otp = otpadaniye = drop-out
352	т = t = treniye = friction
382	н = n = nelineynyy = nonlinear
384	с = s = samolet = airplane
384	р = r = rul' = control surface, rudder
386	р.м = r.m = rulevaya mashinka = control-surface servo
404	п = p = puskovoy = starting
414	тр = tr = treniye = friction
430	ЭМУ = EMU = elektromashinnyy usilitel' = amplidyne
430	ОУ = OU = obmotka upravleniye = control winding
430	ОДП = ODP = obmotka dobavochnykh polyusov = interpole winding
430	н = n = nagruzka = load

Chapter 5

PASSAGE OF SLOWLY VARYING SIGNALS IN SELF-OSCILLATING SYSTEMS. ASYMMETRICAL SELF-OSCILLATIONS

§5.1. ASYMMETRICAL SELF-OSCILLATIONS AND PRINCIPLES GOVERNING THEIR DETERMINATION

In the preceding sections we have investigated symmetrical self-oscillations resulting from free motion of the system (without external action) with symmetrical nonlinearities. However, as will be shown below, the most important practical value attaches also to an examination of asymmetrical self-oscillations. The simplest illustrations of this premise were already presented in §§1.6 – 1.8.

Asymmetrical self-oscillations can be the result of various causes such as:

1) asymmetry of the nonlinear characteristic both in the presence and in the absence of external action;

2) the presence of a constant or slowly varying external action in the case of symmetrical nonlinearities;

3) the presence of a constant or slowly varying rate of change of the external action in the case of symmetrical nonlinearities (for those cases when the constant action does not cause the center of the oscillations to shift; this is usually the case in servomechanisms and in astatic systems in general).

In fact, if an asymmetrical nonlinear characteristic is present (for example, Fig. 5.1a, b), then even symmetrical oscillations of the

variable $x = A \sin \Omega t$ give rise to oscillations of the variable F which are asymmetrical in amplitude (Fig 5.1b). On the other hand, if the nonlinearity is symmetrical (for example, Fig. 5.2a, b), then in the presence of a constant external action (or, in the case of astatic systems, in the presence of a constant rate of change of the external action), the oscillation center of the variable $x = x^0 + A \sin \Omega t$ shifts, so that the oscillations of the variable F become asymmetrical in amplitude and time (Fig. 5.2a), or in time alone (Fig. 5.2b).

Earlier, in developing the principles of the method of harmonic linearization in Chapter 2, we considered homogeneous equations for the nonlinear systems. Now we shall investigate the total equations of the nonlinear systems referred to in §1.2, including the external signals. As the main equation for nonlinear systems of the first class we choose

$$Q(p)x + R(p)F(x, px) = S(p)f(t), \tag{5.1}$$

which covers the majority of practical problems, after which we generalize the results to include other types of equations. We note here that an astatic system is defined as one in which the polynomial S(p) has p as a common factor, thus, $S(p) = pS_1(p)$.

If there are several (m) external signals, then the right half of (5.1) will contain the sum

$$\sum_{i=1}^{m} S_i(p) f_i(t).$$

The external signal f(t) may be either a disturbance or a control (master) signal. In the present chapter we shall regard this signal f(t), and in astatic systems its rate of change pf(t), as constant or slowly varying. We shall define a function of time as slowly varying if it changes relatively little over one cycle of the periodic solution (self-oscillations), that is, if one of the following inequalities is satisfied

$$|f(t+T) - f(t)| < |f(t)| \text{ for } \left|\frac{df}{dt}\right| T < |f(t)|,$$

where $T = 2\pi/\Omega$, and Ω is the frequency of self-oscillation. For astatic systems, accordingly, a slowly varying rate of change pf(t) will be one for which the following condition is satisfied

$$|pf(t+T)-pf(t)| \ll |pf(t)| \quad \text{or} \quad \left|\frac{dpf}{dt}\right| T \ll |pf(t)|.$$

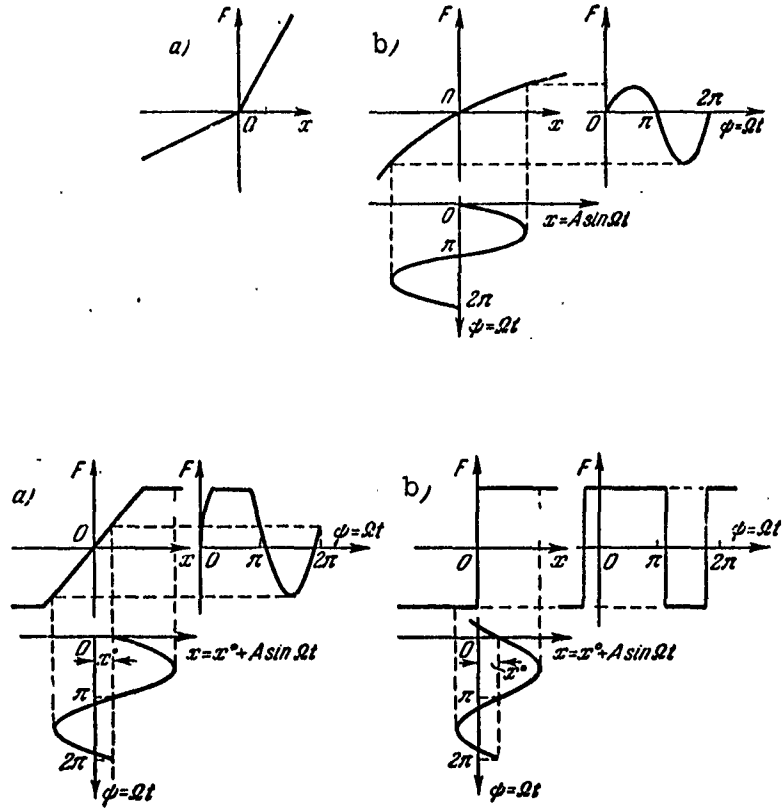


Fig. 5.2.

We shall impose a similar condition on the time variation of the parameters in nonlinear systems with slowly varying parameters contained in the coefficients of the polynominals Q(p) and R(p).

The condition that any function of the time be a slowly varying one can be also expressed in frequency form, namely: a slowly varying function is considered to be one whose possible frequencies of time variations are considerably lower than the possible frequency of the investigated periodic solution.

The assumptions introduced enable us to regard the quantity f(t), or the corresponding quantity pf(t), and the parameters of the system as being constant over each cycle of the investigated self-oscillations. It is assumed besides that the given equation of the nonlinear system satisfies at f(t) = 0 all the conditions indicated in §2.3, with the sole exception of Condition (2.74), which may not be satisfied (in the case of asymmetrical nonlinearity).

We seek the solution of the nonlinear Equation (5.1) in contrast with (2.45), in the form $x = x^0 + x_1 + \varepsilon y(t)$, i.e., we assume it to be nearly sinusoidal with a dc component. Consequently, the first approximation will have in place of (2.73) the form

$$x = x^0 + x^*, \text{ where } x^* = A \sin \Omega t', \tag{5.2}$$

where t' is the time, which is measured within each cycle of oscillations, and x^0, A, Ω are unknown slowly varying functions of the time, which depend on the form of the specified function f(t) in the right half of (5.1). They will be constant when f(t) = const (and in astatic systems when pf(t) = const).

Assuming the variation of the quantities x^0, A, Ω to be so slow that they can be regarded as constants during one cycle of the oscillations (taking their average over the cycle), we shall construct a solution in accordance with same principle of harmonic linearization as was used in Chapter 2, except that account will be taken of the value of the bias x^0. In this connection, the first members of the Fourier expansion must be written not as (2.75) and (2.76), but in the form

$$F(x, px) = F^0 + qx^* + \frac{q'}{\Omega} px^*, \tag{5.3}$$

where putting $\psi = \Omega t'$, we have

$$\left.\begin{aligned} F^0 &= \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\,d\psi, \\ q &= \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\sin\psi\,d\psi, \\ q' &= \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\cos\psi\,d\psi. \end{aligned}\right\} \tag{5.4}$$

We see therefore that in the general case all three coefficients are functions of the three unknowns

$$F^0 = F^0(x^0, A, \Omega),\quad q = q(x^0, A, \Omega),\quad q' = q'(x^0, A, \Omega). \tag{5.5}$$

In particular cases these relationships can be simpler.*

Substitution of (5.2) and (5.3) into the specified differential equation (5.1) yields

$$Q(p)(x^0 + x^*) + R(p)\left(F^0 + qx^* + \frac{q'}{\Omega}px^*\right) = S(p)f(t).$$

If the function f(t) (in astatic systems, pf) and the quantities x^0, A, Ω contained in the coefficients F^0, q, and q', are sufficiently slowly varying, this equation can be separated into two individual equations (168):

$$Q(p)x^0 + R(p)F^0 = S(p)f(t), \tag{5.6}$$

$$Q(p)x^* + R(p)\left(q + \frac{q'}{\Omega}p\right)x^* = 0 \tag{5.7}$$

for the slowly varying and for the oscillatory components, respectively. These separated equations retain essentially the nonlinear properties and the nonsuperposition of the solutions, since the two equations are still nonlinearily related by means of (5.5).

Equation (5.7) coincides with the earlier equation (2.78), the only difference being that now the coefficients q and q' depend, in accordance with (5.5), not only on A and Ω, but also on the bias x^0. Therefore, writing down as before (§2.3) the characteristic equation

$$Q(p)+R(p)\left(q+\frac{q'}{\Omega}p\right)=0, \tag{5.8}$$

replacing p by $j\Omega$ and separating the real and imaginary parts we obtain, unlike (2.83), two algebraic equations with three unknowns

$$X(x^0, A, \Omega)=0, \quad Y(x^0, A, \Omega)=0. \tag{5.9}$$

These equations enable us to determine the amplitude A and the frequency Ω of the self-oscillations as functions of the slowly varying component x^0:

$$A=A(x^0), \quad \Omega=\Omega(x^0). \tag{5.10}$$

To solve this problem we can use any of the methods described in §2.3, depending on which of these fits better the conditions of the given specific problem. The same methods can be used also to determine the dependence of A and Ω not only on x^0 but also on the parameters of the system, in order to be able to select the latter. In those methods of §2.3, which call for plots of q(A) and q'(A), it is necessary to plot these functions in the form of a series of curves for different constant values of x^0 (Fig. 5.3).

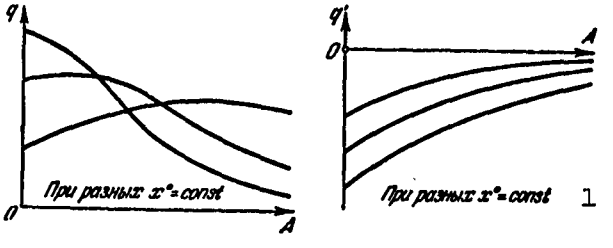


Fig. 5.3. 1) For various x^0 = const.

Once the functions in (5.10) are determined from (5.9) it is possible to use the first of the formulas (5.5) and determine the bias function

$$F^0=\Phi(x^0), \tag{5.11}$$

which will serve as a characteristic of the given nonlinear element

relative to the slowly varying components of the variables F and $\underline{x}$ in the presence of self-oscillations. Usually (5.11) is a continuous function even for relay-type characteristics (see, for example, §1.7). After obtaining this function we can further solve equation (5.6) with account of (5.11) and obtain as a result the solution

$$x^0 = x^0(t) \tag{5.12}$$

for the slowly varying component with specified f(t) in the right half of the equation. By the same token we determine, in accordance with (5.10), the amplitude and the frequency of the self-oscillations, A and Ω, which vary slowly in time together with $x^0(t)$.

If the particular problem calls for finding x^0 = const and the right half of (5.6) is equal to a certain specified constant M^0, we obtain in place of (5.6) the equation

$$Q(0)x^0 + R(0)F^0 = M^0. \tag{5.13}$$

We can suggest two methods for solving the problem in this case.

The first method, in analogy with the forgoing, consists of first solving the equations in (5.9) to determine the dependence of the amplitude A and of the frequency Ω of the self-oscillations on the value of the bias x^0 in the form (5.10). The expression $F^0 = \Phi(x^0)$ is then determined by substituting the values of (5.10) in the first of the expressions (5.5), calculated by means of the first formula of (5.4). Consequently, we obtain in this case in place of the differential equation (5.6) the algebraic equation (5.13) with an unknown x^0, which is contained also in F^0. Most frequently Eq. (5.13) is transcendental in x^0 and is solved graphically. The next step is to determine from (5.10) the amplitude A and the frequency Ω.

The indicated dependence (5.10) of the amplitude and frequency of the self-oscillations on the shift in the center of the oscillations, a dependence which is characteristic percisely of nonlinear

systems (in linear systems the shift of the center of oscillations does not play any role), must always be kept in mind. For some nonlinearities this dependence may be significant, for others it may be less important.

The second method of solving the same problem, which in the case when x^0 = const may turn out to be less simple, consists, on the contrary, of first solving Equation (5.13), where, according to (5.5), F^0 is either a function of x^0, A, and Ω or else frequently a function of x^0 and A. The solution assumes the form

$$x^0 = x^0(A, \Omega), \quad \text{or} \quad x^0 = x^0(A). \tag{5.14}$$

This solution is then substituted in the equations of (5.9) which thus contain only two unknowns, A and Ω. After determining the latter (using and of the methods of §2.3), we use (5.14) to determine the value of x^0. This second method is used in the examples considered in §1.6.

This solves the problem in the general case. In many particular cases, as will be shown latter on, this solution can be greatly simplified.

In the case when there is no external signal (f = 0) and the self-oscillations are determined in a system with asymmetrical nonlinearity, that is, in a nonlinearity F(x) or F(x, px) for which

$$\int_0^{2\pi} F(A \sin \psi, A\Omega \cos \psi)\, d\psi \neq 0, \tag{5.15}$$

we obtain in place of (5.6)

$$Q(0)\, x^0 + R(0)\, F^0 = 0. \tag{5.16}$$

This equation is solved by either of the two methods described above for Equation (5.13). Simultaneously we determine x^0, A, and Ω in accordance with (5.9)

It is known from the foregoing that R(p)/Q(p) is the transfer function of the reduced linear portion of the system. It frequently happens that its denominator Q(p) has a zero root. Then Q(0) = 0, and consequently Equation (5.16) assumes, with allowance for (5.11), the form

$$F^0 = \Phi(x^0) = 0, \tag{3.17}$$

from which we determine x^0. This means that in such systems the bias x^0 of the oscillations of the variable x is such as to balance out exactly the asymmetry of the oscillations of the variable F which is inherent in the given nonlinearity (that is, it guarantees that F^0 = 0); this is illustrated, for example, in Fig. 5.4, in contrast with Fig. 5.1b.

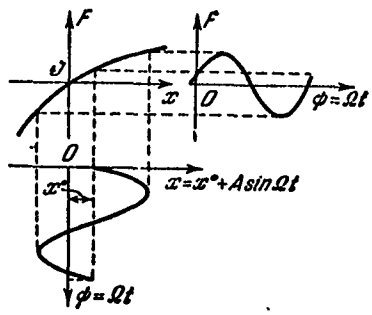


Fig. 5.4.

All our previous considerations applied to the determination of the self-oscillations and of the magnitude of a slowly varying (or dc) component for the argument x of the nonlinear function. It is obvious that the amplitude of the oscillations can be calculated by means of corresponding transfer functions for other system variables, which must be considered in each specific problem. Analogously, the corresponding equations (differential or algebraic) can be used to determine also the slowly varying or dc component for any variable, in particular for the regulated quantity of the given automatic system, the reproduction quality of which is to be determined.

In the present section we have dealt with equations of systems of the first class and of the principal type (5.1). These results, however, in analogy with what was done in Chapter 2, can be generalized also to include equations of the other types and classes of systems

considered in §1.2. In this case there may exist several values of the bias x_1^0, the connection betwen which is written in the form of equations of the corresponding elements for the slowly varying or dc component.

Let us consider, for example, nonlinear systems of the first class of a different type, described by equations of the form (1.56) and (1.57):

$$Q(p)x + R_1(p)F_1(x, px) + R_2(p)F_2(x, px) = S(p)f(t)$$

(the right side may contain several external signals). We separate this into one equation for the slowly varying component

$$Q(p)x^0 + R_1(p)F_1^0 + R_2(p)F_2^0 = S(p)f(t)$$

and an equation for the oscillatory component

$$Q(p)x^* + R_1(p)\left(q_1 + \frac{q_1'}{\Omega}p\right)x^* + R_2\left(q_2 + \frac{q_2'}{\Omega}p\right)x^* = 0.$$

The coefficients of harmonic linearization q_1, q_1', q_2 and q_2' and F_1 and F_2 are determined by the earlier formula (5.4) and depend, like expression (5.5), in general on three sought quantities (x^0, A, Ω), which can be either constant or slowly varying, depending on f(t). The method of solving the problem remains the same.

We can proceed analogously in the investigation of nonlinear systems of the second and third classes, satisfying the condition indicated at the end of §2.2. For example, if the equations of a system with two separated nonlinearities in different variables are specified in the form

$$\begin{aligned} Q_1(p)x_1 - R_1(p)F_2(x_2, px_2) &= S(p)f(t), \\ Q_2(p)x_2 + R_2(p)F_1(x_1, px_1) &= 0, \end{aligned}$$

then the solution is sought in the form

$$x_1 = x_1^0 + A_1 \sin \Omega t', \quad x_2 = x_2^0 + A_2 \sin(\Omega t' + \varphi).$$

It is necessary to carry out harmonic linearization of each of the nonlinearities, using formulas (5.4). The result consists of two systems of equations: one for the slowly varying components

$$Q_1(p)\,x_1^0 - R_1(p)\,F_2^0 = S(p)f(t),$$
$$Q_2(p)\,x_2^0 + R_2(p)\,F_1^0 = 0$$

(or for the dc components if p = 0), and one for the oscillatory components

$$Q_1(p)\,x_1^* - R_1(p)\left(q_2 + \frac{q_2'}{\Omega}p\right)x_2^* = 0,$$
$$Q_2(p)\,x_2^* + R_2(p)\left(q_1 + \frac{q_1'}{\Omega}p\right)x_1^* = 0.$$

In this case the coefficients of harmonic linearization will be functions of the five sought quantities, namely

$$F_1^0(x_1^0, A_1, \Omega),\quad q_1(x_1^0, A_1, \Omega),\quad q_1'(x_1^0, A_1, \Omega),$$
$$F_2^0(x_2^0, A_2, \Omega),\quad q_2(x_2^0, A_2, \Omega),\quad q_2'(x_2^0, A_2, \Omega).$$

We can write down analogously equations for the other cases. The approach to the solution of these equations is in principle analogous in all these cases, but the computational work becomes correspondingly complicated, and this may make it necessary to seek special methods so as to shorten the calculations and make them practically convenient. This is done partially in the examples of Chapter 6.

§5.2. STATIC AND STEADY-STATE ERRORS OF SELF-OSCILLATING SYSTEMS

Assume that we are given a certain automatic system, the dynamics of which is described by Equation (5.1), and that it is known that the given system operates in the self-oscillating mode. We assume that the dependence (5.10) of the amplitude A or of the frequency Ω of the self-oscillations on the value of the bias x^0 has already been determined by means of Equations (5.9)(although a different approach to the solution of the problem, indicated in §5.1, is also possible).

Let us investigate the static and steady-state errors of the system, corresponding to those stationary modes of system operation at

which x^0 = const (hence A = const, Ω = const) [183]. In these cases, the stationary operating mode of the system is determined by the algebraic equation (5.13), that is,

$$Q(0)x^0 + R(0)F^0 = M^0, \tag{5.18}$$

in which we substitute, on the basis of the first of Formulas (5.5) and (5.10), the bias function

$$F^0 = \Phi(x^0). \tag{5.19}$$

If the transfer function of the reduced linear portion of the system, R(p)/Q(p), has a zero root in the denominator, that is, when Q(p) = $pQ_1(p)$, we obtain from (5.18) the equation

$$\Phi(x^0) = \frac{M^0}{R(0)}, \tag{5.20}$$

from which we determine the static deviation $x^0(M^0)$.

The nonlinearity can generally speaking be connected in the system in arbitrary fashion. In particular, if the nonlinearity F(x, px) characterizes such a system control element, in which the variable represents the system error directly, then Formulas (5.18) and (5.19) in the general case, and Formula (5.20) in the particular case, determine the sought value of the steady state error of the system. We note that in a linear system we have $R(0)F^0 = kx^0$, and Q(0) = 1 or Q(0) = 0. This leads to the well-known respective error formula

$$x^0 = \frac{M^0}{1+k} \quad \text{or} \quad x^0 = \frac{M^0}{k}.$$

Consequently, Formulas (5.18) – (5.20) are essentially analogous to the formulas for the linear systems, but they give a nonlinear, usually transcendental dependence of the error x^0 on the value of the right side M^0; thus the error is not proportional here to the value of M^0. In addition, self-oscillations $x^* = A \cos \Omega t$ are imposed on the constant quantity x^0, and represent a supplementary periodic error, the magnitude

of which is determined by Formula (5.10). This error depends through x^0 on the value of the right side M^0, and in the general case this dependence is likewise nonlinear (not proportional).

The right side M^0 can have different meanings. In investigating the given (5.1), we can note three cases in which a constant quantity is obtained in the right member:

1) static system $S(0) \neq 0$, $f(t) = \text{const} = f^0$; then

$$M^0 = S(0) f^0, \tag{5.21}$$

and if there are several (m) external signals, then

$$M^0 = \sum_{i=1}^{m} S_i(0) f_i^0;$$

2) astatic system $S(p) = pS_1(p)$; then $S(0) = 0$ and when $f(t) = \text{const} = f^0$ we have $M^0 = 0$; in the case of a symmetrical nonlinearity, this yields $x^0 = 0$, and for an asymmetrical nonlinearity it leads to Equation (5.16) or (5.17), from which we get $x^0 \neq 0$;

3) astatic system with $f(t) = c^0 t$, where $c^0 = \text{const}$; then

$$M^0 = S_1(0) c^0. \tag{5.22}$$

We can consider analogously also a doubly astatic system, in which $S(p) = p^2 S_2(p)$, etc.

In the first and second cases we calculate x^0 from (5.18) and (5.19) or from (5.20), and obtain the so-called static error, while in the third case we obtain the steady-state error at constant velocity of variation of the external signal (in contradistinction to linear systems, not proportional to the velocity). Superimposed on these is an additional periodic steady-state error in the form of self-oscillations.

Nonlinear system errors of this type can be determined, of course, also in the absence of self-oscillations. It is then necessary to start from the initially specified equation (5.1), which leads to the same formulas, except that in these the bias function $\Phi(x^0)$ is replaced by

the initially specified nonlinear function $F(x, 0)$.

The premise presented that when calculating the static and steady-state errors of nonlinear systems operating in the self-oscillating mode it is necessary to replace the specified nonlinear function $F(x, px)$ by the bias function $\Phi(x^0)$ is exceedingly important. This premise illustrates the breakdown of the superposition principle in nonlinear systems. Whereas in a linear system the static error remains unchanged no matter what oscillations are superimposed on it, in a nonlinear system these oscillations may greatly influence the magnitude of the static error. The same takes place also when forced oscillations are superimposed (see Chapter 9).

In this connection it is important to note the following. If the nonlinear characteristic has a backlash zone (for example, Fig. 5.5a, b, c), then in the absence of self-oscillations the possible static error of the system will in any case be no less than half the backlash zone b. On the other hand, in presence of self-oscillations we introduce in place of the specified nonlinear function the bias function $\Phi(x^0)$, which will not have a backlash zone if the parameters are suitably chosen, and consequently the static error can be decreased theoretically to zero. Analogously, the influence of a hysteresis loop is eliminated when oscillations are superimposed.

Concerning the classification of systems as static or astatic, it must be kept in mind that one and the same system can be either static or astatic, depending on the place where the external signal is applied and on the location of that variable for which the error is determined (in other words, depending on what is assumed to be the input and output in the given closed-loop system). This is clearly seen from the examples considered in §1.6, where the dc components were determined for the all the variables in one and the same system, but with different points of application of the external signals. In the same section, an example

was considered of the calculation of the bias x^0 in the presence of a very simple asymmetrical nonlinearity in the system without external signal.

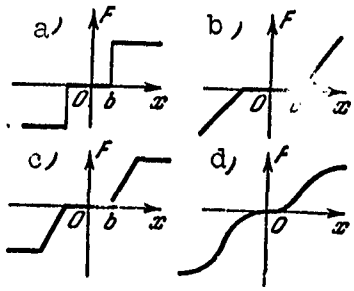


Fig. 5.5.

If, say, we apply to the system two external signals $f_1(t)$ and $f_2(t)$, and if the system is astatic with respect to the first and static with respect to the second, that is, the right side of Equation (5.1) is

$$S_1(p)\,pf_1(t) + S_2(p)f_2(t),$$

then, by calculating the steady-state error of the system for $f_1(t) = c^0t$ and $f_2(t) = \text{const} = f_2^0$, we obtain for the equation (5.18) or (5.20)

$$\Delta f^0 = S_1(0)c^0 + S_2(0)f_2^0.$$

The static and steady-state errors in systems of other classes, considered earlier, are investigated in similar fashion.

In many problems the quantity x^0, which we determined above, is not the steady-state or the static error, inasmuch as in general the variable x, which is the argument of the nonlinearity is not exactly equivalent to the difference between the actual and set values of the controlled quantity (or in general the output variable of the given system). In such a case it is necessary, after first determining x^0 as above, to use the equations of the corresponding system elements to express in terms of x^0 the value of the static or steady state error of the given system. In any problem, such an algebraic expression can be readily obtained.

There are cases, however, when $x^0 = 0$ and there is still a static error in the output variable. This is precisely the case illustrated in one of the examples considered in §1.6. Then the steady-state value of the controlled quantity is expressed directly in terms of the

external signal, for example in the form (1.118). Inasmuch as in that example (Fig. 1.28) $f_1(t)$ is a specified signal which must be duplicated at the output, the static error in that case will be $\Delta x_{st} = x_4^0 - f_1$.

The steady-state error occurring when the rate of change of the external signal in an astatic system is constant is determined quite analogously. It must be kept in mind, however, that certain system variables may not be constant in this case, but may vary in proportion to the time (in particular, the output of a servomechanism operating in the tracking mode with constant speed).

For purposes of illustration, let us continue the analysis of the examples of §1.6. There we investigated the influence of external signals $f_1(t)$ and $f_3(t)$ separately. We now consider a case wherein they act simultaneously (it will be shown below that unlike linear systems, we cannot merely add in our case the static errors resulting from each signal taken separately).

The equations of the automatic system (Fig. 5.6) are specified in the form

$$(T_1 p + 1) x_2 = k_1 x_1, \quad x_1 = f_1(t) - x_4, \tag{5.23}$$

$$x_3 = F(x), \quad x = x_2 - x_{o.c}, \quad x_{o.c} = k_{o.c} x_4, \tag{5.24}$$

$$(T_3 p + 1) p x_4 = k_3 x_3 + f_3(t), \tag{5.25}$$

where F(x) is the very simple symmetrical relay characteristic shown in Fig. 5.6:

$$F(x) = c \operatorname{sign} x. \tag{5.26}$$

In order to use the general formulas derived above,* it is first necessary to reduce the specified system of Equations (5.23) – (5.25) to a single equation of the type (5.1). As a result we obtain

$$(T_1 p + 1)(T_2 p + 1) p x + (k_{o.c} T_1 p + k_1 + k_{o.c}) k_3 F(x) = \\ = k_1 (T_2 p + 1) p f_1(t) + (k_{o.c} T_1 p + k_1 + k_{o.c}) f_3(t). \tag{5.27}$$

Let $f_1(t)$ be the set-point (control) signal, which varies at a constant speed

$$f_1(t) = c^0 t, \tag{5.28}$$

and which must be duplicated at the output of the system in the form $x_4(t)$. Let us assume also that second external signal $f_3(t)$ is a dis-

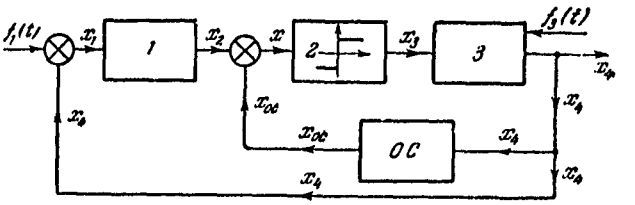


Fig. 5.6.

turbance and has a constant magnitude (for example, a constant load on the output shaft of the system):

$$f_3(t) = \text{const} = f_3^0. \tag{5.29}$$

The influence of the latter must be reduced to a minimum. Let us find the steady-state error at the output of the system. The right side of (5.27) will in this case be constant, and the steady-state solution for $\underline{x}$, with allowance for the self-oscillations, must be sought in the form

$$x = x^0 + x^*, \text{ where } x^* = A \sin \Omega t. \tag{5.30}$$

Harmonic linearization of the nonlinearity (5.26) yields in this case, in accordance with (1.122) and (1.123),

$$F^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A}, \quad q = \frac{4c}{\pi A} \sqrt{1 - \left(\frac{x^0}{A}\right)^2}. \tag{5.31}$$

In the present problem, Equation (5.7) for the periodic component assumes in accord with (5.27) the form

$$(T_1 p + 1)(T_2 p + 1) p x^* + (k_{o.c} T_1 p + k_1 + k_{o.c}) k_2 q x^* = 0, \tag{5.32}$$

and Equation (5.13) for the constant components will be

$$(k_1 + k_{o.c}) k_2 F^0 = M^0, \tag{5.33}$$

where in accord with (5.27), (5.28), and (5.29) we have

$$M^0 = k_1 c^0 + (k_1 + k_{o.c}) f_3^0. \tag{5.34}$$

In §5.1 we indicated two methods for solving this problem. To illustrate both methods, let us solve the problem by each of them.

According to the first method, we first solve Equation (5.32) so as to determine the functions $A(x^0)$ and $\Omega(x^0)$. In this case the characteristic equation will be

$$T_1T_2p^3+(T_1+T_2)p^2+(1+T_1k_2k_{o.c}q)p+(k_1+k_{o.c})k_2q=0, \tag{5.35}$$

and Equation (5.9) will therefore assume the form

$$\left.\begin{aligned} X&=(k_1+k_{o.c})k_2q-(T_1+T_2)\Omega^2=0,\\ Y&=(1+T_1k_2k_{o.c}q)\Omega-T_1T_2\Omega^3=0. \end{aligned}\right\} \tag{5.36}$$

Eliminating q, we obtain the self-oscillation frequency

$$\Omega^2=\frac{k_1+k_{o.c}}{T_1(T_2k_1-T_1k_{o.c})}. \tag{5.37}$$

The frequency Ω in this case was found independent of the bias x^0, and consequently of the value of the external signal.* Then, substituting the expression for q from (5.31) and for Ω^2 from (5.37) into the first equation of (5.36), we obtain a biquadratic equation for determination of the dependence of the self-oscillation amplitude A on the bias x^0 in the form

$$\left(\frac{A}{A_c}\right)^4-\left(\frac{A}{A_c}\right)^2+\left(\frac{x^0}{A_c}\right)^2=0, \tag{5.38}$$

where the quantity

$$A_c=\frac{4ck_2T_1(T_2k_1-T_1k_{o.c})}{\pi(T_1+T_2)} \tag{5.39}$$

represents, in accordance with (1.85), the amplitude of the self-oscillations in the given system in the absence of bias ($x^0=0$). Hence

$$\left(\frac{A}{A_c}\right)^2=\frac{1}{2}+\sqrt{\frac{1}{4}-\left(\frac{x^0}{A_c}\right)^2}. \tag{5.40}$$

The expression obtained can also be written in the form

$$A=A_c\cos\frac{\alpha}{2}, \tag{5.41}$$

if we put

$$\alpha=\arcsin\frac{2x^0}{A_c}. \tag{5.42}$$

The result (5.40) or (5.41) is indeed the sought function $A(x^0)$.

Further, according to the first method of solving the problem, we substitute the value obtained for the amplitude A from (5.41) into the expression (5.31) for F^0, from which we obtain, using (5.42), the bias function

$$F^0 = \Phi(x^0) = \frac{c}{\pi}\alpha = \frac{c}{\pi}\arcsin\frac{2x^0}{A_c}. \quad 0 \leq |x^0| < \frac{A_c}{2}, \tag{5.43}$$

where A_s is determined in terms of the system parameters by Formula (5.39). This is the sought function $\Phi(x^0)$ (Fig. 5.7a), which replaces the specified nonlinear function $F(x)$ in the calculation of the steady-state errors in the nonlinear system (Fig. 5.7b).

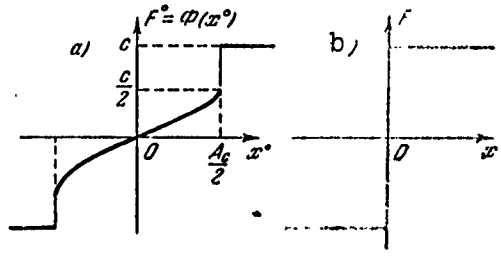


Fig. 5.7.

Substituting the value of (5.43) in Equation (5.33) for the constant components, we obtain with allowance for (5.34) and (5.39)

$$x^0 = \frac{A_c}{2}\sin\left[\frac{\pi}{ck_3}\left(\frac{k_1c^0}{k_1+k_{0.c}}+f_s^0\right)\right]. \tag{5.44}$$

Comparing this with Formula (5.42), we see that for the quantity α which we have previously introduced artificially, we can now write the expression

$$\alpha = \frac{\pi}{ck_3}\left(\frac{k_1c^0}{k_1+k_{0.c}}+f_s^0\right). \tag{5.45}$$

This quantity characterizes the totality of the external signals applied to the system. Taking (5.45) into account, we obtain from (5.41) the amplitude of the self-oscillations

$$A = A_c\cos\left[\frac{\pi}{2ck_3}\left(\frac{k_1c^0}{k_1+k_{0.c}}+f_s^0\right)\right]. \tag{5.46}$$

An important factor is that the amplitude of the self-oscillations depends not only on the system parameters (see (5.39)), but also on the magnitude of the external signals. This dependence is nonlinear. In this case an increase in the external signals causes the amplitude to decrease in accordance with a cosine law, whereas the frequency is independent of the external signals.

It is seen from Formula (5.46) that self-oscillations exist so long as the values of the external signals satisfy the condition

$$0 \leqslant \left| \frac{k_1 c^0}{k_1 + k_{0.c}} + f_s^0 \right| < c k_s \tag{5.47}$$

In this case the amplitude of the self-oscillation lies within the limits $A_s \geq A > 0$.

This is the first method of determining the steady-state values of the bias x^0, the amplitude A, and the frequency Ω of the self-oscillations in the presence of external signals.

Let us illustrate also the second method. According to the second method, as indicated in §5.1, one solves first Equation (5.33). From the first formula of (5.31) and from (5.33) we get

$$\frac{x^0}{A} = \sin \frac{\pi F^0}{2c} = \sin \frac{\pi M^0}{2c (k_1 + k_{0.c}) k_s},$$

or, with allowance for (5.34),

$$\frac{x^0}{A} = \sin \left[\frac{\pi}{2ck_s} \left(\frac{k_1 c^0}{k_1 + k_{0.c}} + f_s^0 \right) \right]. \tag{5.48}$$

To find the amplitude A contained in these formulas we use Equation (5.32). The characteristic equation for the latter is

$$T_1 T_2 p^3 + (T_1 + T_2) p^2 + (1 + T_1 k_2 k_{0.c} q) p + (k_1 + k_{0.c}) k_2 q = 0,$$

and the equations (5.9) therefore assume the form

$$\left.\begin{aligned} X &= (k_1 + k_{0.c}) k_2 q - (T_1 + T_2) \Omega^2 = 0, \\ Y &= (1 + T_1 k_2 k_{0.c} q) \Omega - T_1 T_2 \Omega^3 = 0, \end{aligned}\right\} \tag{5.49}$$

where, in accordance with (5.31) and (5.48):

$$q = \frac{4c}{\pi A} \cos\left[\frac{\pi}{2ck_2}\left(\frac{k_1 c^0}{k_1 + k_{o.c}} + f_2^0\right)\right]. \tag{5.50}$$

Eliminating the quantity q from Equation (5.49), we obtain the frequency of the self-oscillations

$$\Omega^2 = \frac{k_1 + k_{o.c}}{T_1 (T_2 k_1 - T_1 k_{o.c})}. \tag{5.51}$$

Substituting the obtained expressions for q and Ω^2 in the first equation of (5.49), we determine the amplitude of the self-oscillations

$$A = A_c \cos\left[\frac{\pi}{2ck_2}\left(\frac{k_1 c^0}{k_1 + k_{o.c}} + f_2^0\right)\right], \tag{5.52}$$

where the quantity

$$A_c = \frac{4ck_2 T_1 (T_2 k_1 - T_1 k_{o.c})}{\pi (T_1 + T_2)} \tag{5.53}$$

is the amplitude of the self-oscillations in the absence of external signals ($c^0 = 0$, $f_2^0 = 0$).

Substituting the expression obtained for the amplitude (5.52) into (5.48) we obtain finally the value of the bias

$$x^0 = \frac{A_c}{2} \sin\left[\frac{\pi}{ck_2}\left(\frac{k_1 c^0}{k_1 + k_{o.c}} + f_2^0\right)\right]. \tag{5.54}$$

As can be seen, the second method leads in this case to the same result by a much shorter route than the first, something of great importance to practical calculations (in principle both methods are equivalent). Apparently, the greater simplicity of the second method occurs in the majority of other problems, too.

In this second method, the bias function $\Phi(x^0)$ is not determined. Yet this function may be useful later on for other purposes. However, it can also be readily determined. The quantities x^0, A, and their ratio are expressed here in terms of the values of the external signals. On the other hand, the bias function $\Phi(x^0)$ should contain neither the external signals nor the amplitude A, which depends on these signals.

Substituting the bracketed notation from (5.54) into (5.48), we get

$$\frac{x^0}{A} = \sin\left(\frac{1}{2}\arcsin\frac{2x^0}{A_c}\right), \tag{5.55}$$

and substituting this into the first of Formulas (5.31), we obtain directly the sought bias function

$$F^0 = \Phi(x^0) = \frac{c}{\pi}\arcsin\frac{2x^0}{A_c}, \tag{5.56}$$

where A_s is expressed only in terms of the system parameters in accordance with (5.53).

It is important to note that the bias function $\Phi(x^0)$ depends neither on the number of external signals nor on the character of their variation (provided they are constant or slowly varying), as was seen quite clearly from the first method of solving the problem.

Thus, we have determined by two different methods the value of the bias x^0 of the self-oscillations at the input to the relay. Let us determine now the steady-state error at the output of the system, x_4. In as much as the output must duplicate the external signal $f_1(t)$, the error of this system is expressed, in accordance with Fig. 5.6 and the second equation of (5.23), by the quantity x_1, the steady-state solution for which must therefore be sought. We express the variable x_1 in terms of $\underline{x}$, which is already known. From the specified system equations (5.23) and (5.24) we obtain

$$(k_{o.c}T_1p + k_1 + k_{o.c})x_1 = (T_1p + 1)x + k_{o.c}(T_1p + 1)f_1(t).$$

Taking (5.28) and (5.30) into account, we rewrite this equation in the form

$$(k_{o.c}T_1p + k_1 + k_{o.c})x_1 = x^0 + (T_1p + 1)x^* + k_{o.c}T_1c^0 + k_{o.c}c^0t. \tag{5.57}$$

In accordance with the form of the right half of this linear equation, its steady state must be sought in the form

$$x_1 = x_1^0 + c_1t + x_1^*, \tag{5.58}$$

where x_1^0 and c_1 are constants and x_1^* is the periodic component.

Substituting this in (5.57), we obtain three equations for finding the indicated quantities

$$k_{o.c}T_1c_1+(k_1+k_{o.c})x_1^0=x^0+k_{o.c}T_1c^0, \tag{5.59}$$

$$(k_1+k_{o.c})c_1=k_{o.c}c^0, \tag{5.60}$$

$$(k_{o.c}T_1p+k_1+k_{o.c})x_1^*=(T_1p+1)x^*. \tag{5.61}$$

The second of these yields

$$c_1=\frac{k_{o.c}c^0}{k_1+k_{o.c}}. \tag{5.62}$$

We then obtain from (5.59)

$$x_1^0=\frac{1}{k_1+k_{o.c}}\left(x^0+\frac{k_1k_{o.c}T_1}{k_1+k_{o.c}}c^0\right), \tag{5.63}$$

where x^0 is determined by Formula (5.54) in terms of the external signals. Finally, from (5.61) we obtain the amplitude of the self-oscillations of the variable x_1:

$$A_1=A\sqrt{\frac{T_1^2\Omega^2+1}{k_{o.c}^2T_1^2\Omega^2+(k_1+k_{o.c})^2}}, \tag{5.64}$$

where A is determind by Formula (5.52) in terms of the external signal, and Ω is determined by Formula (5.51).

Thus, the present system contains all three error components (5.58) which depend on the magnitude of the external signals and on the system parameters. Most undesirable of these is the component c_1t, which increases in proportion to the time. The system must therefore be modified primarily so as to eliminate this error component, that is, to make $c_1 = 0$. For this purpose one would have to remove the supplementary feedback (Fig. 5.6), for when $k_{o.s} = 0$ we have in accord with (5.62) $c_1 = 0$. This, however, causes an appreciable increase in the amplitude of the self-oscillation (5.64), that is, in the periodic component of the error. Therefore, a more suitable measure would be to replace the proportional feedback $x_{o.s} = k_{o.s}x_4$ by derivative feedback $x_{o.s} = k_{o.s}px_4$.

Then the quantity $k_{o.s}$ in (5.57) is replaced by $k_{o.s}p$, that is,

$$(k_{o.c}T_1p^2+k_{o.c}p+k_1)x_1=x^0+(T_1p+1)x^*+k_{o.c}c^0. \tag{5.65}$$

We see that the component proportional to the time has disappeared from the right half of the equation, and consequently the steady-state solution for the error x_1, unlike (5.58), will be

$$x_1=x_1^0+x_1^*. \tag{5.66}$$

Besides, we obtain from (5.65)

$$x_1^0=\frac{1}{k_1}(x^0+k_{o.c}c^0), \tag{5.67}$$

$$A_1=A\sqrt{\frac{T_1^2\Omega^2+1}{k_{o.c}^2\Omega^2+(k_1-k_{o.c}T_1\Omega^2)^2}}, \tag{5.68}$$

and, of course, the formulas for x^0, A and Ω are also changed (they can be obtained by the same method). By choosing the system parameters we can make the amplitude A_1 of the error self-oscillations quite small.

§5.3. PASSAGE OF SLOWLY VARYING SIGNALS IN SELF-OSCILLATING SYSTEMS

In §5.1, in the analysis of asymmetrical self-oscillations with constant or slowly varying component characterizing the displacement of the center of the oscillations, we introduced the bias function

$$F^0=\Phi(x^0), \tag{5.69}$$

which must be substituted in the equations of the automatic system in lieu of the specified nonlinearity F(x, px) whenever constant or slowly varying components are evaluated.

Consequently, the bias function $\Phi(x^0)$ represents as it were a static characteristic (usually curved), which determines the connection between the output and input quantities of the given nonlinearity for constant or slowly varying signals in the self-oscillating system [187].

For any nonlinearity, including such stepwise nonlinearities as a relay characteristic or dry friction, or such nonlinearities as characteristics of clearances and backlash etc., the bias function $\Phi(x^0)$ can assume under definite conditions the form of a rather smooth curve.

This effect is called vibrational smoothing of the nonlinearity with the aid of self-oscillations, and the bias function $\Phi(x^0)$ can be called the smoothed characteristic.

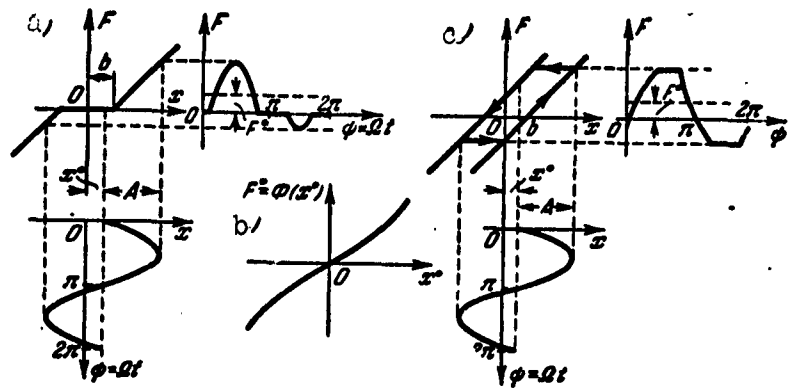


Fig. 5.8.

For example, for nonlinearities with backlash zones (Fig. 5.8a), and also with clearances (Fig. 5.8c) and with hysteresis loops, the signals $x < b$ will not be transmitted in the absence of self-oscillations ($F = 0$). On the other hand, in the presence of self-oscillations, a signal $x^0 < b$ (constant or slowly varying) is transmitted in the form of the dc component F^0. Consequently, for a slowly varying signal we obtain a smooth characteristic (bias function) $\Phi(x^0)$ without a backlash zone (Fig. 5.8b). For this purpose, it is necessary to have first of all a self-oscillation amplitude $A > b - x^0$ or $A > b + x^0$, and, secondly, a sufficiently high self-oscillation frequency, so as to satisfy the condition that the signal x^0 be slow (v.§5.1) and in order that the succeeding elements of the system practically block the self-oscillations. We see that in these examples vibration smoothing of the nonlinearities is advantageous for many practical applications (elimination of backlash zones and hysteresis loops).

In other cases, however, vibration smoothing of nonlinearities may also be harmful. Let us take, for example, a nonlinear characteristic with a saturation zone (limited-linear) as shown in Fig. 5.9. In this case, because the crests of the sine waves are cut off on one side,

the dc component F^0 will be smaller than the value of F itself corresponding to the linear initial portion. Therefore a constant or a slowly varying signal will pass through the given nonlinearity in the presence of self-oscillations as if the gain were smaller than without the self-oscillations, and under certain conditions this can affect adversly the operation of the automatic system as a whole.

Thus, assume that we have some nonlinear automatic system described by the equations (1.61):

$$\left.\begin{array}{c} D_{11}(p)\,x_1+\ldots+D_{1l}(p)\,x_l+\ldots+D_{1m}(p)\,x_m=f_1(t),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ D_{k1}(p)\,x_1+\ldots+D_{kl}(p)\,x_l+F(x_l,\,px_l)+\ldots+D_{km}(p)\,x_m=f_k(t),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ D_{m1}(p)\,x_1+\ldots+D_{ml}(p)\,x_l+\ldots+D_{mm}(p)\,x_m=f_m(t), \end{array}\right\} \qquad (5.70)$$

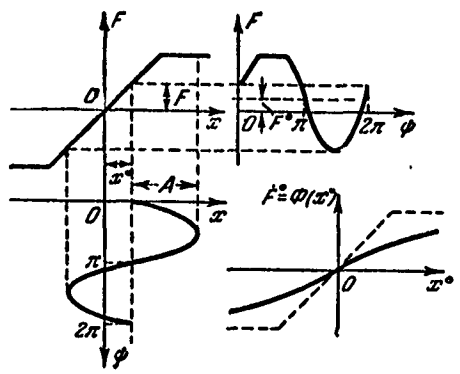


Fig. 5.9.

and which satisfy the conditions of Chapter 2. This system of equations, as shown in §1.2, is equivalent to a single equation (5.1) with one or several terms in the right side, depending on the number of functions $f_1(t)$ which do not equal zero.

If the conditions of vibration smoothing of the nonlinearity are satisfied (these conditions consist, as was already illustrated, of a definite limitation on the amplitude and of a definite requirement on the order of magnitude of the self-oscillation frequency), then all the slowly-varying processes in such a nonlinear system must be determined by solving the same system of equations (5.70), except that the specified nonlinearity $F(x_1, px_1)$ must be replaced by the bias function $\Phi(x_1^0)$, that is,

$$\left.\begin{array}{c} D_{11}(p)\,x_1^0+\ldots+D_{1l}(p)\,x_l^0+\ldots+D_{1m}(p)\,x_m^0=f_1(t),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ D_{k1}(p)\,x_1^0+\ldots+D_{kl}(p)\,x_l^0+\Phi(x_l^0)+\ldots+D_{km}(p)\,x_m^0=f_k(t),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ D_{m1}(p)\,x_1^0+\ldots+D_{ml}(p)\,x_l^0+\ldots+D_{mm}(p)\,x_m^0=f_m(t), \end{array}\right\} \qquad (5.71)$$

where the zero superscripts on all the variables denote that we refer here to the solution of the system without periodic self-oscillating (vibrational) component. The latter are determined separately by means of the equations:

$$\left.\begin{array}{l} D_{11}(p)\,x_1^* + \ldots + D_{1l}(p)\,x_l^* + \ldots + D_{1m}(p)\,x_m^* = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D_{k1}(p)\,x_1^* + \ldots + D_{kl}(p)\,x_l^* + \left(q + \frac{q'}{\Omega} p\right) x_l^* + \ldots + D_{km}(p)\,x_m^* = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D_{m1}(p)\,x_1^* + \ldots + D_{ml}(p)\,x_l^* + \ldots + D_{mm}(p)\,x_m^* = 0, \end{array}\right\} \qquad (5.72)$$

wherein the coefficients q and q' depend, in accordance with (5.4), not only on A and Ω (the latter is not always involved), but must also depend on x_1^0. Therefore the amplitudes A and the frequency Ω of the self-oscillations obtained in this fashion will vary slowly with variation of $x_1^0(t)$ (see, for example, Fig. 5.10 and the example in §1.7).

An analogous approach is used also in systems of other classes, which were considered earlier.

In many cases the calculation of A and Ω is essential only from the point of view of checking whether the conditions of vibration smoothing of the nonlinearity are fulfilled and whether such vibrations are acceptable in the given specific automatic system. Of prime importance for the operating quality of the automatic system in this case will be the slowly varying processes determined by Equations (5.71). Very important from the point of view of simplifying their determination are the following two circumstances.

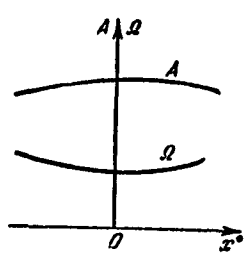


Fig. 5.10.

First, as can be seen from §5.1, the form of the bias function $\Phi(x^0)$ is dependent neither on the number and points of application of the external signals, nor on the character of their variation (provided they are slowly varying). It depends on the form of the nonlinearity and on the structure and parameters of the system. It is therefore possible to use any method of determining

$\Phi(x^0)$ under arbitrary particular simplifying assumptions with respect to the external signals. One can, for example, use the simpler second method of those described in §5.1, as illustrated by the example in §5.2, choosing any external signal that is constant in magnitude (such as f_3^0 in the example of §5.2).

Second, no matter what the specified nonlinearity F(x, px) may be (stepwise, loop-type, etc.), the bias function $\Phi(x^0)$ usually assumes the form of a smooth curve (Fig. 5.11). Therefore, unlike the initially specified nonlinearity, the bias function can be readily linearized by

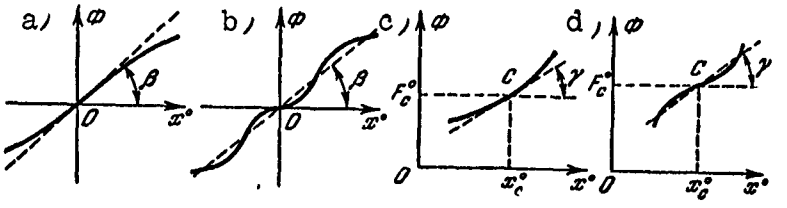


Fig. 5.11.

the usual method (using a tangent or secant at the origin 0 or at some other reference point C). Bearing this property in mind, the term "vibration smoothing" is sometimes replaced by the term "vibration linearization," although the latter term is less appropriate, for what is really done here is ordinary linearization of a vibrationally-smoothed nonlinearity.

Thus, for the case of Fig. 5.11a or b we can assume that in a fixed range

$$F^0 = k_н x^0, \tag{5.73}$$

where

$$k_н = \left(\frac{d\Phi}{dx^0}\right)_{x^0=0} \text{ or } k_н = \operatorname{tg}\beta,$$

and for the case of Fig. 5.11c or d we can assume

$$F^0 = F_c^0 + k_н(x^0 - x_c^0), \tag{5.74}$$

where

$$k_{\text{н}} = \left(\frac{d\Phi}{dx^0}\right)_{x^0 = x_c^0} \quad \text{or} \quad k_{\text{н}} = \operatorname{tg}\gamma.$$

The magnitude of the coefficient k_n depends generally speaking on the relationship of all the parameters of the system.

For example, for a system described by Equations (5.23) - (5.26) we have in accordance with (5.56) and (5.53)

$$k_{\text{н}} = \left(\frac{d\Phi}{dx^0}\right)_{x^0 = 0} = \frac{2c}{\pi A_c} = \frac{T_1 + T_2}{2k_2 T_1 (T_2 k_1 - T_1 k_{0.c})}. \tag{5,75}$$

Therefore, in order to calculate the slow processes in the given system we obtain in accordance with (5.23) - (5.25) and (5.73) the equations

$$\left.\begin{gathered} (T_1 p + 1)\, x_2^0 = k_1 x_1^0, \quad x_1^0 = f_1(t) - x_4^0, \\ x_3^0 = k_{\text{н}} x^0, \quad x^0 = x_{0.c}^0, \quad x_{0.c}^0 = k_{0.c} x_4^0, \\ (T_2 p + 1)\, p x_4^0 = k_2 x_3^0 + f_2(t) \end{gathered}\right\} \tag{5.76}$$

or the single equation (5.27), in which $\underline{x}$ should be replaced by x^0 and $F(x)$ by $k_n x^0$.

The definition of the gain k_n indicated for Formula (5.73) can be greatly simplified in the following fashion. In as much as the bias function $\Phi(x^0)$ is determined, in accordance with (5.11) and (5.5), from the expression $F^0(x^0, A, \Omega)$ in which the function $A(x^0)$ is substituted, the formula for the calculation of k_n can be represented in the form

$$k_{\text{н}} = \left(\frac{d\Phi}{dx^0}\right)_{x^0 = 0} = \left(\frac{\partial F^0}{\partial x^0} + \frac{\partial F^0}{\partial A}\frac{dA}{dx^0} + \frac{\partial F^0}{\partial \Omega}\frac{d\Omega}{dx^0}\right)_{x^0 = 0}.$$

In those cases when nonlinearities $F(x)$ with asymmetry are considered, the value of F^0 is independent of Ω and, furthermore, in accordance with (5.4):

$$\left(\frac{\partial F^0}{\partial A}\right)_{x^0 = 0} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial F}{\partial x}\right)_{x = A\sin\psi} \sin\psi\, d\psi = 0,$$

since the derivative under the integral sign will be an even function.

Hence for oddly symmetrical nonlinearities F(x), both single-valued and loop-type, the quantity k_n can be calculated from the formula

$$k_н = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0}$$

directly using expression (5.5), without determining the bias function $\Phi(x^0)$. In many problems this will appreciably simplify the solution.

This simplification does not apply to asymmetrical nonlinearities, nor does it apply to cases when k_n must be determined not by means of the tangent but by means of the secant (Fig. 5.11b), and to cases corresponding to Formula (5.74) and Fig. 5.11c and d.

Thus, substitution of (5.73) or (5.74) makes the equations (5.71) for the determination of the slow processes ordinary linear equations which can thus be readily solved. Examples are the equations (5.76).

If we express the equations of the nonlinear system in the form (5.1), then substitution of (5.73) in the case of Fig. 5.11a or b yields a linear equation for the slowly varying component in the form

$$[Q(p)+R(p)k_н]x^0=S(p)f(t), \tag{5.77}$$

and in the case of Fig. 5.11c or d with substitution of (5.74) we obtain

$$[Q(p)+R(p)k_н]\Delta x^0=S(p)\Delta f(t), \tag{5.78}$$

where

$$\Delta x^0=x^0-x_c^0, \quad \Delta f(t)=f(t)-f_c^0,$$

and the values of x_c^0 and f_c^0 are defined by the relations

$$Q(0)x_c^0+R(0)F_c^0=S(0)f_c^0, \quad F_c^0=\Phi(x_c^0).$$

We see that the principle of separating the equations for the oscillatory and the slowly varying components, introduced in §5.1, in which the nonlinear properties of the system are essentially retained, leads to results that are very important to the ease of practical calculations. The important fact is that the slowly varying signals pass

through the nonlinearity with different gains k_n than do the self-oscillations [q or q + (q'/Ω)p].

It is particularily important to use the property of vibration smoothing of nonlinearities with subsequent ordinary linearization in the calculation of complex automatic systems.

If, for example, the system for automatic control of an airplane operates in accordance with the diagram shown in Fig. 5.12, then the part of the system enclosed in the dashed box (relay amplifier, drive, and supplementary feedback) can be designed as a separate servomechanism by the method developed above, with allowance for the self-oscillations. The frequency of the latter can be made sufficiently large by suitable choice of the parameters of this part of the system or by introducing correcting devices, so as to make the amplitude of the self-oscillations of the variable x_2 at the output of this part of the system small. On the other hand, if the amplitude x_2 cannot be made small (in which case the rudder will oscillate), it is necessary to see to it that this frequency remains practically unfelt by the hull of the airplane as the latter moves about its center of gravity.

The design of the system will then be as follows. We seek the self-oscillations only in the part inside the dashed box (Fig. 5.12), as if it were a separate independent system, assuming $x_5(t)$ to be an arbitrary slowly varying external input signal and x_2 to be the output. For this simple system we find, by the method developed in §5.2, the bias function $\Phi(x^0)$ and also the frequency and amplitude of the self-oscillations as functions of the magnitude of the external signal. We choose the parameters of this part of the system in such a way as to fulfill the conditions of vibrational smoothing over the entire possible range of variation of the input x_5. At the same time we see to it that the frequency of the self-oscillations (which depends on the system parameters) lies beyond the range of frequencies of possible airplane vibrations (so

that it is practically unfelt by the hull of the airplane).

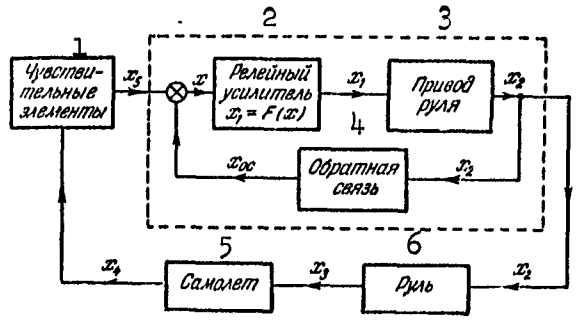


Fig. 5.12. 1) Sensitive elements; 2) relay amplifier; 3) rotor drive; 4) feedback; 5) airplane; 6) rudder.

Following such a design of the internal part of the system, we carry out ordinary linearization of the bias function $\Phi(x^0)$, that is, we replace it by a single straight line $F^0 = k_n x^0$ (where we can use any possible simplification in the determination of k_n). As a result we obtain linear equations for the slow processes in this part of the system. We add to these equations the equation of the entire remaining part of the system (in our case the airplane, the sensitive elements, and the rudder, see Fig. 5.12) and design the entire system as a whole, regarding it as a linear one, using the ordinary methods of automatic-control theory. In this case we no longer pay attention to the self-oscillations, which are localized in the internal loop of the system, which was previously designed. However, their influence is not ignored, since it was taken into account in the determination of the bias function $\Phi(x^0)$ and the coefficient k_n.

The principle developed enables us, first of all, to carry out the calculation of the self-oscillations by means of simpler equations (since only the internal part of the system is separated), and, secondly, to simplify appreciably the design of the entire system as a whole, reducing it to an investigation of ordinary linear equations (but with a coefficient k_n which depends on the self-oscillations, thus, on the parameters of the internal loop of the system). If it is necessary to take

into account variable coefficients and nonlinearities in the airplane itself, then the equations of the system as a whole will no longer be so simple. However, even in this case it is perfectly meaningful to make a preliminary separate design of the internal loop of the system, since the influence of the nonlinearities of the airplane itself will usually extend only over those processes which are slow compared with the self-oscillations of the internal loop.

It is known that the motion of the airplane itself, for example, pitching motion, can be divided into two components, a faster motion about the center of gravity (angular motion), and a slower motion of the center of gravity itself (motion along the trajectory). Both motions are slow compared with the self-oscillations of the internal loop of the control system. They can, however, in turn likewise be considered separately. Consequently, in this case, in addition to the usually employed spatial resolution of the airplane motion into separate channels (pitch, course, bank), the design of the system for each channel (for example, pitch) is broken down into three additional stages with respect to the slowness of the motion in time.

An analogous subdivision of the design, at least into two stages with regard to the degree of slowness in time, can be advantageous also for many other nonlinear automatic control, tracking, stabilization systems and the like. In complicated systems, such measures greatly simplify the entire investigation and are the only ones by which the system design becomes feasible at all. It must be borne in mind that in the proposed principle of breaking down the motions, an essential nonlinear interaction between them still remains.

The concept used here of the slowness of one motion relative to another in the same system is defined in perfect analogy with the concept of slowly varying external signals in §5.1. It is frequently convenient to express this concept in the form of a ratio of the possible

oscillation frequencies of the two motions. For example, the self-oscillation of the airplane's center of gravity along the trajectory are in a fully defined range of possible frequencies within the practical real region of values of parameters for this part of the system. Compared with these oscillations, the motion of the airplane about its center of gravity is slow. The latter circumstance can be expressed in different fashion, by stating that the possible frequencies of the oscillation of the airplane about its center of gravity are lower than the possible self-oscillation frequencies in the internal loop of the system (for example by a factor of 10 or more). The possible frequencies of the oscillation of the airplane's center of gravity along the trajectory are in turn still lower. This is the basis for the separate investigation of each motion. It is possible to plot for each a separate frequency characteristic in both linear and nonlinear form.

In particular, it is possible to determine for each of them separately the forced oscillations and the self-oscillations. Thus, by calculating the self-oscillations and determining the bias functions (the smoothed characteristic) $\Phi(x^0)$ in the internal loop of the system (dashed in Fig. 5.12), we can then determine the self-oscillations at the lower (slowly varying) frequencies in the entire system as a whole (Fig. 5.12), regarding the bias function $\Phi(x^0)$ (Fig. 5.11) as a new nonlinearity of this system (to which one can add also other nonlinearities of the airplane itself). The methods for determining the self-oscillations by harmonic linearization remain the same as before (see Chapter 1), if the system as a whole satisfies the conditions of Chapter 2 with respect to the slowly varying component. We then carry out harmonic linearization of the new nonlinearity $\Phi(x^0)$ and of other nonlinearities if they exist. The formulas for this repeated harmonic linearization will be:

$$\Phi(x^0) = q^0(A^0)\,x^0,$$
$$q^0 = \frac{4}{\pi A^0}\int_0^{\frac{\pi}{2}} \Phi(A^0 \sin\psi)\sin\psi\, d\psi, \quad \psi = \Omega^0 t, \tag{5.79}$$

where A^0 is the amplitude of the self-oscillations of the slowly varying components, which are sought in the form $x^0 = A^0 \sin \Omega^0 t$.

We can thus consider two-frequency self-oscillations with a large difference between the frequencies Ω and Ω^0.

For example, if, in accordance with (5.56), we have

$$\Phi(x^0) = \frac{c}{\pi}\arcsin\frac{2x^0}{A_c} \quad \text{for } x^0 \leqslant \frac{A_c}{2},$$

then

$$q^0 = \frac{4c}{\pi^2 A^0}\int_0^{\frac{\pi}{2}} \arcsin\left(\frac{2A^0}{A_c}\sin\psi\right)\sin\psi\, d\psi \quad \text{for} \quad A^0 \leqslant \frac{A_c}{2}.$$

Introducing the notation

$$k = \frac{2A^0}{A_c},$$

we make the following simple transformations

$$\int_0^{\psi} \arcsin(k\sin\psi)\sin\psi\, d\psi = \cos\psi\arcsin(k\sin\psi) + \int_0^{\psi}\frac{k\cos^2\psi\, d\psi}{\sqrt{1+k^2\sin^2\psi}},$$
$$\int_0^{\psi}\frac{k\cos^2\psi\, d\psi}{\sqrt{1-k^2\sin^2\psi}} = \frac{1}{k}\int_0^{\psi}\frac{k^2-1+1-k^2\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}d\psi =$$
$$= \frac{k^2-1}{k}\int_0^{\psi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} + \frac{1}{k}\int_0^{\psi}\sqrt{1-k^2\sin^2\psi}\, d\psi =$$
$$= \frac{k^2-1}{k}F_{\text{II}}(\psi, k) + \frac{1}{k}E_{\text{II}}(\psi, k),$$

where $F_1(\psi, k)$ and $E_1(\psi, k)$ are elliptic integrals of the first and second kind respectively,* for which detailed numerical tables are available, so that their practical use is just as simple as the use of ordinary trigonometric functions.

As a result we obtain

$$q^0 = \frac{4c}{\pi^2 A^0}\left[\left(\frac{2A^0}{A_c} - \frac{A_c}{2A^0}\right)K\left(\frac{2A^0}{A_c}\right) + \frac{A_c}{2A^0}E\left(\frac{2A^0}{A_c}\right)\right], \quad \left(A^0 \leqslant \frac{A_c}{2}\right), \tag{5.80}$$

where K and E are the complete elliptic integrals of the first and second kind, the values of which are listed usually in mathematical tables as functions of the parameter $a = \arcsin k = \arcsin 2A^0/A_c$, for example

a	0	10°	20°	30°	40°	50°	60°	70°	80°	90°
K	1,57	1,58	1,62	1,69	1,79	1,94	2,16	2,50	3,15	∞
E	1,57	1,56	1,52	1,47	1,39	1,31	1,21	1,12	1,04	1,00

In other cases of repeated harmonic linearization, when the bias function $\Phi(x^0)$ has a still more complicated form, we can use the harmonic method described is §3.8 to determine $q^0(A^0)$ from a specified plot of $\Phi(x^0)$.

§5.4. DEPENDENCE OF THE STABILITY OF A NONLINEAR SYSTEM ON THE EXTERNAL SIGNAL

The stability of a nonlinear system in which self-oscillations may occur can be understood differently. The first concept of stability of a nonlinear system is connected with the absence of self-oscillations (stability of the equilibrium state for all initial conditions, see §2.7). The second concept of stability of a self-oscillating system is connected with the assumption that the self-oscillations have a sufficiently small amplitude, to which all the transients tend no matter what the initial conditions (practical stability of self-oscillating systems, see §2.9). One can speak also of stability of the equilibrium state in a limited region of initial conditions (see §2.9), something that has a somewhat lesser practical value.

We can now introduce still another concept of great practical importance, that of the stability of a nonlinear system operating in the vibration mode, namely stability with respect to the slowly varying component. By this we mean the stability of the equilibrium state of a system, described, for example, by Equations (5.71) or (5.6), that

is, a system in which in place of the initially specified nonlinearity (see (5.7) or (5.1)) we substitute the bias function $F = \Phi(x^0)$. Such a stability denotes the attenuation of the transients with respect to the slowly varying components. The stability of the system with respect to the slowly varying component (that is, with respect to the main signal of the system) must be investigated in addition to the stability of the vibration mode (self-oscillation) itself.

The concept of stability is particularly important in practice for the case when the system is analyzed separately, as was indicated for example in Fig. 5.12. There we calculated the stability of the self-oscillation mode in one demarcated internal part of the system (which was enclosed in a dashed box in Fig. 5.12), and in the entire system as a whole only the slow processes were considered, from which one can consequently judge the stability of the equilibrium state of the entire system as a whole* (it is obvious that the system as a whole may be stable or unstable depending on the stability of the previously calculated internal loop).

The stability of the nonlinear system with respect to the slowly varying component can be investigated by any of the available methods, depending on the type of the equations describing the slow processes. If, for example, all of the operator-polynomial coefficients in the system (5.71) or (5.6) are constant an ordinary linearization of the smooth nonlinearity $F^0 = \Phi(x^0)$ is carried out in the form (5.73) or (5.74); then the stability of the nonlinear system with respect to the slowly varying component will be investigated in analogy with the stability of an ordinary linear system (using the Hurwitz, Mikhaylov, or Nyquist criteria in their ordinary formulation). If the system contains besides time-varying coefficients (which vary little during one cycle of the vibrations), it is necessary to make use of the theory of linear systems with variable coefficients.

Finally, if the system of equations obtained for the slow processes contains essential nonlinearities, the corresponding nonlinear methods must be used. If such a system satisfies the conditions indicated in Chapter 2, then its stability can be again investigated by the method of harmonic linearization (§2.7). In this case the harmonic linearization at the low frequencies that are inherent to the slow processes are now applied to the new nonlinearities, which remain in the system of equations that result for the slowly varying component. In particular, the bias function (smoothed nonlinearity) itself may be this new nonlinearity, as for example $\Phi(x^0)$ in the system (5.71) or (5.6), which has the form of Fig. 5.11. Physically this is connected with the fact that the system as a whole (Fig. 5.12), which has a nonlinearity $F^0 = \Phi(x^0)$ in the slowly varying component, may become unstable by going into self-oscillation, but now at a different frequency, which is possible in this case for the hull of the airplane. This frequency is considerably lower than the vibration self-oscillation mode of the reàays in the internal loop of the system.

As was established earlier in the presence of self-oscillations the bias function (smoothed characteristic) $\Phi(x^0)$ is independent of the character of variation of the external signals, provided the signals are slowly varying, but depends only on the type of nonlinearity and on the ratio of the system parameters. Consequently, the stability of the system relative to the slowly varying component will likewise not depend on the external signals in the presence of self-oscillations.

But a vibrational operating mode of the system can be produced not only with the aid of self-oscillations but also with the aid of forced oscillations of the system induced by an external periodic signal. Then not only the amplitude and frequency of the vibrations themselves, but also the smoothed nonlinear characteristic will depend on this external signal and consequently, the stability of the system with

respect to the slowly varying component may depend quite essentially on this external signal.*

On the other hand, the stability of nonlinear systems in the first and second meanings indicated at the start of the present section may appreciably depend on the magnitude of the constant external signal (and in the case of astatic systems also on the rate of change of the external signal), since this signal governs the amplitude of the self-oscillations and the conditions under which the self-oscillations occur.

The stability of the system in its first meaning (§2.7), that is, the stability of the equilibrium state of the system outside of the self-oscillation region in the presence of a constant external signal (or in the case when the rate of change of the signal is constant, for astatic systems), will be determined by the methods developed previously in §2.7, using Equation (2.206). However, the harmonic linearization coefficients q and q' contained in this equation will now depend on the dc component x^0 (see (5.4) and (5.5)). Consequently, the limits of the possible values of q and q' for each nonlinearity will generally speaking depend on the value of x^0. Consequently the stability limit of the nonlinear system, that is, the boundary separating the stable equilibrium region from the region of periodic solutions, will also depend on the value of x^0.

However, the value of x^0 can be determined as a function of the magnitude or rate of change of the external signal (see §5.2). Therefore, after determining by means of Equation (5.7) and the methods of §2.7 the boundary of the stable-equilibrium region of the system for each specified value of x^0, we can determine by the same token the stable-equilibrium boundary of the nonlinear system for each specified value of a constant external signal (or for each specified value of the constant rate of change of the external signal in the case of an astatic system). We thus obtain the pattern of the displacement of the stability

boundary in the space of the nonlinear system parameters (the change of the outline of its stability region) as a function of the magnitude of the external signal.

There exist, however, exceptions to this rule. The stability limit does not depend on the external signal in every nonlinear system. For example, it obvious that in the case of a single-valued nonlinearity, when $q' = 0$, the stability limit will not depend on the external signal in those particular cases when the possible values of the coefficient q lie in the interval $0 \leq q \leq \infty$. This occurs, for example, for a system with an ideal relay, considered in §5.2. There, in accordance with (5.50), we shall have the same plot of $q(A)$ as in Fig. 2.31a, but with a different scale along the axis of the ordinates, which is equal to the cosine of the constant quantity characterizing the external signal.

Equally independent of the magnitude of the external signal will be all the sufficient stability boundaries (for any single-valued odd nonlinearity), determined during the first stage of the investigation (§2.7), when it is assumed that $0 \leq q \leq \infty$. On the other hand, the necessary conditions obtained during the second stage of the investigation for all nonlinearities with a limited interval of possible values of q (all except a in Fig. 2.31) can depend essentially on the magnitude (or rate of change) of the external signal. In §6.8 we shall give an example of determining such a dependence.

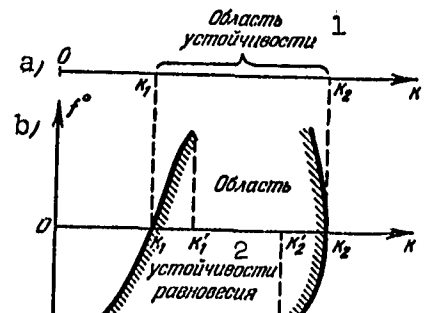


Fig. 5.13. 1) Stability region; 2) stable-equilibrium region.

Consequently, generally speaking, in the design of real automatic systems it is necessary to determine the displacement of the stability region as the external signal is varied, and to choose the system parameters to take this displacement into account. Otherwise the automatic control

or stabilization system, which is made stable in the absence of an external signal, may turn out to be unstable at a definite value of the signal. The stability limit of the servomechanism may shift somewhat in the parameter space with changes in tracking speed, etc.

In such cases one or several of the external signals (or of their rates) is best included among the coordinates of the space in which the stability region is plotted. For example, if the stability region with respect to some parameter $\underline{k}$ (for example the gain, Fig. 5.13a) has been defined in some system and the displacement of the stability-region boundaries as a function of the magnitude of the constant external signal f^0 has been determined, it is possible to plot the stability region in a plane with coordinates $\underline{k}$ and f^0 (Fig. 5.13b). Then, in order to guarantee the stability of the system, it is necessary to specify the value of $\underline{k}$ not within the interval $k_1 < k < k_2$, which would be obtained without account of the external signal, but in a narrower interval $k_1' < k < k_2'$ with account of the possible change in the external signal f^0 within the specified limits.

We have used here the first meaning of system stability, namely stability of the equilibrium state under all initial conditions (§2.7).

The same situation is obtained when we investigate the practical stability of a self-oscillating system in the sense of the permissible amplitude of the self-oscillations (§2.9). Here the amplitude depends on the magnitude or on the rate of change of the external signal. Consequently, the position of the line corresponding to a definite amplitude A_{dop} in the parameter space (see Fig. 2.47), which is regarded in the present case as the practical stability limit of the self-oscillating system, will also depend on the magnitude of the external signal (or on its rate of change in the case of an astatic system). Such a limit will depend on the magnitude of the external signal for all self-oscillating systems, including a system with an ideal relay.

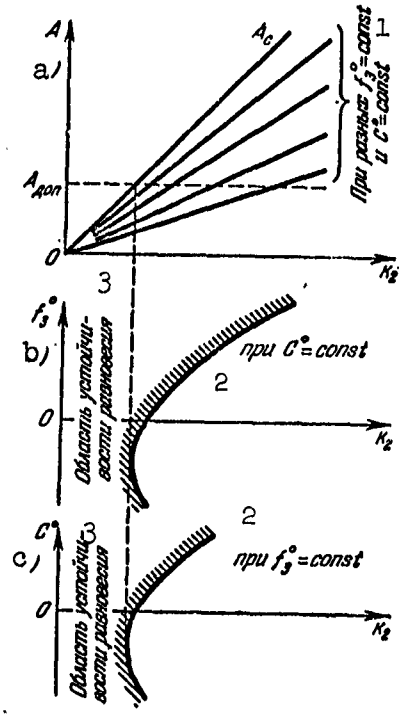


Fig. 5.14. 1) For different f_3^0 = const and c^0 = const; 2) for c^0 = const; 3) stable equilibrium region.

Thus, in the system example considered in §5.2, the amplitude of the self-oscillations in the absence of an external signal is determined by formula (5.53); this is represented as a function of the parameter k_2 by the straight line A_s (Fig. 5.14a). Further, in accordance with (5.52), this quantity must be multiplied by the cosine of a constant quantity that depends on the external signals, something represented by the bundle of rays in Fig. 5.14a. By specifying the value of A_{dop} (Fig. 5.14a) in accord with (5.52) and (5.53), we obtain the following expression for the practical stability limit of the given self-oscillating system:

$$k_2 = \frac{\pi (T_1 + T_2) A_{доп}}{4cT_1 (T_2 k_1 - T_1 k_{o.c})} \sec\left[\frac{\pi}{2ck_2}\left(\frac{k_1 c^0}{k_1 + k_{o.c}} + f_3^0\right)\right]$$

(for $T_2 k_1 > T_1 k_{o.s}$), as shown in Figs. 5.14b and c respectively as a function of the constant disturbance signal f_3^0 (load) and the rate of change c^0 of the set-point signal f_1, which is duplicated by the system (in different tracking modes at constant speed).

Such a dependence of the system stability on the external signal is a characteristic of nonlinear systems only. In linear systems it does not exist at all in this sense. It is most important to take account of this property of nonlinear systems, as shown by experience, in the practice of constructing various types of automatic control systems.

§5.5. SELF-OSCILLATIONS OF SYSTEMS WITH PARAMETERS THAT VARY SLOWLY IN TIME

The application of this same principle of separating processes in accordance with the degree of their slowness in time makes it also possible to investigate by the method of harmonic linearization the self-oscillations of certain nonlinear systems with slowly varying parameters (time-varying gain of the regulator, variable mass of the object or of the moment of inertia, etc.).

In the present section we consider the determination of self-oscillation in nonlinear systems that satisfy the conditions of Chapter 2, but with parameters that vary slowly in time in the same sense as defined in §5.1. The variation of the system parameters with time can be both monotonic and nonmonotonic, but with frequencies considerably lower than the possible frequency of the investigated self-oscillations.

Assume that in the equations of the nonlinear system

$$Q(p)x+R(p)F(x,\ px)=0$$

or

$$Q(p)x+R(p)F(x,\ px)=S(p)f(t)$$

(or else in the expanded form (5.70)) some parameters contained in the coefficients of the operator polynomial Q(p), R(p), or $D_{ij}(p)$ vary slowly in time. Then, assuming them to be constant over the period of the investigated self-oscillations, we can determine, by the methods developed above (Chapters 2 and 5), the amplitude A and the frequency Ω of the self-oscillations, and also the bias x^0 as functions of these slowly varying parameters (and of the external signal if it is present), meaning that we can determine their time variation (see Fig. 5.15, where m denotes a certain variable parameter of the system, the slow variation of which in time is specified).

In the case when the conditions for vibration smoothing of the nonlinearity are satisfied (§5.4), we obtain for the slow processes

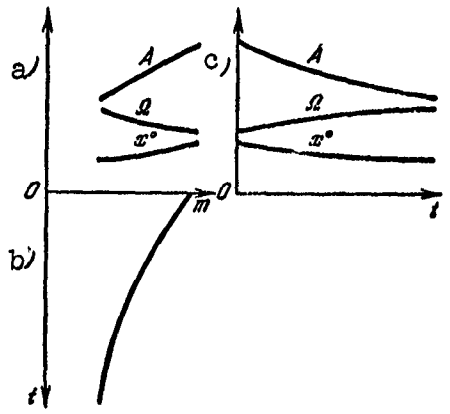


Fig. 5.15.

equations with variable coefficients, of the type (5.71) or (5.6). By linearizing the bias function $F^0 = \Phi(x^0)$ by the usual method, using formula (5.73) or (5.74), we can reduce the problem of investigating the entire system as a whole with respect to the slowly varying component (stability and dynamics of the motion) to a linear problem dealing with the corresponding variable coefficients.

It must be borne in mind here that inasmuch as the coefficient k_n in Formulas (5.73) and (5.74) generally depends on the relation of all the system parameters; then if the parameters of the system are variable this coefficient will in general also be variable. For example, for the system shown in Fig. 5.6 we have derived previously an expression for the coefficient k_n in the form (5.75). Let us assume that in the given system the parameter k_1 is variable (slowly varying in time). Then the coefficient k_n, which is obtained as the result of ordinary linearization of the bias function $\Phi(x^0)$, will also be variable.

Other approximate methods of investigating systems with time varying parameters are given in the book by Yu.A. Mitropol'ski [170] and also in §8 of the book [181].

The following sections of the present chapter are devoted to the calculation of the coefficients of harmonic linearization by means of Formulas (5.4) for various types of nonlinearities.

§5.6. HARMONIC LINEARIZATION OF RELAY CHARACTERISTICS UNDER ASYMMETRICAL OSCILLATIONS

In Chapter 3 we carried out a harmonic linearization of the nonlinearities for the case of symmetrical oscillations in the system.

Now, in order to carry out harmonic linearization of a nonlinear function of F(x, px) under asymmetrical oscillations, we shall assume that the solution for the input quantity x of the nonlinear element is sought in the form

$$x = x^0 + A \sin \psi, \quad \psi = \Omega t.$$

The nonlinear function F(x, px) will in this case be a periodic function of the argument ψ with a steady component F^0.

If we confine ourselves to the calculation of the first harmonic of the Fourier expansion of the periodic function $F(x^0 + A \sin \psi)$, then the nonlinear function F(x, px) will be replaced in the method of harmonic linearization by the following relation:

$$F(x, px) = F^0(A, \Omega, x^0) + \left[q(A, \Omega, x^0) + \frac{q'(A, \Omega, x^0)}{\Omega} p\right](x - x^0); \tag{5.81}$$

the steady component F^0 and the coefficients q and q' are calculated by the formulas (5.4):

$$F^0(A, \Omega, x^0) = \frac{1}{2\pi} \int_0^{2\pi} F(x^0 + A \sin \psi, A\Omega \cos \psi)\, d\psi, \tag{5.82}$$

$$q(A, \Omega, x^0) = \frac{1}{\pi A} \int_0^{2\pi} F(x^0 + A \sin \psi, A\Omega \cos \psi) \sin \psi\, d\psi, \tag{5.83}$$

$$q'(A, \Omega, x^0) = \frac{1}{\pi A} \int_0^{2\pi} F(x^0 + A \sin \psi, A\Omega \cos \psi) \cos \psi\, d\psi. \tag{5.84}$$

If the nonlinear function is independent of the rate of change of the input quantity, then the formula (5.81) for harmonic linearization assumes the form

$$F(x) = F^0(A, x^0) + \left[q(A, x^0) + \frac{q'(A, x^0)}{\Omega} p\right](x - x^0), \tag{5.85}$$

where

$$F^0(A,\ x^0)=\frac{1}{2\pi}\int_0^{2\pi}F(x^0+A\sin\psi)\,d\psi, \tag{5.86}$$

$$q(A,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\sin\psi\,d\psi, \tag{5.87}$$

$$q'(A,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\cos\psi\,d\psi. \tag{5.88}$$

We can linearize analogously the nonlinear functions that depend on the acceleration of the input, as well as nonlinearities which contain as arguments of the nonlinear function both the input and the output quantities (linearities of the second class), as well as for other cases. This will be illustrated by means of specific examples in the next chapter.

Let us calculate the values of F^0, q, and q' for frequently encountered nonlinear functions, so as to simplify the investigation of nonlinear systems by the method of harmonic linearization under asymmetrical oscillations.

Relay characteristic of the general type. A relay characteristic of the general type for asymmetrical oscillations of the input x is shown in Fig. 5.16a. Here m is any fractional number in the interval $-1 \leq m \leq 1$. Let us find the constant component and the coefficients of harmonic linearization subject to the condition $A \geq b + |x^0|$. Calculating the values of F^0, q, and q' in accordance with the form of the function $F(x^0 + A\sin\psi)$, which is represented in Fig. 5.16b, we obtain

$$F^0(A,\ x^0)=\frac{1}{2\pi}\int_0^{2\pi}F(x^0+A\sin\psi)\,d\psi=\frac{c}{2\pi}\left(\int_{\psi_1}^{\pi-\psi_2}d\psi-\int_{\pi+\psi_3}^{2\pi-\psi_4}d\psi\right)=$$

$$=\frac{c}{2\pi}\left(\psi\Big|_{\psi_1}^{\pi-\psi_2}-\psi\Big|_{\pi+\psi_3}^{2\pi-\psi_4}\right)=\frac{c}{2\pi}(\psi_3-\psi_1+\psi_4-\psi_2).$$

With allowance for the values of the corresponding angles we have

$$F^0(A, x^0) = \frac{c}{2\pi}\left(\arcsin\frac{b+x^0}{A} - \arcsin\frac{b-x^0}{A} + \right.$$
$$\left. + \arcsin\frac{mb+x^0}{A} - \arcsin\frac{mb-x^0}{A}\right) \quad \text{for } A \geq b + |x^0|. \tag{5.89}$$

We furthermore obtain for $q(A, x^0)$

$$q(A, x^0) = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi)\sin\psi\, d\psi =$$
$$= \frac{c}{\pi A}\left(\int_{\psi_1}^{\pi-\psi_2}\sin\psi\, d\psi - \int_{\pi+\psi_2}^{2\pi-\psi_1}\sin\psi\, d\psi\right) =$$
$$= \frac{c}{\pi A}\left(\cos\psi\Big|_{\pi+\psi_2}^{2\pi-\psi_1} - \cos\psi\Big|_{\psi_1}^{\pi-\psi_2}\right) =$$
$$= \frac{c}{\pi A}(\cos\psi_1 + \cos\psi_2 + \cos\psi_2 + \cos\psi_1).$$

Taking into account the values of the angles $\psi_1, \ldots, \psi_4$, we obtain:

$$q(A, x^0) = \frac{c}{\pi A}\left[\sqrt{1-\frac{(b+x^0)^2}{A^2}} + \sqrt{1-\frac{(b-x^0)^2}{A^2}} + \right.$$
$$\left. + \sqrt{1-\frac{(mb+x^0)^2}{A^2}} + \sqrt{1-\frac{(mb-x^0)^2}{A^2}}\right] \text{for } A \geq b + |x^0|. \tag{5.90}$$

Finally, we have for $q'(A, x^0)$:

$$q'(A, x^0) = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi)\cos\psi\, d\psi =$$
$$= \frac{c}{\pi A}\left(\int_{\psi_1}^{\pi-\psi_2}\cos\psi\, d\psi - \int_{\pi+\psi_2}^{2\pi-\psi_1}\cos\psi\, d\psi\right) = \sin\psi_2 - \sin\psi_1 + \sin\psi_1 - \sin\psi_2.$$

Allowing for the corresponding sines we obtain

$$q'(A) = -\frac{2cb}{\pi A^2}(1-m) \text{ for } A \geq b + |x^0|. \tag{5.91}$$

Relay characteristic with shifted hysteresis loop. Steady component and the coefficients of harmonic linearization for a relay characteristic with shifted hysteresis loop (Fig. 3.2a) are obtained for the coefficient of the relay characteristic of the general type. Taking m in (5.89), (5.90) and (5.91) with the minus sign, we obtain

$$F^0(A, x^0) = \frac{c}{2\pi}\left(\arcsin\frac{b+x^0}{A} - \arcsin\frac{b-x^0}{A} - \arcsin\frac{mb-x^0}{A} + \right.$$
$$\left. + \arcsin\frac{mb+x^0}{A}\right) \text{ for } A \geq b + |x^0|, \tag{5.92}$$

$$q(A, x^0) = \frac{c}{\pi A}\left(\sqrt{1-\frac{(b+x^0)^2}{A^2}} + \sqrt{1-\frac{(b-x^0)^2}{A^2}} + \right.$$
$$\left. + \sqrt{1-\frac{(mb-x^0)^2}{A^2}} + \sqrt{1-\frac{(mb+x^0)^2}{A^2}}\right) \text{for } A \geq b + |x^0|, \tag{5.93}$$

$$q'(A) = -\frac{2cb}{\pi A^2}(1+m) \text{ for } A \geq b + |x^0|. \tag{5.94}$$

Relay characteristic with hysteresis loop. Recognizing that a relay characteristic with a hysteresis loop (Fig. 5.16c) is a particular case of the relay characteristic of the general type with m = −1, we obtain from (5.89), (5.90), and (5.91):

$$F^0(A, x^0) = \frac{c}{\pi}\left(\arcsin\frac{b+x^0}{A} - \arcsin\frac{b-x^0}{A}\right) \text{for } A \geq b + |x^0|, \tag{5.95}$$

$$q(A, x^0) = \frac{2c}{\pi A}\left(\sqrt{1-\frac{(b-x^0)^2}{A^2}} + \sqrt{1-\frac{(b+x^0)^2}{A^2}}\right) \text{for } A \geq b + |x^0|, \tag{5.96}$$

$$q'(A) = -\frac{4cb}{\pi A^2} \text{ for } A \geq b + |x^0|. \tag{5.97}$$

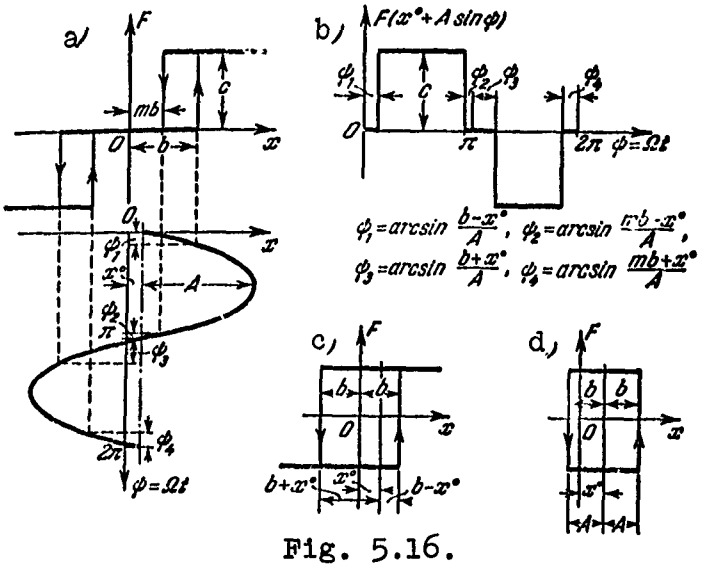


Fig. 5.16.

Relay characteristic with hysteresis loop of variable width. For a relay characteristic with a hysteresis loop of variable width (Fig. 5.16d), it is necessary to bear in mind that it shifts together with the center of the oscillations, and that the oscillation amplitude is equal to half the width of the loop. To get the values of the steady component and of the harmonic linearization coefficients of a nonlinear characteristic of the type shown in Fig. 5.16d by using the formulas for the preceding characteristic, we must bear in mind that when the center of oscillations is shifted (Fig. 5.16c), the segments of the characteristic $b + x^0$ and $b - x^0$ are converted into segments equal to the amplitude A (Fig. 5.16d). Replacing $b + x^0$ and $b - x^0$ in formulas (5.95) and (5.96) by the value of A and replacing in Formula (5.97) 2b by the quantity 2A, we obtain

$$F^0(A, x^0)=0, \quad q(A, x^0)=0, \quad q'(A)=-\frac{4c}{\pi A}. \tag{5.98}$$

As can be seen from (5.98), if the input to the nonlinear element has a sinusoidal variation we obtain for the given characteristic an output that varies as the negative cosine. Consequently, the characteristic given here produces in oscillatory processes an effect equivalent to that of an integrating element, giving rise to a phase lag of the output quantity relative to the input by an angle $B = -\pi/2$.

Relay characteristic with backlash zone. A relay characteristic with a backlash zone (Fig. 3.2d) must be regarded as a particular case of the general-type relay characteristic with m = 1. We then obtain from (5.89) – (5.91) the values of the steady component and the harmonic-linearization coefficients:

$$F^0(A, x^0)=\frac{c}{\pi}\left(\arcsin\frac{b+x^0}{A}-\arcsin\frac{b-x^0}{A}\right) \text{ for } A\geq b+|x^0|, \tag{5.99}$$

$$\left.\begin{aligned} q(A, x^0) &= \frac{2c}{\pi A}\left(\sqrt{1-\frac{(b+x^0)^2}{A^2}}+\sqrt{1-\frac{(b-x^0)^2}{A^2}}\right) \\ &\text{for} \quad A \geqslant b+|x^0|, \\ q'(A) &= 0. \end{aligned}\right\} \tag{5.100}$$

Ideal relay characteristic. For an ideal relay characteristic (Fig. 3.2e) we obtain by putting b = 0 in (5.99) and (5.100):

$$F^0(A, x^0) = \frac{2c}{\pi} \arcsin \frac{x^0}{A} \quad \text{for} \quad A \geqslant |x^0|, \tag{5.101}$$

$$q(A, x^0) = \frac{4c}{\pi A}\sqrt{1-\frac{(x^0)^2}{A^2}} \quad \text{for } A \geqslant |x^0|, \quad q'(A) = 0. \tag{5.102}$$

Asymmetrical relay characteristics. So far we have considered symmetrical relay characteristics. These were obtained as the result of operation of a relay element, used to turn on or switch over the element immediately following it. Sometimes a relay element is used in a system to turn on or off an element following it in the system. In this case all the previously given relay characteristics will be asymmetrical with respect to the origin. Let us calculate the coefficients of harmonic linearization for asymmetrical relay characteristics.

Asymmetrical relay characteristic of the general type. The asymmetrical relay characteristic of the general type for a harmonic variation of the input x with a biased center of oscillation is shown in Fig. 5.17a. Such will be the variation of the voltage on a load controlled by a polarized relay by means of the control current, if the relay applies full voltage to the load when it operates and turns off the voltage when it drops out.

Calculating the dc component and the harmonic linearization coefficients by means of formulas (5.86) and (5.88), we obtain

$$F^0(A,\ x^0)=\frac{1}{2\pi}\int_0^{2\pi}F(x^0+A\sin\psi)\,d\psi=\frac{c}{2\pi}\int_{\psi_1}^{\pi-\psi_2}d\psi=\frac{c}{2\pi}\psi\Big|_{\psi_1}^{\pi-\psi_2}=$$
$$=\frac{c}{2\pi}(\pi-\psi_2-\psi_1).$$

After substituting the values of the corresponding angles we have*

$$F^0(A,\ x^0)=\frac{c}{2}-\frac{c}{2\pi}\left(\arcsin\frac{b-x^0}{A}+\arcsin\frac{mb-x^0}{A}\right)\qquad(5.103)$$
$$\text{for } A\geqslant|b-x^0|,\quad A\geqslant|x^0-mb|.$$

Furthermore

$$q(A,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\sin\psi\,d\psi=\frac{c}{\pi A}\int_{\psi_1}^{\pi-\psi_2}\sin\psi\,d\psi=$$
$$=-\frac{c}{\pi A}\cos\psi\Big|_{\psi_1}^{\pi-\psi_2}=\frac{c}{\pi A}(\cos\psi_1+\cos\psi_2),$$

or, with account of the values of the angles ψ_1 and ψ_2

$$q(A,\ x^0)=\frac{c}{\pi A}\left(\sqrt{1-\frac{(b-x^0)^2}{A^2}}+\sqrt{1-\frac{(mb-x^0)^2}{A^2}}\right)\qquad(5.104)$$
$$\text{for } A\geqslant|b-x^0|,\quad A\geqslant|x^0-mb|.$$

Finally,

$$q'(A,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\cos\psi\,d\psi=\frac{c}{\pi A}\int_{\psi_1}^{\pi-\psi_2}\cos\psi\,d\psi=$$
$$=\frac{c}{\pi A}\sin\psi\Big|_{\psi_1}^{\pi-\psi_2}=\frac{c}{\pi A}(\sin\psi_2-\sin\psi_1),$$

or, with account of the values of the corresponding sines

$$q'(A)=-\frac{cb}{\pi A^2}(1-m)\qquad(5.105)$$
$$\text{for } A\geqslant|b-x^0|,\quad A\geqslant|x^0-mb|.$$

Asymmetrical relay characteristic with shifted hysteresis loop (Fig. 5.17c). Such a characteristic describes, for example, the variation of the voltage on a load controlled by a polarized three-position relay, operating so as to turn the load on and off.

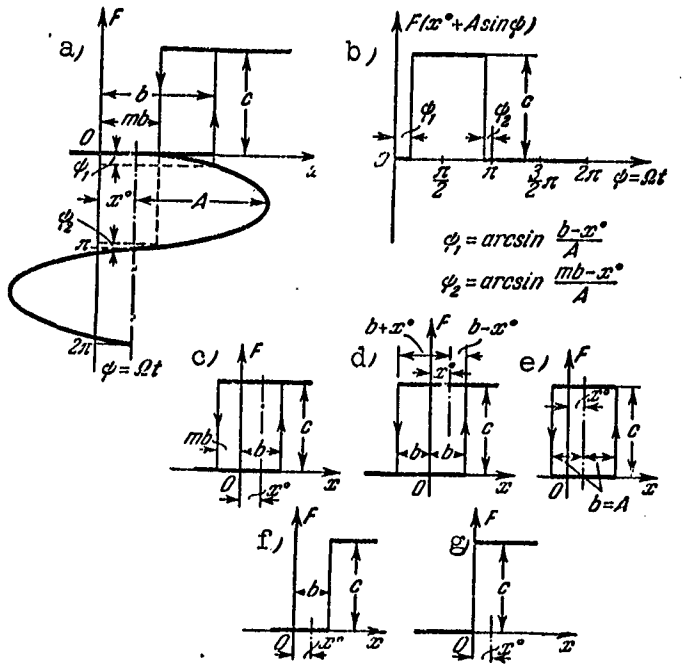


Fig. 5.17

To determine the dc component and the harmonic linearization coefficient of this characteristic, it is necessary to use a minus sign in front of m in Formulas (5.103) and (5.105). We then obtain

$$F^0(A,\ x^0)=\frac{c}{2}-\frac{c}{2\pi}\left(\arcsin\frac{b-x^0}{A}-\arcsin\frac{mb+x^0}{A}\right) \qquad (5.106)$$

for $A\geqslant|b-x^0|,\quad A\geqslant|x^0+mb|,$

$$q(A,\ x^0)=\frac{c}{\pi A}\left(\sqrt{1-\frac{(b-x^0)^2}{A^2}}+\sqrt{1-\frac{(mb+x^0)^2}{A^2}}\right) \qquad (5.107)$$

for $A\geqslant|b-x^0|,\quad A\geqslant|x^0+mb|,$

$$q'(A)=-\frac{cb}{\pi A^2}(1+m) \qquad (5.108)$$

for $A\geqslant|b-x^0|,\quad A\geqslant|x^0+mb|.$

Asymmetrical relay characteristic with hysteresis loop. A relay characteristic with hysteresis loop (Fig. 5.17d) will represent, for example, the dependence of the voltage on a load controlled by a two-position polarized relay on the control current if the relay is used for on-and-off operations.

Setting in Formulas (5.103) - (5.105) the value m = -1, we get

expressions for the dc component and for the harmonic linearization coefficients:

$$F^0(A, x^0) = \frac{c}{2} + \frac{c}{2\pi}\left(\arcsin\frac{b+x^0}{A} - \arcsin\frac{b-x^0}{A}\right) \quad (5.109)$$

$$\text{for } A \geq b + |x^0|,$$

$$q(A, x^0) = \frac{c}{\pi A}\left(\sqrt{1-\frac{(b-x^0)^2}{A^2}} + \sqrt{1-\frac{(b+x^0)^2}{A^2}}\right) \quad (5.110)$$

$$\text{for } A \geq b + |x^0|,$$

$$q'(A) = -\frac{2cb}{\pi A^2} \quad \text{for} \quad A \geq b + |x^0|. \quad (5.111)$$

Asymmetrical relay characteristic with hysteresis loop of variable width. For an asymmetrical relay characteristic with variable-width hysteresis (Fig. 5.17e) we replace in (5.109) and (5.110) $b + x^0$ and $b - x^0$ by the value of A, in accordance with the replacement of the segments $b + x^0$ and $b - x^0$ by A on going over from the characteristic shown in Fig. 5.17d to the characteristic of 5.17e; furthermore we replace 2b in (5.111) by 2A; we then obtain:

$$F^0(A, x^0) = \frac{c}{2}; \quad q(A, x^0) = 0; \quad q'(A) = -\frac{2c}{\pi A}. \quad (5.112)$$

Asymmetrical relay characteristic with backlash zone. By regarding the asymmetrical relay characteristic with backlash zone (Fig. 5.17f) as a particular case of an asymmetrical relay characteris ic of the general type with m = 1, we obtain from (5.103) – (5.105) the values of the steady component and of the harmonic linearization coefficients:

$$F^0(A, x^0) = \frac{c}{2} - \frac{c}{\pi}\arcsin\frac{b-x^0}{A} \quad \text{for} \quad A \geq |b - x^0|, \quad (5.113)$$

$$\left.\begin{aligned} q(A, x^0) &= \frac{2c}{\pi A}\sqrt{1-\frac{(b-x^0)^2}{A^2}} \quad \text{for} \quad A \geq |b - x^0|, \\ q'(A) &= 0. \end{aligned}\right\} \quad (5.114)$$

Asymmetrical ideal relay characteristic. An asymmetrical ideal relay characteristic (Fig. 5.17g) can be regarded as a particular case of the preceding characteristic with b = 0. We then obtain from (5.113) and (5.114) the values of F^0 and q:

$$F^0(A, x^0) = \frac{c}{2} + \frac{c}{\pi} \arcsin \frac{x^0}{A} \text{ for } A \geqslant |x^0|, \tag{5.115}$$

$$\left.\begin{aligned} q(A, x^0) &= \frac{2c}{\pi A} \sqrt{1 - \frac{(x^0)^2}{A^2}} \text{ for } A \geqslant |x^0|, \\ q'(A) &= 0. \end{aligned}\right\} \tag{5.116}$$

§5.7. HARMONIC LINEARIZATION OF PIECEWISE-LINEAR SINGLE-VALUED AND LOOP-TYPE CHARACTERISTICS

In the present section we consider nonlinear characteristics of the saturation type, characteristics with a backlash zone, characteristics with variable gain, with hysteresis and saturation, and also with play (clearance).

Nonlinear characteristics with backlash zone and saturation. A nonlinear characteristic with backlash and saturation, with asymmetrical oscillations of the input to the nonlinear element is shown in Fig. 5.18a. The coefficient q'(A) is equal to zero in this case, since the characteristic is single-valued.

Let us determine the values of the dc component $F^0(A, x^0)$ and of the harmonic-linearization coefficient $q(A, x^0)$ in accordance with the form of the function $F(x^0 + A \sin \psi)$ shown in Fig. 5.18b.

For the dc component $F^0(A, x^0)$ we have:

$$F^0(A, x^0) = \frac{1}{2\pi} \int_0^{2\pi} F(x^0 + A \sin \psi)\, d\psi = \frac{1}{2\pi} \left\{ 2k \int_{\psi_1}^{\psi_2} [A \sin \psi - (b_1 - x^0)]\, d\psi + c \int_{\psi_2}^{\pi - \psi_2} d\psi - 2k \int_{\psi_2}^{\psi_2} [A \sin \psi - (b_1 + x^0)]\, d\psi - c \int_{\pi + \psi_1}^{2\pi - \psi_1} d\psi \right\}.$$

Carrying out the integration and taking into account the values of the corresponding angles, we obtain:

$$F^0(A, x^0) =$$
$$= \frac{k \cdot A}{\pi}\left(\sqrt{1-\frac{(b_1-x^0)^2}{A^2}} - \sqrt{1-\frac{(b_2-x^0)^2}{A^2}} - \sqrt{1-\frac{(b_1+x^0)^2}{A^2}} + \right.$$
$$\left. + \sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\right) + \frac{k}{\pi}(b_1-x^0)\left(\arcsin\frac{b_1-x^0}{A} - \arcsin\frac{b_2-x^0}{A}\right) +$$
$$+ \frac{k}{\pi}(b_1+x^0)\left(\arcsin\frac{b_2+x^0}{A} - \arcsin\frac{b_1+x^0}{A}\right) + \qquad (5.117)$$
$$+ \frac{c}{\pi}\left(\arcsin\frac{b_2+x^0}{A} - \arcsin\frac{b_2-x^0}{A}\right)$$
$$\text{for } A \geqslant b_2 + |x^0|.$$

For the coefficient q(A, x^0) we have

$$q(A, x^0) = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi)\sin\psi\, d\psi =$$
$$= \frac{1}{\pi A}\left\{2k\int_{\psi_1}^{\psi_2}[A\sin\psi - (b_1 - x^0)]\sin\psi\, d\psi + \right.$$
$$\left. + 2k\int_{\psi_2}^{\psi_1}[A\sin\psi - (b_1 + x^0)]\sin\psi\, d\psi + c\int_{\psi_2}^{\pi-\psi_2}\sin\psi\, d\psi - c\int_{\pi+\psi_1}^{2\pi-\psi_1}\sin\psi\, d\psi\right\}.$$

Carrying out the integration and taking into account the values of the corresponding angles we obtain

$$q(A, x^0) = \frac{k}{\pi}\left(\arcsin\frac{b_2-x^0}{A} - \arcsin\frac{b_1-x^0}{A} + \arcsin\frac{b_2+x^0}{A} - \right.$$
$$- \arcsin\frac{b_1+x^0}{A} + \frac{b_1-x^0}{A}\sqrt{1-\frac{(b_1-x^0)^2}{A^2}} - \frac{b_2-x^0}{A}\sqrt{1-\frac{(b_2-x^0)^2}{A^2}} +$$
$$\left. + \frac{b_1+x^0}{A}\sqrt{1-\frac{(b_1+x^0)^2}{A^2}} - \frac{b_2+x^0}{A}\sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\right) +$$
$$+ \frac{2k}{\pi A}(b_1-x^0)\left(\sqrt{1-\frac{(b_2-x^0)^2}{A^2}} - \sqrt{1-\frac{(b_1-x^0)^2}{A^2}}\right) - \qquad (5.118)$$
$$- \frac{2k}{\pi A}(b_1+x^0)\left(\sqrt{1-\frac{(b_1+x^0)^2}{A^2}} - \sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\right) +$$
$$+ \frac{2c}{\pi A}\left(\sqrt{1-\frac{(b_2-x^0)^2}{A^2}} + \sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\right)$$
$$\text{for } A \geqslant b_2 + |x^0|.$$

Nonlinear characteristic with backlash zone without saturation (Fig. 5.18c). The corresponding periodic function $F(x^0 + A\sin\psi)$ for the given characteristic is shown in Fig. 5.18d.

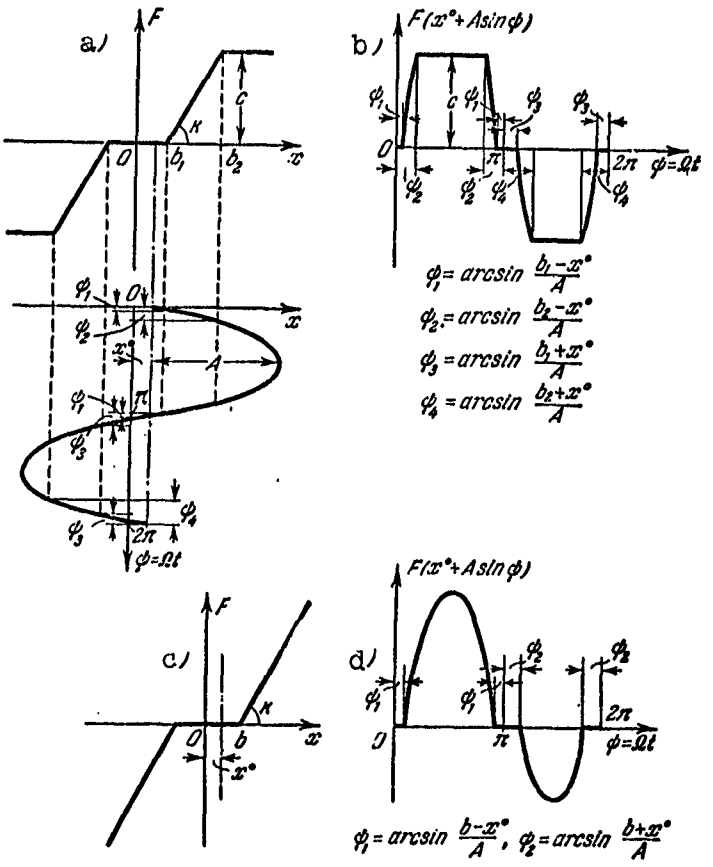


Fig. 5.18.

Carrying out the calculations for the dc component we obtain

$$F^0(A,\ x^0) = \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A\sin\psi)\,d\psi =$$
$$= \frac{1}{2\pi}\left\{\int_{\psi_1}^{\pi-\psi_1} k[A\sin\psi - (b - x^0)]\,d\psi - \int_{\psi_2}^{\pi-\psi_2} k[A\sin\psi - (b + x^0)]\,d\psi\right\} =$$
$$= \frac{kA}{\pi}(\cos\psi_1 - \cos\psi_2) + kx^0 + \frac{k}{\pi}[b(\psi_1 - \psi_2) - x^0(\psi_1 + \psi_2)],$$

which yields after substitution of the values of the corresponding angles

$$F^0(A,\ x^0) = \frac{kA}{\pi}\left(\sqrt{1 - \frac{(b-x^0)^2}{A^2}} - \sqrt{1 - \frac{(b+x^0)^2}{A^2}}\right) +$$
$$+ kx^0 + \frac{k}{\pi}\left[b\left(\arcsin\frac{b-x^0}{A} - \arcsin\frac{b+x^0}{A}\right) - \right.$$
$$\left. - x^0\left(\arcsin\frac{b-x^0}{A} + \arcsin\frac{b+x_0}{A}\right)\right| \qquad (5.119)$$
$$\text{for } A \geq b + |x^0|.$$

Calculating the coefficient $q(A, x^0)$ we obtain

$$q(A, x^0)=\frac{1}{\pi A}\int_0^{2\pi} F(x^0+A\sin\psi)\sin\psi\,d\psi=\frac{1}{\pi A}\left\{\int_{\psi_1}^{\pi-\psi_1} k[A\sin\psi-$$

$$-(b-x^0)]\sin\psi\,d\psi+\int_{\psi_2}^{\pi-\psi_2} k[A\sin\psi-(b+x^0)]\sin\psi\,d\psi\right\}=$$

$$=\frac{k}{\pi}\left[\pi-(\psi_1+\psi_2)+\frac{1}{2}(\sin 2\psi_1+\sin 2\psi_2)\right]-\frac{2k}{\pi A}[(b-x^0)\cos\psi_1+$$

$$+(b+x^0)\cos\psi_2],$$

which yields after allowing for the values of the angles

$$q(A, x^0)=k-\frac{k}{\pi}\left(\arcsin\frac{b-x^0}{A}+\arcsin\frac{b+x^0}{A}+\right.$$

$$\left.+\frac{b-x^0}{A}\sqrt{1-\frac{(b-x^0)^2}{A^2}}+\frac{b+x^0}{A}\sqrt{1-\frac{(b+x^0)^2}{A^2}}\right) \qquad (5.120)$$

$$\text{for } A\geqslant b+|x^0|.$$

Nonlinear characteristic without backlash zone with saturation. For a nonlinear characteristic without backlash zone and with saturation (Fig. 3.5e) with asymmetrical oscillations, it is necessary to put in (5.117) and (5.118) $b_1 = 0$, $b_2 = b$, $c = kb$. We then obtain the following values for the dc component F^0 and the coefficient of harmonic linearization q:

$$F^0(A, x^0)=\frac{k}{\pi}\left[A\left(\sqrt{1-\frac{(b+x^0)^2}{A^2}}-\sqrt{1-\frac{(b-x^0)^2}{A^2}}\right)+\right.$$

$$\left.+(b+x^0)\arcsin\frac{b+x^0}{A}-(b-x^0)\arcsin\frac{b-x^0}{A}\right] \qquad (5.121)$$

$$\text{for } A\geqslant b+|x^0|;$$

$$q(A, x^0)=\frac{k}{\pi}\left(\arcsin\frac{b-x^0}{A}+\arcsin\frac{b+x^0}{A}+\right.$$

$$\left.+\frac{b-x^0}{A}\sqrt{1-\frac{(b-x^0)^2}{A^2}}+\frac{b+x^0}{A}\sqrt{1-\frac{(b+x^0)^2}{A^2}}\right) \qquad (5.122)$$

$$\text{for } A\geqslant b+|x^0|.$$

Let us illustrate with this nonlinear characteristic as an example the plots of $F^0/b = f(x^0/b)$ with $A/b = \text{const}$ and $q = f(A/b)$ with $x^0/b = \text{const}$, calculated with the aid of Formulas (5.121) and (5.122), and shown in Figs. 5.19a and b.

It is seen from the plots of $F^0/b = f(x^0/b)$ (Fig. 5.19a) that in the presence of oscillations of the input to the nonlinear element,

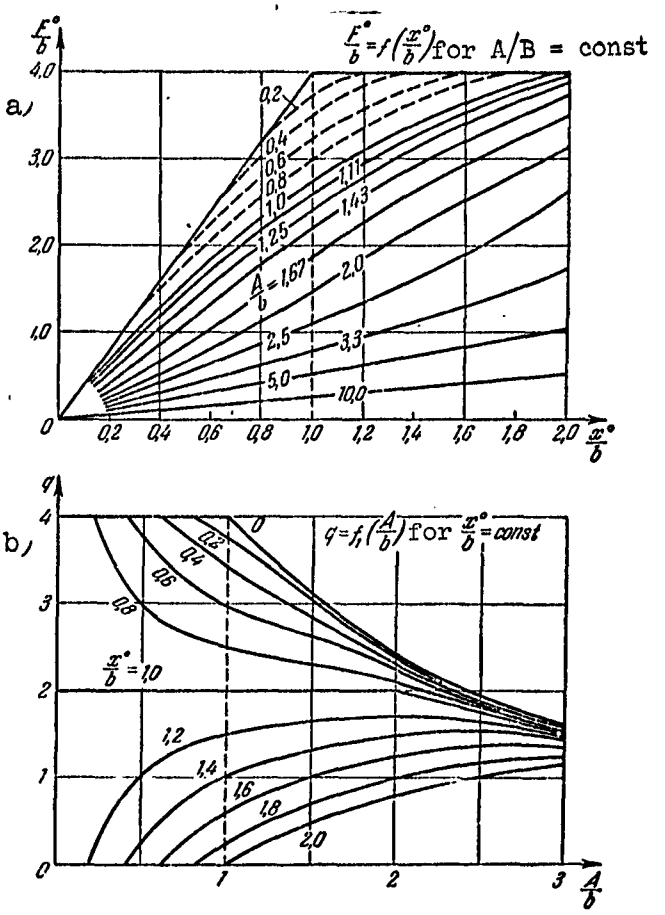


Fig. 5.19.

its static characteristic for a slowly varying function (bias function) becomes smoothed, and an increase in the amplitude of the oscillations of the input leads to a decrease in the gain of the nonlinear element with respect to the constant or slowly varying input signal.

The plots of $q(A) = f_1(A/b)$ (Fig. 5.19b) characterize the passage of the oscillatory component through the nonlinear element, as a function of the amplitude at the input and of the bias of the center of oscillations. We see that an increase in the bias leads to a decrease in the gain for the oscillatory component.

Nonlinear characteristic with variable gain. A nonlinear characteristic with variable gain, for asymmetrical oscillations of the input to the nonlinear element, is shown in Fig. 5.20a. Since the

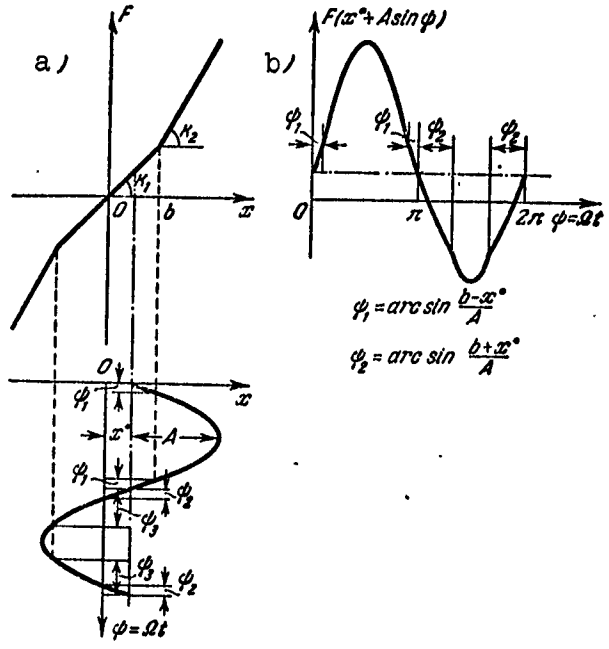


Fig. 5.20

characteristic is single-valued, the coefficient q'(A) = 0. Let us calculate the values $F^0(A, x^0)$ and $q(A, x^0)$ in accordance with the values of the function $F(x^0 + A \sin \psi)$ shown in Fig. 5.20b.

For the dc component $F^0(A, x^0)$ we have

$$F^0(A, x^0)=\frac{1}{2\pi}\int_0^{2\pi}F(x^0+A\sin\psi)\,d\psi=\frac{1}{2\pi}\left\{2\int_0^{\psi_1}k_1(x^0+A\sin\psi)\,d\psi+\right.$$

$$+2\int_{\psi_1}^{\frac{\pi}{2}}k_1 b\,d\psi+2\int_{\psi_1}^{\frac{\pi}{2}}k_2[A\sin\psi-(b-x^0)]\,d\psi+2\int_0^{\psi_2}k_1(x^0-A\sin\psi)\,d\psi-$$

$$\left.-2\int_{\psi_1}^{\frac{\pi}{2}}k_1 b\,d\psi-2\int_{\psi_2}^{\frac{\pi}{2}}k_2[A\sin\psi-(b+x^0)]\,d\psi\right\}.$$

Carrying out the integration and substituting the values of the corresponding angles we obtain

$$F^0(A, x^0)=k_2x^0+\frac{k_2-k_1}{\pi}\left[A\left(\sqrt{1-\frac{(b-x^0)^2}{A^2}}-\sqrt{1-\frac{(b+x^0)^2}{A^2}}\right)+\right.$$

$$\left.+(b-x^0)\arcsin\frac{b-x^0}{A}+(b+x^0)\arcsin\frac{b+x^0}{A}\right] \qquad (5.123)$$

for $A \geq b + |x^0|$.

For the coefficient $q(A, x^0)$ we have

$$q(A, x^0)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\sin\psi\,d\psi=\frac{1}{\pi A}\left\{2\int_0^{\psi_1}k_1(x^0+A\sin\psi)\times\right.$$

$$\times\sin\psi\,d\psi+2\int_{\psi_1}^{\frac{\pi}{2}}k_1 b\sin\psi\,d\psi+2\int_{\psi_1}^{\frac{\pi}{2}}k_2[A\sin\psi-(b-x^0)]\sin\psi\,d\psi-$$

$$-2\int_0^{\psi_2}k_1(x^0-A\sin\psi)\sin\psi\,d\psi+2\int_{\psi_2}^{\psi_2}k_1(A\sin\psi-x^0)\sin\psi\,d\psi+$$

$$\left.+2\int_{\psi_2}^{\frac{\pi}{2}}k_1 b\sin\psi\,d\psi+2\int_{\psi_2}^{\frac{\pi}{2}}k_2[A\sin\psi-(b+x^0)]\sin\psi\,d\psi\right\}.$$

Carrying out the integration and substituting the values of the corresponding angles, we obtain

$$q(A, x_0)=k_2-\frac{k_2-k_1}{\pi A}\left[(b-x^0)\sqrt{1-\frac{(b-x^0)^2}{A^2}}+\right.$$

$$\left.+(b+x^0)\sqrt{1-\frac{(b+x^0)^2}{A^2}}+A\left(\arcsin\frac{b-x^0}{A}+\arcsin\frac{b+x^0}{A}\right)\right] \qquad (5.124)$$

$$\text{for}\quad A\geqslant b+|x^0|.$$

Nonlinear characteristic with hysteresis loop and saturation. A nonlinear characteristic with hysteresis loop and saturation, for asymmetrical oscillations of the input to the nonlinear element, is shown in Fig. 5.21a. Let us calculate the values of the dc component $F^0(A, x^0)$ and of the coefficients $q(A, x^0)$ and $q'(A)$ by means of Formulas (5.86) – (5.88) in accordance with the form of the function $F(x^0 + A\sin\psi)$ (Fig. 5.21b).

For the dc component $F^0(A, x^0)$ we have, in accordance with (5.86),

$$F^0(A, x^0)=\frac{1}{2\pi}\int_0^{2\pi}F(x^0+A\sin\psi)\,d\psi=$$

$$=\frac{1}{2\pi}\left[\int_0^{\psi_2}k(x^0+A\sin\psi-b)\,d\psi+\int_{\psi_2}^{\pi+\psi_1}c\,d\psi+\right.$$

$$\left.+\int_{\pi+\psi_1}^{\pi-\psi_2}k(x^0+A\sin\psi+b)\,d\psi-\int_{\pi+\psi_2}^{2\pi-\psi_2}c\,d\psi+\int_{2\pi-\psi_2}^{2\pi}k(x^0+A\sin\psi-b)\,d\psi\right]=$$

$$=\frac{k}{2\pi}[(x^0-b)(\psi_2+\psi_3)+(x^0+b)(\psi_4-\psi_1)]+\frac{c}{2\pi}(\psi_1-\psi_2+$$

$$+\psi_3+\psi_4)+\frac{kA}{2\pi}(\cos\psi_3-\cos\psi_1-\cos\psi_2+\cos\psi_4).$$

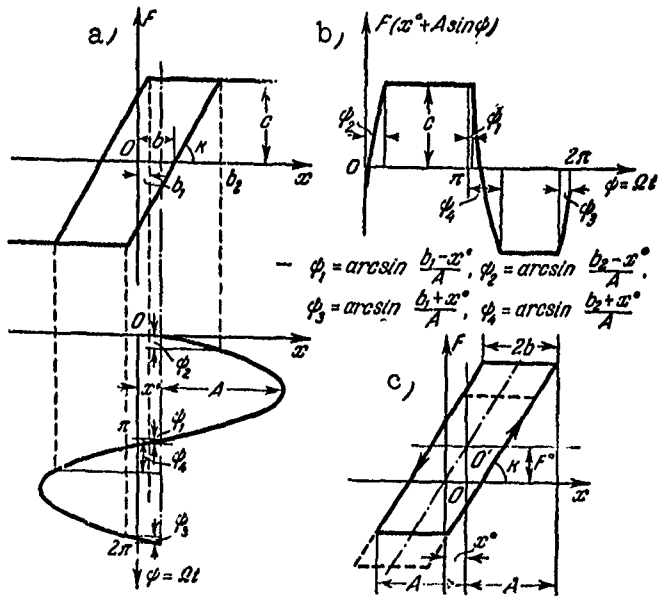


Fig. 5.21.

Substituting the values of the corresponding angles (Fig. 5.21b) we obtain

$$F^0(A, x^0) = \frac{k}{2\pi}\left[(x^0+b)\left(\arcsin\frac{b_1-x^0}{A}+\arcsin\frac{b_2+x^0}{A}\right)+(x^0-b)\times\right.$$
$$\left.\times\left(\arcsin\frac{b_1+x^0}{A}+\arcsin\frac{b_2-x^0}{A}\right)\right]-\frac{c}{2\pi}\left(\arcsin\frac{b_1-x^0}{A}+\right.$$
$$\left.+\arcsin\frac{b_2-x^0}{A}-\arcsin\frac{b_1+x^0}{A}-\arcsin\frac{b_2+x^0}{A}\right)- \qquad (5.125)$$
$$-\frac{kA}{2\pi}\left(\sqrt{1-\frac{(b_1-x^0)^2}{A^2}}+\sqrt{1-\frac{(b_2-x^0)^2}{A^2}}-\sqrt{1-\frac{(b_1+x^0)^2}{A^2}}-\right.$$
$$\left.-\sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\right) \text{ for } A \geqslant b_2+|x^0|.$$

For the coefficient $q(A, x^0)$ we have in accord with (5.87)

$$q(A, x^0) = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi)\sin\psi\, d\psi =$$
$$= \frac{1}{\pi A}\left[\int_0^{\psi_2} k(x^0 + A\sin\psi - b)\sin\psi\, d\psi + \right.$$
$$\left. + \int_{\psi_2}^{\pi+\psi_1} c\sin\psi\, d\psi + \int_{\pi+\psi_1}^{\pi+\psi_2} k(x^0 + A\sin\psi + b)\sin\psi\, d\psi - \right.$$

$$-\int_{\pi+\psi_4}^{2\pi-\psi_3} c\sin\psi\,d\psi+\int_{2\pi-\psi_3}^{2\pi} k(x^0+A\sin\psi-b)\sin\psi\,d\psi\Big]=$$

$$=\frac{k}{\pi A}[(x^0+b)(\cos\psi_4-\cos\psi_1)-(x^0-b)(\cos\psi_2-\cos\psi_3)]+$$

$$+\frac{c}{\pi A}(\cos\psi_1+\cos\psi_2+\cos\psi_3+\cos\psi_4)+$$

$$+\frac{k}{2\pi}\left(-\psi_1+\psi_2+\psi_3+\psi_4+\frac{1}{2}\sin 2\psi_1-\frac{1}{2}\sin 2\psi_2-\frac{1}{2}\sin 2\psi_3-\right.$$

$$\left.-\frac{1}{2}\sin 2\psi_4\right).$$

Substituting the values of the corresponding angles (Fig. 5.21b) we obtain

$$q(A,\ x^0)=\frac{k}{\pi A}\left[(x^0+b)\left(\sqrt{1-\frac{(b_2+x^0)^2}{A^2}}-\sqrt{1-\frac{(b_1-x^0)^2}{A^2}}\right)-\right.$$

$$-(x^0-b)\left(\sqrt{1-\frac{(b_2-x^0)^2}{A^2}}-\sqrt{1-\frac{(b_1+x^0)^2}{A^2}}\right)\Big]+$$

$$+\frac{c}{\pi A}\left(\sqrt{1-\frac{(b_1-x^0)^2}{A^2}}+\sqrt{1-\frac{(b_2-x^0)^2}{A^2}}+\sqrt{1-\frac{(b_1+x^0)^2}{A^2}}+\right.$$

$$+\sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\Big)+\frac{k}{2\pi}\left(\arcsin\frac{b_1-x^0}{A}+\arcsin\frac{b_2-x^0}{A}+\right.$$

$$+\arcsin\frac{b_1+x^0}{A}+\arcsin\frac{b_2+x^0}{A}-\frac{b_1-x^0}{A}\sqrt{1-\frac{(b_1-x^0)^2}{A^2}}-$$

$$-\frac{b_2-x^0}{A}\sqrt{1-\frac{(b_2-x^0)^2}{A^2}}-\frac{b_1+x^0}{A}\sqrt{1-\frac{(b_1+x^0)^2}{A^2}}-$$

$$\left.-\frac{b_2+x^0}{A}\sqrt{1-\frac{(b_2+x^0)^2}{A^2}}\right)\text{for}\quad A\geqslant b_2+|x^0|. \tag{5.126}$$

For the coefficient q'(A) we have in accord with (5.88)

$$q'(A)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\cos\psi\,d\psi=\frac{1}{\pi A}\left[\int_0^{\psi_2}k(x^0+A\sin\psi-b)\times\right.$$

$$\times\cos\psi\,d\psi+\int_{\psi_2}^{\pi+\psi_1}c\cos\psi\,d\psi+\int_{\pi+\psi_1}^{\pi+\psi_4}k(x^0+A\sin\psi+b)\cos\psi\,d\psi-$$

$$-\int_{\pi+\psi_4}^{2\pi-\psi_3}c\cos\psi\,d\psi+\int_{2\pi-\psi_3}^{2\pi}k(x^0+A\sin\psi-b)\cos\psi\,d\psi\Big]=$$

$$=\frac{k}{\pi A}[(x^0-b)(\sin\psi_2+\sin\psi_3)+(x^0+b)(\sin\psi_1-\sin\psi_4)]-$$

$$-\frac{c}{\pi A}(\sin\psi_1+\sin\psi_2-\sin\psi_3+\sin\psi_4)+$$

$$+\frac{k}{2\pi}(-\sin^2\psi_1+\sin^2\psi_2-\sin^2\psi_3+\sin^2\psi_4).$$

Substituting the values of the corresponding angles (Fig. 5.21b) we obtain

$$q'(A)=-\frac{4bc}{\pi A^2}\ \text{for}\ \ A\geqslant b_2+|x^0|. \tag{5.127}$$

Nonlinear characteristic of the clearance or play type. In the

case of asymmetrical oscillations, a nonlinear characteristic of the clearance or play type (Fig. 5.21c) is shifted along the central line so that its previous center 0 goes over into position 0'. The steady component of the function F(x) is determined in this case by the relation

$$F^0 = kx^0.$$

The variation of the oscillatory component of the function $F(x^0 + A \sin \psi)$ relative to the new center of oscillations is independent of the value of the bias x^0 and will proceed as if this bias were nonexistent. Thus, for example, a gear pair with play will transmit motion with the same gear ratio no matter what the angle of rotation of the driving gear. In the case of oscillations in a kinematic transmission including this pair of gears, the play will manifest itself to an equal degree for all angles of rotation.

Consequently we shall have for the harmonic-linearization coefficients of a characteristic of the play or clearance type, in the case of a center of oscillation shifted relative to the origin, the same formulas (3.28) and (3.29) as for the case of symmetrical oscillations:

$$q(A) = \frac{k}{\pi}\left[\frac{\pi}{2} + \arcsin\left(1 - \frac{2b}{A}\right) + 2\left(1 - \frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1 - \frac{b}{A}\right)}\right] \text{ for } A \geq b \qquad (5.128)$$

$$q'(A) = -\frac{4kb}{\pi A}\left(1 - \frac{b}{A}\right) \text{ for } A \geq b. \qquad (5.129)$$

§ 5.8. HARMONIC LINEARIZATION OF POWER-LAW NONLINEAR CHARACTERISTICS

Power-law nonlinear characteristics of the form $F(x) = kx^n$ with n integral and odd, or $F(x) = kx^n \operatorname{sign} x$ with n integral and even, are shown for the case of asymmetrical oscillations of the input to the nonlinear element in the form of the plot of Fig. 5.22a. The periodic function $F(x^0 + A \sin \psi)$ of the argument $\psi = \Omega t$ is shown for this case in Fig. 5.22b.

Since power-law characteristics are single-valued, we have $q'(A) = 0$ for all values of n. When $n = 1$ we obtain a linear characteristic.

Let us calculate the values of the dc component $F^0(A, x^0)$ and the coefficient $q(A, x^0)$ of symmetrical and asymmetrical (with a single branch) power-law characteristics under asymmetrical oscillations of the input to the nonlinear element, for $n = 2$ and $n = 3$.

Power-law symmetrical characteristic $F(x) = kx^2 \operatorname{sign} x$. For the dc component $F^0(A, x^0)$ we obtain in accordance with (5.86) and Fig. 5.22b:

$$F^0(A, x^0) = \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A\sin\psi)\,d\psi =$$

$$= \frac{k}{2\pi}\left[\int_{-\psi_1}^{\pi+\psi_1} (x^0 + A\sin\psi)^2\,d\psi - \int_{\pi+\psi_1}^{2\pi-\psi_1} (x^0 + A\sin\psi)^2\,d\psi\right] =$$

$$= \frac{k}{\pi}\left\{\left[2(x^0)^2 + A^2\right]\psi_1 - \frac{A^2}{2}\sin 2\psi_1 + 4x^0 A\cos\psi_1\right\}.$$

Taking into account the value $\psi_1 = \arcsin x^0/A$, we obtain

$$F^0(A, x^0) = \frac{2k}{\pi}\left\{\left[(x^0)^2 + \frac{A^2}{2}\right]\arcsin\frac{x^0}{A} + \frac{3}{2}x^0 A\sqrt{1 - \frac{(x^0)^2}{A^2}}\right\}. \qquad (5.130)$$

For the coefficient $q(A, x^0)$ we obtain in accordance with (5.87) and Fig. 5.22b:

$$q(A,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F(x^0+A\sin\psi)\sin\psi\,d\psi=$$

$$=\frac{k}{\pi A}\left[\int_{-\psi_1}^{\pi+\psi_1}(x^0+A\sin\psi)^2\sin\psi\,d\psi-\int_{\pi+\psi_1}^{2\pi-\psi_1}(x^0+A\sin\psi)^2\sin\psi\,d\psi\right]=$$

$$=\frac{k}{\pi A}\left[4(x^0)^2\cos\psi_1+4x^0A\psi_1-2x^0A\sin 2\psi_1+4A^2\cos\psi_1-\right.$$

$$\left.-\frac{4}{3}A^2\cos^3\psi_1\right].$$

Taking into account the value $\psi_1 = \arcsin x^0/A$, we obtain

$$q(A,\ x^0)=\frac{4k}{\pi}\left\{x^0\arcsin\frac{x^0}{A}+\left[\frac{2A}{3}+\frac{(x^0)^2}{3A}\right]\sqrt{1-\frac{(x^0)^2}{A^2}}\right\}. \tag{5.131}$$

Power-curve symmetrical characterisitic $F(x) = kx^3$. For the dc component $F^0(A, x^0)$ we obtain in accordance with (5.86) and Fig. 5.22b:

$$F^0(A,\ x^0)=\frac{1}{2\pi}\int_0^{2\pi}F(x^0+A\sin\psi)\,d\psi=\frac{k}{2\pi}\int_0^{2\pi}(x^0+A\sin\psi)^3\,d\psi.$$

Integrating, we get

$$F^0(A,\ x^0)=k\left[(x^0)^3+\frac{3}{2}x^0A^2\right]. \tag{5.132}$$

For the coefficient $q(A, x^0)$ we obtain in accordance with (5.87) and Fig. 5.22b

$$q(A,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}(x^0+A\sin\psi)\,d\psi=\frac{k}{\pi A}\int_0^{2\pi}(x^0+A\sin\psi)^3\sin\psi\,d\psi.$$

Integrating, we get

$$q(A,\ x^0)=3k\left[(x^0)^2+\frac{A^2}{4}\right]. \tag{5.133}$$

Asymmetrical power-law characteristic $F(x) = kx^2\cdot 1(x)$*. An asymmetrical power-law characteristic $F(x) = kx^2\cdot 1(x)$ is shown in Fig. 5.22c.

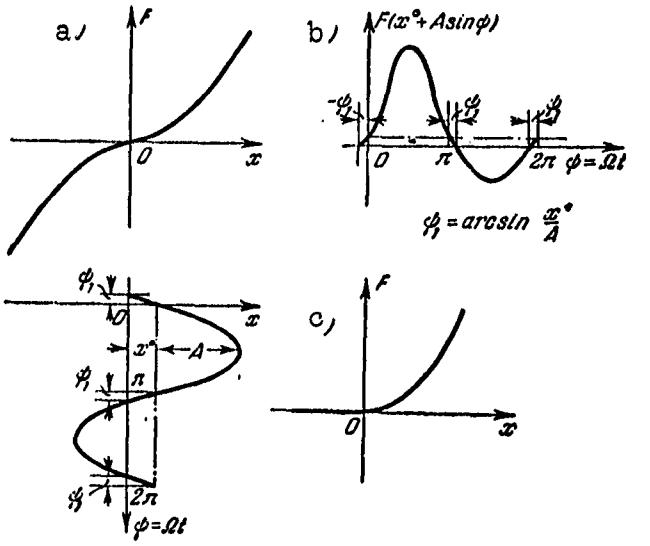


Fig. 5.22

For the dc component $F^0(A, x^0)$ we obtain in accordance with (5.86) and the value of the function $F(x^0 + A \sin \psi)$

$$F^0(A, x^0)=\frac{1}{2\pi}\int_0^{2\pi} F(x^0+A\sin\psi)\,d\psi=\frac{k}{2\pi}\int_{-\psi_1}^{\pi+\psi_1}(x^0+A\sin\psi)^2\,d\psi=$$
$$=\frac{k}{\pi}\left\{\left[(x^0)^2+\frac{A^2}{2}\right]\left(\frac{\pi}{2}+\psi_1\right)+2x^0A\cos\psi_1-\frac{A^2}{4}\sin 2\psi_1\right\}.$$

Taking into account the values of $\psi_1 = \arcsin x^0/A$, we get

$$F^0(A, x^0)=\frac{k}{\pi}\left\{\left[(x^0)^2+\frac{A^2}{2}\right]\left(\frac{\pi}{2}+\arcsin\frac{x^0}{A}\right)+\frac{3}{2}x^0A\sqrt{1-\frac{(x^0)^2}{A^2}}\right\}. \qquad (5.134)$$

For the coefficient $q(A, x^0)$ we obtain in accordance with (5.87) and the value of the function $F(x^0 + A \sin \psi)$

$$q(A, x^0)=\frac{1}{\pi A}\int_0^{2\pi} F(x^0+A\sin\psi)\sin\psi\,d\psi=$$
$$=\frac{k}{\pi A}\int_{-\psi_1}^{\pi+\psi_1}(x^0+A\sin\psi)^2\sin\psi\,d\psi=\frac{k}{\pi A}\Big[x^0\pi A+2(x^0)^2\cos\psi_1+$$
$$+2x^0A\psi_1-x^0A\sin 2\psi_1+2A^2\cos\psi_1-\frac{2}{3}A^2\cos^3\psi_1\Big].$$

Taking into account the value of $\psi_1 = \arcsin x^0/A$, we get

$$q(A, x^0) = \frac{2k}{\pi}\left\{x^0\left(\frac{\pi}{2} + \arcsin\frac{x^0}{A}\right) + \left[\frac{2A}{3} + \frac{(x^0)^2}{3A}\right]\sqrt{1 - \frac{(x^0)^2}{A^2}}\right\}. \quad (5.135)$$

Asymmeterical power-law characteristic $F(x) = kx^3 \cdot 1(x)$. The asymmetrical power-law characteristic $F(x) = kx^3 \cdot 1(x)$, just as the characteristic $F(x) = kx^2 \cdot 1(x)$, has a single branch (Fig. 5.22c).

For the dc component $F^0(A, x^0)$ we obtain in accordance with (5.86) and the value of the function $F(x^0 + A \sin \psi)$

$$F^0(A, x^0) = \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A\sin\psi)\,d\psi = \frac{k}{2\pi}\int_{-\psi_1}^{\pi+\psi_1} (x^0 + A\sin\psi)^3\,d\psi =$$
$$= \frac{k}{2\pi}\left\{[2(x^0)^3 + 3x^0A^2]\left(\frac{\pi}{2} + \psi_1\right) + 3[2(x^0)^2 A\cos\psi_1 - \right.$$
$$\left. - x^0A^2\sin\psi_1\cos\psi_1] + 2A^3\left(\cos\psi_1 - \frac{\cos^3\psi_1}{3}\right)\right\}.$$

Taking into account the value $\psi_1 = \arcsin x^0/A$, we obtain

$$F^0(A, x^0) = \frac{k}{2\pi}\left\{[2(x^0)^3 + 3x^0A^2]\left(\frac{\pi}{2} + \arcsin\frac{x^0}{A}\right) + \right.$$
$$\left. + \left[\frac{4}{3}A^2 + \frac{11}{3}(x^0)^2\right]A\sqrt{1 - \frac{(x^0)^2}{A^2}}\right\}. \quad (5.136)$$

For the coefficient $q(A, x^0)$ we obtain in accordance with (5.87) and the value of the function $F(x^0 + A \sin \psi)$

$$q(A, x^0) = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi)\sin\psi\,d\psi = \frac{k}{\pi A}\int_{-\psi_1}^{\pi+\psi_1} (x^0 +$$
$$+ A\sin\psi)^3\sin\psi\,d\psi = \frac{k}{\pi A}\left\{\left[3(x^0)^2A + \frac{3}{4}A^3\right]\left(\frac{\pi}{2} + \psi_1\right) + \right.$$
$$+ [2(x^0)^3 + 6x^0A^2]\cos\psi_1 - \left[\frac{3}{2}(x^0)^2A + \frac{1}{2}A^3\right]\sin 2\psi_1 -$$
$$\left. - 2x^0A^2\cos^3\psi_1 + \frac{1}{16}A^3\sin 4\psi_1\right\}.$$

Taking into account the value of $\psi_1 = \arcsin x^0/A$ and carrying out the transformations, we obtain

$$q(A, x^0) = \frac{k}{\pi}\left\{\left[3(x^0)^2 + \frac{3}{4}A^2\right]\left(\frac{\pi}{2} + \arcsin\frac{x_0}{A}\right) + \right.$$
$$\left. + \left(\frac{13}{4}x^0A + \frac{(x^0)^3}{2A}\right)\sqrt{1 - \frac{(x^0)^2}{A^2}}\right\}. \quad (5.137)$$

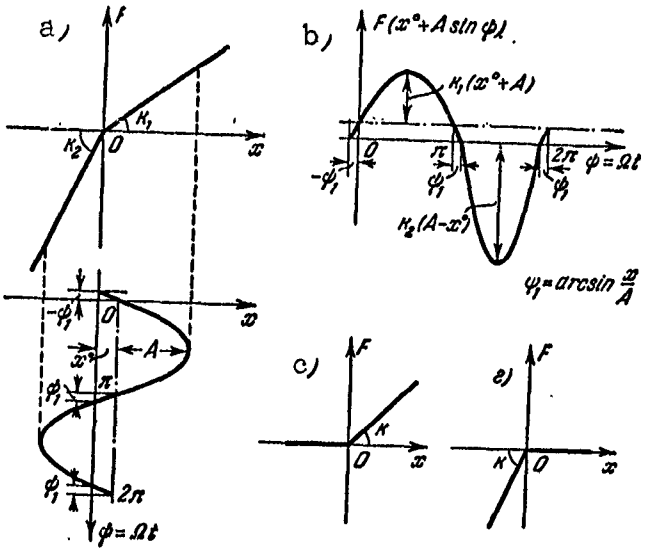


Fig. 5.23.

§5.9. HARMONIC LINEARIZATION OF ASYMMETRICAL PIECEWISE-LINEAR CHARACTERISTICS

The asymmetrical piecewise-linear characteristics shown in Fig. 5.23 are frequently encountered in nonlinear problems when account is taken of the force of reaction of contact springs or other elastic elements. In addition, such characteristics are conveniently used to approximate curvilinear asymmetrical characteristics.

Owing to the asymmetry of the indicated characteristics, the oscillatory motion in a system that has elements with such characteristics will be asymmetrical even in the absence of an external signal input to the system. Harmonic linearization of such characteristics must be carried out for the asymmetrical oscillations.

The characteristics shown in Fig. 5.23 are single-valued, and therefore the coefficient of harmonic linearization $q'(A)$ will vanish. Let us calculate the values of the dc component $F^0(A, x^0)$ and of the coefficient $q(A, x^0)$ for piecewise-linear characteristics.

Characteristics of the type of bilateral reaction of an elastic element with different stiffness. A characteristic of the type of

bilateral reaction of an elastic element with different stiffness, in response to asymmetrical oscillations of the input x to the nonlinear element is shown in Fig. 5.23a. The periodic function $F(x^0 + A \sin \psi)$ of the argument ψ is shown in Fig. 5.23b.

For the dc $F^0(A, x^0)$ we obtain in accordance with (5.86) and Fig. 5.23b

$$F^0(A, x^0) = \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A \sin \psi)\, d\psi = \frac{1}{2\pi}\left[\int_{-\psi_1}^{\pi+\psi_1} k_1 (x^0 + A \sin \psi)\, d\psi + \right.$$
$$\left. + \int_{\pi+\psi_1}^{2\pi-\psi_1} k_2 (x^0 + A \sin \psi)\, d\psi\right] = \frac{k_1+k_2}{2} x^0 + \frac{k_1-k_2}{\pi}(x^0\psi_1 + A \cos \psi_1).$$

Taking into account the value of $\psi_1 = \arcsin x^0/A$, we obtain

$$F^0(A, x^0) = \frac{k_1+k_2}{2} x^0 + \frac{k_1-k_2}{\pi}\left(x^0 \arcsin \frac{x^0}{A} + A\sqrt{1-\frac{(x^0)^2}{A^2}}\right). \qquad (5.138)$$

For the coefficient $q(A, x^0)$ we obtain in accordance with (5.87) and Fig. 5.23b

$$q(A, x^0) = \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A \sin \psi) \sin \psi\, d\psi = \frac{1}{\pi A}\left[\int_{-\psi_1}^{\pi+\psi_1} k_1 (x^0 + \right.$$
$$\left. + A \sin \psi) \sin \psi\, d\psi + \int_{\pi+\psi_1}^{2\pi-\psi_1} k_2 (x^0 + A \sin \psi) \sin \psi\, d\psi\right] =$$
$$= \frac{k_1+k_2}{2} + \frac{k_1-k_2}{\pi A}(2x^0 \cos \psi_1 + A\psi_1 - \frac{A}{2} \sin 2\psi_1).$$

Taking into account the value of $\psi_1 = \arcsin x^0/A$, we obtain

$$q(A, x^0) = \frac{k_1+k_2}{2} + \frac{k_1-k_2}{\pi}\left(\arcsin \frac{x^0}{A} + \frac{x^0}{A}\sqrt{1-\frac{(x^0)^2}{A^2}}\right). \qquad (5.139)$$

Characteristics of the type of unilateral reaction of an elastic element. Characteristics of the type of unilateral reaction of an elastic element are shown in Figs. 5.23c and d. Such characteristics correspond, for example, to the dependence of the force of reaction of a stationary relay contact to the moving contact on the displacement of the latter.

For the characteristic of Fig. 5.23c we obtain from (5.138) and (5.139) with $k_1 = k$ and $k_2 = 0$ the following values for the dc component and the coefficient of harmonic linearization:

$$F^0(A, x^0) = \frac{kx^0}{2} + \frac{k}{\pi}\left(x^0 \arcsin \frac{x^0}{A} + A\sqrt{1 - \frac{(x^0)^2}{A^2}}\right), \tag{5.140}$$

$$q(A, x^0) = \frac{k}{2} + \frac{k}{\pi}\left(\arcsin \frac{x^0}{A} + \frac{x^0}{A}\sqrt{1 - \frac{(x^0)^2}{A^2}}\right). \tag{5.141}$$

For the characteristic of Fig. 5.23d we obtain, putting $k_1 = 0$ and $k_2 = k$ in (5.138) and (5.139)

$$F^0(A, x^0) = \frac{kx^0}{2} - \frac{k}{\pi}\left(x^0 \arcsin \frac{x^0}{A} + A\sqrt{1 - \frac{(x^0)^2}{A^2}}\right), \tag{5.142}$$

$$q(A, x^0) = \frac{k}{2} - \frac{k}{\pi}\left(\arcsin \frac{x^0}{A} + \frac{x^0}{A}\sqrt{1 - \frac{(x^0)^2}{A^2}}\right). \tag{5.143}$$

Let us determine also the coefficients of harmonic linearization of other nonlinear functions, encountered in the examples of the nonlinear systems investigated in Chapter 6.

Let us denote the asymmetrical ideal relay characteristic by $F_1(x)$ and the characteristic of the unilateral reaction of an elastic element $F_2(x)$ (Figs. 5.24a and b).

The nonlinear function $F(x, px) = F_1(x)px$. Seeking a solution for x in the form

$$x = x^0 + A \sin \psi, \quad \psi = \Omega t, \tag{5.144}$$

we have

$$px = A\Omega \cos \psi$$

and consequently

$$F_1(x)\, px = F_1(x^0 + A \sin \psi)\, A\Omega \cos \psi.$$

Calculating the dc component $F^0(A, \Omega, x^0)$, we obtain

$$F^0(A,\ \Omega,\ x^0)=\frac{1}{2\pi}\int_0^{2\pi}F_1(x^0+A\sin\psi)\,A\Omega\cos\psi\,d\psi=$$
$$=\frac{cA\Omega}{2\pi}\int_{-\psi_1}^{\pi+\psi_1}\cos\psi\,d\psi=\frac{cA\Omega}{2\pi}(-\sin\psi_1+\sin\psi_1)=0.$$

For the coefficient q we have

$$q(A,\ \Omega,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F_1(x^0+A\sin\psi)\,A\Omega\cos\psi\sin\psi\,d\psi=$$
$$=\frac{c\Omega}{2\pi}\int_{-\psi_1}^{\pi+\psi_1}\sin 2\psi\,d\psi=\frac{c\Omega}{4\pi}(-\cos 2\psi_1+\cos 2\psi_1)=0.$$

The coefficient q' is

$$q'(A,\ \Omega,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F_1(x^0+A\sin\psi)\,A\Omega\cos\psi\cos\psi\,d\psi=$$
$$=\frac{c\Omega}{\pi}\int_{-\psi_1}^{\pi+\psi_1}\cos^2\psi\,d\psi.$$

Carrying out the integration and taking into account the value of ψ_1, we obtain

$$q'(A,\ \Omega,\ x^0)=\frac{c\Omega}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{x^0}{A}+\frac{x^0}{A}\sqrt{1-\frac{x_0^2}{A^2}}\right).\qquad(5.145)$$

The nonlinear function $F_1(x)p^2x$. In accordance with (5.144) we have

$$p^2x=-A\Omega^2\sin\psi$$

consequently

$$F_1(x)p^2x=-F_1(x^0+A\sin\psi)\,A\Omega^2\sin\psi.$$

Let us calculate the values of the dc component F^0 and of the coefficients q and q'. For F^0 we get

$$F^0(A,\ \Omega,\ x^0)=-\frac{1}{2\pi}\int_0^{2\pi}F_1(x^0+A\sin\psi)\,A\Omega^2\sin\psi\,d\psi=$$
$$=-\frac{cA\Omega^2}{2\pi}\int_{-\psi_1}^{\pi+\psi_1}\sin\psi\,d\psi.$$

Carrying out the integration and taking into account the value of ψ_1, we obtain

$$F^0(A, \Omega, x^0) = -\frac{cA\Omega^2}{\pi}\sqrt{1-\frac{(x^0)^2}{A^2}}. \tag{5.146}$$

For q(A, Ω, x^0) we get

$$q(A, \Omega, x^0) = -\frac{1}{\pi A}\int_0^{2\pi} F_1(x^0 + A\sin\psi) A\Omega^2 \sin\psi \sin\psi\, d\psi =$$
$$= -\frac{c\Omega^2}{\pi}\int_{-\psi_1}^{\pi+\psi_1} \sin^2\psi\, d\psi.$$

Carrying out the integration and taking into account the value of ψ_1, we obtain

$$q(A, \Omega, x^0) = -\frac{c\Omega^2}{\pi}\left(\frac{\pi}{2} + \arcsin\frac{x^0}{A} - \frac{x^0}{A}\sqrt{1-\frac{(x^0)^2}{A^2}}\right). \tag{5.147}$$

For q'(A, Ω, x^0) we have

$$q'(A, \Omega, x^0) = -\frac{1}{\pi A}\int_0^{2\pi} F_1(x^0 + A\sin\psi) A\Omega^2 \sin\psi\cos\psi\, d\psi =$$
$$= -\frac{c\Omega^2}{2\pi}\int_{-\psi_1}^{\pi+\psi_1} \sin 2\psi\, d\psi = 0.$$

The nonlinear function $F_1(x)p^3x$. In accordance with (5.144) we have

$$p^3x = -A\Omega^2\cos\psi$$

and consequently

$$F_1(x)p^3x = -F_1(x^0 + A\sin\psi) A\Omega^2\cos\psi.$$

Let us calculate the values of F^0, q and q'. For F^0 we obtain

$$F^0(A, \Omega, x^0) = -\frac{1}{2\pi}\int_0^{2\pi} F_1(x^0 + A\sin\psi) A\Omega^2\cos\psi\, d\psi =$$
$$= -\frac{cA\Omega^2}{2\pi}\int_{-\psi_1}^{\pi+\psi_1} \cos\psi\, d\psi = 0.$$

For $q(A, \Omega, x^0)$ we obtain

$$q(A, \Omega, x^0) = -\frac{1}{\pi A}\int_0^{2\pi} F_1(x^0 + A\sin\psi)\, A\Omega^2 \cos\psi \sin\psi\, d\psi =$$
$$= -\frac{c\Omega^2}{2\pi}\int_{-\psi_1}^{\pi+\psi_1} \sin 2\psi\, d\psi = 0.$$

For the coefficient $q'(A, \Omega, x^0)$ we have

$$q'(A, \Omega, x^0) = -\frac{1}{\pi A}\int_0^{2\pi} F_1(x^0 + A\sin\psi)\, A\Omega^2 \cos\psi \cos\psi\, d\psi =$$
$$= -\frac{c\Omega^2}{\pi}\int_{-\psi_1}^{\pi+\psi_1} \cos^2\psi\, d\psi.$$

Carrying out the integration and taking into account the value of ψ_1, we obtain

$$q'(A, \Omega, x^0) = -\frac{c\Omega^2}{\pi}\left(\frac{\pi}{2} + \arcsin\frac{x^0}{A} + \frac{x^0}{A}\sqrt{1-\frac{(x^0)^2}{A^2}}\right). \qquad (5.148)$$

The nonlinear function $F(x, px) = \left[dF_2(x)/dx\right] px$. In the case of a nonlinear function $F_2(x)$ of the form of unilateral reaction of an elastic element, $F_3(x) = dF_2(x)/dx$ is represented in the form of an asymmetrical relay characteristic (Fig. 5.24b). In accordance with (5.144) we have

$$px = A\Omega\cos\psi$$

and consequently

$$F_3(x)\, px = F_3(x^0 + A\sin\psi)\, A\Omega\cos\psi.$$

Let us caluclate the values of F^0, q, and q'. For F^0 we obtain

$$F^0(A, \Omega, x^0) = \frac{1}{2\pi}\int_0^{2\pi} F_3(x^0 + A\sin\psi)\, A\Omega\cos\psi\, d\psi =$$
$$= -\frac{cA\Omega}{2\pi}\int_{\pi+\psi_1}^{2\pi-\psi_1} \cos\psi\, d\psi = 0.$$

For the coefficient $q(A, \Omega, x^0)$ we obtain

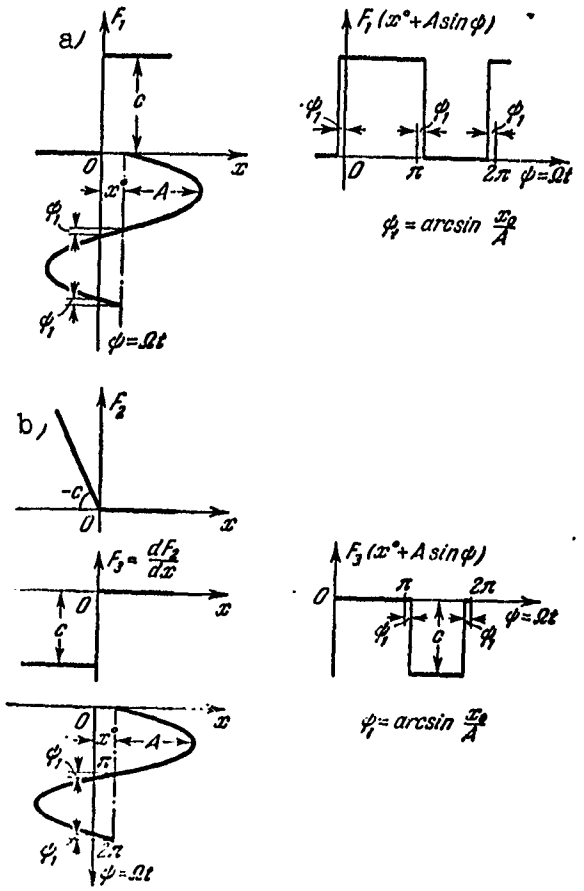


Fig. 5.24.

$$q(A,\ \Omega,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F_2(x^0+A\sin\psi)\,A\Omega\cos\Omega\sin\psi\,d\psi=$$
$$=-\frac{c\Omega}{2\pi}\int_{\pi+\psi_1}^{2\pi-\psi_1}\sin 2\psi\,d\psi=0.$$

For the coefficient $q'(A, \Omega, x^0)$ we have

$$q'(A,\ \Omega,\ x^0)=\frac{1}{\pi A}\int_0^{2\pi}F_2(x^0+A\sin\psi)\,A\Omega\cos\psi\cos\psi\,d\psi=$$
$$=-\frac{c\Omega}{\pi}\int_{\pi+\psi_1}^{2\pi-\psi_1}\cos^2\psi\,d\psi.$$

Carrying out the integration and taking into account the value of ψ_1, we obtain

$$q'(A, \Omega, x^0) = -\frac{c\Omega}{\pi}\left(\frac{\pi}{2} - \arcsin\frac{x^0}{A} - \frac{x^0}{A}\sqrt{1-\frac{(x^0)^2}{A^2}}\right). \qquad (5.149)$$

[Footnotes]

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443	See specific formulas for various nonlinearities in §5.6 and beyond.
454	The problem can also be solved directly without resorting to formulas of the general type, as was done in §1.6.
456	The frequency is independent of the magnitude of the external action only in the first approximation considered here. When the second harmonic of the oscillations is taken into account, a dependence of the frequency on the external action is observed (see §8.6).
473	The index "i" is used here so as not to confuse F_1 with the main symbol F used in the present book for the nonlinearities.
475	Strictly speaking, self-oscillations occur in the entire system, but in practice their amplitude is negligible beyond the limits of the internal loop. In this sense, one can speak of an equilibrium state of the system as a whole, defined by means of an equation such as that of (5.6) or (5.71).
477	For more details concerning this see Chapter 9.
489	The same formulas as given here for F^0, q, and q' will hold true also for oscillations on one general type relay contact (Fig. 5.16a) with addition only of a third condition a.
503	By 1(x) we designate the step function, with value 1 when $x > 0$ and 0 when $x < 0$.

[List of Transliterated Symbols]

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454	O.c = O.s = obratnaya svyaz' = feedback
454	ст = st = staticheskiy = static

[List of Transliterated Symbols (continued)]

466	н = nelineynyy = nonlinear
456	с = s = sistema = system
480	доп = dop = dopustimyy = permissible

Chapter 6

SELF-OSCILLATIONS AND STABILITY OF AUTOMATIC SYSTEMS FOR NONSYMMETRICAL NONLINEARITIES AND IN THE CASE OF AN EXTERNAL DISTURBANCE

§6.1. Analysis of Nonsymmetrical Self-Oscillations

In Chapter 4 we considered the analysis of nonlinear automatic systems for the case of the symmetry of the static characteristics of nonlinear links and the absence of any external disturbance acting on the system. In this case the self-oscillations, if they developed in the system, were symmetrical with respect to the equilibrium state, i.e., there was no constant component in any of the periodically varying variables. The condition for the absence of the constant component

$$\frac{1}{2\pi}\int_0^{2\pi} F(A\sin\psi,\ A\Omega\cos\psi)\,d\psi = 0$$

may be violated only if the characteristic of the nonlinear link is nonsymmetrical with respect to the origin, or when a slowly varying external distrubance is applied to the system for any form of the nonlinear characteristics. By slow variation of the external disturbance we mean here a variation which in practice is imperceptible over one self-oscillation period of the system. In the present chapter, however, we shall not consider the problem of allowing for periodic external disturbances with frequencies close to the natural frequency of the system. This is left for a special chapter devoted to forced oscillations in nonlinear systems.

The disruption of the oscillation symmetry in the system in the presence of nonsymmetrical characteristics in the nonlinear links may be explained as follows. Let us assume that self-oscillations arise in the system (Fig. 6.1a) and that here the angular coordinate of some system element (the shaft V with the lever R) undergoes symmetrical oscillations having an amplitude A (Fig. 6.1c) relative to the equilibrium position OO_1. In what follows we shall refer to the equilibrium position as the center of oscillation. Let us now introduce an elastic arresting device (Fig. 6.1b) into this same system on one side. The reaction moment of the arresting device will be applied to the shaft, and may be allowed for in the form of the static nonsymmetrical characteristic shown in Fig. 6.1e. The shaft oscillations are deformed because of the unidirectional reaction of the arresting device, and the center of oscillation occupies the new position OO_2 (Fig. 6.1d).

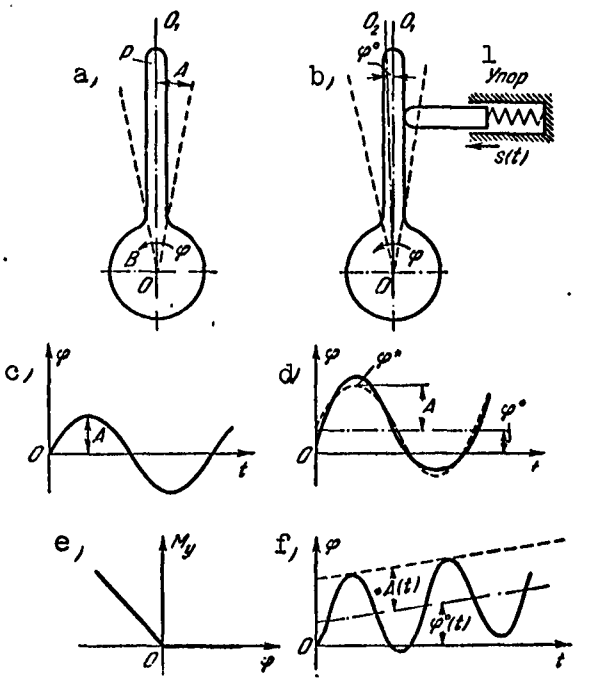


Fig. 6.1. 1) Arresting device.

Relative to the previous equilibrium state, the oscillations may now be considered in approximation as consisting of a constant component φ^0 and a periodic component $\varphi^* = A \sin \Omega t$ (Fig. 6.1d). In the present case the value of the arrester's reaction moment is not an explicit function of time, but is determined by the coordinate φ.

Let us assume that in this same system the arresting device is slowly displaced, together with its base, toward the left according to some relationship s(t). Then an arresting device reaction moment which is an explicit function of time will act upon the shaft. In this case it is impossible to make allowance for it in the form of a single static characteristic; rather we must further introduce into consideration an external disturbance which is applied to the system and varies slowly over time. It is clear that deformation of the oscillation will also take place in this case. In contrast to the case of a simple nonsymmetrical nonlinearity, the frequency, amplitude and displacement of the center of oscillation are now functions not only of the system parameters, but also of time (Fig. 6.1f).

For the case where external disturbances are allowed for, analysis of the self-oscillations presents a more complete problem, since allowance for the nonsymmetry of the static charateristics of the nonlinear links is essentially equivalent to the case where a constant external disturbance is applied. Therefore let us consider the problem of analysis of nonlinear automatic systems in the presence of external disturbances that are functions of time, and do so for arbitrary form of the static characteristics of the nonlinear links.

Let us assume that a slowly varying external (perturbing or setting) disturbance f(t) is introduced at some point of the system (Fig. 6.2); here the equation of the linear part of the system has the form

$$Q(p)x_1 = -R(p)x_2 + S(p)f(t). \qquad (6.1)$$

This is very often the case in self-oscillatory servosystems (and programmed control systems), when f(t) denotes an external setting disturbance at the input of the system, and also in integrators and other computing devices operating in a self-oscillatory mode, where f(t) denotes the function being integrated or another input function. This same case may also occur in ordinary control systems, where f(t) denotes an external perturbing disturbance (a change in the load, etc.).

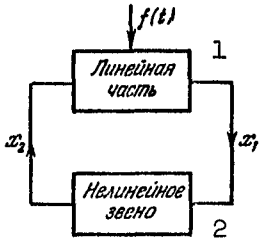


Fig. 6.2. 1) Linear part; 2) nonlinear link.

In the general case, we shall assume that the function f(t) itself and all of its derivatives occurring in Equation (6.1) vary so slowly that in determining the self-oscillations they may all be considered constant for the time of one self-oscillation period. If the function f(t) itself is of an oscillatory nature, then the frequency of its variation is assumed to be many times smaller than the self-oscillation frequency of the system in question. In a particular case, the external disturbance f(t) may be a constant quantity:

$$f(t) = \text{const}$$

or may vary at a constant rate:

$$f(t) = c_1 t + c_2.$$

A nonlinear link of a system may belong to the first or second class, i.e., it may be described, for example, by any equation of the form

$$x_2 = l'(x_1), \quad F(\dot{x}_2, x_2) = c_1 x_1, \quad F_1(\ddot{x}_2, \dot{x}_1) + F_2(x_2) = c_1 x_1,$$
$$x_2 = F(x_1, \dot{x}_1), \quad x_2 = F_1(x_1) + F_2(\dot{x}_1),$$
$$F_2(\dot{x}_2, x_2) = F_1(x_1), \quad F_3(\dot{x}_2) + F_2(x_2) = F_1(x_1),$$
$$F(\dot{x}_2, x_2, \dot{x}_1) = 0, \quad F_2(x_2) + F_1(x_2, x_1) = 0.$$

This same method may also be extended to nonlinear systems of the third class (with several nonlinear links that are separated by linear parts). Here the nonlinear links may have both symmetrical

and nonsymmetrical nonlinear characteristics.

It is convenient to seek a periodic solution in harmonic form for the variable in the nonlinearity. When both the input and the output of the nonlinear link appear in the nonlinear function (nonlinearity of the second class), then just as for symmetrical characteristics, having found the output of the nonlinear link in harmonic form, we convert the input into the output value using the transfer function of the linear part of the system.

Let us assume, for example, that a nonlinear function of the first class has the form

$$x_2 = F(x_1, px_1); \tag{6.2}$$

then, allowing for the equation (6.1) of the linear part, we obtain the equation of the system:

$$Q(p)x_1 + R(p)F(x_1, px_1) = S(p)f(t). \tag{6.3}$$

In contrast to the case of symmetrical nonlinearities and the absence of an external disturbance, here we seek a solution for the variable x_1 (or x_2) in harmonic form allowing for a constant component:

$$x_1 = x_1^0 + A\sin\psi, \quad \psi = \Omega t. \tag{6.4}$$

Allowing for the constant component and the first harmonic of the Fourier-series expansion for the nonlinear function, the formula for the harmonic linearization of the nonlinearities takes the form

$$\begin{aligned} &F(x_1^0 + A\sin\psi) = \\ &\quad = F^0(A, \Omega, x_1^0) + \left[q(A, \Omega, x_1^0) + \frac{q'(A_1, \Omega, x_1^0)}{\Omega}p\right](x_1 - x_1^0), \end{aligned} \tag{6.5}$$

where F^0, q and q' are, respectively, the constant component and the harmonic-linearization coefficients determined by Formulas (5.86) - (5.88). The harmonic linearization is also performed analogously for nonlinearities of other types.

After replacing the nonlinear function in Equation (6.3) by the linear relationship (6.5), we obtain the harmonically linearized equa-

tion of the system:

$$Q(p)x_1 + R(p)\left[F^0 + \left(q + \frac{q'}{\Omega}p\right)(x_1 - x_1^0)\right] = S(p)f(t), \qquad (6.6)$$

or, if we denote the periodic component by $x_1^* = x_1 - x_1^0$:

$$Q(p)(x_1^0 + x_1^*) + R(p)\left[F^0 + \left(q + \frac{q'}{\Omega}p\right)x_1^*\right] = S(p)f(t). \qquad (6.7)$$

Allowing for the fact that S(p)f(t) is a slowly varying quantity, we may represent Equation (6.7) in the form of two equations:

$$\left[Q(p) + R(p)\left(q + \frac{q'}{\Omega}p\right)\right]x_1^* = 0, \qquad (6.8)$$

$$Q(p)x_1^0 + R(p)F^0 = S(p)f(t). \qquad (6.9)$$

Equation (6.8) describes the periodic motion of the system (if such is possible) along the coordinate x_1^* relative to x_1^0, which is the varying position of the center of oscillation. Equation (6.9) de-describes the motion of the center of oscillations relative to the reference origin as a function of the external disturbance f(t). Using the substitution p = jΩ, we obtain two equations from (6.8). As a result, we have three equations:

$$\left.\begin{aligned} X(A, \Omega, x_1^0) &= 0, \\ Y(A, \Omega, x_1^0) &= 0, \\ Q(p)x_1^0 + R(p)F^0 &= S(p)f(t) \end{aligned}\right\} \qquad (6.10)$$

in the three unknowns A, Ω and x_1^0. Equation (6.10) permits us to find these quantities for known system parameters and a given external disturbance. Here we also determine the regions of the periodic solution and the regions where this solution is absent, depending upon the values of the parameters and the external disturbance. The stability of the periodic solutions is determined by the methods introduced earlier.

Let us note that equations of the type of (6.10) may be obtained from Equation (6.6) written for the variable x_1 without resorting to the introduction of the auxiliary variable x_1^*. The first and second equations are obtained from the characteristic equation for (6.6)

after the substitution $p = j\Omega$. The third equation of (6.10) corresponds to the particular solution $x_1 = x_1^0$ of Equation (6.6).

The quantities A, Ω and x_1^0 are determined from Equation (6.10) in the form of time functions for a given slowly varying external disturbance f(t), as described in general form in Chapter 5. However in many cases it is sufficient to find the self-oscillations of the system for the case of various constant values f^0 of the external disturbance. Then Equations (6.10) assume the form

$$\left.\begin{aligned} X(A, \Omega, x_1^0) &= 0, \\ Y(A, \Omega, x_1^0) &= 0, \\ Q(0)\,x_1^0 + R(0)\,F^0 &= S(0)\,f^0, \end{aligned}\right\} \tag{6.11}$$

where Q(0), R(0) and S(0) are the constant terms of the corresponding operator polynomials. In this case the sought quantities A, Ω and x_1^0 are constant and the solution is significantly simplified.

In the case where the external disturbance is absent and we have nonsymmetrical nonlinearities, Equation (6.11) assumes the form

$$\left.\begin{aligned} X(A, \Omega, x_1^0) &= 0, \\ Y(A, \Omega, x_1^0) &= 0, \\ Q(0)\,x_1^0 + R(0)\,F^0 &= 0. \end{aligned}\right\} \tag{6.12}$$

In the presence of nonlinearities of the second class, for example,

$$F_2(px_2,\ x_2) = F_1(x_1), \tag{6.13}$$

the harmonic linearization is performed on the condition that the solution for x_2 and x_1 is sought in the form

$$x_2 = A_2 \sin \Omega t, \qquad x_1 = A_1 \sin(\Omega t + \beta). \tag{6.14}$$

Then the nonlinear equation (6.13) is replaced by the following approximate equation:

$$\begin{aligned} &\left[\frac{q_2'(A_2, \Omega, x_2^0)}{\Omega}\,p + q_2(A_2, \Omega, x_2^0)\right](x_2 - x_2^0) + F_2^0(A_2, \Omega, x_2^0) = \\ &= \left[q_1(A_1, \Omega, x_1^0) + \frac{q_1'(A_1, \Omega, x_1^0)}{\Omega}\,p\right](x_1 - x_1^0) + F_1^0(A_1, \Omega, x_1^0). \end{aligned} \tag{6.15}$$

Denoting $x_2 - x_2^0 = x_2^*$ and $x_1 - x_1^0 = x_1^*$ and omitting the notation for the variables upon which the harmonic linearization coefficients depend, we may rewrite (6.15) in the form

$$\left(\frac{q_2'}{\Omega} p + q_2\right) x_2^* + F_2^0 = \left(q_1 + \frac{q_1'}{\Omega} p\right) x_1^* + F_1^0. \tag{6.16}$$

Equation (6.1) for the linear part of the system is written as follows after the subtitution $x_1 = x_1^0 + x_1^*$ and $x_2 = x_2^0 + x_2^*$

$$Q(p) x_1^* + Q(p) x_1^0 = -R(p) x_2^* - R(p) x_2^0 + S(p) f(t). \tag{6.17}$$

On the basis of (6.16) and 6.17), we write equations for the constant components

$$\left.\begin{aligned} F_2^0 &= F_1^0, \\ Q(p) x_1^0 + R(p) x_2^0 &= S(p) f(t) \end{aligned}\right\} \tag{6.18}$$

and the equations for the periodic components:

$$\left.\begin{aligned} \left(\frac{q_2'}{\Omega} p + q_2\right) x_2^* &= \left(q_1 + \frac{q_1'}{\Omega} p\right) x_1^*, \\ Q(p) x_1^* &= -R(p) x_2^*. \end{aligned}\right\} \tag{6.19}$$

For the last two equations we may write the characteristic equation

$$Q(p)\left(\frac{q_2'}{\Omega} p + q_2\right) + R(p)\left(q + \frac{q_1'}{\Omega} p\right) = 0. \tag{6.20}$$

Making use of the substitution $p = j\Omega$, we obtain two equations from (6.20), to which we join Eqs. (6.18). As a result we obtain the four equations

$$\left.\begin{aligned} X(A_1, A_2, \Omega, x_1^0, x_2^0) &= 0, \\ Y(A_1, A_2, \Omega, x_1^0, x_2^0) &= 0, \\ F_2^0(A_2, \Omega, x_2^0) &= F_1^0(A_1, \Omega, x_1^0), \\ Q(p) x_1^0 + R(p) x_2^0 &= S(p) f(t) \end{aligned}\right\} \tag{6.21}$$

in the five unknowns A_1, A_2, Ω, x_1^0 and x_2^0. The values A_1 and A_2 are connected by the quotient

$$\frac{A_1}{A_2} = \left| W_*(j\omega) \right|_{\omega = \Omega} \text{where } W_*(j\omega) = \frac{R(j\omega)}{Q(j\omega)}.$$

Thus with the use of the amplitude-frequency relationships of the linear part, Equation (6.21) permits us to determine all five unknowns A_2, A_1, Ω, x_1^0 and x_2^0 as functions of the values of the system parameters and of the external disturbance.

The problem is solved analogously for linear systems of the

third class. In this case, in addition to the equations, we use the amplitude and phase relationships for the linear parts separating the nonlinearities occurring in the system.

§6.2. Errors of Self-Oscillatory Systems

The steady-state error of a system operating in a self-oscillatory mode is formed nonlinearly (see Chapter 5) from the error in the displacement of the center of oscillation and the error of the periodic variation for the variable being considered relative to the center of oscillation:

$$x_{\text{уст}} = x^0 + x^*.$$

For the case where symmetrical nonlinearities are present and there is no external disturbance, there remains a steady-state error determined by the periodic component (by the self-oscillations):

$$x_{\text{уст}} = x^*.$$

In this case reduction of the steady-state error is accomplished by choosing the system parameters from the condition for obtaining minimum amplitude and maximum frequency of self-oscillations. Given proper choice of the parameters, we may render x^* negligibly small in practice, so that it does not exert any influence upon the performance of the system. Thus, for example, the self-oscillations of the dc voltage in an aircraft line about a steady-state value with a frequency of 50 – 100 cps are governed by the operation of a vibrational controller and do not in practice influence the operation of the direct current appliances.

The error in the displacement of the center of oscillation is analogous to the static or steady-state error of linear systems. In what follows we shall call it the <u>static error of the self-oscillatory system</u>. This error is almost always undesirable in control sys-

tem.* Thus, in a vibrational voltage-control system, it causes a variation of the voltage by 2 – 3 v depending upon the size of the load. Elimination of the static error in self-oscillatory systems necessitates not only choosing the parameters rationally, but often a modification of the system's structure or the use of special compensation methods.

We call a self-oscillatory system <u>astatic</u> if the condition

$$x^0 = 0$$

is guaranteed by its working principle with an accuracy sufficient for practice for the quantity which is being controlled or corrected in the case of a constant external disturbance $f(t) = \text{const} = f^0$, and <u>static</u> if

$$x^0 \neq 0.$$

In a nonlinear system, the displacement of the center of self-oscillations is analogous to the behavior of the equilibrium state of a linear system. Usually the difference consists only in the presence of the corresponding nonlinear dependences and in the fact that self-oscillations take place about the equilibrium state. Therefore, in order to obtain a static self-oscillatory system, we may use methods analogous to the methods for obtaining a static linear system, although other methods are also possible.

The elimination of the static error is particularly important for automatic measuring and computing systems that operate on the compensation principle, and whose equilibrium state is characterized by zero values of all the variables. In this case, a static error will lead to the instability of the system zero, and this is extremely undesirable for computing and other precision devices.

Let us return now to the same system consisting of a linear part and a nonlinear link (Fig. 6.2). Let us consider the case where a non-

symmetrical nonlinearity is present in the system for the case where there is no external disturbance. We will assume that the variable x_1 is the controlled quantity. It follows from the third equation of (6.12) that the static error of the system will be determined by the relationship

$$x_1^0 = -\frac{R(0)}{Q(0)} F^0. \qquad (6.22)$$

Relationship (6.22) means that the nonsymmetrical nonlinearity for the case of an oscillatory mode leads to separation of the constant component F^0 from the output of the nonlinear link; this is equivalent to a "spurious" signal. This causes a static error in the controlled quantity, with the sign of the error opposite that of the constant component. The magnitude of this error is proportional to the magnitude of the constant component, with the proportionality constant determined by the transfer ratios of the linear links.

For the static error to be equal to zero, we must satisfy one of the two conditions $F^0 = 0$ or $R(0) = 0$.

The condition $F^0 = 0$ denotes (in the absence of a disturbance) the reduction of the characteristic of a nonlinear link to a symmetrical or linear characteristic. This may sometimes be accomplished by simple design methods. Let us assume, for example, that the contact device (Fig. 6.3a) in some nonlinear system provides for control according to an ideal relay characteristic. Owing to the engagement of the contact with the spring on the element carrying the moving contact, the element carrying the moving contact, the reaction moment M_r, which is

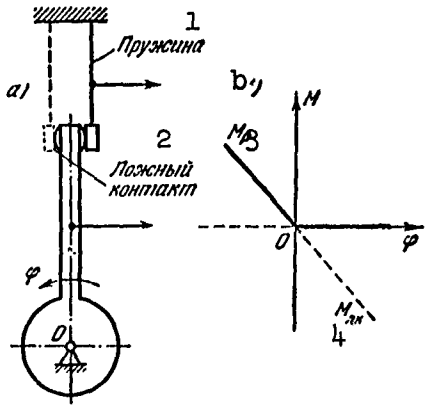


Fig. 6.3. 1) Spring; 2) false contact; 3) R; 4) l.k.

a nonlinear function of the displacement φ (Fig. 6.3b), will operate. The nonlinearity in question is nonsymmetrical and, consequently, it causes a static error in the case of a self-oscillatory mode of operation. If a "false" contact is established on the other side of the moving contact with a spring rigidity equal to that of the operating contact, then the reaction moment $M_{1.k}$ of the false contact (Fig. 6.3b) gives a linear characteristic in combination with the moment of the operating contact. The system will not have a static error.

The condition $R(0) = 0$ signifies that the operator polynomial of the linear part of the system should not have a constant term, i.e., that the operator p is taken out of the parentheses in the polynomial. In this case the linear part must be described by an equation of the form

$$Q(p)\,x_1 = -pR_1(p)\,x_2,$$

or

$$Q(p)\frac{1}{p}\,x_1 = -R_1(p)\,x_2. \tag{6.23}$$

It follows from (6.23) that elimination of the static error resulting in a self-oscillatory system from nonsymmetry of the nonlinearity may be brought about by introducing the integral of the controlled-variable error into the control relationship (i.e., by the method used in linear systems to obtain system astaticism with respect to the external disturbance).

Let us return to the more general case where a constant external disturbance acts upon a self-oscillatory system for any form of the static characteristic of the nonlinear link. From the third equation of (6.11) we have

$$x_1^0 = -\frac{R(0)}{Q(0)}f^0 + \frac{S(0)}{Q(0)}f^0. \tag{6.24}$$

In this case the static error of the system is composed of the error

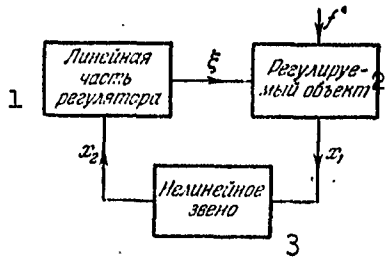


Fig. 6.4. 1) Linear part of controller; 2) controlled object; 3) nonlinear link.

governed by the non-symmetry of the nonlinear characteristic and the error caused by the external disturbance. For the case of a definite combination of nonsymmetry of the static characterisitic with an external disturbance, mutual compensation of these errors is possible:

$$\frac{R(0)}{Q(0)}F^0 = \frac{S(0)}{Q(0)}f^0$$

and then $x_1^0 = 0$.

Such mutual compensation is possible only for one mode of operation, since F^0 is a function of the self-oscillation amplitude and frequency.

A second possible condition for obtaining an astatic self-oscillatory system is

$$R(0)=0 \text{ and } S(0)=0,$$

i.e., in this case the operator polynomial for the input of the linear part and the operator polynomial for the external disturbance may not have constant terms.

Satisfaction of the second condition also involves bringing the error integral of the controlled quantity into the control relationship. Let us show this in an example. Let us assume that the non-linear system consists of a linear controlled object and a controller having a linear part and a nonlinear link (Fig. 6.4). We write the corresponding differential equations in general form:

for the controlled object

$$Q_1(p)x_1 = -R_1(p)\xi + S_1(p)f^0, \qquad (6.25)$$

For the linear part of the controller

$$Q_2(p)\xi = R_2(p)x_2 \qquad (6.26)$$

and for a nonlinear link with the introduction of the input integral*

$$x_2 = F(x_1) + \frac{k}{p} x_1. \qquad (6.27)$$

Combining (6.26) and (6.27), we obtain the equation for the controller:

$$Q_2(p)\xi = R_2(p) F(x_1) + \frac{kR_2(p)}{p} x_1. \qquad (6.28)$$

In accordance with (6.28), the control relationship is

$$\xi = R_2(p) F(x_1) + \frac{kR_2(p)}{p} x_1.$$

Combining the controller equation (6.28) and the equation of the object (6.25) and assuming that the nonlinear function is harmonically linearized according to the formula

$$F(x_1) = F^0 + \left(q + \frac{q'}{\omega} p\right)(x_1 - x_1^0),$$

we obtain the system equation

$$\left[pQ(p) + kR(p)\right] x_1 + \left(q + \frac{q'}{\omega} p\right) R(p) p\,(x_1 - x_1^0) = \\ = -pR(p) F^0 + pS(p) f^0, \qquad (6.29)$$

where

$$Q(p) = Q_1(p) Q_2(p), \quad S(p) = S_1(p) Q_2(p), \quad R(p) = R_1(p) R_2(p).$$

From (6.29), for the case $x_1 = x_1^0$, we obtain an equation for the slowly varying components

$$[pQ(p) + kR(p)] x_1^0 = -pR(p) F^0 + pS(p) f^0.$$

Hence for the case of constant values of F^0 and f^0, the static error of the system will be equal to zero for $R(0) \neq 0$:

$$x_1^0 = \frac{0}{kR(0)} = 0.$$

However, in the case of variation of the external disturbance over time, a steady-state error appears both because of the variation of the quantity f(t) itself and because of the variation of the self-oscillation mode and hence of F^0 (t).

Similarly, we may also obtain a static self-oscillatory system with a high degree of astaticism in order to guarantee a zero static error for the case of a variable external disturbance.

We must keep in mind that the introduction of additional integrating devices into the system causes deterioration of the transient process and the steady-state self-oscillatory mode. If a region of instability lies outside the self-oscillation region, the introduction of integrating devices may lead to system instability. Instead of the introduction of the integral, known methods for the introduction of secondary disturbance-control circuits may prove very useful for avoiding this.

§6.3. Gyropendulum with External Disturbance

In §4.2, we performed an analysis of self-oscillations of a gyropendulum due to the stabilization system of the gyroscope, in the absence of an external disturbance. As a result of the analysis, recommendations were obtained for the choice of gyropendulum parameters from the conditions for obtaining the minimum amplitude and maximum frequency of the self-oscillations. Let us now investigate self-oscillations of a gyropendulum allowing for an external disturbance of constant magnitude μ_0 [185].

In allowing for the external disturbance, we write the equation of the gyroscope about the axis of the inner gimbal in accordance with (4.3) in the form

$$[A_0B_0p^3+(A_0n_1+B_0n_2)p^2+H^2p]\beta=-HF(\beta)+n_0\mu_0. \qquad (6.30)$$

Because of the presence of the external disturbance μ_0, the center of the gyroscope's oscillation in the angle β will be displaced by an amount β_0. Harmonic linearization of the ideal relay static characteristic $F(\beta)$ will now be carried out according to the formula

$$F(\beta)=F^0(A,\beta_0)+q(A,\beta_0)(\beta-\beta_0), \qquad (6.31)$$

where in accordance with (5.101) and (5.102)

$$F' = \frac{2m_0}{\pi} \arcsin \frac{\beta_0}{A},$$

$$q = \frac{4m_0}{\pi A} \sqrt{1 - \frac{\beta_0^2}{A^2}}.$$

Allowing for (6.31), Equation (6.30) is rewritten in the form

$$[A_0B_0p^3 + (A_0n_\beta + B_0n_\alpha)p^2 + H^2p + Hq]\beta = \\ = -HF^0 + Hq\beta_0 + n_\alpha\mu_0. \tag{6.32}$$

Replacing p by $j\Omega$ in the characteristic equation

$$A_0B_0p^3 + (A_0n_\beta + B_0n_\alpha)p^2 + H^2p + Hq = 0,$$

we obtain two equations from the conditions $X(A, \Omega, \beta_0) = 0$ and $Y(A, \Omega, \beta_0) = 0$. We write the third equation as a particular solution of the differential equation (6.32) for the case $\beta = \beta_0$. The three equations indicated for determination of the self-oscillation amplitude and frequency are

$$\left.\begin{aligned} Hq - (A_0n_\beta + B_0n_\alpha)\Omega^2 &= 0, \\ H^2 - A_0B_0\Omega^2 &= 0, \\ HF^0 &= n_\alpha\mu_0. \end{aligned}\right\} \tag{6.33}$$

Let us analyze these equations. It is evident from the second equation of (6.33) that the self-oscillation frequency of the gyropendulum is not a function of the external disturbance and remains the same as in the case $\mu_0 = 0$, i.e.

$$\Omega = \frac{H}{\sqrt{A_0B_0}}.$$

The first and third equations permit us to find the variation in the amplitude A and the oscillation-center displacement β_0 as functions of the system parameters and the external disturbance μ_0. The influence of the system parameters on the self-oscillation amplitude in the absence of an external disturbance was determined earlier.

Substituting the values of the harmonic-linearization coefficients

into (6.33), we write two equations for determining the functions $A = A(\mu_0)$ and $\beta_0 = \beta_0(\mu_0)$:

$$\left.\begin{aligned}(A_0 n_\beta + B_0 n_\alpha)\Omega^2 &= \frac{4Hm_0}{\pi A}\sqrt{1-\frac{\beta_0^2}{A^2}},\\ \frac{2Hm_0}{\pi}\arcsin\frac{\beta_0}{A} &= n_\alpha\mu_0.\end{aligned}\right\} \qquad (6.34)$$

From the condition for breakoff of the self-oscillations, let us use the second equation to determine the admissible maximum measurable moment applied to the gyropendulum. The self-oscillations of the gyropendulum will be broken off at some value $\mu_0 = \mu_{max}$ when the displacement of the center of oscillations becomes equal to the amplitude, i.e., $\beta_0 = A$. Then, using the second equation of (6.34) we obtain a formula for determining the limiting admissible moment applied to the gyropendulum:

$$\mu_{max} = \frac{Hm_0}{n_\alpha}. \qquad (6.35)$$

For $\mu_0 > \mu_{max}$, self-oscillations in the system are impossible and hence the operating principle of the gyropendulum is violated.

It is evident from the first equation of (6.34), allowing for th the value of Ω, that the self-oscillation amplitude will have its maximum value for the case $\beta_0 = 0$ when $\mu_0 = 0$:

$$A_{max} = \frac{4m_0 A_0 B_0}{\pi(A_0 n_\beta + B_0 n_\alpha)H}. \qquad (6.36)$$

The value of the amplitude for any value $0 \leq \mu_0 \leq \mu_{max}$ is determined from the first equation of (6.34) by the formula

$$A = A_{max}\cos\arcsin\frac{\beta_0}{A}, \qquad (6.37)$$

since

$$\sqrt{1-\frac{\beta_0^2}{A^2}} = \cos\arcsin\frac{\beta_0}{A}.$$

Substituting the value

$$\arcsin\frac{\beta_0}{A} = \frac{\pi n_\alpha \mu_0}{2Hm_0},$$

from the second equation of (6.34) into Equation (6.37), we obtain a formula for determining the self-oscillation amplitude of a gyropendulum as a function of the external moment:

$$A = A_{max} \cos \frac{\pi n_\alpha \mu_0}{2Hm_0} \text{ for } 0 \leqslant \mu_0 \leqslant \mu_{max}. \tag{6.38}$$

As we see, the self-oscillation amplitude varies according to a cosine law as a function of the applied external moment.

In order to determine $\beta_0 = \beta_0 (\mu_0)$, let us write an expression for β_0 from the second equation of (6.34):

$$\beta_0 = A \sin \frac{\pi n_\alpha \mu_0}{2Hm_0}. \tag{6.39}$$

Substituting the value of A from (6.38) into (6.39) and performing transformations, we obtain a formula for determining the displacement of the center of oscillations as a function of the external moment applied:

$$\beta_0 = \frac{A_{max}}{2} \sin \frac{\pi n_\alpha \mu_0}{Hm_0} \text{ for } 0 \leqslant \mu_0 \leqslant \mu_{max}. \tag{6.40}$$

As may be seen from (6.40), the displacement of the center of oscillation varies according to a sine law as a function of the external moment μ_0, but with a period half the variation period of the self-oscillation amplitude. The maximum displacement of the center of oscillation is

$$\beta_{max} = \frac{A_{max}}{2} \text{ for } \mu_0 = \frac{\mu_{max}}{2}.$$

The relationships (6.38) and (6.39) are shown graphically in Fig. 6.5 for the case of the values taken earlier for the gyropendulum parameters: H = 400 g·cm·sec, n_α = 2 g·cm·sec, A_0 = 2 g·cm·sec^2, n_β = 1 g·cm·sec, B_0 = 2 g·cm·sec^2, m_0 = 30 g·cm. The limiting admissible measurable moment for the case of the chosen parameter values is

$$\mu_{max} = 6000 \text{ g·cm}.$$

The presence of a displacement in the center of oscillation will cause an error in the measurement of the moment μ_0 even in the case where the gyropendulum is placed upon a horizontally stabilized base, since a gravitational force component will act upon the pendulum.

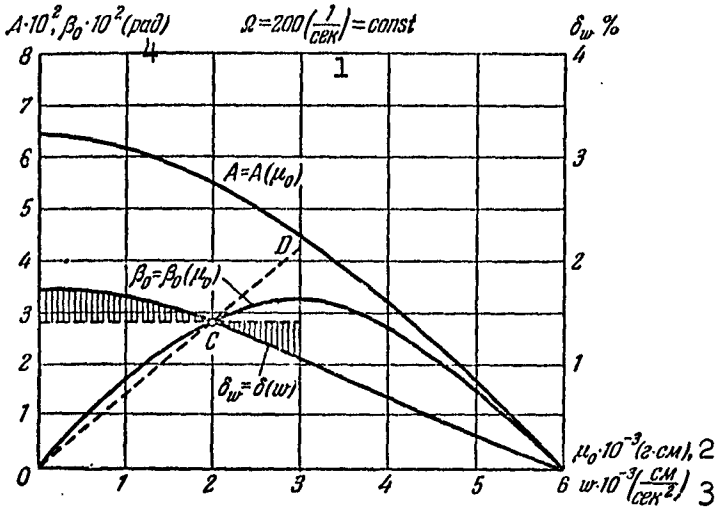


Fig. 6.5. 1) Sec; 2) g.cm; 3) cm/sec^2; 4) radians

For small angles of displacement of the oscillation center, given the condition

$$\beta_0 = \sin \beta_0$$

the absolute error of the gyropendulum in measurement of the moment is

$$\Delta\mu = mgl\beta_0, \qquad (6.41)$$

where m is the mass of the gyroscope together with the inner gimbal, l is the arm of the gyropendulum, and g is the acceleration of gravity.

The relative error of the gyropendulum is

$$\delta_\mu = \frac{\Delta\mu}{\mu_0}. \qquad (6.42)$$

If the gyropendulum is used to measure horizontal inertial accelerations, the external moment applied to the gyropendulum will be proportional to the inertial acceleration:

$$\mu_0 = ml\omega, \tag{6.43}$$

where $\underline{w}$ is the inertial acceleration.

Allowing for (6.40), (6.41), and (6.43), we obtain from (6.42) a formula for determining the relative error of the gyropendulum in measurement of inertial accelerations

$$\delta_w = \frac{gA_{\max}}{2\omega} \sin \frac{\pi n_a ml\omega}{Hm_0}. \tag{6.44}$$

The maximum value of the relative error will prevail at $w = 0$ ($\mu_0 = 0$). This value is determined as the limit of the quantity δ_w for the case where $\underline{w}$ tends toward zero and the value A_{max} (6.36) is taken into account:

$$\delta_{w\max} = \lim_{w \to 0} \delta_w = \frac{2n_a mlgA_0B_0}{(A_0n_\beta + B_0n_a)H^2}. \tag{6.45}$$

It follows from (6.45) that to reduce the relative gyropendulum error it is desirable to reduce the friction in the suspension axis of the outer gimbal, the gyroscope mass with the inner gimbal, and the imbalance arm, and also to increase the angular momentum of the gyroscope. It is most expedient to increase the angular momentum of the gyroscope without increasing its mass, since H occurs squared in the denominator of Formula (6.45).

In Fig. 6.5 we showed the curve of the gyropendulum's relative error, which is calculated from Formula (6.44) for the parameter values taken earlier and the quantity $m\underline{l} = 1$ g·sec^2. In accordance with Formula (6.45), the maximum relative error $\delta_{w\ max} \approx 1.7\%$.

It is evident from the curves of $A(\mu_0)$ and $\beta_0(\mu_0)$ that for the case of large values of μ_0, the magnitudes of the amplitude A and the displacement of the center of oscillation β_0 are close to each other. Because of this it is impossible to guarantee reliable operation of the device for large values of μ_0, since breakoff of the self-oscillations is possible as a result of random disturbances acting upon the

gyropendulum. We may recommend

$$\mu_{0\ xon} \approx \frac{\mu_{0\ max}}{2}. \qquad (6.46)$$

as the admissible value of the measured moment.

With this choice of the calibration point for the device, the relative error of the gyropendulum may be reduced. If, given Condition (6.46), the gyropendulum is calibrated for the case of $\mu_{0\ tar} = 1/3\ \mu_{0\ max}$ (in our case $\mu_{0\ tar} = 200$ g·cm), then for $\mu_0 = \mu_{0\ tar}$ the absolute and relative errors will be equal to zero. For other values of the moments being measured, the relative error will be determined by the difference between the curve of $\delta_w = \delta_w(w)$ and the horizontal straight line passing through the point C (crosshatched ordinates). For this case the relative error is reduced by a factor of approximately five. In our case the maximum relative error is equal to 0.34%.

We must keep in mind that the computation shown was not carried out for optimum values of the gyropendulum parameters. In practice it is possible to obtain self-oscillations of the gyropendulum with an amplitude of a few minutes of arc. For the case where the gyro's maximum oscillation amplitude is $A_{max} = 6'$, the relative error, assuming choice of an advantageous calibration point, does not exceed 0.01% in the present case.

If the choice of the parameters does not accomplish reduction of the error due to displacement of the oscillation center, then it may be compensated, since we know the relationship for its variation as a function of the measured quantity.

The error due to instability of the zero represents the greatest annoyance in devices of this types. Because of the erosion and oxidation that take place during operation of the contacts controlling the stabilizing motor, it is impossible to obtain a stable relay character-

istic. Over the course of time the characteristic will be displaced to the right or to the left of its initial position (Fig. 6.6). The displacement of the characteristic is irregular and hence an irregular "zero drift" of the gyropendulum will take place. The gyropendulum error due to displacement of the characteristic cannot be reduced by calibration or by compensation. In order to reduce the error due to "zero drift," we must use highly stable contacts or else make use of a contactless switching device with a stable ideal relay characteristic.

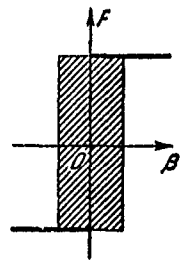


Fig. 6.6.

We must keep in mind that the errors caused by the stabilization inaccuracy of the base upon which the gyropendulum is mounted and the error due to the stabilization inaccuracy of the gyroscope's natural turning rate will be added to the above errors.

§6.4. Vibrational Accelerometer

Operating principle of the accelerometer. The vibration accelerometer is shown schematically in Fig. 6.7. The magnetic measuring system consists of the permanent cylindrical magnet 1 with the magnetic circuits 2 and 3. In the cylindrical air gap we have the sensitive coil 5 suspended on the arm 4. The axis of the coil rides on the bearing 6, so that the coil can oscillate in the plane of the diagram. The contact 7 with the spring 9 is fastened to the lever 4 and insulated from the instrument housing. The moving contact 7 operates as a pair with the fixed contact 8. The winding of the sensitive coil is fed from the direct-current source 11 through the electronic commutator 10. The electronic commutator is controlled by the contact pair 7, 8. The commutator feeds voltages of different polarities to the

coil, depending on the state (closed or opened) of the electronic-commutator contacts. The device is adjusted so that making of the contacts 7 and 8 corresponds to the equilibrium position of the coil. The slightest deflection of the coil in either direction leads to the appearance of a voltage of the polarity corresponding to this position of the coil. Interacting with the magnetic field, current flowing in the coil sets up an electromagnetic moment which is applied to the coil. The coil is connected so that the electromagnetic moment produced by closing of the contacts sets it in motion in the direction that opens the contacts. Thus it is evident from the operating principle of the accelerometer that the coil will undergo oscillations about the equilibrium state.

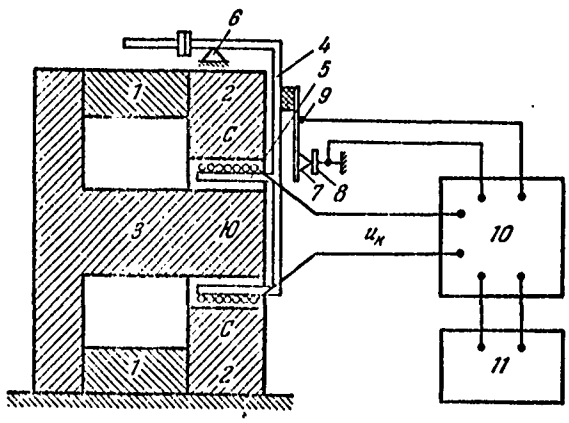


Fig. 6.7.

In order to measure accelerations in a certain direction, the device is usually mounted on a platform stabilized in the horizontal plane. If the platform with the device is placed on a moving object, inertial forces will act upon the coil in the direction of its axis of symmetry on accelerated motion of the object. An external disturbance applied to the self-oscillatory system deforms the oscillations

and varies the constant component of the periodically varying coil current. Since the coil undergoes oscillatory motion relative to some equilibrium state, the moment caused by the inertial forces is compensated by the constant current component generated over an oscillation period.

For the case of a uniform magnetic field, the moment set up by the constant current component in the coil is

$$M_э = cB_\delta W_к i_к^0, \tag{6.47}$$

where B_δ is the induction in the air gap, W_k is the number of turns of the coil, i_k^0 is the constant component of the current flowing in the coil and c is a proportionality coefficient including the design quantities of the accelerometer.

The moment due to the inertial force is

$$M_и = mlw(t), \tag{6.48}$$

where m is the mass of the coil, l is the inertial-force application arm, and w (t) is the acceleration of the moving object in the direction of the coil's axis of symmetry.

For the case of exact mutual compensation of the moments indicated, we may set (6.47) and (6.48) equal to each other; as a result we obtain

$$i_к^0 = kw(t), \tag{6.49}$$

where

$$k = \frac{ml}{cB_\delta W_к}.$$

Hence the measurement of the acceleration is possible by measurement of the constant component of the coil current.

The center of oscillation will be displaced because of the application of the external disturbance and the presence of a nonsymmetrical nonlinearity in the form of the unilateral action of the fixed contact upon the moving contact. Because of the appearance of secondary

moments due to the projection of the force of gravity and the spring reaction, this displacement causes an error in Relationship (6.49).

In the self-oscillatory mode of operation, the displacement of the center of oscillations may not exceed the value of the oscillation amplitude of the sensitive coil (otherwise the contacts will not touch each other and the oscillation will be broken off). Hence in order to guarantee a small error in measurement of the acceleration we must secure small self-oscillation amplitudes for the coil. Here in order to increase the admissible rates of change for the acceleration being measured, it is desirable to increase the self-oscillation frequency.

Let us carry out an analysis of the choice of optimum values for the instrument's parameters on the basis of the conditions for obtaining a small self-oscillation amplitude and a high self-oscillation frequency; let us also analyze the errors which are possible for the instrument.

We will not perform a stability analysis for the periodic solution obtained, since in this case the steady-state self-oscillatory mode of operation is evident for the device from the operating principle itself.

Differential equations of motion of the sensitive coil. Let us assume that the device is shifted in space in the horizontal plane to the right and to the left with the linear acceleration $\underline{w}(t)$. The angular position of the coil in motion about the suspension axis is denoted by φ; here we will take the deflection of the coil from the equilibrium state in the direction of contact closing as positive values of the angle.

We will assume that the coil executes its motion within the limits of small angles. We will neglect the forces of dry friction in

the coil suspension axis. This is justified by the fact that dry friction is significantly reduced for the case of coil oscillations. Then allowing for the above assumptions, the moment equation for the motion of the coil is written in the form

$$J\frac{d^2\varphi}{dt^2}+k_\varphi\frac{d\varphi}{dt}+k_g\varphi+F_2(\varphi)=-k_e i_k+k_w w(t), \qquad (6.50)$$

where J [g·cm·sec^2] is the moment of inertia of the coil relative to the suspension axis; k_φ [g·cm·sec] is the damping coefficient; k_g [g·cm] is the coefficient of proportionality between the deflection angle of the coil and the moment set up by the gravitational force component; k_e [g·cm/amp] is the proportionality coefficient between the current in the coil and the electromagnetic moment; k_w [g·sec^2] is the proportionality coefficient of the inertial force; i_k [amp] is the current flowing in the coil; $F_2(\varphi)$ [g·cm] is the nonlinearly varying moment due to the reaction of the fixed contact upon the moving contact.

The equilibrium equation for the voltages operating in the closed loop incorporating the coil is

$$L_k\frac{di_k}{dt}+R_k i_k+k_e\dot{\varphi}=-F_1(\varphi), \qquad (6.51)$$

where L_k [hy] is the inductance of the coil circuit, R_k [ohms] is the active resistance of the coil circuits, k_e [v·sec] is the proportionality coefficient between the rate of coil displacement and the inductive emf induced in the coil and $F_1(\varphi)$ [v] is the nonlinearly varying voltage from the electronic commutator applied to the coil.

Introducing new notation, we rewrite Equation (6.50) in the form

$$(T_1^2p^2+T_1p+1)\varphi=\frac{k_e}{k_g}i_k-\frac{F_2(\varphi)-k_w w(t)}{k_g}, \qquad (6.52)$$

where

$$T_1^2=\frac{J}{k_g}[\text{sec}^2];\ T_1=\frac{k_\varphi}{k_g}\ [\text{sec}],$$

while Equation (6.51) is rewritten in the form

$$(T_к p + 1) i_к = -\frac{F_1(\varphi) + k_e p \varphi}{R_к}, \qquad (6.53)$$

where $T_k = (L_k/R_k)$ [sec] is the electric [time] constant of the coil circuit.

Combining (6.52) and (6.53), we obtain the equation of motion of the coil for its deflection angle from the initial position:

$$[T_э^2 T_к p^3 + (T_э^2 + T_1 T_к) p^2 + (T_1 + T_к + T_e) p + 1] \varphi = \\ = k_1 F_1(\varphi) - (T_к p + 1)[k_2 F_2(\varphi) - k_3 w(t)], \qquad (6.54)$$

where

$$T_e = \frac{k_э k_e}{k_г R_к} \text{ [sec]}, \quad k_1 = \frac{k_э}{R_к k_г} \left[\frac{1}{\text{г}}\right],$$
$$k_2 = \frac{1}{k_г} \left[\frac{1}{\text{г} \cdot \text{см}}\right], \quad k_3 = \frac{k_w}{k_г} \left[\frac{\text{сек}^2}{\text{см}}\right].$$

The nonlinearly varying voltage from the commutator that is applied to the sensitive coil will be allowed for in the form of the static characteristic shown in Fig. 6.8a, while the nonlinearly varying moment caused by the reaction force of the fixed contact upon the moving contact will be allowed for in the form of the static characteristic shown in Fig. 6.8b. In these diagrams we denote by U the constant value of the voltage supplied from the commutator, by c_1 the rigidity of the movable-contact spring with respect to the coil rotation angle.

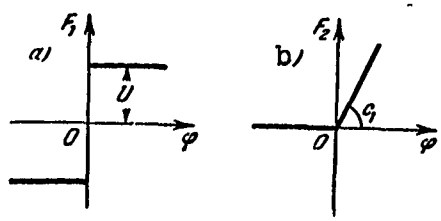


Fig. 6.8.

Since there is a nonsymmetrical nonnonlinearity in the system and the characteristics $F_1(\varphi)$ and $F_2(\varphi)$ are single-valued, the harmonic linearization of the nonlinear static characteristics will be carried out according to the formula

$$\left.\begin{aligned} F_1(\varphi) &= F_1^0(A, \varphi_0) + q_1(A, \varphi_0)(\varphi - \varphi_0), \\ F_2(\varphi) &= F_2^0(A, \varphi_0) + q_2(A, \varphi_0)(\varphi - \varphi_0), \end{aligned}\right\} \qquad (6.55)$$

where, in accordance with Formulas (5.101), (5.102), (5.141) and (5.142), the constant components and the harmonic-linearization coefficients will have the values

$$\left.\begin{aligned} F_1^0(A, \varphi_0) &= \frac{2U}{\pi}\arcsin\frac{\varphi_0}{A}, \\ q_1(A, \varphi_0) &= \frac{4U}{\pi A}\sqrt{1-\frac{\varphi_0^2}{A^2}}, \\ F_2^0(A, \varphi_0) &= \frac{cA}{\pi}\sqrt{1-\frac{\varphi_0^2}{A^2}}+\frac{c\varphi_0}{2\pi}\left(\pi+2\arcsin\frac{\varphi_0}{A}\right), \\ q_2(A, \varphi_0) &= \frac{c}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\varphi_0}{A}+\frac{\varphi_0}{A}\sqrt{1-\frac{\varphi_0^2}{A^2}}\right), \end{aligned}\right\} \qquad (6.56)$$

where A and φ_0 are the oscillation amplitude and the displacement of the oscillation center of the sensitive coil, respectively.

Substituting the expressions for $F_1(\varphi)$ and $F_2(\varphi)$ from (6.55) into (6.54), we obtain the harmonically linearized differential equation for the motion of the sensitive coil:

$$\begin{aligned} &[T_2^2T_\kappa p^3+(T_2^2+T_1T_\kappa)p^2+(T_1+T_\kappa+T_e+k_2T_\kappa q_2)p+ \\ &+k_1q_1+k_2q_2+1]\varphi=-k_1(F_1^0-q_1\varphi_0)-k_2(T_\kappa p+1)(F_2^0-q_2\varphi_0)+ \\ &\qquad +k_3(T_\kappa p+1)w(t). \end{aligned} \qquad (6.57)$$

Equations for determination of self-oscillations. The operator polynomial before φ in (6.57) is the left member of the corresponding characteristic equation. Then on the basis of the substitution $p = j\Omega$, we obtain two equations from the characteristic equation: $X(A, \Omega, \varphi_0) = 0$ and $Y(A, \Omega, \varphi_0)$. We obtain the third equation as a particular solution of the differential equation (6.57) for the case $\varphi = \varphi_0$, where φ_0 is the slowly varying component. The three equations indicated are

$$\left.\begin{aligned} -(T_2^2+T_1T_\kappa)\Omega^2+k_1q_1(A, \varphi_0)+k_2q_2(A, \varphi_0)+1&=0, \\ -T_2^2T_\kappa\Omega^2+T_1+T_\kappa+T_e+k_2T_\kappa q_2(A, \varphi_0)&=0, \\ [T_2^2T_\kappa p^3+(T_2^2+T_1T_\kappa)p^2+(T_1+T_\kappa+T_e)p+1]\varphi_0+ \\ +k_1F_1^0(A, \varphi_0)+k_2(T_\kappa p+1)F_2^0(A, \varphi_0)- \\ -k_3(T_\kappa p+1)w(t)&=0. \end{aligned}\right\} \qquad (6.58)$$

The equations obtained permit us to find the amplitude A, the frequency Ω and the displacement φ_0 of the center of oscillation for

given system parameters and external disturbance (measured acceleration w(t)) as a function of time. But solution of these equations in general form is difficult, since on substitution of the harmonic-linearization coefficients the first two equations of (6.58) will be algebraic and transcendental, while the third will be differential. It is simpler in practice to solve with constant measured accelerations $w(t) = w_0 = \text{const}$ for the values of interest to us. This solution will be just as useful for sufficiently slow variation of the external input disturbance.* In this case the third equation of (6.58) will be algebraic and all three equations will be rewritten in the form

$$\left.\begin{aligned} -(T_3^2 + T_1T_к)\Omega^2 + k_1q_1(A, \varphi_0) + k_2q_2(A, \varphi_0) + 1 = 0, \\ -T_3^2T_к\Omega^2 + T_1 + T_к + T_e + k_2T_кq_2(A, \varphi_0) = 0, \\ \varphi_0 + k_1F_1^0(A, \varphi_0) + k_3F_2^0(A, \varphi_0) - k_3w_0 = 0. \end{aligned}\right\} \qquad (6.59)$$

Instrumental errors. It is evident from the third equation of (6.59) that the constant component F_1^0 of the periodically varying voltage across the coil is proportional to the measured acceleration w_0 with a certain error defined by the first and third terms of the equation. Hence Relationship (6.49) defining the operating principle of the device is not satisfied exactly.

In accordance with the third equation of (6.59), the relative error in the acceleration measurement is

$$\delta_w = \frac{\varphi_0 \pm k_3F_2^0(A, \varphi_0)}{k_3w_0}. \qquad (6.60)$$

The sign $\pm$ in front of F_2^0 means that the constant component governed by the nonlinear function $F_2(\varphi)$ changes its sign when the sign of the acceleration is changed, while φ_0 always has the same sign.

If the object moves with a positive acceleration (the acceleration deflects the coil in the direction of negative angles), then the formula

$$\delta_w = \frac{\varphi_0 + k_3F_2^0(A, \varphi_0)}{k_3w_0},$$

will be valid, while for a negative acceleration

$$\delta_w = \frac{\varphi_0 - k_3F_2^0(A, \varphi_0)}{k_3w_0}. \qquad (6.61)$$

It follows from (6.60) that even if coil control of the commu-

tator coil were accomplished by means of a contactless device with an ideal relay characteristic, then there will be even in this case a relative acceleration-measurement error that is proportional to the deflection of the center of oscillation from the initial position:

$$\delta_w = \frac{\varphi_0}{k_3 w_0}. \tag{6.62}$$

In the case of the measurement of accelerations having one sign, it is advantageous to mount the device so that the coil deflections caused by the accelerations will be in the direction of the fixed contact. In this case, as is evident from (6.61), we may obtain mutual compensation of the relative error components for one value of the acceleration:

$$\frac{\varphi_0}{k_3 w_0} \quad \text{and} \quad \frac{k_3 F_2^0(A, \varphi_0)}{k_3 w_0}.$$

For a quantitative estimate of the instrument's error, we must find φ_0 and F_2^0 as functions of the acceleration.

It is evident from Formula (6.60) that in order to reduce the instrumental error, it is necessary to get rid of the nonsymmetrical nonlinearity $F_2(\varphi)$. In practice this may be accomplished by using a movable-contact spring with a very low rigidity and a false contact and by using a contactless device to guarantee an ideal relay characteristic.

Let us first carry out an analysis of the self-oscillations for the case of the condition $F_2(\varphi) = 0$. In this case, Equation (6.59) is rewritten in the form

$$\left.\begin{aligned} (T_2^2 + T_1 T_к)\,\Omega^2 &= k_1 q_1(A, \varphi_0) + 1, \\ T_2^2 T_к \Omega^2 &= T_1 + T_к + T_c, \\ \varphi_0 + k_1 F_1^0(A, \varphi_0) &= k_3 w_0. \end{aligned}\right\} \tag{6.63}$$

The choice of accelerometer parameters. Equation (6.63) permits us to determine the variation of the amplitude A and the frequency Ω, and the displacement of the center of oscillation φ_0 as functions of

the instrument's parameters and the magnitude of the accelerations to be measured. It is important to obtain the relationship $\varphi_0 = \varphi_0 (w_0)$ for the case of selected system parameters in order to obtain a quantitative estimate of the accelerometer's error.

For choice of the accelerometer parameters it is sufficient to consider the case where $w_0 = 0$. Then the constant component F_1^0 will also be equal to zero (by the operating principle of the device) and hence φ_0 becomes equal to zero.

In accordance with (6.56), the harmonic-linearization coefficient q_1 will have the value

$$q_1(A) = \frac{4U}{\pi A}. \tag{6.64}$$

Allowing for (6.64) we obtain the two equations

$$\left.\begin{aligned}(T_2^2 + T_1 T_k)\Omega^2 &= \frac{4k_1 U}{\pi A} + 1,\\ T_2^2 T_k \Omega^2 &= T_1 + T_k + T_e.\end{aligned}\right\} \tag{6.65}$$

from (6.63). On the basis of Eqs. (6.65), we obtain formulas for determining the frequency and amplitude from the values of the instrument parameters:

$$\Omega = \sqrt{\frac{T_1 + T_k + T_e}{T_2^2 T_k}} = \sqrt{\frac{T_1 + T_e}{T_2^2 T_k} + \frac{1}{T_2^2}}, \tag{6.66}$$

$$A = \frac{4k_1 T_2^2 T_k U}{\pi [T_1 (T_2^2 + T_1 T_k + T_k T_e + T_k^2) + T_2^2 T_e]}. \tag{6.67}$$

It is evident from (6.66) that to increase the self-oscillation frequencies it is desirable to increase the damping constant T_1 and the constant T_e, which is governed by the induced back emf in the coil, and also to reduce the inertial constant T_2 and the electrical constant T_k of the coil. As is evident from (6.67), increasing the constants T_1 and T_e guarantees a reduction of the self-oscillation amplitude of the sensitive coil. To reduce the amplitude it is also desirable to reduce the gain constant in the closed-loop system (the

parameter k_1) and U.

To choose the accelerometer parameters, we may construct the curves of the self-oscillation amplitude and frequency for variation of each parameter with the other parameters held constant. For the calculations we take the following parameter values (these values correspond approximately to the model of the accelerometer that was built): $T_1 = 0.6$ sec, $T_k = 0.0001$ sec, $T^2 = 0.007$ sec^2, $T_e = 0.004$ sec, $k_1 = 0.7$ 1/amp, $k_3 = 1.2 \cdot 10^{-3}$ sec^2/cm and $U^2 = 1$ v.

Figure 6.9 shows curves of $A(T_1)$ and $\Omega(T_1)$ constructed for the values taken for the parameters. As is evident, damping is an effective method for reducing the amplitude and increasing the frequency of self-oscillation.

It is also easy to construct similar curves for other parameters.

For the values taken for the parameters, the self-oscillation frequency will be f = 148 cps and the oscillation amplitude will be $A = 0.148 \cdot 10^{-3}$ radian ≈ 0.5.

Determination of displacement of the center of oscillation and amplitude as functions of acceleration. It is important to determine the displacement of the center of oscillations as a function of the magnitude of the acceleration being measured in order to have a quantitative estimate of the instrument's error.

Substituting the values of $F_1^0(A, \varphi_0)$ and the harmonic-linearization coefficient $q_1(A, \varphi_0)$ from (6.56) into (6.63), we rewrite the equations for determining the self-oscillations in the form

$$\left.\begin{aligned} (T_2^2 + T_1 T_k)\,\Omega^2 &= \frac{4k_1U}{\pi A}\sqrt{1 - \frac{\varphi_0^2}{A^2}} + 1, \\ T_2^2 T_k \Omega^2 &= T_1 + T_k + T_e, \\ \varphi_0 + \frac{2k_1U}{\pi}\arcsin\frac{\varphi_0}{A} &= k_3\omega_0. \end{aligned}\right\} \qquad (6.68)$$

It follows from the second equation of (6.68) that the self-oscillation frequency Ω is not a function of the magnitude of the accelera-

tion being measured and is determined by Relationship (6.66).

The self-oscillations break off at $\varphi_0 = A$. It is evident from the first equation of (6.68) that for the case of constant system parameters and an approach of the ratio φ_0/A to untiy, the amplitude A and the displacement φ_0 of the center of oscillation must both tend toward zero. Here we may obtain the condition for the disruption of the self-oscillations

$$w_{max} = \frac{k_1}{k_2} U. \qquad (6.69)$$

from the third equation of (6.68). For the values taken for the parameters, an acceleration equal to $w_{max} \approx 580$ cm/sec^2 causes a breakoff of the self-oscillations.

To determine the functions $A(w_0)$ and $\varphi_0(w_0)$, we assign values to φ_0/A in the first equation of (6.68) and determine $A(\varphi_0/A)$ and $\varphi_0(\varphi_0/A)$ for the given instrument parameters. Let us determine the corresponding acceleration values w_0 from the third equation of (6.68) for the same φ_0/A and the values obtained for φ_0.

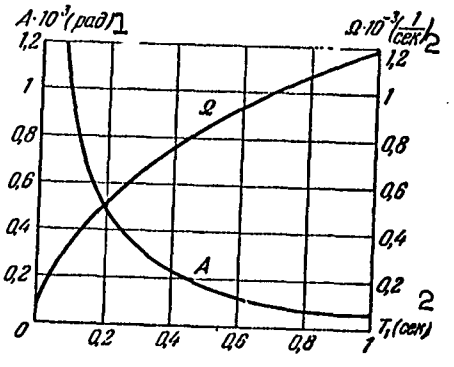


Fig. 6.9. 1) radians; 2) sec.

Figure 6.10 shows curves for the calculations performed with the parameter values taken.

It is evident from the curves that the self-oscillation amplitude has its maximum value at $w_0 = 0$, and that it decreases as the acceleration being measured increases. The deflection of the center of oscillations increases at first and then decreases. Breakoff of the self-oscillations takes place for $A = \varphi_0 = 0$ and $\underline{w}_0 = \underline{w}_{max}$. Figure 6.10 also shows the relative error δ_w% as calculated from Formula (6.62). As is evident, the relative error of the instrument due to its operating principle does not exceed 0.03%.

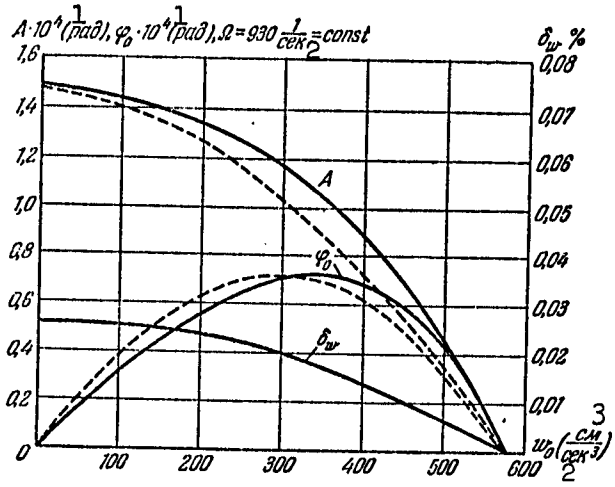


Fig. 6.10. 1) Radians; 2) sec; 3) cm.

For the case of large acceleration values, the displacement of the center of oscillations and the amplitude of the self-oscillation differ only slightly from each other. In this case self-oscillation may be broken off because of random disturbances. Therefore in order to guarantee reliable and stable operation, we must reckon with measurement of accelerations smaller than w_{max}, i.e., we must assume that

$$w_{доп} = (0{,}5 \div 0{,}6)\, w_{max}.$$

In the operating range, the relative error will then be close to a constant value and may be reduced by choosing a calibration acceleration for the device.

Thus, if the oscillation amplitude can be reduced to tenths of a minute of arc or even a few minutes by choosing the parameters, then the error governed by the operating principle of the device will be negligible in practice (thousandths of a percent).

For such devices, the error will be determined basically by the stability of the relay characteristic and the accuracy of stabilization of the device in the horizontal plane. Here, as in the case of

the gyropendulum, it is expedient to convert from the pair of contacts to a contactless controlling apparatus.

Allowance for two nonlinearities. Let us now determine the variation of the amplitude, frequency and displacement of the oscillation center as functions of the magnitude of the acceleration being measured taking the reaction moment of the fixed contact upon the moving contact into account. On substitution of the values of F^0 and the harmonic-linearization coefficients (6.56) into Equation (6.59), we obtain equations for determining the amplitude and frequency of the self-oscillations:

$$\left.\begin{aligned}
&(T_3^2+T_1T_к)\Omega^2=\frac{4k_1U}{\pi A}\sqrt{1-\frac{\varphi_0^2}{A^2}}+\\
&\qquad+\frac{k_2c}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\varphi_0}{A}+\frac{\varphi_0}{A}\sqrt{1-\frac{\varphi_0^2}{A^2}}\right)+1,\\
&T_3^2T_к\Omega^2=T_1+T_к+T_e+\\
&\qquad+\frac{k_2T_кc}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\varphi_0}{A}+\frac{\varphi_0}{A}\sqrt{1-\frac{\varphi_0^2}{A^2}}\right),\\
&\varphi_0+\frac{2k_1U}{\pi}\arcsin\frac{\varphi_0}{A}+\\
&+\frac{k_2c}{\pi}\left[A\sqrt{1-\frac{\varphi_0^2}{A^2}}+\frac{\varphi_0}{2}\left(\pi+2\arcsin\frac{\varphi_0}{A}\right)\right]=k_3w_0.
\end{aligned}\right\}\qquad(6.70)$$

Assigning values $0 \leq \varphi_0/A \leq 1$ for the known parameters we determine the values of the frequency Ω from the second equation of (6.70). For these same values of φ_0/A and the values obtained for Ω from the first equation of (6.70), we determine the values of the amplitude A and consequently, values of the oscillation displacement φ_0.

Substituting the values φ_0/A, A and φ_0 into the third equation of (6.70), we obtain corresponding values for w_0. Thus we determine the functions $A(w_0)$, $\Omega(w_0)$ and $\varphi_0(w_0)$. The corresponding computational formulas take the form

$$\left.\begin{aligned} \Omega^2 &= \frac{T_1+T_\kappa+T_e}{T_2^2 T_\kappa} + \frac{k_2 c}{\pi T_2^2}\left(\frac{\pi}{2} + \arcsin\frac{\varphi_0}{A} + \frac{\varphi_0}{A}\sqrt{1-\frac{\varphi_0^2}{A^2}}\right), \\ A &= \frac{4k_1 U\sqrt{1-\frac{\varphi_0^2}{A^2}}}{\pi\left[(T_2^2+T_1T_\kappa)\Omega^2 - \frac{k_2 c}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\varphi_0}{A}+\frac{\varphi_0}{A}\sqrt{1-\frac{\varphi_0^2}{A^2}}\right)-1\right]}, \\ w_0 &= \frac{\varphi_0}{k_3} + \frac{2k_1U}{\pi k_3}\arcsin\frac{\varphi_0}{A} + \\ &\quad + \frac{k_2 c}{\pi k_3}\left[A\sqrt{1-\frac{\varphi_0^2}{A^2}} + \frac{\varphi_0}{A}\left(\pi + 2\arcsin\frac{\varphi_0}{A}\right)\right]. \end{aligned}\right\} \qquad (6.71)$$

Figure 6.10 (broken-line curves) shows curves for the calculations carried out according to Formulas (6.71) for the values taken earlier for the parameters and a moving-contact spring rigidity coefficient c = 1000 g·cm/rad and $k_2 = (0.0715\cdot10^{-3})$ 1/g·cm. As may be seen from the diagram, the curves for $A(w_0)$ and $\varphi_0(w_0)$ are somewhat deformed because of the reaction of the fixed contact upon the moving contact. However, for a low moving-contact spring rigidity, the instrument's error shows practically no change from the case where the nonlinear function $F_1(\varphi)$ alone is allowed for. Also in this case, the self-oscillation frequency remains approximately constant and is equal to $\Omega \approx (930)$ 1/sec.

§6.5. THE INTEGRATOR

Operating Principle of the Integrator

A schematic diagram of the integrator is shown in Fig. 6.11.

The device is designed for integration of a slowly varying time function. The function to be integrated is introduced in the form of an angle of rotation $\alpha(t)$, while the result of integration is obtained in the form of an angle of rotation $\varphi(t)$ of the magnet axis.

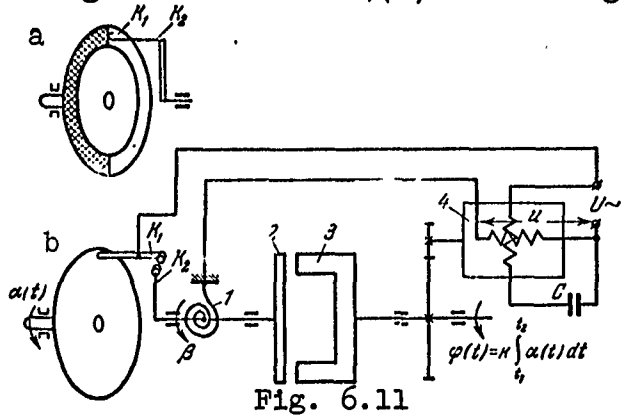


Fig. 6.11

The integrator operates on the follow-up system principle, with a motor controlled in the relay mode and feedback in the form of a magnetic tachometer.

Motor control may be effected either by sliding contacts (Fig. 6.11a) or by means of striking contacts (Fig. 6.11b).

The device is adjusted so that for $\alpha(t) = 0$, the contact K_1, which is mounted on the setting disk, comes into contact with the contact K_2, which rides on the shaft of the magnetic-tachometer sensitive element. When the setting disk turns through an angle $\alpha(t)$, the spring 1 will be twisted by the pressure of the contact K_1 upon contact K_2. The motor 4 is switched on and sets the magnet 3 into motion through a gear drive. When the magnet is turned, a torque will be developed at the disk 2 by eddy currents. When the torque developed at the disk 2 is equal to the moment of the spring which is twisted through an angle $\alpha(t)$, then the contacts K_1 and K_2 are broken. The

motor starts to decelerate and the contacts again close.

As is evident, the integrator operates in a self-oscillatory mode. The mean angular velocity ω_m of the shaft carrying the magnet will be proportional to the value of $\alpha(t)$, while the rotation angle of the magnet shaft (Fig. 6.11) is proportional to the integral of $\alpha(t)$:

$$\varphi = k \int_{t_1}^{t_2} \alpha(t)\, dt.$$

The advantage of such an integrator is that integration of the function, which is given in the form of an angle of rotation (the result is also delivered in the form of an angle of rotation), takes place without intermediate conversions into other physical quantities. The oscillations of the voltage source feeding the motor and the variation of the load applied to the motor do not exert any influence upon the operational accuracy of the integrator.

It is evident that in order to obtain high integration accuracy we must make sure that the oscillations of the tachometer's sensitive element and the motor speed have small amplitude and high frequency. Therefore it is of practical interest to determine how each of the integrator's parameters influences the amplitude and frequency of the self-oscillations, and also to analyze the influence of the setting disturbance $\alpha(t)$ on the self-oscillation parameters.

The use of different types of contact apparatus essentially changes the frequency and amplitude of the self-oscillation. In view of this, we carry out the analysis for both physical versions of the integrator: those with sliding and striking control contacts [174].

Analysis of self-oscillations with sliding control contacts.

In accordance with the diagram of the integrator (Fig. 6.11), neglecting the dry-frictional moment, the motor equation will be

$$J_1 \dot{\omega}_{дв} + n_1 \omega_{дв} + \frac{c}{i}(\omega_{дв} - i\dot{\beta}) = M, \qquad (6.72)$$

where ω_{dv} is the rotational speed of the motor, J_1 is the moment of inertia, reduced to the motor axis, of all masses turned by the motor, c is the torque - transmission factor to the tachometer's sensitive element, i is the transmission ratio from the motor to the magnet shaft, and $\dot{\beta}$ is the angular velocity of the sensitive element.

The motor torque may be approximately represented in the form

$$M = c_1 u - c_2 \omega_{дв}, \tag{6.73}$$

where u is the voltage applied to the motor control winding, $c_1 = (M/u)$[g·cm/v] and $c_2 = (M_0/\omega_0)$ [g·cm.sec] are coefficients obtained from the mechanical characteristics of the motor, M_0 is the torque in the plugging mode and ω_0 is the no-load speed of the motor.

The control voltage across the motor winding will vary according to a nonlinear relationship as a function of the rotation angle of the magnetic-tachometer sensitive element (Fig. 6.12a):

$$u = F_1(\beta) = \begin{cases} U \text{ for } \beta \leq 0 \\ 0 \text{ for } \beta \geq 0 \end{cases} \tag{6.74}$$

where U is the supply voltage to the control winding.

Allowing for (6.73) and (6.74), the motor equation (6.72) is written in the form

$$T_1 \dot{\omega}_{дв} + \omega_{дв} - \eta_1 \dot{\beta} = k_1 F_1(\beta), \tag{6.75}$$

where $T_1 = \dfrac{J_1}{n_1 + \frac{c}{i} + c_2}$ [sec] is the electromechanical time constant of the motor, $\eta_1 = \dfrac{J_1}{c_1 + \frac{c}{i} + c^2}$ is the slip ratio of the tachometer sensitive element and $k_1 = \dfrac{c}{n_1 + \frac{c}{i} + c_2}$ $\left[\frac{1}{\text{sec}\cdot\text{v}}\right]$ is the motor transmission ratio.

Neglecting dry friction, the equation of the tachometer sensitive element is

$$J_2\ddot{\beta} + n_2\dot{\beta} + c_{1p}(\alpha + \beta) = c(\omega_m - \dot{\beta}), \tag{6.76}$$

where α is the input-angle value to be integrated, β is the angle of rotation reckoned from the position of the tachometer sensitive element for which the contacts K_1 and K_2 touch each other without pressure, J_2 is the moment of inertia of the tachometer sensitive element, n_2 is the viscous-friction coefficient, c_{1p} is the rigidity coefficient of the coil spring and ω_m is the rotary speed of the magnet.

If we rewrite Equation (6.76) in the form

$$T_2^2\ddot{\beta} + T_3\dot{\beta} + \beta = k_2\omega_m - \alpha, \tag{6.77}$$

where $T_2 = \sqrt{J_2/c_{1p}}$ [sec] is the inertial time constant of the tachometer sensitive element, $T_3 = (c + n_2/c_{1p})$[sec] is the damping time constant, $k_2 = (c/c_{1p})$ [sec] is the magnetic-tachometer transmission ratio.

In addition we must allow for the kinematic equation of the reducer:

$$\omega_m = k_3\omega_{ab}. \tag{6.78}$$

We reduce Eqs. (6.75), (6.77) and (6.78) to one equation in the variable β, which we write in operator form:

$$[T_1T_2^2p^3 + (T_1T_3 + T_2^2)p^2 + (T_1 + T_3 - k\eta)p + 1]\beta = \\ = kF_1(\beta) - (T_1p + 1)\alpha, \tag{6.79}$$

where $\eta = \eta_1/k_1$ [v·sec], while $k = k_1k_2k_3$[1/v] is the transmission ratio of the integrator.

For a model of the integrator, which was built from the diagram shown with the DID-0.5 motor, we have the value $k\eta \ll T_1 + T_3$, since $T_1 = 0.33$ sec, $T_2^2 = 0.0017$ sec, $T_3 = 0.017$ sec, $k = (0.098)$ 1/v and $\eta = 0.039$ v·sec. Therefore in Equation (6.79) we may neglect the value $k\eta$ as small in comparison with $T_1 + T_3$ and write it in the form

$$[T_1T_2^2p^3+(T_1T_3+T_2^2)p^2+(T_1+T_3)p+1]\beta = kF_1(\beta)-(T_1p+1)\alpha. \qquad (6.80)$$

We shall use Equation (6.80) for analysis of the integrator's self-oscillations. In the present case, due to the presence of a nonsymmetrical nonlinear static characteristic $F_1(\beta)$ and the external disturbance α, the self-oscillations will be nonsymmetrical. But among the nonsymmetrical modes for different constant values of α, we may note one symmetrical self-oscillation mode. Let us denote by α_{max} the maximum value of α, for which breakoff of the self-oscillations takes place. If we set $\alpha_0 = \alpha_{max}/2$, then the constant component of the periodically varying voltage in the motor control winding will be $u_0 = U/2$. For this case the torque at the sensitive element of the magnet due the constant voltage component will be neutralized by the torque of the coil spring, which is twisted through the angle α_0, and hence

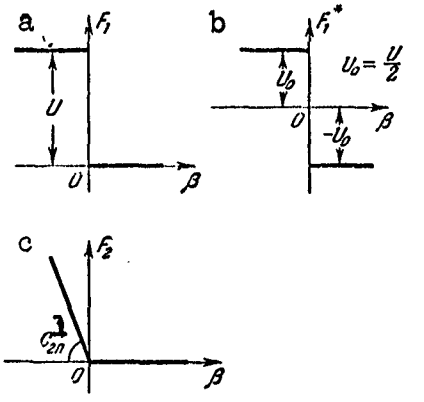


Fig. 6.12. 1) c_{2p}.

$$ku_0 = \alpha_0.$$

Then Equation (6.80) may be written in the form (for the case $\alpha = \alpha_0 = \text{const}$):

$$[T_1T_2^2p^3+(T_1T_3+T_2^2)p^2+(T_1+T_3)p+1]\beta = kF_1^*(\beta), \qquad (6.81)$$

where $F_1^*(\beta) = F(\beta)-u_0$ is represented in the form of a symmetrical ideal relay characteristic (Fig. 6.12b).

Thus for one value $\alpha = \alpha_0$, we may carry out the analysis just as for the case of symmetrical self-oscillations. Investigations for the case of one value of α will be sufficient for the choice of the system parameters from the condition for obtaining small amplitude and

high frequency in the self-oscillations. Where this is possible, recourse should also be taken to this method in other cases for solution of the problem of parameter choice.

We are limited in our choice of parameters for the present type of integrator with sliding control contacts. In what follows we shall carry out the analysis of self-oscillations for the case of various values of α for the case of striking control contacts.

Following the method of harmonic linearization, we shall seek the self-oscillations of the tachometer sensitive element in the form

$$\beta = A \sin \Omega t.$$

Then in accordance with Formula (3.14) we obtain the formula for the harmonic linearization of the nonlinear function $F_{*1}^{*}(\beta)$ in the form

$$F_1^*(\beta) = -\frac{2U}{\pi A}\beta. \tag{6.82}$$

Substituting this into (6.81), we obtain a harmonically linearized equation for determination of the symmetrical self-oscillations:

$$\left[T_1T_2^2p^3 + (T_1T_2 + T_2^2)p^2 + (T_1 + T_2)p + 1 + \frac{2kU}{\pi A}\right]\beta = 0. \tag{6.83}$$

Substituting $p = j\Omega$ into the characteristic polynomial of Equation (6.83) we obtain from the conditions $X = 0$ and $Y = 0$ two equations for determining the frequency and amplitude of the periodic solution:

$$\left.\begin{aligned} 1 + \frac{2kU}{\pi A} - (T_1T_2 + T_2^2)\Omega^2 = 0, \\ T_1 + T_2 - T_1T_2^2\Omega^2 = 0. \end{aligned}\right\} \tag{6.84}$$

Solving Equation (6.84) for Ω and A, we obtain

$$\Omega = \sqrt{\frac{T_1 + T_2}{T_1T_2^2}}, \quad A = \frac{2kUT_1T_2^2}{\pi T_2(T_1^2 + T_1T_2 + T_2^2)}. \tag{6.85}$$

We determine the stability of the periodic solution from the approximate criterion (2.125):

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0.$$

From (6.83) we obtain the values $X(a, \omega)$ and $Y(a, \omega)$:

$$X(a,\ \omega)=\frac{2kU}{\pi a}-(T_1T_3+T_2^2)\omega^2+1,$$
$$Y(a,\ \omega)=(T_1+T_3)\omega-T_1T_2^2\omega^3.$$

The values of the corresponding derivatives will be

$$\left(\frac{\partial X}{\partial a}\right)^*=-\frac{2kU}{\pi A^2}<0,\quad \left(\frac{\partial Y}{\partial a}\right)^*=0,\quad \left(\frac{\partial X}{\partial \omega}\right)^*=-2(T_1T_3+T_2^2)\Omega<0,$$
$$\left(\frac{\partial Y}{\partial \omega}\right)^*=(T_1+T_3)-3T_1T_2^2\Omega^2.$$

Applying the value of Ω^2 from (6.85), we obtain

$$\left(\frac{\partial Y}{\partial \omega}\right)^*=-2(T_1+T_3)<0.$$

Thus the stability criterion is satisfied.

In order to determine the oscillation amplitude A_ω for the angular velocity of the magnet, let us make use of Equation (6.77) without allowing for external disturbance. In accordance with (6.77), the transfer function has the form

$$W(p)=\frac{\beta}{\omega_м}=\frac{k_2}{T_2^2p^2+T_2p+1},$$

and hence the modulus of the complex gain constant for $\omega = \Omega$ is

$$|W(j\omega)|=\frac{k_2}{\sqrt{(1-T_2^2\Omega^2)^2+T_2^2\Omega^2}}.$$

Since $A = |W(j\omega)|A_\omega$, then

$$A_\omega=\frac{A}{k_2}\sqrt{(1-T_2^2\Omega^2)^2+T_2^2\Omega^2}. \qquad (6.86)$$

Determining the self-oscillation frequency and amplitude for the intergrator parameter values taken earlier, U = 30 v and k_2 = 0.0156 sec, we obtain

Ω = 24.85 1/sec; A = 0.53 rad, A_ω = 14.3 1/sec

It is evident that for exact integration we cannot be satisfied with the values obtained for the frequency and amplitudes.

In order to determine the possibilities of obtaining an acceptable mode of self-oscillation for the device being considered, let us construct the curves of the frequency and amplitudes as functions of

the instrument's parameters. Figure 6.13 shows a synthesis carried out according to Formula (7.85) and (6.86) for the parameters T_1, T_2, T_3 and k_1. The curves shown permit us to obtain practical recommendations for improving the integrator. As is evident, the most effective method for reducing the amplitudes and increasing the frequency of self-oscillations is to increase the damping for the tachometer sensitive element (increase the parameter T_3). However, this increases the complexity of the integrator design.

We compared the results obtained for the analysis of self-oscillation by the approximate method of harmonic linearization with the results of synthesizing the settling process of the self-oscillations using the graphoanalytical method of D.A. Bashkirov, and with the results of experimental measurements performed on the model. The results of the calculations and the measurements are shown in Table 6.1.

It is evident from the comparisons shown in Table 6.1 that the method of harmonic linearization gives a result which is sufficiently accurate.

Analysis of self-oscillations of integrator for case of striking control contacts

On replacement of sliding control contacts in the form of a brush and a "yes-no" disk (Fig. 6.11a) by striking contacts (Fig. 6.11b), the instrument parameters being the same as in the previous case, the self-oscillations change sharply in the direction of increasing frequency and decreasing amplitude. This gives us the most simple design method for improving the integrator's characteristics.

In contrast to the preceding case, an additional torque will now act upon the tachometer sensitive element on self-oscillation due to

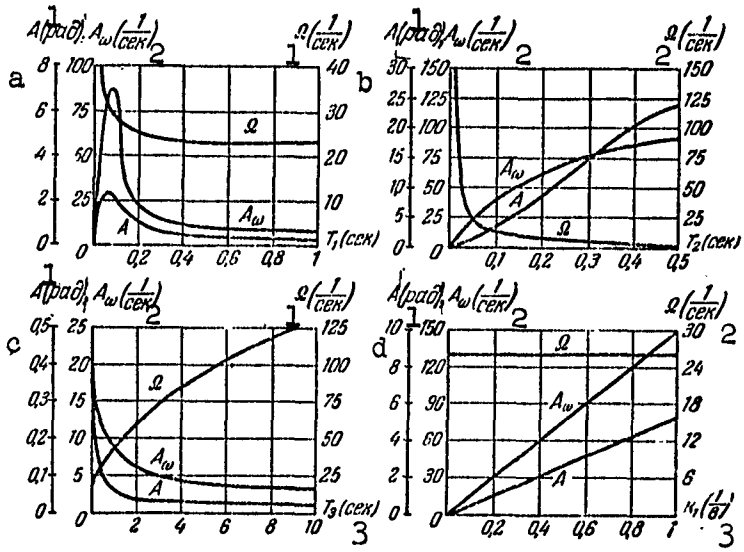


Fig. 6.13. 1) Rad; 2) sec; 3) v.

TABLE 6.1

1 Метод исследования	Ω [1/сек] 5	A [рад] 6	A_ω [1/сек] 7
2 Графоаналитический . .	21,1	0,54	17
3 Гармонической линеаризации	24,8	0,53	14,3
4 Экспериментальные измерения	26,4	0,60	

1) Method of investigation; 2) graphoanalytical; 3) harmonic-linearization; 4) experimental measurements; 5) sec; 6) rad; 7) sec.

the pressure of the contact K_1 upon the contact K_2 and given in the form of a nonlinear function $F_2(\beta)$ (Fig. 6.12c). Here we will assume that the rigidity of the holder of the contact K_1 is incomparably larger for a given deflection angle than the spring rigidity of the tachometer sensitive element, while the rigidity of the holder of contact K_2 is commensurate with the rigidity of the tachomometer spring.

Let us analyze the self-oscillations allowing for an input disturbance α. Let us write the equations for the motor and the tachometer sensitive element and the kinematic equation of the reducer (with the previous method of reckoning the angles β, assuming $\eta_1 \approx 0$) in the form:

$$\left.\begin{aligned}(T_1p+1)\,\omega_{дв} &= k_1F_1(\beta),\\ (T_2^2p^2+T_3p+1)\,\beta &= k_2\omega_м+\frac{F_2(\beta)}{c_{1n}}-\alpha,\\ \omega_м &= k_3\omega_{дв}.\end{aligned}\right\} \qquad (6.87)$$

After eliminating the values ω_{dv} and ω_m from Eq. (6.87) and with the condition α = const, we obtain an equation for the integrator in the variable β:

$$(T_1p+1)(T_2^2p^2+T_3p+1)\beta = kF_1(\beta)+\frac{(T_1p+1)F_2(\beta)}{c_{1n}}-\alpha, \qquad (6.88)$$

where $k = k_1k_2k_3[1/v]$.

Since we have in the closed- loop system of the integrator nonsymmetrical nonlinear functions and an external disturbance is considered, the self-oscillations will be nonsymmetrical. We shall seek a solution for the variable β in the form

$$\beta = \beta_0 + A\sin\psi, \quad \psi = \Omega t. \qquad (6.89)$$

By the method of harmonic linearization, we replace the nonlinear functions in (6.88) by the relationships

$$\left.\begin{aligned}F_1(\beta) &= F_1^0(A,\beta_0)+q_1(A,\beta_0)(\beta-\beta_0),\\ F_2(\beta) &= F_2^0(A,\beta_0)+q_2(A,\beta_0)(\beta-\beta_0),\end{aligned}\right\} \qquad (6.90)$$

where, in accordance with (5.115), (5.116), (5.142) and (5.143) and the static nonlinear characteristics (Fig. 6.12), the values of the constant components and the harmonic-linearization coefficients are

$$\left.\begin{aligned}&F_1^0(A,\beta_0) = \frac{U}{2}-\frac{U}{\pi}\arcsin\frac{\beta_0}{A},\quad q_1(A,\beta_0) = -\frac{2U}{\pi A}\sqrt{1-\frac{\beta_0^2}{A^2}},\\ &F_2^0(A,\beta_0) = -\frac{c_{2n}\beta_0}{2}+\frac{c_{2n}}{\pi}\left(\beta_0\arcsin\frac{\beta_0}{A}+A\sqrt{1-\frac{\beta_0^2}{A^2}}\right),\\ &q_2(A,\beta_0) = -\frac{c_{2n}}{2}+\frac{c_{2n}}{\pi}\left(\arcsin\frac{\beta_0}{A}+\frac{\beta_0}{A}\sqrt{1-\frac{\beta_0^2}{A^2}}\right).\end{aligned}\right\} \qquad (6.91)$$

Substituting the expressions $F_1(\beta)$ and $F_2(\beta)$ from (6.90) into (6.88), we obtain the harmonically linearized system equation:

$$\left[T_1T_2^2p^3+(T_1T_3+T_2^2)p^2+\left(T_1+T_3-\frac{T_1}{c_{1n}}q_2\right)p+\right.$$
$$\left.+1-kq_1-\frac{1}{c_{1n}}q_2\right]\beta=\frac{1}{c_{1n}}T_1p(F_2^0-q_2\beta_0)+k(F_1^0-q_1\beta_0)+$$
$$+\frac{1}{c_{1n}}(F_2^0-q_2\beta_0)-\alpha. \qquad (6.92)$$

Making the substitution $p = j\Omega$ in the characteristic equation corresponding to the differential equation (6.92) and separating the real and imaginary parts, we obtain two equations. We obtain a third euqation as a particular solution of Equation (6.92) for the case $\beta = \beta_0$. We write the three equations indicated for $\beta_0 = \text{const}$, $F_1^0 = \text{const}$ and $F_2^0 = \text{const}$ in the form

$$\left.\begin{aligned}1-kq_1-\frac{1}{c_{1n}}q_2-(T_1T_3+T_2^2)\Omega^2=0,\\ T_1+T_3-\frac{T_1}{c_{1n}}q_2-T_1T_2^2\Omega^2=0,\\ \beta_0=kF_1^0+\frac{1}{c_{1n}}F_2^0-\alpha.\end{aligned}\right\} \qquad (6.93)$$

Equations (6.93) are transcendental and can be solved only graphically. It follows from the first two equations of (6.93) that

$$\Omega^2=-\frac{kq_1(A,\beta_0)}{T_1T_3}-\frac{1}{T_1^2}. \qquad (6.94)$$

Eliminating Ω^2 from the second equation of (6.93), we obtain

$$T_1+T_3+\frac{T_2^2}{T_1}+\frac{kT_2^2}{T_3}q_1(A,\beta_0)-\frac{T_1}{c_{1n}}q_2(A,\beta_0)=0. \qquad (6.95)$$

From Equation (6.95) we may construct the curve of $A(\beta_0)$, and from the third equation of (6.93), we may construct a family of curves of $A(\beta_0)$ for various values of α. The intersection points of the curves indicated give us a solution for $A(\alpha)$ and $\beta_0(\alpha)$, while the values of $\Omega(\alpha)$ are determined from (6.94) for known values of A and β_0. The results of the solution are shown in the form of curves (Fig. 6.14) for the parameter values corresponding to the model built: $T_1 = 0.33$ sec, $T_2^2 = 0.0017$ sec^2, $T_3 = 0.017$ sec, $k = 0.098$ 1/v, $U = 30$v, $\frac{c_{2p}}{2c_{1p}} = 84.4$. In this same figure we show the functions $A(\alpha)$ and $\Omega(\alpha)$ (the

dashed curves) as they were obtained experimentally. If we remember that the parameter values introduced into the equation were measured on the model with an accuracy of the order of 10%, we see that in the present case analysis of the self-oscillations by the method of harmonic linearization furnishes sufficient accuracy.

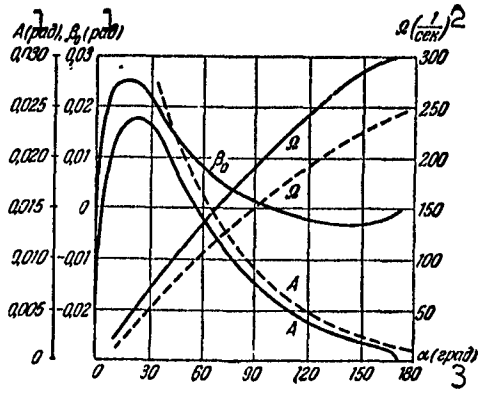


Fig. 6.14. 1) Radians, 2) sec, 3) degrees.

The stability analysis of the periodic solution obtained may be performed in the same way as for the preceding example, but there is no necessity for this analysis, since the steady-state mode of operation is evident from the operating principle of the integrator.

Let us carry out the analysis of the influence of the instrument's parameters upon the self-oscillation frequency and amplitude for the value α for which $\beta_0 = 0$, i.e., let us use the only possible mode of symmetrical oscillation. Setting $\beta_0 = 0$ in (6.92), we obtain the equation

$$\left[T_1T_2^2p^3+(T_1T_2+T_2^2)p^2+\left(T_1+T_2-\frac{T_1}{c_{1n}}q_2\right)p+\right.$$
$$\left.+1-kq_1-\frac{1}{c_{1n}}q_2\right]\beta=kF_1^0+\frac{1}{c_{1n}}F_2^0-\alpha. \qquad (6.96)$$

From Equation (6.96), we obtain three euqations for determination of the self-oscillations: two equations for the case $p = j\Omega$ from the condition for zero real and imaginary parts of the characteristic equations and a third equation as a particular solution of Equation (6.96) for the case $\beta = 0$. The three equations indicated have the form

$$\left.\begin{aligned}1-kq_1-\frac{1}{c_{1n}}q_2-(T_1T_2+T_2^2)\Omega^2=0,\\ T_1+T_2-\frac{T_1}{c_{1n}}q_2-T_1T_2^2\Omega^2=0,\\ kF_1^0+\frac{1}{c_{1n}}F_2^0-\alpha=0,\end{aligned}\right\} \qquad (6.97)$$

where, in accordance with (6.91), the constant components and coefficients of harmonic linearization for the case $\beta_0 = 0$ will have the values

$$\left.\begin{aligned} F_1^0 &= l', \qquad q_1(A) = -\frac{2U}{\pi A}, \\ F_2^0(A) &= \frac{c_{2n}}{\pi} A, \quad q_2(A) = -\frac{c_{2n}}{2}. \end{aligned}\right\} \tag{6.98}$$

Substituting the values of the constant components and the harmonic-linearization coefficients from (6.98) into (6.97), we obtain

$$\left.\begin{aligned} 1 + \frac{2kU}{\pi A} + \frac{c_{2n}}{2c_{1n}} - (T_1 T_3 + T_4^2)\Omega^2 &= 0, \\ T_1 + T_3 + \frac{T_1 c_{2n}}{2c_{1n}} - T_1 T_2^2 \Omega^2 &= 0, \\ \frac{kU}{2} + \frac{c_{2n}}{\pi c_{1n}} A - \alpha &= 0. \end{aligned}\right\} \tag{6.99}$$

Introducing the notation $\delta_p = c_{2p}/2c_{1p}$, we obtain a formula for determining the self-oscillation frequency

$$\Omega = \sqrt{\frac{T_1(1+\delta_n)+T_3}{T_1 T_2^2}}. \tag{6.100}$$

from the second equation of (6.99). Substituting the value of Ω into the first equation of (6.99), we obtain a formula for determining the self-oscillation amplitude:

$$A = \frac{2kUT_1T_2^2}{\pi T_3 [T_2^2(1+\delta_n) + T_1 T_3 + T_2^2]}. \tag{6.101}$$

Without allowing for the external disturbance α and the constant component $F_2^0(A)$ of the harmonically-linearized nonlinear function $F_2(\beta)$, we may obtain the transfer function of the magnetic tachometer's sensitive element

$$W(p) = \frac{\beta}{\omega_M} = \frac{k_2}{T_2^2 p^2 + T_3 p + 1 + \delta_n},$$

from Equation (6.87). Then we obtain the formula

$$A_\omega = \frac{A}{k_2} \sqrt{(1 + \delta_n - T_2^2\Omega^2)^2 + T_3^2\Omega^2}. \tag{6.102}$$

for conversion of the oscillation amplitudes of the tachometer sensitive element into the amplitudes of the magnet's angular velocity.

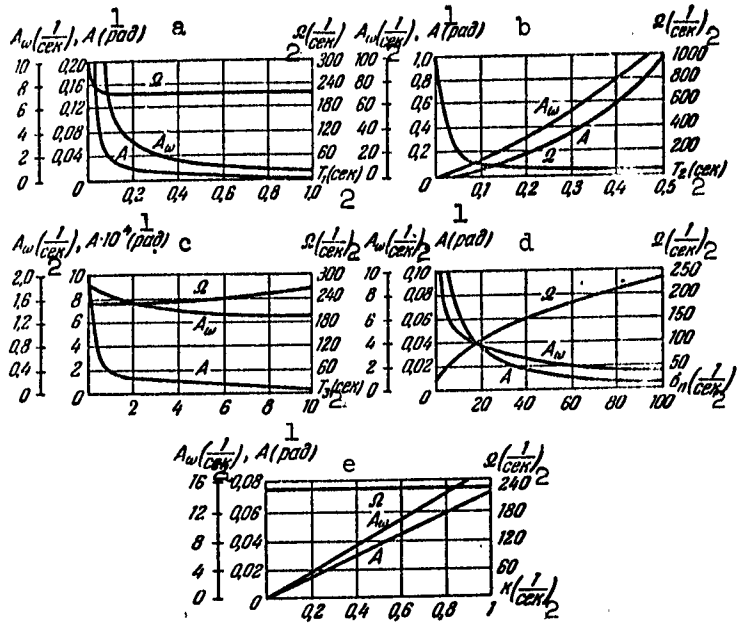


Fig. 6.15. 1) rad; 2) sec.

Formulas (6.100) - (6.102) permit us to construct curves of the frequency Ω and amplitudes A, A_ω as functions of each integrator parameter. Figure 6.15 shows the construction which was carried out. For the parameter values indicated earlier as corresponding to the model, $k_2 = 0.0156$ sec. and $\beta_0 = 0$, we obtain the following values for the self-oscillation frequency and amplitudes: $\Omega = (224)$ 1/sec (f = 35.6 cps), A = 0.0066 radian, $A_\omega = (1.61)$ 1/sec.

As a result of substituting striking contacts for sliding contacts, we obtain a significant increase in the self-oscillation frequency and a decrease in the self-oscillation amplitudes. The curves constructed (Fig. 6.15) permit us to come to practical conclusions for the choice of the integrator parameters.

In order to evaluate the accuracy of the harmonic-linearization method, the process of self-oscillation establishment was sythesized

by the graphoanalytical method of D.A. Bashkirov for the case of one value of α, and the experimental measurements were performed upon a model. Table 6.2 shows the results of the calculations and measurements. It is evident from Table 6.2 that the method of harmonic linearization furnishes sufficient accuracy even in the case where two nonsymmetrical nonlinearities are present in the system.

TABLE 6.2.

1 Метод исследования	Ω [1/сек] (5)	A [рад] (6)	A_ω [1/сек] (7)
2 Графоаналитический	230	0,0083	1,60
3 Гармонической линеаризации	195	0,0077	1,84
4 Экспериментальные измерения	—	0,0070	—

1) Method of analysis; 2) graphoanalytical; 3) harmonic linearization; 4) experimental measurements; 5) sec; 6) radians; 7) sec.

Comparing the results obtained for the self-oscillation analysis of two cases of the integral, we may come to the conclusion that from the point of view of design simplicity, the version of the integrator having striking control contacts is more efficient, since in this case we are certain to obtain a high self-oscillation frequency and low self-oscillation amplitudes by adjusting the rigidity of the control-contact holders without additional damping.

§6.6. SYSTEM FOR CONTROLLING SPEED OF ELECTRIC MOTOR WITH CENTRIFUGAL RELAY CONTROLLER

Operating principle of system. Figure 6.16 shows a schematic of an electric motor with centrifugal relay speed control. The disk S of the centrifugal regulator is mounted on the motor-armature shaft (in the figure the disk is remote from the armature for clarity). The disk carries a contact device with the fixed contact K_1 and the moving contact K_2. Because of the spring P, whose tension is controlled by the setscrew NV, the pair of contacts K_1 and K_2 is held in a closed state at motor speeds less than the nominal speed. If the motor speed exceeds the nominal value, then the preset pair of contacts is opened by the action of the centrifugal force acting upon the mass m

of the moving contact. On opening of the contact, the auxiliary resistance R_d is cut into the armature circuit of the motor, and the motor speed decreases as a result. This leads to the closing of the contacts, and the resistance R_d is shunted by them. Thus the system operates in a self-oscillatory mode by its very principle. Because of the periodic connection and disconnection of the resistance to the armature, the motor speed is controlled with a variable load torque applied to the motor shaft. For the case of constant resistance, control is accomplished by having different closed and open times of the contacts.

Because of their simple construction, centrifugal relay controllers have come into wide use in controlling the speeds of electrical motors. Such controllers maintain a given speed with an accuracy of the order of 1-5% with respect to the relative error.

The operational accuracy of the centrifugal relay controllers may be increased by correct choice of their designs and parameters. This requires analysis of self-oscillations for the motor: controller system [184].

In analysis of the system in question, just as for a nonlinear system with a self-oscillatory steady-state mode of operation, we must use the condition for obtaining small amplitude in the oscillations of the motor speed about its mean value and minimum deviations of the mean speed value oscillatory-system static error) to obtain recommendations for choosing the parameters.

Synthesis of control-system equations. Let us synthesize the equations for the system's links.

The equation for the rotation of the motor shaft is

$$J_1\dot{\omega}_{дв} = M_{дв} - M_c - M_{пр}, \qquad (6.103)$$

where ω_{dv} [1/sec] is the speed of the motor, J_1 [g·cm·sec^2] is the mo-

ment of inertia of all masses turned by the motor, reduced to the motor shaft, M_{dv} [g·cm] is the torque developed by the motor, M_s [g·cm] is the resistance torque, and M_{ng} [g·cm] is the load torque.

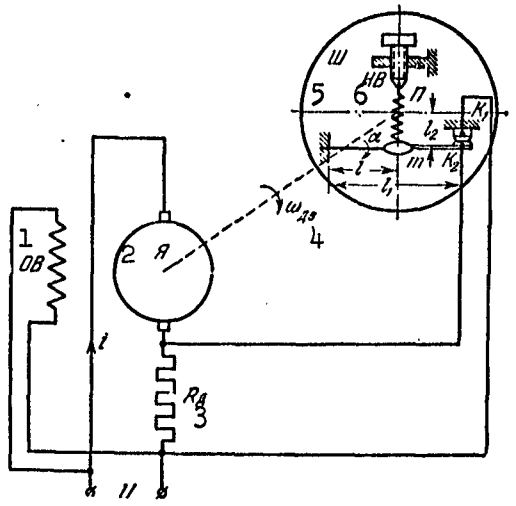


Fig. 6.16. 1) OV; 2) Ya; 3) d; 4) dv; 5) Sh; 6) NV; 7) P.

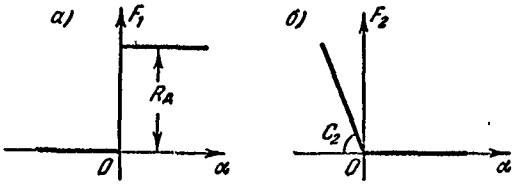


Fig. 6.17. 1) d.

For separate excitation, the motor torque is proportional to the armature current:

$$M_{dv} = k_i i, \qquad (6.104)$$

where i [amp] is the armature current; k_i [g·cm/amp] is a proportionality factor incorporating the motor data. Neglecting armature-circuit inductance in Eq. (6.104)

$$M_{dv} = k_i \frac{U - e}{R + F_1(\alpha)}, \qquad (6.105)$$

where U [v] is the supply voltage, e[v] back emf developed by the armature, R[ohms] the ohmic armature resistance, and $F_1(\alpha)$ [ohms] a nonlinear function of the resistance due to the rotation α of the moving contact that is represented by the static characteristic (Fig. 6.17a). As a reference origin for the angle α we take the movable-contact position at which the fixed contact is touched without pressure.

The load torque M_{ng} is a quantity which is arbitrary over time, and its variation from the nominal value is the basic perturbation in the system. The resistance torque M_s includes the torque due to the dry and viscous frictional forces and varies over time and with the motor speed. Setting $M_{ng} \gg M_s$, we will assume that the total moment

$$M(t) = M_{нг} + M_c \tag{6.106}$$

varies only with respect to time.

The back emf of the motor is proportional to the rotary speed:

$$e = k_e \omega_{дв}, \tag{6.107}$$

where k_e [v·sec] is the proportionality coefficient.

Applying (6.105) - (6.107), we write the motor equation (6.103) in the form

$$\left[J_1 p + \frac{k_1 k_e}{R + F_1(\alpha)}\right] \omega_{дв} = k_1 \frac{U}{R + F_1(\alpha)} - M(t). \tag{6.108}$$

The quantity $T_1(\alpha) = J_1[R + F_1(\alpha)]/(k_1 k_e)$ [sec] is the electromechanical time constant of the motor in the form of (6.108), without reducing it to standard form.

We write the equation of motion of the moving contact in the form

$$J_2 \ddot{\alpha} + k_\alpha \dot{\alpha} + c_1 l^2 \alpha + M_0 = m l l_2 \omega_{дв}^2 + F_2(\alpha), \tag{6.109}$$

where α is the angle of rotation of the moving contact, J_2 [g·cm·sec^2] is the moment of inertia of the moving contact in its motion about the pivot point, k_α [g·cm·sec] is the viscous-friction coefficient, c_1 [g/cm] is the rigidity coefficient of the basic spring, μ_0[g·cm] is the torque due to the pretensioning of the basic spring, m [g·sec.^{2}/cm] is the mass of the movable contact, reduced to the pivot point of the spring, l and l_2[cm] are geometrical dimensions (Fig. 6.16) and F $F_2(\alpha)$ [g·cm] is the reaction torque of the fixed contact upon the moving contact, which is taken into account by the static characateristic (Fig. 6.17b) and varies nonlinearly as a function of α.

The variable ω_{dv} occurs squared in Equation (6.109). Let us replace the torque from the centrifugal forces, which varies nonlinearly as a function of angular velocity, by a linear dependence, i.e., replace the parabola (Fig. 6.18) by a straight line tangent at the point

(M^0_{ts}, ω^0_{dv}) corresponding to the nominal mode of operation. Such a substitution will be satisfactory in investigating the self-oscillation in the neighborhood of the nominal mode. Hence, instead of the term representing the moment of the centrifugal forces, $M_ц = mll_2\omega^2_{дв}$, we take

$$M_ц = 2mll_2\omega^0_{дв}\omega_{дв} - mll_2(\omega^0_{дв})^2. \qquad (6.110)$$

Allowing for (6.110), we write Equation (6.109) in the form

$$J_2\ddot{\alpha} + k_2\dot{\alpha} + c_1l^2\alpha = 2mll_2\omega^0_{дв}\omega_{дв} - (mll_2(\omega^0_{дв})^2 + \mu_0) + F_2(\alpha).$$

Let us introduce the notation

$T_2 = \sqrt{\frac{J_2}{c_1l^2}}$ [sec] — the inertial time constant of the movable contact;

$T_3 = \frac{k_2}{c_1l^2}$ [sec] — the damping time constant;

$k_2 = \frac{2mll_2\omega^0_{дв}}{c_1l^2}$ [sec] — the transmission ratio,

$\gamma = \frac{mll_2(\omega^0_{дв})^2 + \mu_0}{c_1l^2}$ — the precompression angle.

$$(T_2^2p^2 + T_3p + 1)\alpha = k_2\omega_{дв} - \gamma + \frac{F_2(\alpha)}{c_1l^2}. \qquad (6.111)$$

Let us reduce Eqs. (6.108) and (6.111) to one equation in the variable α. We obtain

$$\omega_{дв} = \frac{k_1U - [R + F_1(\alpha)]M(t)}{J_1[R + F_1(\alpha)]p + k_1k_e}.$$

from Equation (6.108). Substituting the value of ω_{dv} into Equation (6.111) and carrying out the transformations, we obtain:

$$\begin{aligned} &J_1R(T_2^2p^2 + T_3p + 1)p\alpha + J_1[T_2^2F_1(\alpha)p^3\alpha + \\ &+ T_3F_1(\alpha)p^2\alpha + F_1(\alpha)p\alpha] + k_1k_e(T_2^2p^2 + T_3p + 1)\alpha = \\ &= k_1(k_2U - k_e\gamma) - k_2RM(t) - k_2F_1(\alpha)M(t) + \\ &+ \frac{J_1R}{c_1l^2}F_1(\alpha)pF_2(\alpha) + \left(\frac{J_1R}{c_1l^2}p + \frac{k_1k_e}{c_1l^2}\right)F_2(\alpha). \end{aligned} \qquad (6.112)$$

Harmonic linearization of nonlinearities and derivation of equations for analysis of self-oscillations. We have the nonlinear func-

tions

$$F_1(\alpha),\quad F_2(\alpha),\quad F_3(\alpha,\ p\alpha)=F_1(\alpha)p\alpha,$$
$$F_4(\alpha,\ p^2\alpha)=F_1(\alpha)p^2\alpha,\quad F_5(\alpha,\ p^3\alpha)=F_1(\alpha)p^3\alpha$$

and the product of the nonlinear functions

$$F_1(\alpha)pF_2(\alpha)=F_1(\alpha)\frac{dF_2(\alpha)}{d\alpha}p\alpha.$$

in Equation (6.112).

Since the nonlinear functions are nonsymmetrical and there is an external disturbance, then we will seek the solution for α in the form

$$\alpha=\alpha_0+A\sin\psi,\quad \psi=\Omega t.$$

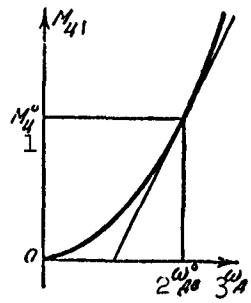


Fig. 6.18. 1) ts; 2) dv; 3) d.

After the harmonic linearization, the nonlinear functions will be replaced by relationships of the form

$$F_n(\alpha)=F_n^0(A,\ \Omega,\ \alpha_0)+\left[q_n(A,\ \Omega,\ \alpha_0)+\frac{q_n'(A,\ \Omega,\ \alpha_0)}{\Omega}p\right](\alpha-\alpha_0). \quad (6.113)$$

As the derivative of the function $F_2(\alpha)$ with respect to α (Fig. 6.17b), the nonlinear function $dF_2(\alpha)/d$ is represented in the form of the static characteristic shown in Fig. 6.19. The product of the function in question by the nonlinear function $F_1(\alpha)$(Fig. 6.17a) is zero, i.e.,

$$F_1(\alpha)pF_2(\alpha)=0. \quad (6.114)$$

In accordance with Formulas (5.115), (5.116), (5.142), (5.143), (5.145), (5.146), (5.147) and (5.148), the constant components in the harmonic linearization coefficients will have the values

$$\begin{aligned}
F_1^0(A,\ \alpha_0)&=\frac{R_3}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}\right),\\
q_1(A,\ \alpha_0)&=\frac{2R_3}{\pi A}\sqrt{1-\frac{\alpha_0^2}{A^2}},\\
F_2^0(A,\ \alpha_0)&=-\frac{c_2\alpha_0}{2}+\frac{c_2}{\pi}\left(\alpha_0\arcsin\frac{\alpha_0}{A}+A\sqrt{1-\frac{\alpha_0^2}{A_2}}\right),\\
q_2(A,\ \alpha_0)&=-\frac{c_2}{2}+\frac{c_2}{\pi}\left(\arcsin\frac{\alpha_0}{A}+\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right),
\end{aligned} \quad (6.115)$$

$$q_3'(A,\ \Omega,\ \alpha_0)=\frac{R_1\Omega}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}+\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right),$$
$$F_1^0(A,\ \Omega,\ \alpha_0)=-\frac{R_1A\Omega^2}{\pi}\sqrt{1-\frac{\alpha_0^2}{A^2}},$$
$$q_4(A,\ \Omega,\ \alpha_0)=-\frac{R_1\Omega^2}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}-\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right),\qquad(6.115)$$
$$q_5'(A,\ \Omega,\ \alpha_0)=-\frac{R_1\Omega^3}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}+\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right).$$

All the other coefficients are zero. Substituting the harmonically linearized values of the nonlinear functions into Equation (6.112), allowing for (6.114), and performing transformations, we obtain:

$$\begin{aligned}&\Big[J_1RT_2^2p^3+(J_1RT_3+k_ik_eT_2^2)p^2+\\&\quad+\left(J_1R+k_ik_eT_3+J_1\frac{q_3'}{\Omega}+J_1T_2^2\frac{q_5'}{\Omega}-\frac{J_1R}{c_1l^2}q_2\right)p+k_ik_e+\\&+J_1T_3q_4-\frac{k_ik_e}{c_1l^2}q_2+k_2q_1M(t)\Big]\alpha=\\&\quad=-J_1T_3F_1^0+J_1T_3q_4\alpha_0+k_i(k_2U-k_e\gamma)-k_2RM(t)+\\&\quad+k_2q_1\alpha_0M(t)-k_2F_1^0M(t)+\frac{k_ik_e}{c_1l^2}F_2^0-\frac{k_ik_e}{c_1l^2}q_2\alpha_0.\end{aligned}\qquad(6.116)$$

From Equation (6.116) we obtain three algebraic equations for analysis of the self-oscillations. Two are given by the vanishing of the real and imaginary parts of the characteristic equation on the substitution $p = j\Omega$ while the third equation is a particular solution of Equation (6.116) for the case $\alpha = \alpha_0$. The three equations indicated are

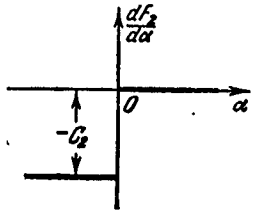


Fig. 6.19.

$$\begin{aligned}&k_ik_e+J_1T_3q_4-\frac{k_ik_e}{c_1l^2}q_2+k_2q_1M(t)-\\&\qquad-(J_1RT_2+k_ik_eT_2^2)\Omega^2=0,\\&J_1R+k_ik_eT_3+J_1\frac{q_3'}{\Omega}+\\&\quad+J_1T_2^2\frac{q_5'}{\Omega}-\frac{J_1R}{c_1l^2}q_2-J_1RT_2^2\Omega^2=0,\\&k_ik_e\left(\alpha_0-\frac{F_2^0}{c_1l^2}\right)+J_1T_3F_1^0-\\&-k_i(k_2U-k_e\gamma)+k_2(R+F_1^0)M(t)=0.\end{aligned}\qquad(6.117)$$

The equations (6.117) obtained permit us to find the values of A, Ω and α_0 for the given system parameters and torque M(t) applied

to the motor.

There is no need to carry out a stability analysis for the periodic solution defined by Eqs. (6.117), since the self-oscillatory steady-state mode of operation is obvious from the operating principle of the relay controller.

Influence of system parameters on self-oscillations. In order to choose the system parameters it is important to define their influence upon the self-oscillation frequency and amplitude. It is sufficient to carry out such an analysis for a constant value of the external disturbance $M(t) = M_0$.

The displacement α_0 of the center of oscillations is determined by the magnitude of the external disturbance $M(t)$. For the nominal mode of operation, we will assume that $M(t) = M_0$, for which $\alpha_0 = 0$. For the nominal mode of operation, let us determine the self-oscillation amplitude and frequency as they depend on the system parameters.

Setting $M(t) = M_0$ and $\alpha_0 = 0$ in Eqs. (6.117), we obtain:

$$\begin{aligned} &k_1k_e + J_1T_3q_1 - \frac{k_1k_e}{c_1l^2}q_2 + k_3q_1M_0 - \\ &\qquad -(J_1RT_3 + k_1k_eT_3^2)\Omega^2 = 0, \\ &J_1R + k_1k_eT_3 + J_1\frac{q_3'}{\Omega} + J_1T_3^2\frac{q_3'}{\Omega} - \frac{J_1R}{c_1l^2}q_2 - J_1RT_3^2\Omega^2 = 0, \\ &-\frac{k_1k_eF_3^0}{c_1l^2} + J_1T_3F_3^0 - k_1(k_3U - k_e\gamma) + k_3(R + F_1^0)M_0 = 0, \end{aligned} \tag{6.118}$$

where, in accordance with (6.115), the constant components and coefficients of harmonic linearization now have the values

$$\begin{aligned} &F_1^0(A) = \frac{R_3}{2}, \qquad q_1(A) = \frac{2R_3}{\pi A}, \qquad F_3^0(A) = \frac{c_3}{\pi}A, \\ &q_2(A) = -\frac{c_3}{2}, \qquad q_3'(\Omega) = \frac{R_3\Omega}{2}, \qquad F_1^0(A, \Omega) = -\frac{R_3A\Omega^2}{\pi}, \\ &q_1(\Omega) = -\frac{R_3\Omega^2}{2}, \qquad q_3'(\Omega) = -\frac{R_3\Omega^2}{2}. \end{aligned} \tag{6.119}$$

Allowing for the values of the constant components and the harmonic linearization coefficients (6.119), we obtain a formula for determining the self-oscillation frequency in terms of the system parameters

$$\Omega=\frac{1}{T_3}\sqrt{1+\left(\frac{k_1k_eT_3}{J_1}+\frac{Rc_2}{2c_1l^2}\right)\frac{1}{R+\frac{R_d}{2}}}. \qquad (6.120)$$

from the second equation of (6.118).

From the first equation of (6.118), we obtain a formula for determing the self-oscillation amplitude in terms of the system parameters for known values of Ω:

$$A=\frac{2k_2R_dM_0}{\pi\left\{\left[J_1T_3\left(R+\frac{R_d}{2}\right)+k_1k_eT_3^2\right]\Omega^2-k_1k_e\left(1+\frac{c_2}{2c_1l^2}\right)\right\}}. \qquad (6.121)$$

The third equation of (6.118) is the equation of the constant component, which determines the position of the center of oscillation for the case of a nominal mode of operation.

Formulas (6.120) and (6.121) permit us to construct the self-oscillation amplitude and frequency as functions of each parameter. In order to synthesize these functions in the region of real parameter values, the parameters were determined experimentally for the MP-15 motor with a centrifugal relay regulator and found to be $J_1 = 0.084$ g·cm·sec, $T_2 = 0.0025$ sec, $T_3 = 0.0053$ sec, $k_1 = 190$ g·cm/amp, $k_e = 0.022$ v·sec, $c_2 = 4600$ g·cm/rad, $c_1 = 1000$ g/cm, $l = 1.8$ cm; $R = 1.7$ ohms, $R_d = 60$ ohms, $M_0 = 100$ g·cm, $\omega_{dv}^0 = 283$ 1/sec^3, $\gamma = 1.5$, $U = 24$ v, $k_2 = 1.8\cdot10^{-3}$ sec.

Figure 6.20 shows curves of the self-oscillation amplitude and frequency as functions of the system parameters, as calculated from Formulas (6.120) and (6.121). The curves shown permit us to make practical recommendations for choosing the system parameters from the condition for obtaining minimum amplitude and maximum frequency in the self-oscillations. For the nominal mode of operation, $A = 0.0029$ rad and $\Omega = (410)1/\text{sec}$.

In the investigation, it is of interest to obtain the variation of the amplitude of the oscillations of the motor speed as a function of the system parameters. To convert the oscillation amplitude of the moving contact to that of the motor speed, we use the equation of motion of the moving contact (6.111). Dropping the constant components and using the value of the nonlinear function $F_2(\alpha) = -(c_2/2)\alpha$ for the nominal mode, we obtain

$$\left(T_3^2p^2+T_3p+1+\frac{c_2}{2c_1l^2}\right)\alpha=k_2\omega_{\Delta\omega}^*,$$

from (6.118); here, ω_{dv}^* [1/sec] is the deviation of the motor speed from the nominal value due to the oscillatory periodic motion of the system. The transfer function from $\Delta\omega^*$ to α is

$$W(p)=\frac{\alpha}{\omega_{\Delta\omega}^*}=\frac{k_2}{T_3^2p^2+T_3p+1+\frac{c_2}{2c_1l^2}}.$$

In accordance with the transfer function, we will have the relationship

$$A_\omega=\frac{A}{k_2}\sqrt{\left(1+\frac{c_2}{2c_1l^2}-T_3^2\Omega^2\right)^2+T_3^2\Omega^2}, \qquad (6.122)$$

for the amplitudes; here, A_ω is the oscillation amplitude of the motor speed.

Since the self-oscillation frequency varies little with changes in the system parameters (other than the parameter T_2), the amplitude variation of the motor speed will duplicate the curves of the oscillation amplitude of the moving contact on a certain scale (except for the curve of Fig. 6.20a). In accordance with (6.122) we obtain

$$A_\omega=1240A=3{,}6 \quad \text{sec}^{-1}$$

for the nominal mode of operation with Ω = 410 1/sec and A = 0.0029 rad, which, for the nominal motor speed ω_{dv}^0 = (283) 1/sec, gives us the periodic relative error δ_{per} = 1.27%.

The experimentally measured value of the moving-contact oscillation frequency agreed with the result of the theoretical investigation to within 20%, which is completely sufficient in the present case.

Steady-state error of system. The total steady-state error of an oscillatory system is composed of the error governed by the displacement of the center of oscillations and the periodic error governed by the oscillations of the controlled quantity relative to the center of

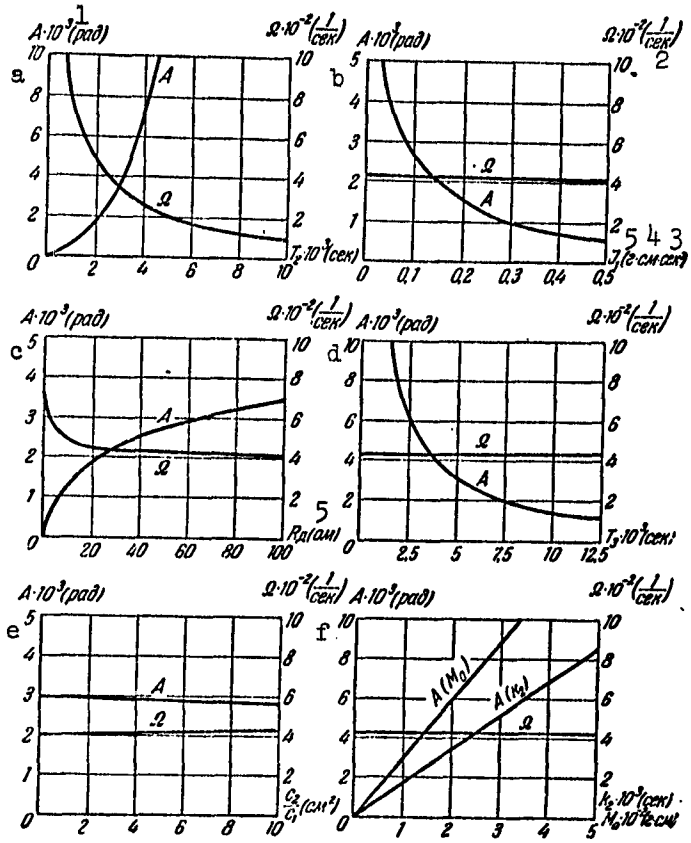


Fig. 6.20. 1) rad; 2) sec; 3) cm; 4) g; 5) R_d(ohms).

oscillation. In order to determine these errors, we must solve the system of three equations (6.117) with respect to the variables A, Ω, α_0 for various values M(t) = M = const. As a result we obtain the functions A(M), Ω(M) and α_0(M) for fixed chosen system parameters. In order to determine the error of the controlled quantity ω_{dv}, we must convert the values of A into values of the motor-speed oscillation amplitudes A_ω and the values of α_0 into values of the static speed error $\Delta\omega_{dv.st}$.

Let us rewrite Eqs. (6.117), substituting the values of the harmonic-linearization coefficients and the values taken earlier for the system parameters. As a result we obtain three transcendental equations:

$$k_1k_e\left[1+\frac{c_2}{2c_1l^2}-\frac{c_2}{\pi c_1l^2}\left(\arcsin\frac{\alpha_0}{A}+\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right)\right]+$$
$$+\frac{2k_2R_\Lambda M}{\pi A}\sqrt{1-\frac{\alpha_0^2}{A^2}}=\left[J_1T_3R+k_1k_eT_3^2+\right.$$
$$\left.+\frac{J_1T_3R_\Lambda}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}-\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right)\right]\Omega^2, \qquad (6.123)$$

$$k_1k_eT_3+J_1\left[R+\frac{R_\Lambda}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}+\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right)+\right.$$
$$\left.+\frac{Rc_2}{\pi c_1l^2}\left(\frac{\pi}{2}-\arcsin\frac{\alpha_0}{A}-\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right)\right]=$$
$$=J_1T_3^2\left[R+\frac{R_\Lambda}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}+\frac{\alpha_0}{A}\sqrt{1-\frac{\alpha_0^2}{A^2}}\right)\right]\Omega^2, \qquad (6.124)$$

$$k_1k_e\left[\alpha_0+\gamma-\frac{k_2}{k_e}U+\frac{c_2\alpha_0}{2c_1l^2}-\frac{c_2}{\pi c_1l^2}\left(\alpha_0\arcsin\frac{\alpha_0}{A}+A\sqrt{1-\frac{\alpha_0^2}{A^2}}\right)\right]+$$
$$+k_2\left[R+\frac{R_\Lambda}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{\alpha_0}{A}\right)\right]M=\frac{J_1T_3R_\Lambda A}{\pi}\sqrt{1-\frac{\alpha_0^2}{A^2}}\,\Omega^2. \qquad (6.125)$$

Where M is variable, it is impossible to solve the transcendental equations (6.123) - (6.125) explicitly for the unknowns A, Ω and α_0. However, without having recourse to graphical solution, we may use the following method here. Assigning values $-1 < \alpha_0/A < 1$, we obtain values of the frequency Ω from Equation (6.124). For these same values of α_0/A and Ω, we obtain the values of the quotient M/A from Equation (6.123). Dividing all the terms of Equation (6.125) by A, we may determine the corresponding values of the amplitude A for known values of α_0/A, Ω and M/A. Multiplying the values of α_0/A by the values the amplitude A, we obtain appropriate values for the displacement α_0 of the center of oscillation. Then, multiplying M/A by A, we obtain the values of the torque M applied to the motor. Figure 6.21a shows curves from the calculation carried out by this method for the parameter values taken earlier.

As is evident from the curves, on a change in the external moment in the direction of a decrease from its nominal value, the oscillation amplitude of the moving contact decreases and the self-oscillations are broken off (A = 0) at some value (M $\approx$ 17 g·cm). The amplitude increases on an increase of the external torque from its nominal value.

For $M < M_0$, the center of oscillation is displaced in the direction of positive α (in the direction of contact breaking) and the displacement of the center of oscillation is zero by the time the oscillations are broken off. For the case $M > M_0$, the center of oscillations is displaced in the direction of negative values of α(in the direction of contact closing). The self-oscillation frequency increases by an insignificant amount on an increase in the external torque.

Since self-oscillation breakoff takes place at $A = \alpha_0 = 0$ on a decrease in the external torque from its nominal value, then in accordance with the third equation of (6.118) we obtain a condition for the self-oscillation breakoff for $\alpha_0 = 0$, $F_2^0 = 0$, $F_4^0 = 0$ and $F_1^0 = R_d$:

$$k_1(k_3U - k_e\gamma) = k_3(R + R_d)M_{min}, \tag{6.126}$$

where M_{min} is the smallest torque required to place the controller in operation.

It follows from 6.126 that in order to insure operation of the controller over the entire range of variation of the external torque, i.e., in the range

$$0 < M < M_{max},$$

the controller must be adjusted so that the condition

$$k_3U = k_e\gamma, \tag{6.127}$$

will be satisified by pretensioning the basic spring, or, if Condition (6.127) cannot be satisfied, the auxiliary resistance R_d must be increased. However it is not advantageous to increase R_d, since this causes an increase in the oscillation ampliutde and hence an increase in the periodic error and static error of the system. Thus, for the motor to have a limited deviation of the external torque from the nominal value, we must always reduce the auxiliary resistance in the interest of increasing the precision of the speed control.

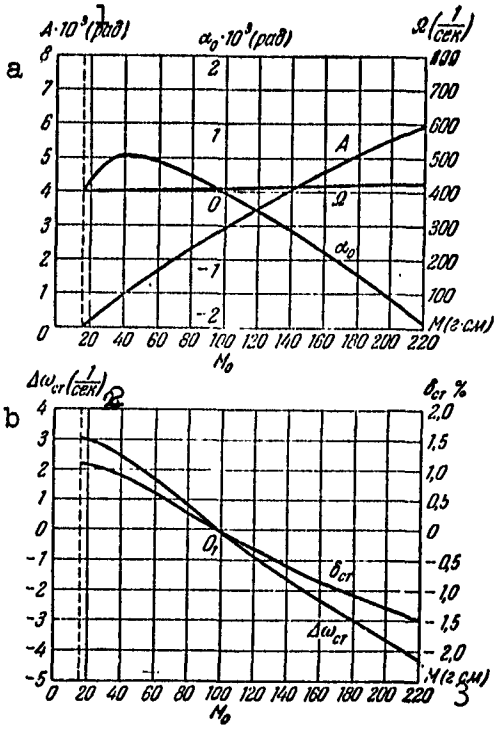


Fig. 6.21. 1) Rad; 2) sec; 3) g·cm.

In order to convert from values of the oscillation-center displacement to the static error in control of the motor speed, let us use Equation (6.111) to write the equation for the steady state of the moving contact allowing only for the constant component of the nonlinear function $F_2(\alpha)$. Allowing for the value of $F_2^0(A, \alpha_0)$ (6.115),

$$\alpha_0 = k_3\omega_{дв} - \gamma - \frac{c_2\alpha_0}{2c_1l^2} + \frac{c_2}{\pi c_1 l^2}\left(\alpha_0 \arcsin\frac{\alpha_0}{A} + A\sqrt{1-\frac{\alpha_0^2}{A^2}}\right). \qquad (6.128)$$

For the steady state in the nominal mode of operation, we have $\alpha_0 = 0$ and $\omega_{dv} = \omega_{dv}^0$ and, consequently,

$$0 = k_3\omega_{дв}^0 - \gamma + \frac{c_2}{\pi c_1 l^2} A_0, \qquad (6.129)$$

where A_0 is the oscillation amplitude for the nominal mode. Subtracting Equation (6.129) from Equation (6.128) and solving the relationship obtained with respect to $\Delta\omega_{dv}$, we obtain a formula for calculating the absolute static error of the motor speed from the known oscillation amplitude and displacement of the oscillation center of the moving contact:

$$\Delta\omega_{дв.\,ст} = \frac{\alpha_0}{k_3}\left[1 + \frac{c_2}{c_1 l^2}\left(\frac{1}{2} - \frac{1}{\pi}\arcsin\frac{\alpha_0}{A}\right)\right] + \frac{c_2}{\pi c_1 l^2}\left(A_0 - A\sqrt{1-\frac{\alpha_0^2}{A^2}}\right). \qquad (6.130)$$

The calculation carried out according to Formula (6.130) is shown by the curves of the absolute and relative static errors (Fig. 6.21b). A periodic error is added to the error mentioned. Since the frequency varies only slightly with varying torque, then we may perform the conversion of the oscillation amplitudes of the moving contact into oscillation amplitudes of the speed according to the relationship for the steady-state rated mode: A_ω = 1240 A.

As a result of the analysis carried out, we have obtained not only curves for the variation of the self-oscillation amplitude and

frequency, but also the periodic and static errors for the controlled quantity - the angular speed of the motor.

§6.7. FOLLOW-UP SYSTEM WITH EXTERNAL DISTURBANCE

Let us investigate the self-oscillations of a follow-up system in the presence of an external nonperiodic setting disturbance. Let us perform the analysis as applicable to the follow-up system whose schematic diagram is shown in Fig. 6.22, where 1 is the setting shaft, 2 is the output shaft, 3 is an amplifier, 4 is a separately-excited direct-current motor and 5 is a reducing gearbox [188]. We will assume that the nonlinearity consists in saturation in the characteristic of the motor torque, which is a function of the armature current (Fig. 6.23).

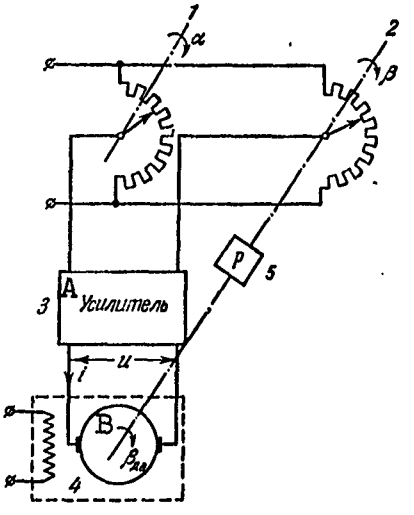


Fig. 6.22. A) Amplifier; B) β_{dv}.

Let us write the equation of moments for the electric motor:

$$J\ddot{\beta} = c_1 i - c_2\dot{\beta},$$

where J is the system moment of inertia reduced to the output shaft, i is the current in the armature winding of the motor and c_1 and c_2 are proportionality coefficients obtained from the mechanical characterisitic of the motor. In standard form, the equation for the motor is written

$$(T_1 p + 1) p\beta = k_1 i, \qquad (6.131)$$

where $T_1 = J/c_2$ is the mechanical time constant of the motor and $k_1 = c_1/c_2$ is the transfer ratio of the motor.

The equation of the amplifier circuit is

$$u = -k'\vartheta, \quad \vartheta = \beta - \alpha, \qquad (6.132)$$

where k' is the gain constant.

For the armature circuit of the motor we obtain

$$(Lp + R)\, i = u - c_3 p \beta_{dv}, \tag{6.133}$$

where L and R are the inductance and resistance of the armature, respectively, and $c_3 p\beta_{dv}$ is the emf induced in the armature.

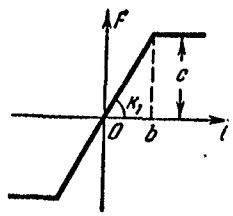


Fig. 6.23

Denoting the transfer ratio of the reducer by c_4 and substituting $\beta_{dv} = \beta/c_4$ in (6.133), applying (6.132) we obtain

$$(T_2 p + 1)\, i = k_2 \alpha - (k_2 + k_3 p)\beta, \tag{6.134}$$

where $T_2 = L/R$ is the electromagnetic time constant of the motor and

$$k_2 = \frac{k'}{R}, \quad k_3 = \frac{c_3}{c_4 R}.$$

Equations (6.131) and (6.134) describe the motion of the servo for armature-current values $i \leq b$ (Fig. 6.23). For certain parameter values, operation of the system is possible with $i \geq b$; this leads to self-oscillations. The equations describing the motion of the servo for this case will, in accordance with (6.131) and (6.134), be

$$\left.\begin{aligned} (T_1 p + 1)\, p\beta &= F(i) \text{ for } i \geq b, \\ (T_2 p + 1)\, i &= k_2 \alpha - (k_2 + k_3 p)\beta, \end{aligned}\right\} \tag{6.135}$$

where F(i) is a nonlinear function expressing the dependence of motor torque on armature current and reduced to the dimensions of speed.

For the case where the setting disturbance α varies at a certain rate $p\alpha$ = const, the self-oscillations will be nonsymmetrical with respect to the equilibrium state. Let us determine the periodic solution for an input of the nonlinear link - the motor-armature current i - in sinusoidal form with a constant component

$$i = i^0 + A \sin \psi, \quad \psi = \Omega t.$$

Harmonic linearization of the nonlinear function F(i) gives

$$F(i) = F^0(A, i^0) + q(A, i^0)\, i^*, \tag{6.136}$$

where $i^* = i - i^0$ is the periodic component of the sought solution. In this case, $\beta = \beta^0 + \beta^*$ at $p\beta^0$ = const.

From (5.121) and (5.122), we obtain values of the constant component and the harmonic-linearization coefficients for the function F(i):

$$F^0(A, i^0)=\frac{k_1}{\pi}\left[A\left(\sqrt{1-\frac{(b+i^0)^2}{A^2}}-\sqrt{1-\frac{(b-i^0)^2}{A^2}}\right)+\right.$$
$$\left.+(b+i_0)\arcsin\frac{b+i^0}{A}-(b-i_0)\arcsin\frac{b-i^0}{A}\right]$$
$$\text{for } A > b+|i^0|. \qquad (6.137)$$

$$q(A, i^0)=\frac{k_1}{\pi}\left(\arcsin\frac{b-i^0}{A}+\arcsin\frac{b+i^0}{A}+\frac{b-i^0}{A}\sqrt{1-\frac{(b-i^0)^2}{A^2}}+\right.$$
$$\left.+\frac{b+i^0}{A}\sqrt{1-\frac{(b+i^0)^2}{A^2}}\right) \text{ for } A\geqslant b+|i^0|. \qquad (6.138)$$

We will assume that the setting disturbance $\alpha(t)$ varies at a constant rate $p\alpha$ = const (Fig. 6.24a), or is some slowly varying function that may be approximated by sections of a broken line (Fig. 6.24b) for practical calculations. In the second case, the solution of the problem for the individual time intervals reduces to the first case of a constant rate of change for the setting disturbance. The minimum length of the time intervals must be larger than the period of the sought periodic solution. In the second case the function $p\alpha$ is piecewise-constant.

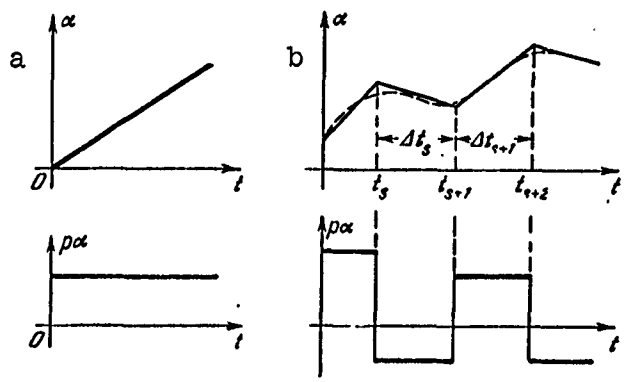


Fig. 6.24.

Thus, on replacement of the external disturbance by a piecewise-linear function, the self-oscillatory process in the time interval being considered may be represented in the form of a sequence of

quasi-stationary processes, the duration of each of which is determined by the time Δt_{s+n}. Here the passage from one quasi-stationary mode of operation to another is, so to speak, accomplished "instantaneously" at the interval boundaries t_s, t_{s+1},..., t_n. The quantities A, Ω and i^0 are assumed constant within each time interval.

Applying (6.136), we obtain from Eq. (6.135) a harmonically linearized equation for the tracking mode with constant speed:

$$[T_1T_2p^3+(T_1+T_2)p^2+(1+k_3q)p+k_2q]i^*=k_2(p\alpha-F^0). \qquad (6.139)$$

Carrying out the substitution $p = j\Omega$ in the characteristic equation corresponding to the differential equation (6.139), we obtain two equations from the condition of equality of the real and imaginary parts to zero. We obtain the third equation as a particular solution of the differential equation (7.139) for $i^* = 0$, applying the condition $p\alpha = \text{const}$. The three equations mentioned for the determination of the amplitude and frequency of the periodic solution are

$$\left.\begin{aligned} k_2q(A,\ i^0)-(T_1+T_2)\Omega^2=0,\\ 1+k_3q(A,\ i^0)-T_1T_2\Omega^2=0,\\ p\alpha-F^0(A,\ i^0)=0. \end{aligned}\right\} \qquad (6.140)$$

Equation (6.140) permits us to determine the amplitude A, the frequency Ω and the displacement i^0 of the oscillation center of the periodic solution for the variable $\underline{i}$ for given system parameters and a given setter rate $p\alpha$.

We will not carry out the analysis of the influence of the system parameters upon the amplitude, frequency and displacement of the center of oscillations in the present case, since this is conveniently carried out for the symmetrical mode where $p\alpha = 0$ and, consequently, $F^0(A, i^0) = 0$ and $i^0 = 0$. Similar analyses were carried out in Chapter 4.

In the present chapter we shall consider a method for finding the amplitude, frequency and displacement of the oscillation center as functions of the setter rate $p\alpha$.

It is evident from the first two equations of (6.140) that for constant parameter values, the frequency of the periodic solution will be constant and determined by the formula

$$\Omega = \sqrt{\frac{k_3}{k_2 T_1 T_2 - k_3 (T_1 + T_2)}}. \qquad (6.141)$$

In order to determine A^0 and i^0, let us eliminate the quantity Ω^2 from the first two equations of (6.140); solving them for $q(A, i^0)$, we obtain

$$q(A, i^0) = \frac{T_1 + T_2}{k T_1 T_2 - k_3 (T_1 + T_2)}. \qquad (6.142)$$

Assigning values of A for various constant values of i^0, we use Formula (6.138) to construct curves of q(A) for the case i^0 = const and, in accordance with (6.142), draw the straight line $q(A, i^0)$ = const (Fig. 6.25). The intersection points of the curves of q(A) with the straight line give values of the amplitude A as a function of the displacement of the center of oscillation. Substituting the values of A and i^0 into Formula (6.137), we determine, according to (6.140), the rates pα of the follow-up system's setter that correspond to these values. Thus we solve the problem of determining the functions $A = A(p\alpha)$, $\Omega = \Omega(p\alpha)$ and $i^0 = i^0(p\alpha)$, which may be represented in the form of certain curves (Fig. 6.26).

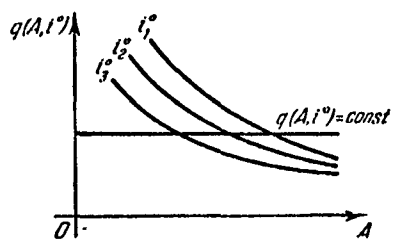


Fig. 6.25.

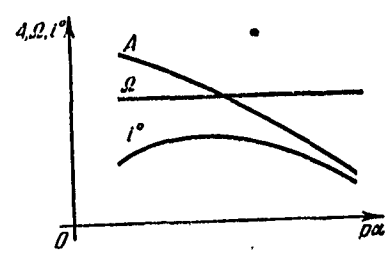


Fig. 6.26.

It is of practical interest to find the oscillation-center displacement $\vartheta = \beta - \alpha$ and the amplitude A_β for the output shaft,

which are the static and periodic errors of the follow-up system. In order to convert values of A into values of A_β, we must make use of the second equation of (6.135), from which we obtain the following transfer function for the periodic components:

$$W(p)=\frac{i^*}{\beta^*}=-\frac{k_2+k_3p}{T_2p+1}. \qquad (6.143)$$

In accordance with (6.135), for a follow-up mode with the constant rate $p\alpha = p\beta = \text{const}$. we obtain

$$i^0=-k_2\vartheta^0-k_3p\alpha,$$

from which we obtain a formula for the steady-state static error in the mismatch angle ϑ:

$$\vartheta^0=-\frac{k_3p\alpha+i^0}{k_2}.$$

In order to convert the amplitudes of the periodic solution, we obtain the formula

$$A_\beta=A\sqrt{\frac{T_2^2\Omega^2+1}{k_2^2+k_3^2\Omega^2}},$$

from (6.143). As a result, we obtain curves of the static and periodic errors of the follow-up system as functions of the follow-up rate for the self-oscillatory mode of operation.

To prove the stability of the periodic solution, we may use the approximate stability criterion

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial\omega}\right)^*-\left(\frac{\partial X}{\partial\omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^*>0.$$

We determine the derivatives occuring in the inequality of the criterion from the expressions

$$X(a,\ \omega)=k_3q-(T_1+T_2)\omega^2,$$
$$Y(a,\ \omega)=(1+k_3q)\omega-T_1T_2\omega^3,$$

corresponding to Equation (6.139).

Calculating the derivatives, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^*=k_3\left(\frac{dq}{da}\right)^*,\quad \left(\frac{\partial X}{\partial\omega}\right)^*=-2(T_1+T_2)\Omega,$$
$$\left(\frac{\partial Y}{\partial a}\right)^*=k_3\left(\frac{dq}{da}\right)^*\Omega,\quad \left(\frac{\partial Y}{\partial\omega}\right)^*=1+k_3q-3T_1T_2\Omega^2.$$

Since

$$1 + k_3 q = T_1 T_2 \Omega^2,$$

from the second equation of (6.140),

$$\left(\frac{\partial Y}{\partial \omega}\right)^* = -2T_1 T_2 \Omega^2.$$

Substituting the values of the derivatives into the left member of the criterion inequality, we obtain

$$-2\left(\frac{dq}{da}\right)^* \Omega^2 [k_3 T_1 T_2 - k_3 (T_1 + T_2)].$$

It follows from (6.141) that

$$k_3 T_1 T_2 - k_3 (T_1 + T_2) = \frac{k_3}{\Omega^2}.$$

Allowing for the sign of the derivative (dq/da)* (see Fig. 3.7), we find that the stability criterion for the periodic solution is satisfied:

$$\left(\frac{\partial X}{\partial a}\right)^* \left(\frac{\partial Y}{\partial \omega}\right) - \left(\frac{\partial X}{\partial \omega}\right)^* \left(\frac{\partial Y}{\partial a}\right)^* = -2k_3 \left(\frac{dq}{da}\right)^* > 0.$$

Thus, the periodic solution which we have obtained is stable, i.e., for $i > b + i^0$ and $p\alpha = \text{const}$, the system has a self-oscillatory steady-state mode of operation.

§6.8. SYSTEM WITH NONSYMMETRICAL NONLINEARITY ON EXTERNAL DISTURBANCE

Let us carry out the investigation for a nonlinear sutomatic control system with a delay, as performed by N.V. Starikova [179]. In this problem, we determine the influence of the system parameters and a slowly-varying perturbing disturbance on the amplitude and frequency of the self-oscillations, together with the stability boundary of the system as a function of the constant external disturbance.

The system considered is represented by a block diagram (Fig. 6.27a, b) where 1 is the controlled object, 2 is the sensitive element, 3 is the magnetic amplifier, 4 is the control relay, 5 is a power re-

lay, 6 is the output device, 7 is the control element and 8 is the feedback link. On combining certain links, the system may be represented by the block diagram shown in Fig. 6.27b.

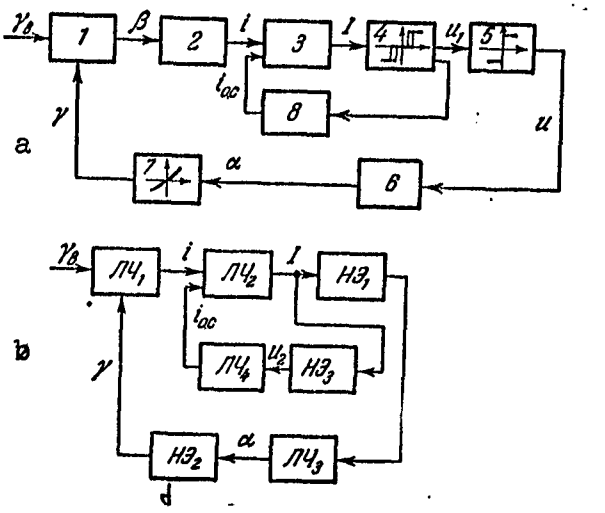


Fig. 6.27

The dynamics of the process in the system are described by the following equations for the links.

The equation for the controlled object is

$$(d_1p^2+d_2p+d_3)\beta=(c_1p+c_2)(\gamma_в-\gamma), \tag{6.144}$$

where γ_v is the perturbing disturbance, γ is the control disturbance, β is the deviation of the controlled quantity and d_1, d_2, d_3, c_1 and c_2 are constant coefficients.

The equation for the sensitive element is

$$(T_1p+1)i=k_1\beta, \tag{6.145}$$

where $\underline{i}$ is the input of the sensitive element, T_1 is the time constant of the sensitive element and k_1 is the transfer ratio.

The equation of the magnetic amplifier is

$$(T_2p+1)I=k_2i-k_3i_{o.c}, \tag{6.146}$$

where I is the amplifier output current, T_2 is the time constant of the amplifier, k_2 is the current gain, and k_3 is the feedback coefficient.

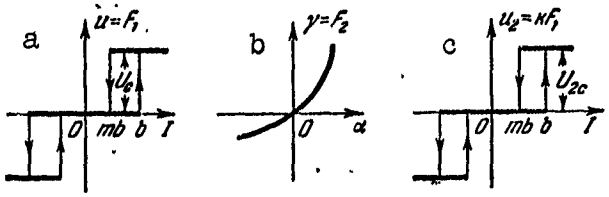


Fig. 6.28

The control and power relays have a nonlinear characteristic (Fig. 6.28a), representing the nonlinear function $F_1(I)$, where U_s is the power voltage to the windings of the motor, which is switched by means of the output relay.

The equation of the output device (motor) is

$$(T_3p+1)p\alpha=k_4e^{-\tau p}u, \qquad u=F_1(I), \tag{6.147}$$

where $k_4 = V/U_s$; here V is the steady-state rate of the output device, U_s is the dc power voltage cut out by the relay, τ is the delay in the output device and the power relay, T_3 is the time constant of the output device and α is the control-element angle of rotation.

As a nonlinear link, the equation of the control element has the form

$$\gamma=r_1\alpha^2+r_2\alpha=F_2(\alpha), \tag{6.148}$$

where r_1, r_2 are coefficients determined by the parameters of the control element. The static characteristic of the control element is shown in Fig. 6.28b.

The equation for the nonlinear-link feedback (Fig. 6.28c) is

$$I_{o.c}=k_{o.c}u_2, \qquad u_2=kF_1(I), \qquad k=\frac{U_{2c}}{U_c}. \tag{6.149}$$

The static characteristic of the nonlinear link NE_3 is shown in Fig. 6.28c.

In the system investigated, the periodic solution will be nonsymmetrical (with a constant component), since there is a nonsymmetrical static characteristic $F_2(\alpha)$. We shall determine the solution for the

variable I in the form

$$I = I^0 + A_1 \sin\psi, \quad \psi = \Omega t$$

and for the variable α in the form

$$\alpha = \alpha^0 + A_2 \sin(\psi + \varphi), \quad \psi = \Omega t.$$

Harmonic linearization of the nonlinear function $F_1(I)$ reduces to its replacement by the relationship

$$F_1(I) = F_1^0(A, I^0) + \left[q_1(A, I^0) + \frac{q_1'(A, I^0)}{\Omega} p\right] I^*, \tag{6.150}$$

where $I^* = I - I^0$ is the periodic component of the solution sought. In accordance with (5.89) - (5.91), the constant component and the harmonic-linearization coefficients are

$$\left.\begin{aligned} F_1^0 &= \frac{U_c}{2\pi}\left(\arcsin\frac{b+I^0}{A_1} - \arcsin\frac{b-I^0}{A_1} + \right. \\ &\qquad \left. + \operatorname{acrsin}\frac{mb+I^0}{A_1} - \arcsin\frac{mb-I^0}{A_1}\right), \\ q_1 &= \frac{U_c}{\pi A_1}\left[\sqrt{1-\frac{(b+I^0)^2}{A_1^2}} + \sqrt{1-\frac{(b-I^0)^2}{A_1^2}} + \right. \\ &\qquad \left. + \sqrt{1-\frac{(mb+I^0)^2}{A_1^2}} + \sqrt{1-\frac{(mb-I^0)^2}{A_1^2}}\right], \\ q_1' &= -\frac{2U_c b}{\pi A_1^2}(1-m) \text{ for } A \geqslant b + |I^0|. \end{aligned}\right\} \tag{6.151}$$

Harmonic linearization of the nonlinear function $F_2(a)$ gives

$$F_2(\alpha) = F_2^0(A_2, \alpha^0) + q_2(A_2, \alpha^0)\alpha^*, \tag{6.152}$$

where, in accordance with (6.148), the constant component and the coefficient of harmonic linearization have the values

$$\left.\begin{aligned} F_2^0 &= r_1\left[(\alpha^0)^2 + \frac{A_2^2}{2}\right] + r_2\alpha^0, \\ q_2 &= 2r_1\alpha^0 + r_2. \end{aligned}\right\} \tag{6.153}$$

From the harmonically linearized equations, we synthesize the characteristic equation and use the linear methods for the analysis. Separating the constant and periodic components in all the variable quantities, i.e., representing them in the form

$$\gamma = \gamma^0 + \gamma^*, \quad i = i^0 + i^*, \quad u = u^0 + u^*,$$
$$i_{o.c} = i_{o.c}^0 + i_{o.c}^*, \quad \alpha = \alpha^0 + \alpha^*, \quad I = I^0 + I^*,$$

we obtain equations for the constant components in the presence of a constant perturbing disturbance γ_v:

$$\left.\begin{array}{c} d_3 i^0 = k_1 c_2 (\gamma_в - \gamma^0), \quad I^0 = k_2 i^0 - k_3 i^0_{o.c}, \\ u^0 = F_1^0(A_1, I^0), \; 0 = k_4 u^0, \; \gamma^0 = F_2^0(A_2, \alpha^0), \; u_2^0 = k F_1^0(A_1, I^0), \\ i^0_{o.c} = k_{o.c} k F_1^0(A_1, I_0) \end{array}\right\} \tag{6.154}$$

and the equations for the periodic components:

$$\left.\begin{array}{l} (d_1 p^2 + d_2 p + d_3)(T_1 p + 1) i^* = -k_1 (c_1 p + c_2) \gamma^*, \\ (T_2 p + 1) I^* = k_2 i^* - k_3 i^*_{o.c}, \quad (T_3 p + 1) p x^* = k_4 e^{-\tau p} u^*, \\ u^* = \left[q_1(A_1, I^0) + \frac{q_1'(A_1, I^0)}{\Omega} p \right] I^*, \\ \gamma^* = q_2(A_2, \alpha^0) \alpha^*, \quad i^*_{o.c} = k_{o.c} k u^*. \end{array}\right\} \tag{6.155}$$

In these equations, F_1^0, q_1 and q_1' are taken from Formulas (6.151). Combining Eqs. (6.155) and substituting the values of q_2, we write the characteristic equation of the system:

$$\begin{array}{l} (d_1 p^2 + d_2 p + d_3)(T_1 p + 1)(T_2 p + 1)(T_3 p + 1) p + \\ \quad + (c_1 p + c_2) k_1 k_2 k_4 e^{-\tau p} \left[q_1(A_1, I^0) + \frac{q_1'(A_1, I^0)}{\Omega} p \right] (2 r_1 x^0 + r_2) + \\ \quad + k k_{o.c} k_3 (d_1 p^2 + d_2 p + d_3)(T_1 p + 1)(T_3 p + 1) p \times \\ \quad \times \left[q_1(A_1, I^0) + \frac{q_1'(A_1, I^0)}{\Omega} p \right] = 0. \end{array} \tag{6.156}$$

From Eqs. (6.154) and (6.151) we obtain

$$u^0 = 0, \quad u_2^0 = 0, \quad F_1^0(A_1, I^0) = 0, \quad I^0 = 0, \quad i^0_{o.c} = 0, \quad i^0 = 0,$$
$$\gamma_в - \gamma^0 = 0, \quad F_2^0(A_2, \alpha^0) = \gamma_в$$

for the constant components.

According to (6.153), we find α^0 for the case of the constant external disturbance γ_v

$$\alpha^0 = \frac{-r_2 + \sqrt{r_2^2 - 4 r_1 \left(\frac{1}{2} r_1 A_2^2 - \gamma_в \right)}}{2 r_1}. \tag{6.157}$$

from the last relationship above. We take the plus sign in front of the square root since $A_2 = 0$ and $\alpha^0 = 0$ for $\gamma_v = 0$.

From the appropriate equations of (6.155), we obtain the relationship between the amplitudes A_1 and A_2:

$$A_2 = \frac{A_1 k_4}{\Omega \sqrt{T_3^2\Omega^2 + 1}} \sqrt{q_1^2(A_1) + [q_1'(A_1)]^2}. \tag{6.158}$$

Substituting $p = j\Omega$ into Equation (6.156), and separating the real and imaginary parts, we obtain the expression $L(j\Omega)$ for the characteristic curve:

$$L(j\Omega) = X(A_1, \Omega, \alpha^0) + jY(A_1, \Omega, \alpha^0). \tag{6.159}$$

The amplitude A_1 and frequency Ω of the periodic solution are determined from the condition $L(j\Omega) = 0$, which gives the equations

$$X(A_1, \Omega, \alpha^0) = 0, \quad Y(A_1, \Omega, \alpha^0) = 0, \tag{6.160}$$

to which we adjoin Formula (6.157).

In order to determine the self-oscillations in the system for given parameters, we represent the left member of Eqs. (6.160) in the form

$$\left.\begin{aligned} X &= X_1(\Omega) + X_2(A_1, \Omega, \alpha^0) + k_{o.c} X_3(A_1, \Omega), \\ Y &= Y_1(\Omega) + Y_2(A_1, \Omega, \alpha^0) + k_{o.c} Y_3(A_1, \Omega), \end{aligned}\right\} \tag{6.161}$$

where

$$\begin{aligned}
X_1(\Omega) &= -T_1T_2T_3d_1\Omega^6 + [(T_1 + T_2 + T_3)d_1 + (T_1T_2 + T_1T_3 + T_2T_3)d_2 + \\
&\quad + T_1T_2T_3d_3]\Omega^4 - [(T_1 + T_2 + T_3)d_3 + d_2]\Omega^2, \\
Y_1(\Omega) &= [(T_1T_2 + T_1T_3 + T_2T_3)d_1 + T_1T_2T_3d_2]\Omega^5 - \\
&\quad - [d_1 + (T_1 + T_2 + T_3)d_2 + (T_1T_2 + T_1T_3 + T_2T_3)d_3]\Omega^3 + d_3\Omega, \\
X_2(A_1, \Omega, \alpha^0) &= k_1k_2k_4(2r_1\alpha^0 + r_2)\{[q_1(A_1)\cos\Omega\tau + q_1'(A_1)\sin\Omega\tau]c_2 - \\
&\quad - [q_1'(A_1)\cos\Omega\tau - q_1(A_1)\sin\Omega\tau]c_1\Omega\}, \\
Y_2(A_1, \Omega, \alpha^0) &= k_1k_2k_4(2r_1\alpha^0 + r_2)\{[q_1(A_1)\cos\Omega\tau + q_1'(A_1)\sin\Omega\tau]c_1\Omega + \\
&\quad + [q_1'(A_1)\cos\Omega\tau - q_1(A_1)\sin\Omega\tau]c_2\}, \\
X_3(A_1, \Omega) &= kk_3q_1'(A_1)\{-d_1T_1T_2\Omega^3 + \\
&\quad + [d_2T_1T_2 + d_2(T_1 + T_2) + d_1]\Omega^3 - d_3\Omega\} + \\
&\quad + kk_3q_1(A_1)\{[d_2T_1T_2 + d_1(T_1 + T_2)]\Omega^4 - [d_2(T_1 + T_2) + d_2]\Omega^2\}, \\
Y_3(A_1, \Omega) &= kk_3q_1(A_1)\{d_1T_1T_2\Omega^3 - \\
&\quad - [d_2T_1T_2 + d_2(T_1 + T_2) + d_1]\Omega^3 + d_3\Omega\} + \\
&\quad + kk_3q_1'(A_1)\{[d_2T_1T_2 + d_1(T_1 + T_2)]\Omega^4 - [d_2(T_1 + T_2) + d_3]\Omega^2\}.
\end{aligned}$$

To evaluate the influence of some system parameter $\underline{z}$, on the other hand, we write the equations in the form

$$\left.\begin{aligned} X &= X^{(1)}(A_1, \Omega, \alpha^0) + X^{(2)}(A_1, \Omega, \alpha^0, z), \\ Y &= Y^{(1)}(A_1, \Omega, \alpha^0) + Y^{(2)}(A_1, \Omega, \alpha^0, z). \end{aligned}\right\} \qquad (6.162)$$

In many cases we may write

$$X = X^{(1)}(A_1, \Omega, \alpha^0) + zX^{(2)}(A_1, \Omega, \alpha^0),$$
$$Y = Y^{(1)}(A_1, \Omega, \alpha^0) + zY^{(1)}(A_1, \Omega, \alpha^0).$$

Because of the complexity of the system equations, it is extremely difficult or impossible to obtain A_1 and Ω as explicit functions of the system parameters. We shall therefore make use of a graphoanalytical method for determining A_1 and Ω. Assigning various numerical values to A_1, let us construct the curves $L(j\omega) = X(\omega) + jY(\omega)$ in the complex plane (X, jY) for A_1 = const, z = const, and interpolate to find the sought value of A_1, for which $L(j\omega)$ passes through the origin, and the unknown Ω at the origin (Fig. 6.29a). The separation of the parameter z in (6.126) permits us to select values of A_1 and Ω such that the quantities (6.162) will be sufficiently close to zero. Therefore we may limit the synthesis of the characteristic curves to the region close to the origin.

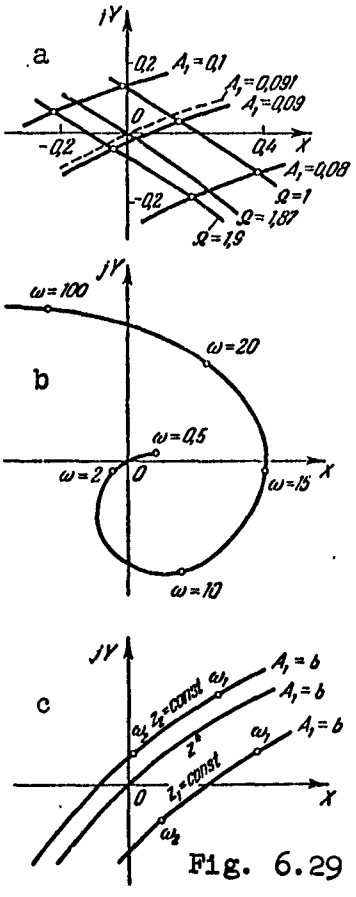


Fig. 6.29

The stability of the periodic solution is approximately determined by the familiar method (see page 125), using the characteristic curve $L(j\omega)$, which is constructed in Fig. 6.29b.

Let us now find the region of equilibrium stability for the system for self-oscillations absent. We determine the value of the parameter z that guaratees the absence of self-oscillations and a stable equilibrium position as follows. Let us choose the boundary value

$z = z^*$, all remaining parameters given, such that for the case $z = z^*$ the curve $L(j\omega)$ will intersect the origin at $A_1 = b$, encompassing $(n - 1)$ quadrants. Here it is necessary that for all $A_1 > b$, the curve of $L(j)$ surround the origin, passing through n quadrants, which corresponds to damping of the oscillations. On the other hand, oscillation amplitudes $A_1 < b$ are impossible, and the system arrives at the stable-equilibrium state inside the dead zone b (Fig. 6.29b).

Self-oscillations in no-feedback system. To determine the influence of the system parameters on A_1 and Ω, let us consider a system without feedback.

Setting $\gamma_v = 0$ and $k_{o.s} = 0$ in Eqs. (6.157), (6.160) and (6.161), we find the self-oscillation parameters of the system for the following values of the coefficients of the equation:

$d_1 = 2.6 \cdot 10^{-3}$ $d_2 = 0.1$ $d_3 = 1.0$ $c_1 = 87$

$k_2 = 20$ $V = 10$ degrees/sec $c_2 = 9.8 \cdot 10^2$ $k_1 k_2 = 2.3$

$k_3 = 0.4$ $T_1 = 0.08$ sec $T_2 = T_3 = 0.2$ sec $\tau = 0.4$ sec

$r_2 = 0.3$

The influence of the various parameters on the self-oscillations was determined by the graphoanalytical method described. Figure 6.30 shows the results obtained in the form of the self-oscillation frequency Ω and amplitude A_1 as functions of: a) the time constant T_2, b) the steady-state rate V, c) the lag τ and d) the gain constant k_2. It is obvious from the curves that k_2, V and τ are most essential in reducing the self-oscillation amplitude. However, self-oscillations in such a system cannot be suppressed by varying the individual parameters without introducing feedback.

Self-oscillations and stability of a system with feedback and without perturbing disturbance.

The feedback is cut in and cut out by a controlling relay which also transmits signals simultaneously to a power relay and a follow-up mechanism (the second contact pair of the power relay). The feed-

back voltage is subtracted from the signal voltage governed by the

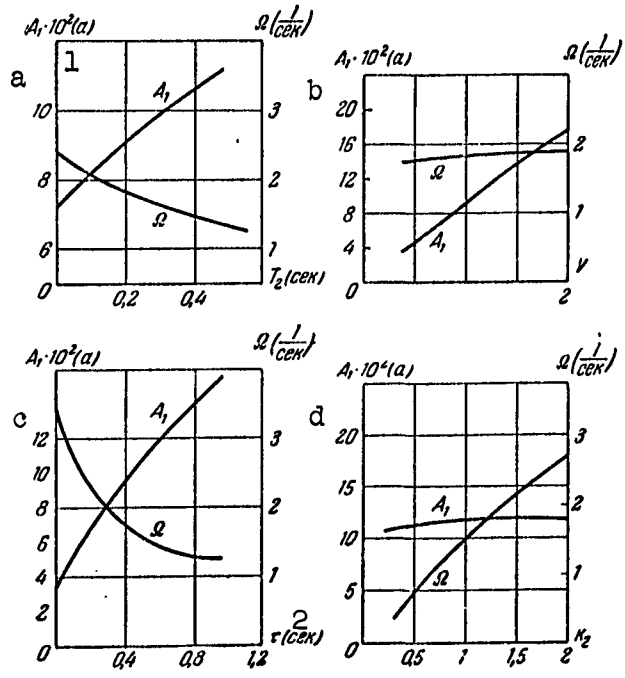


Fig. 6.30. 1) Amperes; 2) sec.

deviation of the controlled quantity in the amplifier. To analyze the self-oscillations, we make use of Eqs. (6.161), (6.158) and (6.157) for $\gamma_V = 0$. We have a periodic solution if

$$\left.\begin{aligned} X_1(\Omega) + X_2(A_1, \Omega, \alpha^0) &= -k_{o.c} X_3(A_1, \Omega), \\ Y_1(\Omega) + Y_2(A_1, \Omega, \alpha^0) &= -k_{o.c} Y_3(A_1, \Omega). \end{aligned}\right\} \qquad (6.163)$$

By the method described above we determine the boundary feedback coefficient as a function of each system parameter, i.e., we find the stability boundary for various system parameters. For example, representing Eqs. (6.163) in the form

$$\begin{aligned} X_1(\Omega) + X_2(A_1, \Omega, \alpha^0, \tau) &= -k_{o.c} X_3(A_1, \Omega), \\ Y_1(\Omega) + Y_2(A_1, \Omega, \alpha^0, \tau) &= -k_{o.c} Y_3(A_1, \Omega). \end{aligned}$$

let us construct the stability boundary $k^*_{o.s} = f(\tau)$. The stability boundary $k_{o.s} = f(\tau)$ is shown in Fig. 6.31a.

We construct the stability boundaries in the parameter planes $k_{o.s}$, b (Fig. 6.31b), $k_{o.s}$, V (Fig. 6.31c) and $k_{o.s}$, k_2 (Fig. 6.31d)

in a similar manner. Eliminating $k_{o.s}$, we may use these curves to construct stability regions for any combination of two parameters of the system [with] $k_{o.s}$.

Self-oscillations and quality of control process in system with constant or slowly varying external disturbance.

In the presence of a constant or slowly varying perturbing disturbance γ_v, which may be assumed constant over the self-oscillation period, the characteristic equation determining the periodic solution remains as before. Only α^0 is a function of the quantity γ_v (see (6.157)). By means of the method developed, and using Eqs. (6.157), (6.158) and (6.160), we may determine the self-oscillation amplitude A_1 and frequency Ω as functions of γ_v. The result of the solution

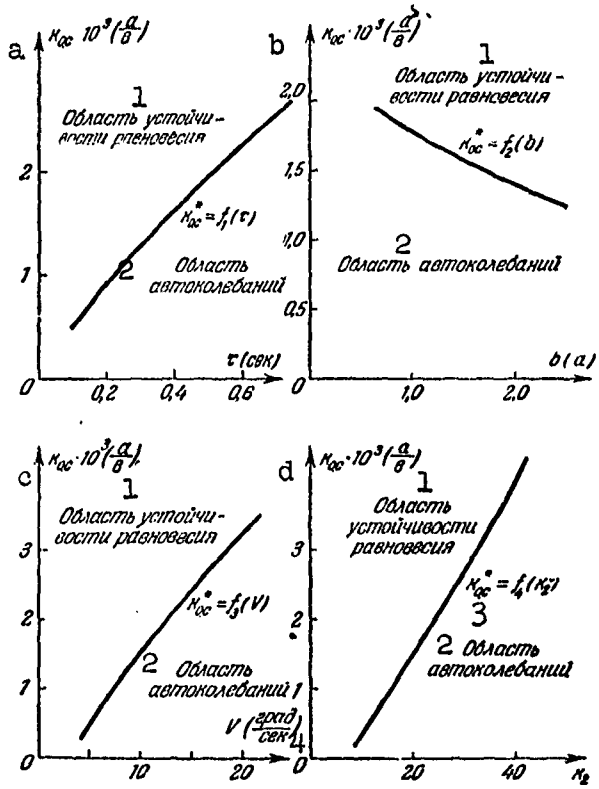


Fig. 6.31. 1) Region of equilibrium stability; 2) region of self-oscillation; 3) o.s; 4) degrees/sec.

is shown in Fig. 6.32a.

In the presence of a slowly varying disturbance γ_v, whose rate of change cannot be disregarded, we must allow for slowly varying components in the equations of all the links (6.144) – (6.149), in which the nonlinear functions are substituted as follows:

$$u = F_1^0(A_1, I^0), \quad \gamma = F_2^0(A_2, \alpha^0), \quad u_2 = kF_1^0(A_1, I^0), \tag{6.164}$$

here the quantities I^0 and α^0 must be regarded as slowly varying over time. Since the functions $u(I^0)$, $\gamma(\alpha^0)$, and $u_2 I^0$ given by Formulas (6.164) for slowly varying I^0, α^0 are represented by smooth curves passing through the origin, they may be linearized (see §5.3) by the formulas

$$u = \left(\frac{\partial F_1^0}{\partial I^0}\right)_{I^0=0} I^0, \quad \gamma = \left(\frac{\partial F_2^0}{\partial \alpha^0}\right)_{\alpha^0=0} \alpha^0, \quad u_2 = k\left(\frac{\partial F_1^0}{\partial I^0}\right)_{I^0=0} I^0. \tag{6.165}$$

Then all the slowly varying components for given $\gamma_v(t)$ will be determined by a system of linear equations. From them, using the ordinary methods of linear control theory, we may determine the quality of the control process in the system for the case of a slowly varying perturbing disturbance $\gamma_v(t)$. Here the self-oscillations are determined by the method set forth above. Since we have already determined the slowly varying component $I^0(t)$, upon which the self-oscillations depend, we also obtain a slow variation of the self-oscillation amplitude $A(t)$ and frequency $\Omega(t)$ corresponding to a given $\gamma_v(t)$.

Stability of system in the presence of a constant external disturbance.

In the case of a nonsymmetrical nonlinear characteristic, the quantity α^0, the constant component F_2^0 and the coefficient q_2^2 vary with variation of the quantity γ_v. In order to determine the position of the system stability boundary in $k_{o.s}$ as a function of the constant external disturbance γ_v, we use the method described above for determining z*. The result is shown in Fig. 6.32b. We may also easily con-

struct the stability boundary for other parameters and find the displacement of the stability boundary in the parameter plane for variation of γ_v. In practice it is very important to allow for this stability-boundary displacement in a nonlinear system with variation of the magnitude of the external disturbance γ_v. Such a specific phenomenon does not occur in linear systems.

The results obtained are confirmed well enough on solution of the intitial nonlinear equations on an electronic simulator.

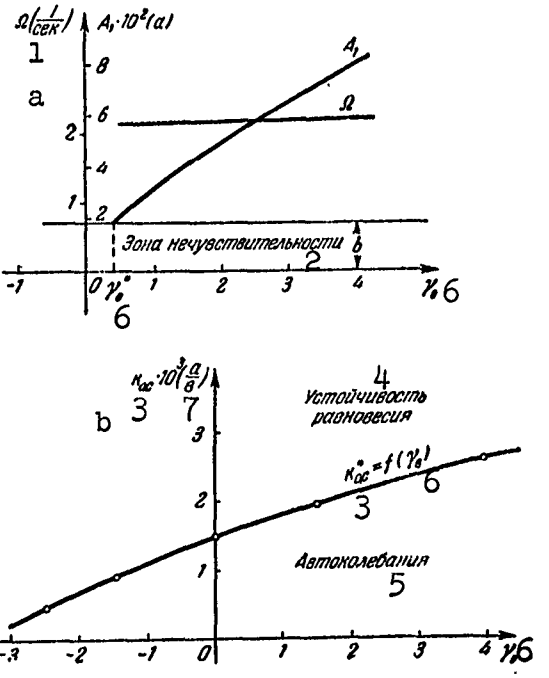


Fig. 6.32. 1) sec; 2) dead zone; 3) o.s; 4) equilibrium stability; 5) self-oscillations; 6) γ_v; 7) amp/v.

§6.9. SMOOTHING OF RELAY CHARACTERISTIC BY SELF-OSCILLATIONS

Let us assume that in an automatic system built according to the diagram shown in Fig. 6.33a, the nonlinearity consists of a three-position relay with a dead zone, the characteristic of which

$$F = F(x) \tag{6.166}$$

is shown in Fig. 6.33b. The relay time delay is referred to linear part I.

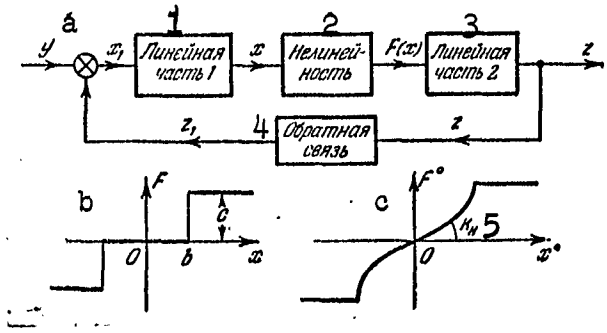


Fig. 6.33. 1) Linear part 1; 2) nonlinearity; 3) linear part 2; 4) 4) feedback; 5) K_n.

The linear part 1 is described by the equation

$$x(t)=k_1x_1(t-\tau) \quad \text{or} \quad x=k_1e^{-\tau p}x_1. \tag{6.167}$$

The equation of the linear part 2 takes the form

$$(T_2p+1)pz=k_2F, \tag{6.168}$$

while the feedback is

$$(T_0p+1)z_1=k_0z, \tag{6.169}$$

where

$$x_1=y-z_1. \tag{6.170}$$

We must first find the self-oscillations in the system in the absence of an external disturbance (y = 0), and then determine the smoothed relay characteristic for a slowly varying component and the variation of the self-oscillation amplitude as a function of this component in the presence of an external disturbance ($y \neq 0$) [187].

In the absence of an external disturbance (y = 0), the system equation is, in accordance with (6.166) – (6.170),

$$Q(p)x+R(p)F(x)=0, \tag{6.171}$$

where

$$Q(p)=(T_0p+1)(T_2p+1)p, \quad R(p)=ke^{-\tau p}, \quad k=k_1k_2k_0. \tag{6.172}$$

Harmonic linearization of the given nonlinearity (Fig. 6.33b) gives

$$F(x)=qx, \tag{6.173}$$

where, according to (3.13),

$$q=\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}}. \tag{6.174}$$

Here the characterisitic equation corresponding to (6.171) is

$$T_0T_2p^3+(T_0+T_2)p^2+p+kqe^{-\tau p}=0. \tag{6.175}$$

Making the substitution p = jΩ and separating the real and imaginary parts, using the expression $e^{-j\tau\Omega} = \cos\tau\Omega - j\sin\tau\Omega$ we obtain two equations:

$$\left.\begin{aligned} X&=kq\cos\tau\Omega-(T_0+T_2)\Omega^2=0,\\ Y&=\Omega-kq\sin\tau\Omega-T_0T_2\Omega^3=0. \end{aligned}\right\} \tag{6.176}$$

Let us assume that the time constant T_2 of the linear part 2 (for example, the motor) and the lag τ in operation of the relay are given, and that we must choose the feedback time constant T_0 and the total loop gain constant for the nonlinear links, $k = k_1k_2k_0$. Let us solve Equation (6.176) for these parameters:

$$\left.\begin{aligned} T_0&=\frac{1-T_2\Omega\operatorname{tg}\tau\Omega}{\Omega(T_2\Omega+\operatorname{tg}\tau\Omega)},\\ k&=\frac{(T_0+T_2)\Omega^2}{q\cos\tau\Omega}=\frac{\Omega(1+T_2^2\Omega^2)}{q(T_2\Omega+\operatorname{tg}\tau\Omega)\cos\tau\Omega}. \end{aligned}\right\} \tag{6.177}$$

From the first formula, assigning various values to Ω and computing T_0 each time, we obtain the frequency Ω of the periodic solution as a function of the constant T_0 (Fig. 6.34a). Here we must assign values to Ω that result in positive values of T_0 and $\underline{k}$. Figure 6.34b shows the graphical determination that follows from (6.177) for intervals of these values.

According to the first equation of (6.177), in each interval a certain frequency Ω of the periodic solution that is not a function of the gain constant $\underline{k}$ corresponds to each value of T_0. For any given value of T_0, therefore, the second formula of (6.177) gives

$$k = \frac{C}{q}, \text{ where } C = \frac{(T_0 + T_2)\Omega^2}{\cos \tau\Omega}. \qquad (6.178)$$

It is evident that the value of C will be different for the different intervals of Ω, whereupon we must take $\cos \tau\Omega > 0$.

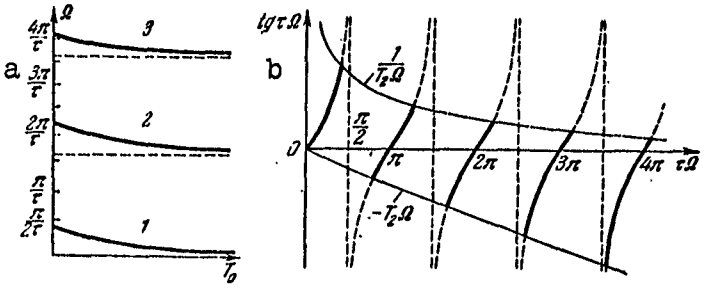


Fig. 6.34

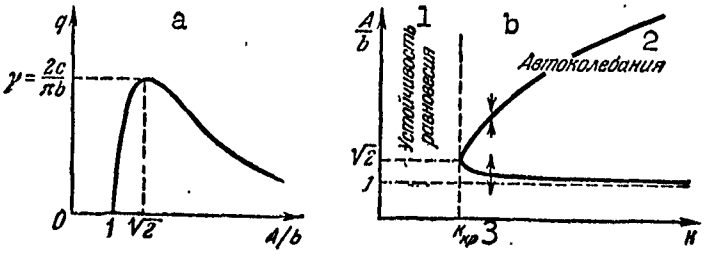


Fig. 6.35. 1) Equilibrium stability; 2) self-oscillations; 3) K_{kr}.

Since according to (6.174), q(A/b) has the form shown in Fig. 6.35a, then the dependence of the periodic-solution amplitude A upon the gain constant k takes the form of Fig. 6.35b, where we show the curve A/b(k) only for the lower frequency interval (curve 1 in Fig. 6.34a). For the remaining intervals it lies much farther to the right in Fig. 6.35b, and it does not have any practical meaning.

The value k_{kr} (Fig. 6.35b) corresponds to the point $q_{max} = \gamma = 2c/\pi b$ (Fig. 6.35a), i.e.,

$$k_{кр} = \frac{(T_0 + T_2)\Omega^2}{\gamma \cos \tau\Omega}. \qquad (6.179)$$

We may call the coefficient γ the shape coefficient of the nonlinearity.

We must isolate a stable value from the two branches obtained

for the amplitude of the periodic solution. For this purpose, substituting the quantities $p = j\omega$ into the characteristic equation (6.175), we obtain

$$X = kq\cos\tau\omega - (T_0 + T_2)\omega^2, \quad Y = \omega - kq\sin\tau\omega - T_0T_2\omega^3.$$

Let us find the partial derivatives of X and Y with respect to a and ω:

$$\left(\frac{\partial X}{\partial a}\right)^* = k\cos\tau\Omega\left(\frac{dq}{da}\right)^*, \quad \left(\frac{\partial X}{\partial\omega}\right)^* = -kq\tau\sin\tau\Omega - 2(T_0 + T_2)\Omega,$$
$$\left(\frac{\partial Y}{\partial a}\right)^* = -k\sin\tau\Omega\left(\frac{dq}{da}\right)^*, \quad \left(\frac{\partial Y}{\partial\omega}\right)^* = 1 - kq\tau\cos\tau\Omega - 3T_0T_2\Omega^2.$$

Here the left-hand side of the expression

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* - \left(\frac{\partial X}{\partial\omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0,$$

determining the stability of the periodic solution obtained assumes the form

$$k\left(\frac{dq}{da}\right)^*[(1 - 3T_0T_2\Omega^2)\cos\tau\Omega - kq\tau - 2(T_0 + T_2)\Omega\sin\tau\Omega].$$

But, according to (6.176)

$$\cos\tau\Omega = \frac{(T_0 + T_2)\Omega^2}{kq}, \quad \sin\tau\Omega = \frac{1 - T_0T_2\Omega^2}{kq}\Omega.$$

Allowing for this, the preceding expression gives

$$k\left(\frac{dq}{da}\right)^*\left[-kq\tau - \frac{(T_0 + T_2)\Omega^2}{kq}(1 + T_0T_2\Omega^2)\right].$$

Here the expression in the square brackets is negative, while from Fig. 6.35a it is evident that

$$\left(\frac{dq}{da}\right)^* > 0 \quad \text{for } 1 < \frac{A}{b} < \sqrt{2},$$
$$\left(\frac{dq}{da}\right)^* < 0 \quad \text{for } \frac{A}{b} > \sqrt{2}.$$

Hence it follows that the stability criterion for the periodic solution is satisfied only when $A/b > \sqrt{2}$. Consequently, only the upper branch in Fig. 6.35b corresponds to self-oscillations (stable periodic soltuion), which take place for the case $k > k_{kr}$. The presence of a nonstable periodic solution with a smaller amplitude testifies to the fact that "hard" excitation of the self-oscillations

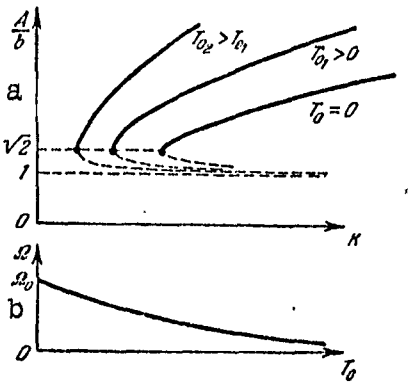


Fig. 6.36

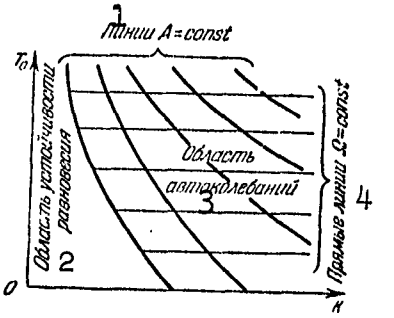


Fig. 6.37. 1) Lines A = const; 2) stable-equilibrium region; 3) self-oscillation region; 4) lines Ω = const.

takes place here. Let us note that for $k < k_{kr}$, the equilibrium state of the system is stable for any initial conditions.

Here we have obtained the magnitude of the self-oscillation amplitude A for the variable x. Let us now determine the self-oscillation amplitude of the output z of the system in question (Fig. 6.33a). It is obvious from the diagram that we may convert from the variable x to z via the transfer functions of the linear part 1 and the feedback. Therefore the self-oscillation amplitude A_z of the output z is

$$A_z = \frac{A}{k_1 k_0} \sqrt{T_0^2 \Omega^2 + 1}. \qquad (6.180)$$

Thus, all the results obtained for A are easily converted for A_z here and in what follows.

Thus, Fig. 6.35b gives us a picture of the self-oscillation amplitude as a function of gain constant for one value of T_0. If, however, T_0 increases, then according to Fig. 6.34, Ω will decrease, and hence the coefficient C will also decrease (see (6.178)), as a consequence of which the curve of A(k) will be shifted to the left (Fig. 6.36); here, the larger the value of T_0, the faster will the self-oscillation amplitude increase.

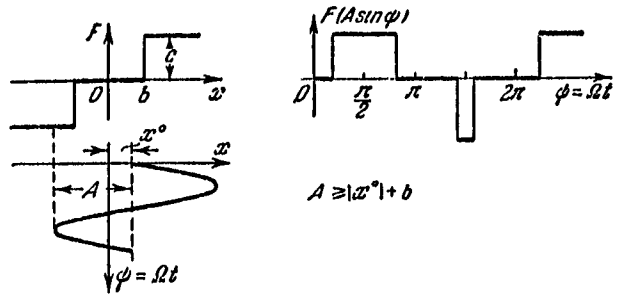


Fig. 6.38

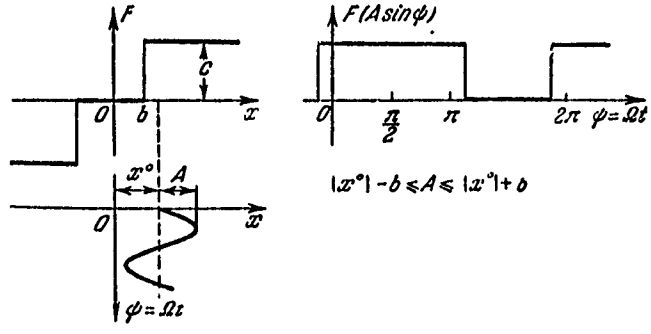


Fig. 6.39

Hence the curves of Fig. 6.36 give us a complete picture of how the self-oscillation amplitude A and the frequency Ω vary on variation of the two system parameters $\underline{k}$ and T_0.

The two diagrams of Fig. 6.36 may be combined into one (Fig. 6.37), as a result of which we delineate the stability region for the equilibrium state of the system in the plane of the two parameters, while in the self-oscillation region we draw lines of equal frequency values (Ω = const) and lines of equal amplitude values (A = const).

On the basis of the diagrams obtained (Fig. 6.36 or 6.37), we may choose the best values of the system parameters T_0 and $\underline{k}$, which provide desirable values for the self-oscillation amplitude A and the frequency Ω.

The position of the lines A = const depends upon the shape of the

nonlinearity, i.e., the magnitude of the amplitude and the width of the equilibrium stability region depend upon the shape of the nonlinearity. According to (6.179), the width of the equilibrium region is inversely proportional to the nonlinearity's shape coefficient $\gamma = 2c/\pi b$.

Let us now pass to determination of the smoothed characteristic of the nonlinear link (relay) for a slowly varying mean component. Since the component is assumed constant for the oscillation period, we perform harmonic linearization of the nonlinearity allowing for the mean component given in Formulas (5.99) and (5.100).

Let us consider two cases separately:

1) in the self-oscillation process, both relay contacts operate (Fig. 6.38), i.e.,

$$A \geqslant |x^0| + b;$$

2) only one relay contact operates in the self-oscillation process (Fig. 6.39), i.e.,

$$|x^0| - b \leqslant A \leqslant |x^0| + b.$$

If, however, $A < |x^0| - b$, then the relay will be permanently cut in in one direction.

In the first case (Fig. 6.38), Formulas (5.99) and (5.100) give $q' = 0$ and

$$\left.\begin{aligned} q &= \frac{2c}{\pi A}\left[\sqrt{1-\left(\frac{x^0+b}{A}\right)^2}+\sqrt{1-\left(\frac{x^0-b}{A}\right)^2}\right], \\ F^0 &= \frac{c}{\pi}\left(\arcsin\frac{x^0+b}{A}+\arcsin\frac{x^0-b}{A}\right). \end{aligned}\right\} \qquad (6.181)$$

In the second case (Fig. 6.39), we will also have $q' = 0$ and

$$\left.\begin{aligned} q &= \frac{2c}{\pi A}\sqrt{1-\left(\frac{|x^0|-b}{A}\right)^2}, \\ F^0 &= \frac{c}{\pi}\left(\frac{\pi}{2}+\arcsin\frac{|x^0|-b}{A}\right)\operatorname{sign} x^0. \end{aligned}\right\} \qquad (6.182)$$

Thus, from the second formulas of (6.181) and (6.182) we deter-

mine the smoothed characteristic $F^0(x^0)$ for the nonlinear link (relay). But the unknown self-oscillation amplitude A still remains in these formulas, and will also vary with the quantity x^0.

Consequently, the slope of the smoothed nonlinear characteristic at any one of its points

$$\frac{dF^0}{dx^0} = \frac{\partial F^0}{\partial x^0} + \frac{\partial F^0}{\partial A}\frac{dA}{dx^0},$$

where the derivative dA/dx^0, like the function $A(x^0)$ itself, is determined only in subsequent solution. However, for oddly symmetrical nonlinear characteristics we will have

$$\left(\frac{\partial F^0}{\partial A}\frac{dA}{dx^0}\right)_{x^0=0} = 0,$$

at the initial point ($x^0 = 0$), as was proven in Chapter 5.

From the second of the formulas of (6.181), we find

$$\frac{\partial F^0}{\partial x^0} = \frac{c}{\pi A}\left[\frac{1}{\sqrt{1-\left(\frac{x^0+b}{A}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x^0-b}{A}\right)^2}}\right] \text{ for } |x^0| < A - b,$$

from which we obtain, e.g., the slope of the smoothed characteristic at the origin ($x^0 = 0$):

$$k_n = \operatorname{tg}\delta = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0} = \frac{2c}{\pi\sqrt{A^2-b^2}}. \qquad (6.183)$$

Hence, for small mismatches, we may write the smoothed characteristic of the nonlinear link in question in the purely linear form

$$F^0 = k_n x^0,$$

where the coefficient k_n is determined by Formula (6.183) and is a function of the self-oscillation amplitude A at $x^0 = 0$, of the system parameters (according to Fig. 6.36).

In a subsequent analysis we will investigate the range of validity of the substitution $F^0 = k_n x^0$, i.e., the limit of x^0 up to which we may use Expression (6.183).

In those cases where Formula (6.183) is valid, according to (6.166) -(6.170), we may investigate the passage through the system of all the

slowly varying components in both steady-state and in transient processes, according to the linear equation

$$[(T_0p+1)(T_2p+1)p+kk_ne^{-\tau p}]x^0 = k_1(T_0p+1)(T_2p+1)e^{-\tau p}py, \qquad (6.184)$$

where k_n is determined from Eq. (6.183) in which we substitute A from the curve of Fig. 6.36.

Let us now consider the variation of the self-oscillation amplitude A with the variation of the mismatch magnitude x^0, and let us construct the smoothed characteristic $F^0(x^0)$ of the nonlinear link allowing for the variation of the amplitude A.

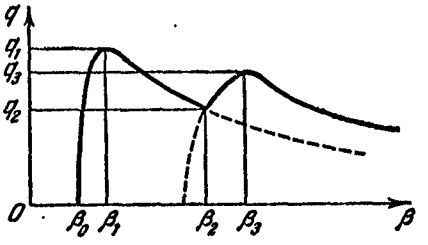


Fig. 6.40

As is evident from Eqs. (6.176), the self-oscillation frequency Ω is not a function of the magnitude of g; the frequency is determined first from Formulas (6.177), to which case the curve of Fig. 6.36b corresponds. Consequently, in the present example of the system, we may take the same self-oscillation frequency Ω from the curve of Fig. 6.36b as was taken in the absence of the input disturbance ($y = 0$, $x^0 = 0$) independently of the magnitude of x^0. Then, on the basis of the first equation of (6.176),

$$q = \frac{(T_0+T_2)\Omega^2}{k\cos\tau\Omega}. \qquad (6.185)$$

For convenience in writing the subsequent formulas, let us introduce the abbreviated notation

$$\beta = \frac{A}{b}, \quad \zeta = \frac{x^0}{b}, \quad \gamma = \frac{2c}{\pi b}, \qquad (6.186)$$

of which the first is the relative amplitude, the second is the relative average component and the third is the shape coefficient of the nonlinearity.

Then Formulas (6.181) and (6.182) for q are written in the form

$$\left.\begin{aligned} q &= \frac{\gamma}{\beta^2}\left[\sqrt{\beta^2-(\zeta+1)^2}+\sqrt{\beta^2-(\zeta-1)^2}\right] \text{ for } |\zeta| \leqslant \beta-1, \\ q &= \frac{\gamma}{\beta^2}\sqrt{\beta^2-(|\zeta|-1)^2} \qquad \text{for } \beta-1 \leqslant |\zeta| \leqslant \beta+1. \end{aligned}\right\} \tag{6.187}$$

The curve of the function $q(\beta)$ for a certain constant value of $|\zeta|$ has the form of Fig. 6.40. The first branch of this curve ($\beta_0 \leq \beta \leq \beta_2$) is determined by the second formula of (6.187), while the second branch ($\beta_2 \leq \beta < \infty$) is determined from the first formula of (6.187). For this case the values

$$\left.\begin{aligned} &\beta_0 = |(|\zeta|-1)|, \quad \beta_1 = \beta_0\sqrt{2}, && q_1 = \frac{\gamma}{2\beta_0^2}, \\ &\beta_2 = |\zeta|+1, && q_2 = \frac{2\gamma\sqrt{|\zeta|}}{(|\zeta|+1)^2}, \\ &\beta_3^2 = \frac{4}{3}(\zeta^2+1)+\frac{2}{3}\sqrt{(\zeta^2+1)^2+12\zeta^2}, && q_3 = q(\beta_3), \end{aligned}\right\} \tag{6.188}$$

correspond to the characteristic points shown in Fig. 6.40; here the last value is determined from the first formula of (6.187). In the particular case where $\zeta = 0$, we obtain $\beta_3 = \sqrt{2}$, $q_3 = \gamma$, i.e., the result known earlier (Fig. 6.35).

From Formulas (6.187), using the values indicated for the char-

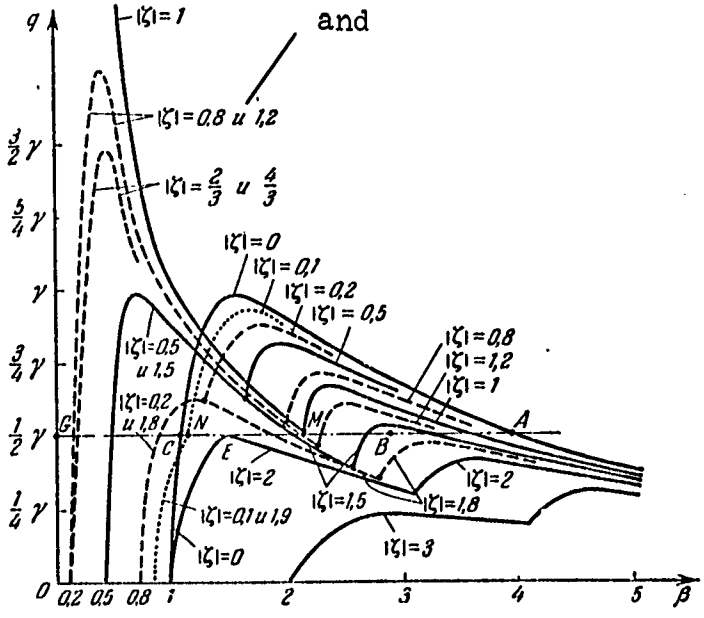


Fig. 6.41

acteristic points, we have constructed a family of curves of $q(\beta)$ for various constant values of ζ (Fig. 6.41).

As we have already said, for given system parameters we may calculate the value of q from Formula (6.185), after which, drawing the appropriate straight line $q = \text{const}$ on the graph of Fig. 6.41, we obtain the sought dependence of the self-oscillation amplitude β (or $A = \beta b$) on the magnitude of the mean component ζ (or $x^0 = \zeta b$).

For example, if the system parameters are such that Formula (6.185) gives us $q = 0.5\gamma$, then we must draw the straight line $q = 0.5\gamma$ on Fig. 6.41 (there it is indicated by the dot-dash line). For this case let us construct the curve $\beta(\zeta)$ (Fig. 6.42) passing along the straight line from the right to the left. The point A in Fig. 6.42 corresponds to A in Fig. 6.41. In Fig. 6.41, the amplitude β decreases from point A to point B, while the magnitude of the average component ζ increases; this is also shown in Fig. 6.42. Then from point B to point M in Fig. 6.41, the amplitude β continues to decrease and the quantity ζ also decreases (line segment BM in Fig. 6.42). Then with the decrease of β (Fig. 6.41) ζ increases anew over the line segment ME (line segment ME in Fig. 6.42), while it decreases over the line segment EN (Fig. 6.41) (line segment EN_1 in Fig. 6.42). Further along on the line segment NG (Fig. 6.41), the amplitude β gradually decreases to zero; here the quantity ζ has two values at each point (curves N_1G and NG in Fig. 6.42, which are symmetrical with each other). In addition, still another branch, the curve NC (Fig. 6.42), where ζ decreases to zero, corresponds to the line segment NC (Fig. 6.41).

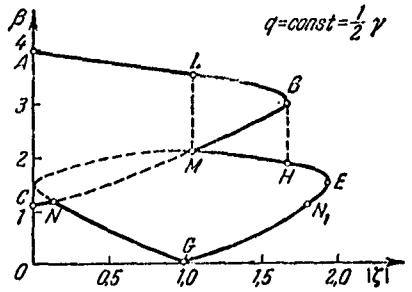


Fig. 6.42

On Fig. 6.42, we have drawn additional curves between the points

M and N; one of them is symmetrical to the curve MEN_1, while the other connects the line segments CN and MB. These secondary curves would be obtained if the curves in Fig. 6.40 had the continuations shown by the broken lines. But these continuations do not satisfy the limitations imposed on the quantity $|\zeta|$, which are indicated in Formulas (6.187). Therefore the broken curves between the points M and N in Fig. 6.42, which are necessary only to make the curve more obvious, do not correspond to real processes.

Thus we have two boundaries: 1) ABMNC, giving us the amplitude function of the periodic solution for the case where both relay contacts operate, and 2) MEGN, giving us the amplitude function of the periodic solution for the case where one relay contact operates.

It is evident that where $\zeta = 0$, the points A and C are equivalent to the two points of the curves in Fig. 6.35b, for the case where one value of $\underline{k}$ is given. Hence point A (Fig. 6.42) corresponds to a stable periodic solution (self-oscillation), while point C corresponds to an unstable periodic solution. Therefore the curve AB also gives us the relationship sought for measuring the self-oscillation amplitude β with increasing average component $|\zeta|$. The amplitude decreases with increasing ζ in the interval

$$\text{from } \beta_A \approx 3.9 \text{ for } \zeta = 0 \text{ to } \beta_B \approx 2.9 \text{ for } \zeta_B \approx 1.7, \tag{6.189}$$

where the abscissa ζ_B of the end point B is determined as that value of ζ for which the value q_3 (Fig. 6.41 and Formula (6.188)) is equal to the given value $q = 0.5\gamma$.

Allowing for this relationship for the variation of β, we may now completely construct the smoothed characteristic $F^0(x^0)$ or $F^0(\zeta)$ obtained by self-oscillations for the case where both contacts operate from the second of Formulas (6.181); this characteristic assumes the

form

$$F^0 = \frac{c}{\pi}\left(\arcsin\frac{\zeta+1}{\beta} + \arcsin\frac{\zeta-1}{\beta}\right) \text{ for } |\zeta| < \beta - 1. \qquad (6.190)$$

in the notation of (6.186).

For the case of the variation of ζ and β in the intervals of (6.189), we obtain:

$$0 \leqslant F^0 \leqslant 0{,}46c.$$

The corresponding smoothed characteristic is shown in the form of the curve OB in Fig. 6.48b.

On a further increase of the average component, $\zeta > \zeta_B = 1.7$, according to Fig. 6.42, self-oscillations become impossible for the case where both relay contacts operate, and either constant closure of one contact occurs (straight line RS in Fig. 6.48b), or self-oscillations are generated, corresponding to the operation of one relay contact and having an amplitude varying as a function of the quantity ζ according to a relationship determined by the curve ME (Fig. 6.42). These self-oscillations are possible in the interval of values

$$\text{from } \zeta_M = 1 \text{ for } \beta_M = 2 \text{ to } \zeta_E = 2 \text{ for } \beta_E = \sqrt{2}; \qquad (6.191)$$

here the point M (Fig. 6.42) corresponds to the point β_2 in Fig. 6.40, as a consequence of which ζ_M and β_M are determined by Formulas (6.188) from the condition that q_2 is equal to the given $q = 0.5\gamma$. However, the point E (Fig. 6.42) corresponds to the point β_1 in Fig. 6.40. Therefore, according to (6.188)

$$|\zeta|_E = \frac{\gamma}{2q_1} + 1, \quad \beta_E = \frac{\gamma\sqrt{2}}{2q_1}, \qquad (6.192)$$

where we must assume that q_1 is equal to the given $q = 0.5\gamma$.

According to the second formula of (6.182) the smoothed characteristic $F^0(\zeta)$ of the nonlinear link in question is, for self-oscillations corresponding to the operation of one relay contact,

$$F^0 = \left(\frac{c}{2} + \frac{c}{\pi}\arcsin\frac{|\zeta|-1}{\beta}\right)\operatorname{sign}\zeta \quad \text{for} \quad -\beta-1 \leqslant \zeta < \beta+1, \qquad (6.193)$$

in the notation of (6.186), which gives us

$$0{,}5c \leqslant F^0 \leqslant 0{,}75c,$$

for ζ and β in the interval of variation (6.191), while on the diagram of Fig. 6.48b the characteristic is shown in the form of the curve ME. Let us note that according to (6.192) and (6.193), the value $F_E^0 = 0.75c$ will always correspond to the point E for any value of $\underline{q}$.

For oscillations at one relay contact, it is obvious that the remaining part EGN of the curve corresponds to an unstable periodic solution. Let us remark that the position of the point N, just as that of M, corresponds to the point β_2 on Fig. 6.40, but for another value of ζ than point M. From Formulas (6.188), where $q_2 = 0.5\gamma$, we find

$$\zeta_N \approx 0{,}08, \quad \beta_N \approx 1{,}08.$$

We obtain this type (Fig. 6.42) of pattern for the curves of $\beta(\zeta)$ and smoothed characteristics $F^0(\zeta)$ of the same type (Fig. 6.48b) for the nonlinear link in question for combinations of the system parameters such that the value of $\underline{q}$ calculated according to Formula (6.185) is in the interval

$$0{,}45\gamma < q < 0{,}67\gamma,$$

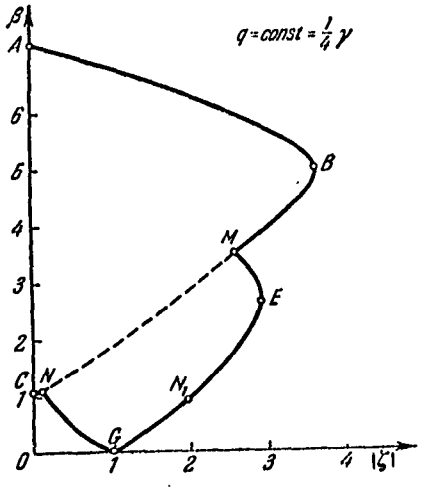


Fig. 6.43

as is indicated in Fig. 6.48b. In this case, as $\underline{q}$ increases, the points B, M, and E on both curves are shifted to the left and downward, while point N is shifted to the right and upward.

When $\underline{q}$ decreases, however, the points B, M and E are shifted to the right and upward; here the point B moves faster than E, and for $q \approx 0.45\gamma$, we obtain $\zeta_B =$

$= \zeta_E$, while for the case $q < 0.45\gamma$ we will have $\zeta_B > \zeta_E$. As a result, for the case $0 < q < 0.45\gamma$, the curve of the smoothed characteristic of the nonlinear link in question acquires the form shown in Fig. 6.48a. Here the range of possible variation of the input (the average component ζ) is expanded, but on the other hand the self-oscillation amplitude increases. For a comparison, the diagram of Fig. 6.48 shows the characteristic of the nonlinear link in question without smoothing, in the form of the broken line OQPRS.

For example, when the system parameters are such that the value q calculated from Formula (6.185) is $q = 0.25\gamma$, we obtain the pattern of curves of $\zeta(\beta)$ shown in Fig. 6.43 (this pattern is synthesized in much the same way as the previous family by drawing the line $q = 0.25\gamma$ in Fig. 6.41). Here we obtain the following results: $\beta_A = 7.9$ (for $\zeta = 0$), $\beta_B \approx 5.2$, $\zeta_B \approx 3.7$, $\beta_M \approx 3.6$, $\zeta_M \approx 2.6$, $\beta_E = 2.83$, $\zeta_E = 3$, $0 \leq F^0 \leq$ $\leq 0.54c$ (curve OB in Fig. 6.48a), $0.65c \leq F^0 \leq 0.75c$ (curve ME in Fig. 6.48a).

If now the system parameters are changed so that the value of q calculated from Formulas (6.185) is larger than 0.67, then the abscissa of the point B becomes less than unity and the two perimeters of the curves shown in the previous Fig. 6.42 will not intersect. In order to illustrate this case Fig. 6.44 shows the pattern of the function $\beta(\zeta)$, for $q = 0.75\gamma$ as obtained on the basis of Fig. 6.41 by the same method as was used on the previous curves.

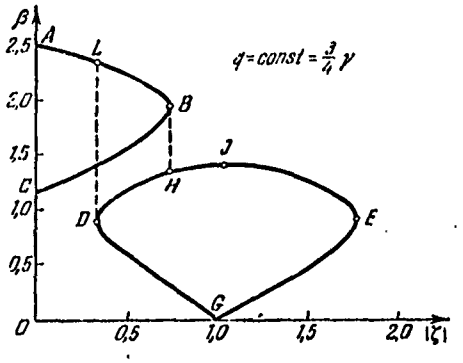


Fig. 6.44

Let us recall that the curve ABC (Fig. 6.44) corresponds to the first of Formulas (6.187), in which case both relay contacts operate in the oscillation process. The curve DJEG, on the other hand, corres-

ponds to the second of Formulas (6.187), in which case only one contact operates in the oscillation process.

The curve AB indicates the law of variation of the self-oscillation amplitude with increasing average component ζ; here

$$\beta_A \approx 2{,}5 \text{ for } \zeta_A = 0, \quad \beta_B \approx 1{,}9 \text{ for } \zeta_B \approx 0{,}7.$$

On a further increase of the average component ($\zeta > 0.7$), we have breakoff of the self-oscillations corresponding to the operation of two relay contacts. But since $\zeta_B < 1$, this disruption takes place within the dead zone of the relay, as is shown on the curve of the smoothed characteristic OB (Fig. 6.48c) synthesized from Formula (6.190); here at the end point B we have $F^0 \approx 0.3c$.

Thus, for $\zeta > \zeta_B = 0.7$, the system is either opened or passes into self-oscillations, for which case only one relay contact operates. Here the self-oscillation amplitude decreases stepwise (in Fig. 6.44 from point B to point H) to the value

$$\beta_H \approx 1{,}3 \text{ for } \zeta = 0{,}7.$$

Now as ζ increases the self-oscillation amplitude (for the case where one contact operates) varies according to the curve HJE. The maximum point J corresponds to the value $\zeta = 1$, when the mean component x^0 is exactly equal to half the width of the dead zone of the relay b; here the quantity β may be determined by substituting $\zeta = 1$ into the second formula of (6.187), i.e.:

$$\beta_J = \frac{1}{q} = 1{,}33.$$

The coordinates of the end point E (Fig. 6.44) are determined from Formula (6.192); hence in the case under consideration (for the case $q_1 = 0.75\gamma$) we obtain

$$\zeta_E = 1{,}67; \quad \beta_E = 0{,}94.$$

The value $\zeta = 0.33$ obtained from the first of Formulas (6.188)

for $q_1 = 0.75\gamma$ determines the position of the point D (Fig. 6.44), i.e.,

$$\zeta_D = 0{,}33; \quad \beta_D = 0{,}94.$$

On a further increase in the mean component ($\zeta > 1.67$), the self-oscillations in the system are broken off altogether, and the relay is cut in in one direction only (lines PS in Fig. 6.48c).

Thus, for the case where the average component varies in the range $\zeta_D < \zeta < \zeta_E$, the self-oscillation amplitude (for one contact operating) will vary according to the curve DJE (Fig. 6.44), while according to Formula (6.193), the smoothed characteristic acquires the form of the curve DJE in Fig. 6.48c; here we obtain at the points D, J, and E

$$F_D^0 = 0{,}25c; \quad F_J^0 = 0{,}5c; \quad F_E^0 = 0{,}75c. \tag{6.194}$$

respectively.

A stepwise change in the self-oscillation amplitude (Fig. 6.44) and in the points of the smoothed characteristic (Fig. 6.48c) is obviously possible at any points of the curvilinear segment LB and DH for $0.33 < \zeta < 0.7$, when stable oscillations are possible at either one or two relay contacts.

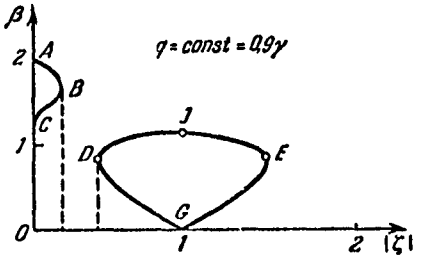


Fig. 6.45

If we vary the system parameters further in such a fashion that the value of q calculated from Formula (6.185) will increase, then the two curvilinear contours ABC and DJEG shown in Fig. 6.44 will contract. For example, for $q = 0.9\gamma$, they assume the form shown in Fig. 6.45. Here for the case where both relay contacts operate, self-oscillations are possible only in a relatively small range of variation of the constant component: from zero to $\zeta_B \approx 0.2$, where $\beta_B \approx 1.7$, $F_B^0 \approx 0.11c$. These oscillations are broken off at the point B (Fig. 6.45) before oscillations are able to develop on

one relay contact. The latter are possible in the interval $\zeta_D < \zeta < \zeta_E$; here, from the first of Formulas (6.188) we find

$$\zeta_D = 0{,}445, \quad \zeta_E = 1{,}555, \quad \beta_D = \beta_E = 0{,}778.$$

for the case $q_1 = 0.9\gamma$.

As a result, the smoothed characteristic (6.190) and (6.193) will have two separate segments OB and DE (Fig. 6.48c), the first of which is obtained for self-oscillations on both relay contacts, while the second is obtained for the case of self-oscillations at one relay contact. Between the points B and D the relay is open, since this interval lies within the dead zone of the relay.

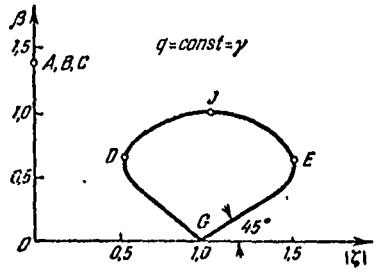


Fig. 6.46

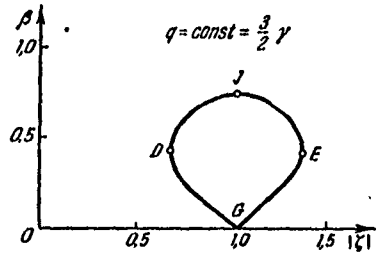


Fig. 6.47

Finally, where $q = \gamma$, the first contour contracts into a single point A (Fig. 6.46) and then for $q > \gamma$ (for example, where $q = 1.5\gamma$ in Fig. 6.47), only one contour remains; this corresponds to self-oscillations at one relay contact. Hence, for system parameters corresponding to the value $q \geq \gamma$ and small values of the constant component ζ we may not obtain a smoothed characteristic by means of self-oscillation. This smoothed characteristic is possible only far from the zero point in the interval $\zeta_D < \zeta < \zeta_E$ (Fig. 6.48e), here the first three of Formulas (6.188) give us

$$\text{for } q = \gamma \quad \zeta_D = 0{,}5, \quad \zeta_E = 1{,}5, \quad \beta_D = \beta_E = 0{,}707;$$

$$\text{for } q = 1{,}5\gamma \quad \zeta_D = \frac{2}{3}, \quad \zeta_E = \frac{4}{3}, \quad \beta_D = \beta_E = 0{,}471.$$

At the points D, J, and E the quantity F^0 has the values given in

one relay contact. The latter are possible in the interval $\zeta_D < \zeta < \zeta_E$; here, from the first of Formulas (6.188) we find

$$\zeta_D = 0{,}445, \quad \zeta_E = 1{,}555, \quad \beta_D = \beta_E = 0{,}778.$$

for the case $q_1 = 0.9\gamma$.

As a result, the smoothed characteristic (6.190) and (6.193) will have two separate segments OB and DE (Fig. 6.48c), the first of which is obtained for self-oscillations on both relay contacts, while the second is obtained for the case of self-oscillations at one relay contact. Between the points B and D the relay is open, since this interval lies within the dead zone of the relay.

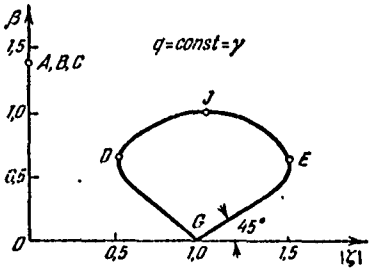


Fig. 6.46

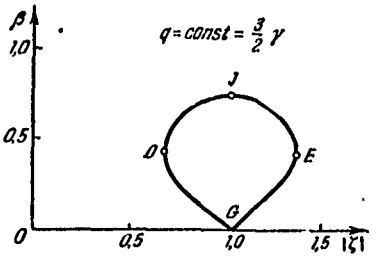


Fig. 6.47

Finally, where $q = \gamma$, the first contour contracts into a single point A (Fig. 6.46) and then for $q > \gamma$ (for example, where $q = 1.5\gamma$ in Fig. 6.47), only one contour remains; this corresponds to self-oscillations at one relay contact. Hence, for system parameters corresponding to the value $q \geq \gamma$ and small values of the constant component ζ we may not obtain a smoothed characteristic by means of self-oscillation. This smoothed characteristic is possible only far from the zero point in the interval $\zeta_D < \zeta < \zeta_E$ (Fig. 6.48e), here the first three of Formulas (6.188) give us

$$\text{for } q = \gamma \quad \zeta_D = 0{,}5, \quad \zeta_E = 1{,}5, \quad \beta_D = \beta_E = 0{,}707;$$

$$\text{for } q = 1{,}5\gamma \quad \zeta_D = \frac{2}{3}, \quad \zeta_E = \frac{4}{3}, \quad \beta_D = \beta_E = 0{,}471.$$

At the points D, J, and E the quantity F^0 has the values given in

(6.194) for any value of g.

Thus, for system-parameter relationships corresponding to values $q \geq \gamma$ and a slowly varying mean component ζ of the signal at the input of the nonlinear link, we obtain, in contrast to the previous case, a dead zone

$$0 \leqslant |\zeta| \leqslant \zeta_D,$$

which is wider the larger g.

Thus, the smoothed characteristic of the nonlinear link extends over a range of the slowly varying input ζ which is larger the larger the self-oscillation amplitude established. But in this case the output self-oscillation amplitude will be larger than is desirable. In addition, this reduces the gain constant k_n of the link, which is determined by the steepness (slope) of the smoothed characteristic.

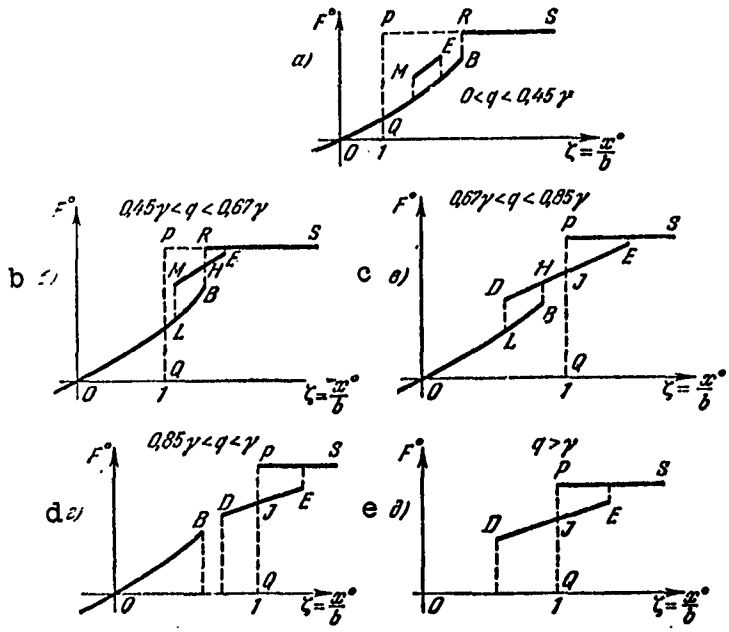


Fig. 6.48

The investigation performed permits us to choose the structure and parameters of the system so as to obtain the shape desired for the smoothed characteristic with an admissible self-oscillation amplitude

and frequency, and then to analyze all the steady-state and transient processes for slowly varying components using the linear equation (6.184), where k_n is taken from Fig. 6.48 as the slope of the curves on their replacement on the operating segments by the appropriate straight lines.

For example, let us find the self-oscillation amplitude A and the mean component x^0 as functions of a slowly varying input disturbance $\underline{y}$ acting upon the system in the steady-state mode of its variation at a constant rate.

According to Eq. (6.184), in a steady-state mode with a constant input ($y = \text{const}$), we obtain $x^0 = 0$ and hence $F^0 = 0$, i.e., the constant input is presented at the system output without a steady-state error (as is usually the case in follow-up systems).

If, however, the input signal varies at the constant rate $\dot{y} = \text{const}$, we have according to (6.184) for a steady-state mode

$$x^0 = \frac{k_1}{kk_n}\dot{y} = \text{const.} \qquad (6.195)$$

Since the functions $F^0(x^0)$, being smooth characteristics, already exist for various relationships of the system parameters, then we may use them on the basis of (6.195) to construct curves of the mean component of the mismatch x^0 (i.e., of the steady-state error), as a function of the constant input-signal rate $\dot{y}$. Essentially these will be the same as the curves of Fig. 6.48, but with the roles of the abscissa and ordinate axes different and with a changed scale.

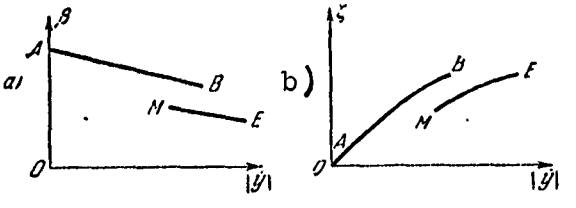


Fig. 6.49

However, in order to obtain the self-oscillation amplitude β (or $A = \beta b$) as a function of the constant input-disturbance rate $\dot{y}$, we must make use of the curves of $\beta(\zeta)$ (Fig. 6.42-6.47), and then find for each ζ the value of F^0 from Fig. 6.48, and also find $\dot{y} = (k/k_1)F^0$ according to (6.195). Thus, the curves in Figs. 6.42-6.47 will be reconstructed in the sought form $\beta(\dot{y})$. For this case we need only make use of the stable-oscillation branch from Figs. 6.42-6.47, for which the curves of Fig. 6.48 were constructed.

For example, for the case where the system parameters are such that Formula (6.185) gives $q = 0.5\gamma$, Figs. 6.42 and 6.48b give the curves $\zeta(\dot{y})$ and $\beta(\dot{y})$ shown in Fig. 6.49. The curves AB are obtained for the case of self-oscillations encompassing the operation of both relay contacts, while the curve ME is obtained for the case of self-oscillations at one contact.

§6.10. Smoothing of Loop Nonlinearities by Self-Oscillations

Let us investigate a system [187] whose diagram is shown in Fig. 6.50a, with the linear part 1 described by the equation

$$(T_1 p + 1) px = k_1 x_1, \tag{6.196}$$

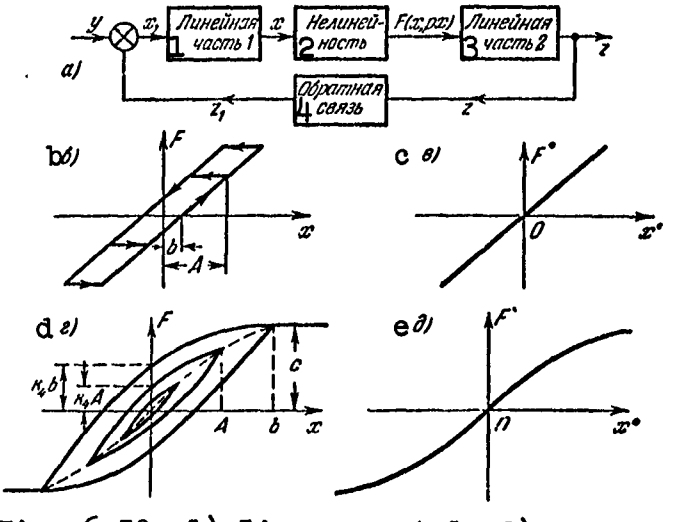


Fig. 6.50. 1) Linear part 1; 2) nonlinearity; 3) linear part 2; 4) feedback.

the linear part 2 by

$$(T_2 p + 1) z = k_2 F \tag{6.197}$$

and the feedback by

$$z_1 = k_0 z, \quad x_1 = y - z_1. \tag{6.198}$$

Let us consider the nonlinearity (Fig. 6.50a) in two forms: 1) the backlash-type loop characteristic (Fig. 6.50b) with constant loop width 2b; 2) the curvilinear hysteresis characteristic with variable loop width and saturation (Fig. 6.50d). Generally speaking, hysteresis phenomena are very complex. Depending upon the properties of the material and the initial conditions, various motions are possible within the limiting hysteresis loop. Let us assume here that for the case of steady-state symmetrical oscillations, the pattern shown in Fig. 6.50d prevails. In both cases we will assume that the steepness of the middle line (the dotted line) is equal to unity at the origin, referring the over-all gain constant to the linear part of the system.

Let us first find the solution for the symmetrical self-oscillations for y = 0 in first approximation: x = A sin Ωt. Harmonic linearization of the nonlinearity gives us

$$F = qx + \frac{q'}{\Omega} px. \tag{6.199}$$

Here in the case of a system with backlash (Fig. 6.50b), according to (3.28), we obtain

$$\left.\begin{aligned} q &= \frac{1}{2} + \frac{1}{\pi} \arcsin\left(1 - \frac{2b}{A}\right) + \frac{2}{\pi}\left(1 - \frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1 - \frac{b}{A}\right)}, \\ q' &= -\frac{4b}{\pi A}\left(1 - \frac{b}{A}\right); \end{aligned}\right\} \tag{6.200}$$

for k = 1 and A $\geq$ b; these functions q(A) and q'(A) are shown in Fig. 6.51a.

In the case of a system with hysteresis and saturation (Fig. 6.50d), we describe the nonlinearity analytically in the following form:

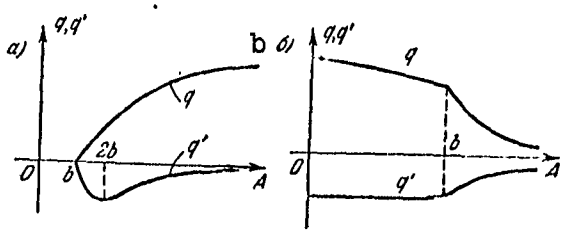


Fig. 6.51

for $|x| \leq b$

$$F=(1-k_3x^2)x-k_4A\left(1-\frac{x^2}{A^2}\right)^2\operatorname{sign}px, \tag{6.201}$$

while for $|x| \geq b$

$$F=c\operatorname{sign}x. \tag{6.202}$$

In Formula (6.201), the first term describes the center line (the broken line in Fig. 6.50d), while the second term describes the deviation of the actual hysteresis characteristic in either direction from the center line.

For the case of symmetrical oscillations, $x = A \sin \Omega t$, harmonic linearization of this nonlinearity leads to Expression (6.199), where according to Formulas (3.32) and (3.42) and Table 3.1 we have for $n = 4$

$$q=1-\frac{3}{4}k_3A^2, \quad q'=-\frac{32}{15\pi}k_4\approx-0{,}68k_4. \tag{6.203}$$

for $A \leq b$. For $A > b$, we obtain

$$\left.\begin{aligned} q&=\left(1-\frac{3}{4}k_3A^2\right)\frac{2}{\pi}\left(\arcsin\frac{b}{A}-\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right)+\\ &\quad+\frac{3k_3b^2}{\pi A}\sqrt{1-\frac{b^2}{A^2}},\\ q'&=-\frac{32k_4b^2}{15\pi A^2}=-0{,}68k_4\frac{b^2}{A^2}; \end{aligned}\right\} \tag{6.204}$$

these functions $q(A)$ and $q'(A)$ are shown in Fig. 6.51b.

In both cases the characteristic equation of System (6.196)-(6.198) is

$$T_1T_2p^3+(T_1+T_2)p^2+p+k\left(q+\frac{q'}{\Omega}p\right)=0, \text{ where } k=k_0k_1k_2. \tag{6.205}$$

After the substitution $p = j\Omega$, we find the real and imaginary parts:

$$\left.\begin{aligned} X &= kq - (T_1 + T_2)\Omega^2 = 0, \\ Y &= kq' + \Omega - T_1 T_2 \Omega^3 = 0. \end{aligned}\right\} \qquad (6.206)$$

Let us assume that the time constants T_1 and T_2 are given. For the present we must choose the gain constant $\underline{k}$ from the condition of admissible values of the self-oscillation amplitude A and frequency Ω (subsequently, the shape of the smoothed characteristic will be allowed for in choosing $\underline{k}$). For this purpose we construct the amplitude and the frequency as functions of the quantity $\underline{k}$.

Eliminating the quantity $\underline{k}$ from Eqs. (6.206), we obtain the frequency Ω as a function of the amplitude A in the form

$$\Omega = \frac{T_1 + T_2}{2T_1T_2}\frac{q'}{q} + \sqrt{\left(\frac{T_1 + T_2}{2T_1T_2}\frac{q'}{q}\right)^2 + \frac{1}{T_1T_2}}. \qquad (6.207)$$

Here in the general solution we will have $\pm$ in front of the root; however, $q' < 0$, and therefore for a minus sign in front of the root, we obtain $\Omega < 0$, which is not real; consequently, in the problem being considered, only the plus sign in front of the square root has physical meaning. From Formula (6.207), assigning a value to the quantity A and taking the corresponding $\underline{q}$ and q' from Fig. 6.51 or from Formulas (6.200), (6.203), and (6.204), we construct the curve $\Omega(A)$ (see Fig. 6.52a for a system with backlash and Fig. 6.53a for a system with hysteresis).

After this we may calculate

$$k = \frac{(T_1 + T_2)\Omega^2}{q(A)} \qquad (6.208)$$

according to (6.206), for any point of the resulting curve $\Omega(A)$, which we have and, consequently, we may plot the amplitude as a function of the quantity $\underline{k}$ (see Fig. 6.52b for a system with backlash and Fig. 6.53b for a system with hysteresis).

In order to ascertain which of the two branches of the curve A(k) corresponds to self-oscillations, let us investigate the stability of

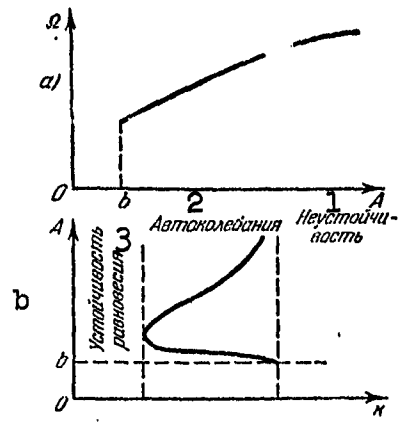


Fig. 6.52. 1) Instability; 2) self-oscillations; 3) equilibrium stability.

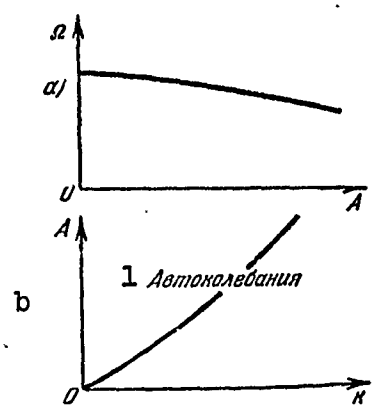


Fig. 6.53. 1) Self-oscillations.

the periodic solution. On the basis of the characteristic equation (6.405), we write

$$X = kq - (T_1 + T_2)\omega^2,$$
$$Y = \left(k\frac{q'}{\Omega} + 1\right)\omega - T_1T_2\omega^3,$$

hence

$$\left(\frac{\partial X}{\partial a}\right)^* = k\left(\frac{dq}{da}\right)^*, \quad \left(\frac{\partial X}{\partial \omega}\right)^* = -2(T_1 + T_2)\Omega,$$
$$\left(\frac{\partial Y}{\partial a}\right)^* = k\left(\frac{dq'}{da}\right)^*, \quad \left(\frac{\partial Y}{\partial \omega}\right)^* = k\frac{q'}{\Omega} + 1 - 3T_1T_2\Omega^2.$$

Noting that according to the second of Eqs. (6.206):

$$k\frac{q'}{\Omega} + 1 = T_1T_2\Omega^2,$$

we may write the stability criterion of the periodic solution

$$\left(\frac{\partial X}{\partial a}\right)^*\left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^*\left(\frac{\partial Y}{\partial a}\right)^* > 0$$

in the form

$$-T_1T_2\Omega\left(\frac{dq}{da}\right)^* + (T_1 + T_2)\left(\frac{dq'}{da}\right)^* > 0. \quad (6.209)$$

Let us rewrite this with the substitution (6.207) in the form

$$\left(\frac{dq'}{da}\right)^* > \left[\frac{q'}{2q} + \sqrt{\left(\frac{q'}{2q}\right)^2 + \frac{T_1T_2}{(T_1 + T_2)^2}}\right]\left(\frac{dq}{da}\right)^*. \quad (6.210)$$

Since in a system with backlash $(dq/da)^* > 0$ (Fig. 6.51a), Inequality (6.210) may be satisfied only for the case $(dq'/da)^* > 0$. But for $A < 2b$, according to Fig. 6.51a, the quantity $(dq'/da)^*$ is negative. Consequently, the lower branch of the curve of A(k) in Fig. 6.52b clearly corresponds to an unstable periodic solution. However, Inequality (6.210) is satisfied for the upper branch, since for large values of A, the quantities $(dq'/da)^*$ and $(dq/da)^*$ are approximately equal, while the quantity $q' \to 0$. For this case (6.210) is replaced by the

approximate inequality

$$\frac{\sqrt{T_1 T_2}}{T_1 + T_2} < 1,$$

which is always satisfied. Hence the self-oscillation amplitude is determined by the upper branch of the curve in Fig. 6.52b.

For systems with hysteresis, however, according to Fig. 6.51b, $(dq/da)^* < 0$ and $(dq'/da)^* \geq 0$. Therefore the stability condition for a periodic solution is always satisfied and consequently the curve of Fig. 6.53 gives us a solution for the self-oscillation amplitude and frequency as functions of the gain constant k.

By means of transfer functions, we may easily convert the self-oscillation amplitude A obtained for the variable x (Fig. 6.50a) into the self-oscillation amplitude of another quantity, for example the output z.

Let us now find the self-oscillation amplitude and frequency variation in the presence of an input disturbance y(t) that varies slowly over time.

The variable x will be composed of an average slowly varying component and a periodic component, i.e.,

$$x = x^0 + x^*, \quad x^* = A \sin \Omega t.$$

Then the harmonic linearization of the nonlinearity assumes the form

$$F = F^0 + \left(q + \frac{q'}{\Omega} p\right) x^*,$$

instead of (6.199); here we obtain from Formulas (5.128) and (5.129) the previous values of q and q' (6.200) and

$$F^0 = x^*. \tag{6.211}$$

for the backlash-type loop characteristic (Fig. 6.54a). We obtain new values for the hysteresis characteristic, i.e. (Fig. 6.54b),

$$\left.\begin{aligned} q &= 1 - 3k_3\left[(x^0)^2 + \frac{A^2}{4}\right], \\ q' &= -\frac{32}{15\pi}k_4 = -0{,}68k_4, \\ F^0 &= \left\{1 - k_3\left[(x^0)^2 + \frac{3}{2}A^2\right]\right\}x^0 \\ &\text{for } |x^0| + A \leqslant b. \end{aligned}\right\} \tag{6.212}$$

Similarly, we may obtain formulas for the case $A - |x^0| > b$ (Fig. 6.54c) and for the case $|x^0| + A > b$, but $|A - (x^0)| < b$ (Fig. 6.54d). Thus, we obtain in the latter case

$$\left.\begin{aligned} q &= \left\{1 - 3k_3\left[(x^0)^2 + \frac{A^2}{4}\right]\right\}\frac{1}{\pi}\left[\frac{\pi}{2} + \arcsin\frac{b-x^0}{A} - \right. \\ &\left. - \frac{b-x^0}{A}\sqrt{1-\left(\frac{b-x^0}{A}\right)^2}\right] + \frac{2x^0}{\pi A}\sqrt{1-\left(\frac{b-x^0}{A}\right)^2} \times \\ &\times\left\{\frac{c}{x^0} - 1 + k_3\left[(x^0)^2 + 2A^2 + (b-x^0)^2 - \frac{(b-x^0)^3}{4x^0}\right]\right\}, \\ q' &= -\frac{32k_4}{15\pi}\left(\frac{b-x^0+A}{2A}\right)^2 = -0{,}68k_4\left(\frac{b-x^0+A}{2A}\right)^2, \\ F^0 &= \left\{1 - k_3\left[(x^0)^2 + \frac{3}{2}A^2\right]\right\}x^0\left(\frac{1}{2} + \frac{1}{\pi}\arcsin\frac{b-x^0}{A}\right) + \\ &+ c\left(\frac{1}{2} - \frac{1}{\pi}\arcsin\frac{b-x^0}{A}\right) - \\ &- \frac{A}{\pi}\sqrt{1-\left(\frac{b-x^0}{A}\right)^2}\left\{1 - k_3\left[3(x^0)^2 + \frac{2}{3}A^2\right] - \right. \\ &\left. - \frac{3}{2}k_3x^0(b-x^0) - \frac{1}{3}k_3(b-x^0)^2\right\}. \end{aligned}\right\} \tag{6.213}$$

It is obvious from Formula (6.211) that in the presence of a backlash-type loop nonlinearity, the self-oscillatory mode of system operation makes the system purely linear for slowly varying signals (Fig. 6.50c) in accordance with the linearity of the center line (Fig. 6.54a). Here q and q' remain as before (they are not functions of x^0). In a system with backlash, therefore, the self-oscillation amplitude and frequency will not be functions of a slowly varying average component (nor, therefore, of the external disturbance), and in the present case no additional calculations are required.

As regards systems with hysteresis characteristics (Fig. 6.54b, c, and d), however, as a consequence of the curvilinearity of the center line, the smoothed characteristic of the nonlinear link will also be curvilinear (Fig. 6.50e); here, as is evident from the third formula

of (6.212) and from (6.201), in contrast to the preceding case, the smoothed characteristic $F^0(x^0)$ does not coincide with the center line of the loop, but differs from it by an amount equal to $3k_3A^2x^0/2$ for $|x^0| + A \leq b$ and by a more complex quantity (see (6.213)) for $|x^0| + A > b$. The difference between them is a function of the self-oscillation amplitude and the latter is a function of the magnitude of the mean component x^0, since x^0 occurs in Expressions (6.212) and (6.213). Before finding the smoothed characteristic in this case, therefore (in contrast to a backlash system), we are obliged first to determine the self-oscillation amplitude A as a function of the mean component x^0, as in §6.9.

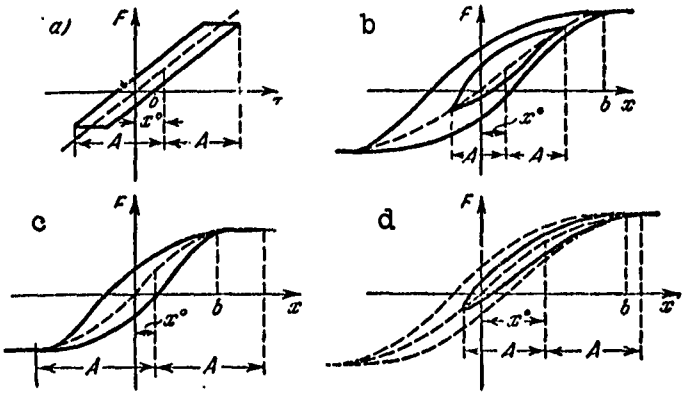


Fig. 6.54

For the case being considered, we arrive at the same equations (6.206), but with a new value g determined by (6.212) or (6.213), i.e.,

$$\left.\begin{aligned} X &= kq(A, x^0) - (T_1 + T_2)\Omega^2 = 0, \\ Y &= kq' + \Omega - T_1T_2\Omega^2 = 0. \end{aligned}\right\} \qquad (6.214)$$

The first equation gives us the formula

$$q(A, x^0) = \frac{(T_1 + T_2)\Omega^2}{k}. \qquad (6.215)$$

Allowing for (6.212), we find the equation of the frequency for $|x^0| + A \leq b$:

$$\Omega^3 - \frac{1}{T_1T_2}\Omega + \frac{32k_1k}{15\pi T_1T_2} = 0. \qquad (6.216)$$

from the second equality of (6.214). The resulting self-oscillation frequency is not a function of either the amplitude A or the mean component x^0. It is determined by the system parameters T_1, T_2, $\underline{k}$ and the coefficient k_4 (Fig. 6.50d), which characterize the dependence of the width of the hysteresis loop upon the amplitude. However, for the case $|x^0| + A > b$, according to (6.214) and (6.213), the equation for the frequency assumes the form

$$\Omega^3 - \frac{1}{T_1 T_2}\Omega + \frac{32 k_4 k}{15\pi T_1 T_2}\left(\frac{b - x^0 + A}{2A}\right)^4 = 0, \qquad (6.217)$$

i.e., the self-oscillation frequency is a function of the mean component x^0 and the self-oscillation amplitude A, which is, in turn, also a function of x^0. Using the same approach to solution of the problem as in §6.9, our result is that with an increase in the mean component x^0, the self-oscillation amplitude decreases (Fig. 6.55a) and vanishes for some value of x^0 (the self-oscillations are broken off). Allowing for the variation of the self-oscillation amplitude, the smoothed characteristic for the nonlinear hysteresis link in question, as synthesized from the third formulas of (6.212) and (6.213), assumes the form shown in Fig. 6.55b.

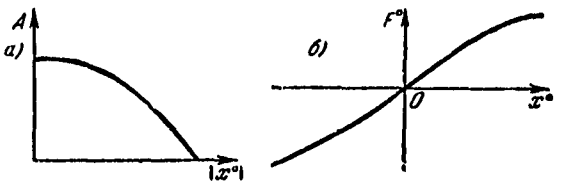


Fig. 6.55

According to the third formula of (6.212), the slope of this smoothed characteristic at the origin is

$$k_н = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0} = 1 - 3k_3\left[(x^0)^2 + \frac{A^2}{2}\right]_{x^0=0} = 1 - \frac{3}{2}k_3 A_0^2,$$

where A_0 is the value of the self-oscillation amplitude A, as determined for various gain constants $\underline{k}$ from the curve of Fig. 6.53b. It is

important to note that $k_n < 1$, i.e., the slowly varying component passes through with a gain constant less than the slope of the middle line in Fig. 6.50d.

Now we have obtained all data necessary for even this type of nonlinear system to calculate the steady-state and transient processes in the system as a whole for slowly varying components, using linear equations as indicated in the preceding sections.

Thus, the specific character and complexity of obtaining a characteristic smoothed by self-oscillations for any nonlinear link consists in the need to allow for a possible variation of the self-oscillation amplitude with variation of a slowly varying mean component (displacement of the center of oscillations) and provision for an admissible self-oscillation amplitude and frequency by choosing the system structure and parameters, together with provision for other qualities required for this system as a whole. We obtain a significant simplification in the solution of the problem in those cases where the linearization $F^0 = k_n x^0$ described in §5.3 is possible.

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[Footnotes]

524 In general, however, the quantity x^0 also incorporates the useful control signal in servo- and control systems.

527 If, however, we introduce the integral into the linear part, i.e., $PQ_2(p)\xi = R_2(p)x_2$, in the usual manner, we shall attain astaticism with respect to the external disturbance, but we retain the static error originating from the asymmetry of the nonlinearity.

543 The full calculation of the system dynamics for a slowly varying, arbitrarily assigned time function w(t) may be conducted by the method set forth in §5.3.

Manuscript Page No.	[List of Transliterated Symbols]
516	В = V = Val = shaft
516	Р = R = Rychag = lever, arm
516	у = u = upor = arrester
522	л = l = lineynyy = linear
523	уст = ust = ustanovivshiyaya = steady-state
525	р = r = reaktsiya = reaction (letter symbol M)
525	л.к = l.k = lozhnyy kontakt = false contact
535	доп = dop = dopustimyy = admissible
535	тар = tar = tarirovannyy = calibrated
537	к = k = katushka = coil
538	э = e = elektrodvizhushchiy = electromotive
538	и = i = inertsiya = inertia
552	м = m = magnit = magnet
552	дв = dv = dvigatel' = motor
554	п = p = pruzhina = spring
565	К = K = kontakt = contact
565	П = P = pruzhina = spring
565	НВ = NV = nastroyechnyy vint = setscrew
566	д = d = dobavochnyy = auxiliary
566	с = s = soprotivleniye = resistance
566	нг = ng = nagruzka = load
567	Ш = Sh = shayba = [illegible]sk
567	Я = Ya = Yakor' = armature
567	ОВ = OV = obmotka vozbuzhdeniya = energizing winding
568	ц = ts = tsentrobezhnyy = centrifugal
574	пер = per = periodicheskiy = periodic
575	ст = st = staticheskiy = static
579	Р = R = Reduktor = reducing gear
586	о.с = o.s = obratnaya svyaz' = feedback
586	ЛЧ = LCh = Lineynaya Chast' = linear part
586	НЭ = NE = Nelineynyy Element = nonlinear element
589	в = v = vneshniy = external
597	н = n = nelineynyy = nonlinear
599	кр = kr = kriticheskiy = critical

Chapter 7

EVALUATION OF QUALITY OF TRANSIENT PROCESSES

§7.1. Generalization of the Krylov-Bogolyubov Asymptotic Method for Analysis of Transient Processes

The asymptotic method of Krylov and Bogolyubov, the first approximation of which was presented by the authors themselves in the form of the so-called equivalent linearization of a nonlinear equation (see Chapter I of the book by N.N. Bogolyubov and Yu.A. Mitropol'skiy [221]), served as a basis for development of the harmonic-linearization method in all the preceding chapters in analysis of self-oscillations, including those with slowly varying components (when the amplitude and frequency vary slowly over time).

Initially, this asymptotic method was developed for a second-order differential equation of the form

$$\frac{d^2x}{dt^2}+\omega_0^2x=\varepsilon f\left(x,\ \frac{dx}{dt}\right), \tag{7.1}$$

and then also for higher-order systems (Chapter IV of the book [221]).

We seek the solution of Eq. (7.1) in the form (see [221], page 37)*

$$x=a\sin\psi+\varepsilon u_1(a,\psi)+\varepsilon^2u_2(a,\psi)+\ldots+\varepsilon^mu_m(a,\psi)\ (m=1,2,\ldots), \tag{7.2}$$

where

$$\left.\begin{aligned}\frac{da}{dt}&=\varepsilon A_1(a)+\varepsilon^2A_2(a)+\ldots+\varepsilon^mA_m(a),\\ \frac{d\psi}{dt}&=\omega_0+\varepsilon B_1(a)+\varepsilon^2B_2(a)+\ldots+\varepsilon^mB_m(a).\end{aligned}\right\} \tag{7.3}$$

In first approximation we assume that

$$x=a\sin\psi,\ \frac{da}{dt}=\varepsilon A_1(a),\quad \frac{d\psi}{dt}=\omega_0+\varepsilon B_1(a). \tag{7.4}$$

It is evident from this form of the solution that the amplitude a

and the frequency $\omega = \omega_0 + \varepsilon B_1(a)$ are assumed to be slowly varying time functions. Thus the possibility arises of investigating slowly attenuating or slowly diverging nonlinear oscillatory transient processes, and in general investigating the oscillations of nonlinear systems that are close to ordinary harmonic systems:

$$x = a_0 \sin \omega_0 t, \tag{7.5}$$

but with amplitude and frequency slowly varying about the values a_0 and ω_0.

In the design of automatic systems, however, we are often obliged to deal with slowly attenuating oscillatory processes that are close not to linear harmonic processes (7.5), but to linear attenuating (or diverging) processes:

$$x = a_0 e^{\xi_0 t} \sin \omega_0 t, \tag{7.6}$$

but with a damping exponent and frequency varying slowly about the values ξ_0 and ω_0 for a fixed bounded time interval. The solution (7.6) corresponds to the second-order differential equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + c^2x = 0, \tag{7.7}$$

where

$$\xi_0 = -b, \qquad \omega_0 = \sqrt{c^2 - b^2}. \tag{7.8}$$

For the purposes of designing automatic systems, therefore, we are interested in the structure of the asymptotic solution for a nonlinear system differential equation that has a form different from (7.1), i.e. [207]:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + c^2x = \varepsilon f\left(x, \frac{dx}{dt}\right), \tag{7.9}$$

where ε is a small parameter, b and c are constant real numbers and f(x, dx/dt) is the given nonlinear function.

Carrying out the generalization of the asymptotic Krylov-Bogolyubov method as applied to an equation of the form (7.9), we shall

seek its solution in the following form:

$$x = a\sin\psi + \varepsilon\varphi_1(a, \psi) + \varepsilon^2\varphi_2(a, \psi) + \ldots + \varepsilon^m\varphi_m(a, \psi)\ (m = 1, 2, \ldots), \qquad (7.10)$$

where

$$\left.\begin{aligned} \frac{da}{dt} &= -ba + \varepsilon\Phi_1(a) + \varepsilon^2\Phi_2(a) + \ldots + \varepsilon^m\Phi_m(a), \\ \frac{d\psi}{dt} &= \omega_0 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots + \varepsilon^m B_m(a). \end{aligned}\right\} \qquad (7.11)$$

In first approximation we obtain

$$x = a\sin\psi, \quad \frac{da}{dt} = -ba + \varepsilon\Phi_1(a), \quad \frac{d\psi}{dt} = \omega_0 + \varepsilon B_1(a), \qquad (7.12)$$

where the quantity ω_0 is determined by Formula (7.8).

It differs essentially from the previous solution (7.4) in that the expression for the derivative da/dt contains a finite term -ba which is not small. In contrast to the previous case, therefore, it now becomes possible to analyze fast-damping (or rapidly diverging) nonlinear oscillations which are close to linear oscillations with a finite damping exponent $\xi_0 = -b$, since where $\varepsilon = 0$, Eqs. (7.10) and (7.11), and also Eqs. (7.12) give us the solutions (7.6) and (7.8). Here, in contrast to the preceding case, we assume that the slowly varying time functions are not a and ω, but the damping index ξ and the frequency ω:

$$\xi = -b + \frac{\varepsilon\Phi_1(a)}{a}, \quad \omega = \omega_0 + \varepsilon B_1(a); \qquad (7.13)$$

the amplitude a may, on the other hand, vary according to (7.12) at a finite rate da/dt as a function of the numerical value of b.

Thus, we will seek the solution of the nonlinear equation (7.9) in the form (7.10) and (7.11).

Considering (7.10) and (7.11) as series expansions, let us limit ourselves to a small finite number m of terms therein and seek the functions $\varphi_1(a, \psi)$, $\varphi_2(a, \psi)$, ..., $\Phi_1(a)$, $\Phi_2(a)$, ..., $B_1(a)$, $B_2(a)$, ... so that the a(t) and $\psi(t)$ figuring in (7.10) and (7.11) satisfy the

given differential equation (7.9) with an accuracy to within a quantity of the order ε^{m+1}.

In order to simplify the calculations, let us introduce the new variable y by the substitution

$$x = ye^{-bt}. \tag{7.14}$$

It follows from (7.14) that

$$\frac{dx}{dt} = e^{-bt}\left(\frac{dy}{dt} - by\right). \tag{7.15}$$

Then the differential equation (7.9) being investigated is changed to the form

$$e^{-bt}\left(\frac{d^2y}{dt^2} + \omega_0^2 y\right) = \varepsilon f\left(x, \frac{dx}{dt}\right), \quad \omega_0^2 = c^2 - b^2, \tag{7.16}$$

while Eqs. (7.10) and (7.11) assume the form

$$y = a_* \sin\psi + \varepsilon u_1(a, \psi) + \varepsilon^2 u_2(a, \psi) + \ldots + \varepsilon^m u_m(a, \psi), \tag{7.17}$$

$$\left.\begin{aligned} \frac{da_*}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^m A_m(a), \\ \frac{d\psi}{dt} &= \omega_0 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots + \varepsilon^m B_m(a), \end{aligned}\right\} \tag{7.18}$$

where

$$\left.\begin{aligned} a &= a_* e^{-bt}, \quad \frac{da}{dt} = -ba + \frac{da_*}{dt} e^{-bt}, \\ \varphi_i &= u_i e^{-bt}, \quad \Phi_i = A_i e^{-bt} \quad (i = 1, 2, \ldots). \end{aligned}\right\} \tag{7.19}$$

Differentiating (7.17) twice with respect to t and using Formulas (7.18) and (7.19), we arrive at the following expressions:

$$\begin{aligned}
\frac{dy}{dt} &= a_* \omega_0 \cos\psi + \varepsilon\left(A_1 \sin\psi + a_* B_1 \cos\psi + \omega_0 \frac{\partial u_1}{\partial \psi} - ba\frac{\partial u_1}{\partial a}\right) + \\
&+ \varepsilon^2\left(A_2 \sin\psi + a_* B_2 \cos\psi + B_1 \frac{\partial u_1}{\partial \psi} + \omega_0 \frac{\partial u_2}{\partial \psi} + e^{-bt} A_1 \frac{\partial u_1}{\partial a} - ba\frac{\partial u_2}{\partial a}\right) + \varepsilon^3 \ldots, \\
\frac{d^2y}{dt^2} &= -a_* \omega_0^2 \sin\psi + \varepsilon\left(-2\omega_0 B_1 a_* \sin\psi - ba\frac{dA_1}{da}\sin\psi + 2\omega_0 A_1 \cos\psi - \right. \\
&\left. - ba\frac{dB_1}{da} a_* \cos\psi + \omega_0^2 \frac{\partial^2 u_1}{\partial \psi^2} - 2ba\omega_0 \frac{\partial^2 u_1}{\partial a \partial \psi} + b^2 a^2 \frac{\partial^2 u_1}{\partial a^2} + b^2 a \frac{\partial u_1}{\partial a}\right) + \\
&+ \varepsilon^2\left(-B_1^2 a_* \sin\psi - 2\omega_0 B_2 a_* \sin\psi - ba\frac{dA_2}{da}\sin\psi + 2\omega_0 A_2 \cos\psi + \right. \\
&+ 2A_1 B_1 \cos\psi + A_1 \frac{dA_1}{da} e^{-bt} \sin\psi + A_1 \frac{dB_1}{da} e^{-bt} a_* \cos\psi - ba\frac{dB_2}{da} a_* \cos\psi + \\
&+ 2\omega_0 B_1 \frac{\partial^2 u_1}{\partial \psi^2} + \omega_0^2 \frac{\partial^2 u_2}{\partial \psi^2} - ba\frac{dB_1}{da}\frac{\partial u_1}{d\psi} + 2\omega_0 A_1 e^{-bt}\frac{\partial^2 u_1}{\partial a \partial \psi} - 2baB_1 \frac{\partial^2 u_1}{\partial a \partial \psi} - \\
&\left. - 2ba\omega_0 \frac{\partial^2 u_2}{\partial a \partial \psi} - 2bA_1 e^{-bt}\frac{\partial u_1}{\partial a} - ba\frac{dA_1}{da} e^{-bt}\frac{\partial u_1}{\partial a} + b^2 a \frac{\partial u_2}{\partial a} - \right.
\end{aligned}$$

$$-2bae^{-bt}A_1\frac{\partial^2 u_1}{\partial a^2}+b^2a^2\frac{\partial^2 u_2}{\partial a^2}\Big)+\varepsilon^2\ldots$$

Now, according to (7.15), we may write dx/dt in the form

$$\frac{dx}{dt}=a\omega_0\cos\psi-ba\sin\psi+$$
$$+\varepsilon e^{-bt}\Big(A_1\sin\psi+a_*B_1\cos\psi-bu_1-ba\frac{\partial u_1}{\partial a}+\omega_0\frac{\partial u_1}{\partial\psi}\Big)+\varepsilon^2\ldots$$

Therefore, the Taylor-series expansion of the right member of Eq. (7.16) gives us

$$\varepsilon f\Big(x,\frac{dx}{dt}\Big)=\varepsilon f(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)+$$
$$+\varepsilon^2e^{-bt}\Big[u_1\frac{\partial f}{\partial x}(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)+\Big(A_1\sin\psi+a_*B_1\cos\psi-$$
$$-bu_1-ba\frac{\partial u_1}{\partial a}+\omega_0\frac{\partial u_1}{\partial\psi}\Big)\frac{\partial f}{\partial\dot{x}}(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)\Big]+\varepsilon^2\ldots$$

Substituting the expressions obtained for d^2y/dt^2 and $\varepsilon f(x, dx/dt)$, and also Expression (7.17) for y into Eq. (7.16) and then equating the coefficients before equal powers of ε, and applying (7.19), we obtain the series of equations

$$-2\omega_0B_1a\sin\psi-ba\frac{d\Phi_1}{da}\sin\psi+2\omega_0\Phi_1\cos\psi-ba^2\frac{dB_1}{da}\cos\psi+$$
$$+\omega_0\dot{\varphi}_1+\omega_0^2\frac{\partial^2\varphi_1}{\partial\psi^2}-2ba\omega_0\frac{\partial^2\varphi_1}{\partial a\,\partial\psi}+b^2a^2\frac{\partial^2\varphi_1}{\partial a^2}+b^2a^2\frac{\partial\varphi_1}{\partial a}=$$
$$=f(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi),\qquad(7.20)$$
$$-B_1^2a\sin\psi-2\omega_0B_2a\sin\psi-ba\frac{d\Phi_2}{da}\sin\psi+2\omega_0\Phi_2\cos\psi+2\Phi_1B_1\cos\psi+$$
$$+\Phi_1\frac{d\Phi_1}{da}\sin\psi+a\Phi_1\frac{dB_1}{da}\cos\psi-ba^2\frac{dB_2}{da}\cos\psi+\omega_0^2\dot{\varphi}_2+2\omega_0B_1\frac{\partial^2\varphi_1}{\partial\psi^2}+$$
$$+\omega_0^2\frac{\partial^2\varphi_2}{\partial\psi^2}-ba\frac{dB_1}{da}\frac{\partial\dot{\varphi}_1}{\partial\psi}+2\omega_0\Phi_1\frac{\partial^2\varphi_1}{\partial a\,\partial\psi}-2baB_1\frac{\partial^2\varphi_1}{\partial a\,\partial\psi}-2ba\omega_0\frac{\partial^2\varphi_2}{\partial a\,\partial\psi}-$$
$$-2b\Phi_1\frac{\partial\varphi_1}{\partial a}-ba\frac{d\Phi_1}{da}\frac{\partial\varphi_1}{\partial a}+b^2a\frac{\partial\varphi_2}{\partial a}-2ba\Phi_1\frac{\partial^2\varphi_1}{\partial a^2}+b^2a^2\frac{\partial^2\varphi_2}{\partial a^2}=$$
$$=\dot{\varphi}_1\frac{\partial f}{\partial x}(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)+\Big(\Phi_1\sin\psi+aB_1\cos\psi-b\varphi_1-$$
$$-ba\frac{\partial\varphi_1}{\partial a}+\omega_0\frac{\partial\varphi_1}{\partial\psi}\Big)\frac{\partial f}{\partial\dot{x}}(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi),\ldots\qquad(7.21)$$

We shall assume that the function $\varphi_1(a, \psi)$ does not contain the terms sin ψ and cos ψ, since it is natural to include them in their entirety in the first term of the solution (7.10). Then, expanding the right member of (7.20) in trigonometric series,* and equating the corresponding coefficients of both parts of Eq. (7.20), we obtain

$$\left.\begin{aligned}-2\omega_0\Phi_1+ba^2\frac{dB_1}{da}&=h_1(a),\\-2a\omega_0B_1-ba\frac{d\Phi_1}{da}&=g_1(a),\end{aligned}\right\}\qquad(7.22)$$

$$ba^2\frac{\partial^2\varphi_1}{\partial a^2}+b^2a\frac{\partial\varphi_1}{\partial a}+\omega_0^2\frac{\partial^2\varphi_1}{\partial\psi^2}-2ba\omega_0\frac{\partial^2\varphi_1}{\partial a\,\partial\psi}+\omega^2\varphi_1=$$
$$=g_0(a)+\sum_{r=2}^{\infty}[g_r(a)\sin r\psi+h_r(a)\cos r\psi], \qquad (7.23)$$

where

$$g_0(a)=\frac{1}{2\pi}\int_0^{2\pi}f(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)\,d\psi,$$
$$g_r(a)=\frac{1}{\pi}\int_0^{2\pi}f(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)\sin r\psi\,d\psi,$$
$$h_r(a)=\frac{1}{\pi}\int_0^{2\pi}f(a\sin\psi,\ a\omega_0\cos\psi-ba\sin\psi)\cos r\psi\,d\psi$$
$$(r=1,\ 2,\ \ldots,\ \infty).$$

The functions $\Phi_1(a)$ and $B_1(a)$ are determined by particular solutions of the two equations (7.22). The first approximation (7.12) is found by the same means. Further, from Eq. (7.23) we determine the function $\varphi_1(a, \psi)$, seeking the latter in the form of the series

$$\varphi_1(a,\ \psi)=v_0(a)+\sum_{r=2}^{\infty}[v_r(a)\sin r\psi+w_r(a)\cos r\psi],$$

the coefficients of which are unknown. Substitution of this expression into (7.23) gives us a possibility of finding all the coefficients v_0, v_r, w_r in terms of the quantities g_0, g_r, h_r.

After this we deal similarly with Eq. (7.21), from which we determine $\Phi_2(a)$, $B_2(a)$, which gives us the second approximation and the function $\varphi_2(a, \psi)$, and so forth.

Let us limit ourselves henceforth to consideration of only the first approximation, representing it according to (7.12) and (7.13) in the form

$$x=a\sin\psi, \qquad (7.24)$$

where

$$\frac{da}{dt}=a\xi(a),\quad \frac{d\psi}{dt}=\omega(a),$$

where

$$\left.\begin{aligned} \xi(a) &= -b + \varepsilon \frac{\Phi_1(a)}{a}, \\ \omega(a) &= \omega_0 + \varepsilon B_1(a), \quad \omega^2(a) = \omega_0^2 + \varepsilon 2\omega_0 B_1(a) \end{aligned}\right\} \tag{7.25}$$

(in the expression for ω^2, the term in ε^2 is discarded).

Here ξ and ω are, so to speak, instantaneous values of the damping exponent and the frequency for the nonlinear process being investigated; they vary with the time variation of the oscillation amplitude $\underline{a}$. Here $\underline{b}$ and $\omega_0 = \sqrt{c^2 - b^2}$ are the damping exponent and oscillation frequency, respectively, of the linear system (7.7) obtained from the nonlinear system (7.9) for $\varepsilon = 0$.

Equations (7.22) for determination of the functions $\Phi_1(a)$ and $B_1(a)$ may be represented in the form

$$\left.\begin{aligned} 2\omega_0 \Phi_1(a) &= \beta(a) + ba^2 \frac{dB_1}{da}, \\ 2\omega_0 B_1(a) &= \gamma(a) - b \frac{d\Phi_1}{da}, \end{aligned}\right\} \tag{7.26}$$

where

$$\left.\begin{aligned} \beta(a) &= -h_1(a) = -\frac{1}{\pi} \int_0^{2\pi} f(a\sin\psi, a\omega_0\cos\psi - ba\sin\psi)\cos\psi\, d\psi, \\ \gamma(a) &= -\frac{g_1(a)}{a} = -\frac{1}{\pi a} \int_0^{2\pi} f(a\sin\psi, a\omega_0\cos\psi - ba\sin\psi)\sin\psi\, d\psi. \end{aligned}\right\} \tag{7.27}$$

In many specific problems, the functions $\beta(a)$ and $\gamma(a)$ are represented geometrically by smooth curves with small curvature. In these cases we may assume that the second and higher derivatives of β and γ with respect to $\underline{a}$ are small quantities of the order of ε (in comparison with the quantities β and γ and their first derivatives), and may be neglected in the first approximation, introducing an allowance for them if required in the synthesis of the subsequent approximations. Then the necessary particular solutions of Eqs. (7.26) are written in the form

$$\left.\begin{aligned} 2\omega_0 \Phi_1(a) &= \beta(a) + \frac{ba^2\omega_0}{2\omega_0^2 + b^2} \frac{d\gamma}{da}, \\ 2\omega_0 B_1(a) &= \gamma(a) - \frac{b}{2\omega_0} \frac{d\beta}{da} - \frac{b^2 a}{2\omega_0^2 + b^2} \frac{d\gamma}{da}. \end{aligned}\right\} \tag{7.28}$$

These expressions are obtained by successive approximations. They satisfy Eqs. (7.26), as may be verified by making a substitution.

In the general case, therefore, to determine ξ and ω^2, according to (7.25), we obtain the following formulas from (7.26) (after discarding the terms in ε^2):

$$\left.\begin{aligned}\xi &= -b + \varepsilon\frac{\beta(a)}{2a\omega_0} + \frac{ba}{2\omega_0}\frac{d\omega}{da} = -b + \varepsilon\frac{\beta(a)}{2a\omega_0} - \frac{1}{2\omega_0}\frac{d\omega}{dt},\\ \omega^2 &= \omega_0^2 + \varepsilon\gamma(a) - ba\frac{d\xi}{da} - b\xi - b^2 = \omega_0^2 + \varepsilon\gamma(a) + \frac{d\xi}{dt} - b\xi - b^2.\end{aligned}\right\} \quad (7.29)$$

It is interesting to note that these last expressions agree to within ε^2 with the results of P.Ye. Grensted [212] which were obtained by a completely different, nonasymptotic method without introducing the small parameter. Since these formulas are differential equations that are in the majority of cases unsuitable for practical use, then it is expedient to represent them in a simpler final form.

In the case where $\beta(a)$ and $\gamma(a)$ are curves with small curvature, Eqs. (7.26) assume the form (7.28), and then according to (7.25) the solution for the damping exponent ξ and the frequency ω is:

$$\left.\begin{aligned}\xi &= -b + \varepsilon\frac{\beta(a)}{2a\omega_0} + \varepsilon_2\frac{ba}{2(2\omega_0^2 + b^2)}\frac{d\gamma}{da},\\ \omega^2 &= \omega_0^2 + \varepsilon\gamma(a) - \varepsilon\frac{b}{2\omega_0}\frac{d\beta}{da} - \varepsilon\frac{b^2a}{2\omega_0^2 + b^2}\frac{d\gamma}{da}.\end{aligned}\right\} \quad (7.30)$$

Finally, in the case where $\beta(a)$ and $\gamma(a)$ are slowly varying functions of amplitude, i.e., their derivatives with respect to a may be assumed to be small and of the order of ε in comparison with the quantities β and γ, we obtain

$$\left.\begin{aligned}\xi &= -b + \varepsilon\frac{\beta(a)}{2a\omega_0},\\ \omega^2 &= \omega_0^2 + \varepsilon\gamma(a) \quad (\omega_0^2 = c^2 - b^2).\end{aligned}\right\} \quad (7.31)$$

For this last case, we may propose the following simple formal method in first approximation for solution of the nonlinear equation (7.9). Let us perform the "equivalent linearization" of Eq. (7.9). For this purpose, using the substitution (7.24), we expand the right mem-

ber of this equation in trigonometric series, restricting ourselves to terms in sin ψ and cos ψ:

$$\varepsilon f\left(x, \frac{dx}{dt}\right) = \varepsilon\left[g_1(a)\sin\psi + h_1(a)\cos\psi\right].$$

Noting that, according to (7.24),

$$\sin\psi = \frac{x}{a}, \quad \cos\psi = \frac{1}{a\omega_0}\left(\frac{dx}{dt} + bx\right),$$

and introducing the notation (7.27), we may write the differential equation (7.9) in the form

$$\frac{d^2x}{dt^2} + \left[2b - \varepsilon\frac{\beta(a)}{a\omega_0}\right]\frac{dx}{dt} + \left[c^2 + \varepsilon\gamma(a) - \varepsilon\frac{b\beta(a)}{a\omega_0}\right]x = 0. \qquad (7.32)$$

Let us write its characteristic equation formally, just as for the linear equation, in the form

$$p^2 + \left[2b - \varepsilon\frac{\beta(a)}{a\omega_0}\right]p + c^2 + \varepsilon\gamma(a) - \varepsilon\frac{b\beta(a)}{a\omega_0} = 0. \qquad (7.33)$$

The solution of this quadratic equation gives us

$$p = -b + \varepsilon\frac{\beta(a)}{2a\omega_0} \pm j\sqrt{c^2 + \varepsilon\gamma(a) - \varepsilon\frac{b\beta(a)}{a\omega_0} - \left[b - \varepsilon\frac{\beta(a)}{2a\omega_0}\right]^2}, \qquad (7.34)$$

hence the "damping exponent" and "frequency" are

$$\xi = -b + \varepsilon\frac{\beta(a)}{2a\omega_0}, \quad \omega^2 = c^2 - b^2 + \varepsilon\gamma(a), \qquad (7.35)$$

with an accuracy to within ε^2; this agrees with Expression (7.31), obtained earlier by a rigorous method. Here the solution has the form (7.24). Consequently, the equivalent linearization described may be used for the investigation of fast-damping nonlinear processes if $\beta(a)$ and $\gamma(a)$ are slowly varying functions of the amplitude a. In more general cases, we must make use of Formulas (7.30) or (7.29) for the same form of the solutions (7.24).

Thus we have obtained a generalization of the Krylov-Bogolyubov method, which they developed for Eq. (7.1), to the complete second-order nonlinear equation of the form (7.9). Ordinarily, however, the dynamics of automatic systems are described by an equation of higher

order. Therefore the results obtained must be extended to a nonlinear equation of high order, just as this was done in Chapter 2 for the periodic solution, and then it must also be extended to oscillatory processes with a slowly varying component.

In the following section this will be done as it is applicable to equations typical for automatic systems. Here we will only indicate that N.N. Bogolyubov (see [195] or [221], Chapter 4) developed an asymptotic method for finding a solution in the two-parameter form (7.2) and (7.3) close to the sinusoidal (7.5) for the high-order system

$$\frac{dx_k}{dt} - \sum_{i=1}^{n} c_{ki}x_i = \varepsilon f_k^{(1)}(x_1, \ldots, x_n) + \varepsilon^2 f_k^{2}(x_1, \ldots, x_n) + \cdots \qquad (k = 1, 2, \ldots, n) \tag{7.36}$$

this made it possible to find single-frequency free oscillations with slowly varying amplitude and frequency in high-order systems. Here we assume that in the linear system obtained from (7.36) for $\varepsilon = 0$,

$$\frac{dx_k}{dt} - \sum_{i=1}^{n} c_{ki}x_i = 0 \qquad (k = 1, 2, \ldots, n), \tag{7.37}$$

nondamping harmonic oscillations of the form

$$x = a_0 \sin \omega_0 t, \tag{7.38}$$

are possible.

It is natural that in this case definite limitations are imposed upon the system to ensure that the general solution for this system will tend toward the two-parameter solution (7.2)-(7.3) for some range of initial conditions.

The generalization, which is analogous to that set forth above, will consist here in seeking the solution for the system (7.36) in the form (7.10) and (7.11), assuming that the linear system (7.37) has a solution close to the sinusoidal attenuating solution:

$$x = a_0 e^{-bt} \sin \omega_0 t, \tag{7.39}$$

for some fixed range of initial conditions; this ensures the appropriate distribution of the roots of the characteristic equation for this system. As a result we are able to seek approximate fast-damping or diverging transient oscillatory processes with slowly varying damping exponent and frequency in certain high-order nonlinear systems.

In conclusion, let us recall that an unexplained expansion of the function $f(a \sin \psi, a\omega_0 \cos \psi - ba \sin \psi)$ in trigonometric series has remained in the derivation of Eqs. (7.22) and (7.23). The fact is that here, according to (7.12), $\underline{a}$ is a variable, close to $a_0 e^{-bt}$, where the variable ψ with respect to which the expansion takes place is, according to (7.12), also connected with $\underline{t}$ by means of a relationship close to $\psi = \omega_0 t$. This differs essentially from the ordinary expansion of the nonlinear function $F(A \sin \psi, A\Omega \cos \psi)$ in trigonometric Fourier series for constant A, as used in the previous chapters.

We may say that, in contrast to the previous Fourier-series expansion with respect to one variable ψ, we must now expand the function

$$f[a(t) \sin \psi,\ a(t)\omega_0 \cos \psi - ba(t) \sin \psi] \qquad (7.40)$$

in the plane of the two variables ψ and $\underline{t}$ (Fig. 7.1), into a series with respect to trigonometric functions of the variable ψ. Let us imagine for the present that the variables ψ and $\underline{t}$ are independent of each other. Then for every fixed value of $\underline{t}$, the expansion of the function (7.40) in a trigonometric series will have the form of the ordinary Fourier series used earlier in Chapter 2. Geometrically, in the

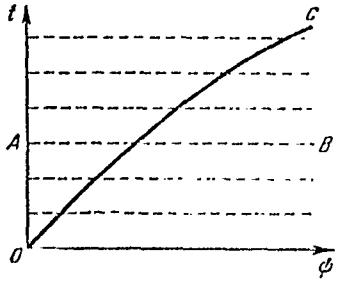


Fig. 7.1

(ψ, t) plane this will correspond to the broken line t = const (for example, AB in Fig. 7.1). It is evident that the indicated Fourier-series expansion, which is valid for the entire line AB, will give us the correct value of the function (7.40) at each point of this line.

It is further evident that the same general formulas for the Fourier-series coefficients for the function (7.40) will be valid for all the lines t = const (the broken straight lines in Fig. 7.1), on substitution of the appropriate value of a(t) into these formulas. Hence it follows that the ordinary formula for the Fourier-series expansion of the function (7.40) in the form (7.23) is valid for any value of a(t) for all points of the plane (Fig. 7.1).

Now let us take into consideration that the variables ψ and t are not independent of each other, but related, according to (7.12), by a definite dependence that is close (at least in the origin) to $\psi = \omega_0 t$. This means that now we will make use of the trigonometric-series expansion of the function (7.40), not over the entire plane (ψ, t), but rather along some line OC (Fig. 7.1) corresponding to the function $\psi(t)$ mentioned. Since it has been shown that using the ordinary formulas for the Fourier-series coefficients, as in (7.23) makes the expansion of the function in trigonometric series valid for all points of the plane, it is also valid along the line OC, i.e., for any function $\psi(t)$.

Thus we have indicated justification for applying the formulas of the ordinary method of trigonometric-series expansion, which was used in (7.23), to nonsteady nonlinear oscillations with amplitude a(t) variable over time. The general idea of the proof set forth was communicated to the author by Professor Kh.L. Smolitskiy.

The formal justification of such a trigonometric-series expansion for any variation of a(t) and $\psi(t)$ does not, of course, indicate that all the properties of the Fourier series and the usual properties of the zeroth, first, and higher harmonics are retained for these series, as was the case for constant amplitude a = A, since for any law of amplitude variation we obtain nonperiodic expansion components of arbit-

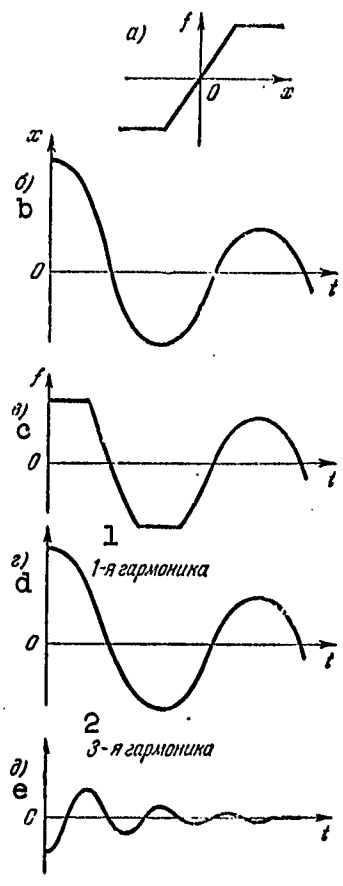


Fig. 7.2. 1) First harmonic; 2) third harmonic.

rary form, with the consequence that the expansion may not have any practical value.

However, as we have already said, in the problem being considered the dependence of the quantity a on t in the expanded function (7.40) is close, at least over a finite time interval, to the function $a = a_0 e^{-bt}$, while the function $\psi(t)$ is close to $\psi = \omega_0 t$. Therefore, in contrast to the ordinary expansion of periodic nonlinear oscillations into Fourier series in harmonics with constant amplitude and frequencies, here we obtain the expansion of the nonlinear attenuating oscillations (for example, Fig. 7.2c) into attenuating "harmonics" with frequencies which are multiples of each other and which vary slowly over time (Fig. 7.2d, e). This corresponds to progressive passage along the line OC (Fig. 7.1) from one broken line to another, with a corresponding gradual decrease of the amplitude according to a fixed relationship as a function of the variable t, which is laid off along the axis of ordinates. The same consideration may also be applied to diverging oscillations (for $b < 0$).

As a result, it is obvious that in the problem being solved here, the expansion of nonstationary nonlinear oscillations in trigonometric series is not only valid formally, but also has a completely clear physical meaning which, on the same basis as in Chapter 2, permits us to take the first "harmonic" of this expansion into consideration as the fundamental harmonic and neglect the higher "harmonics" in first

approximation.

§7.2. Analysis of Symmetrical Oscillatory Transient Processes

In the next two sections, we shall consider oscillatory transient processes in a nonlinear automatic system that are symmetrical with respect to the time axis and may be described in first approximation by an attenuating or diverging sinusoid with both the damping exponent and the frequency varying slowly over time (Fig. 7.3).

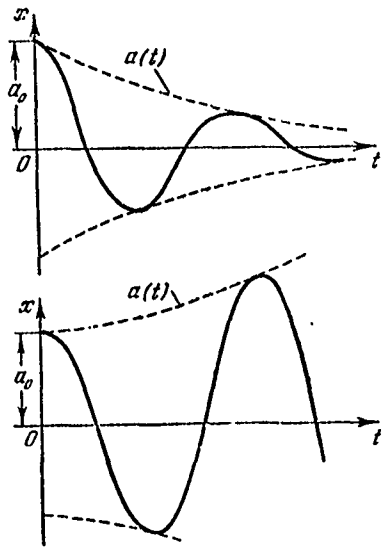


Fig. 7.3

Before describing this mathematically, let us focus our attention on two essential circumstances. For linear systems, when the damping exponent ξ = const and the frequency ω = const, we write

$$x = a_0 e^{\xi t} \sin(\omega t + \varphi). \qquad (7.41)$$

If, however, the frequency ω and the damping exponent ξ in the oscillation process vary with the course of time, then we must write the solution in another form. Firstly, as we already know from Chapter 5, we must write sin $\psi(t)$ and determine the current value of the frequency at an arbitrary time instant in the form

$$\omega = d\psi/dt, \qquad (7.42)$$

where

$$\psi = \int_0^t \omega\, dt + \psi_0, \qquad (7.43)$$

where ψ_0 is constant (initial phase). From this, for a linear system where ω = const, we obtain the expression under the "sin" in Formula (7.41) as a particular case. Let us note that instead of variability of the quantity ω, it would be possible to introduce another variable φ into this type of nonlinear system with ω = const, i.e., $\psi = \omega_0 t +$

+ $\varphi(t)$, which would ultimately give us the same result; here, according to (7.42), the current value of the frequency would assume the form

$$\omega = \frac{d\psi}{dt} = \omega_0 + \frac{d\varphi}{dt}. \qquad (7.44)$$

In what follows, however, it will be expedient to confine ourselves to the first representation (7.42) and (7.43).

Secondly, for a damping exponent variable over time, we must determine the current value of the amplitude a (Fig. 7.3) not in the form $a_0 e^{\xi t}$, as was done in (7.41), but in the form of the differential relation

$$da/dt = a\xi. \qquad (7.45)$$

Then in the case of a linear system where ξ = const, we obtain

$$\frac{da}{a} = \xi\, dt, \quad a = a_0 e^{\xi t},$$

as a particular case, while in the case of a nonlinear system, when ξ varies in the process of the oscillations, the current value of the amplitude according to (7.45) is

$$\frac{da}{a} = \xi\, dt, \quad a = a_0 e^{\int_0^t \xi\, dt}, \qquad (7.46)$$

i.e., the oscillation envelope (Fig. 7.3) consists of elementary segments of exponentials with continuously varying exponent ξ.

In the case indicated at the beginning of the section, therefore, it is expedient to seek a solution for a transient process in a nonlinear system as a first approximation in the form

$$x = a \sin \psi, \qquad (7.47)$$

$$\frac{da}{dt} = a\xi, \quad \omega = \frac{d\psi}{dt}, \qquad (7.48)$$

assuming that the unknowns being sought are the slowly varying quantities ξ and ω. These quantities are called the damping exponent and the frequency, respectively. Their variability over time attests to the conditionality of the use of these terms here, a fact which must be

kept in mind all the time, and we must differentiate them from the corresponding constant quantities for linear systems, and also from the constant frequency of the periodic solutions (self-oscillations in the previous chapters and forced oscillations in Chapter 9). Therefore, in contrast to the latter cases, where we use the letter Ω, we shall use the new symbol ω here. For this same reason we introduce a new symbol here for the amplitude: $\underline{a}$ in place of the previous A, which was used for periodic solutions.

The "damping exponent" may characterize not only the damping rate but also the divergence rate of the oscillations. In fact, according to (7.45), we have

$$\left.\begin{aligned} \frac{da}{dt} > 0 \text{ for } \xi > 0, \\ \frac{da}{dt} < 0 \text{ for } \xi < 0, \end{aligned}\right\} \tag{7.49}$$

i.e., diverging oscillations correspond to positive values of the "damping exponent" ξ, while damping oscillations correspond to negative values.

As we have already stated, the quantities ξ and ω are assumed to be slowly varying functions. However, it is very important to note that this by no means signifies consideration of slowly attenuating oscillations, since the attenuation speed of the oscillations is determined by the full quantity ξ at the moment in question, and not by its variation. Since in linear systems constant values of ξ may correspond to either slow or fast attenuation of the oscillations, then we may also characterize the slowly varying values of ξ just as we characterized the other processes. Hence Eqs. (7.47) and (7.48) with slowly varying ξ and ω are in themselves suitable for describing both slowly (for small ξ) and rapidly (for large ξ) damping processes, i.e., the rate of change of the amplitude $\underline{a}$ is not limited in the processes. Lim-

itations in this respect may be imposed only by the properties of the nonlinear system itself if they lead to violation of the premise set forth at the beginning of the present section.

Earlier (in Chapter 2) we assumed the periodic solution in the nonlinear system to be close to sinusoidal, at least for one variable $\underline{x}$ under the nonlinear function sign (or several nonlinearities). Analogously, we assume here, too, that the transient for the same variable $\underline{x}$ in the nonlinear system is close to either a damped or diverging process (including also a rapidly damped one) in a linear system with slowly varying parameters within the same finite interval of amplitude variation over which this is possible.

Thus, starting from considerations of the most advantageous form for a definite type of transients in nonlinear automatic systems, we arrive at Expressions (7.47) and (7.48), which coincide exactly with the form (7.12) and (7.13) assumed in §7.1 for the solution. For this form of the solution it was proved in §7.1 that in cases when the quantities $\beta(a)$ and $\gamma(a)$ are slowly varying functions of the amplitude $\underline{a}$ we can use equivalent linearization of the specified nonlinear equation of the system. It is obvious in this case that the quantities $\gamma(a)$ and $\beta(a)$ which were introduced there are analogous to the previous coefficients of harmonic linearization $\underline{q}$ and q'.

On this basis, we shall consider [208] the corresponding harmonically linearized differential equation of the nonlinear system, finding its solution in the form (7.47) and (7.48).

The formulas of harmonic linearization of the nonlinearity will have a certain singularity compared with the previous formulas (2.75) and (2.76). In fact, if the value of the damping exponent ξ is not small, then by differentiating (7.47) with respect to tne time as a product of two functions, we obtain with account of (7.48)

$$px = a\omega \cos \psi + a\xi \sin \psi. \tag{7.50}$$

From this and from (7.47) we get

$$\sin \psi = \frac{x}{a}, \quad \cos \psi = \frac{px}{a\omega} - \frac{\xi x}{a\omega} = \frac{p - \xi}{a\omega} x. \tag{7.51}$$

Therefore the first "harmonic" (damped or divergent) of the nonlinear function F(x, px) with x = a(t) sin ψ(t) will now be, in lieu of (2.75)

$$F(x, px) = qx + q' \frac{p - \xi}{\omega} x = \left(q - \frac{\xi}{\omega} q'\right) x + \frac{q'}{\omega} px, \tag{7.52}$$

where

$$\left.\begin{aligned} q &= \frac{1}{\pi a} \int_0^{2\pi} F(a \sin \psi, a\omega \cos \psi + a\xi \sin \psi) \sin \psi \, d\psi, \\ q' &= \frac{1}{\pi a} \int_0^{2\pi} F(a \sin \psi, a\omega \cos \psi + a\xi \sin \psi) \cos \psi \, d\psi. \end{aligned}\right\} \tag{7.53}$$

Here and in general the coefficients of harmonic linearization will depend on three unknowns: a, ω, and ξ. If on the other hand we consider the nonlinearity F(x), as is most frequently the case, then q and q' retain their previous form

$$\left.\begin{aligned} q &= \frac{1}{\pi a} \int_0^{2\pi} F(a \sin \psi) \sin \psi \, d\psi, \\ q' &= \frac{1}{\pi a} \int_0^{2\pi} F(a \sin \psi) \cos \psi \, d\psi, \end{aligned}\right\} \tag{7.54}$$

and in this case we can use the material of Chapter 3 in its entirety, in the form of ready expressions for q(a) and q'(a) for different specific nonlinearities, taking into account, however, the new form (7.52) for the replacement of the nonlinear function.

The new more complicated formula for harmonic linearization (7.52) for oscillatory transients is characterized by the presence of a term $-\xi q' x/\omega$, which was not present in the previously used simple form of harmonic linearization. The ratio ξ/ω, according to linear oscillation theory, determines the speed of damping ($\xi < 0$) or divergence ($\xi > 0$)

of the amplitude of the oscillations per period.*

In fact, at constant ξ and ω the ratio of the two oscillation amplitudes separated by the time equal to one period T will be

$$\frac{a(t+T)}{a(t)} = \frac{a_0 e^{\xi(t+T)}}{a_0 e^{\xi t}} = e^{\xi T} = e^{2\pi\frac{\xi}{\omega}}.$$

At constant values of ξ and ω (in linear systems), the rate of decrease or increase of the oscillation amplitude per period is characterized by the following table:

ξ/ω	$-0,5$	$-0,4$	$-0,3$	$-0,2$	$-0,1$	0	$+0,1$	$+0,2$	$+0,3$
$\frac{a(t+T)}{a(t)}$	0,043	0,081	0,152	0,285	0,534	1	1,87	3,51	6,58

It is obvious that in case of slowly varying values of ξ and ω (i.e., in the nonlinear systems considered here), the result will be somewhat different, depending on the law of variation of ξ and ω with time, but nevertheless close to the figures given in this table.

Thus, if the values of $|\xi/\omega|$ are small, slowly varying or diverging oscillations are observed, and if the values $|\xi/\omega|$ are sufficiently large, rapidly damped or diverging values are observed. The introduction of the more complicated form of harmonic linearization (7.52) is meaningful in practice only when we are considering an oscillatory process that attenuates or diverges comparatively rapidly. In the investigation of oscillatory transient processes with slowly varying amplitude, when the damping exponent ξ is small (more accurately $|\xi| \ll \omega$), we can, as a result of the general approximate nature of the method, neglect the value of the ratio ξ/ω in Expressions (7.50)-(7.53) and carry out harmonic linearization of the nonlinearity by means of the formulas

$$\left.\begin{aligned} q &= \frac{1}{\pi a}\int_0^{2\pi} F(a\sin\psi,\ a\omega\cos\psi)\sin\psi\,d\psi, \\ q' &= \frac{1}{\pi a}\int_0^{2\pi} F(a\sin\psi,\ a\omega\cos\psi)\cos\psi\,d\psi, \end{aligned}\right\} \qquad (7.55)$$

i.e., using the same formulas as for the periodic solution (2.76) except that the constant A is replaced by the variable a. In this connection, in particular, self-oscillating processes with slowly varying amplitude (due to the influence of slowly varying external signals) were considered in Chapter 5 on the basis of the same harmonic linearization formulas as for the periodic solution.

It is interesting to note that in the case of a single-valued odd-symmetry nonlinearity F(x), when q' = 0, the harmonic linearization formula (7.52) assumes the form

$$F(x) = q(a)\,x, \qquad (7.56)$$

which coincides with the form for the periodic solution in which the constant A is replaced by the variable a, something already used in the analysis of the simplest example in §1.5.

To illustrate the foregoing method of harmonic linearization, let us consider a second-order nonlinear equation of the Van der Pol type:

$$\frac{d^2x}{dt^2} - b(1 - x^2)\frac{dx}{dt} + x = 0$$

or

$$(p^2 - bp + 1)x + bx^2px = 0. \qquad (7.57)$$

Unlike in the Van der Pol equation, the coefficient b will not be regarded as small. For the nonlinearity

$$F(x, px) = bx^2px$$

the coefficients of harmonic linearization calculated by Formulas (7.53) will be

$$q = \frac{1}{\pi a}\int_0^{2\pi} ba^2\sin^2\psi\,(a\omega\cos\psi + a\xi\sin\psi)\sin\psi\,d\psi = \frac{3b}{4}a^2\xi,$$

$$q' = \frac{1}{\pi a}\int_0^{2\pi} ba^2 \sin^2\psi\,(a\omega\cos\psi + a\xi\sin\psi)\cos\psi\,d\psi = \frac{b}{4}a^2\omega.$$

Therefore, according to (7.52), it is necessary to make in the given equation the following substitution

$$bx^2px = \left(\frac{3b}{4}a^2\xi - \frac{\xi}{\omega}\frac{b}{4}a^2\omega\right)x + \frac{b}{4}a^2px,$$

after which Eq. (7.57) assumes the form

$$\ddot{x} - \left[p^2 - b\left(1 - \frac{a^2}{4}\right)p + 1 + \frac{b}{2}a^2\xi\right]x = 0.$$

Solving its characteristic equation, we obtain the roots

$$p_{1,2} = \frac{b}{2}\left(1 - \frac{a^2}{4}\right) \pm j\sqrt{1 + \frac{b}{2}a^2\xi - \frac{b^2}{4}\left(1 - \frac{a^2}{4}\right)^2}.$$

The real part of the root will represent the damping exponent of the oscillations, i.e.,

$$\xi = \frac{b}{2}\left(1 - \frac{a^2}{4}\right), \tag{7.58}$$

while the imaginary part will represent the oscillation frequency, namely

$$\omega^2 = 1 - \frac{b^2}{4}\left(1 - \frac{3}{2}a^2 + \frac{5}{16}a^4\right). \tag{7.59}$$

It is seem from this that when $\xi = 0$ we have

$$a = A = 2, \quad \omega = \Omega = 1.$$

These will be the amplitude and frequency of the self-oscillations. They are stable because $\xi < 0$ when $a > 2$ and $\xi > 0$ when $a < 2$. On the basis of Formulas (7.48) and (7.58) we can determine the law of variation of the oscillation amplitude during the transient, namely

$$t = \int_{a_0}^{a}\frac{da}{a\,(a)} = \frac{8}{b}\int_{a_0}^{a}\frac{da}{a\,(4 - a^2)} = \frac{1}{b}\left(\ln\frac{a^2}{|4 - a^2|} - \ln\frac{a_0^2}{|4 - a_0^2|}\right),$$

hence

$$a = \frac{2a_0}{\sqrt{a_0^2 + (4 - a_0^2)e^{-bt}}} = \frac{a_0 e^{\frac{1}{2}bt}}{\sqrt{1 + \frac{1}{4}a_0^2(e^{bt} - 1)}}.$$

In the solution obtained here for the problem all the quantities coincide with the solution obtained with the first approximation of the Krylov and Bogolyubov asymptotic method (see [221], page 70), except for the frequency ω, which there was independent of the amplitude and set equal to unity. On the other hand in the solution obtained here (with the coefficient $\underline{b}$ finite) we see the presence of a dependence (7.59) of the frequency on the amplitude. On the other hand if $b = \varepsilon$ is small, then the quantity $b^2 = \varepsilon^2$ in the expression for the frequency can be neglected, and we also obtain $\omega = 1$.

In the general case, for linear systems of the first class of the main type (see §1.2), the differential equation for the transient will have a high order:

$$Q(p)x + R(p)F(x, px) = 0. \tag{7.60}$$

After harmonic linearization this equation in the general case, in accordance with (7.52), assumes the form

$$Q(p)x + R(p)\left(q + \frac{p-\xi}{\omega}q'\right)x = 0, \tag{7.61}$$

and in simpler cases, according to (7.56):

$$Q(p)x + q(a)R(p)x = \bar{0}. \tag{7.62}$$

The oscillatory process in the linear system, described by the solution (7.41), corresponds to a pair of complex roots of the characteristic equation $p = \xi \pm j\omega$ with constant values of ξ and ω. Analogously, the oscillatory process in the nonlinear system, described approximately by Formulas (7.47) and (7.48), is determined by slowly varying values of ξ and ω, which can be determined by finding the pair of complex roots $p = \xi \pm j\omega$ of the characteristic equation of the harmonically linearized system (7.61) or (7.62).

Accordingly, we substitute in the characteristic equation

$$Q(p) + R(p)\left(q + \frac{p-\xi}{\omega}q'\right) = 0 \tag{7.63}$$

$p = \xi + j\omega$ for the determination of the values of ξ and ω, satisfying this equation. We obtain

$$Q(\xi + j\omega) + R(\xi + j\omega)(q + jq') = 0. \tag{7.64}$$

The value $\xi + j\omega$ is substituted in place of $\underline{p}$ in any polynomial conveniently by expanding it in powers of $j\omega$, for example:

$$Q(\xi + j\omega) = Q(\xi) + \left(\frac{dQ}{dp}\right)_\xi j\omega + \\ + \frac{1}{2!}\left(\frac{d^2Q}{dp^2}\right)_\xi (j\omega)^2 + \ldots + \frac{1}{n!}\left(\frac{d^nQ}{dp^n}\right)_\xi (j\omega)^n, \tag{7.65}$$

where the index ξ denotes that in the expressions for the derivative one must substitute ξ in place of $\underline{p}$. The same formula is used to expand also the polynomial $R(\xi + j\omega)$.

At small values of ξ (for slowly damped processes) it is more convenient to use in place of (7.65) expansion in powers of ξ, and then retain only the first-degree term, namely:

$$\left.\begin{aligned} Q(\xi + j\omega) &= Q(j\omega) + \left(\frac{dQ}{dp}\right)_{j\omega}\xi, \\ R(\xi + j\omega) &= R(j\omega) + \left(\frac{dR}{dp}\right)_{j\omega}\xi, \end{aligned}\right\} \tag{7.66}$$

where the index $j\omega$ denotes that $j\omega$ is substituted for $\underline{p}$ in the expression for the derivatives.

For the equation (7.62), in the case of a single-valued odd-symmetry nonlinearity, we obtain

$$Q(\xi + j\omega) + R(\xi + j\omega)q(a) = 0, \tag{7.67}$$

and depending on the value of ξ we employ either Formula (7.65) or (7.66).

The complex equation (7.64) or (7.67) contains three unknowns: ξ, ω, and $\underline{a}$, while the latter enters in $\underline{q}$ and q'. Therefore this complex equation enables us to find two variables in terms of the third

$$\xi = \xi(a) \text{ and } \omega = \omega(a), \tag{7.68}$$

i.e., the variation of the damping exponent ξ and the frequency ω with change in amplitude $\underline{a}$ of a damped or divergent oscillatory process in

a nonlinear system.

After the functions (7.68) are determined, it is possible to use the two first-order differential equations (7.48) and determine a(t) and $\psi(t)$ for the first approximation of the sought solution of the nonlinear equation (7.60) in the form (7.47). The integrals of Eq. (7.48) have for specified initial conditions ($a = a_0$, $\psi = \psi_0$ with $t = 0$) the following forms:

$$\int_{a_0}^{a} \frac{da}{a\xi(a)} = t, \quad \psi = \int_0^t \omega(a)\,dt + \psi_0, \tag{7.69}$$

where $\xi(a)$ and $\omega(a)$ are the previously obtained functions (7.68). From the first equation of (7.69) we determine a(t), and from the second $\psi(t)$ after first substituting in it the value of a(t) from the first equation. As a result we obtain the solution

$$x = a(t)\sin\psi(t). \tag{7.70}$$

The operation of integration (7.69) is not needed in many cases in order to estimate the quality of the transient processes in automatic systems. In most cases it is perfectly sufficient to merely determine the functions (7.68) from the complex algebraic equation (7.64), since the quality of a symmetrical oscillatory transient can be completely described by means of the quantities ξ and ω and their ratio ξ/ω, and also by the character of their variation as a function of the oscillation amplitude and the system parameters. This will be discussed in detail in §7.3.

We must emphasize here once more that the quantities ξ and ω are assumed to be slowly varying functions of the time. Consequently, the roots $p = \xi \pm j\omega$ of the characteristic equation of the harmonically linearized system should also possess this property. This imposes definite limitations on the variation of the coefficients of the characteristic equation, which depend on the amplitude a through g and q'.

It is first necessary that $\underline{q}$ and q' be continuously dependent on $\underline{a}$, but more stringent conditions may also be imposed. Conditions of this kind limit the length of the time interval (or of the amplitude interval) over which the transient in the nonlinear system can be described with the aid of the quasilinear relations (7.47) and (7.48) with slowly varying ξ and ω, and also limits the degree of speed of attenuation of the oscillation amplitudes, i.e., the upper limit of the quantity $|\xi/\omega|$.

Another important fact is that the solution of a high-order differential equation is sought here in the form (7.70) with two arbitrary constants a_0 and ψ_0, whereas the number of constants should in general be $\underline{n}$ (equal to the total order of the system), i.e., the behavior of a system with several degrees of freedom is assumed to be close to the behavior of the system with one degree of freedom. This of course can occur only under certain conditions, when the system displays symmetrical clearly pronounced damped or diverging oscillations.

In linear automatic control systems and in linear servomechanisms, particularly electromechanical ones, this is frequently encountered. Usually for this purpose it is considered sufficient in linear theory to have the two roots of the characteristic equation closest to the imaginary axis complex, and the remaining roots can be arbitrary but located considerably farther to the left. In this case the initial conditions are likewise not immaterial. The required conditions are best satisfied if only the initial deviation difference from zero and the values of all the derivatives are zero (where the system is called upon to eliminate an initial error only).

We shall assume similar initial conditions also for nonlinear systems in the investigation of transients in the form of symmetrical damped oscillations, as shown in Fig. 7.3. Then the initial value of

the amplitude a_0 will coincide with the initial deviation, and the initial phase will be $\psi_0 = \pi/2$. However, such an approach to the solution of a problem, as will be made clear later on by examples, gives good results even for several other types of initial conditions.

§7.3. Diagrams of Damping Quality of Nonlinear Oscillations

The quality of symmetrical oscillatory transients in nonlinear automatic systems subject to investigation by the methods developed in §§7.1 and 7.2 are characterized in main outline by the values of the attenuation exponent ξ and the frequency ω as functions of the oscillation amplitude $\underline{a}$. It is most important in this case to be able to trace the variation of these quality exponents as the main parameters of the system are varied, so as to make the best choice of these parameters in the design of the automatic system (just as we previously determined the amplitude A and the frequency Ω of the self-oscillations as functions of the system parameters).

This purpose is attained by plotting so-called quality diagrams for the attenuation of the symmetrical nonlinear oscillations [208]. The diagram of Fig. 7.4 represents the family of lines ξ = const and the line ω = const in a plane with coordinates $\underline{k}$ and $\underline{a}$, where $\underline{k}$ denotes any one of the system parameters to be selected (gain or some other).

For a linear system, the lines ξ = const and ω = const, plotted in the same coordinate, would have the form of vertical lines, since the damping exponent and the frequency of the oscillatory transients in a linear system are independent of the amplitude of the oscillations $\underline{a}$ and vary only with variation of the system parameters (in this case, $\underline{k}$). On the other hand, in a nonlinear system, these lines become bent (Fig. 7.4) or are simply inclined, depending on the form of the nonlinearity and on the general structure of the system. This is manifest in the variation of the damping exponent ξ and the frequency ω of the

nonlinear oscillatory transients with change in the amplitude of the oscillation a.

The value $\xi = 0$ corresponds to the absence of damping, i.e., to an amplitude $a = A$ that is constant in time. For example, the point C (Fig. 7.4) corresponds to oscillations with constant amplitude A_c (self-oscillations). Therefore the line $\xi = 0$ represents on the damping quality diagram (Fig. 7.4) none other than the dependence of the self-oscillation amplitude A on the parameter k of the system, a dependence already determined in Chapters 2 and 4. On one side of this line lie the lines $\xi = \text{const} > 0$, and on the other the lines $\xi = \text{const} < 0$. The former correspond to diverging oscillations and the latter to damped ones.

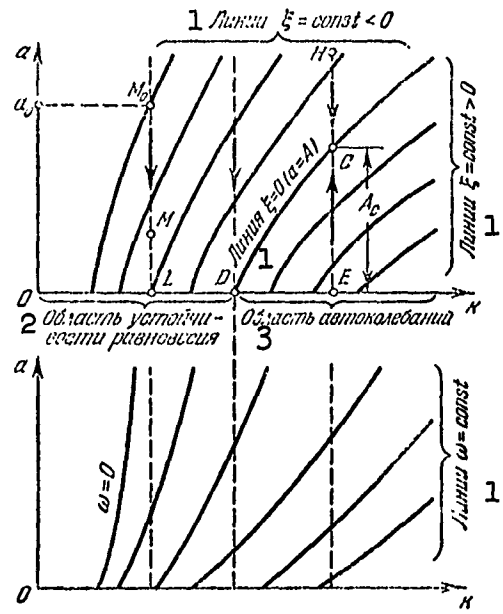


Fig. 7.4. 1) Lines; 2) stable equilibrium region; 3) self-oscillation region.

The course of the transient in time corresponds to motion of the representative point M in a vertical direction (since the amplitude a changes in the transient, and the gain k remains constant), as shown in Fig. 7.4 by the dashed lines and the arrows. For example, the value

of $\underline{k}$ at the point L corresponds to the vertical line M_0L. Inasmuch as this line crosses lines that have only negative values of ξ, the oscillations in the transient will be damped, i.e., the representative point M will move from a certain initial position M_0 (where the initial amplitude a_0 is specified) downward. The variation of the amplitude with time is shown in Fig. 7.5a. The variation of the frequency $\omega(a)$ is determined in this case in accordance with the corresponding vertical line on the lower Fig. 7.4.

In the case when the parameter $\underline{k}$ in the investigated system has the value corresponding to the point E (Fig. 7.4), two variants of the transient are obtained. If the initial position of the representative point lies below the point C ($a_0 < A_c$), then $\xi > 0$, i.e., the oscillations diverge and the representative point moves as shown by the arrow along the line EC, approaching asymptotically the point C. This corresponds to the time variation of the oscillation amplitude shown in Fig. 7.5b. On the other hand if $a_0 > A_c$, then $\xi < 0$ and the representative point follows the line HC downward (Fig. 7.4), corresponding to a damped transient (Fig. 7.5c) which approaches asymptotically self-oscillations with amplitude A_c.

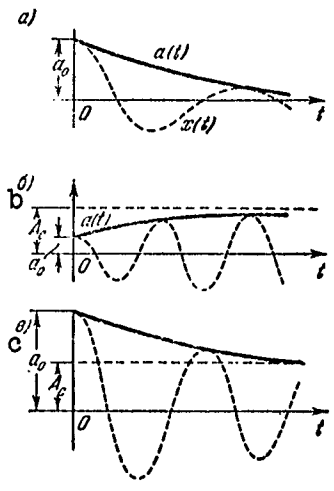


Fig. 7.5

Processes analogous to this will occur for any value of the parameter $\underline{k}$ to the right of the point D (Fig. 7.4). Consequently, the region of values of the parameter $\underline{k}$ lying to the right of the point D is the region where self-oscillations exist, to which the oscillatory transients converge from both sides (from below and from above). At the same time, the equilibrium position (any point $a = 0$ on the abscissa

axis) in the given region of values of the parameter k is unstable, since the oscillations in the transient diverge away from it, tending to another stable state, namely the self-oscillation mode.

To the left of the point D (Fig. 7.4) lie values of the parameter k for which the transient attenuates from any initial amplitude a_0 to zero. This is the stability region of the equilibrium state of the system. To the left of the line $\omega = 0$ (Fig. 7.4) lies usually a region of monotonic transients, which are not investigated by the harmonic linearization method developed here.

Thus, if the damping quality diagrams of nonlinear oscillations for different structural diagrams of any automatic system are plotted as functions of different parameters (k and others), they can serve as good material for the selection of the best system parameters during the course of the system design or synthesis [231].

We now turn to methods of plotting these diagrams [215].

First method. Let a nonlinear system of the first class be described by the high-order differential equation

$$Q(p)x + R(p)F(x, px) = 0. \qquad (7.71)$$

Carrying out harmonic linearization of the nonlinearity we obtain in accord with (7.52) the characteristic equation

$$Q(p) + R(p)\left(q + \frac{p-\xi}{\omega}q'\right) = 0. \qquad (7.72)$$

The substitution $p = \xi + j\omega$ yields the expression

$$Q(\xi + j\omega) + R(\xi + j\omega)(q + jq') = 0, \qquad (7.73)$$

where $Q(\xi + j\omega)$ and $R(\xi + j\omega)$ have in the general case, according to (7.65), the form of finite series in powers of $j\omega$ with coefficients that depend on ξ. The expresssions for q and q' contain the amplitude a and in addition may also contain the damping exponent ξ and the frequency ω.

Separating in (7.73) the real part X and the imaginary part Y, similar to what was done in §2.3, we obtain two equations:

$$\left.\begin{aligned} X(a, \omega, \xi) &= 0, \\ Y(a, \omega, \xi) &= 0; \end{aligned}\right\} \tag{7.74}$$

unlike in §2.3, these equations contain three unknowns. Solution of these equations with respect to ω and ξ makes it possible in principle to determine the functions $\omega(a)$ and $\xi(a)$. Actually, however, such a solution can be obtained in explicit form only in rare cases (essentially for second-order equations, such as (7.58) in the example with the Van der Pol equation, and for certain high-order systems with small ξ, such as Solution (1.102), when only the first power of ξ was taken into consideration).

In the general case it is advantageous for high-order systems to proceed in the following fashion. Assume that we are required to plot a damping quality diagram for nonlinear oscillations with respect to a certain system parameter k, which is contained in the coefficients of the equations (7.74). Obtaining from one of the equations of (7.74) the quantity

$$\omega = f_1(a, \xi, k) \tag{7.75}$$

and substituting into the second equation of (7.74), we get

$$k = f_2(a, \xi). \tag{7.76}$$

Then, assigning different constant values to ξ, we can readily plot with the aid of (7.76) a family of lines ξ = const on the quality diagram (Fig. 7.4). Then, using (7.75), we can plot also the family of lines ω = const.

Second method. The characteristic equation (7.72) of a system of the first class can be written in expanded form

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1} p + A_n = 0, \tag{7.77}$$

where all the coefficients A_1, A_2, ..., A_n or some of them are func-

tions of the unknown quantities $\underline{a}$, ω, and ξ (in the simplest problems, of $\underline{a}$ only). Let us expand the left half of (7.77) in two factors

$$p^n + A_1 p^{n-1} + \ldots + A_n = \\ = (p^{n-2} + C_1 p^{n-3} + \ldots + C_{n-2})(p^2 + B_1 p + B_2), \quad (7.78)$$

the second of which corresponds to the principal pair of complex roots $p_{1,2} = \xi \pm j\omega$, determining the oscillatory transient in the investigated system. We then obtain

$$\xi = -\frac{B_1}{2}, \quad \omega^2 = B_2 - \xi^2. \quad (7.79)$$

The first factor of (7.78) should have roots with much greater absolute values than the second, so that the oscillatory solution corresponding to the sought roots $p_{1,2}$ under the assumed initial conditions ($x = x_0$, $px = p^2x = \ldots = 0$ with $t = 0$) be the principal one.

The expansion coefficients (7.78) are related by the following equations:

$$A_1 = C_1 + B_1, \quad A_2 = B_2 + C_2 + B_1 C_1, \ \ldots, \quad A_n = C_{n-2} B_2.$$

To find the values of ξ and ω it is necessary obviously to express the coefficients B_1 and B_2 in (7.79) in terms of the coefficients of the initial equation (7.77).

In particular, for the third-degree characteristic equation

$$p^3 + A_1 p^2 + A_2 p + A_3 = (p + C_1)(p^2 + B_1 p + B_2)$$

we have

$$A_1 = C_1 + B_1, \quad A_2 = B_2 + B_1 C_1, \quad A_3 = C_1 B_2. \quad (7.80)$$

In order for the values of ξ and ω (7.79) to define the principal part of the solution, and in order to be able to disregard the third root of the equation, we must have

$$C_1 \geqslant \left|\frac{B_1}{2}\right| \quad \text{or} \quad A_1 \geqslant |\xi|, \quad (7.81)$$

which determines the upper limit for the values of $|\xi|$, which must be chosen in plotting the damping quality diagram of the nonlinear oscil-

lations.

Let us set up the penultimate Hurwitz determinant

$$H_{n-1} = A_1A_2 - A_3 = (C_1 + B_1)(B_2 + B_1C_1) - C_1B_2 = \\ = B_1(B_2 + C_1^2 + C_1B_1).$$

But since it follows from (7.80) and (7.79) that $B_2 + C_1B_1 = A_2$, $C_1^2 = (A_1 - B_1)^2$, $B_1 = -2\xi$, the expression obtained above can be rewritten as

$$\xi = -\frac{H_{n-1}}{2[A_2 + (A_1 + 2\xi)^2]} = -\frac{A_1A_2 - A_3}{2[A_2 + (A_1 + 2\xi)^2]}. \qquad (7.82)$$

Further, since it follows from (7.80) that

$$B_2 = \frac{A_3}{C_1} = \frac{A_3}{A_1 - B_1},$$

we obtain from (7.79) the following formula for the square of the frequency

$$\omega^2 = \frac{A_3}{A_1 + 2\xi} - \xi^2. \qquad (7.83)$$

Formulas (7.82) and (7.83) enable us to plot the damping quality diagrams for the nonlinear oscillatory transients of third-order systems.

Analogously, for a fourth-order system we obtain

$$p^4 + A_1p^3 + A_2p^2 + A_3p + A_4 = \\ = (p^2 + C_1p + C_2)(p^2 + B_1p + B_2), \qquad (7.84)$$

with

$$\left.\begin{aligned} A_1 &= C_1 + B_1, & A_2 &= C_2 + B_2 + C_1B_1, \\ A_3 &= C_1B_2 + B_1C_2, & A_4 &= C_2B_2. \end{aligned}\right\} \qquad (7.85)$$

Here we must satisfy the same condition (7.81).

Starting from the expression for the penultimate Hurwitz determinant we obtain by the same method the formulas

$$\xi = -\frac{H_{n-1}}{2(A_1 + 2\xi)\{[A_2 + (A_1 + 2\xi)2\xi]^2 - 4A_4 + A_1A_3\}}, \qquad (7.86)$$

where

$$H_{n-1} = A_3(A_1A_2 - A_3) - A_1^2A_4,$$

and then

$$\omega^2 = \frac{A_1(A_1 + 4\xi)}{(A_1 + 2\xi)(A_2 + 2\xi) - A_3} - \xi^2. \qquad (7.87)$$

Formulas (7.86) and (7.87) enable us to plot quality diagrams for fourth-order nonlinear systems. The use of these formulas will be illustrated by means of examples.

Third method. Whereas in the preceding two methods we spoke of determining the symmetrical oscillatory transients in arbitrary nonlinear systems of the first class, the third method will be given for a simpler but frequently encountered particular case, when the coefficients of harmonic linearization q and q' depend only on the amplitude a and are independent of the frequency ω and the damping exponent ξ, something that occurs for nonlinearities of the type F(x). In this case, after substituting into the characteristic equation $p = \xi + j\omega$ we can rewrite (7.73) in the form

$$W_л(\xi + j\omega) = -\frac{1}{W_н(a)}, \qquad (7.88)$$

where we put

$$W_н(a) = q(a) + jq'(a), \quad W_л(\xi + j\omega) = \frac{R(\xi + j\omega)}{Q(\xi + j\omega)}, \qquad (7.89)$$

with the numerator and denominator of the last expression representing, in accordance with (7.65), polynomials in powers of $j\omega$ with coefficients that depend on ξ.

By specifying various constant values of ξ, we plot a series of curves $W_л(\xi + j\omega)$ as functions of $j\omega$ with ξ = const (Fig. 7.6), similar to the ordinary plotting of amplitude-phase characteristics for the linear part of the system. On the same plot (Fig. 7.6) we draw the line $-1/W_н(a)$. The points where it crosses the lines $W_л(\xi + j\omega)$ determine the solution of Eq. (7.88), namely: for each value of ξ we obtain at these intersection points the corresponding values of a (as given

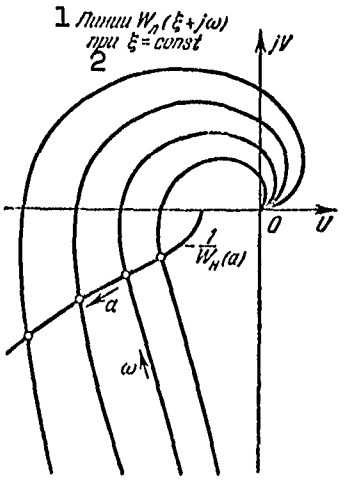


Fig. 7.6. 1) Lines; 2) with.

by the curve of $-1/W_n(a)$) and ω (as given by the curve of $W_1(\xi + j\omega)$). By the same token we determine the quality of the oscillatory transient for all specified parameters of the system, i.e., we determine the points of one vertical line on the quality diagram (Fig. 7.4). By repeating these constructions (Fig. 7.6) for different values of the chosen system parameter k, we can plot the entire quality diagram (Fig. 7.4).

Fourth method. Assume that we have a nonlinear system of the second class (see §1.2), described, for example, by the equations

$$Q_1(p)\,x_1 = -R_1(p)\,x_2, \tag{7.90}$$

$$Q_2(p)\,x_2 = R_2(p)\,x_1 + R_3(p)\,F(x_1,\ px_1,\ x_3), \tag{7.91}$$

in which there are two variables x_1 and x_2 under the nonlinearity sign (their derivatives may also be included), whereas in first-class systems only one variable was involved (in general with its derivative).

The two variables x_1 and x_2 are related here by the linear equation (7.90). Therefore if a solution is found for the first of these in the form

$$x_1 = a\sin\psi, \tag{7.92}$$

we have for the second

$$x_2 = a_2\sin(\psi + \varphi), \tag{7.93}$$

where the quantity φ and the ratio of a_2 to a are completely determined by the coupling equation (7.90). Let us find this relation, recalling that we are considering unsteady oscillations in a transient, when $a = a(t)$, and according to (7.48):

$$\frac{da}{dt} = a\,\xi(t), \quad \frac{d\psi}{dt} = \omega(t). \tag{7.94}$$

We use the symbolic notation*

$$x_1 = ae^{j\psi}, \quad x_2 = a_2 e^{j(\psi+\varphi)}. \tag{7.95}$$

From (7.95) we obtain with account of (7.94)

$$px_1 = \frac{da}{dt}e^{j\psi} + ja\frac{d\psi}{dt}e^{j\psi} = ae^{j\psi}(\xi + j\omega),$$

$$p^2x_1 = \frac{da}{dt}e^{j\psi}(\xi + j\omega) + ja\frac{d\psi}{dt}e^{j\psi}(\xi + j\omega) + ae^{j\psi}\left(\frac{d\xi}{dt} + j\frac{d\omega}{dt}\right) =$$

$$= ae^{j\psi}\left[(\xi + j\omega)^2 + \frac{d\xi}{dt} + j\frac{d\omega}{dt}\right],$$

$$p^3x_1 = ae^{j\psi}\left[(\xi + j\omega)^3 + 3(\xi + j\omega)\left(\frac{d\xi}{dt} + j\frac{d\omega}{dt}\right) + \frac{d^2\xi}{dt^2} + j\frac{d^2\omega}{dt^2}\right]$$

etc. But as can be seen from §§7.1 and 7.2, we consider here such nonlinear oscillatory transients, in which ξ and ω are slowly varying functions. Therefore their derivatives can be neglected and we can write

$$\begin{aligned} px_1 &= (\xi + j\omega)ae^{j\psi}, & px_2 &= (\xi + j\omega)a_2e^{j(\psi+\varphi)}, \\ p^2x_1 &= (\xi + j\omega)^2ae^{j\psi}, & p^2x_2 &= (\xi + j\omega)^2a_2e^{j(\psi+\varphi)}, \\ p^3x_1 &= (\xi + j\omega)^3ae^{j\psi}, & p^3x_2 &= (\xi + j\omega)^3a_2e^{j(\psi+\varphi)}, \\ &\cdots\cdots\cdots & &\cdots\cdots\cdots \end{aligned}$$

Substituting this in (7.90) we get

$$Q_1(\xi + j\omega)ae^{j\psi} = -R_1(\xi + j\omega)a_2e^{j(\psi+\varphi)},$$

whence

$$a_2 = a\left|\frac{Q_1(\xi + j\omega)}{R_1(\xi + j\omega)}\right|, \tag{7.96}$$

$$\varphi = \arg\frac{-Q_1(\xi + j\omega)}{R_1(\xi + j\omega)}. \tag{7.97}$$

We further obtain from (7.93) and (7.92)

$$x_2 = a_2\cos\varphi\sin\psi + a_2\sin\varphi\cos\psi$$

or, using (7.96) and (7.97)

$$x_2 = U_1(\xi, \omega)a\sin\psi + V_1(\xi, \omega)a\cos\psi, \tag{7.98}$$

where

$$U_1(\xi, \omega) = \mathrm{Re}\frac{-Q_1(\xi + j\omega)}{R_1(\xi + j\omega)}, \quad V_1(\xi, \omega) = \mathrm{Im}\frac{-Q_1(\xi + j\omega)}{R_1(\xi + j\omega)}. \tag{7.99}$$

We shall substitute (7.98) in lieu of x_2 under the nonlinearity sign in (7.91) during the course of its harmonic linearization. Thus,

in analogy with (7.52), we obtain here

$$F(x_1, px_1, x_2) = qx_1 + q'\frac{p-\xi}{\omega}x_1, \tag{7.100}$$

where

$$\left.\begin{aligned} q &= \frac{1}{\pi a}\int_0^{2\pi} F(x_1\ px_1,\ x_2)\sin\psi\, d\psi, \\ q' &= \frac{1}{\pi a}\int_0^{2\pi} F(x_1, px_1, x_2)\cos\psi\, d\psi. \end{aligned}\right\} \tag{7.101}$$

In these expressions $x_1 = a \sin \psi$, $px_1 = a\omega \cos \psi + a\xi \sin \psi$, and in place of x_2 we substitute (7.98).

As a result, the characteristic equation of the system (7.90) and (7.91) assumes the form

$$Q_1(p)Q_2(p) + R_1(p)R_2(p) + R_1(p)R_3(p)\left(q + \frac{p-\xi}{\omega}q'\right) = 0.$$

We see that for nonlinear systems of the second class we obtain after harmonic linearization a characteristic equation of the same form (7.72) as for systems of the first class, with

$$Q(p) = Q_1(p)Q_2(p) + R_1(p)R_2(p), \quad R(p) = R_1(p)R_3(p).$$

Consequently, we can use from now on any of the first two methods described above for the plotting of the quality diagrams.

Fifth method. For the investigation of nonlinear systems of the third class, i.e., systems in which linearities of different variables are present simultaneously and are interrelated by nonlinear equations (see §1.2), it is also necessary to make use of a relation of the type (7.96) between the amplitudes of oscillations of different variables. The only difference here is that Expression (7.96) will contain not only linear but also nonlinear terms in the harmonically linearized form. For example, if a system is described by the equations

$$\left.\begin{aligned} Q_1(p)x_2 &= R_1(p)F_1(x_1, px_1), \\ Q_2(p)x_1 &= -R_2(p)F_2(x_2, px_2), \end{aligned}\right\} \tag{7.102}$$

then by assuming that

$$x_1 = a \sin \psi, \quad x_2 = a_2 \sin (\psi + \varphi), \tag{7.103}$$

we carry out harmonic linearization of each nonlinearity separately:

$$\left.\begin{aligned} F_1(x_1, px_1) &= \left(q_1 + q_1' \frac{p+\xi}{\omega}\right) x_1, \\ F_2(x_2, px_2) &= \left(q_2 + q_2' \frac{p-\xi}{\omega}\right) x_2, \end{aligned}\right\} \tag{7.104}$$

where q_1, q'_1, q_2, and q'_2 are calculated by means of Formulas (7.53) except that $\underline{a}$ is replaced by a_2 in the second case. The quantity a_2 which enters into the second formula of (7.104) must be replaced by $\underline{a}$. For this purpose, in analogy with (7.96), we write on the basis of the first equation of (7.102):

$$a_2 = a \left| \frac{R_1(\xi + j\omega)}{Q_1(\xi + j\omega)} \right| \sqrt{q_1^2 + (q_1')^2}. \tag{7.105}$$

Following this, the characteristic equation of the harmonically linearized system (7.102)

$$Q_1(p) Q_2(p) + R_1(p) R_2(p) \left(q_1 + q_1' \frac{p-\xi}{\omega}\right) \left(q_2 + q_2' \frac{p-\xi}{\omega}\right) = 0$$

can be investigated by any of the first two methods developed above. In this case it is essential that all the conditions indicated at the start of §2.3 be fulfilled for each equation of (7.102) separately.

We note that if the values of the damping exponent are sufficiently small, we can use in all of the foregoing methods the simpler expressions (7.66) in lieu of (7.65).

§7.4. Use of Diagrams to Determine Transient Quality

The damping quality diagrams of nonlinear oscillations always give a very illustrative idea of the influence of some particular nonlinearity on the transient quality in automatic systems, provided these transients are close to being symmetrically damped or divergent oscillations, represented by sufficiently smooth curves (the estimate of the quality of other types of transients will be discussed in subsequent sections).

The form of the nonlinearity and also the general structure of the system greatly influence the location of the lines $\xi = \text{const}$ and $\omega = \text{const}$ on the diagram. For example, if the nonlinearity in an automatic system has a backlash zone of width 2b, then the quality diagram can assume in place of Fig. 7.4 the form of Fig. 7.7. In this case the amplitude in the stability region decreases no longer to zero but to a value $a = b$ with subsequent stoppage of the process at any point within the backlash zone ($|x| < b$).

In a system with a nonlinearity containing a linear portion near the origin with width 2b (Fig. 7.8), the lines $\xi = \text{const}$ and $\omega = \text{const}$ will have at first (when $a \leq b$) the form of vertical straight lines, as for any other linear system, and then ($a > b$) they will become curved as a result of the nonlinearity (in this case, saturation). Figure 7.8 shows one possible version of the damping quality diagram of nonlinear oscillations, when the stability limit of the nonlinear system (point D) coincides with the stability limit of the corresponding linear system, but other cases are also possible.

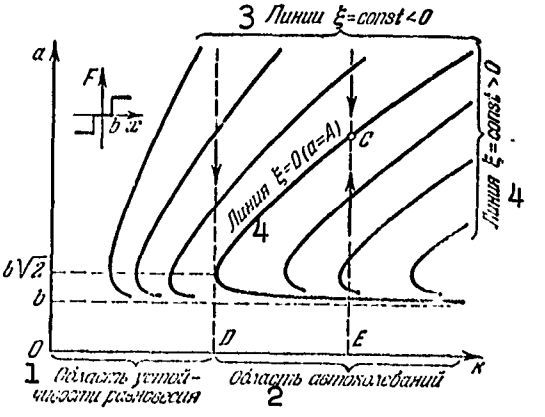


Fig. 7.7. 1) Stable equilibrium region; 2) self-oscillation region; 3) lines; 4) line.

Although in this case the stability limits coincide, the behavior of the nonlinear system on the stability differs appreciably from the

behavior of the linear system. In fact, oscillations with constant amplitude, which occur on the stability limit in the linear system, will be observed here, in accordance with Fig. 7.8, only at initial amplitudes $a_0 \leq b$. At greater initial amplitudes ($a_0 > b$), in accordance with Fig. 7.8, a damped oscillatory process occurs in the given nonlinear system on the boundary of its stability; after termination of this process, oscillations set in with constant amplitude $a = b$, i.e., on the upper limit of the linear portion of the nonlinear characteristic. In other examples, the behavior of the system on the stability limit may be different. In the case of Fig. 7.7 the transient causes oscillations with amplitude $a = b\sqrt{2}$ to set in on the stability limit of the nonlinear system, while in the case of Fig. 7.4 damping of the transient to zero occurs on the stability limit of the system.

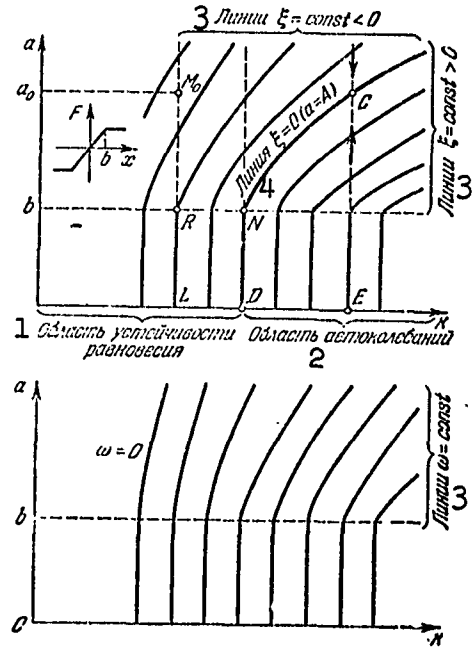


Fig. 7.8. 1) Stable equilibrium region; 2) self-oscillation region; 3) lines; 4) line.

We see that in all cases the behavior of the nonlinear system on the stability limit differs in principle from the behavior of the lin-

ear system on the oscillatory stability limit, where oscillations with constant amplitude that depends on the initial conditions always arise. Thus, in a linear system the stability limit is always dangerous, for the smallest departure beyond it gives rise to the development of oscillations that diverge without limit. On the other hand in nonlinear systems, safe stability limits can occur. A clear-cut example of this is the diagram of Fig. 7.4, where oscillations with large amplitude attenuate not only in the stability region and on its boundary (point D), but also when this boundary is crossed, with safe oscillations of low amplitude established to the right of the point D near this point.

The diagram of Fig. 7.8 can be used to estimate the influence of nonlinearities of the saturation type on the transients, as compared with the processes in the linear system. Assume, for example, that the value of the parameter $\underline{k}$ in this system corresponds to the point L. In the linear system we would have at all times one and the same damping exponent ξ, which corresponds to the line LR. In the nonlinear system, however, owing to saturation, the line $\xi = \text{const}$ bends to the right above the point R. Because of this the values of ξ on the vertical M_0R will have absolute magnitudes that are larger. Consequently, the transient oscillations in the given linear system will be attenuated more rapidly at large deviations than in the linear system, owing to saturation (at small amplitudes, $a < b$, the damping will be the same). In other systems, however, saturation may have an entirely different effect, something that will always be seen from the damping quality diagrams plotted for these systems.

On the other hand, in the instability region of the linear system (for example, at the point E), the oscillations in the linear system will diverge to infinity with a constant exponent ξ, corresponding to the point E. In the nonlinear system, because of the curvature of the

lines ξ = const, the value ξ along the vertical will gradually decrease to zero at the point C, where steady-state self-oscillations with definite amplitudes are established. If these amplitudes are large, then in practice this is also equivalent to instability, and if such an amplitude is tolerable, then we can say that the saturation type nonlinearity extends the possible working limits of the variation of the parameter $\underline{k}$.

The foregoing influence of saturation type nonlinearity can be used in practice to generate stable oscillations with specified amplitude and frequency, something already illustrated in the example of §4.16.

Thus, the quality diagrams (Figs. 7.4, 7.7, and 7.8) give a graphic idea of the character of the symmetrical oscillatory processes, on their rate of damping, or on the time necessary to establish self-oscillations, and also on the oscillation frequency and on the character of its time variation. All these qualities are estimated on the diagrams for arbitrary ranges of system parameters, in which the symmetrical oscillatory transient is observed.

Quantitatively, the diagrams yield the dependence of the damping exponent ξ and of the frequency ω on the amplitude:

$$\xi(a) \text{ and } \omega(a) \qquad (7.106)$$

for any specified set of values of the system parameters (any vertical line on Figs. 7.4, 7.7, and 7.8). This gives an idea of the form of the transient (Fig. 7.5). If a more specific quantitative idea is needed, then the data of (7.106) can be readily used to plot these transient curves by means of the method that will be described below [208].

Let us introduce the running "time constant" of the oscillatory transient

$$T(a) = -\frac{1}{\xi(a)}. \qquad (7.107)$$

In the linear system, T would indeed be the time constant of the exponential curve according to which the amplitude of the oscillations decreases. Here, however, this "time constant" changes slowly with changing amplitude, in accordance with the known variation of $\xi(a)$, defined by Formula (7.106).

The first equation in (7.48) assumes with the notation of (7.107) the form

$$\frac{da}{dt} = -\frac{a}{T(a)}. \qquad (7.108)$$

Therefore, if the initial value of the amplitude is specified ($a = a_0$ when $t = 0$), then by laying off the value of $T(a_0)$ on the time axis (Fig. 7.9) and drawing the line CB, we see that according to (7.108) this line is tangent to the sought a(t) curve at the initial point $t = 0$.

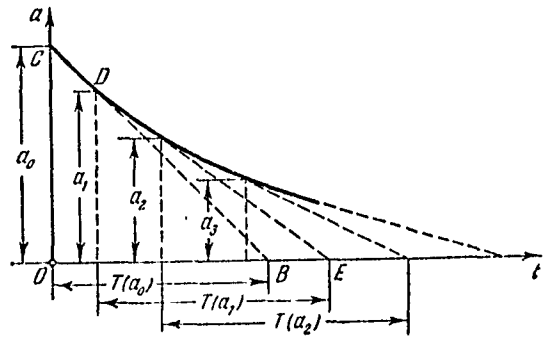


Fig. 7.9

We shall assume that on some segment (up to a certain value of the amplitude a_1) this straight line practically coincides with the sought curve a(t). We shall then use (7.107) to calculate the new value of $T(a_1)$, laid off on the abscissa axis (Fig. 7.9), and draw a new line DE, assuming it to coincide with the sought curve a(t) on the segment up to a certain new point $a = a_2$. We shall continue the plotting in

the same manner. We plot analogously also a diverging process with increasing amplitudes, shown in Fig. 7.5b.

The choice of the length of the segments during each step of the construction is arbitrary. As a result we obtain an approximate a(t) curve, i.e., the curve showing the variation of the oscillation amplitude in the transient. A more accurate construction is not needed, because the entire method as a whole is generally approximate and is aimed only at a preliminary choice of the structure and parameters of the system based on approximate estimates of the transient quality. In some of the simplest cases, the approximate a(t) curve can be obtained analytically by integration, as was demonstrated in §7.2 with a second-order equation (7.57) of the Van der Pol type as an example, and also with an example of a third-order system with small ξ, given at the end of §1.5.

It is important to note that the end of the construction (Fig. 7.9) is determined not only by the very course of this construction, but is monitored in addition by the fact that we know beforehand from the quality diagram (Figs. 7.4, 7.7, and 7.8) that this process tends to: a zero equilibrium point, a definite prescribed steady-state self-oscillation amplitude ($a = A$), or the limit of the backlash zone, etc.

Having obtained the curve showing the amplitude variation a(t) (i.e., the envelope of the oscillations), and knowing from the quality diagram the frequency $\omega(a)$, we can approximately plot the oscillation curve x(t) itself. For this purpose it is necessary first of all to recalculate the dependence $\omega(a)$, taken from the quality diagram, into a dependence $\omega(t)$ with the aid of the already determined (Fig. 7.9) variation of a(t).

Using the curve of $\omega(t)$ obtained (Fig. 7.10a), we determine the characteristic points $\psi = i(\pi/2)$ ($i = 1, 2, \ldots$) on the time axis. Be-

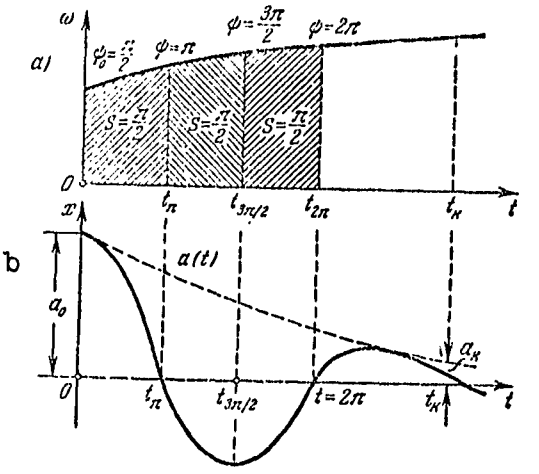


Fig. 7.10

sides we select, as already mentioned in §7.2, fully defined initial conditions for the transients

$$x = x_0, \quad px = p^2x = \ldots = 0 \text{ for } t = 0. \tag{7.109}$$

Then the initial conditions for the assumed form of the solution x = = a sin ψ will obviously be

$$a_0 = x_0, \quad \psi_0 = \frac{\pi}{2} \text{ for } t = 0. \tag{7.110}$$

We furthermore have

$$\psi = \psi_0 + \int_0^t \omega(t)\,dt. \tag{7.111}$$

Consequently, the quantity ψ must be determined as the area under the curve $\omega(t)$ (Fig. 7.10a) to which we add the quantity $\psi_0 = \pi/2$. Therefore the point $\psi = \pi$ and the time $t = t_\pi$ corresponding to it is determined as the point at which the area is $S = \pi/2$ (Fig. 7.10a). The next point, $\psi = 3\pi/2$, and the corresponding time $t_{3\pi/2}$ is determined by the new area increment of the same magnitude, $S = \pi/2$, etc.

We note that in the linear system ω = const, and therefore ψ = = $\psi_0 + \omega t$ and the points t_π, $t_{3\pi/2}$, ... are uniformly distributed over the time axis. In a nonlinear system, on the other hand, they will gen-

erally be unevenly distributed, and will be closer to one another with increasing frequency ω.

Let us transfer the resultant characteristic time points on the t axis of the sought x(t) curve of the transient process (Fig. 7.10b). Inasmuch as $x = a \sin \psi$, we have

$$x=0 \quad \text{for } \psi=\pi, 2\pi, \ldots,$$
$$x=a \quad \text{for } \psi=\frac{\pi}{2}, \frac{5\pi}{2}, \ldots,$$
$$x=-a \quad \text{for } \psi=\frac{3\pi}{2}, \frac{7\pi}{2}, \ldots.$$

This enables us to plot approximately the oscillation curve x(t) for the transient (Fig. 7.10b) with accuracy sufficient for a preliminary estimate of the quality of the process in the system during the course of its design.

From such a sufficiently simple tentative calculation we can obtain the following qualitative indices:

a) the damping time of the transient t_k on a specified interval of amplitude variation, from a certain initial value a_0 to an arbitrarily chosen a_k, will be

$$t_k = t \quad \text{for } a=a_k; \tag{7.112}$$

b) the value of the overshoot $|x_p|$ is

$$|x_n| = a \text{ for } t=t_{\frac{3\pi}{2}}; \tag{7.113}$$

c) the number of oscillations m on the investigated portion from a_0 to a_k is:

$$m=\frac{\psi_k-\psi_0}{2\pi}, \quad \psi_0=\frac{\pi}{2}, \tag{7.114}$$

where ψ_k is the value of the entire area under the $\omega(t)$ curve for $0 \leq t \leq t_k$.

Although the described construction of the oscillation curve during the transient is not complicated, it is nevertheless desirable to be able to obtain a quantitative estimate of the transient quality in-

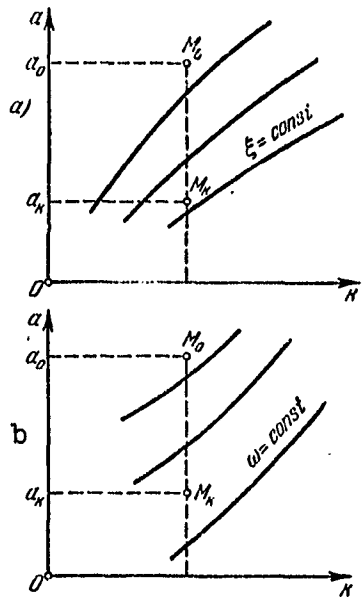


Fig. 7.11

dices directly from the quality diagram, without plotting the process itself.

The damping time of the transient on the specified interval of amplitude variation from a_0 to a_k will, in accordance with (7.69), be

$$t_k = \int_{a_0}^{a_k} \frac{da}{a\xi(a)}, \qquad (7.115)$$

on the basis of which we can make the rough estimate

$$t_k \approx \frac{1}{\xi_c} \ln \frac{a_k}{a_0}, \qquad (7.116)$$

where ξ_s is the average value of the damping exponent ξ, taken tentatively from the quality diagram for the investigated section M_0M_k (Fig. 7.11a).

For a more reliable determination of the damping time t_k it is necessary to break down the section M_0M_k into several (n) segments and to calculate

$$t_k = \sum_{i=1}^{n} \frac{1}{\xi_i} \ln \frac{a_{ki}}{a_{0i}}, \qquad (7.117)$$

where ξ_i is the value of ξ at the center of each segment into which the line M_0M_k is broken; this value is taken from the diagram (Fig. 7.11a); a_{0i} and a_{ki} are the values of the ordinate $\underline{a}$ at the start and at the end of each segment.

To determine the overshoot it is necessary to find the value of the amplitude $\underline{a}$ at $\psi = 3\pi/2$. From (7.48) we have

$$\psi = \psi_0 + \int_{a_0}^{a} \frac{\omega(a)}{a\xi(a)} da. \qquad (7.118)$$

When $\psi = 3\pi/2$ and $\psi_0 = \pi/2$ we arrive at the equation

$$\int_{a_0}^{a} \frac{\omega(a)}{a\xi(a)}\, da = \pi, \tag{7.119}$$

from which we must find $\underline{a}$, which will indeed be the value of the overshoot $|x_p|$. By way of a rough estimate we can write in place of (7.119)

$$\frac{\omega_c}{\xi_c} \ln \frac{a}{a_0} \approx \pi,$$

from which, recognizing that $|x_0| = a_0$, we obtain an estimate for the overshoot in the form

$$\left|\frac{x_n}{x_0}\right| \approx e^{\pi \frac{\xi_c}{\omega_c}} \qquad (\xi_c < 0), \tag{7.120}$$

where ξ_s and ω_s are the average values of ξ and ω, taken from the diagram (Fig. 7.11) for the investigated section M_0M_k.

For a more reliable estimate of the overshoot we must break up the section M_0M_k into several segments and calculate in accordance with (7.118)

$$\psi - \psi_0 = \sum \frac{\omega_i}{\xi_i} \ln \frac{a_{ik}}{a_{i0}}, \tag{7.121}$$

where ω_i and ξ_i are the values of ω and ξ at the center of each segment into which the line M_0M_k is broken up (Fig. 7.11), while a_{i0} and a_{ik} are the values of the ordinate $\underline{a}$ at the start and at the end of each segment. Calculations by means of Formula (7.121) must be carried out, starting with the point M_0, until the sum amounts to $\psi - \psi_0 = \pi$. In this case the overshoot will be

$$|x_p| = a_{ik} \quad (\text{for } \psi - \psi_0 = \pi), \tag{7.122}$$

where a_{ik} is the ordinate of the end of the last of the segments employed in this calculation.

Finally, in order to determine the number of oscillations $\underline{m}$ during the transient time it is necessary, in accordance with (7.114), to carry through the calculations on the basis of Formula (7.121) to conclusion, i.e., to determine

$$m = \frac{1}{2\pi} \sum_{i=1}^{n} \frac{\omega_i}{\xi_i} \ln \frac{a_{ik}}{a_{i0}}. \qquad (7.123)$$

For a rough estimate we can put

$$m \approx \frac{\omega_c}{2\pi\xi_c} \ln \frac{a_k}{a_0}. \qquad (7.124)$$

If, for example, it is required that not more than one oscillation occur on the investigated segment M_0M_k, we must stipulate that

$$\frac{\xi_c}{\omega_c} = \frac{1}{2\pi} \ln \frac{a_k}{a_0}.$$

On the other hand, if the maximum permissible overshoot $|x_p|$ is specified, then, in accordance with (7.120), we should have

$$\frac{\xi_c}{\omega_c} = \frac{1}{\pi} \ln \left| \frac{x_n}{x_0} \right|.$$

Thus, the foregoing construction of the damping quality diagrams for symmetrical nonlinear oscillatory transients is an important tool for the approximate investigation of nonlinear automatic systems.

The quality diagrams (Figs. 7.4, 7.7, and 7.8) are plotted for the variable $\underline{x}$, which is the nonlinearity argument in the system equation. If it is desirable to recalculate it for any other variable, it is possible to employ a formula of the type of (7.96) to determine the amplitude of the new variable, using the corresponding transfer function relating these variables in the investigated system.

Here, as in general in all cases when harmonic linearization is used, we are considering oscillatory transients, i.e., transients in which there is at least one overshoot under the initial conditions (7.109). As regards the investigation of monotonic transients in nonlinear systems, other methods must be used for this purpose. Monotonic processes are best investigated by ordinary numerical-graphical methods. On the other hand, the region of system parameters where monotonic transients occur, can be tentatively determined with the aid of our

diagrams (to the left of the line $\omega = 0$ on Figs. 7.4, 7.6, and 7.8).

We note, however, that monotonic transients in a nonlinear system, resulting from vibration smoothing, and their associated slipping transients, can be investigated by the methods considered in the following sections.

§7.5. Asymmetrical Oscillatory Transients

The symmetrical oscillatory transients investigated above can occur far from always in high-order systems although, as examples indicate (§§7.6 and 7.7) they are encountered frequently. In the general case, in low-order systems, asymmetrical transients were observed, among which we shall consider those having an oscillatory component, including transients in systems with vibration smoothing and slipping processes.

An asymmetrical oscillatory transient can have the form of Fig. 7.12a, b, or c. Assume that in a linear system of the first class the transient is described by a high-order differential equation

$$Q(p)x + R(p)F(x, px) = 0. \qquad (7.125)$$

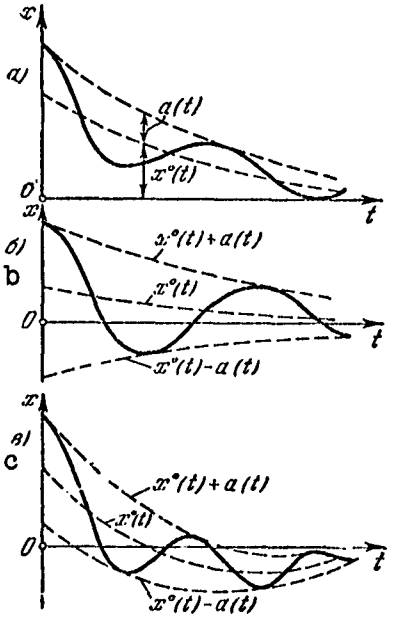


Fig. 7.12

We shall seek the solution in the form (see Fig. 7.12):

$$x = x^0 + x^*, \quad x^* = a\sin\psi, \qquad (7.126)$$

where x^0, a, and $\omega = d\psi/dt$ are the unknown functions of the time.

In finding this solution we shall use the same method as was used in Chapter 5 in the investigation of the flow of a slowly varying signal through a self-oscillating system. Here, however, we deal not with self-oscillations resulting from a slowly varying external signal, but with a tran-

sient described by homogeneous equation and consisting of a damped (or diverging) oscillatory component x* and an aperiodic component x^0, and this transient furthermore can tend either to a steady-state x = 0 or x = const, or else to a steady self-oscillation mode a = const.

Let us consider two versions of the method of solving the problem:

1) the amplitude a of the damped (or diverging) oscillations in the transient process is assumed to be slowly varying in time, together with the aperiodic component x^0, so that the damping exponent ξ is not introduced at all;

2) the value of the damping exponent ξ is introduced when the problem cannot be solved in the first version (the simplest examples of both cases were considered in §1.8).

In the first version of the method, in analogy with §5.1 we carry out, taking (7.126) into account, harmonic linearization of the nonlinearity in the form

$$F(x,\ px) = F^0 + qx^* + \frac{q'}{\omega} px^*, \tag{7.127}$$

where

$$F^0 = \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + a\sin\psi,\ a\omega\cos\psi)\,d\psi, \tag{7.128}$$

$$\left.\begin{aligned} q &= \frac{1}{\pi a}\int_0^{2\pi} F(x^0 + a\sin\psi,\ a\omega\cos\psi)\sin\psi\,d\psi, \\ q' &= \frac{1}{\pi a}\int_0^{2\pi} F^0(x^0 + a\sin\psi,\ a\omega\cos\psi)\cos\psi\,d\psi. \end{aligned}\right\} \tag{7.129}$$

Ready-made expressions for F^0, q, and q' in different standard nonlinearities are given in Chapter 5 (§§5.6-5.9).

After substitution of (7.127), the differential equation (7.125) of the transient breaks up into two, for the aperiodic and the oscillatory components, respectively:

$$Q(p)\,x^0 + R(p)\,F^0 = 0, \tag{7.130}$$

$$Q(p)x^* + R(p)\left(q + \frac{q'}{\omega}p\right)x^* = 0, \qquad (7.131)$$

These equations are interrelated, since in the general case, according to (7.128) and (7.129), all three quantities F^0, $\underline{q}$, and q' depend on all three unknowns x^0, $\underline{a}$, and ω.

For Eq. (7.131) we write down the characteristic equation

$$Q(p) + R\left(q + \frac{q'}{\omega}p\right) = 0, \qquad (7.132)$$

where $\underline{q}$ and q' depend always on $\underline{a}$ and x^0, and sometimes also on ω. Therefore, solving this equation by any of the methods of §2.3 (with the substitution $p = j\omega$) we obtain the relationships

$$a(x^0) \text{ and } \omega(x^0), \qquad (7.133)$$

i.e., the dependence of the amplitude and frequency of the oscillatory component of the sought transient on the ordinate of its aperiodic component x^0 (Fig. 7.13a).

Substituting the obtained functions $a(x^0)$ and $\omega(x^0)$ into Formula (7.128) we obtain the so-called bias function (Fig. 7.13b):

$$F^0 = \Phi(x^0), \qquad (7.134)$$

which is the smoothed characteristic of the nonlinear element for the aperiodic component of the process. As a result the differential equation for the aperiodic component (7.130) will contain one unknown variable, namely

$$Q(p)x^0 + R(p)\Phi(x^0) = 0, \qquad (7.135)$$

Solving this equation we obtain the aperiodic component $x^0(t)$ of the transient process $x(t)$.

However, as soon as this component $x^0(t)$ is determined, we determine thereby according to (7.133) the functions $a(t)$ and $\omega(t)$ for the oscillatory component $x^*(t)$ of the transient. Consequently, we determine the entire solution $x(t)$ for the transient (7.126), where

$$\psi = \psi_0 + \int_0^t \omega(t)\, dt.$$

It is only necessary to make a few practical remarks on the execution of the described procedure for solving the problem.

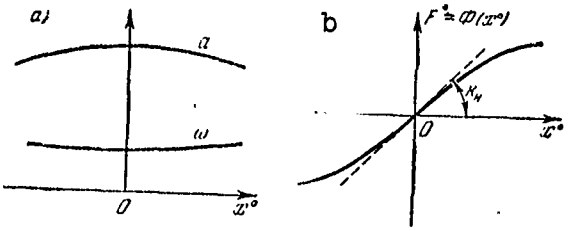


Fig. 7.13

Equation (7.135) is nonlinear. Its solution can be sought by any of the known numerical or graphic methods. However, a simplification is possible here, since most frequently the plot of the function $\Phi(x^0)$ is a smooth curve (Fig. 7.13b). It can therefore be linearized by the usual method, namely replacing the curve by a straight line (tangent, as in Fig. 7.13b, or secant). We then obtain in lieu of the nonlinear function $\Phi(x^0)$ the linear function

$$\Phi = k_n x^0.$$

In this case the nonlinear equation (7.135) is replaced by a linear one, for example

$$Q(p)\, x^0 + k_n R(p)\, x^0 = 0, \qquad (7.136)$$

which can be readily solved.

Summarizing, the most complicated problem is the first stage – determination of the functions $a(x^0)$ and $\omega(x^0)$ from Eq. (7.132). It is rarely possible to carry through this solution analytically to conclusion. Therefore, most frequently it becomes necessary to resort to various graphic procedures indicated in §2.3. The course of the solution will then be as follows.

First, starting from the specified form of the nonlinearity, one

estimates the limits of the variation of x^0, for which the form of the solution can be adopted. One then specifies some single specific value x^0 which is substituted in the expression (7.129) for $\underline{q}$ and q'. Then the characteristic equation (7.132) will contain two unknowns $\underline{a}$ and ω, which are determined in accordance with §2.3. We thus determine a pair of values $\underline{a}$ and ω for one specified value x^0.

Similar calculations are carried out for a whole series of possible values of x^0, as a result of which we obtain the sought plots $a(x^0)$ and $\omega(x^0)$ (Fig. 7.13a).

We note finally that if the problems considered in Chapter 5 have already been solved for the system that is being investigated by this method, it is possible here, in determining the aperiodic component of the transient by means of Eq. (7.135), to use the already evaluated bias function $\Phi(x^0)$ determined in Chapter 5.

Let us turn now to the second version of the method, when the solution for a transient with aperiodic and oscillatory component x^0 and x*, respectively, are sought in the form

$$\left.\begin{aligned} x = x^0 + x^*, \quad x^* = a\sin\psi, \\ \frac{da}{dt} = a\xi, \quad \frac{d\psi}{dt} = \omega. \end{aligned}\right\} \tag{7.137}$$

This form of solution is employed in such problems, when the substitution $p = j\omega$ in the characteristic equation of the harmonically linearized system does not yield a solution (see, for example, §1.8) and it becomes necessary to make use of the substitution $p = \xi + j\omega$.

In this version of the method, the harmonic linearization of the nonlinearity is carried out in analogy with (7.52) in the form

$$F(x,\ px) = F^0 + qx^* + q'\frac{p-\xi}{\omega}x^*, \tag{7.138}$$

and the expressions for F^0, $\underline{q}$, and q' are determined by Formulas (7.128) and (7.129). Here, unlike (7.53), the term $a\xi\sin\psi$ is discarded, since

it has the same order as the derivative dx^0/dt, which is likewise disregarded in the process of harmonic linearization, owing to the slow variation of x^0 over the period.

The differential equation (7.125) for the transient breaks up after harmonic linearization, as before, into two unrelated equations, the second of which (7.131) for the oscillatory component having the form

$$Q(p)\,x^* + R(p)\left(q + q'\frac{p-\xi}{\omega}\right)x^* = 0. \qquad (7.139)$$

Compared with the first version of the method, we have here an additional unknown ξ, and the corresponding supplementary equation $pa = a\xi$, which follows from (7.137).

The characteristic equation

$$Q(p) + R(p)\left(q + \frac{p-\xi}{\omega}q'\right) = 0 \qquad (7.140)$$

differs significantly from the previous one (7.72) in that the quantities q and q' depend here not only on the amplitude a, but also on the coordinate x^0 of the aperiodic component of the transient. Inasmuch as the coordinate x^0, like the amplitude a, decreases in the transient with time,* we can assume for the initial tentative estimate of the quality the condition

$$\frac{x^0}{a} \approx \text{const.} \qquad (7.141)$$

The magnitude of this ratio usually has a limited range, determined by the form of the nonlinearity. For example, for an ideal relay $|x^0/a| < 1$ (Fig. 7.12b), or else the relay will not switch over and the need for this investigation is obviated.

By specifying some average ratio (7.141) or several possible values of this ratio we can plot for it (or for each of them), on the basis of (7.140), by any of the first three methods described in §7.3,

a damping quality diagram for the nonlinear oscillations and thus trace tentatively the influence of different system parameters on the damping quality of the oscillations in the transient (see §7.4) in the presence of an aperiodic component.

If after such a preliminary estimate of the quality it becomes necessary to determine the entire transient curve for some tentatively chosen system parameters, this should be done already directly after a numerical, graphical, or computer solution of the initial nonlinear differential equation (7.125) or another equation, set up for this system more thoroughly than (7.125), since the latter equation was intended for a preliminary estimate aimed at choosing the structure and main parameters of the system.

Let us proceed now to an investigation of slipping processes and processes which occur under conditions of vibration smoothing of nonlinearities.

A slipping process is called a transient in a nonlinear automatic system in which the regulated quantity changes aperiodically (usually with superimposed small vibrations, Fig. 7.14a), while the nonlinear elements operate in the self-oscillation vibration mode (Fig. 7.14b-e) on some boundary of an essential change in its state (for example, on one of the extreme contacts of a three-position polarized relay). These self-oscillations usually can have a small amplitude $\underline{a}$ and a large frequency ω (Fig. 7.14b and c). Therefore in most theoretical investigations in the analysis of slipping processes one assumes $a = 0$ and $\omega = \infty$. Generally speaking, however, slipping processes are also observed in practice with finite amplitudes $\underline{a}$ and frequencies ω (for the variable $\underline{x}$ which is the argument of the nonlinearity; see Fig. 7.14d and e), particularly in the case of loop-type nonlinearities. In these cases the amplitude of the self-oscillating vibrations of the controlled

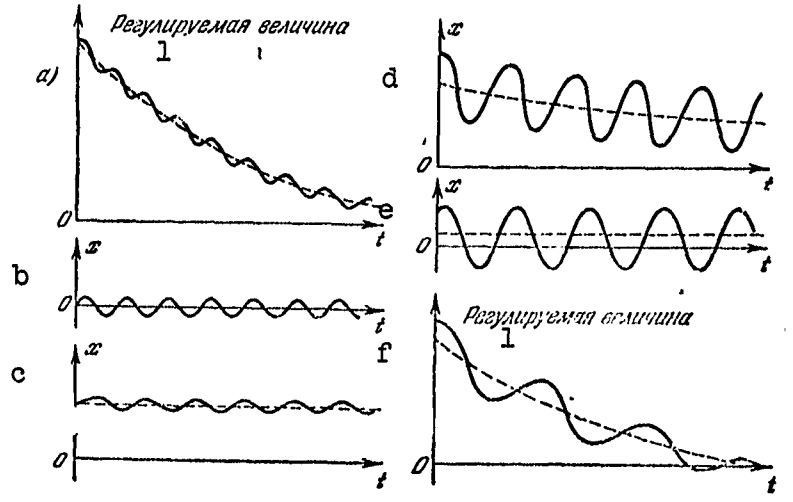


Fig. 7.14. 1) Regulated quantity.

quantity (Fig. 7.14a) can also be very small (owing to the inertia of the elements which relate the regulated quantity with the variable $\underline{x}$), and may practically be unnoticeable. It can also be noticeable but such that nevertheless the aperiodic part of the process remains its main component (Fig. 7.14f). In this case the transient can either tend to an equilibrium state ($a \to 0$) or to a self-oscillation mode ($a \to A = \text{const}$).

With this as a starting point, slipping processes can be approximately investigated in many cases (although of course not in all) by the same methods as asymmetrical oscillatory transients in nonlinear systems, considered above. The solution is sought in the same form (7.126), the same formulas (7.127)-(7.129) are used for harmonic linearization, as well as the same breakdown of the equations into (7.130) and (7.131).

As before, by solving Eq. (7.131) one determines the functions $a(x^0)$ and $\omega(x^0)$, which are then substituted in (7.128) to obtain the bias function $F^0 = \Phi(x^0)$. This will be the smoothed characteristic of the nonlinear element, substituted in (7.130) to calculate the main (aperiodic) component $x^0(t)$ of the slipping process. This indeed is

the solution of the problem.

The distinguishing feature here is that in most cases the slipping process is accompanied by operation of the nonlinear element not over the entire range of its nonlinear characteristic, but only on one narrow portion of the latter (for example, on one of the relay contacts), as shown in Fig. 7.15.

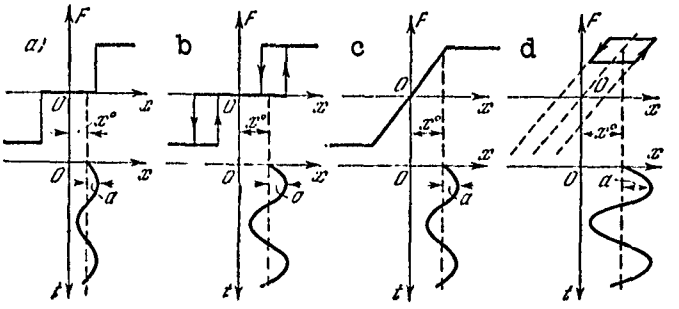


Fig. 7.15

This distinguishing feature appreciably influences the specific expressions (7.129) for the harmonic linearization coefficients, which should be separately derived in accordance with Fig. 7.15. As a result, the functions $a(x^0)$ and $\omega(x^0)$ indicated above, as well as the bias function $\Phi(x^0)$ itself, unlike the preceding one (Fig. 7.13), will be shifted away from the origin (Fig. 7.16). In this case the ordinary method of linearizing the bias function with the aid of the secant (Fig. 7.16b) yields

$$\Phi = \Phi_c + k_н (x^0 - x_c^0)$$

or (Fig. 7.16b)

$$\Phi = \Phi_н + k_н x^0. \qquad (7.142)$$

To determine the main (aperiodic) component of the slipping process x^0, we have in accordance with (7.130) the linear equation

$$Q(p)x^0 + R(p)k_н x^0 = -C, \quad C = R(0)\Phi_н. \qquad (7.143)$$

The method in this form can be applied also to systems with delay.

In precisely the same manner we can investigate transients which

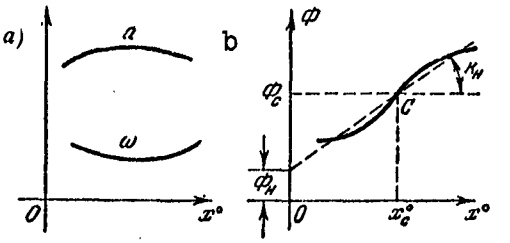


Fig. 7.16

occur under conditions of vibration smoothing of nonlinearities with the aid of self-oscillations. In §5.3 we investigated the passage of slowly varying signals in such systems, due to external action. The only difference here is that to determine the slowly varying component of the process we must solve not (5.77) or (5.78), but a homogeneous equation of the type (7.136) or (7.143).

In Chapter 9 below we shall consider also transients that occur under conditions of vibration smoothing of nonlinearities with the aid of forced oscillations.

The method described in the present section as applied to a system of the first class with one nonlinearity can be extended also to systems with several nonlinearities of all three classes, in exactly the same manner as was used in the investigation of self-oscillations in Chapters 5 and 6.

§7.6. Example of Application of the Asymptotic Method

The construction of the asymptotic solution, proposed in §7.1, will be considered using as an example the motion of the center of gravity of a sea plane in a vertical plane, while landing on a smooth water surface. This problem was considered in [226]. Figure 7.17 shows the coordinate systems chosen for the analysis of the problem: x_k and z_k are axes that are parallel and perpendicular to the keel, respectively; x_w and z_w are axes parallel and perpendicular, respectively, to the surface of the water; $\underline{x}$ and $\underline{z}$ are axes parallel and perpendicu-

lar, respectively, to the keel, but with origin at the point where the keel crosses the rear face. The first two reference frames are stationary and the third is fixed in the sea plane.

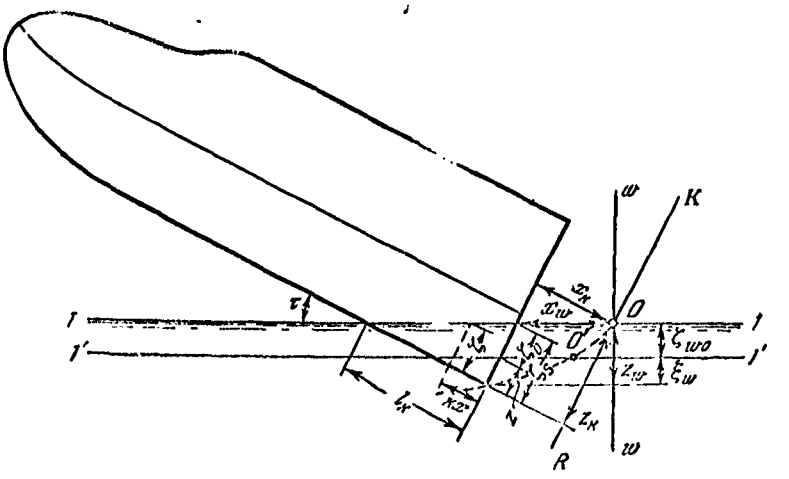


Fig. 7.17

We shall consider motion of the sea plane relative to the position 1'-1' (Fig. 7.17), which is located a distance ζ_0 (measured along the rear face) away from the unperturbed level of the water surface and corresponding to the depth of immersion of the lowest point on the keel of the planing bottom if the motion is steady with given speed.

The differential equation for the case of a rapidly damped oscillatory motion of the center of gravity of the sea plane relative to the water surface has the following form [226]

$$\ddot{z} + 2b\dot{z} + \nu^2 z = \varepsilon f(z, \dot{z}), \qquad (7.144)$$

where z is the displacement of the sea plane in the direction of the z axis of the fixed coordinate system; the average coefficient

$$2b = \frac{G\Gamma^2\zeta_0^2 x_k \operatorname{tg}\tau + D}{I + \Gamma^2\zeta_0^2}$$

characterized the damping of the oscillations; Γ is the coefficient of attached mass in a coordinate system oriented relative to the keel; $D = N\dot{z}_k$; N is a coefficient that takes into account the aerodynamic

damping;

$$\nu^2 = \frac{GT^2\zeta_0 (\dot{z}_k \operatorname{tg} \tau)^2}{1 + T^2\zeta_0^2};$$

the small parameter

$$\varepsilon = \frac{3T^2\zeta_0^2}{1 + T^2\zeta_0^2} < 1. \qquad (7.145)$$

The nonlinear function has the form

$$f(z, \dot{z}) = -\left\{\frac{(\dot{z}_k \operatorname{tg} \tau)^2}{\zeta_0^2} z^2 + \frac{z^2}{\zeta_0^2}\dot{z}^2 + 2\frac{z}{\zeta_0^2}\dot{z}^2 + \frac{1}{\zeta_0}\dot{z}^2 + \right.$$
$$\left. + \frac{2\dot{z}_k z^2 \operatorname{tg} \tau}{\zeta_0^2}\dot{z} + \frac{4\dot{z}_k z \operatorname{tg} \tau}{\zeta_0^2}\dot{z}\right\}. \qquad (7.146)$$

When $\varepsilon = 0$ the solution of (7.144) will be

$$z = a \sin(\omega_0 t + \psi_0),$$

where

$$a = a_0 e^{-bt}, \quad \omega_0 = \sqrt{\nu^2 - b^2}.$$

In the case $\varepsilon \neq 0$ the solution of (7.144) will be sought in the first approximation in the form

$$z = a \sin \psi,$$

and, according to §7.1, we have

$$\frac{da}{dt} = a\xi(a), \quad \frac{d\psi}{dt} = \omega(a), \qquad (7.147)$$

where ξ is the instantaneous damping exponent; ω is the instantaneous frequency of the investigated nonlinear process. In this case to determine $\xi(a)$ and $\omega(a)$ we must first determine the functions $\beta(a)$ and $\gamma(a)$ by means of Formulas (7.27), namely

$$\left.\begin{aligned}\beta(a) &= -\frac{1}{\pi}\int_0^{2\pi} f(a \sin\psi,\ a\omega_0 \cos\psi,\ -ab \sin\psi) \cos\psi\, d\psi,\\ \gamma(a) &= -\frac{1}{\pi a}\int_0^{2\pi} f(a \sin\psi,\ a\omega_0 \cos\psi - ab \sin\psi) \sin\psi\, d\psi.\end{aligned}\right\} \qquad (7.148)$$

Substituting the specified function $f(z, \dot{z})$ from (7.146) and integrating, we obtain

$$\left.\begin{aligned}\beta(a) &= -\frac{a^2}{2\zeta_0^2}[b(3b - 2\omega_0)\zeta_0 + \dot{z}_k \omega_0 \operatorname{tg} \tau],\\ \gamma(a) &= \frac{3ba^2}{2\zeta_0}(\zeta_0 b - \dot{z}_k \operatorname{tg} \tau).\end{aligned}\right\} \qquad (7.149)$$

At the initial parameters that are applicable to a large class of sea planes, the first terms in the brackets of (7.149) amount to 0.5-2% of the values of the second terms. Neglecting these terms, we rewrite (7.149) in the simplified form

$$\left.\begin{aligned}\beta(a) &= -\frac{\omega_0 z_k \operatorname{tg}\tau}{2\zeta_0^3}a^3,\\ \gamma(a) &= -\frac{3bz_k \operatorname{tg}\tau}{2\zeta_0^3}a^2.\end{aligned}\right\} \qquad (7.150)$$

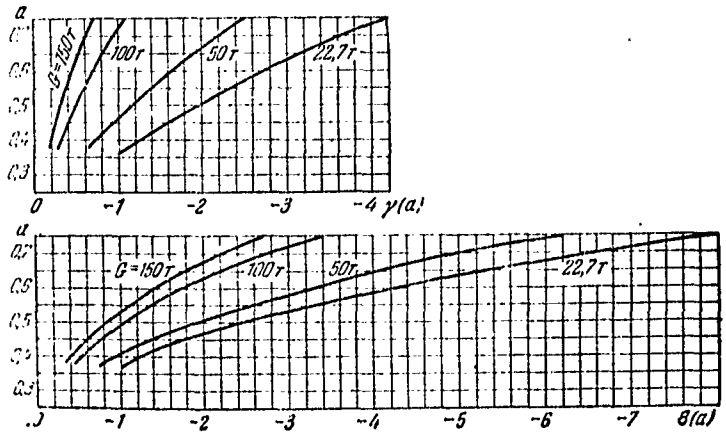


Fig. 7.18

Figure 7.18 shows the $\beta(a)$ and $\gamma(a)$ curves for different types of sea planes. The plots given are smooth curves with small curvature. Therefore, in accordance with (7.30), the formulas for the determination of the damping exponent ξ and the frequency ω have the form

$$\left.\begin{aligned}\xi &= -b + \varepsilon\left[\frac{\beta(a)}{2a\omega_0} + \frac{ba}{2(2\omega_0^2+b^2)}\frac{d\gamma}{da}\right],\\ \omega^2 &= \omega_0^2 + \varepsilon\left[\gamma(a) - \frac{b}{2\omega_0}\frac{d\beta}{da} - \frac{b^2a}{2\omega_0^2+b^2}\frac{d\gamma}{da}\right].\end{aligned}\right\} \qquad (7.151)$$

After determining $\varepsilon(d\gamma/da)$ and $\varepsilon(d\beta/da)$ from (7.150) and (7.145) and substituting in (7.151) we obtain

$$\left.\begin{aligned}\xi &= -b - \frac{1}{4}k_1k_3a^2,\\ \omega^2 &= \omega_0^2 - k_2k_3ba^2,\end{aligned}\right\} \qquad (7.152)$$

where

$$k_1 = 1 + \frac{6b^2}{2\omega_0^2+b^2}, \qquad k_2 = \frac{3}{4} - \frac{3b^2}{2\omega_0^2+b^2}, \qquad k_3 = \frac{3\Gamma^2 z_k \operatorname{tg}\tau}{1+\Gamma^2\zeta_0^2}.$$

According to (7.147) we have

$$\left.\begin{aligned} \frac{da}{dt} &= -a\left(b + \frac{1}{4} k_1 k_3 a^2\right), \\ \omega &= \frac{d\psi}{dt} = \sqrt{\omega_0^2 - k_2 k_3 b a^2}. \end{aligned}\right\} \tag{7.153}$$

Let us write the first equality of (7.153) in the form

$$\frac{da}{a^3 + 4ma} = -\frac{k_1 k_3}{4} dt, \tag{7.154}$$

where

$$m = b/k_1 k_3.$$

Integrating (7.154) and taking account of the fact that $a = a_0$ when $t = 0$, we obtain

$$a^2 = \frac{4m}{\left(1 + \frac{4m}{a_0^2}\right) e^{2bt} - 1}. \tag{7.155}$$

From the second equality of (7.153) we have

$$\omega = \frac{d\psi}{dt} = \sqrt{\omega_0^2 - ca^2}, \tag{7.156}$$

where

$$c = bk_2 k_3.$$

Since $\frac{d\psi}{dt} = \frac{d\psi}{da} \frac{da}{dt}$ we obtain from (7.154), taking (7.156) into account,

$$-k_1 k_3 d\psi = \frac{4\sqrt{\omega_0^2 - ca}}{a(a^2 + 4m)} da. \tag{7.157}$$

Integrating (7.157) with account of the fact that $a = a_0$, and $\psi = \psi_0$ when $t = 0$, we obtain a computation formula for ψ:

$$\psi = \psi_0 + \frac{1}{2b}\left[\omega_0 \ln \frac{(\omega_0 - \sqrt{\omega_0^2 - ca_0^2})(\omega_0 + \sqrt{\omega_0^2 - ca^2})}{(\omega_0 - \sqrt{\omega_0^2 - ca^2})(\omega_0 + \sqrt{\omega_0^2 - ca_0^2})} - \right.$$
$$\left. - \sqrt{\omega_0^2 + 4cm} \ln \frac{(\sqrt{\omega_0^2 + 4cm} - \sqrt{\omega_0^2 - ca_0^2})(\sqrt{\omega_0^2 + cm} + \sqrt{\omega_0^2 - ca^2})}{(\sqrt{\omega_0^2 + 4cm} - \sqrt{\omega_0^2 - ca^2})(\sqrt{\omega_0^2 + 4cm} + \sqrt{\omega_0^2 - ca_0^2})}\right]. \tag{7.158}$$

The formulas (7.155), (7.156), and (7.158) enable us to plot the damped oscillatory motion of the landing sea plane. The results of the theoretical calculations and of the experimental research [226] were in good agreement.

§7.7. Symmetrical Oscillatory Transient in a Servomechanism

Let us plot the transient quality diagram and the transient curve using a nonlinear servomechanism as an example [215].

The block diagram of the servomechanism is shown in Fig. 7.19, where 1 is the error transducer, 2 the amplifier, 3 the relay, 4 the actuating motor, 5 the reduction gear, 6 the controlled object, and 7 the supplementary feedback.

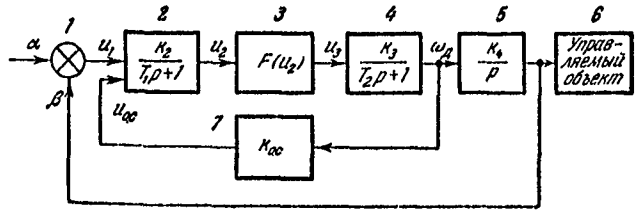


Fig. 7.19. 6) Controlled object.

Systems with such a block diagram are used in those cases, when considerable power is necessary to control the motor, and it is not desirable to increase the amplifier dimensions and weight.

For the error transducer of the system we have the equations:

$$\vartheta = \alpha - \beta, \quad u_1 = k_1\vartheta, \tag{7.159}$$

where ϑ is the system error, α and β are the input and output of the system, respectively, and k_1 is the transfer ratio of the error transducer.

The static characteristics of the nonlinear element, namely the relay, is shown in Fig. 7.20. Carrying out harmonic linearization of the nonlinear characteristic of the relay we obtain the equation

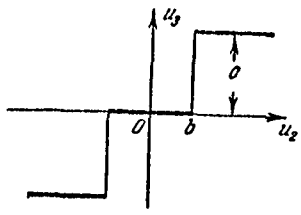


Fig. 7.20

$$u_3 = q(a)\,u_2, \tag{7.160}$$

where in accord with (3.13) the coefficient of harmonic linearization for a single-valued relay characteristic with backlash is determined by the formula

$$q(a)=\frac{4c}{\pi a}\sqrt{1-\frac{b^2}{a^2}}. \qquad (7.161)$$

Taking into account the equation (7.159) for the error transducer, the harmonically linearized equation of the relay (7.160) and the transfer functions of the other linear elements shown in Fig. 7.19, we write down the equation for the motion of the servomechanism system proper ($\alpha = 0$) in the form

$$[(T_1p+1)(T_2p+1)p+k_2k_3k_{o.c}q(a)p+k_1k_2k_3k_4q(a)]u_2=0. \qquad (7.162)$$

The characteristic equation corresponding to the resultant differential equation will be

$$L(p)=(T_1p+1)(T_2p+1)p+k_2k_3k_{o.c}q(a)p+ \\ +k_1k_2k_3k_4q(a)=0. \qquad (7.163)$$

Inasmuch as the static characteristic of the nonlinear element is symmetrical and no external signal is applied to the system, the oscillatory solution (if it exists) corresponding to Eq. (7.162) will be symmetrical.

Let us first plot the quality diagram of the oscillatory process by the first method indicated in §7.3. For this purpose it is necessary to make in (7.163) the substitution

$$p = \xi + j\omega,$$

where ξ is the damping and ω is the frequency of the sought transient. To simplify this problem we can represent the characteristic polynomial by means of a finite series in powers of $j\omega$ with coefficients that depend on ξ.

Calculating the corresponding derivatives of the characteristic polynomial (7.163) we obtain

$$\frac{dL(p)}{dp}=3T_1T_2p^2+2(T_1+T_2)p+[1+k_2k_3k_{o.c}q(a)],$$
$$\frac{1}{2!}\frac{d^2L(p)}{dp^2}=3T_1T_2p+T_1+T_2, \quad \frac{1}{3!}\frac{d^3L(p)}{dp^3}=T_1T_2.$$

The substitution $p = \xi$ in the expressions for L(p) and its derivatives

yields the coefficients of the series expansion of the equation $L(a, \xi, j\omega) = 0$, which breaks up into the following two equations:

$$X = T_1T_2\xi^3 + (T_1 + T_2)\xi^2 + [1 + k_2k_3k_{o.c}q(a)]\xi + k_1k_2k_3k_4q(a) - [3T_1T_2\xi + T_1 + T_2]\omega^2 = 0. \quad (7.164)$$

$$Y = [3T_1T_2\xi^2 + 2(T_1 + T_2)\xi + 1 + k_2k_3k_{o.c}q(a)]\omega - T_1T_2\omega^3 = 0. \quad (7.165)$$

From (7.165) we determine the square of the frequency

$$\omega^2 = \frac{1}{T_1T_2}[3T_1T_2\xi^2 + 2(T_1 + T_2)\xi + 1 + k_2k_3k_{o.c}q(a)]. \quad (7.166)$$

Substituting the value of ω^2 in (7.164) we obtain

$$T_1T_2\xi^3 + (T_1 + T_2)\xi^2 + [1 + k_2k_3k_{o.c}q(a)]\xi + k_1k_2k_3k_4q(a) = \frac{1}{T_1T_2}[3T_1T_2\xi^2 + 2(T_1 + T_2)\xi + 1 + k_2k_3k_{o.c}q(a)][3T_1T_2\xi + T_1 + T_2]. \quad (7.167)$$

Let us plot the quality of the nonlinear process in the servomechanism for the parameter k_1, i.e., for the transfer ratio (the slope of the characteristic) of the error transducer. If the damping ξ is contained in (7.167) in a nonlinear manner, it is convenient to solve this equation with respect to the parameter k_1.

As a result we obtain

$$k_1 = \frac{1}{k_2k_3k_4q(a)}\left\{\frac{1}{T_1T_2}[3T_1T_2\xi^2 + 2(T_1 + T_2)\xi + 1 + k_2k_3k_{o.c}q(a)] \times [3T_1T_2\xi + T_1 + T_2] - [T_1T_2\xi^3 + (T_1 + T_2)\xi^2 + \xi]\right\}. \quad (7.168)$$

To plot the diagram we specify the following values of the other parameters: $T_1 = 0.05$ sec, $T_2 = 0.05$ sec, $k_2 = 1$, $k_3 = 200$ deg/sec-v, $k_4 = 0.01$, $k_{o.s} = 10^{-3}$ sec-v/deg, b = 5 v, c = 120 v.

Substituting the given values of the parameters in (7.168) and specifying different constant values of the damping exponent $\xi = \text{const}$, we plot the $a(k_1)$ curves (Fig. 7.21). On the basis of this construction we plot also the $a(k_1)$, using Formula (7.166) and constant values of the frequency, $\omega = \text{const}$.

The curves shown in Fig. 7.21 represent the quality diagram of the transient for the servomechanism under consideration. The $a(k_1)$

curves with $\xi = 0$ correspond to self-oscillations.

Let us plot now the quality diagram for the oscillatory process in the nonlinear system by the second method indicated in §7.3.

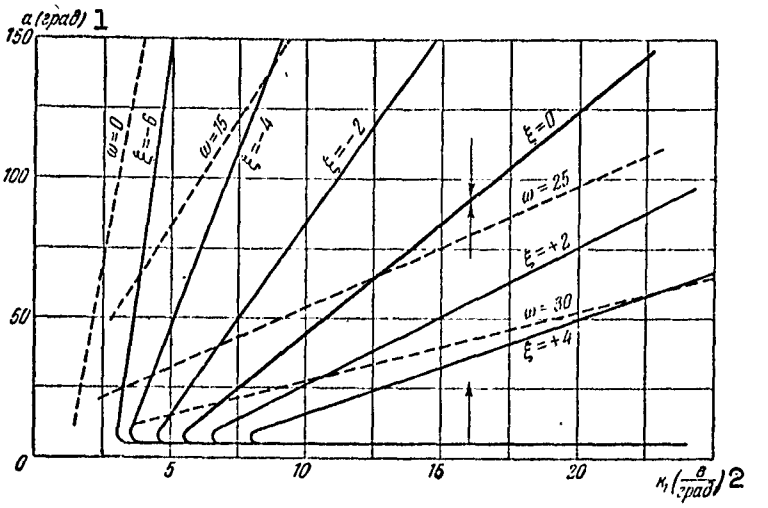


Fig. 7.21. 1) deg; 2) v/deg.

We rewrite (7.163) in the form

$$p^3 + A_1 p^2 + A_2 p + A_3 = 0, \tag{7.169}$$

where

$$A_1 = \frac{T_1 + T_2}{T_1 T_2}; \quad A_2 = \frac{1 + k_2 k_3 k_{o.c} q(a)}{T_1 T_2}; \quad A_3 = \frac{k_1 k_2 k_3 k_4 q(a)}{T_1 T_2}. \tag{7.170}$$

Factoring the left half of (7.169), we obtain the equation

$$(p + C_1)(p^2 + B_1 p + B_2) = 0, \tag{7.171}$$

the coefficients of which are related with the coefficients of (7.169) by the equations

$$A_1 = C_1 + B_1, \quad A_2 = B_2 + B_1 C_1, \quad A_3 = C_1 B_2. \tag{7.172}$$

Assuming the last factor of (7.171) to correspond to a pair of complex roots of the system, which have a real part of much smaller modulus than the root of the first factor, i.e., assuming that

$$C_1 > \left|\frac{B_1}{2}\right|,$$

we write down the formulas for the connection between the damping ξ and the frequency ω with the system parameters in the form

$$\xi = -\frac{B_1}{2}, \quad \omega^2 = B_2 - \xi^2. \tag{7.173}$$

Let us set up in accord with Eq. (7.169) the penultimate Hurwitz determinant (for our example, the second) with allowance for (7.172):

$$H_2 = A_1A_2 - A_3 = (C_1 + B_1)(B_2 + B_1C_1) - C_1B_2 = \\ = B_1(B_2 + C_1^2 + C_1B_1).$$

Since we have from (7.172)

$$B_2 + C_1B_1 = A_2 \text{ and } C_1^2 = (A_1 - B_1)^2,$$

and from (7.173)

$$B_1 = -2\xi,$$

we obtain the following formula for the determination of the damping

$$\xi = -\frac{H_2}{2[A_2 + (A_1 + 2\xi)^2]} = -\frac{A_1A_2 - A_3}{2[A_2 + (A_1 + 2\xi)^2]}. \tag{7.174}$$

Recognizing that we have from (7.172)

$$B_2 = \frac{A_3}{C_1} = \frac{A_3}{A_1 - B_1} = \frac{A_3}{A_1 + 2\xi},$$

and substituting the value of B_2 in (7.173), we obtain a formula for the square of the frequency:

$$\omega^2 = \frac{A_3}{A_1 + 2\xi} - \xi^2. \tag{7.175}$$

Formulas (7.174) and (7.175) enable us to plot the damping of the nonlinear processes relative to any of the system parameters. For this purpose it is sufficient to use Formulas (7.170), which relate the coefficients A_1, A_2, and A_3 with the system parameters. For the parameter k_1, with selected values of the other parameters of the servomechanism, this gives the same result as in the preceding case.

By plotting the transient quality diagram for the same system with the supplementary feedback removed, we obtain the results shown in Fig. 7.22. In this particular case the lines ξ = const and ω = const are superimposed on one another.

Comparing the diagrams obtained for the cases with and without

the additional feedback, we see that the feedback broadens the region of the damped oscillations (the region to the left and above the line $\xi = 0$, corresponding to self-oscillations). In addition, for the same values of the parameter k_1 the damping in the region of damped processes is larger in absolute magnitude in the presence of feedback than without feedback. For example, when $k_1 = 8$ and $a = 90^\circ$ the damping is $\xi = -4$ in the presence of feedback and $\xi = -2$ without feedback. This indicates that feedback results in faster damping of the transient.

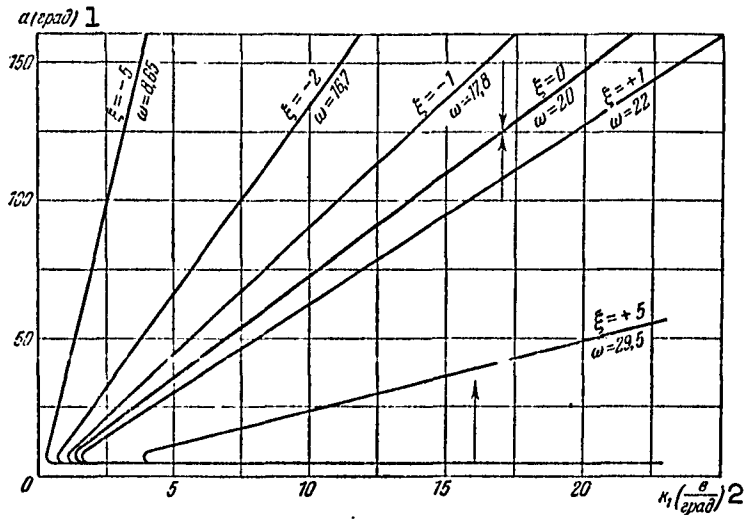


Fig. 7.22. 1) deg; 2) v/deg.

The quality diagrams obtained enable us to estimate the transient in a nonlinear system if the parameters of the system are specified, and also make it possible to solve the inverse problem, i.e., choose the values of the parameters necessary to obtain a given transient quality. In addition, as was shown in §7.4, it is easy to plot from the quality diagrams the envelope of the transient amplitudes and to determine the change in frequency of the process from period to period, i.e., in final analysis, to obtain an approximate plot of the transient. The error in the approximate plot of the transient, as will be shown later on, is small and is quite acceptable for engineering de-

sign of nonlinear automatic systems.

To determine the error of the method, Fig. 7.23 shows the transient in the system under consideration plotted by a numerical-graphic method [198] for a parameter value $k_1 = 5$ v/deg and with initial value of the oscillation amplitude $a_0 = 250$ volts. To improve the accuracy, the plot was drawn on a large scale with a small spacing in the integration.

On the same Fig. 7.23 is shown dotted the envelope of the transient, plotted approximately on the basis of the quality diagram (Fig. 7.21). It is seen from this plot that the approximate calculation based on the harmonic linearization method results in a small error in the determination of the envelope.

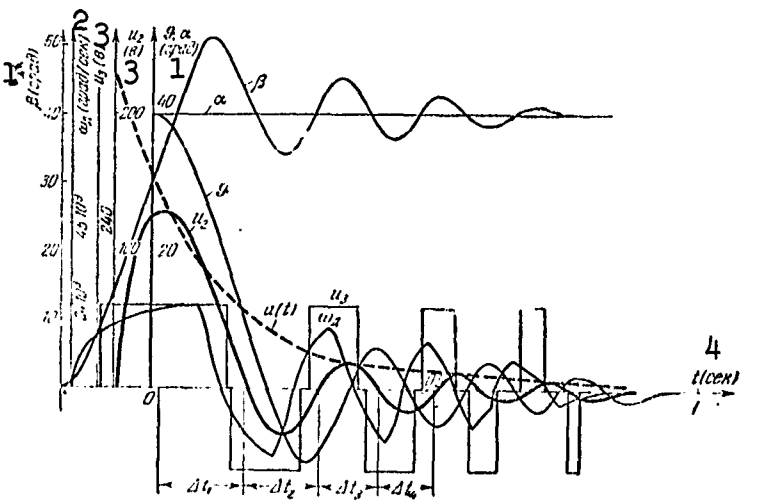


Fig. 7.23. 1) deg; 2) deg/sec; 3) v; 4) sec.

The error in the determination of the frequency ω will be estimated in the following manner. The exact curve, obtained by the numerical-graphic methods, will be broken up into sections 1, 2, 3, 4 (Fig. 7.23) corresponding as it were to quarters or halves of the oscillation period. For each of these sections we determine from the exact curve the quantities

$$\omega_T = \frac{\pi}{2\Delta t} \quad \text{or} \quad \omega_T = \frac{\pi}{\Delta t},$$

where Δt denotes the duration of each section. The error in the determination of the frequency with the aid of this approximate method will be

$$\frac{\Delta\omega}{\omega_T}\% = \frac{\omega - \omega_T}{\omega_T} 100\%,$$

where ω is the value of the frequency, taken from the quality diagram of the investigated nonlinear system (7.21) for the average amplitude a on each section. The results of the calculations are listed in the table:

Δt	ω_T	ω	$\frac{\Delta\omega}{\omega_T}\%$
Δt_1	10,5	9,5	9,5
Δt_2	22	21	4,6
Δt_3	28	28	0
Δt_4	31	30	3,2

Let us estimate the errors in the determination of the damping time t_1 of the transient from the initial amplitude $a_0 = 250$ v to the amplitude $a_1 = 15$ v. The exact value of t_1, determined from the plot of $u_2(t)$ (Fig. 7.23), will be $t_1 = 0.85$ sec. Approximate calculation by means of the quality diagram (Fig. 7.21), using a rough calculation in which the entire process is broken up into three parts, within each of which we assume $\xi = \text{const}$, yields $t_1 = 0.98$ sec. A more careful calculation leads to better agreement between the obtained transient time and the exact value. It is more likely here that the error is due to the method of calculating the transient time directly from the quality diagram, since it is seen from Fig. 7.23 that in practice the transient time is the same for both the envelope obtained by the approximate method and for the transient curve obtained by the numerical-graphic method.

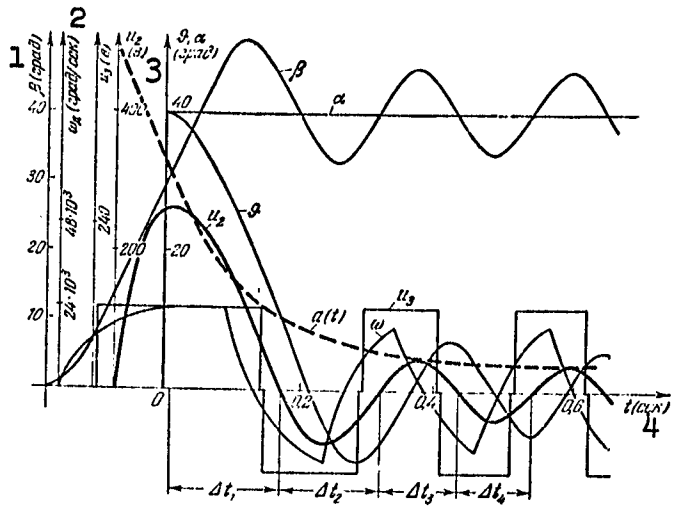


Fig. 7.24. 1) deg; 2) deg/sec; 3) v; 4) sec.

Figure 7.24 shows a plot of the transient in the same system, with an increased transfer ratio of the error transducer $k_1 = 10$ v/deg and for $a_0 = 500$ v. In this case self-oscillations with amplitude A = = 42 v occur in the steady-state mode.

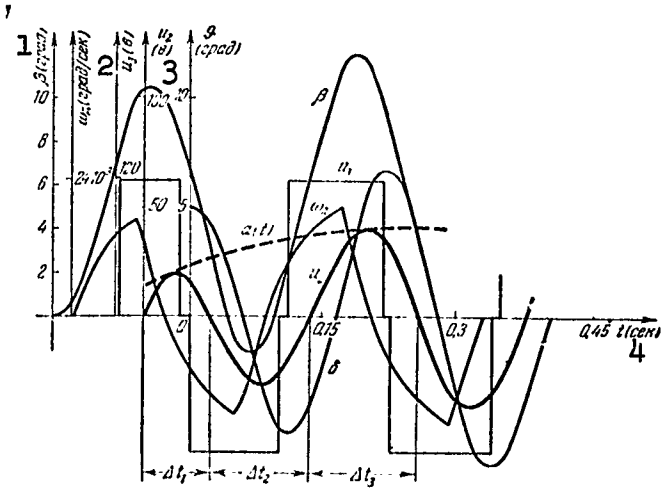


Fig. 7.25. 1) deg; 2) deg/sec; 3) v; 4) sec.

Figure 7.25 shows the transient in the same system with $k_1 = 10$ v/deg for the case when the system arrives at the self-oscillation modes from small initial deviations ("from below"), i.e., as a result of a diverging process. The same figure shows the envelope a(t), obtained by the harmonic linearization method on the basis of the quality

diagram.

The method gives sufficiently good results also in the case when the oscillations are damped in practice within a single period (Fig. 7.26).

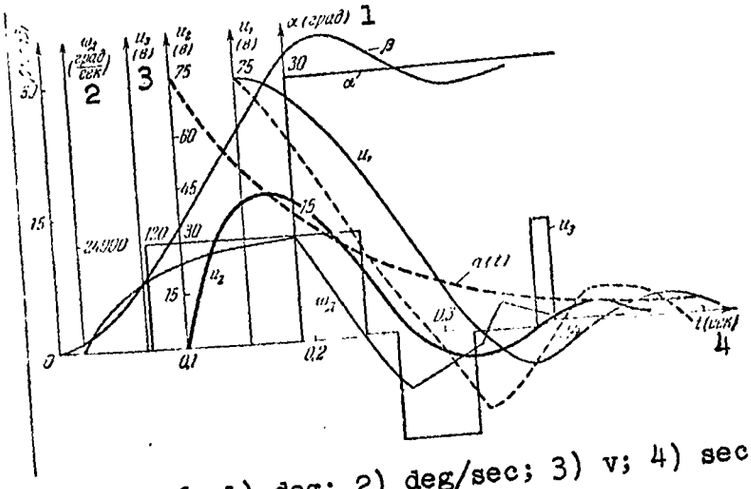


Fig. 7.26. 1) deg; 2) deg/sec; 3) v; 4) sec.

§7.8. Transient in a Servomechanism of the Second Class

In the preceding section we considered a nonlinear servomechanism of the first class, where the argument of the linear function was only a single variable. In the present section we carry out an investigation of the transient for a nonlinear servomechanism of the second class, where the arguments of the nonlinear function are both the input and the output of the nonlinear element [228].

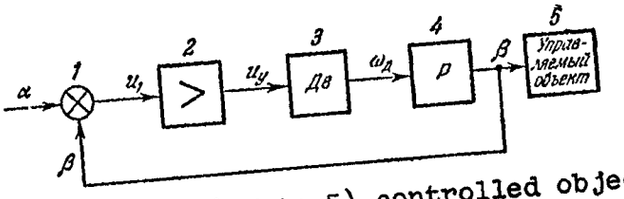


Fig. 7.27. 3) Mot; 5) controlled object.

The block diagram of the servomechanism is shown in Fig. 7.27. The system comprises an error transducer 1, with transfer ratio k_1, an amplifier 2, a two-phase induction motor 3, and a reduction gear 4.

The figure shows also the controlled object 5.

The elements of the system are described by the following equations:

error transducer

$$\vartheta = \alpha - \beta, \quad u_1 = -k_1\vartheta, \tag{7.176}$$

amplifier

$$(T_1 p + 1)\, u_y = k_2 u_1, \tag{7.177}$$

reduction gear

$$p\beta = k_4 \omega_д. \tag{7.178}$$

In these formulas ϑ is the error, α and β are the input and the output of the nonlinear element, u_1 the voltage at the amplifier input, u_u the voltage at the amplifier output, k_2 the amplifier gain, T_1 the amplifier time constant, ω_d the angular velocity of the motor shaft, and β the position angle of the reduction gear output shaft.

The two-phase induction motor with hollow rotor is the nonlinear element. A feature of the mechanical characteristics of such motors is that when they are fed from a low-power source, the slip is a function of the control voltage. The motor characteristics can in this case be approximated by straight lines with different slopes, and the slope of the characteristic is a function of the control voltage (Fig. 7.28).

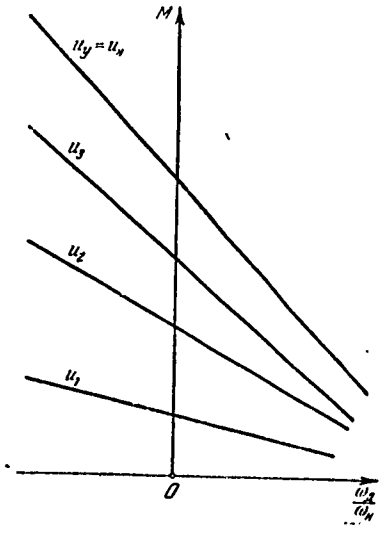


Fig. 7.28

The equation of the motor can in this case be written in the form

$$Jp\omega_д + c(u_y)\,\omega_д = k_3 u_y, \tag{7.179}$$

where J is the moment of inertia of all the rotating masses, referred to the motor shaft. With accuracy sufficient for engineering calculations we can assume that the slope of the

mechanical characteristics changes in proportion to the control voltage u_u, i.e.,

$$c(u_y) = m - (m-n)\left|\frac{u_y}{u_n}\right|,$$

where u_n is the nominal value of the control voltage, while m and n are the slopes of the characteristics at $u_u = 0$ and $u_u = u_n$.

In this case Eq. (7.179) is written in the form

$$\omega_д + m\omega_д - (m-n)\left|\frac{u_y}{u_n}\right|\omega_д = k_3 u_y. \qquad (7.180)$$

The nonlinearity in Eq. (7.180) can be represented in the following fashion:

$$y = \frac{m-n}{u_n}|u_y|\omega_д. \qquad (7.181)$$

This is a linearity of the second class, since it contains the product of the input $|u_u|$ and of the output ω_d.

To investigate the transients we represent in this case, in accord with (7.52), the nonlinear function in terms of the coefficients of harmonic linearization in the form

$$y = \left[q(a, \xi, \omega) - \frac{\xi}{\omega}q'(a, \xi, \omega)\right]u_y + \frac{q'(a, \xi, \omega)}{\omega}pu_y. \qquad (7.182)$$

The solution for u_u is sought in the form of a damped or diverging sine wave

$$u_y = a(t)\sin\psi(t). \qquad (7.183)$$

The motor speed ω_d will oscillate with amplitude $a_2(t)$ and a phase shift $\varphi(\xi, \omega)$:

$$\omega_д = a_2(t)\sin[\psi(t) + \varphi(\xi, \omega)]. \qquad (7.184)$$

Let us express the output of the nonlinear element ω_d in terms of the input u_u, using the connection between these variables through the linear part of the system. The equation of the linear part of the system will be written, in accordance with the element equations (7.176)-(7.178) for $\alpha(t) = 0$ in the form

$$\omega_{\text{д}} = -\frac{(T_1 p + 1) p}{k} u_y, \quad k = k_1 k_2 k_4. \tag{7.185}$$

We can then write on the basis of (7.98)

$$\omega_{\text{д}} = U_1(\xi, \omega) a \sin\psi + V_1(\xi, \omega) a \cos\psi, \tag{7.186}$$

where U_1 and V_1 are, respectively, the real and imaginary parts of the complex expression obtained from the transfer function (7.185) upon substitution of $p = \xi + j\omega$.

Making this substitution, we obtain

$$U_1 = -\frac{\xi + T_1\xi^2 - T_1\omega^2}{k}, \quad V_1 = -\frac{2T_1\xi\omega + \omega}{k}. \tag{7.187}$$

Consequently, the nonlinearity (7.181), with account of (7.186), (7.183), and (7.187) is written in the form

$$y = -\frac{m-n}{u_{\text{н}}} |a \sin\psi| a\left[\frac{\xi + T_1\xi^2 - T_1\omega^2}{k} \sin\psi + \frac{2T_1\xi\omega + \omega}{k} \cos\psi\right]. \tag{7.188}$$

On the basis of the expression obtained for the nonlinearity, we determine the values of the harmonic linearization coefficients q and q' entering in Formula (7.182).

For the coefficient q we have

$$q = \frac{1}{\pi a} \int_0^{2\pi} \frac{m-n}{u_{\text{н}}} |a \sin\psi| a\left[\frac{\xi + T_1\xi^2 - T_1\omega^2}{k} \sin\psi + \frac{2T_1\xi\omega + \omega}{k} \cos\psi\right] \sin\psi \, d\psi =$$

$$= \frac{8(m-n)a}{3\pi u_{\text{н}}} \cdot \frac{\xi + T_1\xi^2 - T_1\omega^2}{k}.$$

Carrying out the calculation of the coefficient q', we obtain

$$q' = \frac{1}{\pi a_1} \int_0^{2\pi} \frac{m-n}{u_{\text{н}}} |a \sin\psi| a\left[\frac{\xi + T_1\xi^2 - T_1\omega^2}{k} \sin\psi + \frac{2T_1\xi\omega + \omega}{k} \cos\psi\right] \cos\psi \, d\psi =$$

$$= -\frac{4(m-n)a}{3\pi u_{\text{н}}} \cdot \frac{2T_1\xi\omega + \omega}{k}.$$

The harmonically linearized expression for the nonlinearity then assumes the form

$$y = \left[\frac{8(m-n)a}{3\pi u_{\text{н}}} \cdot \frac{\xi + T_1\xi^2 - T_1\omega^2}{k} + \frac{4(m-n)a}{3\pi u_{\text{н}}} \cdot \frac{(2T_1\xi + 1)\xi}{k}\right] u_y - \frac{4(m-n)a}{3\pi u_{\text{н}}} \cdot \frac{2T_1\xi + 1}{k} p u_y.$$

The equation of the motor, with account of the value obtained for y, will in accordance with (7.180) be

$$\frac{J}{m}p\omega_{\text{д}}+\omega_{\text{д}}=\left[\frac{k_3}{m}-\frac{4(m-n)a}{3u_{\text{н}}\pi km}(4T_1\xi^2+3\xi-2T_1\omega^2)+\right.$$
$$\left.+\frac{4(m-n)a}{3u_{\text{н}}\pi km}(2T_1\xi+1)p\right]u_y.$$

We introduce the notation

$$k'=-\frac{4(m-n)a}{3u_{\text{н}}\pi km}(4T_1\xi^2+3\xi-2T_1\omega^2),$$
$$k''=\frac{4(m-n)a}{3u_{\text{н}}\pi km}(2T_1\xi+1),\quad T=\frac{J}{m}.$$

The harmonically linearized motor equation is then written in the form

$$(Tp+1)\omega_{\text{д}}=\left(\frac{k_3}{m}+k'+k''p\right)u_y. \qquad (7.189)$$

We see from this equation that the nonlinearity of the motor characteristics reduces in this case to the presence in (7.189) of the operator coefficient

$$k^*=\frac{k_3}{m}+k'+k''p,$$

which is a function of the amplitude a of the oscillations of the voltage u_u, of the damping coefficient ξ, and of the system oscillation frequency ω.

Taking into account the equation of the linear part (7.185) and the harmonically linearized motor equation (7.189), we write for the characteristic equation of the system

$$p^3+A_1p^2+A_2p+A_3=0, \qquad (7.190)$$

where

$$A_1=\frac{T_1+T}{T_1T},\qquad A_2=\frac{1+kk''}{T_1T},\qquad A_3=\left(\frac{k_3}{m}+k'\right)\frac{k}{T_1T}. \qquad (7.191)$$

To determine the transient it is necessary to plot a quality diagram, i.e., the dependence of the amplitude on the parameter of interest at constant values of the damping ξ and of the frequency ω. For this purpose we make use of Formulas (7.82) and (7.83) for a third-order system. These formulas determine the connection of the damping

exponent and the frequency with the system parameters and have the form

$$\xi = -\frac{A_1A_2 - A_3}{2[A_2 + (A_1 + 2\xi)^2]}, \quad \omega^2 = \frac{A_3}{A_1 + 2\xi} - \xi^2.$$

Substituting in the given formulas the values of the coefficients A_1, A_2, and A_3 we obtain

$$\xi = -\frac{(T_1 + T)(1 + kk'') - T_1T\left(\frac{k_3}{m} + k'\right)k}{2[T_1T(1 + kk'') + (T_1 + T + 2\xi T_1T)^2]}, \tag{7.192}$$

$$\omega^2 = \frac{\left(\frac{k_3}{m} + k'\right)k}{T_1 + T + 2\xi T_1T} - \xi^2. \tag{7.193}$$

From these equations it is necessary to determine the functions $\xi(a)$ and $\omega(\xi)$ in order to plot the transient. For this purpose we solve Eqs. (7.192) and (7.193) graphically.

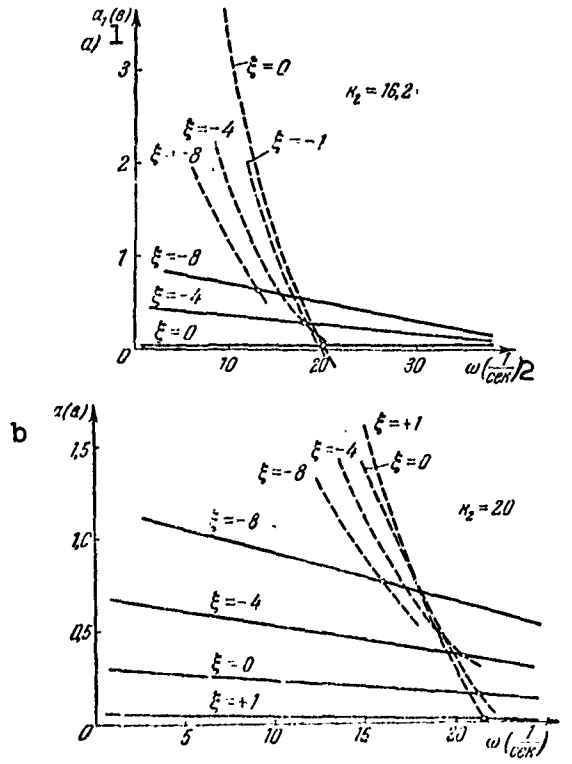


Fig. 7.29. 1) v; 2) sec.

Obtaining from each equation the oscillation amplitude a (which is expressed in terms of k' and k''), we plot for each of the equations the functions $a = a(\omega)$ with $\xi_i = \text{const}$ ($i = 1, 2, 3, \ldots$). The points

where curves having the same index ξ_1 cross correspond to the simultaneous solution of Eqs. (7.192) and (7.193).

Figures 7.29a and b show graphical solutions for $k_2 = 16.2$ and $k_2 = 20$ for the following values of the other system parameters: $T = 0.05$ sec, $T_1 = 0.05$ sec, $u_n = 110$ v, $k_1 = 57$ v, $k_3 = 1.5$ g-cm/v, $m = 0.34$ g-cm-sec, $n = 1$ g-cm-sec, and $k_4 = 0.01$. The solid lines are plotted on the basis of the transformed equation (7.192) and the dashed ones on the basis of Eq. (7.193).

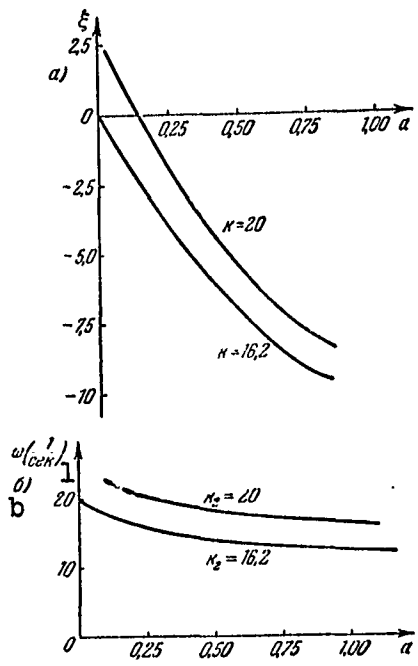


Fig. 7.30. 1) sec.

With the aid of the plots shown in Fig. 7.29 we construct the main functions $\xi(a)$ and $\omega(a)$ (Figs. 7.30a and b) for $k = 16.2$ and 20. The obtained functions make it possible to estimate approximately the character of the transient for the known system parameters.

On the basis of the preceding construction we can obtain a transient quality diagram (Fig. 7.31). The quality diagram shows how the time constant of the transient $\tau = 1/\xi$ and the oscillation frequency ω change with changing oscillation amplitude in the region of parameter values of interest (in this case $16.2 < k_2 < 20$).

The quality diagram for the transient in the nonlinear system enables us to determine, for given values of the system parameters, the overshoot and the duration of the transient. For example, when $k_2 = 20$, the duration of the transient for an initial oscillation amplitude $a_0 = 135$ volts and a final amplitude $a_k = 35$ volts will be, in accordance with (7.115),

$$t_1 = \int_{a_0}^{a_k} \frac{da}{\dot{a}(a)} \approx \sum_{l=1}^{k} \frac{1}{\dot{a}(a_{0l} + 0{,}5\Delta a_l)} \ln \frac{a_{0l} + \Delta a_l}{a_l} =$$
$$= \frac{1}{9{,}5} \ln \frac{135}{100} + \frac{1}{7{,}5} \ln \frac{100}{66} + \frac{1}{5} \ln \frac{66}{35} = 0{,}209 \text{ sec}.$$

The transient is broken down here into three sections, on each of which the average value of ξ was taken. We note that the exact value for this case is $t_1 = 0.240$ sec, i.e., the error in the determination of the duration of the transient amounts to 12.5%.

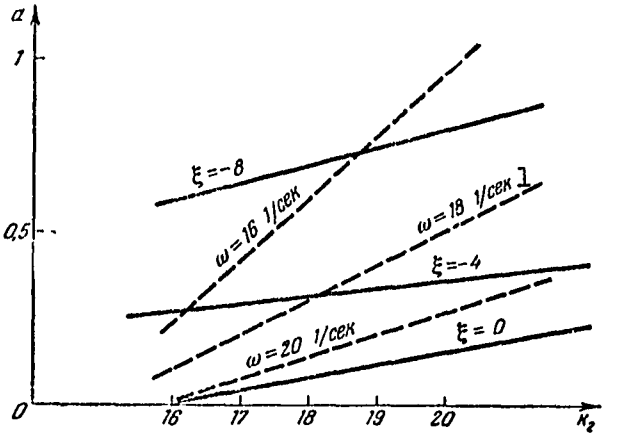


Fig. 7.31. 1) sec.

To estimate the errors of the method, Fig. 7.32 shows the exact graphoanalytic solution of the initial equations. The dashed line is based on the plots of Fig. 7.30 and represent the variation of the transient envelope as obtained from the approximate solution. We see that for nonlinear systems of the second class, too, the approximate

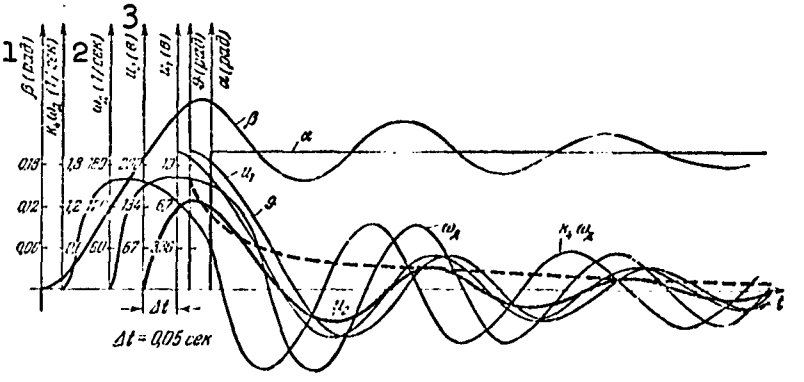


Fig. 7.32. 1) rad; 2) sec; 3) v.

determination of the transient by the harmonic linearization method results in a small error which is perfectly acceptable in technical calculations.

§7.9. Transient in a System of the Third Class

Let us consider the application of the method of harmonic linearization for the construction of the transients in nonlinear systems of the third class, i.e., systems containing several nonlinear elements which are separated by linear parts.

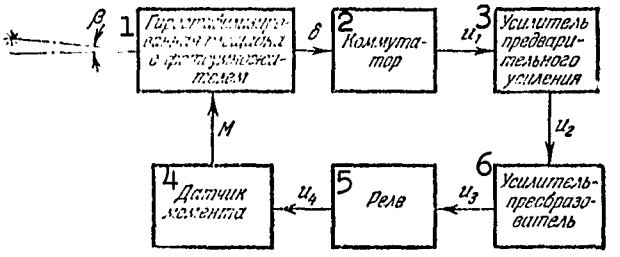


Fig. 7.33. 1) Gyrostabilized platform with photomultiplier; 2) commutator; 3) preamplifier; 4) torque transmitter; 5) relay; 6) amplifier-converter.

By way of the first example let us take a system for tracking a radiation source with a sensitive element in the form of a photomultiplier. A block diagram of the system is shown in Fig. 7.33.

The photomultiplier is installed on a gyrostabilized platform. The polarity and the magnitude of the signal at the output of the photomultiplier does not change with changing sign of the deviation of the optical axis of the photomultiplier from the direction to the radiation source. To determine the direction of the deviation, a commutator is used. The photomultiplier together with the commutator can be represented in the form of an ideal relay element with an ideal relay static characteristic (Fig. 7.34).

The low power signal from the output of the sensitive element is fed to a preamplifier with gain k_1 and time constant T_1.

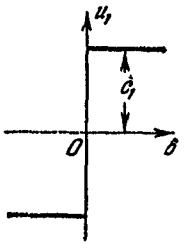


Fig. 7.34

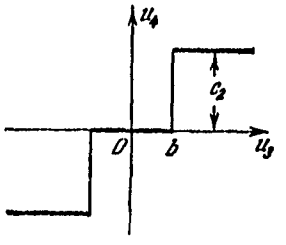


Fig. 7.35

The signal then flows to the amplifier-converter, which amplifies the signal additionally and converts the pulses into a DC voltage. The amplifier-converter has a gain k_2 and a time constant T_2. The voltage u_3 from the amplifier-converter is fed to the control winding of the relay, whose static characteristic is shown in Fig. 7.35.

The relay contact applies the voltage u_4 to the torque transmitter. The latter applies a torque to the gyroscope, which precesses together with the sensitive element in a direction so as to reduce the error δ (the angle in the vertical plane between the optical axis of the sensitive element and the direction to the radiation source).

The system parameters should ensure the following: a) either the absence of self-oscillations or self-oscillations with an amplitude that does not exceed a definite value; b) a rapidly damped transient when the system deviates from the zero position.*

The system elements in their own motion are described by the following equations:

sensitive element

$$\delta = -\beta, \quad u_1 = F_1(\delta),$$

preamplifier

$$(T_1 p + 1) u_2 = k_1 u_1,$$

amplifier-converter

$$(T_2 p + 1) u_3 = k_2 u_2,$$

relay

$$u_4 = F_2(u_3),$$

torque transmitter

$$M = k_3 u_4,$$

gyrostabilized platform

$$\delta = -\frac{k_4}{p} M,$$

where $F_1(\delta)$ and $F_2(u_3)$ are the nonlinear functions for the corresponding elements.

To investigate the system it is necessary first to carry out harmonic linearization of the nonlinearities in accordance with Formulas (7.104).

For the characteristic of the commutator and photomultiplier combination (Fig. 7.34), the harmonic linearization coefficients will be, in accordance with (3.14),

$$q_1(a) = \frac{4c_1}{\pi a}; \quad q_1'(a) = 0.$$

Thus, harmonic linearization yields

$$u_1 = q_1(a)\,\delta = \frac{4c_1}{\pi a}\,\delta. \tag{7.194}$$

Analogously, for the characteristic of the relay (Fig. 7.35) we obtain in accord with (3.13)

$$q_2(a_2) = \frac{4c_2}{\pi a_2}\sqrt{1 - \frac{b^2}{a_2^2}}, \quad q_2'(a_2) = 0$$

and, consequently, the result of the harmonic linearization is

$$u_4 = q_2(a_2)\,u_3 = \frac{4c_2}{\pi a_2}\sqrt{1 - \frac{b^2}{a_2^2}}\,u_3. \tag{7.195}$$

Taking into account the equations of the linear elements and the harmonically linearized equations (7.194) and (7.195) for the nonlinear elements, we write the general system equation for the error angle δ in the form

$$[(T_1 p + 1)(T_2 p + 1)p + k_1 k_2 k_3 k_4 q_1(a)\,q_2(a_2)]\,\delta = 0.$$

Thus, the characteristic equation of the system has the form

$$p^3 + A_1 p^2 + A_2 p + A_3 = 0, \qquad (7.196)$$

where

$$A_1 = \frac{T_1 + T_2}{T_1 T_2}, \quad A_2 = \frac{1}{T_1 T_2}, \quad A_3 = \frac{k_1 k_2 k_3 k_4 q_1(a) q_2(a_2)}{T_1 T_2}.$$

The connection between the oscillation amplitudes at the input of the nonlinear elements $\underline{a}$ and a_2, in accordance with the block diagram of the system, is defined by a suitable transfer function:

$$\frac{b}{a_2} = \frac{4c_2}{\pi a_2} \sqrt{1 - \frac{b^2}{a_2^2}} \frac{k_3 k_4}{p}.$$

This results in an expression for the amplitude of the oscillations of the gyrostabilized platform

$$a = \frac{4c_2}{\pi} \frac{k_3 k_4}{\sqrt{\xi^2 + \omega^2}} \sqrt{1 - \frac{b^2}{a_2^2}}. \qquad (7.197)$$

If we replace in the expressions for the coefficients of the characteristic equation (7.196) the value of $\underline{a}$ by a_2, in accordance with Formula (7.197), then our investigation will determine the amplitude of the voltage oscillations at the input of the relay element. However, great practical interest attaches to an investigation of the oscillations of the sensitive elements, and therefore it is advantageous to use Eq. (7.197) in the form

$$a_2 = \frac{4c_2 k_3 k_4 b}{\sqrt{(4c_2 k_3 k_4)^2 - a^2 \pi^2 (\xi^2 + \omega^2)}}. \qquad (7.198)$$

The equations for the investigation of the transients will have in this case, according to (7.82) and (7.83), the form

$$\left.\begin{aligned} \xi &= -\frac{\frac{T_1 + T_2}{T_1 T_2} - k_1 k_2 k_3 k_4 q_1(a) q_2(a_2)}{2T_1 T_2 \left(\frac{T_1 + T_2}{T_1 T_2} + 2\xi\right)}, \\ \omega^2 &= \frac{k_1 k_2 k_3 k_4 q_1(a) q_2(a_2)}{T_1 + T_2 + 2T_1 T_2 \xi} - \xi^2. \end{aligned}\right\} \qquad (7.199)$$

The coefficient characterizing the parameters of the nonlinear element will be, in accordance with the values of $q_1(a)$, $q_2(a_2)$, and (7.197),

$$q_1(a)\,q_2(a_2)=\frac{4c_1}{\pi k_3 k_1 a_2}\sqrt{\xi^2+\omega^2}.$$

Replacing in the formula obtained the oscillation amplitude a_2 by its value from (7.198), we get

$$q_1(a)\,q_2(a_2)=\frac{4c_1(\xi^2+\omega^2)}{\pi b k_3 k_1}\sqrt{\frac{1}{\xi^2+\omega^2}-\left(\frac{a\pi}{4c_2k_3k_1}\right)^2}. \qquad (7.200)$$

To investigate the transients, it is necessary to substitute in (7.199) the resultant value of $q_1(a)q_2(a_2)$ and to solve the equations by successive approximations. In this case, however, the expressions obtained are cumbersome and relatively difficult to solve. Consequently, before solving the formulas it is best to simplify them somewhat, without making at the same time excessive assumptions or approximations.

In accordance with the idea underlying the method, one of the conditions under which the method is applicable is that the transient be sufficiently oscillatory. In practice the oscillations must not be damped faster than within 1-3 periods. Even when the oscillations damp out within one period, the ratio $\omega/\xi = 4$. In this case ξ^2 amounts to less than 7% of ω^2. Consequently, taking account of the approximate nature of the method, we can neglect in all formulas the quantity ξ^2 compared with ω^2. Simultaneously, to abbreviate the notation, we introduce the symbols

$$k_r=k_3k_1, \quad k_y=k_1k_2.$$

In this case the main formulas (7.199) become

$$\left.\begin{aligned}\xi&=-\frac{\frac{T_1+T_2}{T_1T_2}-\frac{4c_1\omega}{\pi b}k_y\sqrt{1-\left(\frac{a\pi\omega}{4c_2k_r}\right)^2}}{2T_1T_2\left(\frac{T_1+T_2}{T_1T_2}+2\xi\right)^2},\\ \omega^2&=\left(\frac{4c_1c_2k_yk_r}{\pi}\right)^2\frac{1}{(c_2bk_r)^2(T_1+T_2+2T_1T_2\xi)^2+(ac_1k_y)^2}.\end{aligned}\right\} \qquad (7.201)$$

It is seen from (7.201) that the damping coefficient ξ of the transients and the frequency ω of the oscillations are complicated functions of the system parameters and of the oscillation amplitude a.

In addition, the damping coefficient is a function of the oscillation frequency, and the oscillation frequency in turn is a function of the damping coefficient.

The first step in the solution of the general problem, that of investigating the transients in the nonlinear system, is to determine the self-oscillations. In this case $\xi = 0$, $a = A$, $\omega = \Omega$, and the formulas (7.201) assume the form

$$\left.\begin{aligned} &\frac{T_1+T_2}{T_1T_2} - \frac{4c_1\Omega k_y}{\pi b}\sqrt{1-\left(\frac{A\pi\Omega}{4c_2k_r}\right)^2} = 0,\\ &\Omega = \frac{4c_1c_2k_yk_r}{\pi}\,\frac{1}{\sqrt{(c_2bk_r)^2(T_1+T_2)^2+(Ac_1k_y)^2}}. \end{aligned}\right\} \qquad (7.202)$$

From the first and second formulas in (7.202) we obtain expressions for the oscillation amplitude

$$\left.\begin{aligned} A^2 &= \left(\frac{c_2k_r}{\Omega^2}\right)^2\left[1{,}62 - \left(\frac{T_1+T_2}{T_1T_2}\right)^2\frac{b^2}{(c_1\Omega k_y)^2}\right],\\ A^2 &= \left(\frac{4c_2k_r}{\pi\Omega}\right)^2 - \frac{(c_2bk_r)^2(T_1+T_2)^2}{(c_1k_y)^2}. \end{aligned}\right\} \qquad (7.203)$$

Equating the resultant expressions for the squares of the amplitude and canceling, we obtain

$$\Omega^2 = 1/T_1T_2,$$

i.e., in this nonlinear system the self-oscillation frequency is determined only by the time constants of the system and is independent of the gains.

Substituting the resultant value of Ω in the first formula of (7.203) we obtain a final expression for the self-oscillation amplitude

$$A^2 = (c_2k_r)^2T_1T_2\left[1{,}62 - \frac{(T_1+T_2)^2b^2}{T_1T_2c_1^2k_y^2}\right]. \qquad (7.204)$$

Inasmuch as the coefficient $(c_2k_g)^2T_1T_2$ is obviously positive, self-oscillations are possible in the system only if

$$\frac{(T_1+T_2)^2b^2}{T_1T_2c_1^2k_y^2} < 1{,}62.$$

Therefore the critical value of the gain will be

$$k_{y.\,кр} = 0{,}786\,\frac{b}{c_1}\,\frac{T_1+T_2}{\sqrt{T_1T_2}}.$$

It follows from (7.204) that to reduce the self-oscillation amplitude it is necessary first of all to reduce the gain of the system and the time constants.

Figure 7.36 shows, for specific numerical values of the parameters $T_1 = 0.1$ sec, $c_1 = 5 \cdot 10^{-3}$ volts, $k_g = 1.6 \cdot 10^{-5}$ rad/v-sec, $T_2 = 0.2$ sec, $c_2 = 40$ volts, $k_u = 4700$, $b = 10$ volts, $k_1 = 1.6 \cdot 10^{-5}$ $v^{-1}sec^{-1}$, plots of the amplitude and self-oscillation frequency as functions of several system parameters.

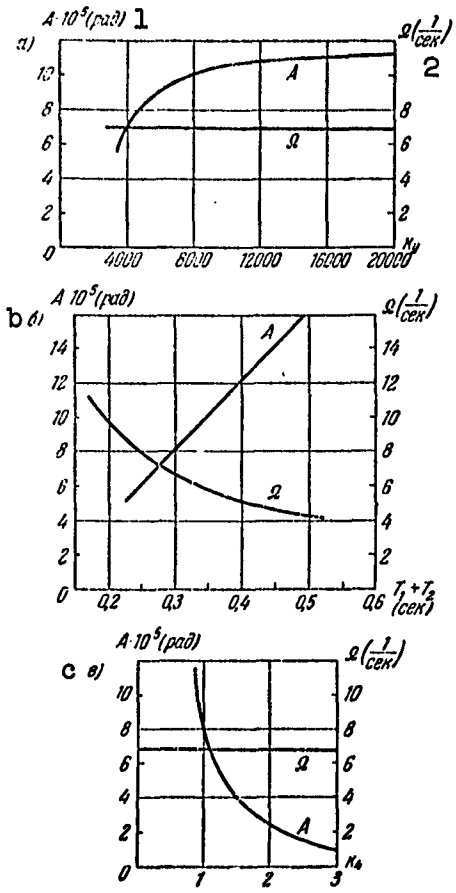


Fig. 7.36. 1) rad; 2) sec.

It follows from Fig. 7.36a that the change in the self-oscillation amplitude is most strongly effected by the gain in the region where the gain is small. From Fig. 7.36b it is seen that the amplitude of the self-oscillations changes in direct proportion to the "over-all time constant" $T_1 + T_2$. Figure 7.36c illustrates the influence of the coefficient k_4, which is contained in the coefficient k_g, on the amplitude of the self-oscillations. The coefficient k_4 can be varied by changing the kinetic torque of the gyroscope H.

The plots presented enable us to choose approximately the main parameters of the system. We can then solve the next problem, that is, investigate the transients.

Under the conditions prevailing when the system is first turned on, or under some other conditions, the receiver may deviate appreciably from its zero position. The magnitude of the deviation can exceed

by several times the amplitude of the self-oscillations. After the system is turned on, a transient sets in, in which the amplitude of the oscillations will decrease to the value of the self-oscillation amplitude. The damping time is determined by the system parameters and is of appreciable significance for its operating quality. In practice it is necessary that the transient terminate at any rate after a few seconds. Therefore, the next calculation should yield, first, the possibility of determining the time constant and the frequency of the oscillations during the transient, and, secondly, recommendations on the choice of system parameters such as to obtain the required transients, if the transient is obtained by the first calculation are not acceptable.

The variation of the amplitude and oscillation frequency in the transient is determined in the case of the investigated system by Formulas (7.201). The solution of these formulas, as indicated above, can be carried out by successive approximation. Inasmuch as in accordance with the conditions under which the method is applicable the transient must be sufficiently oscillatory, we assume that $(T_1 + T_2)/T_1T_2 \gg 2\xi$. Under these assumptions the solution of the formulas (7.201) is carried out in the following sequence.

We substitute into the formulas the system parameters which are either selected or specified by the operating conditions. In particular, for example, $T_1 = 0.1$ sec, $c_2 = 10$ volts, $k_g = 1.6 \cdot 10^{-5}$ sec^{-1}, $T_2 = 0.2$ sec, $k_u = 4700$, $a_0 = 82 \cdot 10^{-5}$ rad, $c_1 = 5 \cdot 10^{-3}$ volts, and $b = 10$ volts. Here a_0 is the initial deviation of the system from the zero position.

As a result of this, there are left in the right halves of Formulas (7.201) two unknown quantities: the oscillation frequency ω and the system damping coefficient ξ. In first approximation we put in the

right halves of the formulas $\xi = 0$ and determine from the second formula of (7.201) the first approximation for the oscillation frequency ω_1. We substitute the resultant value of ω_1 in the first formula of (7.201) and determine the first approximation for the damping coefficient ξ. Substituting the value of ξ in the second formula of (7.199), we determine the second approximation ω_2. In the same way we determine the second approximation ξ_2. If the resultant value of ξ_2 differs appreciably from ξ_1, a third approximation is determined.

In practice, owing to the small value of ξ_1, the solution is obtained quite rapidly. We can thus obtain the values of ξ and ω for one of the values of the oscillation amplitude a. In particular, for the foregoing values of the system parameters and for $a_{11} = 82 \cdot 10^{-5}$ radians we have $\xi \approx -2.3$ sec^{-1}, $\omega \approx 1$ sec^{-1}.

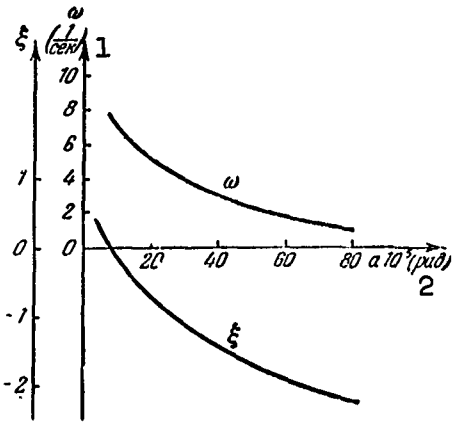


Fig. 7.37. 1) sec; 2) rad.

The calculation is then repeated for the other necessary values of the oscillation amplitudes.

Figure 7.37 shows plots of the damping coefficient ξ and of the oscillation frequency ω as functions of the oscillation amplitude a. The plots obtained enable us to estimate the transient approximately.

If similar calculations are carried out for several other values of one of the system parameters, for example the gain, then we can plot a quality diagram for the nonlinear process, which enables us to evaluate the transients over the entire range of variation of the parameter of interest.

The system considered above was a single-loop system. This simplified appreciably all the derivations. However, even for more com-

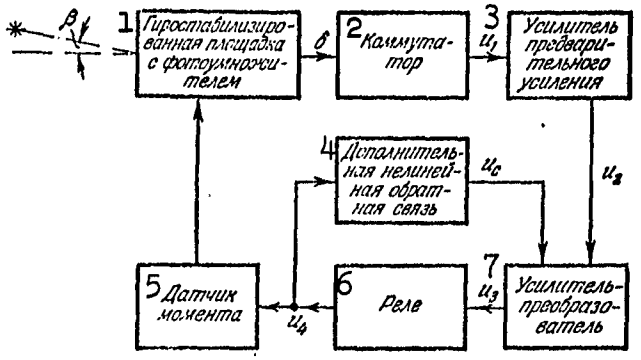


Fig. 7.38. 1) Gyrostabilized platform with photomultiplier; 2) commutator; 3) preamplifier; 4) additional feedback loop; 5) torque transmitter; 6) relay; 7) amplifier-converter.

plicated systems, the investigation of the transients entails no difficulties.

Let us assume, for example, that the calculation has shown that a system with a similar block diagram does not provide the necessary transient parameters. It is therefore necessary to introduce into the system additional correcting signals, for example derivative feedback. It is difficult to pick off a signal proportional to the velocity or the acceleration of the precessional motion of the platform, in view of the slow speed of the platform. Consequently, it is advantageous to consider, for example, a version of a system with a nonlinear derivative feedback loop encompassing the relay and the amplifier stages (Fig. 7.38).

In this case the transients of the systems own motion are described by the following equations:

$$\left.\begin{array}{l} \delta = -\beta;\ u_1 = F_1(\delta);\ u_2 = k_1 u_1;\ (T_1 p + 1) u_3 = k_2 (u_2 - u_c); \\ u_4 = F_2(u_3);\ M = k_3 u_4;\ (T_c p + 1) u_c = p k_c F_3(u_4);\ \delta = -\frac{k_1}{p} M. \end{array}\right\} \quad (7.205)$$

Unlike the preceding example, the preamplifier is assumed here to be not subject to lag.

We assume that the nonlinear term in the feedback equation (the

last equation of (7.205)) is defined by the cubic equation $F_3 = u_4^3$.

The coefficients of harmonic linearization of the nonlinearities will be, in accordance with (3.14), (3.13), and (3.32):

$$q_1(a) = \frac{4c_1}{\pi a}; \quad q_2(a_2) = \frac{4c_2}{\pi a_2}\sqrt{1 - \frac{b^2}{a_2^2}}; \quad q_3(a_3) = \frac{3a_3^2}{4}.$$

Combining the equation of the system elements we obtain with account of the harmonic linearization of the nonlinearities a general system equation for the error δ in the form

$$\{(T_1p + 1)(T_cp + 1) + k_2k_cq_2(a_2)q_3(a_3)p]p + \\ + k_1k_2k_3k_4q_1(a)q_2(a_2)(T_cp + 1)\}\delta = 0.$$

The characteristic equation of the system will be

$$p^3 + A_1p^2 + A_2p + A_3 = 0,$$

where

$$A_1 = \frac{T_1 + T_c}{T_1T_c}; \quad A_2 = \frac{1 + k_2k_cq_2(a_2)q_3(a_3) + k_1k_2k_3k_1q_1(a)q_3(a_3)T_c}{T_1T_c};$$

$$A_3 = \frac{k_1k_2k_3k_4q_1(a_1)q_2(a_2)}{T_1T_c}.$$

To abbreviate the notation we introduce the symbols

$$k_c' = k_2k_c, \qquad q_2(a_2)q_3(a_3) = q_4(a_2, a_3),$$
$$k = k_1k_2k_3k_4, \qquad q_1(a_1)q_2(a_2) = q_5(a, a_2).$$

The equations for the investigation of the transients will then assume, in accordance with (7.82) and (7.83), the following form

$$\xi = -\frac{\frac{T_1 + T_c}{T_1T_c}[1 + k_c'q_4(a_2, a_3) + kq_5(a, a_2)] - kq_5(a, a_2)}{2\left[1 + k_c'q_4(a_2, a_3) + kq_5(a, a_2) + T_1T_c\left(\frac{T_1 + T_c}{T_1T_c} + 2\xi\right)^2\right]},$$

$$\omega^2 = \frac{kq_5(a, a_2)}{T_1 + T_c + 2\xi T_1T_c} - \xi^2.$$

According to (7.205), the connection between a, a_2, and a_3 is determined by the formulas

$$a = \frac{k_3k_4}{\omega}\frac{4c_2}{\pi a_2}\sqrt{1 - \frac{b^2}{a_2^2}}, \quad a_2 = \frac{k_3k_4}{\omega}\frac{4c_2}{\pi}\sqrt{1 - \frac{b^2}{a_2^2}},$$

$$a_3 = \frac{4c_2}{\pi}\sqrt{1 - \frac{b^2}{a_2^2}}.$$

From there on the investigation is similar to the one made in the case of the single-loop system.

To determine the errors of the method, Fig. 7.39 shows a plot of the transient in the system obtained by numerical-graphical solution of the nonlinear equations of this system with initial value of the oscillation amplitude $a_0 = 80 \cdot 10^{-5}$ radian and with the previously assumed parameters. The dashed curve in the same figure is the envelope of the transients, plotted approximately on the basis of the $\xi(a)$ plots.

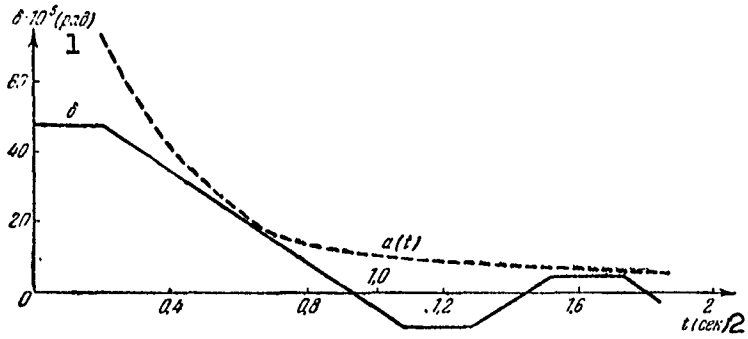


Fig. 7.39. 1) rad; 2) sec.

It is seen from the figure that in determining the envelope of the transient in nonlinear systems of the third class the approximate calculation by the harmonic linearization method results in a small error which is acceptable in technical calculations.

§7.10. Example of Slipping Transient

Let us consider an investigation of a slipping mode in a nonlinear automatic system, as carried out by M.V. Starikova [227]. The self-oscillations in such a system were investigated in §6.8.

The slipping mode is regarded as a monotonic variation of the coordinate on which self-oscillations are superimposed.

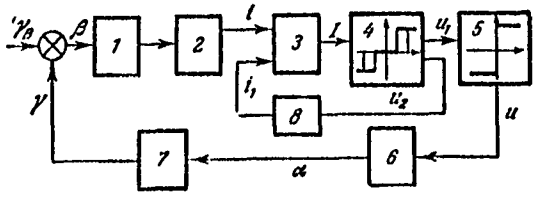


Fig. 7.40

The investigated system is represented by block diagrams (Figs. 7.40 and 7.41) in which 1 is the regulated object, 2 the sensitive element, 3 the amplifier, 4 the control relay, 5 the power relay, 6 the actuating mechanism, 7 the regulating organ, and 8 the feedback element. The system is described by the following equations:

object with sensitive element

$$(d_1p^2+d_2p+d_3)(T_1p+1)l=(c_1p+c_2)(\gamma_B-\gamma), \quad (7.206)$$

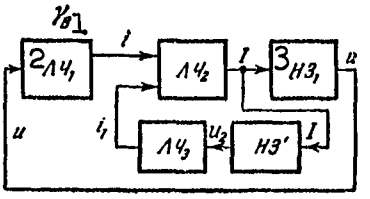


Fig. 7.41. 1) v; 2) LCh; 3) NZ.

amplifier

$$(T_2p+1)I=k_1i-k_2i_1, \quad (7.207)$$

nonlinear element NZ_1

$$u=F_1(I), \quad (7.208)$$

actuating mechanism

$$(T_3p+1)p\alpha=k_3e^{-\tau p}u, \quad (7.209)$$

regulating organ

$$\gamma=s\alpha, \quad (7.210)$$

feedback with nonlinear element NZ'

$$i_1=k_{o.c}u_2, \quad u_2=F'(I)=kF_1(I), \quad (7.211)$$

where γ is the regulating signal, γ_v the disturbance signal, τ the delay in the actuating mechanism and in the relay, and s the slope of the characteristic of the regulating organ.

We shall seek a solution for the variable I in the form of two components

$$I=I^0+I^*,$$
$$I^*=a_1\sin\omega t,$$

where I^0 is the slowly varying aperiodic component of the current and I* is the oscillatory component.

Figure 7.42 shows the characteristics of the nonlinear elements NZ_1 (Fig. 7.42a) and NZ' (Fig. 7.42b).

Harmonic linearization of Eq. (7.208) yields

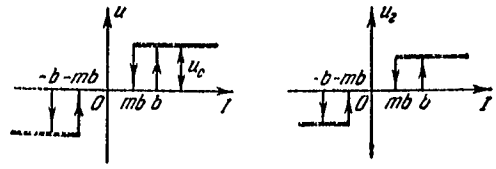


Fig. 7.42

$$u = F_1^0(a_1, l^0) + \left[q_1(a_1, l^0) + \frac{q_1'(a_1)}{\omega} p\right] l^*, \tag{7.212}$$

where the component F_1^0 and the coefficients q_1 and q'_1 for oscillations in which only one relay contact is in operation (slipping mode), i.e., under the condition

$$|b - l^0| < a_1 < b + l^0,$$

have the following values

$$\left.\begin{aligned} F_1^0(a_1, l^0) &= \frac{u_c}{2\pi}\left[\pi - \arcsin\frac{mb - |l^0|}{a_1} - \arcsin\frac{b - |l^0|}{a_1}\right]\operatorname{sign} l^0, \\ q_1(a_1, l^0) &= \frac{u_c}{\pi a_1}\left[\sqrt{1 - \left(\frac{b - |l^0|}{a_1}\right)^2} + \sqrt{1 - \left(\frac{mb - |l^0|}{a_1}\right)^2}\right], \\ q_1'(a_1) &= \frac{u_c}{\pi a_1}[m - 1]. \end{aligned}\right\} \tag{7.213}$$

To investigate the self-oscillations we use the method of harmonic linearization. For this purpose we derive from the linear equations of the system elements and from the harmonically linearized equation of the nonlinear element a characteristic equation, using the oscillatory components of the alternating quantities.

The characteristic equation has the form

$$\begin{aligned} &(d_1p^2 + d_2p + d_3)(T_1p + 1)(T_2p + 1)(T_3p + 1)p + \\ &\quad + sk_1k_3(c_1p + c_2)\left[q_1(a_1, l^0) + \frac{q_1'(a_1)}{\omega} p\right]e^{-\tau p} + \\ &\quad + k_{0.c}k_2k(d_1p^2 + d_2p + d_3)(T_1p + 1)(T_3p + 1)p\left[q_1(a_1, l^0) + \right. \\ &\qquad \left. + \frac{q_1'(a_1)}{\omega} p\right] = 0. \end{aligned} \tag{7.214}$$

The substitution $p = j\omega$ into the characteristic polynomial (7.214) yields the equation

$$L(j\omega) = X(a_1, \omega, l^0) + jY(a_1, \omega, l^0) = 0,$$

which we break up into two equations

$$X(a_1, \omega, I^0) = 0, \quad Y(a_1, \omega, I^0) = 0. \tag{7.215}$$

To determine the amplitude and frequency of the vibrations for specified system parameters, we rewrite the left halves of Eqs. (7.215) in the form

$$\left.\begin{aligned} X(a_1, \omega, I^0) &= X_1(\omega) + sX_2(a_1, \omega, I^0) + k_{0.c} X_3(a_1, \omega, I^0), \\ Y(a_1, \omega, I^0) &= Y_1(\omega) + sY_2(a_1, \omega, I^0) + k_{0.c} Y_3(a_1, \omega, I^0). \end{aligned}\right\} \tag{7.216}$$

The stability of the periodic solution is determined by means of the known methods.

Let us determine the conditions for the existence of a slipping mode. In the case of a constant or slowly varying perturbation, the slipping mode will occur if the slowly varying component I^0 and the amplitude a_1 change within such limits, that only one and the same contact operates (Fig. 7.43), i.e., the slipping mode occurs when the following conditions are satisfied

$$b - a_1 \leqslant |I^0| \leqslant a_1 + mb; \quad |I^0| > a_1 - b. \tag{7.217}$$

The first condition guarantees here the pull in or the drop out of one relay contact; the second condition guarantees that the second contact of the relay will not pull in.

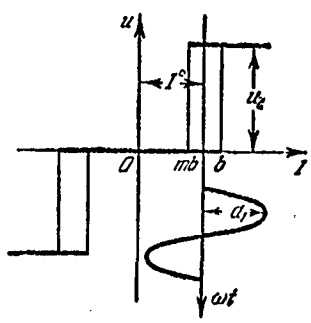


Fig. 7.43

Let us plot the vibration-smoothed characteristic of the nonlinear element. Variation of the value of I^0 brings about changes in the values of a_1 and ω, since the coefficient $q_1(a_1, I^0)$ contained in the characteristic equation depends on I^0.

Let us find the dependence of the amplitude and frequency of the vibrations, a_1 and ω, on the slowly varying component I^0. For this purpose, by specifying different values of I^0, we determine $q_1(a_1, I^0)$; we then find for the same values of I^0 the values of a_1 and ω from the equations $X = 0$, $Y = 0$.

From the condition (7.217) it follows that the limiting values of I^0, at which a slipping mode is possible, are as follows:

$$|I_1^0| = b - a_1; \quad |I_2^0| = a_1 + mb \quad (|I^0| > a_1 - b). \tag{7.218}$$

Let us find the amplitude and frequency of the vibrations for these values of I^0 with the system parameters specified. For convenience, we represent the terms of the characteristic equation (7.214) in the following form:

$$\left.\begin{aligned}
&N_1(\omega) = T_1T_2T_3d_1\omega^6 + \\
&+ [(T_1T_2T_3)d_1 + (T_1T_2 + T_1T_3 + T_2T_3)d_2 + T_1T_2T_3d_3]\omega^4 - \\
&\qquad - [(T_1 + T_2 + T_3)d_3 + d_2]\omega^2, \\
&Y_1(\omega) = [(T_1T_2 + T_1T_3 + T_2T_3)d_1 + T_1T_2T_3d_2]\omega^5 - \\
&- [d_1 + (T_1 + T_2 + T_3)d_2 + (T_1T_2 + T_1T_3 + T_2T_3)d_3]\omega^3 + \\
&\qquad + d_3\omega, \\
&N_2(a_1, \omega, I^0) = \\
&\qquad = k_1k_3\{[q_1(a_1, I^0)\cos\omega\tau + q_1'(a_1)\sin\omega\tau]c_2 - \\
&\qquad\qquad - [q_1'(a_1)\cos\omega\tau - q_1(a_1, I^0)\sin\omega\tau]c_1\omega\}, \\
&Y_2(a_1, \omega, I^0) = \\
&\qquad = k_1k_3\{[q_1(a_1, I^0)\cos\omega\tau + q_1'(a_1)\sin\omega\tau]c_1\omega + \\
&\qquad\qquad + [q_1'(a_1)\cos\omega\tau - q_1(a_1, I^0)\sin\omega\tau]c_2\}, \\
&N_3(a_1, \omega, I^0) = k_2q_1(a_1, I^0)\{[T_1T_2d_2 + (T_1 + T_2)d_1]\omega^4 + \\
&\qquad + [(T_1 + T_2)d_3 + d_2]\omega^2\} + kk_2q_1'(a_1) \times \\
&\times\{-d_1T_1T_2\omega^5 + [T_1T_2d_3 + (T_1 + T_2)d_2 + d_1]\omega^3 - d_3\omega\}, \\
&Y_3(a_1, \omega, I^0) = kk_2q_1(a_1, I^0)\{T_1T_2d_1\omega^5 - \\
&\qquad - [T_1T_2d_3 + (T_1 + T_2)d_2 + d_1]\omega^3 + d_3\omega\} + \\
&+ kk_2q_1'(a_1)\{[T_1T_2d_2 + (T_1 + T_2)d_1]\omega^4 - \\
&\qquad - [(T_1 + T_2)d_3 + d_2]\omega^2\}.
\end{aligned}\right\} \tag{7.219}$$

From Eqs. (7.215) we determine the amplitude and frequency of the self-oscillations for constant values of I^0 within the limits of interest to us, and obtain the dependence $a_1 = a_1(I^0)$.

As a result we can readily determine the influence of the slowly varying component I^0 on the component F_1^0 of the harmonically linearized equation (7.212) in accordance with the first formula of (7.213), using the dependence $a_1 = a_1(I^0)$. The resultant dependence $F_1^0 = F_1^0(I^0)$ is the characteristic of the nonlinear element with respect to a slowly varying input signal I^0, or the so-called bias function, corresponding to the slipping transient process.

By way of an example let us use the following numerical values:

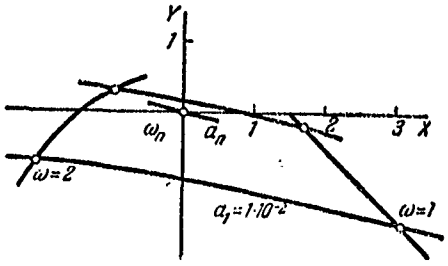


Fig. 7.44

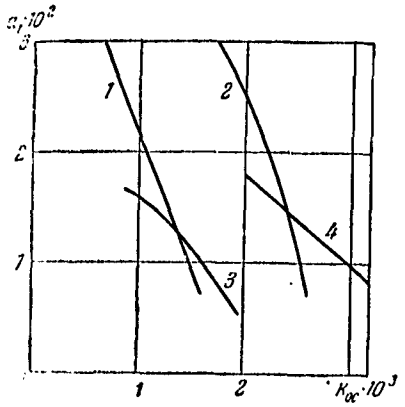


Fig. 7.45

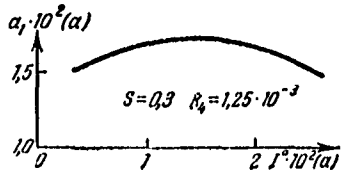


Fig. 7.46

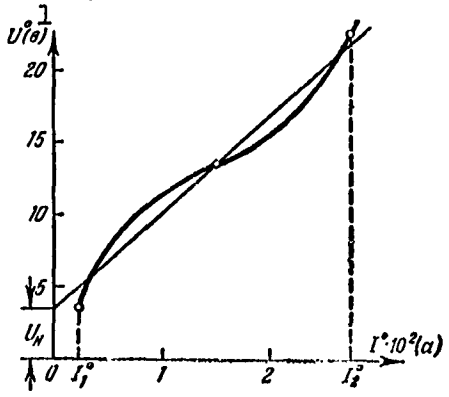


Fig. 7.47

backlash zone of the nonlinear element $b = 1.8\cdot10^{-2}$ amp, relay resetting ratio $m = 0.65$, $u_s = 26$ volts. In this case we have

$$|I_1^0| = 1{,}8\cdot10^{-2} - a_1, \quad |I_2^0| = a_1 + 1{,}18\cdot10^{-2},$$
$$|I_{cp}^0| = \frac{mb+b}{2} = 1{,}5\cdot10^{-2}.$$

Figure 7.44 shows an example of the graphoanalytic method of determining the amplitude and frequency of the self-oscillations (see §6.8) for parameter values $s = 0.3$, $k_{o.s} = 1.25\cdot10^{-3}$, and the obtained values of I_1^0, and I_2^0. Figure 7.45 shows the dependence $a_1 = a_1(k_{o.s})$ for certain values of the parameters: 1 – for $s = 0.3$, I_{sr}^0; 2 – for $s = 0.5$, I_{sr}^0; 3 – for $s = 0.3$, I_1^0, I_2^0; 4 – for $s = 0.5$, I_1^0, I_2^0.

Let us determine the parameters of the sliding mode for values $s = 0.3$ and $k_{o.s} = 1.25\cdot10^{-3}$.

For the vibration amplitude we obtain values $a_1 = 1.7\cdot10^{-2}$ for I_{sr}^0 and $a_1 = 1.5$ for the limiting values $I_1^0 = 1.8\cdot10^{-2} - a_1$ and $I_2^0 = 1.18\cdot10^{-2} + a_1$. Using these data, we obtain

$$I_1^0 = 0{,}3\cdot10^{-2}a \text{ and } I_2^0 = 2{,}68\cdot10^{-2}a.$$

Figure 7.46 shows a plot of $a_1 = a_1(I^0)$. From the second condition of (7.217) it follows that $a_1 < |I^0| + b$.

As can be seen from the figure, this condition is satisfied, i.e., both Conditions (7.217) are fulfilled and the slipping mode does take place. From the obtained relation $a_1 = a_1(I^0)$ we determine $F_1^0 = F_1^0(I^0)$, using the first formula of (7.213) and the relation

$$u = F_1^0 + u^*.$$

The smooth characteristic of the nonlinear element is shown in Fig. 7.47.

Once we have the smooth characteristic we can plot the transient for the case of a slowly varying external signal.

In order to be able to use linear methods in plotting the transient, we linearize the smooth relay characteristic and describe it by the following equation

$$U^0 = k_{\text{н}} I^0 + U_{\text{н}}.$$

Putting $I^0 = I$, $U^0 = U$, we obtain an expression for the slowly varying component

$$I = \frac{1}{k_{\text{н}}}(U - U_{\text{н}}). \qquad (7.220)$$

We can now plot the transient by the methods of the linear theory, for example by the method of trapezoidal frequency characteristics. For this purpose, using Eqs. (7.206)-(7.211) and (7.220), we obtain the equation of the closed system. The motion of the system, described by the indicated system of differential equations, will be slipping so long as the conditions (7.217) will be satisfied, i.e., so long as I^0 varies within the limits

$$b - a_1 \leqslant |I^0| \leqslant a_1 + mb, \quad |I^0| > a_1 - b.$$

The equation of the closed system for the variable $\underline{i}$ will have the form

$$\begin{aligned}&\{(d_1p^2 + d_2p + d_3)(T_1p + 1)[(T_2p + 1)\cdot + k_{\text{н}}k_2kk_{\text{o.c}}](T_3p + 1)p + \\ &+ k_1k_{\text{н}}k_3se^{-\tau p}(c_1p + c_2)\}\, i = (T_1p + 1)[(T_2p + 1) + \\ &+ k_{\text{н}}k_2k_{\text{o.c}}k](T_3p + 1)p(c_1p + c_2)\gamma_{\text{н}} - sk_3c_3U_{\text{н}}.\end{aligned} \qquad (7.221)$$

The solution of the equation consists of two parts:

$$i = i(t) + i_н.$$

The value of i_n is determined from the algebraic equation

$$c_2 k_1 k_н s k_3 i_н = s k_3 c_2 U_н,$$

i.e.,

$$i_н = \frac{1}{k_1 k_н} U_н.$$

The variable i(t) is determined from the differential equation

$$\begin{aligned}\{(d_1 p^2 + d_2 p + d_3)(T_1 p + 1)(T_3 p + 1) p [(T_2 p + 1) + k_н k_2 k k_{о.с}] + \\ + k_1 k_н s k_3 e^{-\tau p} (c_1 p + c_2)\} i = \\ = (T_1 p + 1)(T_3 p + 1) p [(T_2 p + 1) + k_н k_2 k_{о.с} k] (c_1 p + c_2) \gamma_н.\end{aligned} \qquad (7.222)$$

The result of solving the equation by the method of trapezoidal characteristics [202] for certain values of the system parameters and for the current i at the input of the amplifier is shown in Fig. 7.48.

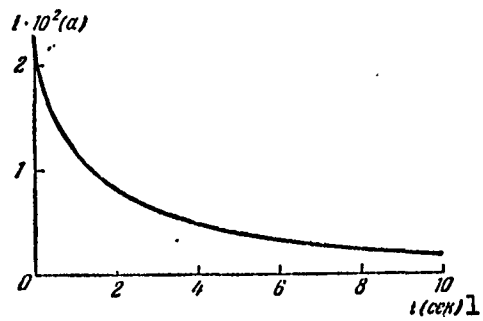


Fig. 7.48. 1) sec.

The previously obtained self-oscillations are superimposed on the obtained transient.

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[Footnotes]

627 Here we have taken a sin ψ instead of a cos ψ, but this introduces no essential change.

631 See below (end of this section) for the justification of this expansion.

645 The term "period," like "frequency," has no perfectly clear physical meaning for damped oscillations and is used in an arbitrary sense.

[Footnotes (Continued)]

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661	More accurately, it would be necessary to place the sign Im in front of both complex expressions, but this does not influence the result.
680	In this version of the method we are speaking of an estimate of the quality of the transient within the stability region of the nonlinear system, inasmuch as in the region where self-oscillations exist one can always employ the simpler first version of the methods, which was described above.
707	This problem was solved by Ye.I. Khlypalo.

[List of Transliterated Symbols]

Manuscript Page No.	
659	л = l = lineynyy = linear
659	н = n = nelineynyy = nonlinear
671	к = k = kolebaniya = oscillations
671	п = p = pereregulirovaniye = overshoot
672	с = s = signal = signal
684	к = k = kil' = keel
689	о.с = o.s = obratnaya svyaz'
698	д = d = dvigatel' = motor
696	т = t = tochnyy = exact
698	у = u = usilitel', usileniya = amplifier, amplification, gain
698	Дв = Dv = Dvigatel' = motor
700	н = n = nominal'nyy = nominal
711	кр = kr = kriticheskiy = critical
717	в = v = vozdeystviye = disturbance
718	ЛЧ = LCh = Lineynaya Chast' = linear part
718	НЗ = NZ = Nelineynoye Zveno = nonlinear link
722	ср = sr = sredniy = average

Chapter 8

CALCULATION OF HIGHER HARMONICS OF SELF-OSCILLATIONS

§8.1. Determination of a Finite Number of Higher Harmonics and a More Accurate First Harmonic of the Self-Oscillations

So far we have sought the first approximation of the periodic solution (self-oscillation) for nonlinear systems in the form*

$$x = A \sin \Omega t, \tag{8.1}$$

which corresponded to an approximate value of the first harmonic of the periodic solution. All the higher harmonics were discarded here as being too small. In §2.2 we derived conditions that ensured the smallness of all the higher harmonics of the periodic solution for the variable $\underline{x}$ in the presence of an essential nonlinearity F(x, px).

Leaving aside all the conditions imposed there on the system, let us determine [242] the small higher harmonics defined by Formula (2.47):

$$\varepsilon x_{\text{II}}(t) = \varepsilon \sum_{k=2}^{\infty} A_k \sin (k\Omega_1 t + \varphi_k), \tag{8.2}$$

for the periodic solution (2.45):

$$x = x_{\text{I}} + \varepsilon x_{\text{II}}(t), \tag{8.3}$$

which makes it possible also to obtain subsequently a more exact value of the amplitude and frequency of the first harmonic (2.46):

$$x_{\text{I}} = A_1 \sin \Omega_1 t \tag{8.4}$$

as compared with its first approximation (8.1).

All the previously discarded higher harmonics were written out in solution (8.3) in the form of a single additional term $\varepsilon x_v(t)$. Now, using the expansion (8.2), we introduce a separate symbol for each $\underline{k}$-th

harmonic

$$x_k = \delta_k A \sin(k\Omega_1 t + \varphi_k), \tag{8.5}$$

where the amplitude $\delta_k A$ of the $\underline{k}$-th harmonic, previously denoted εA_k, is now expressed in terms of the amplitude of the first harmonic A, with the coefficient δ_k being a small quantity (since the amplitude of the higher harmonic is assumed to be small compared with the amplitude of the first harmonic). The quantity δ_k, which plays in the present problem the role of a small parameter, can be called the relative amplitude of the $\underline{k}$-th harmonic.

After calculating the higher harmonics (8.5) on the basis of the previously obtained (§2.3) first approximation for the first harmonic (8.1), and after using these higher harmonics to obtain more accurate values for the amplitude and frequency of the first harmonic (8.4), we can then obtain also better values for the higher harmonics, obtaining instead of (8.5)

$$x_k = \delta_k' A_1 \sin(k\Omega_1 t + \varphi_k'),$$

as will be discussed in detail later on.

Inasmuch as the coefficients of the terms in the series expansion (8.2) tend to zero as $k \to \infty$, we shall take into consideration only a finite number (n) of harmonics. Then the sought solution (8.3), with account of the notation (8.5), will be written in the form

$$x = x_1 + \sum_{k=2}^{n} x_k. \tag{8.6}$$

The exact equation for the first harmonic x_1 has, in accord with (2.57), (2.51), and (2.76), the form

$$Q(p)x_1 + R(p)\left(q + \frac{q'}{\Omega_1}p\right)x_1 + R(p)\varepsilon\Phi_1 = 0. \tag{8.7}$$

When we considered previously Eq. (2.78) for the first approximation, we have discarded the small term $R(p)\varepsilon\Phi_1$ from the exact equation (8.7) for the first harmonic. Now we take account in this term of components

of first order of smallness, and discard only terms of higher order of smallness. According to (2.53), the quantity $\varepsilon\Phi_1$ is the first harmonic of the complicated periodic function (2.49). Confining ourselves in (2.49), as already stated, to terms of first order of smallness and taking account of the fact that $x_1 = A_1 \sin \Omega_1 t$, $\cos \Omega_1 t$, $\cos \Omega_1 t = (1/\Omega_1)p \sin \Omega_1 t$, we can rewrite the first harmonic $\varepsilon\Phi_1$ in the form

$$\varepsilon\Phi_1 = \left(C_1 + \frac{D_1}{\Omega_1} p\right) \sin \Omega_1 t, \tag{8.8}$$

where, in accordance with the formulas of the Fourier series coefficients for (2.49) we have

$$C_1 = \frac{1}{\pi} \int_0^{2\pi} \varepsilon \left[\frac{\partial}{\partial x} F(x_1, px_1)\, x_n + \frac{\partial}{\partial px} F(x_1, px_1)\, px_n \right] \sin \psi \, d\psi,$$

$$D_1 = \frac{1}{\pi} \int_0^{2\pi} \varepsilon \left[\frac{\partial}{\partial x} F(x_1, px_1)\, x_n + \frac{\partial}{\partial px} F(x_1, px_1)\, px_n \right] \cos \psi \, d\psi.$$

Substituting (8.8) in (8.7) and taking into account the previously adopted substitution

$$\varepsilon x_n = \sum_{k=2}^{n} x_k,$$

we obtain a more exact linearized equation for the first harmonic of the periodic solution x_1 in the form

$$Q(p)\, x_1 + R(p) \left(q + \Delta q + \frac{q' + \Delta q'}{\Omega_1} p \right) x_1 = 0, \tag{8.9}$$

where we have the following main coefficients that are analogous to the previous first approximation formulas (2.76) (with $\psi = \Omega_1 t$):

$$\left.\begin{aligned} q &= \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1 \sin \psi, A_1 \Omega_1 \cos \psi) \sin \psi \, d\psi, \\ q' &= \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1 \sin \psi, A_1 \Omega_1 \cos \psi) \cos \psi \, d\psi \end{aligned}\right\} \tag{8.10}$$

as well as new additions to these coefficients, calculated, unlike in (8.10), in terms of the first approximation (8.1):

$$\Delta q = \frac{1}{\pi A}\int_0^{2\pi}\left[\frac{\partial}{\partial x}F(x_1, px_1)\sum_{k=2}^{n}x_k + \frac{\partial}{\partial px}F(x_1, px_1)\sum_{k=2}^{n}px_k\right]\sin\psi\, d\psi,$$

$$\Delta q' = \frac{1}{\pi A}\int_0^{2\pi}\left[\frac{\partial}{\partial x}F(x_1, px_1)\sum_{k=2}^{n}x_k + \frac{\partial}{\partial px}F(x_1, px_1)\sum_{k=2}^{n}px_k\right]\cos\psi\, d\psi,$$

which give more precise values for the first harmonic x_1, owing to an account of the higher harmonics of the sought periodic solution. The formulas for Δq and $\Delta q'$, with account of (8.5) and using the expressions

$$\left.\begin{aligned}\sin(k\psi + \varphi_k) &= \cos\varphi_k \sin k\psi + \sin\varphi_k \cos k\psi,\\ \cos(k\psi + \varphi_k) &= \cos\varphi_k \cos k\psi - \sin\varphi_k \sin k\psi,\end{aligned}\right\} \qquad (8.11)$$

can be transformed to the following form which is more convenient in computation

$$\left.\begin{aligned}\Delta q &= \sum_{k=2}^{n}(I_{k1}\delta_k\cos\varphi_k + I_{k2}\delta_k\sin\varphi_k),\\ \Delta q' &= \sum_{k=2}^{n}(I_{k3}\delta_k\cos\varphi_k + I_{k4}\delta_k\sin\varphi_k),\end{aligned}\right\} \qquad (8.12)$$

where

$$\left.\begin{aligned}I_{k1} &= \frac{1}{\pi}\int_0^{2\pi}\Psi_k(\psi)\sin\psi\, d\psi, \qquad I_{k2} = \frac{1}{\pi}\int_0^{2\pi}\Theta_k(\psi)\sin\psi\, d\psi,\\ I_{k3} &= \frac{1}{\pi}\int_0^{2\pi}\Psi_k(\psi)\cos\psi\, d\psi, \qquad I_{k4} = \frac{1}{\pi}\int_0^{2\pi}\Theta_k(\psi)\cos\psi\, d\psi,\end{aligned}\right\} \qquad (8.13)$$

with

$$\Psi_k(\psi) = \frac{\partial}{\partial x}F(A\sin\psi, A\Omega\cos\psi)\sin k\psi + \frac{\partial}{\partial px}F(A\sin\psi, A\Omega\cos\psi)\, k\Omega\cos k\psi,$$

$$\Theta_k(\psi) = \frac{\partial}{\partial x}F(A\sin\psi, A\Omega\cos\psi)\cos k\psi - \frac{\partial}{\partial px}F(A\sin\psi, A\Omega\cos\psi)\, k\Omega\sin k\psi.$$

Bearing in mind that the new harmonically linearized equation (8.9) contains more precise values of the previous harmonic linearization coefficients q and q' in the form of additions Δq and $\Delta q'$, determined in accordance with (8.12) in terms of the higher harmonics of

the sought periodic solution (8.6), we can call the new expression which we have introduced here in the calculation of the first harmonic in place of the older expression (2.75), namely

$$F(x,\ px) \to \left(q + \Delta q + \frac{q' + \Delta q'}{\Omega_1} p\right) x_1$$

the more exact harmonic linearization of the nonlinearity. A complete expression for the nonlinear function F(x, px) is (8.15) below.

Were we to solve Eq. (8.9), we would obtain a more exact value of the amplitude A_1 and of the frequency Ω_1 of the first harmonic of self-oscillations, resulting from an account of the higher harmonics. The solution of Eq. (8.9) can be carried out, generally speaking, by any of the methods described in §2.3. For this, however, we must already know the higher harmonics (i.e., the values of δ_k and φ_k), so as to be able to calculate the additions Δq and $\Delta q'$ by means of Formulas (8.12). Consequently, before we solve (8.9), we must first turn to the equations (2.60) for the higher harmonics.

In Eqs. (2.60) we also confine ourselves to terms of first order of smallness. As was shown in §2.2, in view of the generalized property of the filter (2.64), the term $R(p)F_k$ is of the first order of smallness, while the term $R(p)\varepsilon\Phi_k$ is consequently of second order. Discarding the last term and expanding the expressions for F_k in accordance with the Fourier series (2.50), we represent Eqs. (2.60), with account of (8.5), in the form

$$Q(p)\,x_k + R(p)\,\frac{1}{\delta_k}\left(q_k + \frac{q'_k}{k\Omega}\,p\right)x_k = 0 \qquad (k = 2,\ 3,\ \ldots,\ n), \qquad (8.14)$$

where

$$q_k = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi,\ A\Omega\cos\psi)\sin(k\psi + \varphi_k)\,d\psi,$$

$$q'_k = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi,\ A\Omega\cos\psi)\cos(k\psi + \varphi_k)\,d\psi.$$

As a result we obtain that the initial nonlinear differential equation (2.71) is equivalent in a certain sense to a system of n linear differential equations (8.9) and (8.14), constructed in accordance with a definite procedure, and that the previous single linear equation (2.78) is their first approximation.

If we add Eqs. (8.9) and (8.14) term by term and compare the result with the initial nonlinear equation (2.71), we see that in the more exact situation considered here the nonlinear function F(x, px) is replaced by the following expression in terms of the first harmonic x_1 and the higher harmonics x_k of the sought variable x:

$$F(x,\ px)=\left(q+\Delta q+\frac{q'+\Delta q'}{\Omega_1}p\right)x_1+\sum_{k=2}^{n}\frac{1}{\delta_k}\left(q_k+\frac{q'_k}{k\Omega}p\right)x_k. \qquad (8.15)$$

It is very important to note that the small parameter δ_k in the denominator of each of the terms of the sum (8.15) is simultaneously a factor in each of the quantities x_k (8.5). Therefore the quantity δ_k actually cancels out in each term of the sum (8.15) and we obtain here not small but finite values of the higher harmonics of the nonlinear function F(x, px). This is in full agreement with the nature of the problem involved in investigating automatic systems with essential nonlinearities. Furthermore, whereas in §2.3 mention was made of the arbitrariness of Eq. (2.75), we can state here that Expression (8.15) approximates in the best fashion the nonlinear function F(x, px) together with its finite higher harmonics.

The number n − 1 of Eqs. (8.14) is determined by the number of higher harmonics of the sought periodic solution (8.6) that are included. Consequently, each of the higher harmonics is now determined separately from a corresponding equation (8.14).

Let us transform to the form most convenient for calculation the harmonic linearization coefficients for the higher harmonics, q_k and

q'_k, using Relations (8.11), so that we get

$$\left.\begin{aligned} q_k &= r_k \cos \varphi_k + s_k \sin \varphi_k, \\ q'_k &= s_k \cos \varphi_k - r_k \sin \varphi_k, \end{aligned}\right\} \tag{8.16}$$

where we introduce the auxiliary coefficients

$$\left.\begin{aligned} r_k &= \frac{1}{\pi A}\int_0^{2\pi} F(A \sin \psi,\ A\Omega \cos \psi) \sin k\psi \, d\psi, \\ s_k &= \frac{1}{\pi A}\int_0^{2\pi} F(A \sin \psi,\ A\Omega \cos \psi) \cos k\psi \, d\psi. \end{aligned}\right\} \tag{8.17}$$

On the basis of Eqs. (8.14) we obtain for each of the higher harmonics its own characteristic equation

$$Q(p) + \frac{1}{\delta_k} R(p)\left(q_k + \frac{q'_k}{k\Omega} p\right) = 0 \qquad (k = 2, 3, \ldots, n). \tag{8.18}$$

In analogy with the method of §2.3, we can now solve each of these equations separately by means of the substitution $p = jk\Omega$, i.e.,

$$Q(jk\Omega) + \frac{1}{\delta_k} R(jk\Omega)(q_k + jq'_k) = 0 \qquad (k = 2, 3, \ldots, n). \tag{8.19}$$

Using Expressions (8.16), we can readily check the correctness of the following relation:

$$q_k + jq'_k = (r_k + js_k)(\cos \varphi_k - j \sin \varphi_k) = (r_k + js_k) e^{-j\varphi_k}.$$

We substitute this expression in (8.19) and obtain

$$\delta_k e^{j\varphi_k} = -\frac{R(jk\Omega)}{Q(jk\Omega)}(r_k + js_k), \tag{8.20}$$

from which we obtain in explicit form the relative amplitude and phase of each of the higher harmonics

$$\left.\begin{aligned} \delta_k &= \left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| \sqrt{r_k^2 + s_k^2}, \\ \varphi_k &= \arg \frac{-R(jk\Omega)}{Q(jk\Omega)} + \operatorname{arctg} \frac{s_k}{r_k} \end{aligned}\right\} \quad (k = 2, 3, \ldots, n). \tag{8.21}$$

Inasmuch as the coefficients r_k and s_k have, in accordance with (8.17), already been expressed in terms of the quantities A and Ω, which are known from the first approximation solution of the equation (§2.3), now each pair of Formulas (8.21) enables us to determine for each harmonic the value of the relative amplitude δ_k and of the phase φ_k.

Consequently, we have obtained all the higher harmonics (8.5) of the variable $\underline{x}$:

$$x_k = \delta_k A \sin(k\Omega t + \varphi_k).$$

We can now calculate also the more exact values of the amplitude A_1 and of the frequency Ω_1 of the first harmonic. For this purpose we write down, in accordance with (8.9), the characteristic equation

$$Q_1(p) + R(p)\left(q + \frac{q'}{\Omega_1} p\right) = 0, \tag{8.22}$$

where we introduce

$$Q_1(p) = Q(p) + R(p)\left(\Delta q + \frac{\Delta q'}{\Omega} p\right) \tag{8.23}$$

(the replacement of Ω_1 by Ω in the small added terms does not play any essential role). The introduction of this notation is convenient for two reasons. First, we determine the sought A_1 and Ω_1 which enter into $\underline{q}$ and q', from the known values of Δq and $\Delta q'$, which are calculated here by means of Formulas (8.12) in terms of the previously obtained values δ_k and φ_k and in terms of A and Ω, which are known from the first approximation (§2.3). Second, Eq. (8.22) for the more exact first harmonic $x_1 = A_1 \sin \Omega_1 t$ has been reduced to a form that coincides formally with Eq. (2.79), which determines the first approximation. This enables us to employ for the determination of the more exact first harmonic precisely the same methods as used in §2.3 for the first approximation. In addition, according to (8.10), we can use here all the ready-made expressions for the harmonic linearization coefficients $\underline{q}$ and q' for specific prescribed nonlinearities, except that A and Ω are replaced by A_1 and Ω_1.

It must be remembered that when we apply any of the methods of §2.3 to some specified problem we must everywhere replace Q(p) by the new polynomial $Q_1(p)$, which differs from Q(p) by several added terms in its coefficients, determined by Formula (8.23).

An important feature of the more exact solution is also the fact that the polynomial $Q_1(p)$, unlike the previous polynomial $Q(p)$, depends not only on the parameters of the linear part of the system, but in accord with (8.23) and (8.12) also on the form of the nonlinearity $F(x, px)$, owing to the added terms Δq and $\Delta q'$. However, while the main coefficients q and q' are given by ready-made expressions for each nonlinearity (see Chapter 3), here we cannot employ previously calculated specific formulas for the quantities Δq and $\Delta q'$, since the quantities δ_k and φ_k contained in Formula (8.12) depend, in accordance with (8.21), on the parameters and structure of the linear portion of the system. We can, however, calculate beforehand the auxiliary quantities r_k and s_k for the different specific forms of the nonlinearities, something that will be done for the simplest technical cases in §8.3 below.

Thus, we have completely determined the sought more exact solution for the self-oscillations (8.6) in the form

$$x = A_1 \sin \Omega_1 t + \sum_{k=2}^{n} \delta_k A \sin (k\Omega_1 t + \gamma_k),$$

where the first harmonic is given in the more exact form, and the higher harmonics are given in the first approximation.

We can then make more exact also the values of the higher harmonics, on the basis of the fact that we now already know the more exact first harmonic. For this purpose we substitute in (8.17) and (8.21) the new more exact values of A_1 and Ω_1 in place of the previous A and Ω. We obtain from this new more exact values for the relative amplitude δ'_k and phase φ'_k of each higher harmonic. Then the more exact solution for the self-oscillations assumes the form

$$x = A_1 \left[\sin \Omega_1 t + \sum_{k=2}^{n} \delta'_k \sin (k\Omega_1 t + \varphi'_k) \right]. \qquad (8.24)$$

We could follow this by determining with the aid of the new values

δ'_k and φ'_k the more exact values of Δq and $\Delta q'$ by means of Formulas (8.12) and find a second correction for the first harmonic by means of Eq. (8.22). The result would be a development of a method of successive approximations which, however, hardly makes sense in practice. We therefore confine ourselves to Formula (8.24) or even to the preceding one.

How many of the higher harmonics (8.6) should be taken into consideration in each specific problem can be judged from the expansion (2.50) of the specified nonlinear function $F(x_1, px_1)$ in a Fourier series. Thus, for example, in the case frequently encountered in practice of single-valued odd-symmetry nonlinearity $F(x)$, the most significant of the higher harmonics is the third. Taking this harmonic into account, we represent the sought periodic solution (self-oscillations) in accordance with (8.6), in the form

$$\left.\begin{aligned} x = x_1 + x_3, \quad x_1 = A_1 \sin \Omega_1 t, \\ x_3 = \delta_3 A \sin (3\Omega_1 t + \varphi_3) \quad \text{or} \quad x_3 = \delta'_3 A_1 \sin (3\Omega_1 t + \varphi'_3). \end{aligned}\right\} \tag{8.25}$$

In this case the coefficient q' will vanish in the equation for the first harmonic (8.9), as before. The characteristic equation (8.22) for the more exact first harmonic will consequently be

$$Q_1(p) + R(p)\, q = 0, \tag{8.26}$$

where

$$Q_1(p) = Q(p) + R(p)\left(\Delta q + \frac{\Delta q'}{\Omega} p\right),$$

and the expressions for the coefficient

$$q = \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1 \sin \psi) \sin \psi \, d\psi \tag{8.27}$$

can be taken from Chapter 3 with replacement of A by A_1. The formulas for the additional coefficients Δq and $\Delta q'$ become much simpler, since many terms drop out from Formulas (8.12) and (8.13), while in the formulas (8.16) the coefficient $s_k = 0$. As a result we obtain in place of

(8.12) and (8.13), with allowance for (8.17), the very simple relations

$$\Delta q = h_3 \delta_3 \cos \varphi_3, \quad \Delta q' = 3 r_3 \delta_3 \sin \varphi_3, \tag{8.28}$$

where a new abbreviated symbol h_3 has been introduced. In (8.12) and (8.17), in view of the odd symmetry of the single-valued nonlinearity F(x), we can replace the integral between the limits (0; 2π) by the integral with limits (0; $\pi/2$) multiplied by four. We then obtain

$$\left.\begin{aligned} h_3 &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \frac{d}{dx} F(A \sin \psi) \sin 3\psi \sin \psi \, d\psi, \\ r_3 &= \frac{4}{\pi A} \int_0^{\frac{\pi}{2}} F(A \sin \psi) \sin 3\psi \, d\psi. \end{aligned}\right\} \tag{8.29}$$

The coefficients h_3 and r_3 for specific nonlinearities are calculated beforehand and given in §8.3 below. From Formulas (8.21) we determine the relative amplitude and phase of the third harmonic:

$$\delta_3 = r_3 \left| \frac{R(j3\Omega)}{Q(j3\Omega)} \right|, \quad \varphi_3 = \arg \frac{R(j3\Omega)}{Q(j3\Omega)}. \tag{8.30}$$

Thus, it is quite simple to determine the more exact periodic solution for the case of a single-valued nonlinearity F(x) with inclusion of the third harmonic, in the form

$$x = A_1 \sin \Omega_1 t + \delta_3 A \sin (3\Omega_1 t + \varphi_3). \tag{8.31}$$

If we use Formulas (8.30) once more, substituting in them A_1 and Ω_1 in lieu of A and Ω, then a more exact value is obtained also for the third harmonic, and the solution assumes the form

$$x = A_1 [\sin \Omega_1 t + \delta_3' \sin (3\Omega_1 t + \varphi_3')].$$

After determining the periodic solution for the variable x, which is contained under the nonlinearity sign in the equations of the automatic system, we can then determine also the solution for all the other variables, with inclusion of the higher harmonics, using the corresponding transfer functions. If some variable z is related with x by means of a transfer function

$$z = W(p)\,x, \tag{8.32}$$

then in the general case, when according to (8.6) we have

$$x = x_1 + \sum_{k=2}^{n} x_k,$$

we obtain

$$z = z_1 + \sum_{k=2}^{n} z_k, \tag{8.33}$$

In accordance with (8.4) we obtain for the first harmonic

$$z_1 = A_1 |W(j\Omega_1)| \sin [\Omega_1 t + \arg W(j\Omega_1)] \tag{8.34}$$

and, in accordance with (8.5), for the higher harmonics

$$z_k = \delta_k A |W(jk\Omega_1)| \sin [k\Omega_1 t + \varphi_k + \arg W(jk\Omega_1)]. \tag{8.35}$$

On the other hand, for the variable that represents the nonlinear function itself, y = F(x, px), we obtain

$$y = F(x, px) = y_1 + \sum_{k=2}^{n} y_k, \tag{8.36}$$

where according to (8.5) we have for the first harmonic

$$y_1 = \left(q + \Delta q + \frac{q' + \Delta q'}{\Omega_1} p\right) x_1 \tag{8.37}$$

and for the higher harmonic

$$y_k = \frac{1}{\delta_k}\left(q_k + \frac{q'_k}{k\Omega_1} p\right) x_k, \tag{8.38}$$

with q_k and q'_k determined from Formulas (8.16) and (8.17), while x_1 and x_k are given by (8.4) and (8.5).

§8.2. More Exact First Approximation

The method developed above for determining a more exact first approximation of the self-oscillations by calculating a finite number of higher harmonics, enables us to find the periodic solution for the variable x in the form

$$x = A_1 \sin \Omega_1 t + \sum_{k=2}^{n} \delta_k A \sin (k\Omega_1 t + \varphi_k) \tag{8.39}$$

or else in the form

$$x = A_1\left[\sin \Omega_1 t + \sum_{k=2}^{n} \delta_k' \sin (k\Omega_1 t + \varphi_k')\right]. \qquad (8.40)$$

The determination of this more exact solution has two purposes. First, by finding the values of δ_k we obtain a quantitative estimate of the amplitudes of the higher harmonics for the variable $\underline{x}$. If they are actually sufficiently small, as was proposed in the start of the solution of the problem, then Expression (8.39) or (8.40) actually represents the more exact solution for the self-oscillations in the given nonlinear system, and can be used in practice as such.

Second, the solution for the self-oscillations, obtained in the form (8.39) or (8.40) with inclusion of the higher harmonics, has certain principally new properties compared with the first approximation $x = A \sin \Omega t$. We can thus correct certain qualitative shortcomings of the first approximation [251] (which may quantitatively be small). These include, for example, the following:

1) in systems with one single-valued odd-symmetry nonlinearity of any configuration, the solution in the first approximation ($x = A \sin \Omega t$) results in a self-oscillation frequency Ω which is perfectly independent of the form of the nonlinearity;

2) the first approximation solution does not result in a phase shift upon passage of the oscillations through the single-valued odd-symmetry nonlinearity, although actually such a phase shift may be brought about by the higher harmonics.

Let us show in general form that the foregoing two circumstances do indeed take place.

The differential equation of the dynamics of the natural oscillations of any system (with constant lumped parameters) with one nonlinearity F(x) can be written in accordance with (2.42) in the form

$$Q(p)x + R(p)F(x) = 0. \qquad (8.41)$$

To find the self-oscillations in the approximate form $x = A \sin \Omega t$, we use for a single-valued odd-symmetry nonlinearity the following characteristic equation

$$Q(p) + R(p)q(A) = 0. \qquad (8.42)$$

Making the substitution $p = j\Omega$ and denoting

$$Q(j\Omega) = X_Q(\Omega) + jY_Q(\Omega), \quad R(j\Omega) = X_R(\Omega) + jY_R(\Omega), \qquad (8.43)$$

we obtain two equations

$$\left.\begin{aligned} X &= X_Q(\Omega) + X_R(\Omega)q(A) = 0, \\ Y &= Y_Q(\Omega) + Y_R(\Omega)q(A) = 0. \end{aligned}\right\} \qquad (8.44)$$

Eliminating q(A) we obtain an equation for the frequency Ω of the self-oscillations (with $X_R \neq 0$):

$$X_R(\Omega)Y_Q(\Omega) - X_Q(\Omega)Y_R(\Omega) = 0, \qquad (8.45)$$

after which we determine the amplitude by using one of the equations (8.44).

In the case of a loop-type nonlinear characteristic F(x), we obtain in place of (8.42), as is well known, the characteristic equation

$$Q(p) + R(p)\left[q(A) + \frac{q'(A)}{\Omega}p\right] = 0, \qquad (8.46)$$

and consequently we are unable to obtain for the frequency an equation of the type (8.45), containing no parameters of the nonlinear characteristic.

Thus, Expression (8.45) shows that actually in any system with one single-valued nonlinearity the result of the solution for the self-oscillation frequency depends only on the parameters of the linear portion of the system and is perfectly independent of the form of the nonlinearity (it is independent of the coefficient g). This is indeed the first of the aforementioned shortcomings of the first approximation. In fact, however, the frequency of the self-oscillations will depend in many cases on the form of the nonlinearity (for example, on the

width of the backlash zone in a relay etc.), although in other cases this dependence may actually be missing even from the exact solution [253].

We note that the presence of a constant time delay τ in the system, i.e., the presence of a factor $e^{-\tau p}$ in R(p), does not change the situation.

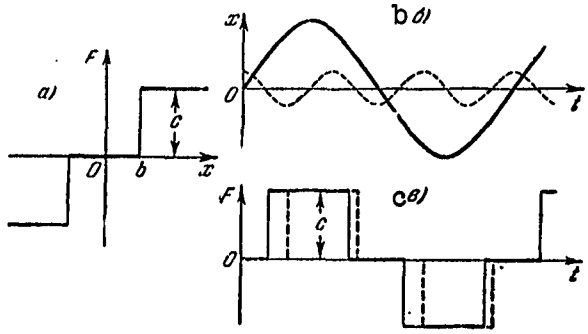


Fig. 8.1

The second of the foregoing shortcomings is even more obvious. In fact, whereas for a loop-type nonlinearity the first harmonic has, in accordance with (8.46), a phase shift given by

$$\varphi = \operatorname{arctg} \frac{q'(A)}{q(A)}, \qquad (8.47)$$

for the single-valued nonlinearity, in accordance with (8.42), there is no phase shift. Actually, however, even small higher harmonics present at the "input" $\underline{x}$ of any nonlinearity give rise to a phase shift of the first harmonic at the "output" F, the magnitude of this phase shift being dependent on the form of the nonlinearity. Let us illustrate this with a very simple example.

Let the nonlinearity have the form of Fig. 8.1a. Were the relay switching being governed by the first harmonic of $\underline{x}$ (solid line in Fig. 8.1b), the output F would have the form of the solid line in Fig. 8.1c, the first harmonic of which has no phase shift. But it is clear from the figure that even a small third harmonic of $\underline{x}$ (dashed in Fig. 8.1b)

results in a shift in the switching time of the relay (dashed in Fig. 8.1c). Besides, owing to the difference in the phase shift of the front and rear jumps, the first harmonic of the output F shifts not only in phase but changes also in amplitude.

What is principally new compared with the first approximation solution is in this case the appearance of the phase shift. The shift of the first harmonic of the quantity F, and its change in amplitude as well, resulting from the presence of small higher harmonics of the variable $\underline{x}$, should depend here not only on the parameters of the linear part, but also on the form of the nonlinearity, for example, in the present case on the width of the backlash zone 2b.

The method developed in §8.1 enables us to take into account any finite number of higher harmonics. However, to investigate the problems that are being studied here, we confine ourselves to the third harmonic only,* seeking self-oscillations of the system in the form

$$x = A_1 \sin \Omega_1 t + \delta_3 A \sin (3\Omega_1 t + \varphi_3), \qquad (8.48)$$

where A_1 and Ω_1 are the new (more exact) amplitude and frequency of the first harmonic of the variable $\underline{x}$, which were modified (compared with the previous values A and Ω) by the inclusion of the third harmonic; δ_3 is the relative amplitude of the third harmonic (small); φ_3 is its phase relative to the new first harmonic. All these quantities are to be calculated.

The new first harmonic of the self-oscillations of the variable $\underline{x}$ is determined, in accord with (8.26), by the following characteristic equation in place of (8.42):

$$Q_1(p) + R(p)\, q(A_1) = 0, \quad Q_1(p) = Q(p) + R(p)\left(\Delta q + \frac{\Delta q'}{\Omega_1} p\right), \qquad (8.49)$$

where $\underline{q}$, Δq, and $\Delta q'$ are determined by Formulas (8.27)-(8.29), and for the third harmonic we have, in accordance with (8.30)

$$\delta_3 = r_3 \left| \frac{R(j3\Omega_1)}{Q(j3\Omega_1)} \right|, \quad \varphi_3 = \arg \frac{-R(j3\Omega_1)}{Q(j3\Omega_1)}, \qquad (8.50) \qquad (8.50)$$

with, unlike (8.30), the more exact frequency Ω_1 introduced here directly so as to improve the solution.

We thus determine completely the more exact periodic solution (8.48) with inclusion of the third harmonic. Besides, using the previous notation in (8.43), we obtain here from (8.49) in place of (8.44) a new pair of computation equations

$$\left.\begin{aligned} X_1 &= X_{Q1}(\Omega_1) + X_R(\Omega_1)\, q(A_1) = 0, \\ Y_1 &= Y_{Q1}(\Omega_1) + Y_R(\Omega_1)\, q(A_1) = 0. \end{aligned}\right\} \qquad (8.51)$$

Therefore the equation for the determination of the more exact value of the self-oscillation frequency assumes in place of (8.45) the form

$$X_R(\Omega_1)\, Y_{Q1}(\Omega_1) - X_{Q1}(\Omega_1)\, Y_R(\Omega_1) = 0, \qquad (8.52)$$

where in accordance with (8.49) we have

$$\left.\begin{aligned} X_{Q1}(\Omega_1) &= X_Q(\Omega_1) + X_R(\Omega_1)\,\Delta q - Y_R(\Omega_1)\,\Delta q', \\ Y_{Q1}(\Omega_1) &= Y_Q(\Omega_1) + Y_R(\Omega_1)\,\Delta q + X_R(\Omega_1)\,\Delta q'. \end{aligned}\right\} \qquad (8.53)$$

It is obvious that, first, the more exact value of the first harmonic frequency Ω_1 determined from this will already depend on the form of the nonlinearity, since this form governs the additional coefficients Δq and $\Delta q'$. Second, even in the case of a single valued nonlinearity a phase shift of the first harmonic of the quantity F is observed (shown, for example, by the dashed line in Fig. 8.1c), in the form

$$\varphi = \operatorname{arctg} \frac{\Delta q'}{q + \Delta q}. \qquad (8.54)$$

If the third harmonic of the variable $\underline{x}$ is small, the quantity $\Delta q'$ and the shift φ will also be small (in this case the arc tangent symbol in Formula (8.54) can be left out).

The following order of calculations can be proposed. We first determine in first approximation the self-oscillations by means of Eqs.

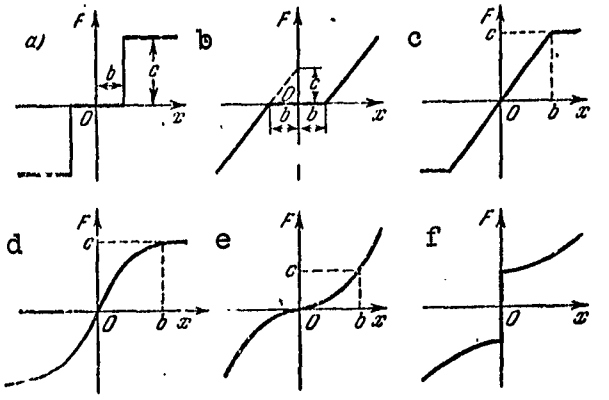


Fig. 8.2

(8.44), which yield the values of A and Ω ($x = A \sin \Omega t$). The result obtained for Ω does not depend on the form of the nonlinear characteristic. We then determine the dependence of the frequency (and amplitude) of the self-oscillations on the form of the nonlinearity, using the more exact solution (and a procedure which will be developed below). The form of the nonlinearity (Fig. 8.2) and its connection with the amplitude of the first harmonic of the variable x will be characterized here by the following main coefficients:

$$\gamma = \frac{2c}{\pi b}, \quad \beta = \frac{A}{b} \text{ and } \beta_1 = \frac{A_1}{b}, \tag{8.55}$$

where γ is the nonlinearity form coefficient, β and β_1 are the relative amplitudes of the first harmonics. For example, for the nonlinearities shown in Figs. 8.2a, b, and c, we obtain, respectively, by using the formulas of Chapter 3 and the notation of (8.55):

$$\left.\begin{aligned} &\text{a)}\ q = \frac{2\gamma}{\beta^2}\sqrt{\beta^2 - 1}, \\ &\text{b)}\ q = \gamma\left(\arccos\frac{1}{\beta} - \frac{1}{\beta^2}\sqrt{\beta^2 - 1}\right), \\ &\text{c)}\ q = \gamma\left(\arcsin\frac{1}{\beta} + \frac{1}{\beta^2}\sqrt{\beta^2 - 1}\right) \end{aligned}\right\} \tag{8.56}$$

(the case b = 0 on Figs. 8.2a and c, for which the notation of (8.55) is meaningless, will be considered separately later on).

In exactly the same way, for example, for a quadratic or cubic

characteristic (Fig. 8.2b), when

$$F = k|x|x = kx^2 \operatorname{sign} x \quad \text{and} \quad F = kx^3, \tag{8.57}$$

we can use the coordinates of any specified point (b, c) and readily obtain, respectively, in accordance with Chapter 3 and the notation of (8.55),

$$q = \frac{4}{3}\gamma\beta \quad \text{and} \quad q = \frac{3\pi}{8}\gamma\beta^2. \tag{8.58}$$

We proceed analogously with the other nonlinearities.

Introducing the concept of the nonlinearity form coefficient γ, we shall define the sought dependence of the self-oscillation frequency on the nonlinearity form by means of the function

$$\Omega_1 = \Omega_1(\gamma). \tag{8.59}$$

Let us transform all the formulas obtained above for the more exact solution to a form that is convenient for finding the function (8.59), recalling that in the first approximation solution the frequency Ω is independent of the nonlinearity form coefficient γ.

We first write down the auxiliary coefficients h_3 and r_3, determined by Formulas (8.29) as functions of γ, i.e.,

$$h_3 = h_3(\gamma) \text{ and } r_3 = r_3(\gamma). \tag{8.60}$$

We can use here the ready-made values of h_3 and r_3 given in §8.3. For example, for a relay characteristic with a backlash zone (Fig. 8.2a) we obtain

$$h_3 = 2\gamma\frac{\beta^2}{\beta^2}\sqrt{\beta^2-1}, \quad r_3 = 2\gamma\frac{\beta^2}{3\beta^2}\sqrt{\beta^2-1}, \tag{8.61}$$

where $\beta = A/b$ should be known from the first approximation as a function of γ. Consequently, to solve this problem, the result of finding the self-oscillations by means of the first approximation (§2.3) should be represented in the form of a plot such as Fig. 8.3a. Then the formulas (8.61), and in the general case Formulas (8.29), make it possible to determine the functions (8.60) which are needed for the subsequent

solution, for example, in the form of Fig. 8.3b.

We can then calculate by means of Formulas (8.50) the relative amplitude δ_3 and phase φ_3 of the third harmonic of the self-oscillations as a function of the sought frequency Ω_1 and of the nonlinearity form coefficient, after which we determine from Formulas (8.28) Δq and $\Delta q'$:

$$\Delta q = U_3(3\Omega_1)\, h_3(\gamma)\, r_3(\gamma), \quad \Delta q' = 3V(3\Omega_1)\,[r_3(\gamma)]^3, \tag{8.62}$$

where

$$U_3(3\Omega_1) = \mathrm{Re}\left[\frac{-R(j3\Omega_1)}{Q(j3\Omega_1)}\right], \quad V_3(3\Omega_1) = \mathrm{Im}\left[\frac{-R(j3\Omega_1)}{Q(j3\Omega_1)}\right], \tag{8.63}$$

which represent the values of the real and imaginary parts of the amplitude-phase frequency characteristic of the reduced linear part of the system with its sign inverted, at a frequency 3Ω.

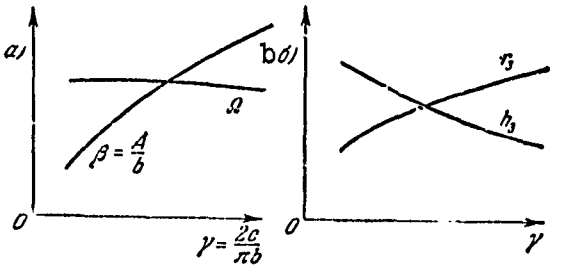


Fig. 8.3

Let us write down Eqs. (8.51) for the determination of the more exact value of the frequency Ω_1 with account of (8.53), in the form

$$\left.\begin{aligned} X_Q(\Omega_1) + X_R(\Omega_1)(q + \Delta q) - Y_R(\Omega_1)\Delta q' = 0, \\ Y_Q(\Omega_1) + Y_R(\Omega_1)(q + \Delta q) + X_R(\Omega_1)\Delta q' = 0. \end{aligned}\right\} \tag{8.64}$$

Eliminating the quantity $(q + \Delta q)$ we obtain

$$[X_R^2(\Omega_1) + Y_R^2(\Omega_1)]\,\Delta q' = X_Q(\Omega_1)\, Y_R(\Omega_1) - Y_Q(\Omega_1)\, X_R(\Omega_1),$$

which can be reduced upon substitution of (8.62) to the form

$$[r_3(\gamma)]^2 = \Phi_1(\Omega_1), \tag{8.65}$$

where

$$\Phi_1(\Omega_1) = \frac{X_Q(\Omega_1)\, Y_R(\Omega_1) - Y_Q(\Omega_1)\, X_R(\Omega_1)}{3V_3(3\Omega_1)\,[X_R^2(\Omega_1) + Y_R^2(\Omega_1)]}. \tag{8.66}$$

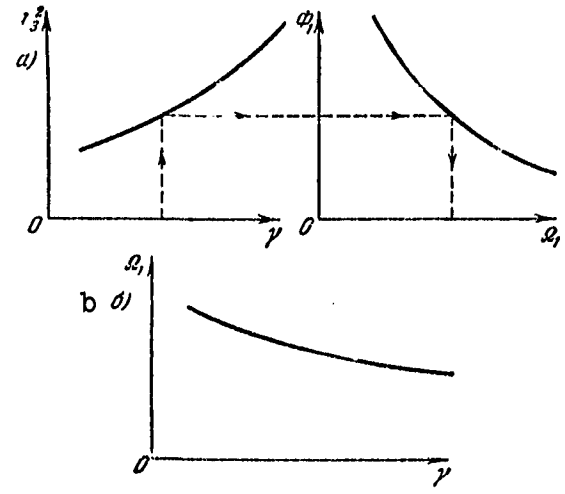


Fig. 8.4

Equation (8.65) can be readily solved graphically, as shown in Fig. 8.4a. In fact, by plotting separately the left and right halves of Eq. (8.65) we can find for each specified value of the nonlinearity form coefficient γ its own more exact value of frequency Ω_1, following the arrows shown in Fig. 8.4a. As a result we can plot the sought dependence (8.59), for example in the form of Fig. 8.4b.

Following this we can, if necessary, determine also the more exact value of the amplitude A_1 of the first self-oscillation harmonic, using the first equation of (8.64). In accordance with the notation (8.55), we shall seek the more exact value of the amplitude in dimensionless form $\beta_1 = A_1/b$. Consequently, the quantity q in Eq. (8.64) will have the form $q(\gamma, \beta_1)$, as for example in (8.56) and (8.58), except that β and β_1 are interchanged. We then obtain from the first equation of (8.64)

$$q(\gamma, \beta_1) = -\frac{X_Q(\Omega_1) + X_R(\Omega_1)\Delta q - Y_R(\Omega_1)\Delta q'}{X_R(\Omega_1)}, \qquad (8.67)$$

where Ω_1 is already determined as a function of γ (Fig. 8.4b), and consequently, Δq and $\Delta q'$ are also determined as functions of γ, in accordance with (8.62). This equation is solved graphically. For each speci-

fied value of γ, the left half of Eq. (8.67) is plotted (as shown in Fig. 8.5a) in the form of curves, while the right half is plotted in the form of straight lines. The points where the curves cross the lines (for equal values of γ) yield the required solution, and this results in the dependence of the more exact amplitude of $\beta_1 = A_1/b$ of the first harmonic of the self-oscillations on the nonlinearity form coefficient γ.

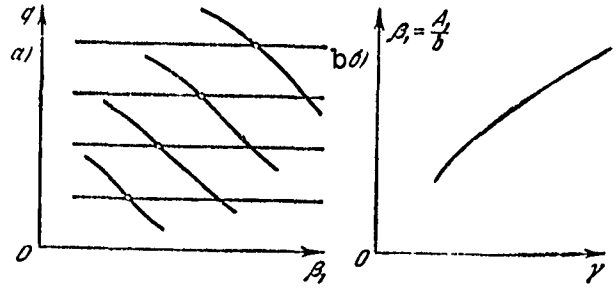


Fig. 8.5

We note that the initial dependence of the amplitude $\beta = A/b$ of the first approximation on γ, shown in Fig. 8.3a, was determined in accordance with the first equation of (8.44) from the following equality:

$$q(\gamma, \beta) = -\frac{X_Q(\Omega)}{X_R(\Omega)}. \qquad (8.68)$$

The left half is represented here by the same curves as the left half of Eq. (8.67), since in both cases use is made of the same formulas, of the type (8.56) and (8.58), except that β and β_1 are interchanged. The right halves of (8.67) and (8.68), on the other hand, differ from each other in the added terms $\Delta q = f_1(\gamma)$ and $\Delta q' = f_2(\gamma)$, and also because Ω and Ω_1 are interchanged, meaning a shift of the horizontal lines of Fig. 8.5a. We can therefore use the same graphic method to determine both the first approximation $\beta(\gamma)$ and the more exact values $\beta_1(\gamma)$.

We next determine, in accordance with (8.50), the relative amplitude δ_3 and the phase φ_3 of the third harmonic of the self-oscillations, also as functions of the nonlinearity form coefficient γ.

We can now finally calculate also the phase shift φ of the first harmonic of the variable F (a phase shift which, as is well known, did not exist in the first approximation solution), using Formula (8.54), in which we must substitute the values of Δq and $\Delta q'$ from (8.62) as functions of γ, as well as the quantity $q(\beta_1, \gamma)$, bearing in mind that the functions $\Omega_1(\gamma)$ and $\beta_1(\gamma)$ have already been determined.

This completes therefore the solution of our problem.*

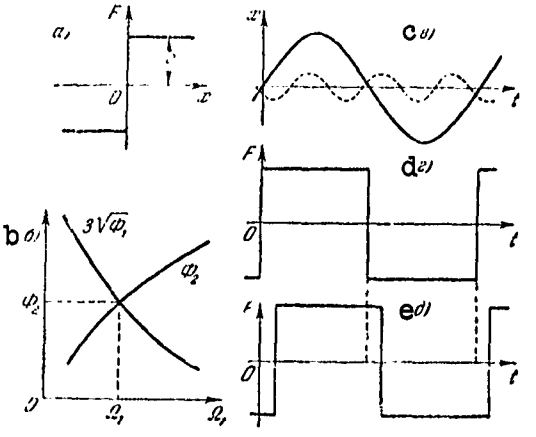


Fig. 8.6

One singular particular case. As already mentioned, the introduction of the nonlinearity form coefficient γ and of the relative amplitudes β and β_1 is meaningless when $b = 0$, i.e., for the case of an ideal relay characteristic (Fig. 8.6a). The general formulas derived above are consequently likewise inapplicable for this case. We shall therefore consider this particular case separately. In analogy with Chapter 3 we have here

$$q = \frac{4c}{\pi A_1}, \tag{8.69}$$

and the formulas (8.29) and (8.28) yield

$$\left.\begin{aligned} h_3 &= 0, \quad r_3 = \frac{4c}{3\pi A_1} = \frac{1}{3}q, \\ \Delta q &= 0, \quad \Delta q' = \frac{1}{3}V_3(3\Omega_1)q^2. \end{aligned}\right\} \tag{8.70}$$

Therefore in place of (8.65) we obtain the equation

$$q^2 = 9\Phi_1(\Omega_1). \tag{8.71}$$

But on the other hand, the quantity g, in view of the fact that $\Delta q = 0$, can be expressed with the aid of Eqs. (8.64) in the following form (by eliminating from them the quantity $\Delta q'$):

$$q = \Phi_2(\Omega_1), \tag{8.72}$$

where

$$\Phi_2(\Omega_1) = -\frac{X_Q(\Omega_1)X_R(\Omega_1) + Y_Q(\Omega_1)Y_R(\Omega_1)}{X_R^2(\Omega_1) + Y_R^2(\Omega_1)}. \tag{8.73}$$

Comparing (8.71) and (8.72) we obtain the following equation for the determination of the more exact frequency of the first self-oscillation harmonic

$$\Phi_2(\Omega_1) = 3\sqrt{\Phi_1(\Omega_1)}, \tag{8.74}$$

where $\Phi_1(\Omega_1)$ and $\Phi_2(\Omega_1)$ are determined by Formulas (8.66) and (8.73).

By simple graphic solution of this equation (Fig. 8.6b) we determine the more exact value of the self-oscillation frequency Ω_1. It is independent of the value of c (Fig. 8.6a), something confirmed, as is well known, also by the exact solution of the problem. We then find, in accordance with (8.69) and (8.72), the more exact value of the first harmonic amplitude in the form

$$A_1 = \frac{4c}{\pi\Phi_2}, \tag{8.75}$$

where the value of Φ_2 is taken from the same plot (Fig. 8.6b).

We can then easily determine with the aid of (8.50) the amplitude and phase of the third harmonic:

$$A_3 = \delta_3 A_1 = \frac{4c}{3\pi}\left|\frac{R(j3\Omega_1)}{Q(j3\Omega_1)}\right|, \quad \varphi_3 = \arg\frac{-R(j3\Omega_1)}{Q(j3\Omega_1)}. \tag{8.76}$$

In view of the fact that the phase of the third harmonic is shifted by φ_3 relative to that of the first (Fig. 8.6c), the relay switching will be governed not by the first harmonic (Fig. 8.6d), as was the case in the solution of the first approximation, but after a

certain shift (Fig. 8.6e), with the phase of this shift for the first harmonic of the variable F being, in accordance with (8.54) and (8.70),

$$\varphi = \operatorname{arctg}(\delta_3 \sin \varphi_3). \tag{8.77}$$

In §8.4 below we shall demonstrate by means of an example that the results of the more exact solution proposed in §§8.1 and 8.2 are quite close to the result of the exact solution determined for this example.

§8.3. Third Harmonic Coefficients for Some Nonlinearities

As was stated in §8.1, in order to effect a more exact harmonic linearization of frequently encountered single-valued* odd-symmetry nonlinear characteristics, it is sufficient to include the third harmonic of the Fourier expansion of the nonlinear periodic function $F(A \sin \psi)$. This account reduces to an introduction of several corrections Δq and $\Delta q'$ into the harmonic linearization coefficients; these corrections are determined in accord with (8.28) from the formulas

$$\Delta q = h_3 \delta_3 \cos \varphi_3, \quad \Delta q' = 3 r_3 \delta_3 \sin \varphi_3, \tag{8.78}$$

where $\delta_3 = A_3/A$ is the relative amplitude of the third harmonic and φ_3 is the phase shift for the third harmonic, determined by Formulas (8.21). The coefficients h_3 and r_3 depend on the form of the nonlinear characteristic and in accordance with (8.29) are determined for single-valued odd-symmetry nonlinear characteristics by means of the formulas

$$\left.\begin{aligned} h_3 &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \frac{d}{dx} F(A \sin \psi) \sin 3\psi \sin \psi \, d\psi, \\ r_3 &= \frac{4}{\pi A} \int_0^{\frac{\pi}{2}} F(A \sin \psi) \sin 3\psi \, d\psi. \end{aligned}\right\} \tag{8.79}$$

Let us calculate the coefficients h_3 and r_3 by means of these formulas for several typical nonlinearities.

1. Single-valued relay characteristic. Let us consider a relay characteristic with a backlash zone (Fig. 8.7a). The value of the derivative dF/dx, which enters under the integral sign in the formula

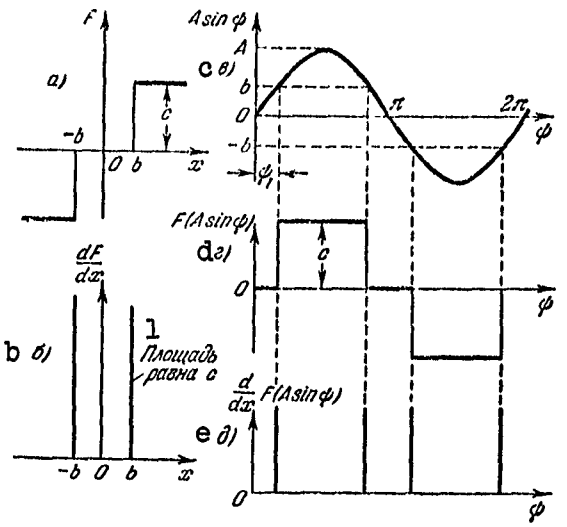


Fig. 8.7. 1) Area equal to c.

for h_3, will vanish everywhere for this nonlinearity, except at the two points $x = \pm b$, where it is equal to the instantaneous pulse whose area is equal to c (Fig. 8.7b). Such a pulse is called a delta function. The expression $\sin\psi d\psi$ contained there can be transformed when $x = A\sin\psi$ into

$$\sin\psi\, d\psi = \frac{\sin\psi}{\frac{dx}{d\psi}}\, dx = \frac{\sin\psi}{A\cos\psi}\, dx = \frac{\operatorname{tg}\psi}{A}\, dx. \qquad (8.80)$$

Inasmuch as the integrand in Formula (8.79) for h_3, will vanish everywhere on the integration interval $(0, \pi/2)$, in accordance with Fig. 8.7e, with the exception of the single point $\psi = \psi_1$, we can rewrite this formula for the present example in the form

$$h_3 = \frac{4}{\pi A}\sin 3\psi_1 \operatorname{tg}\psi_1 \int_0^A \frac{dF}{dx}\, dx = \frac{4}{\pi A}\sin 3\psi_1 \operatorname{tg}\psi_1 [F(A) - F(0)].$$

But from Fig. 8.7c we have

$$\sin\psi_1 = \frac{b}{A}, \quad \operatorname{tg}\psi_1 = \frac{b}{\sqrt{A^2 - b^2}}, \quad \sin 3\psi_1 = \frac{3bA^2 - 4b^3}{A^3}, \qquad (8.81)$$

and from Fig. 8.7a with $A \geq b$ we have

$$F(A) = c, \quad F(0) = 0.$$

We ultimately obtain

$$h_3 = \frac{4cb^2(3A^2 - 4b^2)}{\pi A^4}\sqrt{A^2 - b^2} \qquad (A \geqslant b), \tag{8.82}$$

or in the notation of (8.55):

$$h_3 = 2\gamma\frac{3\beta^2 - 4}{\beta^4}\sqrt{\beta^2 - 1} \qquad (\beta \geqslant 1). \tag{8.83}$$

Formula (8.79) for r_3 assumes, in accordance with Fig. 8.7d, the form

$$r_3 = \frac{4c}{\pi A}\int_{\psi_1}^{\frac{\pi}{2}} \sin 3\psi\, d\psi = \frac{4c}{3\pi A}\cos 3\psi_1,$$

from which we obtain with allowance for (8.81)

$$r_3 = \frac{4c(A^2 - 4b^2)}{3\pi A^4}\sqrt{A^2 - b^2} \qquad (A \geqslant b), \tag{8.84}$$

or in the notation of (8.55)

$$r_3 = 2\gamma\frac{\beta^2 - 4}{3\beta^4}\sqrt{\beta^2 - 1} \qquad (\beta \geqslant 1). \tag{8.85}$$

2. In particular, for an ideal relay characteristic (Fig. 8.6a) we obtain from Formulas (8.82) and (8.84), putting b = 0,

$$h_3 = 0, \quad r_3 = \frac{4c}{3\pi A}. \tag{8.86}$$

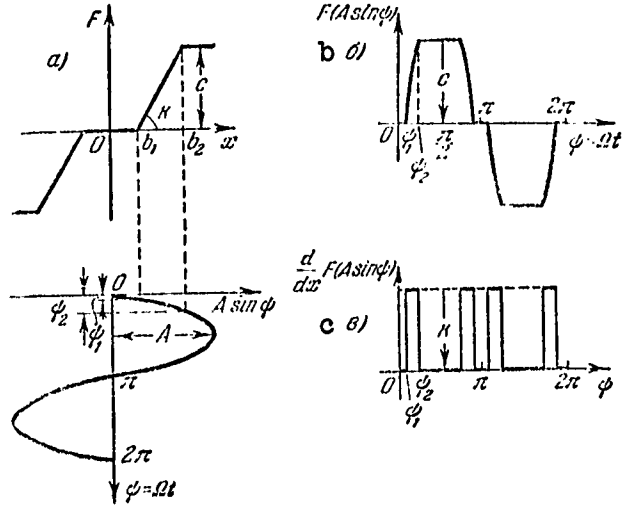


Fig. 8.8

3. Characteristic with backlash zone and saturation. For such a characteristic (Fig. 8.8a), the functions $F(A \sin \psi)$ and $(d/dx)F(A \sin \psi)$ assume when $x = A \sin \psi$ the form shown in Fig. 8.8b and c. Therefore, after calculating the coefficient h_3, we obtain from Formula (8.79)

$$h_3 = \frac{4}{\pi}\int_{\psi_1}^{\psi_2} k \sin 3\psi \sin \psi \, d\psi = \frac{4k}{\pi}\left(\int_{\psi_1}^{\psi_2} 3\sin^2\psi \, d\psi - \int_{\psi_1}^{\psi_2} 4\sin^4\psi \, d\psi\right) =$$
$$= \frac{k}{\pi}(\sin 2\psi_2 - \sin 2\psi_1) - \frac{k}{2\pi}(\sin 4\psi_2 - \sin 4\psi_1). \qquad (8.87)$$

Taking into account the values

$$\sin\psi_1 = \frac{b_1}{A}, \quad \sin 2\psi_1 = \frac{2b_1\sqrt{A^2 - b_1^2}}{A^2}, \quad \sin 4\psi_1 = \frac{4b_1(A^2 - 2b_1^2)\sqrt{A^2 - b_1^2}}{A^4},$$
$$\sin\psi_2 = \frac{b_2}{A}, \quad \sin 2\psi_2 = \frac{2b_2\sqrt{A^2 - b_2^2}}{A^2}, \quad \sin 4\psi_2 = \frac{4b_2(A^2 - 2b_2^2)\sqrt{A^2 - b_2^2}}{A^4},$$

we obtain ultimately from (8.87)

$$h_3 = \frac{4k}{\pi A^4}(b_2\sqrt{A^2 - b_2^2} - b_1\sqrt{A^2 - b_1^2}) \qquad (A \geq b_2). \qquad (8.88)$$

Calculation of the coefficient r_3 in accordance with (8.79) and Fig. 8.8b yields

$$r_3 = \frac{4}{\pi A}\left(\int_{\psi_1}^{\psi_2} kA \sin\psi \sin 3\psi \, d\psi + \int_{\psi_2}^{\frac{\pi}{2}} c \sin 3\psi \, d\psi\right) =$$
$$= \frac{k}{\pi}(\sin 2\psi_2 - \sin 2\psi_1) - \frac{k}{2\pi}(\sin 4\psi_2 - \sin 4\psi_1) + \frac{4c}{3\pi A}\cos 3\psi_2.$$

Taking into account the values written out above, and also the fact that

$$\cos 3\psi_2 = \frac{(A^2 - 4b_2^2)\sqrt{A^2 - b_2^2}}{A^3},$$

we obtain ultimately

$$r_3 = \frac{4k}{3\pi A^4}[(b_2A^2 - b_1A^2 - b_2^3 + 4b_1b_2^2)\sqrt{A^2 - b_2^2} -$$
$$- 3b_1^3\sqrt{A^2 - b_1^2}]. \qquad (8.89)$$

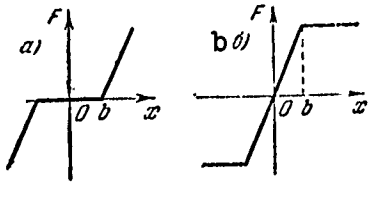


Fig. 8.9

4. Characteristic with backlash zone without saturation. For a characteristic with backlash zone without saturation (Fig. 8.9a), putting $b_1 = b$, $b_2 = A$ in Formulas

(8.88) and (8.89), we obtain

$$h_3 = -\frac{4kb^2}{\pi A^2}\sqrt{A^2 - b^2},$$
$$r_3 = -\frac{4kb^2}{\pi A^2}\sqrt{A^2 - b^2}$$
$$(A \geqslant b), \qquad (8.90)$$

or in the notation of (8.55), where we must put c = kb (in accordance with Fig. 8.2b):

$$h_3 = -\frac{2\gamma}{\beta^2}\sqrt{\beta^2 - 1}, \quad r_3 = -\frac{2\gamma}{\beta^2}\sqrt{\beta^2 - 1} \quad (\beta \geqslant 1). \qquad (8.91)$$

In this case $\gamma = 2k/\pi$.

5. Characteristic with saturation without backlash zone. For a characteristic with saturation without a backlash zone (Fig. 8.9b) we obtain, putting $b_1 = 0$, $b_2 = b$ in Formulas (8.88) and (8.89)

$$h_3 = \frac{4kb^2}{\pi A^2}\sqrt{A^2 - b^2}, \quad r_3 = \frac{4kb(A^2 - b^2)}{3\pi A^2}\sqrt{A^2 - b^2} \quad (A \geqslant b), \qquad (8.92)$$

or in the notation of (8.55)

$$h_3 = \frac{2\gamma}{\beta^2}\sqrt{\beta^2 - 1}, \quad r_3 = \frac{2\gamma(\beta^2 - 1)}{3\beta^2}\sqrt{\beta^2 - 1} \quad (\beta \geqslant 1). \qquad (8.93)$$

6. Characteristic with variable gain. For a nonlinear characteristic with variable gain (Fig. 8.10a) and for $\underline{x}$ varying sinusoidally, the function $F(A \sin \psi)$ is plotted in Fig. 8.10b and the function $(d/dx)F(A \sin \psi)$ is plotted in Fig. 8.10c.

Calculating the coefficient h_3 in accordance with Formula (8.79) and Fig. 8.10c, we obtain

$$h_3 = \frac{4}{\pi}\left(\int_0^{\psi_1} k_1 \sin 3\psi \sin \psi \, d\psi + \int_{\psi_1}^{\frac{\pi}{2}} k_2 \sin 3\psi \sin \psi \, d\psi\right) =$$
$$= \frac{k_1 - k_2}{\pi}\left(\sin 2\psi_1 - \frac{1}{2}\sin 4\psi_1\right).$$

Taking into account the values

$$\sin \psi_1 = \frac{b}{A}, \quad \sin 2\psi_1 = \frac{2b\sqrt{A^2 - b^2}}{A^2}, \quad \sin 4\psi_1 = \frac{4b(A^2 - 2b^2)}{A^4}\sqrt{A^2 - b^2},$$

we obtain

$$h_3 = \frac{4(k_1 - k_2)b^2}{\pi A^2}\sqrt{A^2 - b^2} \quad (A \geqslant b), \qquad (8.94)$$

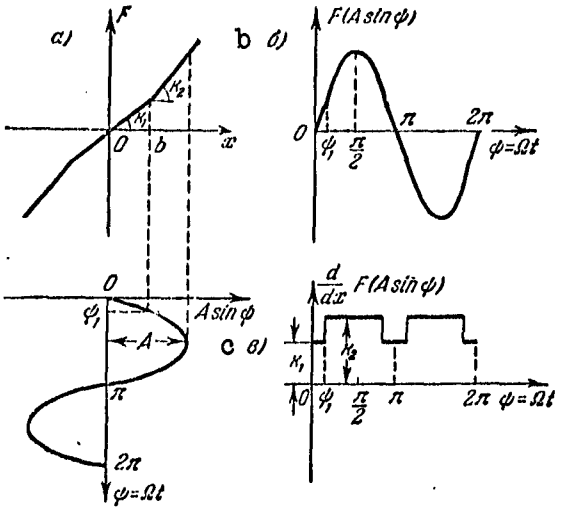


Fig. 8.10

or

$$h_3=\frac{2\gamma}{\beta^2}\sqrt{\beta^2-1}\qquad\left(\gamma=\frac{2(k_1-k_2)}{\pi},\ \beta=\frac{A}{b}\geq 1\right). \tag{8.95}$$

Calculating the coefficient r_3 in accordance with Formula (8.79) and Fig. 8.10b, we obtain

$$r_3=\frac{4}{\pi A}\left\{\int_0^{\psi_1}k_1 A\sin\psi\sin 3\psi\,d\psi+\int_{\psi_1}^{\frac{\pi}{2}}[k_2(A\sin\psi-b)+k_1 b]\sin 3\psi\,d\psi\right\}=$$
$$=\frac{k_1-k_2}{\pi}\left(\sin 2\psi_1-\frac{1}{2}\sin 4\psi_1\right)+\frac{4(k_1-k_2)b}{3\pi A}\cos 3\psi_1.$$

Taking into account the values written out above, and also

$$\cos 3\psi_1=\frac{A^2-4b^2}{A^3}\sqrt{A^2-b^2},$$

we ultimately get

$$r_3=\frac{4(k_1-k_2)(A^2-b^2)b}{3\pi A^4}\sqrt{A^2-b^2}\qquad(A\geq b), \tag{8.96}$$

or

$$r_3=\frac{2\gamma(\beta^2-1)}{3\beta^4}\sqrt{\beta^2-1}\qquad\left(\gamma=\frac{2(k_1-k_2)}{\pi},\ \beta=\frac{A}{b}\geq 1\right). \tag{8.97}$$

7. The nonlinear function $F = kx^2$ sign x. For this nonlinear function (Fig. 8.11a) and with x having a sinusoidal variation, the function $F(A\sin\psi)$ has the form shown in Fig. 8.11b, while the function $(d/dx)F(A\sin\psi)$ is represented by two positive half sine waves with

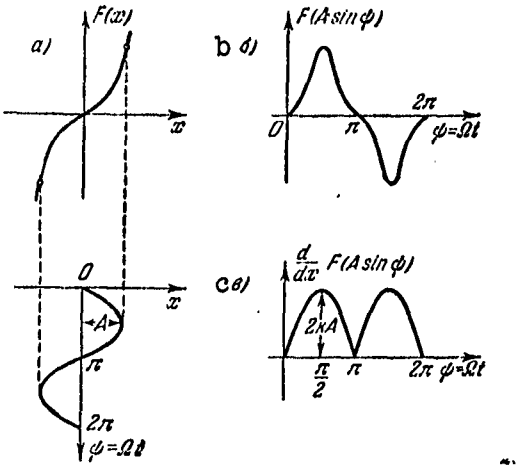


Fig. 8.11

amplitude 2kA (Fig. 8.11c).

Calculating the coefficient h_3 in accordance with Formula (8.79) and Fig. 8.11c we obtain

$$h_3=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}2kA\sin\psi\sin 3\psi\sin\psi\,d\psi=\frac{8kA}{\pi}\int_0^{\frac{\pi}{2}}\sin 3\psi\sin^2\psi\,d\psi.$$

As a result of integration we find

$$h_3=-\frac{16kA}{15\pi}=-\frac{8}{15}\gamma\beta\quad\left(\gamma=\frac{2kb}{\pi},\quad\beta=\frac{A}{b}\right),\qquad(8.98)$$

where b denotes any fixed value of x, selected to obtain the dimensionless expression for the amplitude β.

Calculating the coefficient r_3 in accordance with Formula (8.79) and Fig. 8.11b we obtain

$$r_3=\frac{4}{\pi A}\int_0^{\frac{\pi}{2}}kA^2\sin^2\psi\sin 3\psi\,d\psi=\frac{4kA}{\pi}\int_0^{\frac{\pi}{2}}\sin 3\psi\sin^2\psi\,d\psi.$$

As a result of integration we get

$$r_3=-\frac{8kA}{15\pi}=-\frac{4}{15}\gamma\beta\quad\left(\gamma=\frac{2kb}{\pi},\quad\beta=\frac{A}{b}\right).\qquad(8.99)$$

8. Nonlinear function $F = kx^3$. For this nonlinear function the calculation of the coefficients h_3 and r_3 will be carried out in analogy with the preceding case. The function $F(A\sin\psi)$ is in this case equal

to $kA^3 \sin^3 \psi$, while $(d/dx)F(A \sin \psi) = 3kA^2 \sin^2 \psi$.

Calculations yield for the coefficient h_3

$$h_3 = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} 3kA^3 \sin^3 \psi \sin 3\psi \sin \psi \, d\psi.$$

As a result of integration we have

$$h_3 = -\frac{3kA^3}{4} = -\frac{3\pi}{8} \gamma\beta^3 \qquad \left(\gamma = \frac{2kb^3}{\pi}, \quad \beta = \frac{A}{b}\right). \tag{8.100}$$

For the coefficient r_3 we have

$$r_3 = \frac{4}{\pi A} \int_0^{\frac{\pi}{2}} kA^3 \sin^3 \psi \sin 3\psi \, d\psi,$$

which yields

$$r_3 = -\frac{kA^3}{4} = -\frac{\pi}{8} \gamma\beta^3 \qquad \left(\gamma = \frac{2kb^3}{\pi}, \quad \beta = \frac{A}{b}\right). \tag{8.101}$$

We can similarly calculate the coefficients h_3 and r_3 for other single-valued odd-symmetry nonlinear characteristics.

9. For loop-type nonlinearities F(x) and for nonlinearities of more general form F(x, px) it is necessary to employ in place of Formulas (8.78) and (8.79) the general formulas (8.12) and (8.17). By way of an illustration we shall consider below the calculation of the third harmonic coefficients for two types of loop nonlinearities, which can serve as an example of the calculations for all other nonlinearities.

To calculate the third harmonic of the self-oscillations it is necessary, in accordance with (8.21), to know the coefficients r_3 and s_3, which are defined by Formulas (8.17), which in turn can be written for odd-symmetry loop-type nonlinearities F(x) in the form

$$\left.\begin{aligned} r_3 &= \frac{2}{\pi A} \int_0^{\pi} F(A \sin \psi) \sin 3\psi \, d\psi, \\ s_3 &= \frac{2}{\pi A} \int_0^{2\pi} F(A \sin \psi) \cos 3\psi \, d\psi. \end{aligned}\right\} \tag{8.102}$$

To obtain a more exact first harmonic of the self-oscillations with

inclusion of the third harmonic it is necessary to calculate the additional terms Δq and $\Delta q'$ by means of Formulas (8.12), namely:

$$\left.\begin{aligned} \Delta q &= I_{31}\delta_3 \cos\varphi_3 + I_{32}\delta_3 \sin\varphi_3, \\ \Delta q' &= I_{33}\delta_3 \cos\varphi_3 + I_{34}\delta_3 \sin\varphi_3, \end{aligned}\right\} \qquad (8.103)$$

where

$$\left.\begin{aligned} I_{31} &= \frac{2}{\pi}\int_0^{\pi} \frac{d}{dx}F(A\sin\psi)\sin 3\psi \sin\psi\, d\psi, \\ I_{32} &= \frac{2}{\pi}\int_0^{\pi} \frac{d}{dx}F(A\sin\psi)\cos 3\psi \sin\psi\, d\psi, \\ I_{33} &= \frac{2}{\pi}\int_0^{\pi} \frac{d}{dx}F(A\sin\psi)\sin 3\psi \cos\psi\, d\psi, \\ I_{34} &= \frac{2}{\pi}\int_0^{\pi} \frac{d}{dx}F(A\sin\psi)\cos 3\psi \cos\psi\, d\psi. \end{aligned}\right\} \qquad (8.104)$$

Formulas (8.102)-(8.104) are suitable for all loop-type nonlinearities F(x).

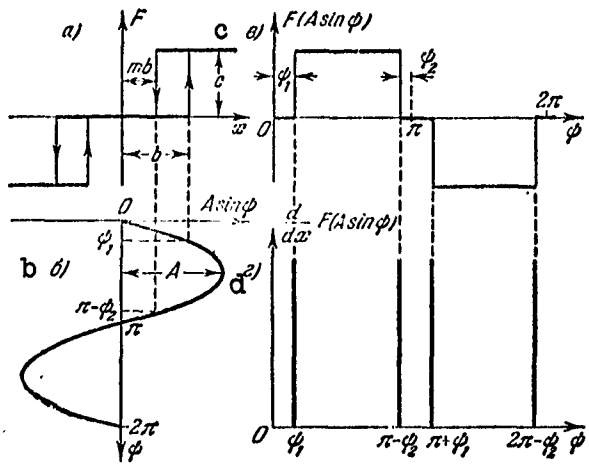


Fig. 8.12

10. Relay characteristic of general type. For a loop relay characteristic of the general type (Fig. 8.12a) we obtain from (8.102) with account of Fig. 8.12c:

$$r_3 = \frac{2c}{\pi A}\int_{\psi_1}^{\pi-\psi_2} \sin 3\psi\, d\psi = \frac{2c}{3\pi A}(\cos 3\psi_1 + \cos 3\psi_2),$$

$$s_3 = \frac{2c}{\pi A}\int_{\psi_1}^{\pi-\psi_2} \cos 3\psi\, d\psi = -\frac{2c}{3\pi A}(\sin 3\psi_1 - \sin 3\psi_2).$$

But since

$$\psi_1 = \arcsin\frac{b}{A}, \qquad \psi_2 = \arcsin\frac{mb}{A}, \tag{8.105}$$

we ultimately obtain

$$\left.\begin{aligned} r_3 &= \frac{2c}{3\pi A}\left[\left(1-4\frac{b^2}{A^2}\right)\sqrt{1-\frac{b^2}{A^2}}+\left(1-4\frac{m^2b^2}{A^2}\right)\sqrt{1-\frac{m^2b^2}{A^2}}\right], \\ s_3 &= -\frac{2cb}{\pi A^2}\left[1-\frac{4b^2}{3A^2}-m\left(1-\frac{4m^2b^2}{3A^2}\right)\right] \qquad (A \geqslant b). \end{aligned}\right\} \tag{8.106}$$

We now calculate the integrals (8.104) contained in the formulas of (8.103). In this case it is convenient to use the transformation (8.80), i.e.,

$$\sin\psi\, d\psi = \frac{\operatorname{tg}\psi}{A}\, dx, \qquad \cos\psi\, d\psi = \frac{1}{A}\, dx. \tag{8.107}$$

Noting that in the integration interval (0, π), in accordance with Fig. 8.12d, the value of dF/dx does not vanish only at the two points $\psi = \psi_1$ and $\psi = \pi - \psi_2$ as $\underline{x}$ varies from 0 to A and from A to 0, respectively, we obtain in accordance with (8.104) together with the substitution (8.107), the following expressions:

$$\begin{aligned} I_{31} &= \frac{2}{\pi A}\sin 3\psi_1 \operatorname{tg}\psi_1 \int_0^A \frac{dF}{dx}\, dx - \frac{2}{\pi A}\sin 3\psi_2 \operatorname{tg}\psi_2 \int_A^0 \frac{dF}{dx}\, dx = \\ &= \frac{2}{\pi A}(\sin 3\psi_1 \operatorname{tg}\psi_1 + \sin 3\psi_2 \operatorname{tg}\psi_2)[F(A) - F(0)], \\ I_{32} &= \frac{2}{\pi A}(\cos 3\psi_1 \operatorname{tg}\psi_1 - \cos 3\psi_2 \operatorname{tg}\psi_2)[F(A) - F(0)], \\ I_{33} &= \frac{2}{\pi A}(\sin 3\psi_1 - \sin 3\psi_2)[F(A) - F(0)], \\ I_{34} &= \frac{2}{\pi A}(\cos 3\psi_1 + \cos 3\psi_2)[F(A) - F(0)]. \end{aligned}$$

From this, taking (8.81) into account, we obtain

$$\left.\begin{aligned} I_{31} &= \frac{2cb^2}{\pi A^4}\left(\frac{3A^2-4b^2}{\sqrt{A^2-b^2}}+m^2\frac{3A^2-4m^2b^2}{\sqrt{A^2-m^2b^2}}\right), \\ I_{32} &= \frac{2cb}{\pi A^2}\left[1-4\frac{b^2}{A^2}-m\left(1-4\frac{m^2b^2}{A^2}\right)\right], \\ I_{33} &= \frac{2cb}{\pi A^2}\left[\left(3-4\frac{b^2}{A^2}\right)-m\left(3-4\frac{m^2b^2}{A^2}\right)\right], \\ I_{34} &= \frac{2c}{\pi A}\left[\left(1-4\frac{b^2}{A^2}\right)\sqrt{1-\frac{b^2}{A^2}}+\left(1-4\frac{m^2b^2}{A^2}\right)\sqrt{1-\frac{m^2b^2}{A^2}}\right]. \end{aligned}\right\} \tag{8.108}$$

11. Loop-type relay characteristic. As a particular case of the preceding formulas, with m = −1, we can obtain formulas for a pure

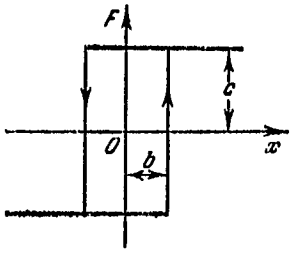


Fig. 8.13

loop-type relay characteristic without a central zero position (Fig. 8.13). Thus we have from Formula (8.106) with m = −1

$$r_3=\frac{4c}{3\pi A}\left(1-4\frac{b^2}{A^2}\right)\sqrt{1-\frac{b^2}{A^2}},$$
$$s_3=-\frac{4cb}{\pi A^2}\left(1-\frac{4b^2}{3A^2}\right)\quad (A\geqslant b), \tag{8.109}$$

or in the notation of (8.55)

$$r_3=\frac{2\gamma(3\beta^2-4)}{3\beta^4}\sqrt{\beta^2-1},\quad s_3=-\frac{2\gamma(3\beta^2-4)}{3\beta^4}\quad (\beta\geqslant 1). \tag{8.110}$$

From (8.108) with m = −1 we get

$$\left.\begin{aligned}
l_{31}&=\frac{4cb^2(3A^2-4b^2)}{\pi A^4\sqrt{A^2-b^2}}=\frac{2\gamma(3\beta^2-4)}{\beta^4\sqrt{\beta^2-1}},\\
l_{32}&=\frac{4cb}{\pi A^2}\left(1-4\frac{b^2}{A^2}\right)=\frac{2\gamma}{\beta^4}(\beta^2-4),\\
l_{33}&=\frac{4cb}{\pi A^2}\left(3-4\frac{b^2}{A^2}\right)=\frac{2\gamma}{\beta^4}(3\beta^2-4),\\
l_{34}&=\frac{4c}{\pi A}\left(1-4\frac{b^2}{A^2}\right)\sqrt{1-\frac{b^2}{A^2}}=\frac{2\gamma}{\beta^4}(\beta^2-4)\sqrt{\beta^2-1}.
\end{aligned}\right\} \tag{8.111}$$

It is also interesting to note that all the formulas derived at the start of the present section for a single-valued relay characteristic are obtained from the general case given above by putting m = +1.

§8.4. Examples of the Calculation of the Third Harmonic and of More Exact Values of the First Harmonic

In the present section we shall present two examples illustrating the determination of the higher harmonics in self-oscillations, and also the determination of the more exact value of the first harmonic by inclusion of the higher harmonics [246].

Example 1. We consider a servomechanism with a nonlinearity of the saturation type. The equation of the system, a diagram of which is shown in Fig. 8.14a, in terms of the variable $\underline{x}$ with $\alpha = 0$, will be

$$(T_1p+1)(T_2p+1)px+[k_1+(T_1p+1)k_{o.c}p]k_2F(x)=0, \tag{8.112}$$

where the nonlinearity F(x) has the form shown in Fig. 8.14b.

We first find the first approximation x = A sin Ωt. For this pur-

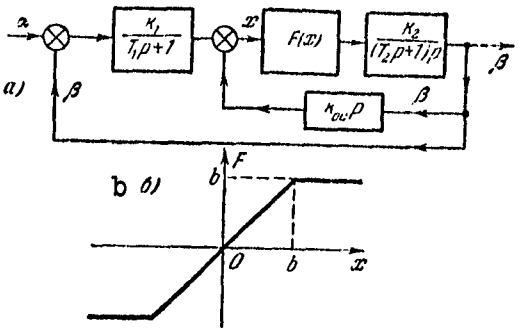


Fig. 8.14

pose we set up the characteristic equation of the harmonically linearized system:

$$(T_1p+1)(T_2p+1)p+[k_1+(T_1p+1)k_{o.c}p]k_2q(A)=0, \qquad (8.113)$$

where in accordance with Fig. 8.14b and Chapter 3 we have:

$$\left.\begin{aligned} q&=1 \text{ for } A\leqslant b,\\ q&=\frac{2}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right) \text{ for } A\geqslant b. \end{aligned}\right\} \qquad (8.114)$$

Substituting in (8.113) $p = j\Omega$ and separating the real and imaginary parts, we obtain

$$\begin{aligned} k_1k_2q(A)-[T_1+T_2+T_1k_{o.c}k_2q(A)]\Omega^2&=0,\\ [1+k_{o.c}k_2q(A)]\Omega-T_1T_2\Omega^3&=0. \end{aligned}$$

From the last equation it follows that

$$\Omega^2=\frac{1+k_{o.c}k_2q(A)}{T_1T_2}, \qquad (8.115)$$

and from the first that

$$k_1=\left[\frac{T_1+T_2}{k_2q(A)}+T_1k_{o.c}\right]\Omega^2. \qquad (8.116)$$

By specifying different values of the amplitude A we determine with the aid of Formulas (8.114) and (8.115) the values of q(A) and Ω^2, and then by means of Formula (8.116) the value of k_1. This makes it possible to plot the amplitude of the periodic solution A as a function of the gain k_1 (with all other system parameters specified). This plot has the form of Fig. 8.15a or b under restrictively the following relationships between the system parameters (see §4.4):

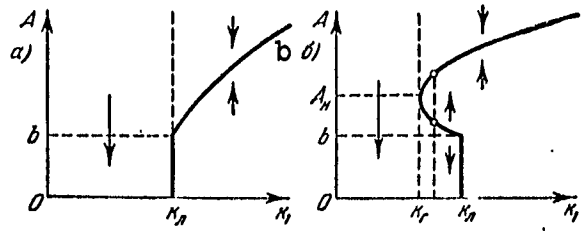


Fig. 8.15

$$k_2 k_{o.c} \leqslant \sqrt{1+\frac{T_2}{T_1}} \quad \text{or} \quad k_2 k_{o.c} > \sqrt{1+\frac{T_2}{T_1}}, \tag{8.117}$$

with

$$k_n = \left(\frac{T_1+T_2}{k_2}+T_1 k_{o.c}\right)\frac{1+k_2 k_{o.c}}{T_1 T_2},$$
$$k_r = \frac{k_{o.c}}{T_2}\left(1+\sqrt{1+\frac{T_2}{T_1}}\right)^2. \tag{8.118}$$

Assume that the following system parameters are specified:

$$T_1 = 0.005 \text{ sec}, \quad T_2 = 0.4 \text{ sec}, \quad k_1 = 140,$$
$$k_2 = 100 \text{ sec}^{-1}, \quad k_{o.s} = 0.5 \text{ sec}.$$

They satisfy the second relation in (8.117). Consequently we have here the case shown in Fig. 8.15b, and in accordance with (8.118) $k_1 = 166$ and $k_g = 125$. The specified value of k_1 lies in between these two, corresponding to the region where there are two periodic modes, designated by the points in Fig. 8.15b. The first approximation formulas (8.115) and (8.116) derived above give in this case for the unstable mode $A = 2.29$ v and $\Omega = 118.2 \text{ sec}^{-1}$, and for the stable mode $A = 21.4$ v and $\Omega = 44.8 \text{ sec}^{-1}$, with $A_n = 7.08$ v.

Of greatest interest is the first (unstable) periodic solution. It delineates the region of initial conditions, outside of which the transient in the system will diverge and will tend to self-oscillations with very large amplitude, $A = 21.4$ v, which in practice can be regarded as instability of the system in the large. Therefore, we shall determine more exact solutions with calculation of the higher harmonics only for the first periodic solution.

For the given nonlinearity (Fig. 8.14b) we find in §8.3 ready-made expressions (8.92):

$$h_3 = \frac{4b^2}{\pi A^2}\sqrt{1-\frac{b^2}{A^2}}, \quad r_3 = \frac{4b}{3\pi A}\left(1-\frac{b^2}{A^2}\right)^{\frac{3}{2}}. \tag{8.119}$$

We therefore obtain from (8.30) and (8.113) the relative amplitude δ_3 and the phase φ_3 of the third harmonic in the form

$$\delta_3 = \frac{k_2 r_3}{3\Omega}\sqrt{\frac{(k_1 - 9k_{o.c}T_1\Omega^2)^2 + 9k_{o.c}^2\Omega^2}{(9T_1^2\Omega^2+1)(9T_2^2\Omega^2+1)}},$$

$$\varphi_3 = \frac{\pi}{2} + \operatorname{arctg}\frac{3k_{o.c}\Omega}{k_1 - 9T_1k_{o.c}\Omega^2} - \operatorname{arctg} 3T_1\Omega - \operatorname{arctg} 3T_2\Omega.$$

Calculation by means of these formulas yields

$$\delta_3 = 0{,}0317, \quad \varphi_3 = -1{,}875.$$

To obtain a more exact value of the first harmonic using the just calculated third harmonic we determine, in accordance with (8.28), the added terms for the harmonic linearization coefficients

$$\Delta q = h_3\delta_3\cos\varphi_3, \quad \Delta q' = 3r_3\delta_3\sin\varphi_3,$$

the substitution of which in (8.26) yields, in accordance with (8.113), the more exact characteristic equation

$$(T_1p+1)(T_2p+1)p + [k_1 + (T_1p+1)k_{o.c}p]k_2\left(\Delta q + \frac{\Delta q'}{\Omega}p\right) + \\ + [k_1 + (T_1p+1)k_{o.c}p]k_2q(A_1) = 0, \tag{8.120}$$

where in analogy with (8.114) we have

$$\left.\begin{aligned} q &= 1 \quad \text{for } A_1 \leqslant b, \\ q &= \frac{2}{\pi}\left(\arcsin\frac{b}{A_1} + \frac{b}{A_1}\sqrt{1-\frac{b^2}{A_1^2}}\right) \quad \text{for } A_1 > b. \end{aligned}\right\} \tag{8.121}$$

Substituting in (8.120) $p = j\Omega_1$ and separating the real and imaginary parts, we obtain two equations

$$k_1k_2q(A_1) + k_1k_2\Delta q - [T_1 + T_2 + T_1k_{o.c}k_2q(A_1)]\Omega_1^2 - \\ - \left(T_1\Delta q + \frac{\Delta q'}{\Omega}\right)k_{o.c}k_2\Omega_1^2 = 0,$$

$$[1 + k_{o.c}k_2q(A_1)]\Omega_1 + \left(k_{o.c}\Delta q + k_1\frac{\Delta q'}{\Omega}\right)k_2\Omega_1 - T_1T_2\Omega_1^3 - \\ - T_1k_{o.c}k_2\frac{\Delta q'}{\Omega}\Omega_1^2 = 0.$$

These more exact equations differ from the previous first approximation equations in that they contain several additional terms, but the method

of solving them remains the same. From the last equation we get

$$\Omega_1^2 = \frac{1 + k_{o.c}k_2q(A_1) + \left(k_{o.c}\Delta q + k_1\frac{\Delta q'}{\Omega}\right)k_2}{T_1T_2 + T_1k_{o.c}k_2\frac{\Delta q'}{\Omega}}, \qquad (8.122)$$

and from the first equation

$$k_1 = \frac{T_1 + T_2 + T_1k_{o.c}k_2q(A_1) + \left(T_1\Delta q + \frac{\Delta q'}{\Omega}\right)k_{o.c}k_2}{k_2q(A_1) + k_2\Delta q}\Omega_1^2. \qquad (8.123)$$

By specifying different values of the amplitude A_1 and calculating each time the values of $q(A_1)$, Ω_1^2, and k_1 by means of Formulas (8.121), (8.122), and (8.123), we obtain plots of $A_1(k_1)$ such as shown in Fig. 8.15, but now for the more exact value of the amplitude A_1 of the first harmonic of the periodic solution.

For a specified value $k_1 = 140$, the more exact values are $A_1 = 2.39$ v and $\Omega_1 = 117.8$ sec^{-1}. These values are sufficiently close to those of the first approximation, and the amplitude of the third harmonic, calculated above, is sufficiently small.

Example 2. Assume that an automatic control system incorporates a two-phase induction motor described by the nonlinear equation (1.27), namely

$$Jpx + Jc_3|x|px + c_2x + (c_2c_3 + c_4)|x|x + c_3c_4x^3 = c_1u,$$

where $\underline{x}$ is the angle of velocity of the motor shaft and $\underline{u}$ is the control voltage. We rewrite this equation in the form

$$(T_1p + 1)x + F(x, px) = k_1u,$$

where

$$T_1 = \frac{J}{c_2}, \quad k_1 = \frac{c_1}{c_2},$$

$$F(x, px) = T_1c_3|x, px + \left(c_3 + \frac{c_4}{c_2}\right)|x|x + \frac{c_3c_4}{c_2}x^3.$$

Harmonic linearization of this nonlinearity for $x = A \sin \Omega t$ leads to a motor equation in the form

$$[(T_1p + 1) + (T_1b_1Ap + b_2A + b_3A^2)]x = k_1u, \qquad (8.124)$$

where

$$b_1 = \frac{4c_1}{3\pi}, \quad b_2 = 2b_1 + \frac{8c_1}{3\pi c_2}, \quad b_3 = \frac{3c_2c_1}{4c_2}. \tag{8.125}$$

The equations of the remaining elements of the automatic control system are assumed to be linear:

$$px_1 = k_2x, \quad (T_2p + 1)x_2 = -k_3x_1, \quad u = k_4x_2, \tag{8.126}$$

where x_2 is the controlled quantity.

We first find the first approximation for the self-oscillations. The characteristic equation, in accordance with (8.124) and (8.126), will be

$$[(T_1p + 1) + (T_1b_1Ap + b_2A + b_3A^2)](T_2p + 1)p + k = 0, \tag{8.127}$$

where $k = k_1k_2k_3k_4$. Substituting $p = j\Omega$ and separating the real and imaginary parts we obtain

$$\left.\begin{aligned} k - [T_1(1 + b_1A) + T_2(1 + b_2A + b_3A^2)]\Omega^2 = 0, \\ (1 + b_2A + b_3A^2)\Omega - T_1T_2(1 + b_1A)\Omega^3 = 0. \end{aligned}\right\} \tag{8.128}$$

From the second equation we have

$$\Omega^2 = \frac{1 + b_2A + b_3A^2}{T_1T_2(1 + b_1A)}, \tag{8.129}$$

and from the first equation

$$k = [T_1(1 + b_1A) + T_2(1 + b_2A + b_3A^2)]\Omega^2. \tag{8.130}$$

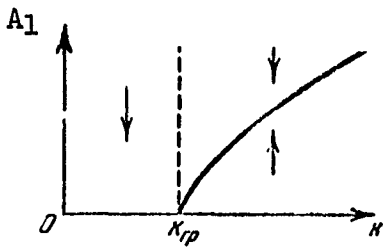


Fig. 8.16

By specifying different values of A and calculating Ω^2 and $\underline{k}$ we obtain the function A(k) plotted in Fig. 8.16. Let us consider the following numerical example

$$T_1 = 0.5 \text{ sec}, \quad T_2 = 0.1 \text{ sec},$$
$$k = 26.5 \text{ sec}^{-1}$$

with two types of nonlinearity:

a) weak nonlinearity

$$b_1 = 0{,}01, \quad b_2 = 0{,}1, \quad b_3 = 0{,}002;$$

b) strong nonlinearity

$$b_1 = 0{,}1, \quad b_2 = 1, \quad b_3 = 0{,}166.$$

Calculations by means of the first approximation formulas (8.129) and (8.130) yield self-oscillations in the form $x = A \sin \Omega t$, where for the weak nonlinearity we have

$$A = 8.14, \quad \Omega = 6 \text{ sec}^{-1},$$

and for the strong nonlinearity

$$A = 0.834, \quad \Omega = 6 \text{ sec}^{-1}.$$

We now calculate the higher harmonics. To take into account the second and third harmonics we make use of Formula (8.17). For the nonlinearity $F(x, px)$ considered in the present example, the coefficients r_2 and s_2, calculated by Formulas (8.17), are found to be equal to zero. Therefore, we are left with only the third harmonic, for which we obtain for the given nonlinearity with allowance for the notation of (8.125), using the formulas (8.17),

$$r_3 = -\frac{2}{5} b_2 A - \frac{1}{3} b_3 A^2, \quad s_3 = -\frac{3}{5} T_1 b_1 A \Omega. \tag{8.131}$$

We then obtain from the formulas (8.21), allowing for the fact that according to (8.127) we have

$$Q(p) = (T_1 p + 1)(T_2 p + 1)p + k, \quad R(p) = (T_2 p + 1)p,$$

the relative amplitude and phase of the third harmonic

$$\delta_3 = 3\Omega \sqrt{\frac{(9T_2^2\Omega^2 + 1)(r_3^2 + s_3^2)}{[k - 9(T_1 + T_2)\Omega^2]^2 + 9\Omega^2(1 - 9T_1T_2\Omega^2)^2}},$$

$$\varphi_3 = -\frac{\pi}{2} + \operatorname{arctg} 3T_2\Omega - \operatorname{arctg} \frac{3\Omega(1 - 9T_1T_2\Omega^2)}{k - 9(T_1 + T_2)\Omega^2} + \operatorname{arctg} \frac{s_3}{r_3}.$$

For the data indicated above we obtain for the weak nonlinearity

$$\delta_3 = 0{,}041, \quad \varphi_3 = -0{,}377,$$

and for the strong nonlinearity

$$\delta_3 = 0{,}042, \quad \varphi_3 = -0{,}0384.$$

We then obtain a more exact first harmonic of the self-oscillations $A_1 \sin \Omega_1 t$. For this purpose we determine from Formulas (8.12) the values of the added terms Δq and $\Delta q'$ in the harmonic linearization

coefficients:

$$\Delta q = \frac{3}{5} T_1 b_1 A \Omega \delta_3 \sin \varphi_3 - \left(\frac{2}{5} b_2 + b_3 A\right) A \delta_3 \cos \varphi_3,$$

$$\Delta q' = -\frac{2}{5} T_1 b_1 A \Omega \delta_3 \cos \varphi_3 - \left(\frac{3}{5} b_2 + b_3 A\right) A \delta_3 \sin \varphi_3.$$

Therefore, the new characteristic equation for the determination of the more exact first harmonic will be

$$[(T_1 p + 1) + (T_1 b_1 A_1 p + b_2 A_1 + b_3 A_1^3)](T_2 p + 1) p + k + \\ + \left(\Delta q + \frac{\Delta q'}{\Omega} p\right)(T_2 p + 1) p = 0.$$

Substituting $p = j\Omega_1$ and separating the real and imaginary parts we obtain

$$k - [T_1(1 + b_1 A_1) + T_2(1 + b_2 A_1 + b_3 A_1^2)]\,\Omega_1^2 - \left(T_2 \Delta q + \frac{\Delta q'}{\Omega}\right)\Omega_1^2 = 0,$$

$$(1 + b_2 A_1 + b_3 A_1^2)\,\Omega_1 + \Delta q \Omega_1 - T_1 T_2 (1 + b_1 A_1)\,\Omega_1^3 - T_2 \frac{\Delta q'}{\Omega} \Omega_1^2 = 0.$$

These equations are solved by the same method as (8.128), namely, we have from the second equation:

$$\Omega_1^2 = \frac{1 + b_2 A_1 + b_3 A_1^2 + \Delta q}{T_1 T_2 (1 + b_1 A_1) + T_2 \frac{\Delta q'}{\Omega}},$$

and from the first

$$k = \left[T_1(1 + b_1 A_1) + T_2(1 + b_2 A_1 + b_3 A_1^2) + \left(T_2 \Delta q + \frac{\Delta q'}{\Omega}\right)\right]\Omega_1^2.$$

These equations lead also to a plot of $A_1(k)$ in the form of Fig. 8.16.

For the numerical values of the system parameters given above we obtain the following more exact values of the amplitude and frequency of the self-oscillations:

for the weak nonlinearity

$$A_1 = 8.03, \quad \Omega_1 = 5.99 \text{ sec}^{-1},$$

and for the strong nonlinearity

$$A_1 = 0.820, \quad \Omega_1 = 5.98 \text{ sec}^{-1}.$$

We see that a strong nonlinearity* greatly reduces the amplitude of the self-oscillations (in a linear system we would have $A_1 = \infty$).

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This result was obtained above in the solution based on the first approximation, and is now confirmed by the more exact solution.

§8.5. Examples of Determination of the Dependence of the Frequency on the Form of the Nonlinearity

Let us illustrate the procedure developed in §8.2 for disclosing the dependence of the self-oscillation frequency on the form of the nonlinearity for certain single-valued nonlinearities by taking into account the higher harmonics [251].

Example 1. We consider a relay-type automatic control system, in which the dynamic processes are described by the third order nonlinear equation

$$(T_1 p + 1)(T_2 p + 1) p x + k F(x) = 0, \tag{8.132}$$

where F(x) is specified in the form of the plot of Fig. 8.17a (relay characteristic with backlash zone).

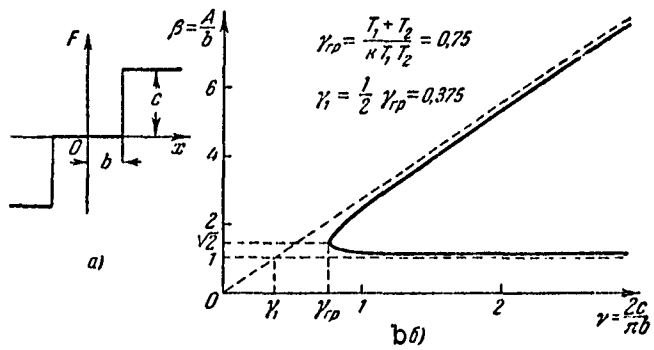


Fig. 8.17

The characteristic equation of the harmonically linearized system for the first approximation will be in this case

$$T_1 T_2 p^3 + (T_1 + T_2) p^2 + p + kq = 0, \tag{8.133}$$

where in accordance with (8.56) we have in the notation of (8.55)

$$q = \frac{2\gamma}{\beta^2}\sqrt{\beta^2 - 1}.$$

Substituting $p = j\Omega$ and separating the real and imaginary parts, we obtain two equations of the type of (8.44) in the form

$$X = kq - (T_1 + T_2)\Omega^2 = 0,$$
$$Y = \Omega - T_1 T_2 \Omega^3 = 0.$$

From this we get

$$\Omega = \frac{1}{\sqrt{T_1 T_2}}, \quad \gamma = \frac{T_1 + T_2}{2kT_1T_2} \frac{\beta^2}{\sqrt{\beta^2 - 1}} \quad (\beta \geq 1). \qquad (8.134)$$

The last expression gives the dependence of the relative self-oscillation amplitude $\beta = A/b$ on the nonlinearity form coefficient $\gamma = 2c/\pi b$, shown in Fig. 8.17b. The upper branch of the curve of Fig. 8.17b corresponds to a stable periodic solution (self-oscillations), and the lower branch to an unstable one. To plot the curve we assume the following values of the system parameters:

$$T_1 = 0.1 \text{ sec}, \quad T_2 = 0.2 \text{ sec}, \quad k = 20, \quad c = 1. \quad (8.135)$$

If the results of the first approximation calculations, given in Fig. 8.17b, are recalculated for these data into the dependence of the self-oscillation amplitude A (stable branch of the periodic solutions) on the backlash zone b, then we obtain

b	0	0,1	0,2	0,3	0,4	0,[illegible]	0,325	0,349
A	1,698	1,695	1,686	1,670	1,647	1,550	1,331	1,200

As can be seen from Fig. 8.17b, when the system parameter ratio corresponds to the inequality

$$\gamma = \frac{2c}{\pi b} < \frac{T_1 + T_2}{kT_1T_2} = 0{,}75,$$

self-oscillations are impossible. The given relay system will then be stable.

The self-oscillation frequency Ω in the first approximation solution, as can be seen from (8.134), is independent of the form of the nonlinearity and is equal to 7.07 sec^{-1}.

In order to establish the dependence of the frequency of the self-oscillations on the nonlinearity form coefficient γ, let us turn to

the more exact solution. Here we determine simultaneously the new more exact value of the first harmonic amplitude, and find the amplitudes and phase of the third harmonic.

Let us find the more exact value of the frequency Ω_1 as a function of the nonlinearity form, for which purpose we solve Eq. (8.65). We first set up an expression for $\Phi_1(\Omega_1)$ by means of Formula (8.66). Noting that in accord with (8.133) we have in the present example

$$Q(p)=T_1T_2p^3+(T_1+T_2)p^2+p, \quad R(p)=k,$$

we get

$$X_Q(\Omega_1)=-(T_1+T_2)\Omega_1^2, \quad X_R(\Omega_1)=k,$$
$$Y_Q(\Omega_1)=\Omega_1-T_1T_2\Omega_1^3, \quad Y_R(\Omega_1)=0.$$

In addition, we write in accordance with (8.63)

$$V_3(3\Omega_1)=\operatorname{Im}\left\{\frac{-k}{j3\Omega_1[1+(T_1+T_2)j3\Omega_1-T_1T_2 9\Omega_1^2]}\right\}=$$
$$=\frac{-k(9T_1T_2\Omega_1^2-1)}{3\Omega_1[(9T_1T_2\Omega_1^2-1)^2+9(T_1+T_2)^2\Omega_1^2]}.$$

Substituting all of these expressions into (8.66) we get

$$\Phi_1(\Omega_1)=\frac{\Omega_1^2(1-T_1T_2\Omega_1^2)[(9T_1T_2\Omega_1^2-1)^2+9(T_1+T_2)^2\Omega_1^2]}{k^2(9T_1T_2\Omega_1^2-1)}. \qquad (8.136)$$

In order to solve the equation (8.65) we must also know $r_3(\gamma)$. From §8.3 we have for the given nonlinearity (Fig. 8.17a), in accordance with (8.85),

$$r_3=2\gamma\frac{\beta^2}{\omega_1^2}-\frac{4}{\pi}\sqrt{\beta^2-1}. \qquad (8.137)$$

Now we can easily solve (8.65) graphically, as shown in Fig. 8.4a. Using the previous values of the system parameters and the functions $\beta(\gamma)$ or A(b) obtained above as the result of the more exact calculations, we find the dependence of the frequency Ω_1 on the backlash zone b, as shown in Fig. 8.18 (curve 2). The upper branch of this curve corresponds to the stable periodic solution (i.e., the upper branch of the curve on Fig. 8.17b), and the lower one to the unstable solution

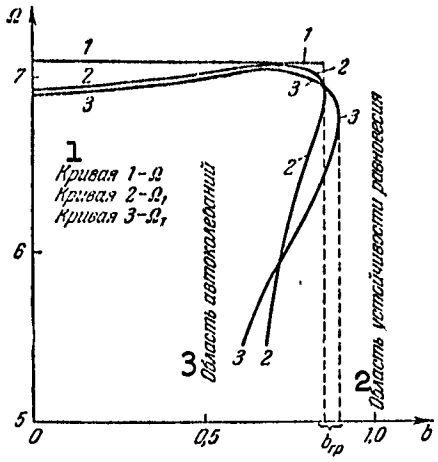


Fig. 8.18. 1) Curve; 2) stable equilibrium region; 3)

(i.e., to the lower branch on Fig. 8.17b).

On the other hand, in the first approximation solution we have a frequency $\Omega = \text{const} = 7.07\ \text{sec}^{-1}$ for both branches (line 1 on Fig. 8.18). We see that the more exact solution (curve 2) gives a result that is quite close to that of the first approximation in the case of the stable branch of the periodic solutions, but differs greatly for the unstable branch. However, the latter case does not interest us in practice.* In the first case, on the other hand, the fact of principal importance is that we are able to detect here the dependence of the frequency on the form of the nonlinearity (in this case, on the dimension of the backlash zone), and at the same time the sufficiently high accuracy of the first approximation is confirmed.

It is further of great interest to compare the more exact solution with the rigorously exact solution of this problem. Let us determine the periodic solutions in the given nonlinear system by the exact method of G.S. Pospelov [245]. According to this method, it is first necessary to expand the transfer function of the linear part of the system in the form

$$W_л(p) = \frac{k}{(T_1 p + 1)(T_2 p + 1) p} = \frac{c_1}{p + p_1} + \frac{c_2}{p + p_2} + \frac{k}{p}, \qquad (8.138)$$

where

$$c_1 = -\frac{kT_1}{T_2 - T_1}, \quad c_2 = \frac{kT_2}{T_1 - T_2}, \quad p_1 = \frac{1}{T_1}, \quad p_2 = \frac{1}{T_2}.$$

For a transfer function of this type, with a specified nonlinearity (Fig. 8.17a), we determine from Tables 1 and 2 of [245] the equa-

tions for the periods:

$$F_1(t)_{t=\theta_1}=\frac{k}{2}\theta_1+c_1T_1\frac{1-e^{-\frac{\theta_1}{T_1}}}{1+e^{-\frac{\theta_1+\theta_2}{T_1}}}+c_2T_2\frac{1-e^{-\frac{\theta_1}{T_2}}}{1+e^{-\frac{\theta_1+\theta_2}{T_2}}}=-b,$$

$$F_2(t)_{t=\theta_2}=\frac{k}{2}\theta_1+c_1T_1\frac{(1-e^{-\frac{\theta_1}{T_1}})e^{-\frac{\theta_2}{T_1}}}{1+e^{-\frac{\theta_1+\theta_2}{T_1}}}+c_2T_2\frac{(1-e^{-\frac{\theta_1}{T_2}})e^{-\frac{\theta_2}{T_2}}}{1+e^{-\frac{\theta_1+\theta_2}{T_2}}}=b,$$

where $\theta_1 + \theta_2$ is the half-period, and the accurate value of the frequency is consequently

$$\Omega_t=\frac{\pi}{\theta_1+\theta_2}. \qquad (8.139)$$

Adding this pair of equations and then rewriting the second equation, we arrive at the following two equations:

$$\theta_1+\frac{T_1^2}{T_2-T_1}\frac{(1-e^{-\frac{\theta_1}{T_1}})(1+e^{\frac{\theta_2}{T_1}})}{e^{-\frac{\theta_1}{T_1}}+e^{\frac{\theta_2}{T_1}}}+\frac{T_2^2}{T_1-T_2}\frac{(1-e^{-\frac{\theta_1}{T_2}})(1+e^{\frac{\theta_2}{T_2}})}{e^{-\frac{\theta_1}{T_2}}+e^{\frac{\theta_2}{T_2}}}=0, \qquad (8.140)$$

$$b=\frac{k}{2}\theta_1+c_1T_1\frac{1-e^{-\frac{\theta_1}{T_1}}}{e^{-\frac{\theta_1}{T_1}}+e^{\frac{\theta_2}{T_1}}}+c_2T_2\frac{1-e^{-\frac{\theta_1}{T_2}}}{e^{-\frac{\theta_1}{T_2}}+e^{\frac{\theta_2}{T_2}}}. \qquad (8.141)$$

The solution of these equations for specified system parameters (8.135) will be obtained in the following fashion. We assign some numerical value to θ_1 and find graphically, from Eq. (8.140), the value of θ_2. This enables us to calculate directly the accurate value of the frequency Ω_t (8.139) and determine then by means of Formula (8.141) the backlash zone b to which this corresponds. As a result we obtain the accurate dependence $\Omega_t(b)$.

For convenience in graphical solution of Eq. (8.140), we transform the latter to the form

$$\theta_2=\frac{T_1}{\lg e}[\lg(B_2e^{\frac{\theta_2}{T_2}}-B_1)-\lg(B_2-e^{\frac{\theta_2}{T_2}})], \qquad (8.142)$$

and plot the left and right halves as functions of θ_2. In Formula (8.142) we have

$$B_1 = -\frac{\alpha_1\alpha_2 - M\alpha_2 - N\alpha_1}{M+N-1}, \quad B_2 = \alpha_1 - \frac{M(\alpha_1-1)}{M+N-1},$$
$$B_3 = -\frac{N(\alpha_2-1)}{M+N-1} - \alpha_2, \quad M = -\frac{T_2^2}{T_2-T_1}\frac{\alpha_1-1}{\theta_1},$$
$$N = -\frac{T_1^2}{T_1-T_2}\frac{\alpha_2-1}{\theta_1}, \quad \alpha_1 = e^{-\frac{\theta_1}{T_1}}, \quad \alpha_2 = e^{-\frac{\theta_1}{T_2}}.$$

The accurate dependence $\Omega_t(b)$ obtained in this manner is represented by curve 3 of Fig. 8.18. We see that it is sufficiently close to curve 2, which corresponds to the more exact solution. This confirms, first, the good accuracy of both the more exact solution and of the first approximation for the stable branch of the periodic solutions and, second, the good accuracy of the more exact solution for a definite region of the unstable branch of the periodic solutions.

It is also very important that the accurate solution confirms the breakdown of the straight line representing the variation of the parameter b, obtained by the method of harmonic linearization, into two regions: the region where there are two periodic solutions ($b < b_{gr}$) and the region of stable equilibrium of the system ($b > b_{gr}$). With this, both the first approximation and the more exact solution give for the stability limit

$$b_{гр} = 0{,}85,$$

while the accurate solution yields

$$b_{гр} = 0{,}89.$$

We give also a table of the numerical values of $\Omega(b)$, used to plot the curves of Fig. 8.18:

1) first approximation $\Omega = 7.07$ sec^{-1};

2) more exact solution $\Omega_1(b)$:

b	0	0,25	0,50	0,65	0,82	0,85	0,81	0,75	0,68
Ω_1	6,92	6,95	7,02	7,06	7,04	6,92	6,60	6,20	5,40

3) accurate solution $\Omega_t(b)$:

b	0	0,25	0,48	0,63	0,82	0,88	0,88	0,83	0,60
Ω_r	6,90	6,94	6,98	7,06	6,98	6,89	6,61	6,27	5,43

Figures 8.19a, b, and c show the forms of the oscillations of x(t) for the following sizes of the backlash zone: b = 0, b = 0.633, and

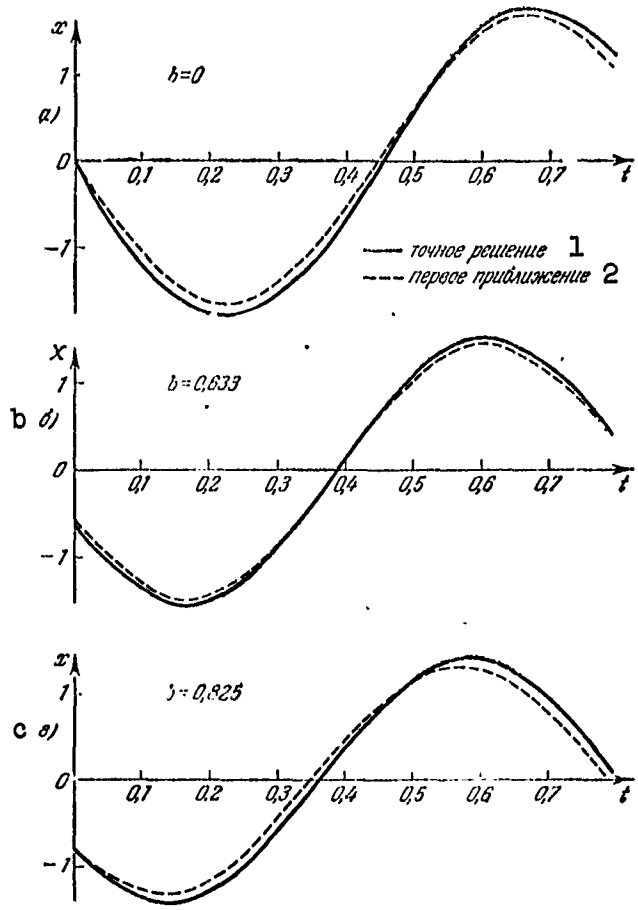


Fig. 8.19. 1) Accurate solution; 2) first approximation.

b = 0.825, with the continuous line representing everywhere the accurate solution x(t), and the dashed line representing the first approximation x = A sin Ωt. The figure shows the sufficiently good agreement between the accurate solution and even the first approximation which is not made more exact.

Finally, by means of Formulas (8.50) we can easily calculate the

relative amplitude and phase of the third harmonic of the self-oscillations

$$\delta_3 = \frac{4c\,(A^2 - 4b^2)\sqrt{A^2 - b^2}}{3\pi A^4} \cdot \frac{k}{3\Omega_1 \sqrt{(1 - 9T_1T_2\Omega_1^2)^2 + 9(T_1 + T_2)^2\Omega_1^2}}, \qquad (8.143)$$

$$\varphi_3 = \frac{\pi}{2} - \operatorname{arctg}\frac{3(T_1 + T_2)\Omega_1}{1 - 9T_1T_2\Omega_1^2},$$

and then also the more exact value of the amplitude A_1 of the first harmonic, similar to what was already done in §8.4. As a result we can plot the more exact solution

$$x = A_1 \sin \Omega_1 t + \delta_3 A \sin(3\Omega_1 t + \varphi_3).$$

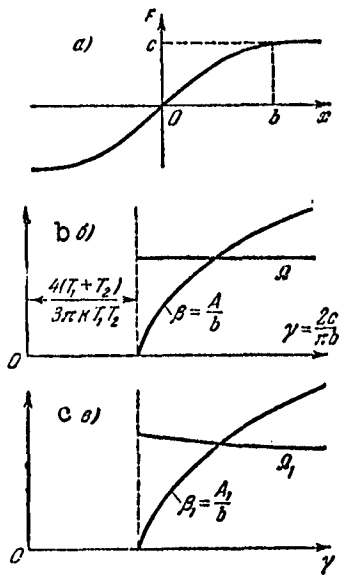


Fig. 8.20

Example 2. Let the dynamics of the system be likewise described by a third-order nonlinear equation

$$(T_1p + 1)(T_2p + 1)px + kF(x) = 0,$$

but the nonlinearity F(x) is specified in the form of the curve (Fig. 8.20a):

$$F = k_1(1 - k_2x^2)x \quad \text{for} \quad x \leq b,$$

$$F = c \quad \text{for} \quad x \geq b.$$

From the conditions

$$F = c \quad \text{and} \quad dF/dx = 0 \quad \text{for} \quad x = b$$

we get

$$k_1 = \frac{3c}{2b}, \quad k_2 = \frac{1}{3b^2}.$$

The formula for harmonic linearization yields in this case, using the notation of (8.55),

$$q = \gamma f_1(\beta),$$

where for $A \leq b$ we have

$$f_1(\beta) = \frac{3\pi}{4}\left(1 - \frac{1}{4}\beta^2\right) \qquad (\beta \leq 1), \qquad (8.144)$$

and for $A \geq b$

$$f_1(\beta) = \frac{3}{2}\left(1 - \frac{1}{4}\beta^2\right)\arcsin\frac{1}{\beta} + \frac{3}{8}\left(1 - \frac{2}{\beta^2}\right)\sqrt{\beta^2 - 1}. \qquad (\beta \geq 1) \qquad (8.145)$$

In analogy with the preceding example, we obtain here the first approximation in the form

$$\Omega = \frac{1}{\sqrt{T_1 T_2}}, \qquad \gamma = \frac{T_1 + T_2}{kT_1 T_2 f_1(\beta)}.$$

Consequently, the self-oscillation frequency remains the same in the first approximation, independently of the form of the nonlinearity (since it is independent of the value of γ). The second of these formulas gives the dependence of the relative amplitude $\beta = A/b$ on the nonlinearity form coefficient $\gamma = 2c/\pi b$. This dependence can be easily plotted (Fig. 8.20b) by assigning different values of β and calculating γ every time.

As can be seen from Fig. 8.20b, when the ratio of the system parameters corresponds to the inequality

$$\gamma = \frac{2c}{\pi b} < \frac{4(T_1 + T_2)}{3\pi k T_1 T_2},$$

no self-oscillations are possible in the system. In this case the given nonlinear system will be stable (as is also the corresponding linear system obtained when $k_2 = 0$).

To find the more exact value of the frequency Ω_1 as a function of the form of the nonlinearity, we obtain here the previous expression (8.136) for the function $\Phi_1(\Omega_1)$. It is now merely necessary to find the value of r_3 for the given nonlinearity (Fig. 8.20a) by means of Formula (8.29). As the result we get

$$f_2(\beta) = \frac{1}{8}\beta^2 \arcsin\frac{1}{\beta} - \left(\frac{1}{8} - \frac{7}{12\beta^2} + \frac{1}{3\beta^4}\right)\sqrt{\beta^2 - 1} \qquad (\beta \geqslant 1). \qquad (8.146)$$

where for $A \leq b$ we have

$$r_3 = \gamma f_2(\beta),$$

and for $A \geq b$

$$f_2(\beta) = \frac{\pi}{16}\beta^2 \qquad (\beta \leqslant 1).$$

We thus are able to employ in this problem, too, a graphical con-

struction (Fig. 8.4a) which gives us the sought dependence of the self-oscillation frequency Ω_1 on the nonlinearity form coefficient γ (Fig. 8.20c).

In addition, we can calculate also the relative amplitude of the third harmonic by means of the formula

$$\delta_3 = \frac{k\gamma f_3(\beta)}{3\Omega_1 \sqrt{(1-9T_1T_2\Omega_1^2)^2 + 9(T_1+T_2)^2\Omega_1^2}}, \qquad (8.147)$$

and then, if necessary, obtain also the more exact value of the relative amplitude of the first harmonic $\beta_1 = A_1/b$ as a function of the nonlinearity form coefficient γ (Fig. 8.20c). The second formula in (8.143) for the phase of the third harmonic retains the same form for this example, too.

§8.6. Determination of the Higher Harmonics of Asymmetrical Self-Oscillations in the Presence of an External Signal

In the calculation of the higher harmonics of symmetrical self-oscillations we used in §8.1 expressions for the total solution in the forms of Taylor and Fourier series, as set up in §2.2. Since we did not have in the preceding chapters analogous material for asymmetrical oscillations, we must start here with a formulation of these expressions [255].

We start out with the equation for the nonlinear system

$$Q(p)x + R(p)F(x) = S(p)f(t), \qquad (8.148)$$

where f(t) is an external signal that varies slowly in time. To simplify the general formulas we choose here a nonlinear function F(x) in place of the previous F(x, px). It is assumed, however, that the function F(x) may be not only single valued but also of the loop type, and also asymmetrical.

In the case of a constant external signal (in a static system) or in the case of a signal with constant rate of change (in an astatic system), Eq. (8.148) has the form

$$Q(p)x + R(p)F(x) = M^0, \tag{8.149}$$

where M^0 is a specified constant, defined as in Chapter 5 by means of Formulas (5.21) and (5.22).

We seek the periodic solution in the form

$$x = x^0 + x_1 + \sum_{k=2}^{n} x_k, \tag{8.150}$$

$$x_1 = A \sin \Omega t, \quad x_k = \delta_k A \sin (k\Omega t + \varphi_k), \tag{8.151}$$

where x^0, x_1, and x_k are the corresponding harmonics of the periodic solution x(t), expanded in a Fourier series; n is an arbitrary positive integer, with the harmonics having $k > n$ assumed to be insignificant. We denote by δ_k the relative amplitudes of the higher harmonics x_k (relative to the first).

We expand the nonlinear function F(x) in a Taylor series, using (8.150):

$$F(x) = F(x^0 + x_1) + \frac{dF}{dx}(x^0 + x_1) \sum_{k=2}^{n} x_k + \dots, \tag{8.152}$$

and then in a Fourier series

$$F(x) = F^0 + F_1 + \sum_{k=2}^{n} F_k + \sum_{k=0}^{n} F_k, \tag{8.153}$$

where

$$\left.\begin{aligned} F^0 &= \frac{1}{2\pi} \int_0^{2\pi} F(x^0 + A \sin \psi)\, d\psi, \\ F_1 &= q(x^0, A) A \sin \Omega t + q'(x^0, A) A \cos \Omega t, \\ q &= \frac{1}{\pi A} \int_0^{2\pi} F(x^0 + A \sin \psi) \sin \psi\, d\psi, \\ q' &= \frac{1}{\pi A} \int_0^{2\pi} F(x^0 + A \sin \psi) \cos \psi\, d\psi, \end{aligned}\right\} \tag{8.154}$$

$$
\left.
\begin{aligned}
F_k &= r_k(x^0, A) A \sin k\Omega t + s_k(x^0, A) A \cos k\Omega t \quad (k=2, 3, \ldots, n).\\
r_k &= \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A \sin \Omega t) \sin k\psi \, d\psi,\\
s_k &= \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A \sin \psi) \cos k\psi \, d\psi,
\end{aligned}
\right\} \qquad (8.155)
$$

$$
\left.
\begin{aligned}
F^0 &= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{dF}{dx}(x^0 + A \sin \psi)\sum_{k=2}^{n} x_k + \ldots\right] d\psi,\\
F_k &= Q_k A \sin k\Omega t + H_k A \cos k\Omega t,\\
Q_k &= \frac{1}{\pi A}\int_0^{2\pi}\left[\frac{dF}{dx}(x^0 + A \sin \psi)\sum_{k=2}^{n} x_k + \ldots\right] \sin k\psi \, d\psi,\\
H_k &= \frac{1}{\pi A}\int_0^{2\pi}\left[\frac{dF}{dx}(x^0 + A \sin \psi)\sum_{k=2}^{n} x_k + \ldots\right] \cos k\psi \, d\psi
\end{aligned}
\right\} (k=1, 2, \ldots, n). \qquad (8.156)
$$

Each harmonic of the nonlinear function F(x) is broken up into two components F_k and E_k where, in accordance with (8.153), F_k is defined in terms of the zeroth x^0 and first x_1 harmonics of the sought solution, amounting usually to the principal part of the solution, and E_k is defined in terms of the higher x_k harmonics of the sought solution. Consequently, the higher harmonic components F_k of the nonlinear function F(x) cannot be regarded small (if we deal with an arbitrary nonlinearity) independently of the magnitude of the higher harmonics x_k of the sought solution. As regards the other higher harmonic components E_k of the nonlinear function F(x), they will be small if the higher harmonics x_k are small, provided the derivative dF/dx is finite or represents a delta function (for example, in the case of relay characteristics), and the additional terms contained in E_k and represented by the dots, will in this case be small quantities of higher order, if the higher derivatives of F with respect to x are finite (for arbitrary nonlinearities) or are delta functions (for piecewise-linear characteristics). We shall assume that the form F(x) satisfies the foregoing conditions; this does not impose in practice large limitations on this function, but is of importance to what follows.

Substituting (8.150) and (8.153) into the specified nonlinear equation (8.149), we obtain a series of equations which are nonlinearly interrelated:

$$Q(0)\,x^0 + R(0)\,F^0(x^0, A) + R(0)\,E^0 = M^0, \tag{8.157}$$

$$Q(p)\,x_1 + R(p)\left[q(x^0, A) + \frac{q_1(x^0, A)}{\Omega}\,p\right]x_1 + R(p)\,E_1 = 0, \tag{8.158}$$

$$Q(p)\,x_k + R(p)\,F_k + R(p)\,E_k = 0 \quad (k = 2, 3, \ldots, n). \tag{8.159}$$

We see that the equations (8.159) interrelate the quantities F_k and x_k. It follows therefore that if the higher harmonics F_k are not small, we can have small higher harmonics x_k in the solution only if

$$\frac{R(jk\Omega)}{Q(jk\Omega)} < \left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \quad (k = 2, 3, \ldots, n). \tag{8.160}$$

It is assumed here that $|R(jk\Omega)/Q(jk\Omega)| \to 0$ as $k \to \infty$, for which it is sufficient to have the degree of R(p) lower than that of Q(p). In addition, we shall assume, as before, that Q(p) does not have pure imaginary roots or roots with positive real parts.

Consequently, the filter condition (8.160) guarantees, if the polynomials Q(p) and R(p) have a suitable structure, the smallness of the higher harmonics of the sought solution for an arbitrary form of the nonlinearity (i.e., for not small F_k), satisfying only the requirement stated above with respect to its derivatives.

In such a case, as is already known, all the quantities E_k ($k = 0, 1, \ldots, n$) will have the same order of smallness as the higher harmonics of the solution $\sum_{k=2}^{n} x_k$. In the last quantity, in accordance with (8.151), all the δ_k will play the role of a small parameter. Then, in accordance with (8.157) and (8.158), we can determine the zeroth and the first harmonics of the solution approximately from the equations

$$\left.\begin{aligned} Q(0)\,x^0 + R(0)\,F^0(x^0, A) &= M^0, \\ Q(p)\,x_1 + R(p)\left[q(x^0, A) + \frac{q'(x^0, A)}{\Omega}\,p\right]x_1 &= 0. \end{aligned}\right\} \tag{8.161}$$

This is the result of the harmonic linearization considered in Chapter 5.

After substituting $p = j\Omega$ in the second equation of (8.161) and separating the real and imaginary parts we obtain in place of (8.161) three algebraic equations

$$\left.\begin{aligned} Q(0)\,x^0 + R(0)\,F^0(x^0, A) &= M^0, \\ X_Q(\Omega) + X_R(\Omega)\,q(x^0, A) - Y_R(\Omega)\,q'(x^0, A) &= 0, \\ Y_Q(\Omega) + Y_R(\Omega)\,q(x^0, A) + X_R(\Omega)\,q'(x^0, A) &= 0, \end{aligned}\right\} \qquad (8.162)$$

from which we determine the three unknowns x^0, A, and Ω. Here X_Q, Y_Q, X_R, Y_R denote the real and imaginary parts of the polynomials $Q(j\Omega)$ and $R(j\Omega)$.

This approximation can be satisfactory if the filter property (8.160) is well satisfied for the given system, or, in the opposite case if the F_k are themselves small. The worse the filter property (8.160) is satisfied with not small F_k, the more important it becomes to take into account the higher harmonics x_k of the sought solution.

Using the result of the solution of Eqs. (8.162), i.e., the quantities x^0, A, and Ω, as the first approximation we obtain the higher harmonics (δ_k and φ_k) and then also the more accurate values of the zeroth and first harmonics (x_1^0, A_1, and Ω_1). Following this we can if necessary obtain also more exact higher harmonics (δ'_k and φ'_k).

The problem is solved then in the following manner. The first approximation for each of the higher harmonics is determined in accordance with Eqs. (8.159) and (8.151) in the form

$$Q(p)\,x_k + \frac{1}{\delta_k} R(p)\left[r_k(x^0, A)\left(\cos\varphi_k - \frac{\sin\varphi_k}{k\Omega}\,p\right) + s_k(x^0, A)\left(\sin\varphi_k + \frac{\cos\varphi_k}{k\Omega}\,p\right)\right] x_k = 0,$$

from which we obtain as a result of the substitution $p = jk\Omega$

$$\left.\begin{aligned} \delta_k &= \sqrt{\frac{X_{kR}^2 + Y_{kR}^2}{X_{kQ}^2 + Y_{kQ}^2}}\,[r_k^2(x^0, A) + s_k^2(x^0, A)], \\ \varphi_k &= \operatorname{arctg}\frac{Y_{kR}}{X_{kR}} - \operatorname{arctg}\frac{Y_{kQ}}{X_{kQ}} + \operatorname{arctg}\frac{s_k(x^0, A)}{r_k(x^0, A)}, \end{aligned}\right\} \qquad (8.163)$$

where X_{kR}, Y_{kR}, X_{kQ}, Y_{kQ} denote the real and imaginary parts of the polynomials $R(jk\Omega)$ and $Q(jk\Omega)$.

The more exact values of the first and second harmonics are then determined in accordance with (8.157) and (8.158) by the equations

$$\left.\begin{aligned} &Q(0)\,x_1^0 + R(0)\,F^0(x_1^0,\ A_1) = M^0 - R(0)\,E^0(x^0,\ A,\ \delta_k,\ \varphi_k),\\ &Q(p)\,x_1 + R(p)\left[q(x_1^0,\ A_1) + \frac{q'(x_1^0,\ A_1)}{\Omega_1}\,p\right]x_1 + \\ &\qquad + R(p)\,E_1(x^0,\ A,\ \delta_k,\ \varphi_k)\,x_1 = 0, \end{aligned}\right\} \qquad (8.164)$$

where

$$E^0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{dF}{dx}(x^0 + A\sin\psi)\sum_{k=2}^{n} x_k\,d\psi, \qquad E_1 = \left(G_1 + \frac{H_1}{\Omega}\,p\right)x_1,$$

$$G_1 = \frac{1}{\pi A}\int_0^{2\pi}\frac{dF}{dx}(x^0 + A\sin\psi)\sum_{k=2}^{n} x_k\sin\psi\,d\psi,$$

$$H_1 = \frac{1}{\pi A}\int_0^{2\pi}\frac{dF}{dx}(x^0 + A\sin\psi)\sum_{k=2}^{n} x_k\cos\psi\,d\psi.$$

If we rewrite (8.164) in the form

$$Q(0)\,x_1^0 + R(0)\,F^0(x_1^0,\ A_1) = M_1^0, \qquad (8.165)$$

$$Q_1(p)\,x_1 + R(p)\left[q(x_1^0,\ A_1) + \frac{q'(x_1^0,\ A_1)}{\Omega_1}\,p\right]x_1 = 0, \qquad (8.166)$$

where

$$M_1^0 = M^0 - E^0(x^0,\ A,\ \delta_k,\ \varphi_k),$$

$$Q_1(p) = Q(p) + R(p)\left[G_1(x^0,\ A,\ \delta_k,\ \varphi_k) + \frac{1}{\Omega}\,H_1(x^0,\ A,\ \delta_k,\ \varphi_k)\,p\right],$$

then it becomes clear that the more exact solution based on Eqs. (8.165) and (8.166) can be determined in exactly the same manner as the approximate solution (8.161), except that the constant M^0 and the coefficients of the polynomial $Q(p)$ must be corrected in suitable fashion by taking into account the higher harmonics of the sought solution.

More exact values for the higher harmonics (δ'_k, φ'_k) can be obtained by means of the same formulas (8.163), except that new values of Ω_1, A_1, and x_1^0 must be substituted. However, in this case one can also make the solution even more exact by adding the term $R(p)E_k$ in

Eq. (8.159). The second correction for the zeroth and first harmonics (the need for which arises very rarely) can then be obtained by means of equations of the same type, (8.165) and (8.166), in which x_1^0, A_1, and Ω_1 are substituted in place of x^0, A, and Ω. If necessary we can include in the expressions for E^0, G_1, and H_1 the second terms of the Taylor series, depending on the order of magnitude of $\sum_{k=2}^{n} x_k$, i.e., on the order of magnitude of δ_k compared with unity.

Thus, for a system described by means of Eq. (8.149) with a constant right half, we determine the higher harmonics of the asymmetrical oscillations and obtain more exact zeroth and first harmonics, along with determining their dependence on the magnitude of the right half, i.e., on the magnitude of the external signal. We solve in analogous fashion also Eq. (8.148) in the presence of a slowly varying external signal f(t). In this case we have in place of the algebraic equation (8.165) the differential equation

$$Q(p)\,x_1^0 + R(p)\,F^0(x_1^0,\ A_1) = S(p)\,f(t) - E^0,$$

where x_1^0, A_1, and E^0 are slowly varying functions that depend on f(t). The method of solving this equation together with Eq. (8.166) remains perfectly the same as in Chapter 5, namely it is necessary to find the bias function $\Phi(x_1^0)$ and substitute it in place of $F^0(x_1^0, A_1)$. We then determine from this equation $x_1^0(t)$. Ordinary linearization of the bias function $\Phi(x_1^0) = k_n x_1^0$ is also possible.

Let us consider an example in which the higher harmonics of asymmetrical self-oscillations of a relay system are determined with $F(x) = c \operatorname{sign} x$, $R(p) = p + b_1$, $Q(p) = p^3 + a_1 p^2 + a_2 p$.

We take into account the second and third harmonics of the self-oscillations, seeking the solution of (8.149) in the form

$$\left.\begin{aligned} x &= x^0 + x_1 + x_2 + x_3, \\ x_1 &= A \sin \Omega t, \quad x_2 = \delta_2 A \sin (2\Omega t + \varphi_2), \\ x_3 &= \delta_3 A \sin (3\Omega t + \varphi_3). \end{aligned}\right\} \qquad (8.167)$$

Let us calculate the necessary quantities:

$$F^0 = \frac{1}{2\pi}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\,d\psi = \frac{2c}{\pi}\arcsin\frac{x^0}{A}, \tag{8.168}$$

$$\left.\begin{aligned} f &= \frac{1}{\pi A}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\sin\psi\,d\psi = \frac{4c}{\pi A}\sqrt{1-\left(\frac{x^0}{A}\right)^2}, \\ g &= \frac{1}{\pi A}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\cos\psi\,d\psi = 0, \end{aligned}\right\} \tag{8.169}$$

$$\left.\begin{aligned} f_2 &= \frac{1}{\pi A}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\sin 2\psi\,d\psi = 0, \\ g_2 &= \frac{1}{\pi A}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\cos 2\psi\,d\psi = \frac{4c}{\pi A}\frac{x^0}{A}\sqrt{1-\left(\frac{x^0}{A}\right)^2}, \end{aligned}\right\} \tag{8.170}$$

$$\left.\begin{aligned} f_3 &= \frac{1}{\pi A}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\sin 3\psi\,d\psi = \\ &= \frac{4c}{3\pi A}\left[1 - 4\left(\frac{x^0}{A}\right)^2\right]\sqrt{1-\left(\frac{x^0}{A}\right)^2}, \\ g_3 &= \frac{1}{\pi A}\int_0^{2\pi} c\,\mathrm{sign}\,(x^0 + A\sin\psi)\cos 3\psi\,d\psi = 0, \end{aligned}\right\} \tag{8.171}$$

$$\begin{aligned} E^0 &= \frac{1}{2\pi}\int_0^{2\pi} 2c\Delta\,(x^0 + A\sin\psi)\,[\delta_2 A\sin(2\psi+\varphi_2) + \\ &\quad + \delta_3 A\sin(3\psi+\varphi_3)]\,d\psi = \\ &= -\frac{2c}{\pi}\left\{2\delta_2\frac{x^0}{A}\cos\varphi_2 + \delta_3\left[1-4\left(\frac{x^0}{A}\right)^2\right]\sin\varphi_3\right\}, \end{aligned} \tag{8.172}$$

$$\begin{aligned} G_1 &= \frac{1}{\pi A}\int_0^{2\pi} 2c\Delta\,(x^0 + A\sin\psi)\,[\delta_2 A\sin(2\psi+\varphi_2) + \\ &\quad + \delta_3 A\sin(3\psi+\varphi_3)]\sin\psi\,d\psi = \\ &= \frac{4c}{\pi A}\frac{x^0}{A}\left\{2\delta_2\frac{x^0}{A}\cos\varphi_2 + \delta_3\left[1-4\left(\frac{x^0}{A}\right)^2\right]\sin\varphi_3\right\}, \end{aligned} \tag{8.173}$$

$$\begin{aligned} H_1 &= \frac{1}{\pi A}\int_0^{2\pi} 2c\Delta\,(x^0 + A\sin\psi)\,[\delta_2 A\sin(2\psi+\varphi_2) + \\ &\quad + \delta_3 A\sin(3\psi+\varphi_3)]\cos\psi\,d\psi = \\ &= \frac{4c}{\pi A}\left\{\delta_2\left[1-2\left(\frac{x^0}{A}\right)^2\right]\sin\varphi_2 - \delta_3\frac{x^0}{A}\left[3-4\left(\frac{x^0}{A}\right)^2\right]\cos\varphi_3\right\}, \end{aligned} \tag{8.174}$$

where $\Delta(x^0 + A\sin\psi)$ stands for the delta function. The evaluation of integrals containing this function was illustrated previously (see, for example, §8.3).

The first approximation equations (the result of the harmonic lin-

earization) will, in accord with (8.162), be in this case

$$\frac{2cb_1}{\pi}\arcsin\frac{x^0}{A}=M^0,$$

$$-a_1\Omega^2+b_1\frac{4c}{\pi A}\sqrt{1-\left(\frac{x^0}{A}\right)^2}=0,$$

$$-\Omega^3+a_2\Omega+\frac{4c\Omega}{\pi A}\sqrt{1-\left(\frac{x^0}{A}\right)^2}=0.$$

Hence

$$\left.\begin{aligned}\frac{x^0}{A}&=\sin\frac{\pi M^0}{2cb_1}, & \Omega&=\sqrt{\frac{a_2b_1}{b_1-a_1}},\\ A&=\frac{4c}{\pi a_2}\left(\frac{b_1}{a_1}-1\right)\cos\frac{\pi M^0}{2cb_1}, & x^0&=\frac{2c}{\pi a_2}\left(\frac{b_1}{a_1}-1\right)\sin\frac{\pi M^0}{cb_1}.\end{aligned}\right\} \qquad (8.175)$$

This approximate solution determines the dependence of the constant component x^0 and of the self-oscillation amplitude A on the system parameters and on the magnitude of the external constant signal M^0 (dashed curves in Figs. 8.21 and 8.22). On the other hand, the self-

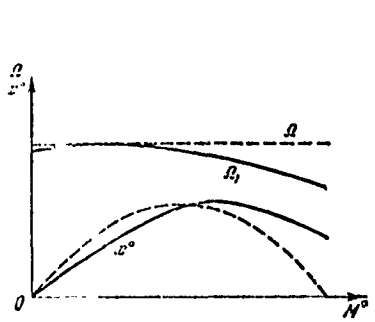


Fig. 8.21

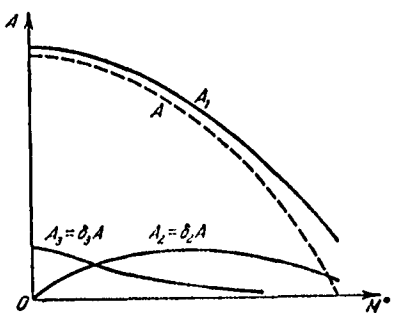


Fig. 8.22

oscillation frequency Ω turns out to be independent of M^0. However, an analog solution of this problem shows that the self-oscillation frequency Ω also depends on the magnitude of the constant external signal M^0, and that with increasing M^0 the form of the self-oscillations is not merely subject to a displacement x^0, as in the approximate solution, but is also distorted and becomes asymmetrical within the half cycle. This is evidence of the occurrence of a second harmonic.

In order to obtain a correct qualitative picture of the phenomenon and a correct quantitative result, let us solve the problem with inclusion of the second and third harmonics. By means of Formulas (8.163),

(8.170), and (8.171) we obtain, with allowance for (8.175), the relative amplitudes and phases of the second and third harmonics:

$$\delta_2 = \frac{1}{2\Omega}\sqrt{\frac{b_1^2+4\Omega^2}{4a_1^2\Omega^2+(a_2-4\Omega^2)^2}}\frac{a_1a_2}{b_1-a_1}\sin\frac{\pi M^0}{2cb_1}, \tag{8.176}$$

$$\varphi_2 = \operatorname{arctg}\frac{2\Omega}{b_1} - \operatorname{arctg}\frac{a_2-4\Omega^2}{2a_1\Omega} + \frac{\pi}{2}, \tag{8.177}$$

$$\delta_3 = \frac{1}{9\Omega}\sqrt{\frac{b_1^2+9\Omega^2}{9a_1^2\Omega^2+(a_2-9\Omega^2)^2}}\frac{a_1a_2}{b_1-a_1}\left(1-4\sin^2\frac{\pi M^0}{2cb_1}\right), \tag{8.178}$$

$$\varphi_3 = \operatorname{arctg}\frac{3\Omega}{b_1} - \operatorname{arctg}\frac{a_2-9\Omega^2}{3a_1\Omega}. \tag{8.179}$$

We see from (8.176) that the second harmonic of the self-oscillations does not arise ($\delta_2 = 0$) if there is no external signal ($M^0 = 0$). In this case the oscillations are symmetrical and only the third harmonic arises ($\delta_3 \neq 0$ when $M^0 = 0$). On the other hand, in the presence of a constant external signal M^0, a second harmonic arises, the amplitude of which first increases with increasing M^0 (Fig. 8.22), thereby distorting the form of the self-oscillation curve, as in the case of the analog solution. The amplitude of the third harmonic δ_3 first decreases with increasing M^0.

In order to find the dependence of the self-oscillation frequency on the magnitude of the external constant signal, let us introduce the corrections for the zeroth and first harmonics. We set up Eqs. (8.165) and (8.166) for the given problem with allowance for Eqs. (8.168) to (8.179). From (8.165) we obtain

$$\frac{x^0}{\lambda} = \sin\left[\frac{\pi M^0}{2cb_1} - f_2(\Omega)\sin^2\frac{\pi M^0}{2cb_1}\cos\varphi_2(\Omega) - f_3(\Omega)\left(1-4\sin^2\frac{\pi M^0}{2cb_1}\right)^3\sin\varphi_3(\Omega)\right]. \tag{8.180}$$

where Ω, $\varphi_2(\Omega)$, and $\varphi_3(\Omega)$ are determined by Formulas (8.175), (8.177), and (8.179) in terms of the system parameters, and in addition

$$f_2(\Omega) = \frac{a_1a_2}{b_1\Omega(b_1-a_1)}\sqrt{\frac{b_1^2+4\Omega^2}{4a_1^2\Omega^2+(a_2-4\Omega^2)^2}},$$

$$f_3(\Omega) = \frac{a_1a_2}{9b_1\Omega(b_1-a_1)}\sqrt{\frac{b_1^2+9\Omega^2}{9a_1^2\Omega^2+(a_2-9\Omega^2)^2}}.$$

The more exact dependence $x^0(M^0)$ is represented by the continuous line in Fig. 8.21.

We write down Eq. (8.166) directly in algebraic form, namely

$$-a_1\Omega_1^2+b_1G_1'(x^0, A, \delta_k, \varphi_k)-$$
$$-\Omega_1^2\frac{1}{\Omega}H_1'(x^0, A, \delta_k, \varphi_k)+b_1q(x_1^0, A_1)=0,$$
$$-\Omega_1^2+a_2\Omega_1+\Omega_1\left[G_1'(x^0, A, \delta_k, \varphi_k)+\right.$$
$$\left.+\frac{b_1}{\Omega}H_1'(x^0, A, \delta_k, \varphi_k)\right]+\Omega_1q(x_1^0, A_1)=0.$$

From this we obtain, with account of (8.173)-(8.179)

$$\Omega_1=\sqrt{\frac{a_2b_1+b_1^2f_4(M^0)}{b_1-a_1-f_4(M^0)}}, \tag{8.181}$$

$$q(x^0, A_1)=\frac{[a_2+f_5(M^0)+b_1f_4(M^0)][a_1+f_4(M^0)]-b_1f_5(M^0)}{b_1-a_1-f_4(M^0)}, \tag{8.182}$$

where

$$f_4(M^0)=\frac{a_1a_2}{2(b_1-a_1)}\left[f_2(\Omega)\operatorname{tg}\frac{\pi M^0}{2cb_1}\cos\frac{\pi M^0}{cb_1}\sin\varphi_2(\Omega)-\right.$$
$$\left.-f_3(\Omega)\sec^2\frac{\pi M^0}{2cb_1}\sin\frac{3\pi M^0}{cb_1}\cos\varphi_3(\Omega)\right],$$
$$f_5(M^0)=-\frac{a_1a_2}{b_1-a_1}\operatorname{tg}\frac{\pi M^0}{2cb_1}\left[f_2(\Omega)\sin^2\frac{\pi M^0}{2cb_1}\cos\varphi_2(\Omega)-\right.$$
$$\left.-f_3(\Omega)\left(1-4\sin^2\frac{\pi M^0}{2cb_1}\right)^2\sin\varphi_3(\Omega)\right].$$

Thus, the formula of the more exact solution (8.181) gives the sought dependence of the self-oscillation frequency Ω not only on the system parameters, but also on the magnitude of the external constant signal M^0 (Fig. 8.21), unlike the formula (8.175) for the first approximation of Ω, something which is a very important practical result.

From Eq. (8.182) we determine with account of (8.169) and (8.180) the more exact value of the amplitude of the first harmonic of the self-oscillations A_1 (Fig. 8.22). The form of the self-oscillations is determined by the expression

$$x=x^0+A_1[\sin\Omega_1t+\delta_2\sin(2\Omega_1t+\varphi_2)+\delta_3\sin(3\Omega_1t+\varphi_3)],$$

where all the quantities have been defined above.

For the sake of simplicity and clarity we have chosen in this example an ideal relay characteristic. It is obvious, however, that the

method developed for solving the problem is applicable also to an arbitrary nonlinearity F(x), satisfying only several conditions with respect to its derivatives. The limitations imposed here on the polynomials Q(p) and R(p) are fulfilled in many real systems, particularly in systems for the automatic control of the motion of mechanical objects.

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[Footnotes]

725 We consider here the case of symmetrical oscillations. In §8.6 below we shall discuss also the determination of the higher harmonics in the case of asymmetrical oscillations and in the presence of a constant or slowly varying external signal.

740 The even harmonics are disregarded because the nonlinear characteristic F(x) is odd. An account of the higher odd harmonics does not add anything that is principally new.

747 The examples are given in the following sections of the present chapter.

749 At the end of this section we shall give also formulas for loop-type nonlinearities.

766 The nonlinearity in this example characterizes the degree of deviation of the real curvilinear characteristic of the two-phased induction motor from the straight-line characteristic.

770 The discrepancy is explained here by the fast change in the function q(A) in the given amplitude interval, that is, violation of the condition stipulated in Chapter 2 that the function q(A) be smooth.

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[List of Transliterated Symbols]

725 в = v = vysshiye = higher

760 о.с = o.s = obratnaya svyaz' = feedback

761 л = l = lineynyy = linear

761 н = n = nelineynyy = nonlinear

764 гр = gr = granichnyy = boundary

772 т = t = tochnyy = accurate, exact

Chapter 9

FORCED OSCILLATIONS OF NONLINEAR SYSTEMS

§9.1. Symmetrical Single Frequency Forced Oscillations

The problem of determining forced oscillations of nonlinear systems is in general very complicated and varied. Inasmuch as the principle of superposition of the solutions does not hold here, one cannot in general add particular solutions resulting from different external signals and determined separately, nor can we add the free and forced oscillations. Special nonlinear addition of solutions is possible if the solutions differ in the degree of slowness of their course in time (i.e., in the values of the possible oscillation frequencies), in analogy with what was already done in Chapter 5. Besides, each of the added solutions depended essentially on the other solutions, namely, the amplitude of the self-oscillations depended essentially on the magnitude of the displacement characterizing the slow processes. A similar separation of the solutions for forced oscillations will be considered in §§9.2 and 9.3 below, where the possibility will be demonstrated of considering also nonlinear two-frequency oscillations with a large frequency difference.

Without touching upon complicated forms of forced oscillations of nonlinear systems (although their investigation also has great practical significance), we confine ourselves in the present section to a determination of single frequency forced oscillations, in which the system oscillations occur at the same frequency as the external periodic signal [270]. The form of the oscillations will be considered as

before nearly sinusoidal for the variable $\underline{x}$, which is the argument of the nonlinear function. At the same time, we retain as before all the limitations that delineate the classes of systems under consideration, as indicated in Chapter 2. We add to these, in many cases, additional limitations imposed on the amplitude and frequency of the external periodic signal (which depend also on the system parameters), which bring about the existence of single-particle forced oscillations in the nonlinear system. We shall call them for short the locking conditions (taking this term to have not a special but a general meaning in the broad sense indicated). These conditions become particularly important for self-oscillating systems at frequencies close to the self-oscillation frequency and above.

In the present section the external periodic signal is assumed to be sinusoidal. In §9.4 below we shall consider also a nonsinusoidal periodic external signal.

Thus, assume that we have a certain nonlinear automatic system to which we apply at an arbitrary point a sinusoidal signal

$$f(t) = B \sin \Omega_B t. \tag{9.1}$$

We consider first a system of the first class (see §§1.2 and 2.3) of the principal type, for which the dynamic equations reduce to the single form (2.71):

$$Q(p)\, x + R(p) F(x, px) = S(p) f(t). \tag{9.2}$$

If the conditions indicated at the start of §2.3 for Q(p), R(p), and F(x, px) are satisfied, as are also the locking conditions to be derived below (where necessary), we can seek in first approximation a solution for the steady state forced oscillations of the system in sinusoidal form

$$x = A_B \sin (\Omega_B t + \varphi), \tag{9.3}$$

where the sought unknown constants will be the amplitude A_V and the

phase shift φ, whereas the frequency Ω_v is already specified here by means of the expression (9.1). Unlike this typical formulation of the problem we can, of course, solve later on the inverse problem of determining the required frequency Ω_v or amplitude B of the external signal for a given amplitude A_v of the forced oscillations, etc. But for the time being we shall, in the derivation of the general formulas for the determination of the forced oscillations, assume here, unlike the problem of finding the self-oscillations (§2.3), that A_v and φ are unknown and the frequency Ω_v is given.

In order to be able to employ the same general approach to the solution of the problem as was used in finding the self oscillations, let us express in (9.2) the variable $\underline{f}$ in terms of $\underline{x}$. According to (9.1)

$$f(t) = B \sin[(\Omega_v t + \varphi) - \varphi] = \\ = B \cos\varphi \sin(\Omega_v t + \varphi) - B \sin\varphi \cos(\Omega_v t + \varphi).$$

Hence, taking into consideration the expression (9.3) for $\underline{x}$ and for its derivative

$$px = A_v \Omega_v \cos(\Omega_v t + \varphi),$$

we obtain ultimately

$$f(t) = \frac{B}{A_v}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_v} p\right) x. \qquad (9.4)$$

Substituting this expression into the specified differential equation of the system (9.2), we obtain

$$\left[Q(p) - S(p)\frac{B}{A_v}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_v} p\right)\right] x + R(p) F(x, px) = 0. \qquad (9.5)$$

Thus, the inhomogeneous nonlinear equation (9.2) with a specified external signal (9.1) and an assumed form (9.3) for the solution has been reduced to a homogeneous nonlinear equation (9.5), which contains an additional term in the left half. Equation (9.5) is analogous to the previous equation (2.71) from which it differs only in that the

operator polynomial Q(p) has been replaced by a new operator polynomial contained in the square brackets in (9.5). Using formally here for the determination of the sinusoidal periodic solution the same method as in Chapter 2, we must satisfy all the conditions indicated in §2.3, in which Q is replaced by the content of the square bracket in (9.5). Consequently, the polynomial

$$Q(p) - S(p)\frac{B}{A_в}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_в}p\right) \qquad (9.6)$$

should not have pure imaginary roots or roots with a positive real part; before starting to solve this problem we check on this condition for the polynomial Q(p) itself, and after obtaining the solution we check on the polynomial (9.6). In accordance with (2.72) the generalized filter property must also be satisfied:

$$\left|\frac{R(jk\Omega_в)A_в}{Q(jk\Omega_в)A_в - S(jk\Omega_в)B}\right| < \left|\frac{R(j\Omega_в)A_в}{Q(j\Omega_в)A_в - S(j\Omega_в)B}\right|, \qquad (9.7)$$

which is checked likewise in the previous form (2.72) prior to the start of the solution, and in the form (9.7) after completing the solution.

The specified nonlinearity F(x, px) should admit of symmetrical oscillations, i.e., the condition (2.74) must be satisfied:

$$\int_0^{2\pi} F(A_в \sin\psi,\ A_в\Omega_в\cos\psi)\,d\psi = 0. \qquad (9.8)$$

Thus, after obtaining the homogeneous equation (9.5) for the determination of the forced oscillations, we can as in §2.3 carry out the harmonic linearization of the nonlinearity in the form

$$F(x,\ px) = qx + \frac{q'}{\Omega_в}px, \qquad (9.9)$$

where

$$\left.\begin{aligned} q &= \frac{1}{\pi A_в}\int_0^{2\pi} F(A_в\sin\psi,\ A_в\Omega_в\cos\psi)\sin\psi\,d\psi, \\ q' &= \frac{1}{\pi A_в}\int_0^{2\pi} F(A_в\sin\psi,\ A_в\Omega_в\cos\psi)\cos\psi\,d\psi, \end{aligned}\right\} \qquad (9.10)$$

whereas here, in contrast with §2.3 and in accordance with (9.3), we have

$$\psi = \Omega_в t + \varphi, \tag{9.11}$$

which, however, does not influence the results of calculating $\underline{g}$ and q'. Therefore, in determining the symmetrical single-valued forced oscillations we can make complete use of the ready-made expressions for $\underline{g}$ and q' given in Chapter 3, except that we replace A and Ω by A_v and Ω_v. Thus, for each nonlinearity we obtain in the general case the relationships

$$q(A_в, \Omega_в), \quad q'(A_в, \Omega_в),$$

and in many particular cases (see Chapter 3)

$$q(A_в), \; q'(A_в).$$

As a result we obtain from (9.5) and (9.9) the characteristic equation for the first approximation:

$$Q(p) - S(p)\left(\frac{B}{A_в}\cos\varphi - \frac{\sin\varphi}{\Omega_в}p\right) + R(p)\left(q + \frac{q'}{\Omega_в}p\right) = 0. \tag{9.12}$$

Substituting here the pure imaginary value $p = j\Omega_v$, which corresponds to finding the sinusoidal solution (9.3), we obtain

$$Q(j\Omega_в) - S(j\Omega_в)\frac{B}{A_в}(\cos\varphi - j\sin\varphi) + R(j\Omega_в)(q + jq') = 0. \tag{9.13}$$

Noting that

$$\cos\varphi - j\sin\varphi = e^{-j\varphi},$$

we obtain from (9.13)

$$A_в \frac{Q(j\Omega_в) + R(j\Omega_в)(q + jq')}{S(j\Omega_в)} = Be^{-j\varphi}. \tag{9.14}$$

The problem can then be solved further by two methods. These methods remain valid also for nonlinear systems with time delay τ, when Expression (9.14) assumes the form

$$A_в \frac{Q(j\Omega_в) + R(j\Omega_в)(q + jq')e^{-j\tau\Omega_в}}{S(j\Omega_в)} = Be^{-j\varphi} \tag{9.15}$$

or another analogous form containing τ.

The graphic method. For each specified value of the frequency and with the system parameters specified, we plot on the complex plane the curve (Fig. 9.1):

$$Z(A_v) = A_v \frac{Q(j\Omega_0) + R(j\Omega_0)(q + jq')}{S(j\Omega_0)}. \qquad (9.16)$$

This curve corresponds to the left half of (9.14).* The right half of (9.14) or (9.15) is plotted in the form of a circle of radius B. The point where the circle crosses the curve $Z(A_v)$ yields the solution of the problem, in that the circular arc up to the point of intersection determines the phase shifts φ and the curve $Z(A_v)$ determines the amplitude A_v of the forced oscillations.

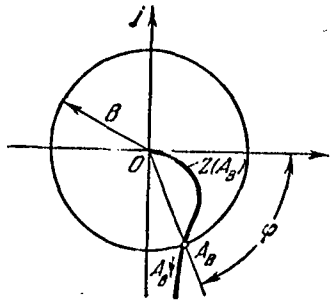


Fig. 9.1

The dependence of the amplitude A_v of the forced oscillations on the frequency Ω_v

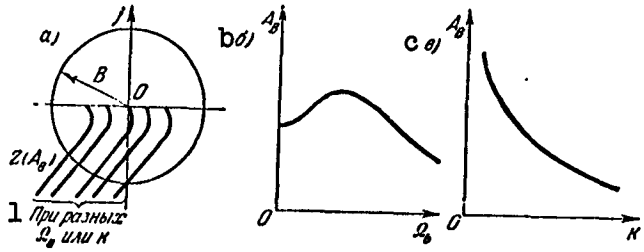


Fig. 9.2. 1) For different Ω_v or $\underline{k}$.

(Fig. 9.2b) can be obtained by plotting on Fig. 9.1 a series of $Z(A_v)$ curves for different constant values of Ω_v (Fig. 9.2a). In the same manner, by plotting $Z(A_v)$ curves for different constant values of some parameter $\underline{k}$ (Fig. 9.2a), we can determine the dependence of A_v on any system parameter $\underline{k}$ (Fig. 9.2c) contained in the expression (9.16) for $Z(A_v)$.

To find the dependence of A_v on the amplitude B of the external signal we must plot a series of concentric circles of different radii

B (Fig. 9.3a). Two cases are possible here: the first occurs when there exists a point of intersection of the circle with the $Z(A_v)$ curve for any value of the radius B, starting with zero; this gives the dependence $A_v(B)$ for example in the form of Fig. 9.3b. The second case occurs when the point of intersection of the circle with the $Z(A_v)$ curve exists only for values of the radius B exceeding a certain threshold value B_{por} (Fig. 9.3a), which leads to an $A_v(B)$ dependence of the type shown in Fig. 9.3c.

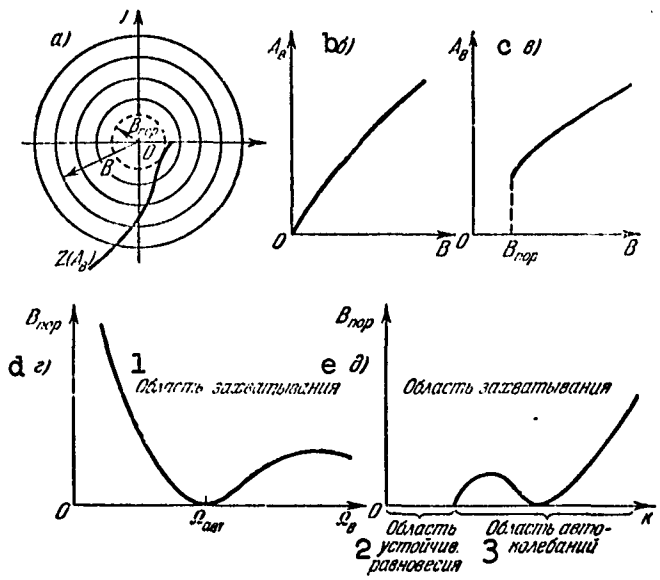


Fig. 9.3. 1) Locking region; 2) stable equilibrium region; 3) self-oscillation region.

The graphic determination of B_{por} is clear from the figure. It is possible to plot the dependence of the threshold amplitude B_{por} of the external signal on the frequency Ω_v for specified system parameters (Fig. 9.3d) or for any parameter $\underline{k}$ at a given frequency Ω_v (Fig. 9.3e). The latter dependence can be obtained with the aid of Fig. 9.3a, plotted for a series of $Z(A)_v$ curves corresponding to different $\underline{k}$.

The second considered case, in which the system goes over to single-frequency oscillations with frequency Ω_v only when $B > B_{por}$, occurs

most frequently in such nonlinear systems, which operate in the self-oscillating mode prior to the application of the external periodic signal. Furthermore, the quantity B_{por} vanishes in the case when the frequency Ω_v coincides with the self-oscillation frequency Ω_{avt} of the given system (Fig. 9.3d). B_{por} vanishes also usually in the region where there are no self-oscillations (stable equilibrium region of the system, Fig. 9.3e).

In this case the values of the amplitude B above the curves of Figs. 9.3d and e will correspond to an external signal resulting in a single-frequency mode of forced oscillations with frequency Ω_v (the locking region), while values of B below the curve will correspond to a more complicated forced motion of the system. This indeed is a definition (for the time being, graphic) of the locking conditions referred to above.

In other nonlinear systems we can have $B_{por} = 0$ as in the case of Fig. 9.3b.

Analytic method. From Eq. (9.14) or (9.15) we can obtain analytic expressions for the determination of the amplitude A_v and the phase shift φ of single frequency forced oscillations of a nonlinear system. For this purpose we separate the real and imaginary parts of the numerator and denominator and write down the equations for the moduli and arguments of both parts of Eq. (9.14) or (9.15). As a result we obtain

$$A_v^2 \frac{X^2(A_v, \Omega_v) + Y^2(A_v, \Omega_v)}{X_s^2(\Omega_v) + Y_s^2(\Omega_v)} = B^2, \tag{9.17}$$

$$\varphi = -\operatorname{arctg} \frac{Y(A_v, \Omega_v)}{X(A_v, \Omega_v)} + \operatorname{arctg} \frac{Y_s(\Omega_v)}{X_s(\Omega_v)}, \tag{9.18}$$

where X and Y are the real and imaginary parts of the numerator of (9.14) or (9.15) and X_s and Y_s are the real and imaginary parts of the denominator, i.e., $S(j\Omega_v)$. In this case X and Y correspond to the left

half of the specified nonlinear equation (9.2), i.e., they are the same expressions X and Y which were used in the investigation of self-oscillations (§2.3), while X_s and Y_s are new expressions, corresponding to the right half of the specified nonlinear equation (9.2).

We see that Expression (9.17) can generally speaking turn out to be a rather complicated algebraic equation in A_v. What is important, however, is that this equation contains only a single unknown A_v which consequently can be determined in one manner or another. After this we can readily calculate the phase shift φ by means of Formula (9.18). We recall that in finding the self-oscillations (Chapter 4) we also obtained an equation that was complicated with respect to A, but this did not entail any great difficulties. Indeed, in most cases we are interested in the variation of the amplitude of the forced oscillations A_v as a function of the frequency and of the amplitude of the external signal, and also as a function of variation of any particular system parameter. These quantities may enter into (9.17) in a simpler manner than the amplitude A_v. Then the equation (9.17) can be solved in explicit form relative to any of these quantities and then, by specifying different values of A_v and calculating in accordance with the resultant formula the quantity on which A_v depends, we can plot the sought functions $A_v(B)$, $A_v(\Omega_v)$, or $A_v(k)$, etc.; then Formula (9.18) can also be used to calculate the phase shift φ for each case.

For example, the following simple procedure for solving Eq. (9.17) is possible. For each specified external signal frequency Ω_v we assign different values of A_v and calculate the value of B in every case. From the results of these calculations we can readily plot a curve (Fig. 9.4) which indeed is the sought solution of Eq. (9.17).

As regards the analytic determination of the locking condition, i.e., of the value B_{por} as a function of the frequency Ω_v and of the

system parameters, it is defined as the condition for the existence of a real positive solution for A_v in Eq. (9.17). An example of the analytic determination of the locking condition was given for the very simple case in §1.9 (Formula (1.181)). In other problems this condition manifests itself automatically during the course of plotting the curve of the type of Fig. 9.4.

We have thus obtained the amplitude A_v and the phase shift φ of the forced oscillations for the variable $\underline{x}$ which is the argument of the nonlinear function. We can then calculate the amplitude and phase of the first harmonic of the forced oscillations for any other variable of the investigated system on the basis of the corresponding equations or transfer functions of the elements relating this variable with the variable $\underline{x}$.

In similar fashion we can consider also more complicated nonlinear systems of all other classes described in §1.2.

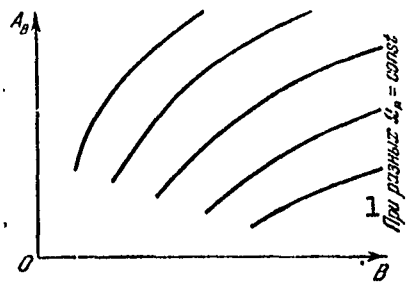


Fig. 9.4. 1) For different Ω_v = const.

It is sometimes necessary to investigate the stability of single frequency forced oscillations, particularly if two equations are obtained. As in §2.4, we first turn here to the classical method of stability investigation, namely comparison of the linearized differential equation expressed in terms of small deviations from the investigated solution (9.3). We introduce a small deviation from the investigated periodic solution, i.e., the variable

$$\Delta x = x - x^*, \text{ where } x^* = A_v \sin(\Omega_v t + \varphi).$$

Substituting this in the initial nonlinear equation of the given system (9.2) and expanding $F(x, px)$ in a Taylor series, we obtain

$$Q(p)(x^* + \Delta x) + R(p)\left[F(x^*, px^*) + \left(\frac{\partial F}{\partial x}\right)^* \Delta x + \right.$$
$$\left. + \left(\frac{\partial F}{\partial px}\right)^* p\Delta x + \ldots\right] = S(p) f(t), \qquad (9.19)$$

where the asterisk denotes the substitution $x = A_v \sin(\Omega_v t + \varphi)$. But inasmuch as the solution x* satisfies Eq. (9.2), we have

$$Q(p) x^* + R(p) F(x^*, px^*) = S(p) f(t).$$

Subtracting this expression from (9.19) and discarding small quantities of higher order (denoted by the dots), we arrive ultimately at a linear equation in the small deviations:

$$Q(p)\Delta x + R(p)\left[\left(\frac{\partial F}{\partial x}\right)^* \Delta x + \left(\frac{\partial F}{\partial px}\right)^* p\Delta x\right] = 0. \qquad (9.20)$$

Inasmuch as the investigation of this linear equation with periodic coefficients is difficult in the overwhelming majority of cases, we can resort, as in §2.4, to a method of averaging the periodic coefficients, replacing (9.20) by an equation with constant coefficients

$$[Q(p) + R(p)(\varkappa + \varkappa' p)]\Delta x = 0, \qquad (9.21)$$

where

$$\varkappa = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial F}{\partial x}\right)^* d\psi, \quad \varkappa' = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial F}{\partial px}\right)^* d\psi, \quad \psi = \Omega_v t + \varphi.$$

We then can apply to Eq. (9.21) any of the ordinary linear stability criteria (Hurwitz, Mikhaylov, Nyquist). It must be stated that there is as yet no rigorous solution to the problem of the class of system for which such an averaging method of the periodic coefficients will give a correct answer. One can expect, however, as in the case of §2.4, that this does hold true for the automatic systems under consideration.

Thus, for the example considered in §1.9, the value of the average coefficient $\varkappa$ will have a form analogous to (2.113), i.e.,

$$\varkappa = \frac{2c}{\pi A_v},$$

and in accordance with the characteristic equation (2.114) the condi-

tion for the stability of the forced oscillations will be

$$\pi A_v (T_1 + T_2) + 2cT_1k_2(T_1k_{o\cdot c} - T_2k_1) > 0,$$

where A_v is the amplitude of the forced oscillations, which depends on B and Ω_v and also on the system parameters.

Other approximate methods of investigating the stability of forced oscillations of nonlinear systems can be found in References [274] and [301]. Concerning forced oscillations of relay systems, see [272] and [279].

It should be noted that in most practical problems one need not resort to an investigation of the stability of the obtained forced oscillations of the system.

§9.2. Asymmetrical Forced Oscillations. Determination of the Bias Function

The forced oscillations will be asymmetrical in the following cases:

1) if the nonlinear characteristics of the system are asymmetrical;

2) in the presence of a constant or slowly varying external signal (in static systems);

3) in the presence of a constant or slowly varying rate of change of the external signal (in astatic systems).

All the general considerations associated with this, and also the concept of slowly varying signals, remain the same as in Chapter 5 (§5.1), except that we deal here not with self-oscillations but with forced oscillations which are imposed on the constant or slowly varying (in time) component (the shift of the center of the forced oscillations) [293].

In the general case we shall assume that two external signals are applied to the nonlinear system, as the result of which its equation has in lieu of (9.2) the form

$$Q(p)x + R(p)F(x, px) = S_1(p)f_1(t) + S_2(p)f_2(t), \tag{9.22}$$

where $f_1(t)$ is the slowly varying external signal and $f_2(t)$ is the periodic external signal:

$$f_2(t) = B \sin \Omega_v t \tag{9.23}$$

The slowly varying signal $f_1(t)$ is assumed to change little over the period $T_v = 2\pi/\Omega_v$, i.e., it is assumed that the possible frequencies of the variation of $f_1(t)$ are considerably lower than the frequency Ω_v.

We shall seek the solution of (9.22) in the form

$$x = x^0 + x^*, \quad x^* = A_v \sin(\Omega_v t + \varphi), \tag{9.24}$$

where $x^0(t)$ is the slowly varying component and x* is the oscillatory component with amplitude A_v and phase φ which in general also varies slowly in time.

Then the harmonic linearization of the nonlinearity F(x, px) can be carried out by means of a formula analogous to (5.3):

$$F(x, px) = F^0 + qx^* + \frac{q'}{\Omega_v} px^*, \tag{9.25}$$

where

$$\left.\begin{aligned} F^0 &= \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A_v \sin\psi, A_v\Omega_v \cos\psi)\, d\psi, \\ q &= \frac{1}{\pi A_v}\int_0^{2\pi} F(x^0 + A_v \sin\psi, A_v\Omega_v \cos\psi)\sin\psi\, d\psi, \\ q' &= \frac{1}{\pi A_v}\int_0^{2\pi} F(x^0 + A_v \sin\psi, A_v\Omega_v \cos\psi)\cos\psi\, d\psi, \end{aligned}\right\} \tag{9.26}$$

with $\psi = \Omega_v t + \varphi$. From a comparison of these formulas with (5.4) we see that when finding the forced oscillations we can use in their entirety all the specific expressions for F^0, q, and q' which were derived for the different nonlinearities in Chapter 5 (§§5.6-5.9). Thus, for each specific nonlinearity we have the ready-made expressions:

$$F^0(x^0, A_v, \Omega_v), \quad q(x^0, A_v, \Omega_v, \quad q'(x^0, A_v, \Omega_v), \tag{9.27}$$

from which frequency the quantity Ω_v is missing. By way of an example, Fig. 9.5 shows these relationships for a nonlinearity of the saturation type, plotted on the basis of Formulas (5.121) and (5.122).

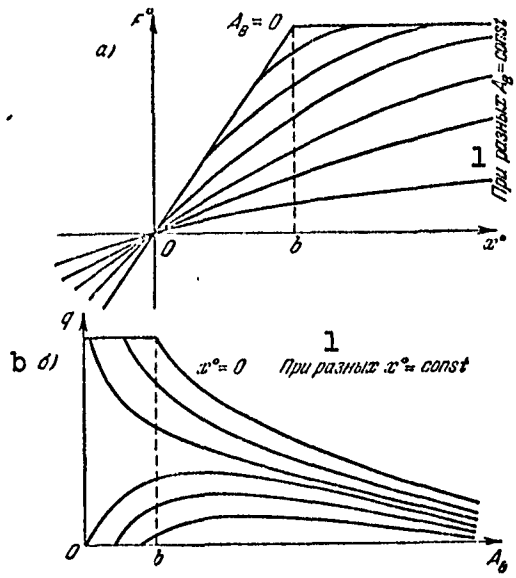


Fig. 9.5. 1) For different.

In analogy with Formula (9.4) we write

$$f_2(t) = \frac{B}{A_B}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_B} p\right) x^*. \qquad (9.28)$$

Substituting the expressions for F(x, px), $f_2(t)$, and $\underline{x}$ in the given differential equation of the nonlinear system (9.22), we obtain the equation

$$Q(p)(x^0 + x^*) + R(p)\left(F^0 + qx^* + \frac{q'}{\Omega_B} px^*\right) = \\ = S_1(p) f_1(t) + S_2(p) \frac{B}{A_B}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_B} p\right) x^*,$$

which breaks up in nonlinear fashion (see Chapter 5) into two equations for the slowly varying and for the oscillatory components, respectively:

$$Q(p) x^0 + R(p) F^0 = S_1(p) f_1(t), \qquad (9.29)$$

$$\left[Q(p) - S_2(p) \frac{B}{A_B}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_B} p\right)\right] x^* + R(p)\left(q + \frac{q'}{\Omega_B} p\right) x^* = 0. \qquad (9.30)$$

Both equations contain all three unknowns A, φ, and x^0.

The second of these equations (9.30) coincides with the previous

equation (9.5), except that it has different harmonic linearization coefficients $\underline{q}$ and q', which depend on the magnitude of the bias x^0. Therefore, Eq. (9.30) is solved to the end only simultaneously with (9.29), although, as will be shown later on, simpler cases are also possible. For the time being we can write down a characteristic equation of the form (9.12) and then substitute $p = j\Omega_v$ and reduce Eq. (9.30) to the form

$$A_n \frac{Q(j\Omega_n) + R(j\Omega_n)(q + jq')}{S_2(j\Omega_n)} = Be^{-j\varphi}, \qquad (9.31)$$

which when solved by any of the two methods (graphic or analytic) described in §9.1 yield the dependence of the amplitude A_v and the phase shift φ on the magnitude of the bias x^0, i.e.,

$$A_n(x^0, \Omega_n, B), \quad \varphi(x^0, \Omega_n, B), \qquad (9.32)$$

where x^0 is for the time being unknown.

To use the graphic method of §9.1 in order to find the dependence $A_v(x^0)$ by means of Eq. (9.31), we must plot in Fig. 9.1 a series of $Z(A_v)$ curves for different values $x^0 = \text{const}$, which in accord with (9.26) enters into the expressions for $\underline{q}$ and q'. The equation (9.17) of the analytic method assumes the form

$$A_n^2 = \frac{X_2^2(\Omega_n) + Y_2^2(\Omega_n)}{X^2(\Omega_n, A_n, x^0) + Y^2(\Omega_n, A_n, x^0)} B^2, \qquad (9.33)$$

where X_2, Y_2 and X, Y denote the real and imaginary parts, respectively, for $S_2(j\Omega_v)$ and for

$$Q(j\Omega_n) + R(j\Omega_n)[q(A_n, \Omega_n, x^0) + jq'(A_n, \Omega_n, x^0)].$$

Equation (9.33) is not solved by simple calculation like (9.17). We can, however, use the following graphic method for solving it. Breaking up (9.33) into two equations

$$A_n^2 = \zeta,$$
$$\frac{X_2^2(\Omega_n) + Y_2^2(\Omega_n)}{X^2(\Omega_n, A_n, x^0) + Y^2(\Omega_n, A_n, x^0)} B^2 = \zeta,$$

we plot the first of these as curve 1 on the (ζ, A_v) plane (Fig. 9.6),

and the second one as a series of curves 2 for different values of x^0 = const for specified B and Ω_v. Transferring the resultant points of intersection of the curves to the right on the x^0, A_v plane, we obtain immediately the sought function $A_v(x^0)$ for a specified external periodic signal, i.e., for a specified pair of values B and Ω_v. This dependence can be readily obtained in the same manner for any other specified values of B and Ω_v.

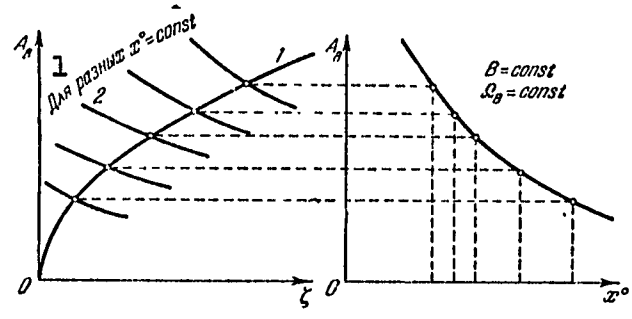


Fig. 9.6. 1) For different.

Substituting now the values of the amplitude A_v in the first of the expressions in (9.27), we obtain the so-called bias function in the form

$$F^0 = \Phi(x^0, \Omega_в, B), \qquad (9.34)$$

which is a characteristic of the given nonlinear element of the system relative to the slowly varying components of the variables F and $\underline{x}$. These slowly varying components are determined then by solving the differential equation (9.29), in which it is necessary to substitute the obtained bias function (9.34).

The fact that the outline of the bias function $\Phi(x^0)$ is independent of the character of the variation and of the location of application of the slowly varying external signals remains in force here as was the case in the self-oscillations (Chapter 5).

However, the principal difference between the bias function (9.34),

which determines the flow of slowly varying signals through a nonlinear system in the presence of forced oscillations, from the bias function (5.11) in the case of self-oscillations, is that the former depend essentially on the frequency and amplitude of the external periodic signal (whereas in the case of self-oscillations the form of the bias function depended only on the structure and on the relations between the parameters of the system itself).

As a result we obtain for each specified forced oscillation frequency Ω_v a series of curves $F^0 = \Phi(x^0)$ for different values of the amplitude B of the external periodic signal $f_2(t)$, as shown for example in Figs. 9.7a and b. For specified Ω_v and B we obtain a fully determined outline of the bias function $\Phi(x^0)$, which depends only on the structure and parameters of the system itself, parameters which are contained in Eq. (9.31).

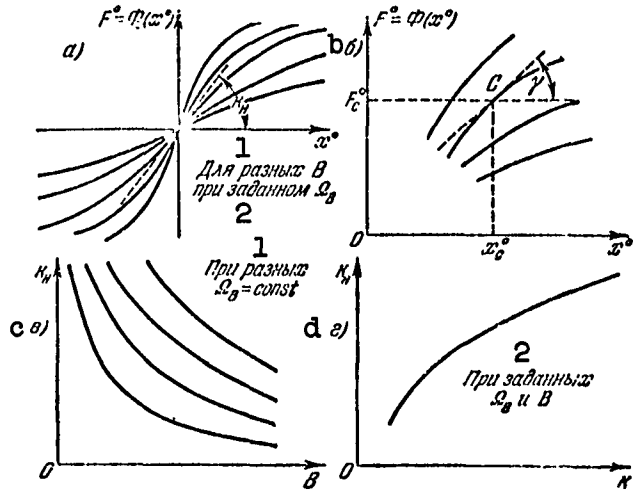


Fig. 9.7. 1) For different; 2) for specified.

Here, as in Chapter 5, a second method of obtaining the bias function is also possible. In this method we determine also incidentally the static and steady-state errors. The method consists of the following.

Inasmuch as the bias function $F^0 = \Phi(x^0)$ is independent of the character of variation and of the point of application of the slowly varying signals, it can be determined for the simplest case $f_1 = \text{const} = f_1^0$ (or in the case of an astatic system for $pf_1 = \text{const} = c_1^0$). Then Eq. (9.29) assumes the form

$$Q(0)x^0 + R(0)F^0 = M^0, \tag{9.35}$$

where $M^0 = S_1(0)f_1^0$, or for astatic systems $M^0 = dS_1/dp(0)c_1^0$. Using the first expression from (9.27), i.e. (for a specified frequency Ω_v),

$$F^0(x^0, A_v), \tag{9.36}$$

from (9.35) we obtain

$$x^0(A_v, M^0). \tag{9.37}$$

Substituting this in the expression for $\underline{q}$ and q', which are determined by the second and third formulas in (9.27), we obtain the relationships

$$q(A_v, M^0) \quad \text{and} \quad q'(A_v, M^0).$$

Inserting them into (9.31), which is equivalent to (9.30), and solving this equation by any of the two methods indicated above, we obtain for specified B and Ω_v the amplitude $A_v(M^0)$ of the forced oscillations. Inserting $A_v(M^0)$ in (9.36) and (9.37) we obtain the relationships

$$F^0(x^0, M^0) \quad \text{and} \quad x^0(M^0). \tag{9.38}$$

These relationships are of interest in themselves, since they determine the static error (or, for an astatic system, the steady-state error at constant velocity) of the nonlinear system with respect to the slowly varying component, on which there is superimposed also the steady-state periodic error of the forced oscillations, with amplitude $A_v(M^0)$. All these errors are determined, as we see, as functions of the magnitude M^0 of the constant right half of Eq. (9.35), i.e., on the magnitude of the external signal (which is constant and equal to

f_1^0 or which varies at a constant rate c_1^0). In addition, however, a fact of great importance for nonlinear systems, the magnitude of the static deviation $x^0(M^0)$ can depend essentially on the amplitude B and the frequency Ω_v of the external periodic signal, since the expressions (9.38) were derived with the aid of Eq. (9.31), which contains B and Ω_v. The amplitude of the forced oscillations A_v depends in turn, through M^0, on the magnitude of the constant external signal. This is a clear-cut example of the failure of the superposition principle for nonlinear systems, and at the same time illustrates the advantages of the method developed here, which makes it possible to detect this failure, in spite of the approximate nature of the solution of the problem.

Further, eliminating from (9.38) the quantity M^0, we obtain the bias function $F^0 = \Phi(x^0)$ for specified B and Ω_v (Fig. 9.7a).

Thus, the presence in the nonlinear system of forced oscillations with the same frequency as the external periodic signal also leads to the effect of vibration smoothing of the nonlinearity (Fig. 9.7a) as in the case of self-oscillations (see §5.3). In this case, according to (9.29), the initial differential equation of the system (9.22) is replaced for the slow processes under forced vibration conditions by the equation

$$Q(p)x^0 + R(p)\Phi(x^0) = S_1(p)f_1(t), \qquad (9.39)$$

i.e., the specified nonlinearity F(x, px) is replaced by the bias function $\Phi(x^0)$ and the external periodic signal $f_2(t)$, by comparison with which $f_1(t)$ is slowly varying, is discarded.

The bias function $\Phi(x^0)$ is usually represented over a definite interval of variation of x^0 by a single-valued smooth curve (Figs. 9.7a and b), whereas the specified nonlinearity F(x, px) or F(x) may be of the discontinuous (relay) or loop type, with backlash zone, etc.

This smoothed characteristic $\Phi(x^0)$ of the nonlinear element with respect to the slowly varying signals makes it possible consequently to eliminate the influence of harmful hysteresis loops, backlash zones, dry friction effects, etc. (see Fig. 5.8). In some cases, however, vibration smoothing may turn out to be a harmful effect, as was the case of Fig. 5.9, where the gain was decreased. In addition to these phenomena, which are analogous to vibration smoothing in self-oscillations, there appear here in principle also new phenomena resulting from the dependence of the characteristic $\Phi(x^0)$ on B and Ω_v, which will be discussed later on.

The smoothness of the bias function $\Phi(x^0)$ (Fig. 9.7a and b) makes it possible to carry out ordinary linearization; namely, for the case of Fig. 9.7a we can assume on some interval near the origin

$$F^0 = k_{\text{н}} x^0, \tag{9.40}$$

where

$$k_{\text{н}} = \left(\frac{d\Phi}{dx^0}\right)_{x^0 \to 0} \tag{9.41}$$

and for the case of Fig. 9.7b

$$F^0 = F_C^0 + k_{\text{н}}(x^0 - x_C^0), \tag{9.42}$$

where x_C^0 is the nominal value of the variable x^0, determined by the conditions of the specific problem. In this case we have

$$F_C^0 = \Phi(x_C^0)$$

and the quantity

$$k_{\text{н}} = \left(\frac{d\Phi}{dx^0}\right)_{x^0 \to x_C^0} \quad \text{or} \quad k_{\text{н}} = \operatorname{tg}\gamma,$$

is determined first by means of Formula (9.34) or by means of a corresponding plot of the bias function prior to its linearization.

All the slow processes in the given nonlinear system can then be calculated not by means of Eq. (9.39), but by means of the linear equation

$$[Q(p)+k_{\text{п}}R(p)]\,x^0=S_1(p)f(t) \tag{9.43}$$

for the case of Fig. 9.7a, or by means of the equation

$$[Q(p)+k_{\text{п}}R(p)]\,\Delta x^0=S_1(p)\,\Delta f_1(t), \tag{9.44}$$

where

$$\Delta x^0=x^0-x_C^0,\quad \Delta f_1=f_1-f_{1C}^0,$$

for the case of Fig. 9.7b, with the quantities x_C^0 and f_{1C}^0 related by the equation

$$Q(0)\,x_C^0+R(0)\,\Phi(x_C^0)=S_1(0)f_{1C}^0.$$

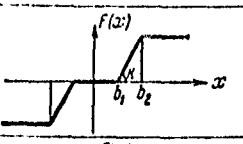
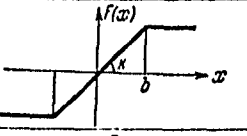
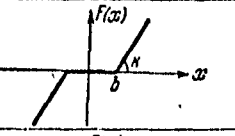
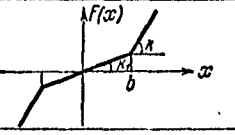
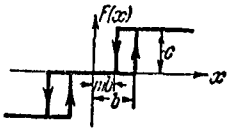
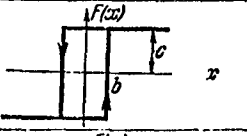
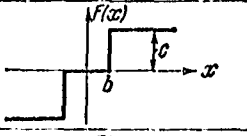
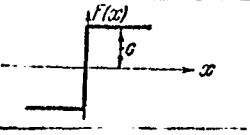
What is very important here is that the gain k_n (Fig. 9.7a) will depend not only on the structure and parameters of the system itself, as was the case in self-oscillations, but also on the amplitude B and frequency Ω_v of the external periodic signal, which can change within certain limits independently of the system itself. Therefore, on the one hand, vibration smoothing of nonlinear characteristics with the aid of forced oscillations has appreciably greater practical possibilities than in the case of self-oscillations, something used quite frequently in engineering, particularly in relay systems for automatic control. In other cases, however, as will be shown below, vibration smoothing can lead to harmful effects, and even to loss of system stability.

From the point of view of simplifying the solution of the problem, it must be kept in mind that for all odd-symmetry nonlinearities F(x), both single valued and of the loop type, the calculation of the coefficient k_n during the linearization of the bias function can be carried out, as was shown in §5.3, not by means of Formula (9.41), but by means of the simpler formula

$$k_{\text{п}}=\left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0}, \tag{9.45}$$

i.e., directly by means of the first expression in (9.27), without de-

TABLE 9.1

№	1 Форма нелинейности	2 Выражение $k_н(A_н)$
1		$k_н = \frac{2k}{\pi}\left(\arcsin\frac{b_2}{A_н} - \arcsin\frac{b_1}{A_н}\right)$.
2		$k_н = \frac{2k}{\pi}\arcsin\frac{b}{A_н}$
3		$k_н = k - \frac{2k}{\pi}\arcsin\frac{b}{A_н}$
4		$k_н = k_2 - \frac{2(k_2 - k_1)}{\pi}\arcsin\frac{b}{A_н}$
5		$k_н = \frac{c}{\pi A_н}\left[\frac{1}{\sqrt{1-\left(\frac{b}{A_н}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{mb}{A_н}\right)^2}}\right]$
6		$k_н = \frac{2c}{\pi A_н}\frac{1}{\sqrt{1-\frac{b^2}{A_н^2}}}$
7		$k_н = \frac{2c}{\pi A}\frac{1}{\sqrt{1-\frac{b^2}{A_н^2}}}$
8		$k_н = \frac{2c}{\pi A_н}$

1) Form of nonlinearity; 2) expression $k_n(A_v)$.

termining the bias function $\Phi(x^0)$ itself. The expressions for $k_n(A_v)$, determined by means of Formula (9.45) for certain nonlinearities, are listed in Table 9.1. Geometrically the quantity k_n is the slope of the curve $F^0(x^0)$ at the origin, for example, the curve $F^0(x^0)$ on Fig. 9.5a at the origin. In order to take in this case a definite curve from among the series of curves shown in Fig. 9.5a for different values of A_v, it is first necessary to use the specified values of the amplitude

B and frequency Ω_v of the external periodic signal to determine the value of the amplitude of the forced oscillations A_v for $x^0 = 0$. But this problem was already solved in §9.1, and the result of the solution is represented by the curve of Fig. 9.4. Consequently, we can merely take the ready-made values of A_v of Fig. 9.4, for specified B and Ω_v and substitute them in Formula (9.45) or use them with Fig. 9.5a.

It is then easy to plot the dependence of the quantity k_n not only on B and Ω_v (Fig. 9.7c), but also on any other system parameter $\underline{k}$ (Fig. 9.7d), the influence of which is desired to investigate, and on which the amplitude of the forced oscillations A_v (Fig. 9.2c) as shown on Fig. 9.5a depends.

The method developed in the present section can also be extended to include more complicated nonlinear systems of other classes, as indicated in §1.2. An account of the time delay τ also entails no difficulties.

§9.3. Dependence of the Stability and Quality of Nonlinear Systems on the External Vibrations

After determining the bias function $F^0 = \Phi(x^0)$, it becomes possible to investigate by means of Eq. (9.39) or by means of the purely linear equations (9.43) or (9.44) any slowly varying process in the system, subject to the corresponding slow variation of $f_1(t)$ or other signals. It is merely necessary to bear in mind here that the bias function $\Phi(x^0)$ and the value of the coefficient k_n which replaces it in the linear equation can depend essentially on the amplitude B and on the frequency Ω_v of the external periodic signal, and also on the structure and parameters of the entire system as a whole. We shall show below that in many particular cases some of these complicated relationships can be disregarded, something that simplifies the solution of the problem.

We have previously considered in §5.4 the stability and quality of a nonlinear system from the point of view of the slowly varying components. It is meaningful to consider the same in this case, too. This is a question of very great practical importance, for in many automatic control systems it is precisely the slow processes that serve as the basis of operation of the automatic system (the useful control signal), and the forced vibrations represent the noise (with the exception of the specially produced vibration smoothing of the nonlinearity, which will be considered below separately in the form of a very simple particular problem).

The stability of the system with respect to the slowly varying component can be regarded, in accordance with (9.39), by investigating the nonlinear equation

$$Q(p)\,x^0 + R(p)\,\Phi(x^0) = 0, \tag{9.46}$$

where $\Phi(x^0)$ is the bias function, or else, in accordance with (9.43), by investigating the linear equation

$$[Q(p) + k_n R(p)]\,x^0 = 0. \tag{9.47}$$

In both cases the stability of the system can be appreciably influenced by the amplitude B and the frequency Ω_v of the external periodic signal, since these determine, as is already known, the form of the bias function $\Phi(x^0)$ and the value of the coefficient k_n. This is a quite new and very important specific nonlinear factor, which we did not encounter in the preceding chapters. In linear systems there is no such phenomenon at all.

In the case when the nonlinear equation (9.46) is used for the slow processes the stability limit can, in particular, be determined by the method of §2.7 as being the limit of occurrence of self-oscillations with repeated harmonic linearization.* On the other hand, if the linear equation (9.47) is used, we can employ the ordinary stability

criteria for linear systems (Hurwitz, Mikhaylov, Nyquist) and the ordinary logarithmic frequency characteristics.

Consequently, the stability region of the nonlinear system can change under the influence of the noise received in the form of sinusoidal vibrations applied from the outside, depending on the amplitude and frequency of the latter. For example, it may turn out that the stability region of the system with respect to some parameter k (Fig. 9.8a) becomes narrower, as shown in Fig. 9.8b, with increasing amplitude B of the external noise, which has the form of vibrations of specified frequency Ω_v. Consequently, for each value of k there can exist for a given frequency of external vibrations its own critical value of the amplitude B, at which the system becomes unstable. Similarly, by varying the vibration frequency Ω_v, it is possible to determine for a given value of the parameter k the dependence of the critical amplitude of the external vibrations on the frequency (Fig. 9.8c).

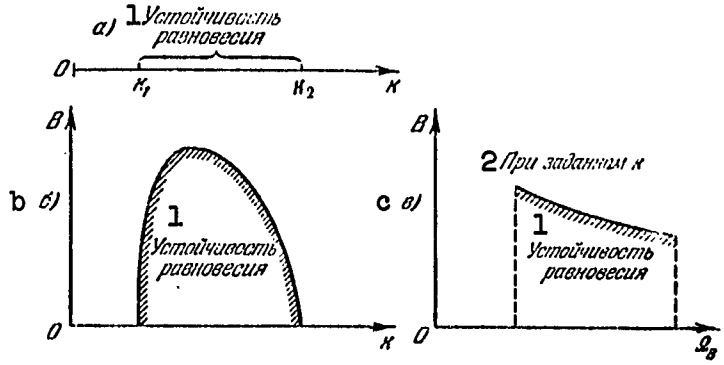


Fig. 9.8. 1) Stable equilibrium; 2) for specified k.

It is important to bear in mind here that when the system parameters change the coefficient k_n also changes, along with the outline of the bias function $\Phi(x^0)$. Therefore, when plotting the system stability regions for any particular parameter k (Fig. 9.8), it is necessary to vary accordingly throughout the entire time the value of k_n in Eq. (9.47) or the value of $\Phi(x^0)$ in (9.46), i.e., in plotting the stability

region it is necessary to take account of the fact that any system parameter $\underline{k}$ can be contained not only in R(p) and Q(p), but also in the quantity k_n. On the other hand, the dependence of the quantity k_n on any system parameter can be easily determined beforehand in accordance with §9.2 (see, for example, Fig. 9.7d).

In addition to investigating the stability of the nonlinear system, we can use Eq. (9.46) or (9.47), and also Eq. (9.39) or (9.43), to carry out a complete analysis of all the dynamic qualities of the nonlinear system subjected to external vibrations (the transient quality, or the static and dynamic errors), under arbitrary external signals $f_1(t)$ which are slow compared with the vibrations. If we use at the same time the linear equations (9.47) and (9.43), we can employ all the methods of the linear theory of automatic control both for the analysis and for the synthesis of the system. Only in the investigation of the influence of different parameters on the static and dynamic qualities of the system it is necessary to bear in mind always that any system parameter can be included not only in Q(p) and R(p), but can also influence the quantity k_n (see, for example, Fig. 9.7d).

The indicated equations can also be used to determine the forced oscillations of the system at low frequencies, provided the slowly varying signal $f_1(t)$ is periodic, i.e., it is possible to investigate two frequency force oscillations of a nonlinear system if the frequency difference is large. Here, too (as in §5.3), we can subdivide the overall motion of the nonlinear system not only into two but into three types of motion with different degrees of slowness.

The result of all these calculations will be a dependence, which is peculiar to nonlinear systems, of all the static and dynamic properties and even of its stability on the magnitude of the amplitude B and of the frequency Ω_v of the external periodic signal (vibrations),

which in some cases may be decisive in practice for the operating quality of the automatic system as a whole.

The investigation of the nonlinear equations (9.46) and (9.39) by the method of harmonic linearization will be discussed at the end of the present section.

The general theory of the behavior of nonlinear automatic systems in the presence of an external periodic signal (vibrations) discussed in §§9.2 and 9.3 [293] can be greatly simplified in different particular problems. We give here a modification of this general theory for the following three most typical particular problems:

1) application of a special external periodic signal to obtain vibration smoothing of nonlinearity (with subsequent linearization of the smooth characteristic in the design of the system as a whole);

2) investigation of the operation of a nonlinear automatic system under high frequency external vibration noise, when not all the system elements pass these vibrations;

3) investigation of the equations of a nonlinear automatic system with respect to the slow processes in the presence of vibration, using repeated harmonic linearization.

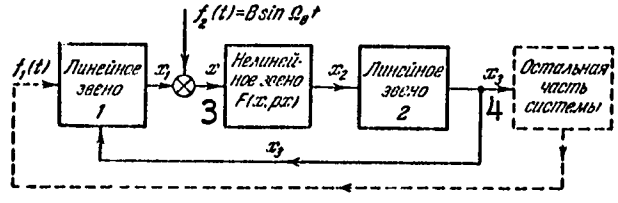


Fig. 9.9. 1) Linear element; 2) linear element; 3) nonlinear element; 4) remaining part of the system.

Problem 1. If an external periodic signal $f_2(t)$ is applied to any automatic system (Fig. 9.9) especially for the purpose of carrying out vibration smoothing of the nonlinearity, the obligatory condition is that the amplitude of the forced oscillations at the output x_3 be prac-

tically negligible. As a result, the variables x_3 and x_1 (Fig. 9.9) will practically contain no oscillatory component, and will be determined in terms of the slowly varying signal $f_1(t)$ by means of equations of the type (9.39) or (9.43). Therefore, the variable $\underline{x}$ at the input of the nonlinear element will contain only the one oscillatory component $f_2(t)$ (Fig. 9.9), i.e.,

$$x = x^0 + x^*, \quad x^* = B \sin \Omega_v t. \tag{9.48}$$

Consequently, in our problem (dealing with vibration linearization of a nonlinearity with the aid of forced oscillations) there is no need for solving (9.30) or (9.31) in order to determine the oscillatory components, since in accord with (9.24) and (9.41) we have a ready-made solution:

$$A_v = B, \quad \varphi = 0. \tag{9.49}$$

Inasmuch as the external periodic signal $f_2(t)$ is assumed to be applied to this system directly in the same place where the variable $\underline{x}$ is applied (Fig. 9.9), we have in Eq. (9.22), which has been set up for the investigated part of the system (excluding the dashed part in Fig. 9.9),

$$S_2(p) = Q(p). \tag{9.50}$$

On the basis of (9.49) we obtain from the first formula in (9.26)

$$F^0 = \frac{1}{2\pi} \int_0^{2\pi} F(x^0 + B \sin \psi, \; B\Omega_v \cos \psi)\, d\psi,$$

which gives us the sought smooth characteristic. We can use here the ready-made formulas of §§5.6-5.9 and their plots as given in Fig. 9.5a for all the typical nonlinearities, except that A and A_v are replaced everywhere by B. We see that the special determination of the bias function $\Phi(x^0)$, described in §9.2, is completely eliminated here.

As a result, the smoothed characteristic $F^0(x^0)$ will have a slope that depends in general on the amplitude B and the frequency Ω_v of the

external vibrations. On the other hand, if the nonlinearity is of a less general type, namely $F(x)$, then the frequency Ω_v will not enter into the expression F^0 as, for example, in the case of Fig. 9.5a. Nevertheless, it is necessary to stipulate in this case too that the frequency be contained within definite limits, so that the signal $f_1(t)$ can be regarded as slowly varying compared with $f_2(t)$.

After determining in this manner the smooth characteristic $F^0(x^0)$, we can then use an equation of the type of (9.29) or (9.43), together with the linearization (9.45), to investigate any slow process in the system as a whole, using the ordinary methods of regulation theory. We note that linearization by means of Formula (9.45) is valid in this problem for arbitrary form of nonlinearity since the partial derivative with respect to x^0 coincides here with the total derivative.

As regards the equation for the oscillatory components (9.30), or what is the same, (9.31), it must be used in the present problem only to determine the desired value of the frequency Ω_v of the external periodic signal $f_2(t)$, necessary to ensure the possibility of obtaining the solution of (9.49) for the forced oscillations and the fulfillment of the assumption made above that the forced oscillations at the output of the system x_3 are small. For this purpose we substitute (9.49) and (9.50) into (9.31). To satisfy the last equation we must stipulate that the modulus of the ratio

$$\left|\frac{R(j\Omega_v)}{Q(j\Omega_v)}\right|$$

be very small. Consequently, the frequency of the external periodic signal Ω_v should lie beyond the limits of the passband of the frequency characteristic of the entire given linear part of the portion of the system under consideration.

In addition, in order for the amplitude of the forced vibrations

at the output of the system x_3 be negligible, it is necessary to choose the frequency Ω_v also beyond the limits of the passband of the separate block 2 of the investigated system (Fig. 9.9).

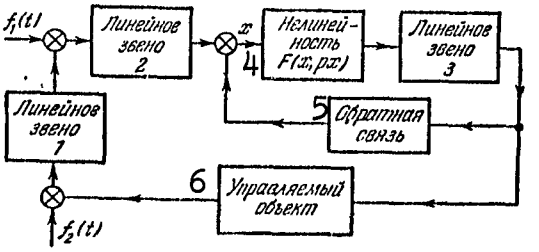


Fig. 9.10. 1) Linear element; 2) linear element; 3) linear element; 4) nonlinearity; 5) feedback; 6) controlled object.

Problem 2. Assume that some automatic control system (Fig. 9.10) is acted upon by an external vibration noise

$$f_2(t) = B \sin \Omega_n t$$

and also by an external control or disturbing signal $f_1(t)$, which is slowly varying with respect to the noise. The equations of the system dynamics can be reduced to the form (9.22).

We seek the solution of (9.22) in the form (9.24), where x^0 is the useful control signal and x* is the vibration noise at the input of the nonlinear element. Breaking down Eq. (9.22) into two, namely (9.29) and (9.31), we must, in accordance with the general method developed above, first determine with the aid of (9.31) and (9.27) the bias function $F^0 = \Phi(x^0)$, after which we can solve the differential equation (9.29) with respect to the variable $x^0(t)$ for a specified function $f_1(t)$. In this problem, however, this general method of solution can be simplified. Let us consider two cases.

In the case when the entire given linear part of the system (Fig. 9.10), defined by a transfer function

$$W_n(p) = \frac{R(p)}{Q(p)}, \qquad (9.51)$$

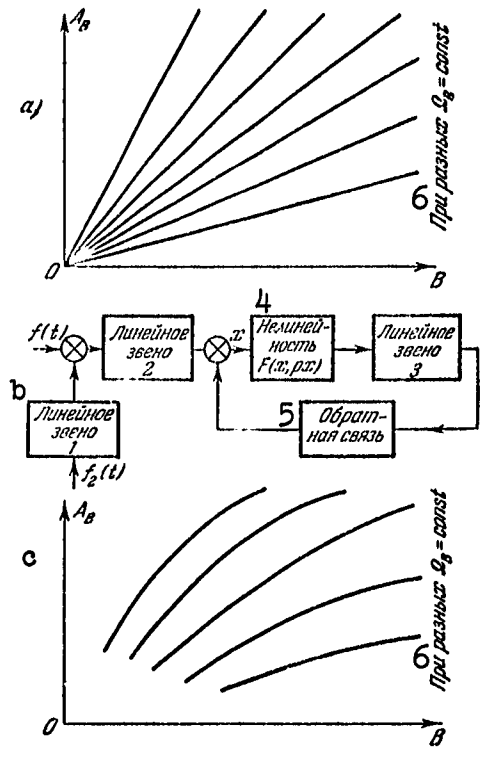


Fig. 9.11. 1) Linear element; 2) linear element; 3) linear element; 4) nonlinearity; 5) feedback; 6) for different.

practically blocks the vibrations with specified frequency Ω_v, the equation (9.31) can be rewritten as

$$A_в = \frac{S_2(j\Omega_в)}{Q(j\Omega_в)} Be^{-j\varphi}.$$

Then the amplitude of the vibrations at the input of the nonlinear element will be determined by the formula

$$A_в^2 = \frac{X_2^2(\Omega_в) + Y_2^2(\Omega_в)}{X_Q^2(\Omega_в) + Y_Q^2(\Omega_в)} B^2, \qquad (9.52)$$

where $X_2(\Omega_v)$, $Y_2(\Omega_v)$ and $X_Q(\Omega_v)$, $Y_Q(\Omega_v)$ denote the real and imaginary parts for the expressions $S_2(j\Omega_v)$ and $Q(j\Omega_v)$, respectively.

Formula (9.52) yields a linear relationship $A_v = (B)$ with different coefficients of proportionality for different vibration frequencies Ω_v (Fig. 9.11a). In particular, for the circuit of Fig. 9.10, this will be determined by the structure of the linear blocks 1 and 2.

Compared with general theory, an important factor is now that the vibration amplitude A_v at the input of the nonlinear element is in this case independent of the magnitude of the useful signal x^0. Therefore here, as in problem 1, there is no need for finding the bias function $\Phi(x^0)$ and the characteristic of the nonlinear element relative to the useful signal $F^0(x^0)$ will be determined directly by the first formula of (9.27), which is represented graphically, for example, in Fig. 9.5a. Here, however, it is necessary to substitute in the expression for F^0 (or take on the plot of Fig. 9.5a) the value of A_v determined by Formula (9.52) (or by the plot of Fig. 9.11a). Therefore, unlike problem 1, here the outline of the characteristic of the nonlinear element with respect to the useful signal $F^0(x^0)$, and its slope

$$k_{\text{н}} = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0 = 0}$$

will depend even for the simplest nonlinearities not only on the amplitude B, but also on the frequency Ω_v of the vibration noise, and also, of course, on the parameters of the linear blocks 1 and 2 (Fig. 9.10), which enter into the used formula (9.52).

Let us consider further a second case, when the first harmonic of the vibrations with specified frequency Ω_v is passed by the linear portion of the system with transfer function (9.51), but is nevertheless not passed by any other block of the system. Assume, for example, that the vibrations in the circuit of Fig. 9.10 are blocked completely only by the controlled object, and that the first harmonic of the vibrations with frequency Ω_v does pass through the internal feedback loop. Then, generally speaking, we can no longer disregard the dependence (9.32) of the vibration amplitude A_v of the variable $\underline{x}$ on the magnitude of the useful signal x^0. However, even in this case a simplification is possible in the solution of the problem, as compared with the general

theory, the simplification consisting of discarding in the determination of the bias function that part of the system, which does not pass the vibrations (Fig. 9.11b).

In this case we must write down the dynamic equations only for the remaining part of the system (Fig. 9.11b):

$$Q_c(p)\,x + R_c(p)\,F(x,\ px) = S_{1c} f_1(t) + S_{2c}(p) f_2(t), \qquad (9.53)$$

which, of course, will be simpler than the general equation (9.22). From this we obtain in analogy with (9.33) an equation for the determination of the vibration amplitude at the input of the nonlinear element in the form

$$A_B^2 = \frac{X_{2c}^2(\Omega_B) + Y_{2c}^2(\Omega_B)}{X_c^2(A_B, \Omega_B, x^0) + Y_c^2(A_B, \Omega_B, x^0)} B^2,$$

where X_{2s}, Y_{2s} and X_s, Y_s denote the real and imaginary parts, respectively, for $S_{2s}(j\Omega_V)$ and for the expression

$$Q_c(j\Omega_B) + R_c(j\Omega_B)\,[q(A_B, \Omega_B, x^0) + jq'(A_B, \Omega_B, x^0)].$$

The equation written down enables us to determine the dependence of the vibration amplitude A_V on the magnitude of the useful signal x^0 at the input of the nonlinear element for each specified external vibration noise (i.e., for a specified B and Ω_V) by means of the graphic procedure described in §9.2 (Fig. 9.6).

The obtained dependence $A_V(x^0)$ is then substituted in the first formula of (9.27) to obtain the bias function $F^0 = \Phi(x^0)$, which in this case will indeed be the characteristic of the nonlinear element with respect to the useful signal. Its form will depend on the specified amplitude B and frequency Ω_V of the external vibrations and on the system parameters contained in the separated part of the loop (Fig. 9.11b).

In both cases considered above, by carrying out the linearization $F^0 = k_n x^0$ of the characteristic $F^0(x^0)$ or $F^0 = \Phi(x^0)$ of the nonlinear

element relative to the useful signal, we can use the ordinary methods of automatic control theory and the linear equations (9.43) to determine the dependence of all the static and dynamic qualities of the given nonlinear automatic control system (and its stability) on the amplitude B and the frequency Ω_v of the vibration noise.

A linear system would go out of order in the presence of noise whenever the useful signal can in practice not be distinguished against the noise background. However, so long as it differs normally, all the static and dynamic properties of the system with respect to the useful signal, provided the system is linear, remain unchanged. In this case the vibration noise is superimposed as a supplementary error. The situation is entirely different in a nonlinear system, and with it all the qualities and even stability of the system depend so much on the noise (on B and Ω_v), that the system may go out of order for this reason before the useful signal can no longer be distinguished against the noise level. This is very important to consider in practice (see example 2 in §9.6).

From the point of view of simplifying the solution of the problem, it is always necessary to bear in mind the simplified linearization formula (9.45), which makes it possible even in the second of the considered cases to get along without determining the bias function. In this case it is necessary to substitute in (9.45) the value of the vibration amplitude at the input of the nonlinear element A_v, determined in the absence of the useful signal ($x^0 = 0$), by any of the two methods described in §9.1, but for the simpler system equation (9.53). The dependence $A_v(B)$ will in this case be represented by a curve (Fig. 9.11c), unlike the first case (Fig. 9.11a).

Problem 3. In those cases where for some reason the ordinary linearization $F^0 = k_n x^0$ of the characteristic of the nonlinear element

with respect to the useful signal is not desirable (for example, because it has an essential nonlinearity on the interval considered), it is necessary to investigate the nonlinear equation (9.39). For this purpose we can use any nonlinear method of automatic control theory, including the approximate harmonic linearization method developed in the present book. The last method can be used here to solve, for Eq. (9.39), all the problems considered in Chapters 2-7 and §9.1.

Inasmuch as Eq. (9.39) was obtained as a result of harmonic linearization (9.25) of the initial system equation (9.22), we are now dealing with a repeated harmonic linearization of the new nonlinear equation (9.39). The difference between them lies in the fact that the first harmonic linearization of Eq. (9.22) was carried out for the high frequency Ω_v of the vibration noise. The second harmonic linearization of Eq. (9.39) will be carried out for the low frequencies Ω^0 of the oscillations of the useful signal x^0.

To be able to employ such a repeated harmonic linearization, in accordance with Chapter 2, the entire linear portion of the system described by the transfer function (9.51) must not pass the higher harmonics of the useful signal oscillations, meaning also the vibration noise. Consequently, we shall deal here with the first case of problem 2, where the characteristic of the nonlinear element with respect to the useful signal is determined directly by the first formula of (9.27) or by a plot of the type of Fig. 9.5a, where A_v is completely defined for each specified noise and is determined from Fig. 9.11a. Therefore the nonlinear differential equation of the automatic system for a useful signal in the presence of vibration noise is written here in the form

$$Q(p)x^0 + R(p)F^0(x^0) = S_1(p)f_1(t). \qquad (9.54)$$

To investigate the stability of the system and to find the self-

oscillations with respect to the useful signal for $f_1(t) = 0$, assuming that the new nonlinearity $F^0(x^0)$ is symmetrical, we put $x^0 = A^0 \sin \Omega^0 t$ and carry out the repeated harmonic linearization of Eq. (9.54) in the form

$$[Q(p) + R(p) q^0(A^0)] x^0 = 0, \tag{9.55}$$

where

$$q^0 = \frac{4}{\pi A^0} \int_0^{\frac{\pi}{2}} F^0(A^0 \sin \psi) \sin \psi \, d\psi, \quad \psi = \Omega^0 t. \tag{9.56}$$

The function $F^0(x^0)$ is defined in the manner described above in terms of the initial nonlinearity $F(x, px)$. Inasmuch as in Chapter 5 we already have ready-made expressions $F^0(x^0)$ for typical nonlinearities, then, by using these expressions, we can compile ready-made expressions also for $q^0(A^0)$ by using Formula (9.56).

For example, for the ideal relay characteristic $F(x) = c \operatorname{sign} x$ we have in accordance with (5.101) the following expression for F^0

$$F^0 = \frac{2c}{\pi} \arcsin \frac{x^0}{A_n} \quad \text{for} \quad x^0 \leqslant A_n.$$

We obtain therefore from Formula (9.56)

$$q^0 = \frac{8c}{\pi A^0} \int_0^{\frac{\pi}{2}} \arcsin \left(\frac{A^0}{A_n} \sin \psi \right) \sin \psi \, d\psi \qquad (A^0 \leqslant A_n).$$

As a result we ultimately obtain in analogy with Formula (5.80)

$$q^0 = \frac{8c}{\pi A^0} \left[\left(\frac{A^0}{A_n} - \frac{A_n}{A^0} \right) K\left(\frac{A^0}{A_n} \right) + \frac{A_n}{A^0} E\left(\frac{A^0}{A_n} \right) \right] \qquad (A^0 \leqslant A_n),$$

where K and E are the complete elliptic integrals of the first and second kind, the values of which are listed in mathematical tables. An abbreviated table of their values was given at the end of §5.3, where in this case it is necessary to take

$$\alpha = \arcsin \frac{A^0}{A_n}.$$

Analogously, we can obtain analytic expressions $q^0(A^0)$ also for

other typical nonlinearities. In more complicated cases it is possible to use the graphical procedure described in §3.8, which is applied to a plot of the corresponding function $F^0(x^0)$, obtained by the method of Chapter 5 (§§5.6-5.9).

Using Eq. (9.55) we can investigate the stability of a nonlinear system relative to the useful signal by the method described in §2.7. If necessary, the same equation can be used to determine also the self-oscillations $x^0 = A^0 \sin \Omega^0 t$ of the system relative to the useful signal, by using any of the methods given in §2.3, and one can likewise answer all the questions raised in §2.9. Consequently, we obtain here again the possibility of investigating nonlinear two-frequency oscillations with a large difference in frequencies, of which one Ω_v corresponds to forced vibrations and the other Ω^0 corresponds to self-oscillations.

Using the same coefficient $q^0(A^0)$ of repeated harmonic linearization, we can use the equation

$$[Q(p) + R(p) q^0(A^0)] x^0 = S_1(p) f_1(t),$$

which follows from (9.54), to investigate also nonlinear forced oscillations of a system with respect to a slowly varying signal with $f_1(t) = B^0 \sin \Omega_v^0 t$ by the method developed in §9.1. These will be nonlinear two-frequency forced oscillations with a large difference in the frequencies Ω_v and Ω_v^0.

In the present section we have referred throughout to Eq. (9.22) for a nonlinear system of the first class. The same method of solving the problems described here can, however, be extended also to include other classes of systems with several nonlinearities, similar to what was done in Chapter 2.

§9.4. Calculation of the Higher Harmonics of Forced Oscillations

Forced oscillations in the locking mode (i.e., having the same

frequency as the external periodic signal) were determined in first approximation in the sinusoidal form (9.3):

$$x = A_n \sin(\Omega_n t + \varphi). \tag{9.57}$$

Leaving in force all the previous conditions imposed on the system in §9.1, we shall now seek the forced oscillations with allowance for a finite number of external harmonics, in the form*

$$x = x_1 + \sum_{k=2}^{n} x_k, \tag{9.58}$$

where x_1 is the more exact first harmonic

$$x_1 = A_1 \sin(\Omega_n t + \varphi_1), \tag{9.59}$$

and A_1 and φ_1 denote the values of the amplitude and phase shift of the first harmonic of the forced oscillations, made more exact by taking into account the higher harmonics. By x_k we denote in the solution (9.58) the higher harmonics of the forced oscillations, which we shall seek in the form

$$x_k = \delta_k A_n \sin(k\Omega_n t + \varphi_k) \qquad (k = 2, 3, \ldots, n), \tag{9.60}$$

where δ_k is the relative amplitude of the k-th harmonic (the ratio of its amplitude to the amplitude of the first harmonic A_v, calculated in the first approximation). The quantity δ_k will play the role of a small parameter.

We shall specify the system equation as before in the form

$$Q(p)x + R(p)F(x, px) = S(p)f(t), \tag{9.61}$$

and in connection with the possible inclusion of the higher harmonics we can specify a nonsinusoidal external periodic signal (triangular, rectangular, etc.), expanded in a Fourier series in the form

$$f(t) = B_1 \sin \Omega_n t + \sum_{k=2}^{m} B_k \sin(k\Omega_n t + \vartheta_k) \quad (m \leqslant n,\ B_k < B_1). \tag{9.62}$$

We retain in the expansion a number of harmonics m, which is equal to or smaller than the number of the harmonics n considered in the solu-

tion. It is also assumed that the principal role in the makeup of the function (9.62) is played by the first harmonic $B_1 \sin \Omega_v t$.

As was done in the investigation of self-oscillations (§2.2), let us expand the nonlinear function $F(x, px)$ in a Taylor series and make the substitution (9.58), and in view of the smallness of the sum Σx_k of the higher harmonics we confine ourselves to the first terms of the expansion

$$F(x, px) = F(x_1, px_1) + \frac{\partial}{\partial x} F(x_1, px_1) \sum_{k=2}^{n} x_k + \\ + \frac{\partial}{\partial px} F(x_1, px_1) \sum_{k=2}^{n} px_k. \qquad (9.63)$$

We then expand this expression in a Fourier series. In seeking the more exact solution for the self-oscillations in §8.1 we have shown that the first harmonic of the Fourier expansion must be taken over the entire expression (9.63), while the higher harmonics need be taken of only the first term of $F(x_1, px_1)$ in its first approximation, since inclusion of the higher harmonics in the calculation of the highest harmonics would correspond in the solution to terms of higher order of smallness. Thus, the result of the expansion of Expression (9.63) in a Fourier series is represented in analogy with (8.15) in the form

$$F(x, px) = \left(q + \Delta q + \frac{q' + \Delta q'}{\Omega_v} p\right) x_1 + \sum_{k=2}^{n} \frac{1}{\delta_k} \left(q_k + \frac{q_k'}{k\Omega_v} p\right) x_k, \qquad (9.64)$$

where for $x_1 = A_1 \sin \psi$, $\psi = \Omega_v t + \varphi_1$ we have

$$\left.\begin{aligned} q &= \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1 \sin \psi, A_1 \Omega_v \cos \psi) \sin \psi \, d\psi, \\ q' &= \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1 \sin \psi, A_1 \Omega_v \cos \psi) \cos \psi \, d\psi, \end{aligned}\right\} \qquad (9.65)$$

and, in addition, for $x_1 = A_v \sin \psi$, $\psi = \Omega_v t + \varphi$ we have

$$\left.\begin{aligned}\Delta q &= \frac{1}{\pi A_n}\int_0^{2\pi}\left[\frac{\partial}{\partial x}F(x_1, px_1)\sum_{k=2}^{n}x_k + \right.\\ &\qquad \left. + \frac{\partial}{\partial px}F(x_1, px_1)\sum_{k=2}^{n}px_k\right]\sin\psi\, d\psi,\\ \Delta q' &= \frac{1}{\pi A_n}\int_0^{2\pi}\left[\frac{\partial}{\partial x}F(x_1, px_1)\sum_{k=2}^{n}x_k + \right.\\ &\qquad \left. + \frac{\partial}{\partial px}F(x_1, px_1)\sum_{k=2}^{n}px_k\right]\cos\psi\, d\psi,\end{aligned}\right\} \tag{9.66}$$

$$\left.\begin{aligned}q_k &= \frac{1}{\pi A_n}\int_0^{2\pi}F(A_n\sin\psi, A_n\Omega_n\cos\psi)\sin(k\psi - k\varphi + \varphi_k)\,d\psi,\\ q_k' &= \frac{1}{\pi A_n}\int_0^{2\pi}F(A_n\sin\psi, A_n\Omega_n\cos\psi)\cos(k\psi - k\varphi + \varphi_k)\,d\psi.\end{aligned}\right\} \tag{9.67}$$

The principal coefficients q and q' (9.65) have their previous form (9.10). We can therefore use here in its entirety the material of Chapter 3, except that in the formulas there we replace A and Ω by A_1 and Ω_v. The new additions Δq and $\Delta q'$, which give the correction to the first harmonic x_1 resulting from the higher harmonics Σx_k of the sought periodic solution, and also the coefficients q_k and q'_k for the higher harmonics, can be transformed by the procedures used above (see §8.1) to a form more convenient in computation.

Substituting x_k from (9.60) into (9.66) and introducing the quantity $\psi = \Omega_v t + \varphi$, we make use of the substitution

$$\begin{aligned}\sin(k\Omega_n t + \varphi_k) &= \sin(k\psi - k\varphi + \varphi_k) = \sin(k\psi + \psi_k) =\\ &= \cos\psi_k\sin k\psi + \sin\psi_k\cos k\psi,\\ \cos(k\Omega_n t + \varphi_k) &= \cos\psi_k\cos k\psi - \sin\psi_k\sin k\psi,\end{aligned}$$

where

$$\psi = \Omega_n t + \varphi, \qquad \psi_k = \varphi_k - k\varphi. \tag{9.68}$$

With this substitution, the formulas of (9.66) are transformed into

$$\left.\begin{aligned}\Delta q &= \sum_{k=2}^{n}[I_{k_1}\delta_k\cos\psi_k + I_{k_2}\delta_k\sin\psi_k],\\ \Delta q' &= \sum_{k=2}^{n}[I_{k_3}\delta_k\cos\psi_k + I_{k_4}\delta_k\sin\psi_k],\end{aligned}\right\} \tag{9.69}$$

where I_{k_1}, I_{k_2}, I_{k_3}, and I_{k_4} are calculated by means of Formulas (8.13) in which Ω is replaced by Ω_y.

On the other hand, the formulas of (9.67) assume the form

$$\left.\begin{aligned} q_k &= r_k \cos \psi_k + s_k \sin \psi_k, \\ q_k' &= s_k \cos \psi_k - r_k \sin \psi_k, \end{aligned}\right\} \tag{9.70}$$

where

$$\left.\begin{aligned} r_k &= \frac{1}{\pi A_n} \int_0^{2\pi} F(A_n \sin \psi, A_n \Omega_n \cos \psi) \sin k\psi \, d\psi, \\ s_k &= \frac{1}{\pi A_n} \int_0^{2\pi} F(A_n \sin \psi, A_n \Omega_n \cos \psi) \cos k\psi \, d\psi. \end{aligned}\right\} \tag{9.71}$$

We now express all the harmonics of the external periodic signal (9.62) in terms of the corresponding harmonics of the sought periodic solution (9.58). According to (9.60) we have

$$x_k = \delta_k A_n \sin (k\Omega_n t + \varphi_k), \quad px_k = k\delta_k A_n \Omega_n \cos (k\Omega_n t + \varphi_k).$$

Each higher harmonics of the external signal (9.62) can be represented in the form

$$\begin{aligned} B_k \sin (k\Omega_n t + \vartheta_k) &= B_k \sin (k\Omega_n t + \varphi_k - \varphi_k + \vartheta_k) = \\ &= B_k \cos (\varphi_k - \vartheta_k) \sin (k\Omega_n t + \varphi_k) - B_k \sin (\varphi_k - \vartheta_k) \cos (k\Omega_n t + \varphi_k) = \\ &= \frac{B_k \cos (\varphi_k - \vartheta_k)}{\delta_k A_n} x_k - \frac{B_k \sin (\varphi_k - \vartheta_k)}{k\delta_k A_n \Omega_n} px_k. \end{aligned}$$

As a result we write down the external periodic signal (9.62) in the following form

$$\begin{aligned} f(t) = \frac{B_1}{A_1} \left(\cos \varphi_1 - \frac{\sin \varphi_1}{\Omega_n} p\right) x_1 + \\ + \sum_{k=2}^{m} \frac{B_k}{\delta_k A_n} \left[\cos (\varphi_k - \vartheta_k) - \frac{\sin (\varphi_k - \vartheta_k)}{k\Omega_n} p\right] x_k. \end{aligned} \tag{9.72}$$

We now substitute the sought solution (9.58), and also the expression (9.64) for the series expansion of the nonlinearity and the formula for the external periodic signal (9.72), into the specified nonlinear differential equation of the automatic system (9.61). It breaks up into n linear equations: one equation for the determination of the more exact first harmonic with allowance for the higher har-

monics,

$$Q(p)x_1 + R(p)\left(q + \Delta q + \frac{q' + \Delta q'}{\Omega_n} p\right) x_1 = \\ = S(p)\frac{B_1}{A_1}\left(\cos\varphi_1 - \frac{\sin\varphi_1}{\Omega_n} p\right) x_1 \qquad (9.73)$$

and n − 1 equations for the determination of the higher harmonics

$$Q(p)x_k + R(p)\frac{1}{\delta_k}\left(q_k + \frac{q_k'}{k\Omega_n} p\right) x_k = \\ = S(p)\frac{B_k}{\delta_k A_n}\left[\cos(\varphi_k - \vartheta_k) - \frac{\sin(\varphi_k - \vartheta_k)}{k\Omega_n} p\right] x_k \quad (k = 2, 3, \ldots, n). \qquad (9.74)$$

In the case when $m < n$, we have $B_k = 0$ for $k > m$, i.e., in the equations numbered $k > m$ there are no right halves. If the external signal is specified in sinusoidal form, then all Eqs. (9.74) will not have any right halves. On the other hand, if the external periodic signal is not sinusoidal and $m = n$, then all terms are present in all equations of (9.74).

Equations (9.73) and (9.74) can be regarded as homogeneous differential equations, which are linear along the periodic solution. We first find the higher harmonics, assuming that the first approximation for the first harmonic (i.e., the values of A_v and φ), in accord with §9.1, is already known. For this purpose, substituting $p = jk\Omega_v$ in (9.74), we write

$$Q(jk\Omega_n) + R(jk\Omega_n)\frac{1}{\delta_k}(q_k + jq_k') = \\ = S(jk\Omega_n)\frac{B_k}{\delta_k A_n}[\cos(\varphi_k - \vartheta_k) - j\sin(\varphi_k - \vartheta_k)]. \qquad (9.75)$$

Recognizing that in accord with (9.70) and (9.68)

$$q_k + jq_k' = (r_k + js_k)(\cos\psi_k - j\sin\psi_k) = (r_k + js_k)e^{-j\psi_k}e^{jk\varphi},$$

and also that

$$\cos(\varphi_k - \vartheta_k) - j\sin(\varphi_k - \vartheta_k) = e^{-j\varphi_k}e^{j\vartheta_k},$$

we obtain from (9.75)

$$\delta_k e^{j\psi_k} = -\frac{R(jk\Omega_n)}{Q(jk\Omega_n)}(r_k + js_k)e^{jk\varphi} + \\ + \frac{S(jk\Omega_n)}{Q(jk\Omega_n)}\frac{B_k}{A_n}e^{j\vartheta_k} \quad (k = 2, 3, \ldots, n). \qquad (9.76)$$

Inasmuch as Ω_v, B_k, and ϑ_k are specified if the external periodic signal (9.62) is specified, and the values of A_v and φ are known from the previously obtained first approximation (see §9.1), we can determine from (9.76) the relative amplitude δ_k and the phase shift φ_k of each of the higher harmonics of the forced oscillations separately, namely

$$\left.\begin{aligned} \delta_k &= \left| -\frac{R(jk\Omega_n)}{Q(jk\Omega_n)}(r_k + js_k)\, e^{jk\varphi} + \frac{S(jk\Omega_n)}{Q(jk\Omega_n)}\frac{B_k}{A_n} e^{j\vartheta_k} \right|, \\ \varphi_k &= \arg\left[-\frac{R(jk\Omega_n)}{Q(jk\Omega_n)}(r_k + js_k)\, e^{jk\varphi} + \frac{S(jk\Omega_n)}{Q(jk\Omega_n)}\frac{B_k}{A_n} e^{j\vartheta_k} \right] \end{aligned}\right\} \quad (k = 2, 3, \ldots, n). \tag{9.77}$$

In particular, in the case of a sinusoidal external signal, when all the $B_k = 0$, we have in place of (9.77)

$$\left.\begin{aligned} \delta_k &= \left|\frac{R(jk\Omega_n)}{Q(jk\Omega_n)}\right| \sqrt{r_k^2 + s_k^2}, \\ \varphi_k &= \arg \frac{R(jk\Omega_n)}{Q(jk\Omega_n)} + \operatorname{arctg}\frac{s_k}{r_k} + k\varphi, \end{aligned}\right\} (k = 2, 3, \ldots, n). \tag{9.78}$$

We have thus calculated the higher harmonics of the forced oscillations. Let us proceed to obtain a more exact first harmonic by taking into account the higher harmonics, i.e., to determine the quantities A_1 and φ_1. For this purpose we substitute the obtained values of δ_k and φ_k into the formulas (9.69) and calculate the additions Δq and $\Delta q'$ to the harmonic linearization coefficients. To determine the more exact first harmonic of the forced oscillations, we write down on the basis of (9.73) the characteristic equation

$$Q_1(p) + R(p)\left(q + \frac{q'}{\Omega_n} p\right) = S(p)\frac{B_1}{A_1}\left(\cos\varphi_1 - \frac{\sin\varphi_1}{\Omega_n} p\right), \tag{9.79}$$

where

$$Q_1(p) = Q(p) + R(p)\left(\Delta q + \frac{\Delta q'}{\Omega_n} p\right) \tag{9.80}$$

is the polynomial Q(p) which has been corrected by additions to its coefficients. The calculated values of Δq and $\Delta q'$ play here the role of constant numbers along with the system parameters, in terms of which

the coefficients of the polynomial Q(p) are calculated.

Thus, in accordance with (9.79), the determination of the more exact first harmonic of the forced oscillations is based on the complex equation

$$A_1 \frac{Q_1(j\Omega_n) + R(j\Omega_n)(q+jq')}{S(j\Omega_n)} = B_1 e^{-j\varphi_1}. \qquad (9.81)$$

This equation has the same outward appearance as the previous equation (9.14). We can therefore solve it in the previous manner, either graphically by determining the points of intersection of the curve

$$Z(A_1) = A_1 \frac{Q_1(j\Omega_n) + R(j\Omega_n)(q+jq')}{S(j\Omega_n)} \qquad (9.82)$$

with the circle of radius B_1 on the complex plane (Fig. 9.12), or else analytically by reducing it to two equations

$$\left.\begin{aligned} A_1^2 \frac{X_1^2(A_1, \Omega_n) + Y_1^2(A_1, \Omega_n)}{X_s^2(\Omega_n) + Y_s^2(\Omega_n)} &= B_1^2, \\ \varphi_1 = -\operatorname{arctg} \frac{Y_1(A_1, \Omega_n)}{X_1(A_1, \Omega_n)} &+ \operatorname{arctg} \frac{Y_s(\Omega_n)}{X_s(\Omega_n)}, \end{aligned}\right\} \qquad (9.83)$$

as was done in §9.1, but with replacement of the polynomial Q(p) by $Q_1(p)$.

We thus find the first harmonic of the forced oscillations in the more exact form (9.59), and consequently also the entire solution for the forced oscillations in the form

$$x = A_1 \sin(\Omega_n t + \varphi_1) + \sum_{k=2}^{n} \delta_k A_1 \sin(k\Omega_n t + \varphi_k). \qquad (9.84)$$

If necessary, we can now obtain more exact values also for the relative amplitudes δ_k and the phase shifts φ_k of the higher harmonics, calculating them by means of Formulas (9.77) or (9.78), but by substituting in them and in (9.71) not the first approximation A_v and φ, as before, but the more exact values A_1 and φ_1 obtained here. Let us denote these more exact values of δ_k and φ_k by δ'_k and φ'_k. Then the

solution for the forced oscillations will be found in the form

$$x = A_1\left[\sin(\Omega_n t + \varphi_1) + \sum_{k=2}^{n} \delta_k' \sin(k\Omega_n t + \varphi_k')\right]. \qquad (9.85)$$

In general we can proceed to develop successive approximations which, however, are hardly necessary for practical calculations.

Let us consider in particular the case most widely used in practice, when the system contains one single-valued odd-symmetry nonlinearity F(x), for which the most significant of the higher harmonics is the third. Assuming that the external periodic signal has the form

$$f(t) = B_1 \sin \Omega_n t + B_3 \sin(3\Omega_n t + \vartheta_3), \qquad (9.86)$$

we shall seek the forced oscillations of the system, described by nonlinear differential equation (9.61), in the form

$$x = A_1 \sin(\Omega_n t + \varphi_1) + \delta_3 A_n \sin(3\Omega_n t + \varphi_3). \qquad (9.87)$$

The formulas for the calculation of the third harmonics of the forced oscillations will be here, in accordance with (9.77) and (9.71),

$$\left.\begin{aligned} \delta_3 &= \left| -\frac{R(j3\Omega_n)}{Q(j3\Omega_n)} r_3 e^{j\vartheta_3} + \frac{S(j3\Omega_n)}{Q(j3\Omega_n)} \frac{B_3}{A_n} e^{j\vartheta_3} \right|, \\ \varphi_3 &= \arg\left[-\frac{R(j3\Omega_n)}{Q(j3\Omega_n)} r_3 e^{j3\varphi} + \frac{S(j3\Omega_n)}{Q(j3\Omega_n)} \frac{B_3}{A_n} e^{j\vartheta_3} \right]. \end{aligned}\right\} \qquad (9.88)$$

In the case of a sinusoidal external signal ($B_3 = 0$) we get

$$\delta_3 = r_3 \left| \frac{-R(j3\Omega_n)}{Q(j3\Omega_n)} \right|, \quad \varphi_3 = \arg \frac{-R(j3\Omega_n)}{Q(j3\Omega_n)} + 3\varphi, \qquad (9.89)$$

where

$$r_3 = \frac{4}{\pi A_n} \int_0^{\frac{\pi}{2}} F(A_n \sin\psi) \sin 3\psi \, d\psi, \qquad (9.90)$$

with A_v and φ now assumed known from the first approximation (§9.1).

In Eq. (9.81) we have for the more exact first harmonic in the present problem $q' = 0$, i.e.,

$$A_1 \frac{Q_1(j\Omega_n) + R(j\Omega_n) q(A_1)}{S(j\Omega_n)} = B_1 e^{-j\varphi_1}, \qquad (9.91)$$

where

$$Q_1(j\Omega_n) = Q(j\Omega_n) + R(j\Omega_n)(\Delta q + j\Delta q'). \qquad (9.92)$$

In this case the formulas for the additions Δq and $\Delta q'$ become much more simple than in the general case. In place of the expressions (9.69) we now obtain

$$\left.\begin{aligned} \Delta q &= h_3\delta_3 \cos(\varphi_3 - k\varphi), \\ \Delta q' &= 3r_3\delta_3 \sin(\varphi_3 - k\varphi), \end{aligned}\right\} \qquad (9.93)$$

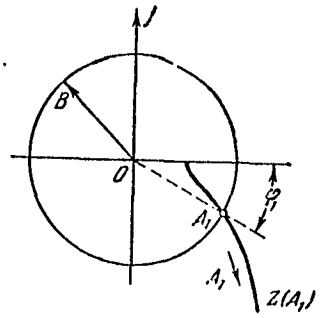


Fig. 9.12

where

$$h_3 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \frac{d}{dx} F(A_n \sin\psi) \sin 3\psi \sin\psi\, d\psi. \qquad (9.94)$$

For r_3 and h_3 we can use the ready-made expressions given in §8.3, and for $q(A_1)$ we can use the material of Chapter 3. The result will be the more exact solution (9.87) for the forced oscillations. We can use here both the graphic method (Fig. 9.12), plotting the curve

$$Z(A_1) = A_1 \frac{Q_1(j\Omega_n) + R(j\Omega_n)q(A_1)}{S(j\Omega_n)}, \qquad (9.95)$$

and the analytic method using two equations (9.83), which in this case can be written in the more expanded form

$$\left.\begin{aligned} A_1^2 \frac{[X_{Q_1}(\Omega_n) + X_R(\Omega_n)q(A_1)]^2 + [Y_{Q_1}(\Omega_n) + Y_R(\Omega_n)q(A_1)]^2}{X_S^2(\Omega_n) + Y_S^2(\Omega_n)} &= B_1^2, \\ \psi = -\operatorname{arctg}\frac{Y_{Q_1}(\Omega_n) + Y_R(\Omega_n)q(A_1)}{X_{Q_1}(\Omega_n) + X_R(\Omega_n)q(A_1)} + \operatorname{arctg}\frac{Y_S(\Omega_n)}{X_S(\Omega_n)}, \end{aligned}\right\} \qquad (9.96)$$

where X_{Q_1}, X_R, Y_{Q_1}, and Y_R are the real and imaginary parts for the expressions $Q_1(j\Omega_v)$ and $R(j\Omega_v)$, respectively.

If necessary we can refine also the third harmonic, writing down the solution in the form

$$x = A_1[\sin(\Omega_n t + \varphi_1) + \delta_3' \sin(3\Omega_n t + \varphi_3')], \qquad (9.97)$$

where δ'_3 and φ'_3 are determined by the same formulas (9.88) and (9.89), but in which we substitute, as we do also in (9.90), the more exact

values A_1 and φ_1 in place of A_v and φ.

§9.5. Examples of Determination of Symmetrical One-Frequency Forced Oscillations

Let us illustrate by means of examples the use of the methods for the determination of forced one-frequency oscillations in nonlinear systems.

Example 1. Relay system of second order with delay. The system consists of a linear part and of a nonlinear (relay) element (Fig. 9.13). Assume that the input to the system is an external sinusoidal signal

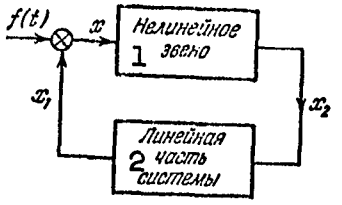


Fig. 9.13. 1) Nonlinear element; 2) linear part of the system.

$$f(t) = B \sin \Omega_0 t. \qquad (9.98)$$

The equation of the linear part has the form

$$(T_1 p + 1) p x_1 = k x_2. \qquad (9.99)$$

The equation of the relay element is the nonlinear function

$$x_2 = F_\tau(x) = e^{-\tau p} F(x), \qquad (9.100)$$

which is specified in the form of an ideal relay characteristic F(x) (Fig. 9.14a), but with the output of the nonlinear element subject to a time delay τ.

The forced oscillations of the input to the nonlinear element will be sought in the form

$$x = A_0 \sin(\Omega_0 t + \varphi). \qquad (9.101)$$

Assuming that the condition for the existence of a single frequency periodic solution (locking condition) are satisfied, let us determine the amplitude A_v and the phase shift φ of the forced oscillations.

Substituting (9.100) in (9.99) with allowance for the fact that

$$x_1 = f(t) - x,$$

we obtain the system equation

$$(T_1p+1)px+kF_\tau(x)=(T_1p+1)pf(t). \qquad (9.102)$$

We represent the sinusoidal external signal (9.98) in the form

$$f(t)=B\sin[(\Omega_\text{н}t+\varphi)-\varphi]=$$
$$=B\cos\varphi\sin(\Omega_\text{н}t+\varphi)-B\sin\varphi\cos(\Omega_\text{н}t+\varphi).$$

Recognizing that in accordance with (9.101) we have

$$px=A_\text{н}\Omega_\text{н}\cos(\Omega_\text{н}t+\varphi),$$

we ultimately obtain

$$f(t)=\frac{B}{A_\text{н}}\left(\cos\varphi-\frac{\sin\varphi}{\Omega_\text{н}}p\right)x. \qquad (9.103)$$

Substituting (9.103) in (9.102), we obtain a homogeneous nonlinear system equation for the variable $\underline{x}$

$$(T_1p+1)\left[1-\frac{B}{A_\text{н}}\left(\cos\varphi-\frac{\sin\varphi}{\Omega_\text{н}}p\right)\right]px+kF_\tau(x)=0. \qquad (9.104)$$

Harmonic linearization of the ideal relay characteristic in accordance with Formula (3.14) and with allowance for the time delay yields

$$F_\tau(x)=-\frac{4c}{\pi A_\text{н}}e^{-\tau p}x. \qquad (9.105)$$

The characteristic equation, corresponding to the equation (9.104) with allowance for (9.105) assumes the form

$$(T_1p+1)\left[1-\frac{B}{A_\text{н}}\left(\cos\varphi-\frac{\sin\varphi}{\Omega_\text{н}}p\right)\right]p+\frac{4kc}{\pi A_\text{н}}e^{-\tau p}=0. \qquad (9.106)$$

Substituting in (9.106) the imaginary value $p = j\Omega_\text{v}$ and taking into account the fact that

$$\cos\varphi-j\sin\varphi=e^{-j\varphi},$$

we obtain

$$A_\text{н}-\frac{4kce^{-j\tau\Omega_\text{н}}}{\pi(T_1\Omega_\text{н}^2-j\Omega_\text{н})}=Be^{-j\varphi}. \qquad (9.107)$$

Let us specify the numerical values of the system parameters, and also the amplitude and frequency of the external signal: $k = 10\ \text{sec}^{-1}$, $c = 10$ volts, $T_1 = 0.01$ sec, $\tau = 0.01$ sec, $\Omega_\text{v} = 10\ \text{sec}^{-1}$, $B = 20$ v.

Substituting the assumed values of the parameters in (9.107), we

obtain

$$A_s - 2{,}5 - j \cdot 12{,}3 = 20e^{-j\varphi}. \qquad (9.108)$$

Let us employ the graphic method described in §9.1. We draw on the complex plane (Fig. 9.14b) a circle with radius R = 20, representing the right half of (9.108), and a line corresponding to the left half of (9.108). On the line we mark the values of the amplitudes of the forced oscillations A_v. The points where the circle and the line cross yield the solution for A_v and φ. We note that the phase shift angle is positive in a clockwise direction, since the right half of (9.108) represents a vector which produces rotation by an angle φ in the opposite direction corresponding with the usual convention for positive angles. As can be seen from Fig. 9.14b, in this case we have $A_v = 18.2$ volts and $\varphi = 38^\circ$. We can conclude from this construction that for the assumed frequency $\Omega_v = 10$ sec^{-1} the lowest threshold amplitude of external signal is $B_{por} = 12.3$ volts, i.e., it is equal to the radius of the circle tangent to the line A_v.

For the given values of the parameters, there is only one point of intersection of the circle and the line with positive values of the amplitude, i.e., we have one periodic solution. At other values of the parameters, however, we can have in principle two periodic solutions, since circles and lines having two points of intersection are possible when $A_v > 0$.

Drawing several circles for the values of the amplitude of the external signal, other than B = 20, we can determine graphically the functions $A_v(B)$ and $\varphi(B)$. Analogously, drawing a series of straight lines for different constant values of Ω_v, τ, and $\underline{k}$, we can determine the functions $A_v(\Omega_v)$, $\varphi(\Omega_v)$, $A_v(\tau)$, $\varphi(\tau)$, $A_v(k)$, and $\varphi(k)$.

Plots of the foregoing functions based on the graphic solution are shown in Figs. 9.15a, b, c. The auxiliary construction for the de-

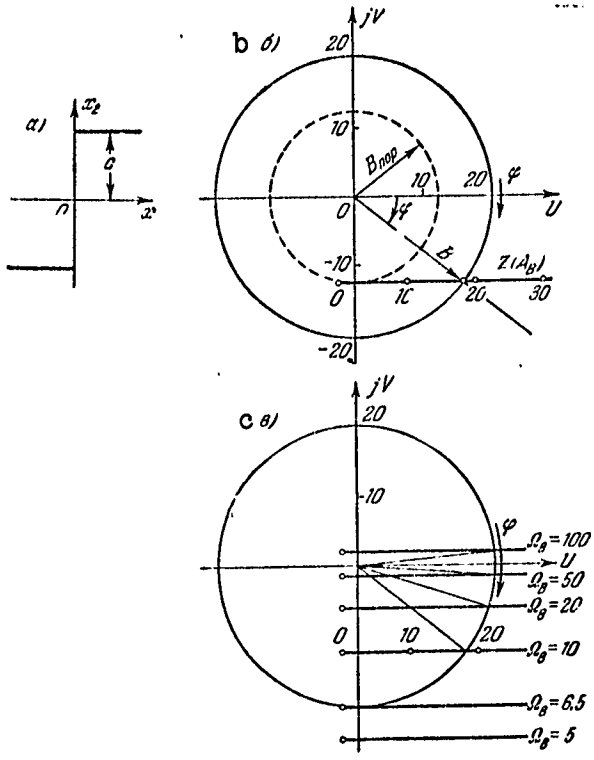


Fig. 9.14

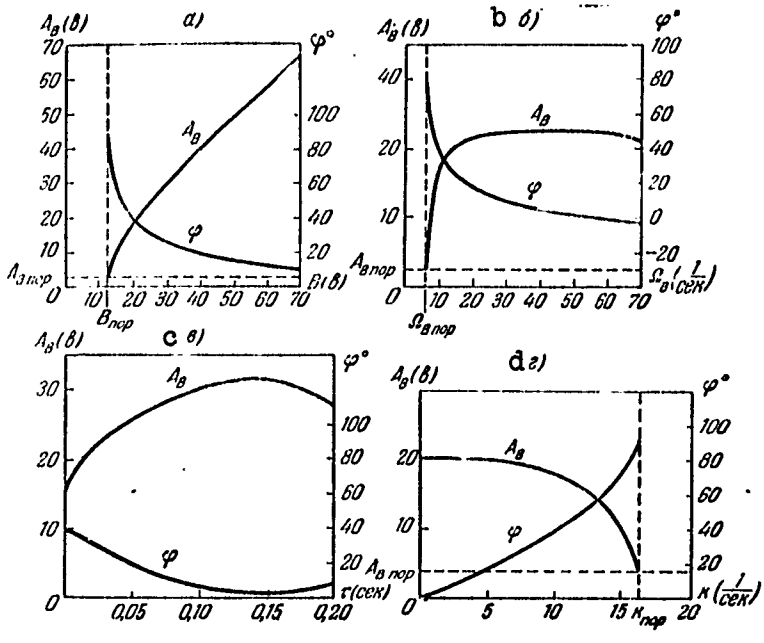


Fig. 9.15

termination of $A_V(\Omega_V)$ and $\varphi(\Omega_V)$ is shown in Fig. 9.14c.

Let us illustrate, using the same example of a relay system with

delay, the use of the analytic method. For this purpose we set up two equations by means of Eq. (9.107) with two unknown A_v and φ: the first equation follows from the equality of the moduli of the complex expressions of the left and right halves of (9.107), and the second expression follows from the equality of the arguments.

Substituting in (9.107) the value

$$e^{-j\tau\Omega_n} = \cos\tau\Omega_n - j\sin\tau\Omega_n$$

and carrying out the transformations, we obtain

$$A_n - \frac{4kc}{\pi}\left(\frac{\alpha - j\beta}{\gamma}\right) = B e^{-j\varphi}, \qquad (9.109)$$

where

$$\alpha = T_1\Omega_n^2\cos\tau\Omega_n + \Omega_n\sin\tau\Omega_n,$$
$$\beta = T_1\Omega_n^2\sin\tau\Omega_n - \Omega_n\cos\tau\Omega_n,$$
$$\gamma = T_1^2\Omega_n^4 + \Omega_n^2.$$

From Eq. (9.109) we obtain the corresponding expressions resulting from the equality of the moduli and arguments:

$$\left(A_n - \frac{4kc\alpha}{\pi\gamma}\right)^2 + \left(\frac{4kc\beta}{\pi\gamma}\right)^2 = B^2, \qquad (9.110)$$

$$\varphi = -\operatorname{arctg}\frac{4kc\beta}{\pi\gamma A_n - 4kc\alpha}. \qquad (9.111)$$

Equation (9.110) can be solved with respect to the amplitude of the forced oscillations. The result of the solution yields

$$A_n = \frac{4kc\alpha}{\pi\gamma} \pm \sqrt{B^2 - \left(\frac{4kc\beta}{\pi\gamma}\right)^2}. \qquad (9.112)$$

As can be seen from (9.112), forced oscillations are possible only if

$$B > \frac{4kc\beta}{\pi\gamma},$$

i.e., when the values of the amplitude are real.

Calculating the amplitudes and phases of the forced oscillations by means of (9.112) and (9.111) with the assumed values of the system parameters and of the external signal amplitude, we obtain

$$A_v = 18.2 \text{ v}, \quad \varphi = 38^\circ.$$

Inasmuch as in this system two forced periodic solutions are possible for certain values of the parameters, frequency, and amplitude of the external signal, the solution obtained for $\underline{x}$ must be checked for stability. Let us do this by the method of averaging the periodically varying coefficients over the linearized equation in terms of small deviations from the periodic solution (see §9.1).

We turn to the initial equation (9.102). The periodic solution for Eq. (9.102) has already been obtained. We introduce a small deviation from the investigated periodic solution for the variable $\underline{x}$:

$$x = x^* + \Delta x, \text{ where } x^* = A_v \sin(\Omega_v t + \varphi).$$

Substituting the value of $\underline{x}$ in (9.102) and expanding the function $F_\tau(x)$ in a Taylor series and then neglecting terms of higher orders of smallness, we obtain

$$(T_1 p + 1) p (x^* + \Delta x) + k\left[F_\tau(x^*) + \left(\frac{\partial F_\tau}{\partial x}\right)^* \Delta x\right] = (T_1 p + 1) p f(t). \quad (9.113)$$

Since the initial equation is satisfied by the value $x = x^*$ in the periodic solution, we have for the steady-state oscillations

$$(T_1 p + 1) p x^* + k F_\tau(x^*) = (T_1 p + 1) p f(t). \quad (9.114)$$

Subtracting (9.114) from (9.113) and representing the function F_τ in the form

$$F_\tau(x) = F(x) e^{-\tau p},$$

we obtain a linear equation in the small deviations from the periodic solution:

$$(T_1 p + 1) p \Delta x + k e^{-\tau p}\left(\frac{\partial F}{\partial x}\right)^* \Delta x = 0. \quad (9.115)$$

The asterisk denotes that after taking the derivative it is necessary to make the substitution

$$x = A_n \sin(\Omega_n t + \varphi).$$

The periodically varying coefficient $(\partial F/\partial x)^*$, which is contained in (9.115), will be replaced by its average over the period

$$\varkappa = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial F}{\partial x}\right)^* d\psi, \quad \psi = \Omega_v t + \varphi. \tag{9.116}$$

With allowance for (9.116), Eq. (9.115) assumes the form

$$[(T_1 p + 1)p + k\varkappa e^{-\tau p}]\Delta x = 0. \tag{9.117}$$

Let us calculate the value of the coefficient κ. The derivative $\partial F/\partial x$, in accordance with Fig. 9.14, is an instantaneous pulse with area 2c. Consequently, the integrand in (9.116) is equal to zero everywhere except at the point x = 0. We represent the integral (9.116) in the form

$$\varkappa = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\partial F}{\partial x}\right)^* d\psi = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{dF}{dx}\right)^* \frac{dx}{\frac{dx}{d\psi}}.$$

Since we have when $x = A_v \sin\psi$ and $\psi = \Omega_v t + \varphi$

$$\frac{dx}{d\psi} = A_v \cos\psi = \sqrt{A_v^2 - x^2},$$

we obtain, by changing the integration limits,

$$\varkappa = \frac{1}{\pi}\int_{-A_v}^{A_v}\left(\frac{dF}{dx}\right)^* \frac{dx}{\sqrt{A_v^2 - x^2}}.$$

When x = 0 we have $\sqrt{A_v^2 - x^2} = A_v$, and the integral of dF/dx with respect to dx is the area of the pulse, equal to 2c. Consequently, we obtain ultimately

$$\varkappa = \frac{2c}{\pi A_v}. \tag{9.118}$$

Substituting the value of κ from (9.118) into (9.117) we obtain the equation

$$[(T_1 p + 1)p + \frac{2kc}{\pi A_v} e^{-\tau p}]\Delta x = 0. \tag{9.119}$$

The characteristic polynomial for Eq. (9.119) can be written in the form

$$L(p) = \frac{\pi A_v}{2kc}(T_1 p + 1)p + e^{-\tau p}. \tag{9.120}$$

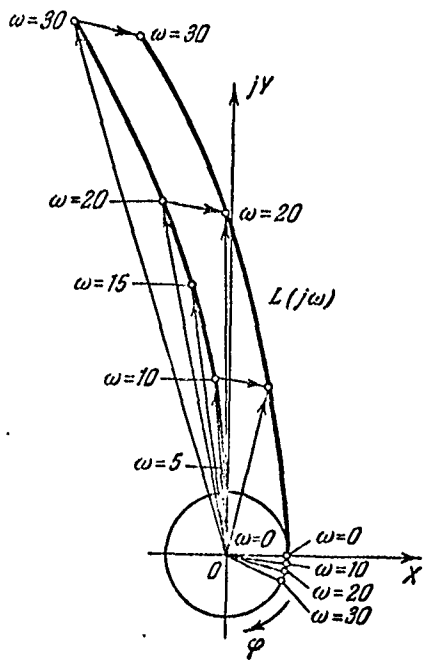


Fig. 9.16

An analytic expression for the Mikhaylov curve will be

$$L(j\omega)=\frac{\pi A_v}{2kc}(-T_1\omega^2+j\omega)+e^{-j\tau\omega}. \quad (9.121)$$

After substituting the numerical values of the parameters in the value of the amplitude A_v we obtain

$$L(j\omega)=-0{,}286\cdot 10^{-2}\omega^2+ j\cdot 0{,}286\,\omega+e^{-j0{,}01\omega}. \quad (9.122)$$

In accordance with (9.122), let us construct the hodograph of the vector $L(j\omega)$. For this purpose we plot the hodograph of the vector corresponding to the first two terms of (9.122), and draw a circle with unit radius corresponding to the third term. As a result of addition of the corresponding vectors for the different frequencies we obtain the Mikhaylov curve (Fig. 9.16). The resultant Mikhaylov curve corresponds to a second-order stable system with delay, and consequently the periodic solution is stable.

Example 2. Relay system of third order. Assume that the linear part of the relay system of Fig. 9.13 is described by the equation

$$(T_1p+1)(T_2p+1)px_1=kx_2, \quad (9.123)$$

and that the relay element has a coordinate delay determined by the characteristics shown in Fig. 9.17a. The external signal is as before sinusoidal and is applied to the input of the nonlinear element:

$$f(t)=B\sin\Omega_0 t.$$

We shall seek the forced oscillations of the variable x as before in the form

$$x=A_0\sin(\Omega_0 t+\varphi).$$

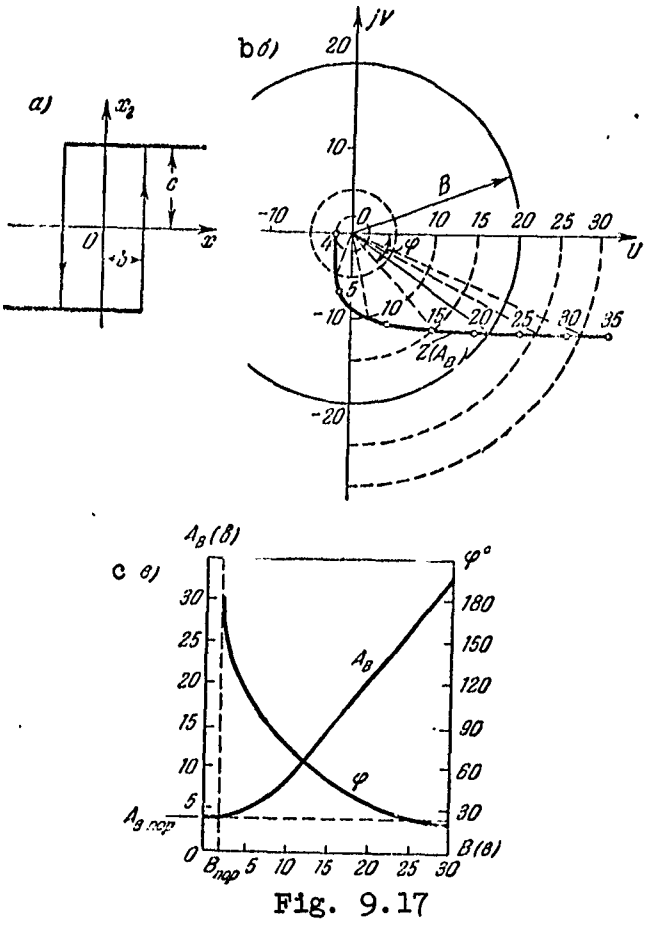


Fig. 9.17

For a relay element of the type considered, the harmonic linearization reduces to a replacement of the nonlinear function by means of a linear relation

$$x_2 = F(x) = \left[q(A_n) + \frac{q'(A_n)}{\Omega_n} p\right] x,$$

where in accord with Formulas (3.9) and (3.10) the values of the harmonic linearization coefficients are

$$q(A_n) = \frac{4c}{\pi A_n} \sqrt{1 - \frac{b^2}{A_n^2}}, \quad q'(A_n) = -\frac{4cb}{\pi A_n^2} \text{ for } A_n \geqslant b.$$

Assuming that the locking condition is satisfied, let us determine the amplitude A_v and the phase shift φ of the forced oscillations for the variable

$$x = f(t) - x_1.$$

Substituting

$$x_2 = F(x)$$

in the equation of the linear portion (9.123) and recognizing that

$$x_1 = f(t) - x,$$

we obtain the system equation

$$(T_1 p + 1)(T_2 p + 1) p x + kF(x) = (T_1 p + 1)(T_2 p + 1) p f(t).$$

Replacing f(t) by its value (9.103), we get

$$(T_1 p + 1)(T_2 p + 1) p \left[1 - \frac{B}{A_n}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_n} p\right)\right] x + kF(x) = 0. \qquad (9.124)$$

The characteristic equation corresponding to (9.124) will be

$$(T_1 p + 1)(T_2 p + 1) p \left[1 - \frac{B}{A_n}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_n} p\right)\right] + k\left(q + \frac{q'}{\Omega_n} p\right) = 0, \qquad (9.125)$$

where the nonlinearity is harmonically linearized by the formula

$$F(x) = \left[q(A_n) + \frac{q'(A_n)}{\Omega_n} p\right] x.$$

Substituting into (9.125) the imaginary value $p = j\Omega_V$ and recognizing that

$$\cos\varphi - j\sin\varphi = e^{-j\varphi},$$

we obtain

$$A_n \left\{1 - \frac{k[q(A_n) + jq'(A_n)]}{(T_1 + T_2)\Omega_n^2 - j(1 - T_1 T_2 \Omega_n^2)\Omega_n}\right\} = Be^{-j\varphi}. \qquad (9.126)$$

Let us specify the numerical values of the system parameters, and also the frequency and amplitude of the external signal: $k = 10\ \text{sec}^{-1}$, $c = 10$ v, $b = 4$ v, $T_1 = 0.01$ sec, $T_2 = 0.02$ sec, $B = 20$ v, and $\Omega_V = 10\ \text{sec}^{-1}$.

Substituting the numerical values in (9.126) and taking the values of $q(A_V)$ and $q'(A_V)$ into account, we obtain

$$A_n - \frac{(3{,}63\sqrt{A_n^2 - 16} + 23{,}7) + j(11{,}8\sqrt{A_n^2 - 16} - 0{,}72)}{A_n} = 20e^{-j\varphi}. \qquad (9.127)$$

Let us employ the graphic method for the determination of A_V and φ. For this purpose we draw on the complex plane (Fig. 2.17b) a circle

with radius R = 20, representing the right half of (9.127), and specifying different values of A_v let us plot the curve corresponding to the left half of (9.127). The point where the circle R = B = 20 crosses the curve $Z(A_v)$ yields the solution of Eq. (9.127), namely A_v = 21 v and $\varphi = 35^\circ$.

Carrying out similar solutions for different values of the amplitude of the forced oscillations, i.e., drawing a series of circles with different radii R = B, we obtain the functions $A_v(B)$ and $\varphi(B)$, represented by the curves of Fig. 9.17c. Similar relations can be obtained also for the external signal frequency Ω_v and for any other system parameter. To obtain these relationships it is necessary to vary one of the parameters while keeping the remaining ones constant, and plot the A_v curves. It is sufficient here to plot only those portions of the curve that cross the circle corresponding to the specified sinusoidal external signal.

An investigation of the stability of the forced oscillations is unnecessary in this case, since we have here a single periodic solution.

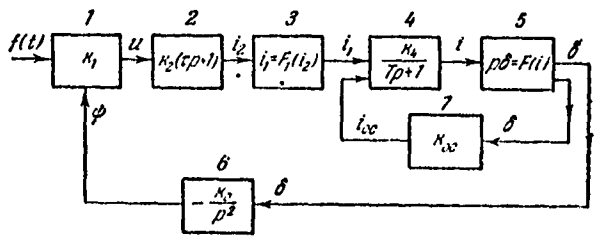


Fig. 9.18

Example 3. System for angular stabilization. By way of the third example of determining the forced oscillations, let us consider a system for the stabilization of the angle of an object in space, represented by the block diagram of Fig. 9.18. The sensitive element of the system is a gyroscope 1, the potentiometer of which yields a voltage u

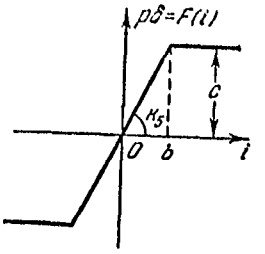


Fig. 9.19

which is applied through filter 2 to the first amplifier 3. The resultant current i_1 is applied to the input of the second amplifier 4, which feeds the rudder engine 5. The rudder engine acts on the object 6 as a result of a change in the rudder deflection δ. The system is provided with an additional proportional feedback loop 7, which encloses the amplifier and the rudder engine.

Let us consider first the case when only one nonlinearity is taken into consideration, namely the saturation in the rudder engine $p\delta = F(i)$, assuming $F_1(i_2) = k_3 i_2$.

Let us assume that an external harmonic signal is applied to the sensitive element of the system (to the gyroscope):

$$f(t) = B \sin \Omega_* t. \tag{9.128}$$

In accordance with the block diagram, the equations for the elements will be as follows:

1) gyroscope

$$u = k_1 [f(t) - \psi]; \tag{9.129}$$

2) filter

$$i_2 = k_2 (\tau p + 1) u; \tag{9.130}$$

3) first amplifier

$$i_1 = k_3 i_2; \tag{9.131}$$

4) second amplifier

$$(Tp + 1) i = k_4 (i_1 - i_{o.c}); \tag{9.132}$$

5) object

$$p^2 \psi = -k_6 \delta; \tag{9.133}$$

6) supplementary feedback

$$i_{o.c} = k_{o.c} \delta. \tag{9.134}$$

The nonlinear link – the control-surface servo (rudder engine) has

the nonlinear static characteristic pδ = F(i) (Fig. 9.19).

We shall determine the forced oscillations for an input to the nonlinear element in the form

$$i = A_в \sin(\Omega_в t + \varphi). \tag{9.135}$$

Employing harmonic linearization for the nonlinear element, we obtain

$$F(i) = q(A_в) i, \tag{9.136}$$

where in accordance with (3.19) the value of the harmonic linearization coefficient is

$$q(A_в) = \frac{2k_з}{\pi}\left(\arcsin\frac{b}{A_в} + \frac{b}{A_в}\sqrt{1-\frac{b^2}{A_в^2}}\right) \quad \text{for } A_в \geqslant b,$$
$$q(A_в) = k_з \quad \text{for } A_в \leqslant b.$$

From the linear element equations (9.129)-(9.134) and from the linearized equation of the nonlinear element (9.136) we obtain the following system equation for the variable i:

$$(Tp^4 + p^3 + k_з k_{о.с} q p^2 + k k_0 \tau q p + k k_0 q) i = k(\tau p + 1) p^3 f(t), \tag{9.137}$$

where $k = k_1 k_2 k_3 k_4$.

To reduce Eq. (9.137) to a homogeneous one, let us express the external signal in terms of the variable i. Representing f(t) in accord with (9.128) in the form

$$f(t) = B\sin[(\Omega_в t + \varphi) - \varphi] =$$
$$= B\cos\varphi\sin(\Omega_в t + \varphi) - B\sin\varphi\cos(\Omega_в t + \varphi)$$

and taking into consideration the fact that

$$i = A_в \sin(\Omega_в t + \varphi) \text{ and } pi = A_в \Omega_в \cos(\Omega_в t + \varphi),$$

we obtain

$$f(t) = \frac{B}{A_в}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_в} p\right) i.$$

Substituting the value of f(t) in (9.137), we obtain the homogeneous equation

$$(Tp^4 + p^3 + k_з k_{о.с} q p^2 + k k_0 \tau q p + k k_0 q) i =$$
$$= k(\tau p + 1) p^3 \frac{B}{A_в}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_в} p\right) i. \tag{9.138}$$

To determine the forced oscillations, let us substitute into the

characteristic equation, corresponding to (9.138), the value $p = j\Omega_v$. As a result, recognizing that

$$\cos\varphi - j\sin\varphi = e^{-j\varphi}$$

we obtain the equation

$$\frac{A_{\text{в}}(T\Omega_{\text{в}}^4 - j\Omega_{\text{в}}^3 - k_4k_{\text{о.с}}q\Omega_{\text{в}}^2 + jkk_0\tau q\Omega_{\text{в}} + kk_0q)}{-k(\tau j\Omega_{\text{в}} + 1)j\Omega_{\text{в}}^3} = Be^{-j\varphi}.$$

From the condition that the moduli of the right and left halves of this complex expression must be equal, we obtain a formula relating the amplitude A_v of the forced oscillations with the amplitude B of the external signal, the frequency Ω_v, and the system parameters:

$$A_{\text{в}}^2\frac{[T\Omega_{\text{в}}^4 + (kk_0 - k_4k_{\text{о.с}}\Omega_{\text{в}}^2)\,q(A_{\text{в}})]^2 + [kk_0\tau q(A_{\text{в}})\Omega_{\text{в}} - \Omega_{\text{в}}^3]^2}{k^2\Omega_{\text{в}}^6(\tau^2\Omega_{\text{в}}^2 + 1)} = B^2; \qquad (9.139)$$

from which we can obtain also a relation for the determination of the phase shift φ. In practice, however, it is frequently of interest to determine only the amplitude of the forced oscillations.

To determine the dependence of the amplitude of the forced oscillations on the amplitude and frequency of the external signal, and also on the system parameters, we shall assume the following numerical values for the parameters: $k = k_1k_2k_3k_4 = 18$ mA/deg, $kk_0 = 9$ mA/sec^{2}deg, $k_4k_{\text{о.с}} = 1.8$ mA/deg, $T = 0.02$ sec, $\tau = 0.5$ sec, $\Omega_v = 12\text{-}95$ sec^{-1}, and $k_5 = 4$ deg/sec·mA.

For the assumed values of the parameters we shall have in Formula (9.139)

$$kk_0 \ll k_4k_{\text{о.с}}\Omega_{\text{в}}^2, \quad kk_0\tau q\Omega_{\text{в}} \ll \Omega_{\text{в}}^3, \quad 1 \ll \tau^2\Omega_{\text{в}}^2.$$

Then Eq. (9.139) can be approximately rewritten in the form

$$\frac{A_{\text{в}}}{k\tau\Omega_{\text{в}}^3}\sqrt{[T\Omega_{\text{в}}^2 - k_4k_{\text{о.с}}q(A_{\text{в}})]^2 + \Omega_{\text{в}}^2} = B. \qquad (9.140)$$

Formula (9.140) was used to calculate the change in the amplitude of the forced oscillations as a function of the amplitude and frequency of the external signal and of the gain k; the results of the calcula-

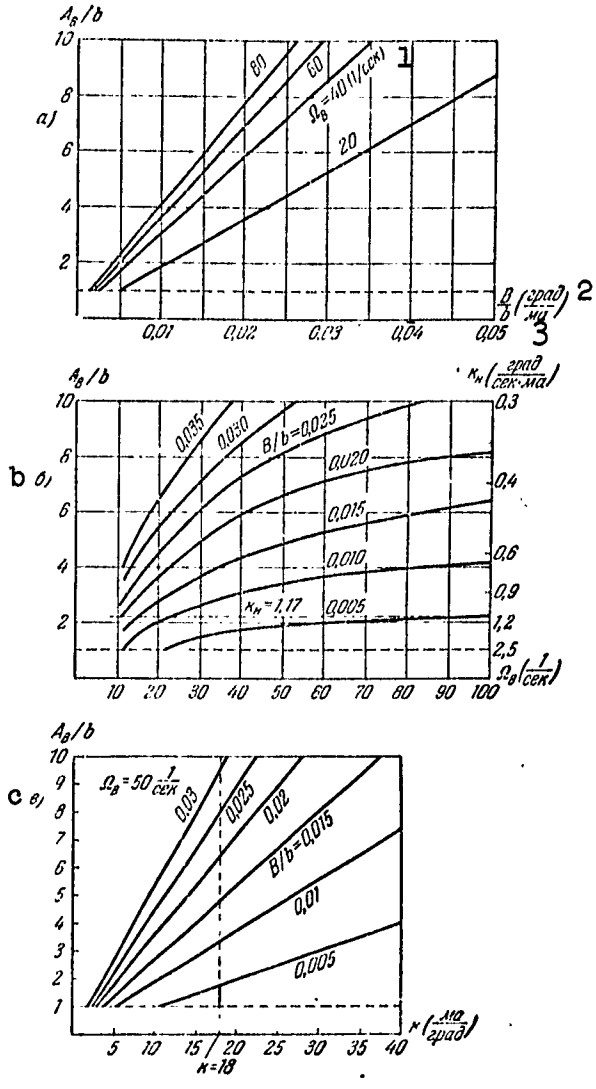


Fig. 9.20. 1) sec; 2) deg; 3) mA.

tions are plotted in Fig. 9.20. The amplitudes are given in the relative values A_v/b and B/b. The scale for the quantity k_n shown in Fig. 9.20b will be used later on in example 2 of §9.6, in the investigation of the dynamic properties of a system with respect to a useful signal under conditions of vibration interference.

For the same system for the stabilization of the angle of an object in space (Fig. 9.18), let us examine the determination of the forced oscillations with allowance for two nonlinearities. Assume that in addition to saturation in the rudder engine we also have a charac-

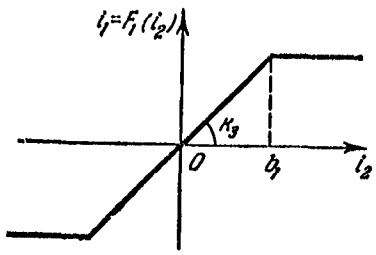


Fig. 9.21

teristic with saturation in the first amplifier.

In this case we have in place of Eq. (9.131) for the first amplifier, the harmonically linearized equation

$$i_1 = F_1(i_2) = q_1(A_{1n})\, i_2, \qquad (9.141)$$

where

$$q_1(A_{1n}) = \frac{2k_3}{\pi}\left(\arcsin\frac{b_1}{A_{1n}} + \frac{b_1}{A_{1n}}\sqrt{1-\frac{b_1^2}{A_{1n}^2}}\right),$$

A_{1_v} is the amplitude of the forced oscillations for the current i_2, while k_3 and b_1 are determined by the nonlinear static characteristic of the first amplifier (Fig. 9.21).

The system equation now assumes in accord with (9.138) the form

$$(Tp^4 + p^3 + k_4 k_{o.c} q p^2 + k^* k_0 \tau q q_1 p + k^* k_0 q q_1)\, i = \\ = k^* q_1 (\tau p + 1) p^3 \frac{B}{A_n}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_n} p\right) i, \qquad (9.142)$$

where $k^* = k_1 k_2 k_4$.

The characteristic equation corresponding to the differential equation (9.142) will be

$$\frac{T}{q_1} p^4 + \frac{1}{q_1} p^3 + k_4 k_{o.c} \frac{q}{q_1} p^2 + k^* k_0 \tau q p + k^* k_0 q = \\ = k^* (\tau p + 1) p^3 \frac{B}{A_n}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_n} p\right). \qquad (9.143)$$

Inserting $p = j\Omega_v$ in (9.143) we obtain

$$\frac{T}{q_1}\Omega_n^4 - j\frac{1}{q_1}\Omega_n^3 - k_4 k_{o.c}\frac{q}{q_1}\Omega_n^2 + jk^* k_0 \tau q \Omega_n + k^* k_0 q = \\ = \Omega_n^3 (k^* \tau \Omega_n^2 - jk^* \Omega_n)\frac{B}{A_n} e^{-j\varphi}. \qquad (9.144)$$

From the condition that the moduli of the complex quantities must be equal we obtain on the basis of (9.144)

$$\frac{A_n^2 (T\Omega_n^4 - k_4 k_{o.c} q \Omega_n^2 + k^* k_0 q q_1)^2 \frac{1}{q_1^2} + (k^* k_0 \tau q_1 q \Omega_n - \Omega_n^3)^2 \frac{1}{q_1^2}}{(k^*)^2 \Omega_n^6 (\tau^2 \Omega_n^2 + 1)} = B^2 \qquad (9.145)$$

For the previously assumed values of the parameters and for $k_3 = q_{1max} = 50$ we obtain

$$k^*k_0q_1 \ll k_4k_{0.c}\Omega_n^2, \quad k^*k_0\tau q_1 q\Omega_n \ll \Omega_n^2, \quad 1 \ll \tau^2\Omega_n^2.$$

Neglecting the small quantities in (9.145), we obtain a simplified formula for the determination of the amplitude of the forced oscillations

$$\frac{A_n}{k^*q_1\tau\Omega_n^2}\sqrt{(T\Omega_n^2 - k_4k_{0.c}q)^2 + \Omega_n^2} = B. \qquad (9.146)$$

Since the coefficients $q(A_v)$ and $q_1(A_{1_v})$ depend on the amplitudes of different variables, then to change over to a single amplitude it is necessary to employ the transfer function of the elements separating the variables i_2 and $\underline{i}$. On the basis of the indicated transfer function

$$W(p) = \frac{i_2}{i} = \frac{-k_2k_1(\tau p + 1)k_0}{p^2}q$$

we obtain, taking the preceding assumptions into account, a formula for the changeover from amplitude A_v to the amplitude A_{1_v}:

$$A_{1_B} = \frac{k_2k_1k_0\tau}{\Omega_n}qA_B. \qquad (9.147)$$

On the basis of Formulas (9.146) and (9.147) we can carry out calculations to determine the amplitudes A_v of the forced oscillations for the current $\underline{i}$ as a function of the amplitude B of the external signal, the

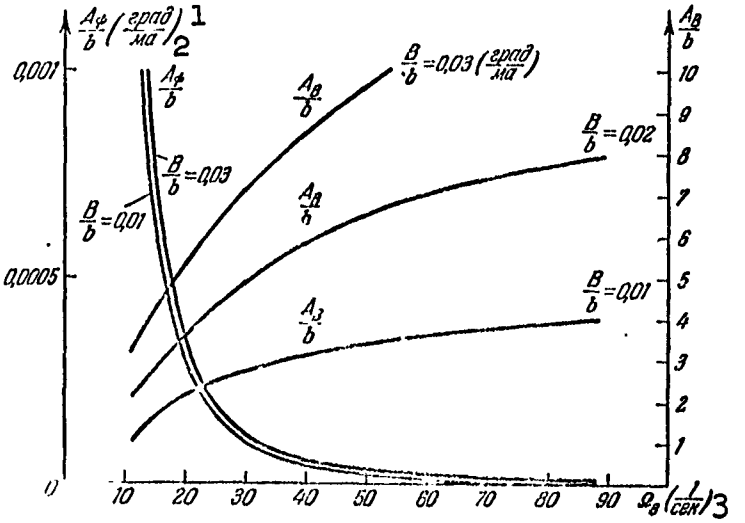


Fig. 9.22. 1) deg; 2) mA; 3) sec.

frequency Ω_v of the external signal, and the system parameters, and then change over, using the corresponding transfer function, to the amplitude A_ψ of the object oscillations.

Figure 9.22 shows in the form of graphs the result of the calculation of the dependence $A_v(\Omega_v)$ for the amplitude of the current $\underline{i}$ and the dependence $A_\psi(\Omega_v)$ for the amplitude of the angle ψ of the oscillatory motion of the object for the previously given values of the system parameters.

Example 4. Combined automatic control system. If the dynamics of an ordinary nonlinear automatic control system, based on the measurement of the deviation of the regulated quantity, is described by an equation

$$Q(p)x + R(p)F(x) = S(p)f(t), \tag{9.148}$$

then the dynamics of a combined nonlinear automatic control system, in which disturbance control is introduced in addition, will be described by the equation

$$Q(p)x + R(p)F(x) = [S(p) - S_1(p)]f(t), \tag{9.149}$$

where $S_1(p)$ is an operator polynomial characterizing the additional disturbance control loop. It is seen from (9.149) that by suitable choice of the polynomial $S_1(p)$, i.e., by introducing a correction proportional to the disturbance, we can greatly reduce the influence of the disturbing action on the automatic system.

Formula (9.14) enables us to determine the amplitude A_v of the forced oscillations of the variable $\underline{x}$ under a sinusoidal disturbance

$$f(t) = B \sin \Omega_v t. \tag{9.150}$$

Let us assume that the nonlinearity $F(x)$ is single valued and has odd symmetry (Fig. 9.23). We denote the amplitude of the forced oscillations in the ordinary system with Eq. (9.148) by A_v^0. According to (9.14), it is determined by the equation

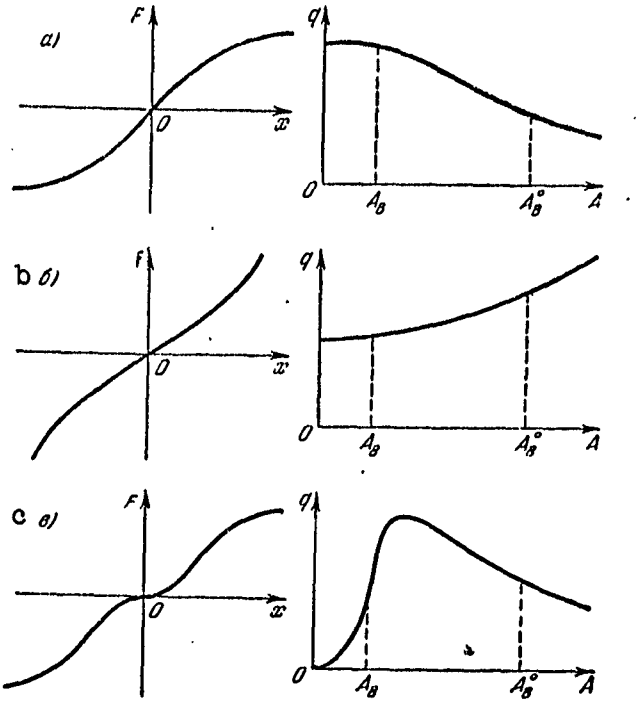


Fig. 9.23

$$A_B^0 |Q(j\Omega_B) + R(j\Omega_B) q(A_B^0)| = B |S(j\Omega_B)|. \qquad (9.151)$$

On the other hand, for a combined system with Eq. (9.149), the amplitude of the forced oscillations A_V will in accord with (9.14) be

$$A_B |Q(j\Omega_B) + R(j\Omega_B) q(A_B)| = B |S(j\Omega_B) - S_1(j\Omega_B)|. \qquad (9.152)$$

In order to determine by how many times the amplitude of the forced oscillations in the combined system is reduced compared with the ordinary system, let us divide (9.152) by (9.151). As a result we obtain

$$\frac{A_B |Q(j\Omega_B) + R(j\Omega_B) q(A_B)|}{A_B^0 |Q(j\Omega_B) + R(j\Omega_B) q(A_B^0)|} = \left| \frac{S(j\Omega_B) - S_1(j\Omega_B)}{S(j\Omega_B)} \right|. \qquad (9.153)$$

For linear systems (combined and ordinary) we have $q(A_V) = q(A_V^0) = k$, and therefore the reduction in the amplitude of the forced oscillations in the combined linear system, compared with the ordinary linear system, is determined by the relation

$$\left(\frac{A_B}{A_B^0}\right)_л = \left| \frac{S(j\Omega_B) - S_1(j\Omega_B)}{S(j\Omega_B)} \right|. \qquad (9.154)$$

On the other hand, in a nonlinear system we have $q(A_v) \neq q(A_v^0)$. To compare the effect of the additional disturbance-proportional control in linear and nonlinear systems, it is more convenient to rewrite Formula (9.153) in the form

$$\frac{A_n}{A_n^0} = \left(\frac{A_n}{A_n^0}\right)_л \left| \frac{Q(j\Omega_n) + R(j\Omega_n) q(A_n^0)}{Q(j\Omega_n) + R(j\Omega_n) q(A_n)} \right|. \qquad (9.155)$$

Unlike the linear system, the reduction in the amplitude of the forced oscillations depends here not only on the oscillation frequency Ω_v and the system parameters, but also on the value of the amplitude A_v itself, and on the form of the nonlinearity.

Inasmuch as $A_v < A_v^0$ (in accord with the sense of the problem), we obtain for a nonlinearity of the type of Fig. 9.23a $q(A_v) > q(A_v^0)$, and for a nonlinearity of the type of Fig. 9.23b, we obtain, to the contrary, $q(A_v) < q(A_v^0)$. For nonlinearities of type 9.23c, on the other hand, any of these two relations may hold true. Consequently, in the first case (Fig. 9.23a), the reduction in the amplitude of the forced oscillations in a combined nonlinear system will be stronger, in accordance with (9.155), than in a linear system, while in the second case (Fig. 9.23b) it will be weaker. In the third case (Fig. 9.23c) either version is possible, depending on the specific numerical values of the parameters.

If the disturbance-proportional correction unit can be constructed such that

$$S_1(j\Omega_n) = S(j\Omega_n), \qquad (9.156)$$

then it becomes possible to cancel out the external signal completely, i.e., the amplitude of the forced oscillations reduces to zero. For one arbitrary frequency Ω_v (for example, the most dangerous one), this can be done almost always. Then the forced oscillations with different frequencies will exist, but within a definite bandwidth (near the

aforementioned frequency) they will be very small (this is called compensation or invariance with accuracy to ε). In practice the satisfaction of this condition is usually quite sufficient.

If, finally, it becomes possible to construct a disturbance-proportional correcting unit such that

$$S_1(p) = S(p), \tag{9.157}$$

then the influence of the investigated disturbance f(t) will be completely canceled out (only the transients remain). This is called total invariance of the system with respect to the given disturbance. The only remaining errors are due to other secondary interferences present in the system, but not taken into account in the formulation of Eqs. (9.148) and (9.149). These errors, however, are eliminated by the principal closed loop of the system, which operates on the basis of the deviation of the controlled quantity (independently of the causes of this deviation).

§9.6. Examples of Determination of Asymmetrical Forced Oscillations

In the examples of the present sections we shall illustrate the solutions, described in §§9.2 and 9.3, of problems involved in the determination of asymmetrical forced oscillations in the presence of a slowly varying signal (Example 1) and the influence of the amplitude and frequency of an external vibrating signal on the position of the stability limit of the nonlinear system with respect to the slowly varying component (Example 2).

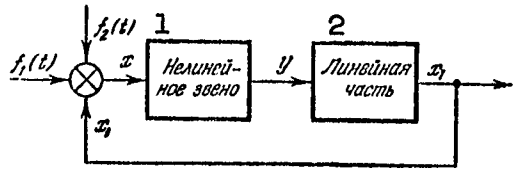


Fig. 9.24. 1) Nonlinear element; 2) linear part.

Example 1. Consider a closed loop system consisting of a linear

part and a nonlinear element (Fig. 9.24). The input to the nonlinear element comprises a slowly varying external signal $f_1(t)$, a periodic signal $f_2(t)$, and the output x_1 of the system applied via a feedback loop.

Let us consider the case when the linear part consists of an inertial magnetic amplifier and motor, while the nonlinear element is a two-position polarized relay which controls the motor through an amplifier.

The linear part is described by the equation

$$(T_1 p + 1)(T_2 p + 1) p x_1 = ky, \tag{9.158}$$

where T_1 is the time constant of the magnetic amplifier, T_2 is the electromechanical time constant of the motor, and k is the gain of the linear part.

For the nonlinear element we have the static characteristic $y = F(x)$ shown in Fig. 9.25a.

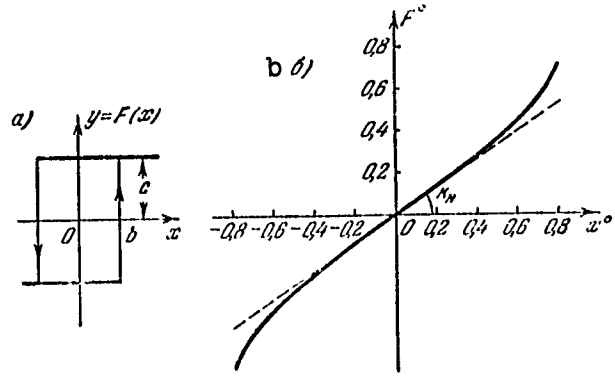


Fig. 9.25

In addition, we shall take into account the equation for the summation of the signals at the input of the nonlinear element

$$x = f_1(t) + f_2(t) - x_1. \tag{9.159}$$

Since the input to the nonlinear element contains a slowly varying signal and a periodic signal, the solution for the variable x will be represented in the form of a sum of a constant component x^0 (dis-

placement of the center of the oscillations) and a periodic sinusoidal component x*, i.e.,

$$x = x^0 + x^*, \quad x^* = A_н \sin(\Omega_н t + \varphi). \tag{9.160}$$

We assume also that the external periodic signal is sinusoidal

$$f_1(t) = B \sin \Omega_н t,$$

and, just as in the determination of the symmetrical oscillations, we reduce it to the form

$$f_1(t) = \frac{B}{A_н}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_н} p\right) x^*. \tag{9.161}$$

Harmonic linearization of the nonlinear function F(x), with account of the asymmetry of the oscillations, yields

$$y = F(x) = F^0 + qx^* + \frac{q'}{\Omega_н} px^*, \tag{9.162}$$

where in accord with (5.95)-(5.97) the DC component F^0 and the coefficients q and q' are

$$F^0 = \frac{c}{\pi}\left(\arcsin\frac{b + x^0}{A_н} - \arcsin\frac{b - x^0}{A_н}\right) \quad \text{for} \quad A_н \geq b + |x^0|,$$

$$q = \frac{2c}{\pi A_н}\left(\sqrt{1 - \frac{(b + x^0)^2}{A_н^2}} + \sqrt{1 - \frac{(b - x^0)^2}{A_н^2}}\right) \text{for } A_н \geq b + |x^0|,$$

$$q' = -\frac{4cb}{\pi A_н^2} \quad \text{for} \quad A_н \geq b + |x^0|.$$

Combining the equation of the linear part (9.158) and the linearized equation of the nonlinear element (9.162) we obtain with account of (9.159) and (9.161) the linearized equation for the entire system:

$$[T_1T_2p^3 + (T_1 + T_2)p^2 + p](x^0 + x^*) + k\left(F^0 + qx^* + \frac{q'}{\Omega_н} px^*\right) =$$
$$= [T_1T_2p^3 + (T_1 + T_2)p^2 + p]\left[f_1(t) + \frac{B}{A_н}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_н} p\right) x^*\right] \tag{9.163}$$

Equation (9.163) breaks up into two equations for the slowly varying and for the oscillatory components, respectively:

$$[T_1T_2p^3 + (T_1 + T_2)p^2 + p]x^0 + kF^0 =$$
$$= [T_1T_2p^3 + (T_1 + T_2)p^2 + p]f_1(t), \tag{9.164}$$

$$[T_1T_2p^3 + (T_1 + T_2)p^2 + p]\left[1 - \frac{B}{A_н}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_н} p\right)\right] x^* +$$
$$+ k\left(q + \frac{q'}{\Omega_н} p\right) x^* = 0. \tag{9.165}$$

Upon substitution of $p = j\Omega_v$ into the characteristic equation corresponding to the differential equation (9.165), and taking account of the fact that

$$\cos\varphi - j\sin\varphi = e^{-j\varphi},$$

we obtain

$$A_v \frac{-jT_1T_2\Omega_v^3 - (T_1+T_2)\Omega_v^2 + j\Omega_v + k(q+jq')}{-jT_1T_2\Omega_v^3 - (T_1+T_2)\Omega_v^2 + j\Omega_v} = Be^{-j\varphi}. \qquad (9.166)$$

Let us consider first a case in which the input to the nonlinear element contains harmonic oscillations aimed at vibration smoothing of the nonlinearity, i.e., to ensure a smooth dependence, amenable to ordinary linearization, of the output of the nonlinear element on the slowly varying input. In this case the system changes over into a linear one for the slowly varying processes. The frequency and the amplitude of the external periodic signal are chosen here such as to make the periodic component of the output quantity (or its rate of change) have an amplitude that is small, practically close to zero.

On the basis of the statements made above, we have for $x^*_1 \approx 0$ equality of the amplitudes of the external periodic signal $f_2(t)$ and of the variable $\underline{x}$, i.e.,

$$A_v \approx B \quad \text{for } \varphi \approx 0.$$

In accordance with this, we can write the formula for the constant component of the nonlinear function contained in (9.162) in the form

$$F^0 = \frac{c}{\pi}\left(\arcsin\frac{b+x^0}{B} - \arcsin\frac{b-x^0}{B}\right) \text{for } B \geq b + |x^0|, \qquad (9.167)$$

i.e., we obtain in this case the function $F^0(x^0, B)$ directly, which gives, for a specified value of the amplitude B of the external periodic signal, the smoothed characteristic $F^0(x^0)$ for the slowly varying component. This enables us to use Eq. (9.164) to investigate the slow processes.

Let us specify the values B = 1 v, c = 1 v, and b = 0.2 v. Then

for the case considered here we obtain the smoothed characteristic shown in Fig. 9.25b. We see that for a wide range of variation of x^0 the smoothed characteristic $F^0 = F^0(x^0)$ is close to linear. The fact that the smooth characteristic is close to linear for a definite type of nonlinearity, is due to the form of the external periodic signal. As was shown in §1.9, by applying an external periodic signal of special form it is possible to obtain a linear smoothed characteristic over the entire range of variation of x^0.

Equation (9.166) is used in this case to determine the desirable frequency Ω_v of the external periodic signal, so as to guarantee the condition $A_v \approx B$. It follows from (9.166) that to satisfy this condition it is required to have the modulus of the ratio

$$\frac{k(q+jq')}{-j\Omega_v(T_1T_2\Omega_v^2-1)-(T_1+T_2)\Omega_v^2}$$

small compared with unity. Let us assume that we are required to satisfy the condition

$$\frac{k\sqrt{q^2+(q')^2}}{\sqrt{(T_1+T_2)^2\Omega_v^2+\Omega_v^2(T_1T_2\Omega_v^2-1)^2}}<0{,}01. \qquad (9.168)$$

We specify the values: $T_1 = 0.02$ sec, $T_2 = 0.1$ sec, $k = 10$ sec^{-1}. Taking into account the previously assumed values $c = 1$ v, $b = 0.2$ v, $B = 1$ v, we obtain for $x^0 = 0$, from the formulas for the coefficients of harmonic linearization, values $q = 1.25$ and $q' = 0.25$. Then the value of the frequency satisfying the inequality (9.168) will be $\Omega_v \approx 100$ sec^{-1} and higher.

Let us turn to Eq. (9.164). On the basis of the smoothed characteristic (Fig. 9.25b) obtained for values $|x^0| < 0.5$ v, we can assume this characteristic to be linear and consequently we can put $F^0 = k_n x^0$. Then Eq. (9.164) is rewritten

$$[T_1T_2p^3+(T_1+T_2)p^2+p+kk_n]x^0 = [T_1T_2p^3+(T_1+T_2)p^2+p]f_1(t); \qquad (9.169)$$

with allowance for the fact that $x^0 = f_1(t) - x_1$ we can rewrite Eq. (9.169) in terms of the output of the system in the form

$$[T_1T_2p^3 + (T_1 + T_2)p^2 + p + kk_{11}]x_1 = kk_{11}f_1(t). \qquad (9.170)$$

Thus, owing to vibration smoothing of the nonlinearity by means of forced oscillations, the nonlinear system for the slowly varying signal has turned into a linear system, the investigation of which must be carried out further by the methods of the linear theory.

If one applies to the system an arbitrary sinusoidal external signal with frequency at which the condition that the amplitude be small at the output of the system can no longer be used, the determination of the forced oscillations becomes somewhat more complicated.

In the example considered here, in order to determine the forced oscillations in this case we must solve Eq. (9.166) graphically or analytically with respect to A_v and φ. We then obtain the dependences $A_v(x^0, \Omega_v, B)$ and $\varphi(x^0, \Omega_v, B)$, and for specified values of the amplitude B and of the frequency Ω_v of the external periodic signal we obtain the dependences $A_v(x^0)$ and $\varphi(x^0)$. Substitution of A_v into the formula for the constant component of the nonlinear function yields the bias function $F^0 = \Phi(x^0)$.

Assume that the system considered has the same parameters as before and that the external periodic signal with amplitude B = 1 volt has a frequency $\Omega_v = 10$ sec^{-1} in place of the $\Omega_v \approx 100$ sec^{-1} used in the case of vibration smoothing. Then, in accord with (9.166) and with allowance for the values of $\underline{q}$ and q', we obtain

$$A_n + \frac{20}{\pi A_n}\frac{\left[\sqrt{A_n^2 - (0{,}2 + x^0)^2} + \sqrt{A_n^2 - (0{,}2 - x^0)^2} - j \cdot 0{,}4\right]}{-12 + j8} = e^{-j\varphi}. \qquad (9.171)$$

Solution of Eq. (9.171) enables us to determine the dependence $A_v = A_v(x^0)$, for example, by the method indicated in Fig. 9.6; substitution of the latter in the formula for F^0 gives the bias function $F^0 =$

$= \Phi(x^0)$. Carrying out linearization of the bias function, we can investigate the slowly occurring processes by means of Eq. (9.164) as for a linear system. Knowing the values of x^0 corresponding to definite values of the external signal $f_1(t)$, we can determine by means of the same bias function the values of the corresponding amplitude, and from the relation $\varphi = \varphi(x^0)$ also the phase shift of the forced oscillations for the variable $\underline{x}$ relative to the external periodic signal $f_2(t)$.

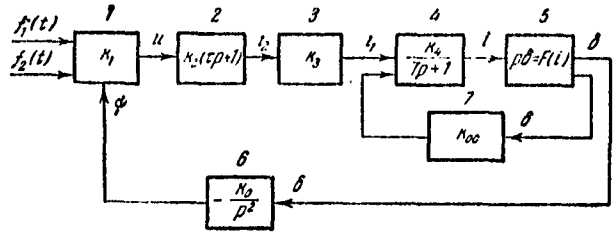


Fig. 9.26

Example 2. As the second example let us take a system for the stabilization of the angle of an object in space, considered in §9.5 (Example 3). The block diagram of the system is shown in Fig. 9.26. Here, as before, we assume that there has been applied to the sensitive element – gyroscope 1 – an external periodic signal

$$f_2(t) = B \sin \Omega_0 t. \qquad (9.172)$$

In addition, a slowly varying signal $f_1(t)$ is applied in this case to the gyroscope. By periodic signal we mean a certain vibrational interference, which affects the way that the system follows up the control signal $f_1(t)$. In particular, let us see how the vibration interference changes the position of the stability limit of the nonlinear system with respect to slowly varying useful signal.

We now write for the gyroscope equation in place of (9.129)

$$u = k_1 [f_1(t) + f_2(t) - \psi]. \qquad (9.173)$$

The equations of the other elements remain the same as before, i.e., they have the form (9.130)-(9.134). The nonlinear element – the rudder

engine – is accounted for by its static characteristic (Fig. 9.19).

Since an external slowly varying signal is applied to the system, we seek the solution for the variable $\underline{i}$ in the form

$$i = i^0 + i^*, \quad i^* = A_u \sin(\Omega_u t + \varphi), \tag{9.174}$$

where i^0 is the slowly varying component (useful signal) and i* is the periodic (vibrational) component.

The harmonic linearization of the nonlinear function $F = p\delta$ (Fig. 9.19) yields

$$F = F^0 + qi^*, \tag{9.175}$$

where the constant component and the coefficient of harmonic linearization, in accord with Formulas (5.121) and (5.122), are given by the expressions

$$F^0(A_u, i^0) = \frac{k_5 A_u}{\pi}\left(\sqrt{1-\frac{(b+i^0)^2}{A_u^2}} - \sqrt{1-\frac{(b-i^0)^2}{A_u^2}}\right) + \frac{k_5 i^0}{\pi}\left(\arcsin\frac{b-i^0}{A_u} + \arcsin\frac{b+i^0}{A_u}\right) + \frac{c}{\pi}\left(\arcsin\frac{b+i^0}{A_u} - \arcsin\frac{b-i^0}{A_u}\right) \tag{9.176}$$

for $A_v \geq b + |i_0|$;

$$q(A_u, i^0) = \frac{k_5}{\pi}\left(\arcsin\frac{b-i^0}{A_u} + \arcsin\frac{b+i^0}{A_u} - \frac{b-i^0}{A_u}\sqrt{1-\frac{(b-i^0)^2}{A_u^2}} - \frac{b+i^0}{A_u}\sqrt{1-\frac{(b+i^0)^2}{A_u^2}}\right) - \frac{2k_5 i^0}{\pi A_u}\left(\sqrt{1-\frac{(b-i^0)^2}{A_u^2}} - \sqrt{1-\frac{(b+i^0)^2}{A^2}}\right) + \frac{2c}{\pi A_u}\left(\sqrt{1-\frac{(b-i^0)^2}{A_u^2}} + \sqrt{1-\frac{(b+i^0)^2}{A_u^2}}\right) \tag{9.177}$$

for $A_v \geq b + |i^0|$.

Taking into account the gyroscope equation (9.173), the equations of the other linear elements (9.130)-(9.134), and the equation of the nonlinear element (9.175), we obtain the harmonically linearized equation of the investigated system

$$(Tp^4 + p^3)(i^0 + i^*) + (k_1 k_{0.c} p^2 + kk_0 \tau p + kk_0)(F^0 + qi^*) = k(\tau p + 1)p^3[f_1(t) + f_2(t)], \tag{9.178}$$

where $k = k_1 k_2 k_3 k_4$.

The vibration interference $f_2(t)$ we represent in accord with

(9.172) in the form

$$f_2(t) = B \sin[(\Omega_н t + \varphi) - \varphi] =$$
$$= B \cos\varphi \sin(\Omega_н t + \varphi) - B \sin\varphi \cos(\Omega_н t + \varphi).$$

Recognizing that

$$i^* = A_н \sin(\Omega_н t + \varphi) \text{ and } pi^* = A_н \Omega_н \cos(\Omega_н t + \varphi),$$

we obtain

$$f_2(t) = \frac{B}{A_н}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_н} p\right) i^*. \qquad (9.179)$$

Substituting in (9.178) the value obtained for $f_2(t)$, we represent it in the form of two equations:

$$(Tp^3 + p^2) i^* + (k_1 k_{0.c} p^2 + kk_0\tau p + kk_0) qi^* =$$
$$= k(\tau p + 1) p^2 \frac{B}{A_н}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_н} p\right) i^*; \qquad (9.180)$$

$$(Tp^3 + p^2) i^0 + (k_1 k_{0.c} p^2 + kk_0\tau p + kk_0) F^0 =$$
$$= k(\tau p + 1) p^2 f_1(t). \qquad (9.181)$$

The first of the equations obtained corresponds to the vibrational component and the second to the useful control signal.

Equation (9.180) enables us to determine the amplitude A_v of the vibrations at the input of the nonlinear element as a function of the amplitude B and the frequency Ω_v of the external periodic interference applied to the system, for specified system parameters.

Let us make the substitution $p = j\Omega_v$ in the characteristic equation corresponding to the differential equation (9.180).

As a result of this substitution and with account of the fact that

$$\cos\varphi - j\sin\varphi = e^{-j\varphi},$$

we obtain

$$\frac{A_н(T\Omega_н^3 - j\Omega_н^2 - k_1 k_{0.c} q\Omega_н^2 + jkk_0\tau q\Omega_н + kk_0 q)}{-k(\tau j\Omega_н + 1) j\Omega_н^2} = Be^{-j\varphi}. \qquad (9.182)$$

The complex ratio (9.182) is analogous to Eq. (9.139), the only difference being that the harmonic linearization coefficient q depends here not only on the amplitude A_v, but also on the bias i^0 of the input to the nonlinear element.

The phase relations between the input periodic signal $f_2(t)$ and the variable $\underline{i}$ are not of practical interest. We therefore derive an equation relating the amplitudes A_v and B from the condition that the moduli of the left and right halves of the complex expression (9.182) must be equal:

$$A_v^2 \frac{[T\Omega_v^2 + (kk_0 - k_4 k_{0.c}\Omega_v^2)\, q]^2 + [kk_0 \tau q \Omega_v - \Omega_v^2]^2}{k^2\Omega_v^4(\tau^2\Omega_v^2 + 1)} = B^2. \qquad (9.183)$$

For the investigated system we assume the following numerical values of the parameters: k = 18 mA/deg, kk_0 = 9 mA/deg·sec^2, $k_4 k_{0.s}$ = = 1.8 mA/deg, T = 0.02 sec, τ = 0.5 sec, Ω_v = 12-95 sec^{-1}, and k_5 = 4 deg/sec·mA.

For the assumed values of the parameters we have

$$kk_0 \ll k_4 k_{0.c}\Omega_v^2, \quad kk_0 \tau q \Omega_v \ll \Omega_v^2, \quad 1 \ll \tau^2 \Omega_v^2.$$

Then, neglecting the corresponding terms we rewrite (9.183) in the form

$$\frac{A_v}{k\tau\Omega_v^2}\sqrt{(T\Omega_v^2 - k_4 k_{0.c} q)^2 + \Omega_v^2} = B. \qquad (9.184)$$

We use the formula (9.184) to plot the dependences $A_v = A_v(B)$ for different constant values of the frequency Ω_v. In this case, as shown by calculation, the value of i_0 entering into the expression for the coefficients $\underline{q}$ makes practically no noticeable change in the position of the plotted curves, so that we can use the curves plotted during the course of the investigation of the same system for the case $i^0 = 0$ in Example 3 of §9.5 (Fig. 9.20a).

Likewise unchanged are the plots showing the dependence of the variation of the amplitude of the forced oscillations A_v/B at the input of the nonlinear element on the frequency Ω_v of the periodic signal applied to the system, for constant values of the amplitude of the external interference B/b (Fig. 9.20b).

To investigate the stability of the system relative to the useful

control signal, let us turn to Eq. (9.181). This equation can be reduced to the single variable i^0, by plotting the bias function $F^0(i^0)$. In our case this will be a family of curves for different constant values of the amplitudes A_v of the vibrations at the input of the nonlinear element. Bias functions plotted for A_v/b = const in accord with Formula (9.176) are shown in Fig. 9.27.

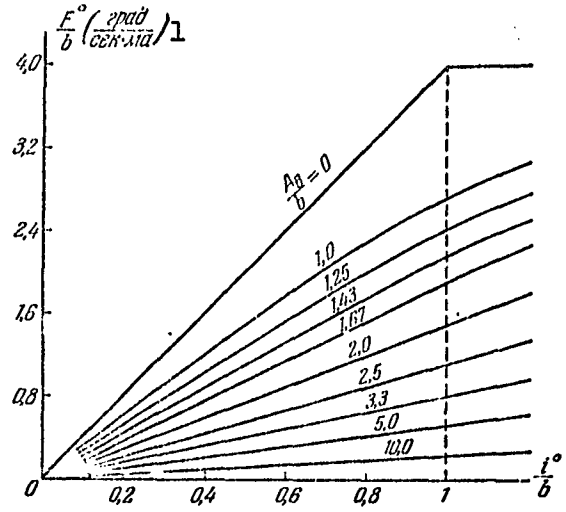


Fig. 9.27. 1) deg/sec·mA.

We see that the bias functions are close to linear. In accordance with the plot of Fig. 9.20a, the amplitude A_v can be converted into the amplitude of the interference B at the input of the system for a specified frequency Ω_v. The plot of Fig. 9.20b can be used to determine the corresponding value of the interference frequency Ω_v for a specified interference amplitude B. It is seen from the plot of Fig. 9.27 that an increase in the interference amplitude for a specified frequency leads to a decrease in the gain of the nonlinear element with respect to the useful signal. The maximum value of the gain occurs in the absence of vibration interference, i.e., $k_{nmax} = k_5$.

Within the range of variation of the useful signal $0 < i^0 < b$ we shall represent the bias function in the form

$$\frac{F^0}{b} = k_n\left(\frac{A_v}{B}\right)\frac{i^0}{b}, \qquad (9.185)$$

where k_n is the gain of the useful signal in the nonlinear element (the slope of the bias function), which depends on the vibration amplitude A_v. But since A_v depends on B and Ω_v, we can write Formula (9.185) in the form

$$\frac{F^0}{b} = k_n\left(\frac{B}{b},\ \Omega_v\right)\frac{i^0}{b}. \qquad (9.186)$$

In order to determine the dependence of k_n on B/b and Ω_v, let us plot the curve $k_n = f(A_v/b)$ (Fig. 9.28) on the basis of the bias functions (Fig. 9.27). The resultant curve enables us to provide the curves of Fig. 9.20b with a suitable scale for k_n, and thus determine with the aid of the same curves $A_v/b = f(\Omega_v)$ the function $k_n = k_n(B/b, \Omega_v)$. On the basis of the dependence obtained we can now determine the influence of the amplitude and frequency of the vibration interference on the stability of the nonlinear system relative to the useful signal.

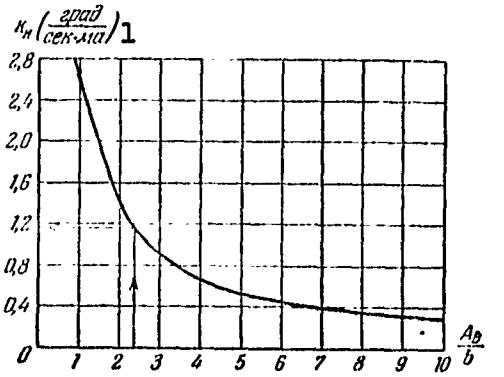


Fig. 9.28. 1) deg/sec·mA.

Indeed, using (9.186) we can represent the quantity F^0, which is contained in Eq. (9.181), in the form $F^0 = k_n(B/b, \Omega_v)i^0$ and rewrite Eq. (9.181) in the form

$$[Tp^4 + p^3 + k_n(k_4k_{o.c}p^2 + kk_0\tau p + kk_0)]i^0 = k(\tau p + 1)p^3 f_1(t). \qquad (9.187)$$

The equation obtained is linear, and the stability of the system with respect to the useful signal can be investigated by means of the

Hurwitz stability criterion. The characteristic equation corresponding to (9.187) will be

$$Tp^4 + p^3 + k_н (k_4 k_{о.с} p^2 + k k_0 \tau p + k k_0) = 0. \qquad (9.188)$$

Equating the penultimate Hurwitz determinant to zero, we obtain an equation for the stability limit:

$$k_н \tau (k_4 k_{о.с} - T k k_0 \tau) = 1. \qquad (9.189)$$

Thus, the value of the gain corresponding to the stability limit is determined by the relation

$$k_{н.гр} = \frac{1}{\tau (k_4 k_{о.с} - T k k_0 \tau)}, \qquad (9.190)$$

so that when we substitute the assumed values of the parameters we obtain for the investigated system $k_{n.gr} = 1.17$ deg/sec·mA.

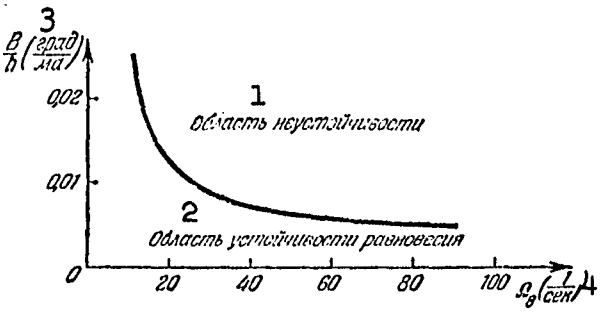


Fig. 9.29. 1) Instability region; 2) stable equilibrium region; 3) deg/mA; 4) 1/sec.

Drawing on Fig. 9.20b the line corresponding to $k_{n.gr} = 1.17$ we obtain from the points of intersection between this line and the curves $A_V/b = f(\Omega_V)$ the values of A_V/b and Ω_V corresponding to the stability limit. Using the curves on Fig. 9.20a, we recalculate the values of A_V/b into the values of B/b. As a result we plot (Fig. 9.29) the stability limit on a plane with coordinates Ω_V (frequency) and B (amplitude of vibration interference).

At large values of the useful control signal i^0, when the plots of the bias function can no longer be regarded as straight lines, the

latter can be subjected to repeated harmonic linearization (see Problem 3 in §9.3). In this system will behave like an essentially nonlinear one also with respect to the useful signal, but with a nonlinear element characteristic that has been deformed under the influence of the vibration interference.

§9.7. Example of Inclusion of Higher Harmonics of the Forced Oscillations

Let us consider [292] a relay automatic control system of third order with a general form relay characteristic and with a constant time

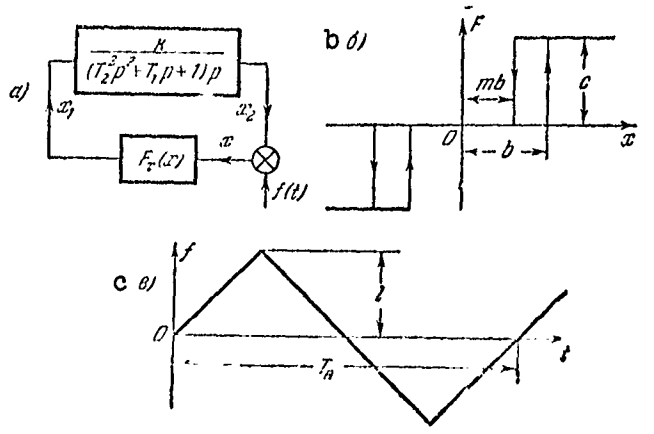


Fig. 9.30

delay τ (Fig. 9.30a). In accordance with the specified system circuit, we write down its equation in the form

$$\left.\begin{aligned}(T_2^2p^2+T_1p+1)px_2 &= kx_1,\\ x_1 = F_\tau(x) &= e^{-\tau p}F(x), \quad x = f(t)-x_2.\end{aligned}\right\} \tag{9.191}$$

The nonlinearity F(x) is shown in Fig. 9.30b. Reducing the equations (9.191) to the form (9.2), we obtain

$$(T_2^2p^2+T_1p+1)px + ke^{-\tau p}F(x) = (T_2^2p^2+T_1p+1)pf(t). \tag{9.192}$$

Let the external signal f(t) have a sawtooth form (Fig. 9.30c), with a Fourier expansion

$$f(t) = \frac{8l}{\pi^2}\left(\sin\Omega_n t - \frac{1}{9}\sin 3\Omega_n t + \ldots\right). \tag{9.193}$$

According to (9.62) we have

$$B_1 = \frac{8l}{\pi^2}, \quad B_3 = -\frac{8l}{9\pi^2}, \quad \ldots, \quad \Omega_n = \frac{2\pi}{T_n}. \tag{9.194}$$

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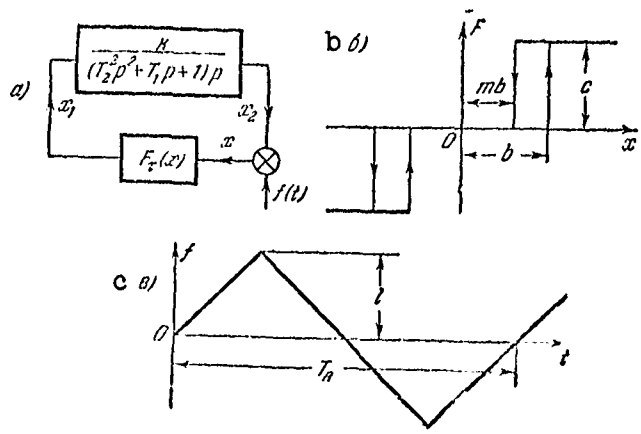


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Let the external signal f(t) have a sawtooth form (Fig. 9.30c), with a Fourier expansion

$$f(t)=\frac{8l}{\pi^2}\left(\sin\Omega_1t-\frac{1}{9}\sin 3\Omega_1t+\ldots\right). \tag{9.193}$$

According to (9.62) we have

$$B_1=\frac{8l}{\pi^2},\ B_3=-\frac{8l}{9\pi^2},\ \ldots,\ \Omega_n=\frac{2\pi}{T_n}. \tag{9.194}$$

Let us confine ourselves from now on to an inclusion of the third harmonic only. We first find the first approximation based on the first harmonic only, seeking the solution in the form

$$x_n = A_n \sin(\Omega_n t + \varphi).$$

The first approximation equation for the forced oscillations will be, in accordance with (9.14) and (9.192)

$$A_n\left\{1 - \frac{ke^{-j\tau\Omega_n}[q(A_n) + jq'(A_n)]}{j\Omega_n(1 - T_2^2\Omega_n^2) - T_1\Omega_n^2}\right\} = B_1 e^{-j\varphi}, \qquad (9.195)$$

where in accord with Fig. 9.30b and Formulas (3.5) and (3.6)

$$q = \frac{2c}{\pi A_n}\left(\sqrt{1 - \frac{b^2}{A_n^2}} + \sqrt{1 - \frac{m^2b^2}{A_n^2}}\right), \quad q' = -\frac{2cb}{\pi A_n^2}(1 - m). \qquad (9.196)$$

From this we obtain graphically (Fig. 9.1) or analytically (Formulas (9.17) and (9.18)) the amplitude A_v and the phase φ of the first approximation.

Let us find the amplitude and phase of the third harmonic. For this purpose, in accordance with (9.77), it is first necessary to find the coefficients r_3 and s_3 by means of Formulas (9.71). However, for the specified nonlinearity (Fig. 9.30b) these were already calculated in §8.3. We therefore use the ready-made formulas (8.106)

$$\left.\begin{aligned} r_3 &= \frac{2c}{3\pi A_n}\left[\left(1 - 4\frac{b^2}{A_n^2}\right)\sqrt{1 - \frac{b^2}{A_n^2}} + \left(1 - 4\frac{m^2b^2}{A_n^2}\right)\sqrt{1 - \frac{m^2b^2}{A_n^2}}\right], \\ s_3 &= -\frac{2cb}{\pi A_n^2}\left|1 - \frac{4b^2}{3A_n^2} - m\left(1 - \frac{4m^2b^2}{3A_n^2}\right)\right| (A_n \geq b). \end{aligned}\right\} \qquad (9.197)$$

The values of r_3 and s_3 can be readily obtained in numerical form, since all the quantities contained there are already known.

The relative amplitude δ_3 and the phase φ_3 of the third harmonic of the forced oscillations (with inclusion of the third harmonic of the external signal) will be, in accord with (9.77), (9.192), and (9.194)

$$\left.\begin{aligned} \delta_3 &= \left|\frac{ke^{-j\cdot 3\tau\Omega_n}(r_3 + js_3)}{j\cdot 3\Omega_n(1 - 9T_2^2\Omega_n^2) - 9T_1\Omega_n^2}e^{j\cdot 3\varphi} + \frac{8l}{9\pi^2 A_n}\right|, \\ \varphi_3 &= \arg\left[\frac{ke^{-j\cdot 3\tau\Omega_n}(r_3 + js_3)}{j\cdot 3\Omega_n(1 - 9T_2^2\Omega_n^2) - 9T_1\Omega_n^2}e^{j\cdot 3\varphi} + \frac{8l}{9\pi^2 A_n}\right] - \pi. \end{aligned}\right\} \qquad (9.198)$$

The values of δ_3 and φ_3 are readily calculated if the system parameters are specified.

As a result we know the first and third harmonics of the forced oscillations, i.e.,

$$x = A_n [\sin(\Omega_n t + \varphi) + \delta_3 \sin(3\Omega_n t + \varphi_3)]. \qquad (9.199)$$

However, the first harmonic was determined here independently of the presence of the third and with inclusion of only the first harmonic of the external signal (i.e., with replacement of the sawtooth signal by a sinusoidal one), whereas in the determination of the third harmonic account was already taken of the sawtooth form of the signal.

Consequently, it remains here to determine more exactly the first harmonic in (9.199), i.e., to determine in place of A_v and φ the more exact values A_1 and φ_1 with account of the presence of the third harmonic both in the solution itself and in the external sawtooth signal. For this purpose it is necessary, as can be seen from (9.80), to calculate the additions Δq and $\Delta q'$ to the harmonic linearization coefficient by means of Formulas (9.69), namely:

$$\Delta q = I_{31}\delta_3 \cos\psi_3 + I_{32}\delta_3 \sin\psi_3, \quad \Delta q' = I_{33}\delta_3 \cos\psi_3 + I_{34}\delta_3 \sin\psi_3, \qquad (9.200)$$

where

$$\psi_3 = \varphi_3 - 3\varphi.$$

The quantities I_{31}, I_{32}, I_{33}, and I_{34} have already been determined for the given nonlinearity (Fig. 9.30b) in §8.3. In accordance with (8.108) they are expressed in the following manner:

$$I_{31} = \frac{2cb^2}{\pi A_n^4}\left(\frac{3A_n^2 - 4b^2}{\sqrt{A_n^2 - b^2}} + m^2\frac{3A_n^2 - 4m^2b^2}{\sqrt{A_n^2 - m^2b^2}}\right),$$

$$I_{32} = \frac{2cb}{\pi A_n^2}\left[\left(1 - 4\frac{b^2}{A_n^2}\right) - m\left(1 - 4\frac{m^2b^2}{A_n^2}\right)\right],$$

$$I_{33} = \frac{2cb}{\pi A_n^2}\left[\left(3 - 4\frac{b^2}{A_n^2}\right) - m\left(3 - 4\frac{m^2b^2}{A_n^2}\right)\right],$$

$$I_{34} = \frac{2c}{\pi A_n}\left[\left(1 - 4\frac{b^2}{A_n^2}\right)\sqrt{1 - \frac{b^2}{A_n^2}} + \left(1 - 4\frac{m^2b^2}{A_n^2}\right)\sqrt{1 - \frac{m^2b^2}{A_n^2}}\right].$$

All the quantities contained here are already known. Therefore the ad-

ditions Δq and $\Delta q'$ can be readily calculated numerically from Formulas (9.200).

After calculating them, we obtain in accord with (9.80) and (9.192)

$$Q_1(p)=(T_1^2p^2+T_1p+1)p+ke^{-\tau p}\left(\Delta q+\frac{\Delta q'}{\Omega_в}p\right). \qquad (9.201)$$

As a result we obtain on the basis of (9.81) an equation for the determination of the more exact values of the amplitude A_1 and the phase φ_1 of the first harmonic of the forced oscillations in the form

$$A_1\left\{1+D-\frac{ke^{-j\tau\Omega_в}[q(A_1)+jq'(A_1)]}{j\Omega_в(1-T_2^2\Omega_в^2)-T_1\Omega_в^3}\right\}=B_1e^{-j\varphi_1}, \qquad (9.202)$$

where

$$D=\frac{ke^{-j\tau\Omega_в}(\Delta q+j\Delta q')}{j\Omega_в(1-T_2^2\Omega_в^2)-T_1\Omega_в^3},$$

$$q=\frac{2c}{\pi A_1}\left(\sqrt{1-\frac{b^2}{A_1^2}}+\sqrt{1-\frac{m^2b^2}{A_1^2}}\right), \quad q'=-\frac{2cb}{\pi A_1^2}(1-m).$$

We see that Eq. (9.192) with respect to the more exact values of A_1 and φ_1 has in general the same form as the preceding equation (9.195) with respect to the values of the first approximation A_v and φ. Therefore, solving it now by the same method as used to solve the previous Equation (9.195), we obtain the more exact values of A_1 and φ_1.

The difference between the new equation (9.202) and the previous one (9.195) lies in the fact that it contains in place of the real unity the complex quantity 1 + D, which effects the sought correction to the first harmonic of the forced oscillations by inclusion of the third harmonic of the external signal and of the solution itself.

As a result we obtain in place of (9.199) a more exact solution for the forced oscillations, in the form

$$x=A_1\sin(\Omega_вt+\varphi_1)+\delta_3A_в\sin(3\Omega_вt+\varphi_3). \qquad (9.203)$$

In the case of necessity we can make more exact also the quantities δ_3 and φ_3 for the third harmonic, calculating them now by means

of Formulas (9.198) and (9.197), in which we replace the old values A_v and φ by the new values A_1 and φ_1, which yields the new more exact equation

$$x = A_1[\sin(\Omega_n t + \varphi_1) + \delta'_3 \sin(3\Omega_n t + \varphi'_3)].$$

On the other hand, in most practical cases the previous solution (9.203) will be perfectly adequate.

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[Footnotes]

793 Similarly also for expression (9.15).

811 This is discussed in greater detail at the end of the present section.

825 As in §8.6, we could calculate here, too, the higher harmonics of the forced oscillations in the presence of a slowly varying signal x^0.

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[List of Transliterated Symbols]

789 в = v = vynuzhdennyy = forced (subscript)

794 пор = por = porogovoy = threshold

795 авт = avt = avtokolebaniya = self-oscillations

807 н = n = nelineynyy = nonlinear

820 с = s = signal = signal

837 сек = sek = sekunda = second

837 в = v = vol't = volts (in parentheses)

844 ос (о.с) = os = obratnaya svyaz' = feedback

852 л = l = lineynyy = linear

866 гр = gr = granitsa = limit, boundary

Chapter 10

RANDOM PROCESSES IN NONLINEAR SYSTEMS

§10.1. Oscillatory Systems under Slowly Varying Random Signals

In many cases of analysis and synthesis of automatic control and regulation systems it becomes necessary to investigate random processes occurring when external signals of random character, either noise or useful control signals, are applied to the system. In the present chapter we shall give a few approximate methods of investigating random processes in nonlinear automatic systems; these methods are similar to those developed in the preceding chapters and are at the same time the most effective from the point of view of carrying out the engineering calculations.

The present section is devoted to two types of nonlinear systems under slowly varying random signals:

1) systems operating in the self-oscillation mode;

2) systems operating in the forced oscillation mode.

Let us start with nonlinear systems operating in the self-oscillation mode. Their investigation is based on the materials of Chapter 5, where we dealt with the flow of slowly varying signals through self-oscillating systems [311]. Here, however, these signals will be random functions of the time.

Let the nonlinear automatic system be described by an equation satisfying all the conditions of §2.3

$$Q(p)x + R(p)F(x, px) = S(p)f(t), \qquad (10.1)$$

where the notation in the left half is the same as before and pertains

to systems of the first class (see §1.2), while the right half contains the external signal f(t) representing a random function of time. We shall assume further that we are dealing with a stationary random process, and the variations of f(t) within the limits of the self-oscillation cycle are insignificant, i.e., the conditions indicated in §5.1, that the external signal be a slowly varying quantity, are satisfied with a probability close to unity. In other words, it is assumed that the frequency spectrum of the random signal f(t) lies appreciably below the self-oscillation frequency.

Therefore the solution, as in Chapter 5, will be sought in the form

$$x = x^0 + x^*, \quad \text{where} \quad x^* = A \sin \Omega t, \qquad (10.2)$$

and both the slowly varying component $x^0(t)$ itself and the amplitude of the oscillatory component A are random quantities, which depend on the character of variation of the random external signal f(t).

It is important to note here the following. If, for example, a random quantity f(t) has a normal distribution, then as it passes through a nonlinearity F(x, px), this distribution is subject to distortion manifest in the form of supplementary components, which will actually occur in the variable F. However, on passing through the linear part R(p)/Q(p), which has the properties of a filter (§2.3), these supplementary components will be suppressed (similar to the suppression of the higher harmonics, see §2.2), so that the distribution law for the random variable $\underline{x}$ again becomes close to normal. This very important property of the nonlinear automatic systems considered in the present book will be used to simplify the investigation in what follows.

Thus, on the same basis as in Chapter 5, let us carry out harmonic linearization of the nonlinearity by means of the formula

$$F(x,\ px) = F^0 + qx^* + \frac{q'}{\Omega} px^*, \tag{10.3}$$

where

$$F^0 = \frac{1}{2\pi} \int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\, d\psi, \tag{10.4}$$

$$\left.\begin{aligned} q &= \frac{1}{\pi A} \int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\sin\psi\, d\psi, \\ q' &= \frac{1}{\pi A} \int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\cos\psi\, d\psi. \end{aligned}\right\} \tag{10.5}$$

The expressions for F^0, q, and q' pertaining to specific nonlinearities are taken in ready-made form from §§5.6-5.9.

Substituting (10.4) in the specified system equation (10.1), we break down the latter into two equations

$$Q(p)\, x^0 + R(p)\, F^0 = S(p) f(t), \tag{10.6}$$

$$Q(p)\, x^* + R(p) \left(q + \frac{q'}{\Omega} p\right) x^* = 0 \tag{10.7}$$

for the slowly varying and for the oscillatory components, respectively. It must be noted, however, that these equations cannot be solved with respect to each other, and are mutually interrelated, since the quantities F^0, q, and q', as can be seen from (10.4) and (10.5), depend in general on all three unknowns x^0, A, and Ω, or at any rate on the first two of them.

From Eq. (10.7) we determine the dependence of the amplitude A and of the frequency Ω on the slowly varying random component x^0 (which is for the time being unknown), which enters into the expression for the coefficients of harmonic linearization q and q'. We thus determine the relationships

$$A(x^0) \text{ and } \Omega(x^0), \tag{10.8}$$

for which we use any of the methods of §2.3. This was illustrated by means of examples in Chapter 6.

The obtained relationships (10.8) are substituted in the expres-

sion obtained from (10.4)

$$F^0(x^0, A, \Omega), \tag{10.9}$$

as a result we obtain the bias function*

$$F^0 = \Phi(x^0), \tag{10.10}$$

which, generally speaking, is also nonlinear, but differs essentially from the specified nonlinearity F(x, px) in that it represents usually a smoothed curve (Fig. 10.1). This smoothed characteristic lends itself in many cases to ordinary linearization, i.e., to replacement by a straight line tangent at the origin (Fig. 10.1), namely

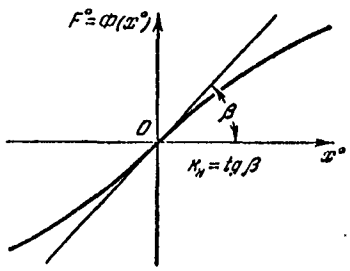


Fig. 10.1

$$F^0 = k_н x^0, \text{ where}$$

$$k_н = \left(\frac{d\Phi}{dx^0}\right)_{x^0=0} = \operatorname{tg}\beta. \tag{10.11}$$

In the case of odd-symmetry nonlinearities, as was shown in Chapter 5, we can assume that

$$k_н = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0}, \tag{10.12}$$

i.e., we can use the function (10.9) directly without reducing it to the form (10.10).

As a result of this ordinary method of linearization of a nonlinear characteristic (10.10) smoothed with the aid of self-oscillations, we obtain in place of (10.6) a purely linear differential equation for the slowly varying component

$$[Q(p) + k_н R(p)]\, x^0 = S(p) f(t), \tag{10.13}$$

where f(t) is a stationary random function of the time.

Let us note that in §10.4 below we shall consider also a case when ordinary linearization (10.11) is either impossible or undesirable.

Thus, to determine the component $x^0(t)$ we must solve the linear

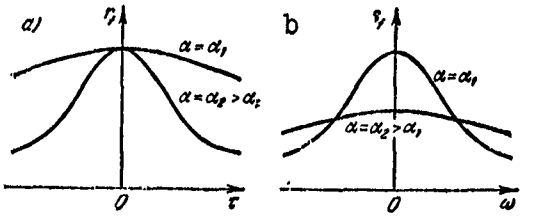


Fig. 10.2

equation (10.13) with stationary random function f(t) in the right half.

We shall assume that the spectral density $s_f(\omega)$ of the external signal f(t) is specified and that its mathematical expectation is equal to zero. If we are given not the spectral density but the correlation function $r_f(\tau)$, we can always determine

$$s_f(\omega) = 2\int_0^\infty r(\tau)\cos\omega\tau d\tau. \qquad (10.14)$$

For example, a frequently used expression for the correlation function of the external signal has the form (Fig. 10.2a)

$$r_f(\tau) = e^{-\alpha|\tau|}, \qquad (10.15)$$

for which we have in accordance with Formula (10.14) (Fig. 10.2b)

$$s_f(\omega) = \frac{2\alpha}{\alpha^2+\omega^2}. \qquad (10.16)$$

Having a definite spectral density $s_f(\omega)$ of the external signal f(t), we can use the known formula to calculate on the basis of (10.13) the spectral density of the slowly varying component x^0 in the form

$$s_{x^0}(\omega) = \left|\frac{S(j\omega)}{Q(j\omega)+k_\mu R(j\omega)}\right|^2 s_f(\omega), \qquad (10.17)$$

and knowing this, we can also determine in the familiar manner the dispersion (or the mean value of the square) of the slowly varying component

$$\sigma^2_{x^0} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} s_{x^0}(\omega)\,d\omega, \qquad (10.18)$$

where its square root, i.e., σ_{x^0}, will represent the mean square value of the slowly varying component $x^0(t)$. If the latter has the meaning of the system error, then σ_{x^0} yields therefore the mean square error of the given system relative to the slowly varying component.

Usually in real problems the spectral density (10.17) of the variable x^0 is a ratio of polynomials in the form

$$s_{x^0}(\omega) = h \frac{G(\omega)}{|H(j\omega)|^2}, \tag{10.19}$$

where h is a constant factor.

In this case the polynomial $H(j\omega)$ satisfies after the substitution $j\omega = p$ the stability criterion,* and the polynomial $G(\omega)$ has a degree lower than 2n if n is the degree of $H(j\omega)$.

Inasmuch as the denominator of (10.19) is an even function, all the terms of the numerator with odd powers of ω yield zero upon integration of (10.18). Consequently, we need consider in the polynomial $G(\omega)$ only even powers of ω, and all the odd powers can be discarded.

Thus,

$$\sigma_{x^0} = h \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{G(\omega)}{|H(j\omega)|^2} d\omega, \tag{10.20}$$

where the polynomials under the integral signs have the form

$$\left.\begin{aligned} H(j\omega) &= a_0(j\omega)^n + a_1(j\omega)^{n-1} + \ldots + a_n, \\ G(\omega) &= b_0\omega^{2n-2} + b_1\omega^{2n-4} + \ldots + b_{n-1}. \end{aligned}\right\} \tag{10.21}$$

The integrals

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{G(\omega)}{|H(j\omega)|^2} d\omega$$

have been previously calculated for all degrees of n up to the seventh inclusive (see, for example, the appendix to the book [322]). We give the ready-made formulas

$$
\left.
\begin{aligned}
I_1 &= \frac{b_0}{2a_0a_1}, \quad I_2 = \frac{-b_0 + \frac{a_0b_1}{a_2}}{2a_0a_1}, \quad I_3 = \frac{-a_2b_0 + a_0b_1 - \frac{a_0a_1b_2}{a_2}}{2a_0(a_0a_3 - a_1a_2)}, \\
I_4 &= \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_3 + \frac{a_0b_2}{a_4}(a_0a_3 - a_1a_2)}{2a_0(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3)}, \\
I_5 &= \frac{N_5}{2a_0\Delta_5}, \\
N_5 &= b_0(-a_0a_4a_5 + a_1a_4^2 + a_2^2a_5 - a_2a_3a_4) + \\
&\quad + a_0b_1(-a_2a_5 + a_3a_4) + a_0b_2(a_0a_5 - a_1a_4) + \\
&\quad + a_0b_3(-a_0a_3 + a_1a_2) + \frac{a_0b_4}{a_5}(-a_0a_1a_5 + a_0a_3^2 + a_1^2a_4 - a_1a_2a_3), \\
\Delta_5 &= a_0^2a_5^2 - 2a_0a_1a_4a_5 - a_0a_2a_3a_5 + a_0a_3^2a_4 + a_1^2a_4^2 + \\
&\quad + a_1a_2^2a_5 - a_1a_2a_3a_4.
\end{aligned}
\right\} \qquad (10.22)
$$

As a result we obtain the following formula for the practical calculations

$$\sigma_{x0}^2 = hI_n. \qquad (10.23)$$

There is also another method (approximate) for the calculation of the integral of the spectral density, based on the use of its transform in logarithmic coordinates, with the curves replaced by broken lines (in analogy with the logarithmic frequency characteristics).

After calculating in this manner the dispersion σ_{x0}^2, meaning also the mean square value σ_{x0} of the slowly varying random component $x^0(t)$ of the sought solution (10.2), we can determine also the probability characteristics of the amplitude A of the self-oscillation component, which in accordance with (10.8) has already been expressed in terms of the slowly varying component x_0. These characteristics are the mathematical expectation, i.e., the mean value of the amplitude $\tilde{A}$ and the dispersion σ_A^2, or the mean square deviation of the amplitude σ_A, which characterizes the scatter of the values of the amplitude about its mean value $\tilde{A}$. From the known formulas we get

$$\tilde{A} = \int_{-\infty}^{+\infty} A(x^0)\, w(x^0)\, dx^0, \qquad (10.24)$$

$$\sigma_A^2 = \int_{-\infty}^{+\infty} [A(x^0) - \tilde{A}]^2 w(x^0)\, dx^0, \qquad (10.25)$$

where $w(x^0)$ is the differential distribution of the random quantity x^0. Assuming it to be normal and recognizing that the mathematical expectation of x^0 is equal to zero,* we obtain (Fig. 10.3):

$$w(x^0) = \frac{1}{\sigma_{x^0}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x^0}{\sigma_{x^0}}\right)^2}. \qquad (10.26)$$

We note that the integration limits in Formulas (10.24) and (10.25) will actually not be infinite, inasmuch as the limit is a variation of the quantity x^0 in the expression $A(x^0)$, within which the self-oscillation and the smoothed characteristic occur, are bounded.

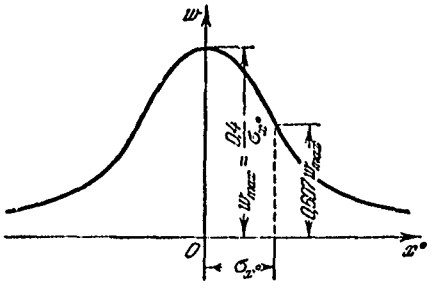


Fig. 10.3

An example of the application of the method developed here will be given in §10.5 below.

The same method of solving the problem can be extended to include also other systems of different classes (§1.2), including a system with several nonlinearities, similar to what was done in the study of other phenomena in the preceding chapters.

Let us turn now to nonlinear systems that operate in the forced oscillation mode. Their investigation will be based on the material of §9.2. Assume that two external signals are applied to the system at two different places, and that the system equation has the form

$$Q(p)x + R(p)F(x, px) = S_1(p)f_1(t) + S_2(p)f_2(t), \qquad (10.27)$$

where $f_1(t)$ is a slowly varying external signal, representing a stationary random process, and $f_2(t)$ is a nonrandom periodic external signal

$$f_2(t) = B \sin \Omega_v t, \qquad (10.28)$$

with B and Ω_v specified quantities. Such a periodic signal can be used, for example, in real systems for vibration smoothing of nonlinearities.

Another example is the vibration noise produced, for example, by the vibrations of the housing of the controlled object or by other factors. Nonlinear systems subject to random noise will be considered in §10.3 below.

Assuming that we have single-frequency forced oscillations (the locking mode), we seek a solution in the form

$$x = x^0 + x^*, \quad \text{where} \quad x^* = A_v \sin(\Omega_v t + \varphi), \quad (10.29)$$

with $x^0(t)$ the slowly varying random component. The amplitude A_v and the phase φ of the oscillatory component, which depend on x^0, will also be random quantities.

Carrying out harmonic linearization of the nonlinearities by means of Formulas (9.25) and (9.26), and using Expression (9.28), we break up the specified system equation (10.27) into two

$$Q(p)\,x^0 + R(p)\,F^0 = S_1(p)\,f_1(t), \quad (10.30)$$

$$\left[Q(p) - S_2(p)\,\frac{B}{A_v}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_v}p\right)\right]x^* + \\ + R(p)\left(q + \frac{q'}{\Omega_v}p\right)x^* = 0. \quad (10.31)$$

The second of these equations, as in §9.2, is represented in the form

$$A_v\,\frac{Q(j\Omega_v) + R(j\Omega_v)(q + jq')}{S_2(j\Omega_v)} = Be^{-j\varphi}. \quad (10.32)$$

From this we can determine by any of the two methods described in §9.1 the dependence of the amplitude A_v and the phase φ of the forced oscillations on the random slowly varying component x^0 (for the time being unknown and included in the coefficients $\underline{g}$ and q') and on the specified numbers B and Ω_v, namely

$$A_v(x^0, \Omega_v, B), \quad \varphi(x^0, \Omega_v, B). \quad (10.33)$$

Let us turn now to the first expression in (9.26), which has the form

$$F^0(x^0, A_v, \Omega_v), \quad (10.34)$$

and let us substitute in it the expression for the amplitude (10.33). We then obtain the bias function*

$$F^0 = \Phi(x^0, \Omega_в, B). \tag{10.35}$$

We employ further ordinary linearization (and if this is impossible we must resort to the methods developed in §10.4) of this bias function (Fig. 10.1) in the form (10.11) or (10.12), where F^0 is given by (10.35) and (10.34). The essential feature here is that according to (10.11) and (10.35) the coefficient k_n will depend on the amplitude B and on the frequency Ω_v of the external vibration signal

$$k_н(B, \Omega_в), \tag{10.36}$$

whereas in the case of self-oscillations it is dependent only on the parameters of the system itself.

In the case of useful applications of forced oscillations, this provides the engineer with greater capabilities in the design of the system, and in the presence of vibration noise this can lead to harmful influence of the vibration on the stability of the entire system, as was illustrated in §9.6.

Thus, on the basis of (10.30) we can now write a purely linear equation for the determination of the slowly varying random component

$$[Q(p) + k_н R(p)]\, x^0 = S_1(p) f_1(t). \tag{10.37}$$

Further solution of the problem proceeds via the use of exactly the same formulas that follow Eq. (10.13), up to Formula (10.26) inclusive. It is merely necessary to replace in Formulas (10.24) and (10.25) the value of A by the value of A_v defined by the first of the expressions in (10.33). Here, unlike the preceding, all the results of the solution will depend not only on the system parameters and on the spectral density of the random signal $f_1(t)$, but also on the specified values of the amplitude B and frequency Ω_v of the external periodic signal $f_2(t)$.

723

§10.2. Statistic Linearization of Nonlinearities

The simplicity of the solution of the problem in the preceding section was brought about by the fact that we investigated slowly varying random processes in a system whose nonlinear characteristics were smoothed by means of self-oscillations or forced oscillations, and were then subjected to ordinary linearization. We could therefore employ in its entirety the linear theory of random processes. On the other hand, the nonlinear part of the solution of the problem coincided with that of Chapter 5 and of §9.2, and consisted of finding the smoothed characteristic (bias function) itself and the dependence of the amplitude and frequency of the oscillatory component on the magnitude of the slowly varying component.

We now turn to a solution of other problems, when there are no self-oscillations or forced oscillations, or when they are present but the bias function cannot be or is best not subjected to ordinary linearization, or else finally, when high-frequency random interference is present in the system.

In many such cases, in analyzing nonlinear automatic systems of the same classes as in all the preceding chapters but under random signals, it will be convenient to employ the so-called statistical linearization of nonlinearities, developed by I.Ye. Kazakov [309]. Its gist is as follows.

As in the preceding section, to estimate the dynamic accuracy of automatic systems under random signals we shall determine the first two probability moments of the random processes: the mathematical expectation (mean value) and the dispersion (or mean square deviation). The latter, as was already shown, is equivalent to determining the spectral density or the correlation function.

If the nonlinear system is described by the differential equation

$$Q(p)x + R(p)F(x, px) = S(p)f(t), \qquad (10.38)$$

then we can schematically visualize the signal flow as shown in Fig. 10.4. Upon passing through the linear part, the random process f(t),

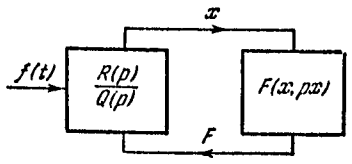


Fig. 10.4

specified in terms of the first two probability moments, is transformed into a variable x, which can also be defined by the first two moments. However, the determination of the further transformation of the random process x(t) in the nonlinear element F(x, px) is essentially related with the higher probability moments (just as in §2.2 we had to deal with higher harmonics). Since this is a closed loop system, this situation leaves its imprint on all the processes in the system. Consequently, an exact solution of the problem is in most cases unattainable.

A first approximation which is sufficiently good for engineering calculations, as applied to the system classes considered under the same limitations as previously (see §§2.2 and 2.3), neglects the higher moments, i.e., the nonlinear element is replaced by an equivalent linear element, which like the given nonlinear element converts the first two probability moments, namely the mathematical expectation (mean value) and dispersion (or mean square deviation). This is what we call statistical linearization of the nonlinearities.

This operation is analogous in the general idea (but not in the specific details) to the manner in which in Chapter 5 (see also Formula (10.3)) we replaced the nonlinear element with the aid of harmonic linearization by an equivalent linear element, which in exactly the same way as the given nonlinear element transformed the constant (or slowly varying) component and the first harmonic of the oscillatory component, i.e., the first two terms of the Fourier series were taken into account and all the higher harmonics discarded.

Thus, let us represent the variable $\underline{x}$ under the nonlinearity sign F(x, px) in the form

$$x = \tilde{x} + x^{sl}, \tag{10.39}$$

where $\tilde{x}$ is the mathematic expectation (mean value) which is a regular function of the time, and x^{sl} is a random component with zero mathematical expectation (centered random function of the time). This representation is analogous to the one given in Chapter 5 in the harmonic linearization (see also Formula (10.2)), but here it has an entirely different, probabilistic meaning. Further, we represent also the variable F(x, px) in the form

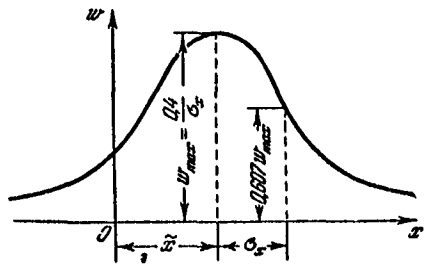


Fig. 10.5

$$F(x,\ px) = \tilde{F} + q^{sl} x^{sl}, \tag{10.40}$$

where $\tilde{F}$ is the mathematical expectation (mean value) of the nonlinear function F, which is the regular component, and q^{sl} is the equivalent gain of the random (centered) component. This expression is also analogous in form to the one used in Chapter 5 (see Formula (10.3)), but with a different concrete content.

The value of the regular component $\tilde{F}$ is determined consequently from the known formula for the mathematical expectation. In the case of a single-valued nonlinear function F(x) this will be

$$\tilde{F} = M[F(\tilde{x} + x^{sl})] = \int_{-\infty}^{+\infty} F(\tilde{x} + x^{sl})\, w(x)\, dx, \tag{10.41}$$

where M stands for the operation of taking the mathematical expectation and w(x) is the differential distribution of the random component, for example, a normal distribution (Fig. 10.5):

$$w = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\tilde{x}}{\sigma_x}\right)^2}. \tag{10.42}$$

For a nonlinearity of general form F(x, px) we have the more complicated expression

$$\tilde{F}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(\tilde{x}+x^{c.a},\ p\tilde{x}+px^{c.a})\,w(x,\ px)\,dx\,dpx, \qquad (10.43)$$

which for loop-type nonlinearities F(x) with a symmetrical distribution law (including normal distribution) becomes simpler. For example, for the nonlinearity shown in Fig. 10.6 we have

$$\tilde{F}=\int_{-\infty}^{-b_1}F(\tilde{x}+x^{c.a})\,w(x)\,dx+ \\ +\int_{-b_1}^{b_2}\frac{1}{2}[F_1(\tilde{x}+x^{c.a}+F_2(\tilde{x}+x^{c.a})]\,w(x)\,dx+ \\ +\int_{b_2}^{+\infty}F(\tilde{x}+x^{c.a})\,w(x)\,dx. \qquad (10.44)$$

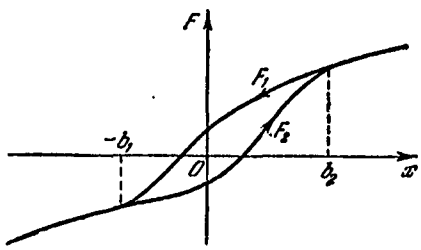


Fig. 10.6

It is recommended [309] that the value of the equivalent gain q^{sl} of the random component in (10.40) be determined by one of the following two methods.

The first method starts directly from the values of the mean square deviations σ_x and σ_F of the variable $\underline{x}$ and of the nonlinear function F, namely

$$q^{c.a}=\frac{\sigma_F}{\sigma_x}=\sqrt{\frac{M[(F^{c.a})^2]}{M[(x^{c.a})^2]}}, \qquad (10.45)$$

which in the case of a single-valued nonlinearity F(x) yields

$$q^{c.a}=\frac{1}{\sigma_x}\sqrt{\int_{-\infty}^{+\infty}F^2(\tilde{x}+x^{c.a})\,w(x)\,dx-\tilde{F}^2}, \qquad (10.46)$$

and for the general case F(x, px) or for the case of a loop-type nonlinearity F(x) more complicated expressions are obtained, which can be set up for q^{sl} by generalizing (10.46) using the same model as was used to generalize Expressions (10.43) and (10.44) as compared with (10.41).

The second method consists of determining the coefficient q^{sl} from the condition of minimum mathematical expectation of the square of the difference of the true nonlinear function F(x, px) and the equation (10.40) that replaces it, i.e., the minimum of the mean squared deviation. Writing down this condition

$$M\{[F(x, px) - \tilde{F} - q^{c.s} x^{c.s}]^2\} = \min,$$

we obtain

$$q^{c.s} = \frac{M[F^{c.s} x^{c.s}]}{M[(\dot{x}^{c.s})^2]} = \frac{r_{Fx}}{\sigma_{\dot{x}}^2}, \qquad (10.47)$$

where r_{F_x} is the mutual correlation function of the variables F and $\underline{x}$. From this we obtain in the case of a single-valued nonlinearity F(x)

$$q^{c.s} = \frac{1}{\sigma_{\dot{x}}^2} \int_{-\infty}^{+\infty} F(\tilde{x} + x^{c.s})\, x w(x)\, dx. \qquad (10.48)$$

In analogy with the foregoing, we can readily set up an expression also for the coefficient q^{sl} for the general case F(x, px) and for a loop-type nonlinearity F(x).

The second method of determining the coefficient q^{sl} leads to simpler calculation formulas. From this point of view, its use is preferable. In accuracy both are approximately equivalent and correspond to the general degree of approximation of the entire method as a whole. It is noted that in many cases when the first of these methods gives overestimates of the correlation function of the nonlinear process F(t) compared with the exact value, the second method gives underestimates. One can therefore obtain a very good approximation by choosing for the value of q^{sl} the arithmetic mean of both (10.47) and (10.45).

It must be kept in mind that the quantities $\tilde{F}$ and q^{sl} are interrelated by the fact that each depends on both considered characteristics of the random process $\tilde{x}$ and σ_x (which is contained in the distribution law $\underline{w}$). The very existence of these relationships and their

mutual relationship make it indeed possible, in spite of the linearization of the problem, to detect the essentially nonlinear properties of the random processes, just as in the preceding chapters the dependence of the quantities F^0, $\underline{q}$, and q' on all three unknowns x^0, A, and Ω (or at least on the first two of them) and the interrelationships between these quantities made it possible to investigate essentially nonlinear features of regular processes in time by using the harmonic linearization method.

We present expressions for the quantities $\tilde{F}$ and q^{sl} as well as their plots for several typical nonlinearities, as obtained from Formulas (10.41), (10.46), and (10.48) by I.Ye. Kazakov [317] under the condition that the random variable $\underline{x}$ has a normal distribution (10.42) (in the case of other distribution laws, the expressions for $\tilde{F}$ and q^{sl} would be different).

1. Ideal relay characteristic (Fig. 10.7a). From Formula (10.41) we get

$$\tilde{F}=c\Phi(u), \quad u=\frac{\tilde{x}}{\sigma_x\sqrt{2}}, \tag{10.49}$$

where

$$\Phi(u)=\frac{2}{\sqrt{2}}\int_0^u e^{-y^2}dy \tag{10.50}$$

(numerical values of this probability integral are contained in the book [322] and in some collections of mathematical tables). The dependence (10.49) of the quantity F/c on the ratio $\tilde{x}/\sigma_x$ is plotted in Fig. 10.7b.

From (10.46) and (10.48) we obtain, respectively,

$$q^{c,l}=\frac{c}{\sigma_x}\varphi^{(1)}(\tilde{x},\ \sigma_x) \text{ and } q^{c,l}=\frac{c}{\sigma_x}\varphi^{(2)}(\tilde{x},\ \sigma_x), \tag{10.51}$$

where

$$\varphi^{(1)}=\sqrt{1-\Phi^2(u)}, \quad \varphi^{(2)}=\frac{2}{\sqrt{2\pi}}e^{-u^2}. \tag{10.52}$$

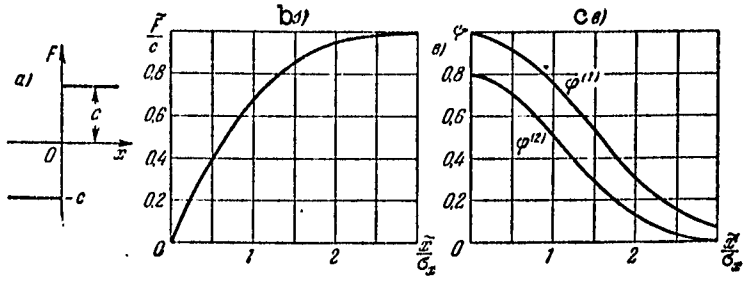


Fig. 10.7

The dependences of $\varphi^{(1)}$ and $\varphi^{(2)}$ on $\tilde{x}/\sigma_x$ are shown in Fig. 10.7c.

2. Single-valued relay characteristic with backlash zone (Fig. 10.8a). Using Formula (10.41) with the notation (10.50) we obtain

$$\tilde{F} = \frac{c}{2}[\Phi(u_1) - \Phi(u_2)], \tag{10.53}$$

where

$$u_1 = \frac{1+\tilde{x}_1}{\sigma_1\sqrt{2}}, \quad u_2 = \frac{1-\tilde{x}_1}{\sigma_1\sqrt{2}}, \quad \tilde{x}_1 = \frac{\tilde{x}}{b}, \quad \sigma_1 = \frac{\sigma_x}{b}. \tag{10.54}$$

The function $\tilde{F}/c$ is plotted in Fig. 10.8b as a function of $\tilde{x}_1$ for different values of σ_1.

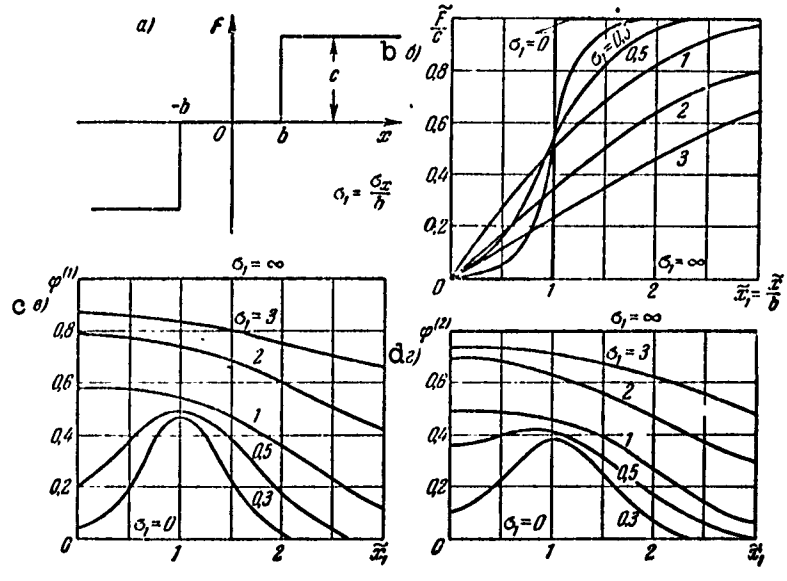


Fig. 10.8

From Formulas (10.46) and (10.48) we obtain expressions similar to (10.51), where

$$\varphi^{(1)} = \sqrt{1 - \left(\frac{\tilde{F}}{c}\right)^2} - \frac{1}{2}[\Phi(u_1) + \Phi(u_2)], \tag{10.55}$$

$$\varphi^{(2)} = \frac{1}{\sqrt{2\pi}}(e^{-u_1^2} + e^{-u_2^2}), \tag{10.56}$$

which are plotted in Figs. 10.8c and d.

3. Loop-type relay characteristic of general form (Fig. 10.9a). From Formulas (10.44) we get

$$\tilde{F} = \frac{c}{4}[\Phi(u_1) - \Phi(u_2) + \Phi(u_3) - \Phi(u_4)], \tag{10.57}$$

where, in addition to (10.54) and (10.50), we introduce also the notation

$$u_3 = \frac{m + \tilde{x}_1}{\sigma_1\sqrt{2}}, \quad u_4 = \frac{m - \tilde{x}_1}{\sigma_1\sqrt{2}}. \tag{10.58}$$

The dependence of $\tilde{F}/c$ for the case m = 0.5 is shown in Fig. 10.9b.

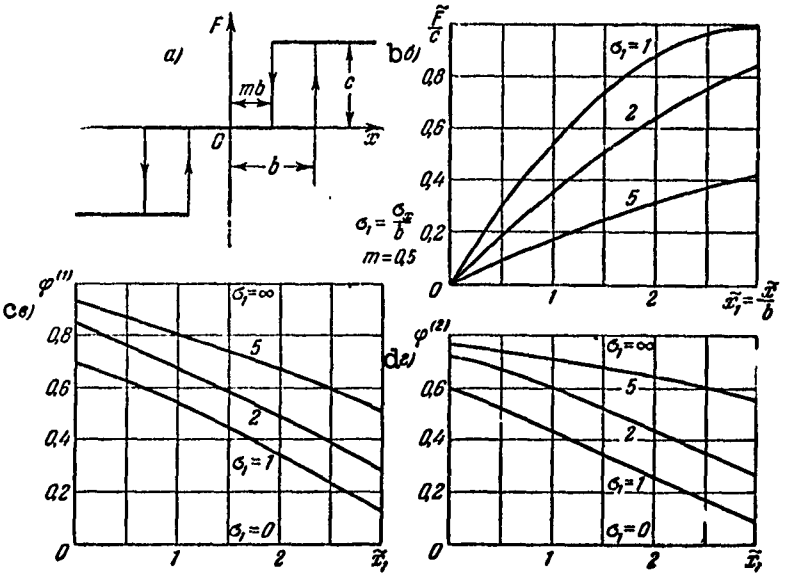


Fig. 10.9

We further obtain expressions similar to (10.51), where

$$\varphi^{(1)} = \sqrt{1 - \left(\frac{\tilde{F}}{c}\right)^2} - \frac{1}{4}[\Phi(u_1) + \Phi(u_2) + \Phi(u_3) + \Phi(u_4)], \tag{10.59}$$

$$\varphi^{(2)} = \frac{1}{2\sqrt{2\pi}}(e^{-u_1^2} + e^{-u_2^2} + e^{-u_3^2} + e^{-u_4^2}). \tag{10.60}$$

These functions are plotted for the case m = 0.5 in Figs. 10.9c and d.

4. Characteristic of the saturation type (Fig. 10.10a). From For-

mula (10.41) using the notation of (10.50) and (10.54) we get

$$\frac{\tilde{F}}{c}=\frac{\tilde{x}_1+1}{2}\Phi(u_1)+\frac{\tilde{x}_1-1}{2}\Phi(u_2)+$$
$$+\frac{\sigma_1}{\sqrt{2\pi}}(e^{-u_1^2}-e^{-u_2^2}), \qquad (10.61)$$

which is plotted as a function of $\tilde{x}_1$ for different σ_1 in Fig. 10.10b.

On the other hand, from Formulas (10.46) and (10.48) we obtain Expression (10.51), in which

$$\varphi^{(1)}=\left\{1-\left(\frac{\tilde{F}}{c}\right)^2+\frac{1}{2}(\tilde{x}_1^2+\sigma_1^2+1)[\Phi(u_1)+\Phi(u_2)]-\right.$$
$$\left.-\sigma_1^2(u_2e^{-u_1^2}+u_1e^{-u_2^2})\right\}^{\frac{1}{2}}, \qquad (10.62)$$

$$\varphi^{(2)}=\frac{\sigma_1}{2}[\Phi(u_1)+\Phi(u_2)], \qquad (10.63)$$

which is plotted in Fig. 10.10c and d.

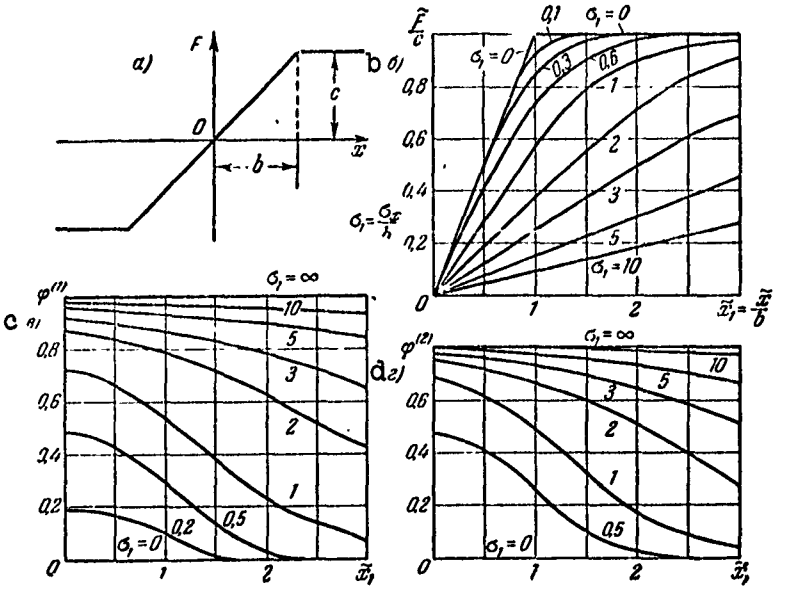


Fig. 10.10

5. Characteristic with backlash zone and saturation (Fig. 10.11a). In the notation of (10.50) and (10.54) we obtain here

$$\frac{\tilde{F}}{c}=\frac{\tilde{x}_1}{2(1-m)}[\Phi(u_2)-\Phi(u_1)+\Phi(u_4)-\Phi(u_3)]+$$
$$+\frac{1}{2}[\Phi(u_1)-\Phi(u_2)]+\frac{\sigma_1}{(1-m)\sqrt{2\pi}}(e^{-u_1^2}-e^{-u_2^2}+e^{-u_1^2}-e^{-u_3^2}), \qquad (10.64)$$

and in the expressions of (10.51) we have

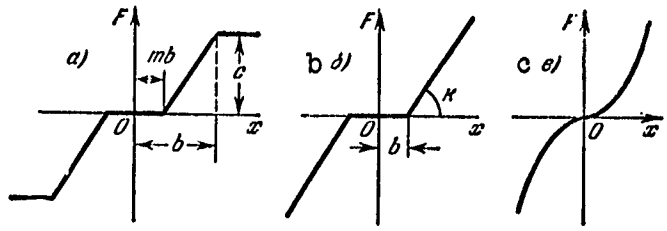


Fig. 10.11

$$\varphi^{(1)}=\left\{1-\left(\frac{\tilde{F}}{c}\right)^2-\frac{1}{2}[\Phi(u_1)+\Phi(u_2)]+\right.$$
$$+\frac{\tilde{x}_1^2-\sigma_1^2}{2(1-m^2)}[\Phi(u_1)+\Phi(u_2)-\Phi(u_3)-\Phi(u_4)]-$$
$$\left.-\frac{\sigma_1^2}{(1-m^2)\sqrt{\pi}}(u_2e^{-u_1^2}+u_1e^{-u_2^2}-u_4e^{-u_3^2}-u_3e^{-u_4^2})\right\}^{\frac{1}{2}}, \qquad (10.65)$$

$$\varphi^{(2)}=\frac{\sigma_1}{2(1-m)}[\Phi(u_1)+\Phi(u_2)-\Phi(u_3)-\Phi(u_4)]-$$
$$-\frac{1}{\sqrt{2\pi}}(e^{-u_1^2}+e^{-u_2^2})-\frac{1}{(1-m)\sqrt{2\pi}}(e^{-u_1^2}+e^{-u_2^2}-me^{-u_3^2}-me^{-u_4^2}). \qquad (10.66)$$

6. Characteristic with backlash zone (Fig. 10.11b). For this characteristic we have

$$\tilde{F}=kb\left\{\tilde{x}_1+\frac{1-\tilde{x}_1}{2}[\Phi(u_1)-\Phi(u_2)]-\frac{\sigma_1}{\sqrt{2\pi}}(e^{-u_1^2}-e^{-u_2^2})\right\}, \qquad (10.67)$$

and from Formulas (10.46) and (10.48) we obtain, respectively,

$$q^{c.s}=\frac{kb}{\sigma_x}\varphi^{(1)}(\tilde{x},\sigma_x)\ \text{and}\ q^{c.s}=\frac{kb}{\sigma_x}\varphi^{(2)}(\tilde{x},\sigma_x), \qquad (10.68)$$

where

$$\varphi^{(1)}=\left\{\sigma_1^2-1-\left(\frac{\tilde{F}}{kb}\right)^2+\tilde{x}_1^2-\sigma_1^2\left[\left(\frac{1}{2}+u_1^2\right)\Phi(u_1)+\right.\right.$$
$$\left.\left.+\left(\frac{1}{2}+u_2^2\right)\Phi(u_2)\right]+\frac{u_1}{\sqrt{\pi}}e^{-u_1^2}+\frac{u_2}{\sqrt{\pi}}e^{-u_2^2}\right\}^{\frac{1}{2}}, \qquad (10.69)$$

$$\varphi^{(2)}=\sigma_1-\frac{\sigma_1}{2}[\Phi(u_1)+\Phi(u_2)]+\frac{1}{\sqrt{2\pi}}(e^{-u_1^2}+e^{-u_2^2}). \qquad (10.70)$$

7. Quadratic characteristic (Fig. 10.11c):

$$F=kx^2\operatorname{sign}x=k|x|x. \qquad (10.71)$$

For this characteristic we obtain from (10.41)

$$\tilde{F}=k\sigma_x^2\left[(1+2u^2)\Phi(u)+\frac{2u}{\sqrt{2\pi}}e^{-u^2}\right], \qquad (10.72)$$

where

$$u=\frac{\tilde{x}}{\sigma_x\sqrt{2}}, \qquad (10.73)$$

and from Formulas (10.46) and (10.48) we obtain, respectively,

$$q^{c.l} = \frac{k}{\sigma_x}\varphi^{(1)}(\tilde{x}, \sigma_x) \text{ and } q^{c.l} = \frac{k}{\sigma_x}\varphi^{(2)}(\tilde{x}, \sigma_x), \tag{10.74}$$

where

$$\varphi^{(1)} = \sigma_x^2\sqrt{4u^4 + 12u^2 + 3 - \left(\frac{\tilde{F}}{k\sigma_x^3}\right)^2}, \tag{10.75}$$

$$\varphi^{(2)} = \frac{\sigma_x^2}{\sqrt{2}}\left[u\Phi(u) + \frac{4}{\sqrt{\pi}}(1 - 3u^2)e^{-u^2}\right]. \tag{10.76}$$

8. Cubic characteristic (Fig. 10.12a):

$$F = kx^3. \tag{10.77}$$

Using the notation of (10.73) we obtain here

$$\tilde{F} = k\sigma_x^3(3 + 2u^2)u\sqrt{2}, \tag{10.78}$$

and also an expression of the form (10.74) in which

$$\varphi^{(1)} = \sigma_x^2\sqrt{15 + 72u^2 + 36u^4}, \tag{10.79}$$

$$\varphi^{(2)} = 3\sigma_x^2(1 + 2u^2), \tag{10.80}$$

which is plotted in Figs. 10.12b and c.

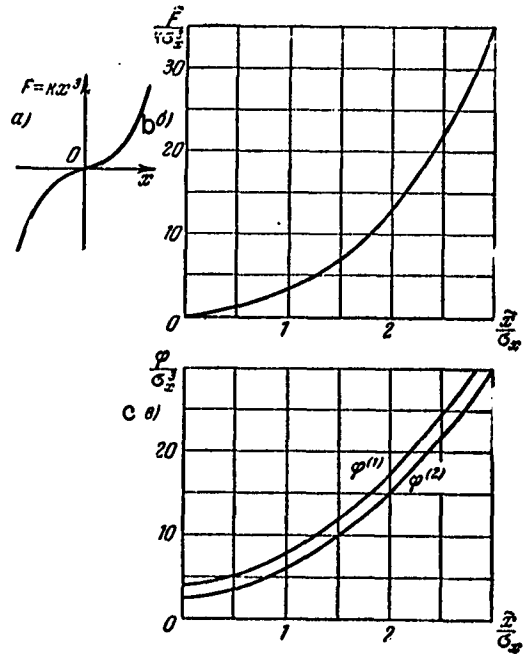


Fig. 10.12

The use of the statistical linearization described here for an investigation of random process will be described in the following sec-

tions.

§10.3. High-Frequency Random Processes

In the present section we consider problems in which the regular component $\tilde{x}$ of the process (mathematical expectation) is constant or varies in time slowly compared with the fundamental frequencies of the spectrum of the random component x^{sl}. We first turn to nonlinear systems of the first class* (see §1.2), the dynamics of which is described by equations of the form

$$Q(p)x + R(p)F(x, px) = S(p)f(t), \tag{10.81}$$

where f(t) is the external signal, representing a random process, with

$$f(t) = \tilde{f} + f^{cs}(t). \tag{10.82}$$

Here $\tilde{f}$ is the specified mathematical expectation (regular component), and f^{sl} is the centered random component.

Let the system parameters be such that there are no self-oscillations and the system is stable relative to the equilibrium state. Using the statistical linearization (10.40) and substituting the expression obtained into the given equation (10.81), we break up the latter into two equations

$$Q(p)\tilde{x} + R(p)\tilde{F} = S(p)\tilde{f}, \tag{10.83}$$

$$[Q(p) + R(p)q^{cs}]x^{cs} = S(p)f^{cs}, \tag{10.84}$$

for the regular (mathematical expectations) and random (centered) components, respectively. Here

$$\tilde{F}(\tilde{x}, \sigma_x), \quad q^{cs}(\tilde{x}, \sigma_x)$$

are determined for each specified nonlinearity, as indicated in §10.2.

Let us consider several principally different problems in general form (specific examples will be given in §§10.6 and 10.7).

First problem. If a stationary process takes place, then the quantities $\tilde{f}$, $\tilde{x}$, σ_x are constant (a certain steady-state mode is established) and Eq. (10.83) assumes the algebraic form

$$Q(0)\tilde{x}+R(0)\tilde{F}(\tilde{x},\ \sigma_x)=S(0)\tilde{f}. \qquad (10.85)$$

It contains two unknowns, $\tilde{x}$ and σ_x. Therefore in principle we can only express the value of $\tilde{x}$ as a function of σ_x:

$$\tilde{x}(\sigma_x). \qquad (10.86)$$

We then use the linear theory of random processes, described in §10.1 (see the formulas (10.14) through (10.23), in which x^0 is replaced by x^{sl}) to investigate the equation (10.84). In this equation the quantity f^{sl} is specified in terms of a spectral density $s_f(\omega)$ or a correlation function $r_f(\tau)$, with Formulas (10.17) and (10.18) yielding

$$\sigma_x^2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{S(j\omega)}{Q(j\omega)+q^{ca}R(j\omega)}\right|^2 s_f(\omega)\,d\omega, \qquad (10.87)$$

where in the expression

$$q^{ca}(\tilde{x},\ \sigma_x) \qquad (10.88)$$

it is necessary to replace $\tilde{x}$ by the previously obtained function (10.86). Then one unknown σ_x remains in (10.87). Taking (10.23) and (10.22) into account, Eq. (10.87) can be written in the form

$$\sigma_x^2=hI_n(\tilde{x},\ \sigma_x), \qquad (10.89)$$

where $\underline{h}$ is a constant factor, which is taken outside the integral sign.

Thus, by solving (10.89) with the substitution (10.86), we obtain the mean square deviation σ_x, and then calculate by means of Formula (10.86) also the mathematical expectation $\tilde{x}$, i.e., we determine completely the sought solution* of Eq. (10.81) in the approximate form

$$x=\tilde{x}+x^{ca} \qquad (10.90)$$

for the case of a steady-state for a stationary random process.

However, the function $\tilde{x}(\sigma_x)$ cannot always be expressed by means of Eq. (10.85) in explicit form, owing to the complexity of the expression $\tilde{F}(\tilde{x},\ \sigma_x)$. In most cases, therefore, it becomes necessary to solve the two equations (10.85) and (10.89) simultaneously either by successive approximation numerically, or graphically.

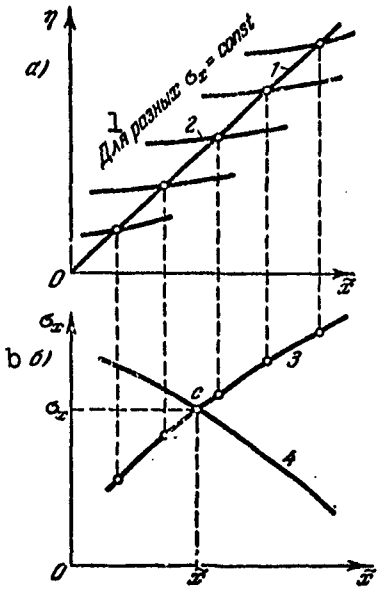


Fig. 10.13. 1) For different.

One can use, for example, the following graphic procedure [322]. We represent (10.85) in the form of two equations

$$\left.\begin{aligned}\eta &= \tilde{x},\\ \eta &= \frac{S(0)}{Q(0)}\tilde{f} + \frac{R(0)}{Q(0)}\tilde{F}(\tilde{x}, \sigma_x).\end{aligned}\right\} \qquad (10.91)$$

The first yields the line 1 (Fig. 10.13a), and the second a series of curves for different constant values of σ_x. Transferring all the points where these curves cross the line 1 to the plane of the coordinates $\tilde{x}$, σ_x (Fig. 10.13b), we obtain the dependence $\sigma_x(\tilde{x})$ in the form of curve 3, since each point of intersection on the upper plot corresponds to a definite value of σ_x. We then plot on Fig. 10.13b still another dependence $\sigma_x(\tilde{x})$ in the form of curve 4, using Formula (10.89), in the right half of which we substitute the values of σ_x taken for each $\tilde{x}$ from the curve 3. It is obvious that the coordinates of the points of intersection c of curves 3 and 4 represent the sought result of the simultaneous solution of (10.85) and (10.89).

Second problem. Let us proceed now to solve the second problem, in which we investigate a nonstationary process [336].

Frequently in automatic control systems the resolution of the sought solution (10.90) into $\tilde{x}$ and x^{sl} corresponds to its resolution into a useful regular signal $\tilde{x}$ and a random noise x^{sl}. When the useful control signal $\tilde{x}$ varies in time, the process is no longer stationary. However, if the noise (fluctuations) are characterized by a spectrum of appreciably higher frequencies than the useful signal, we can assume the latter to be slowly varying. We can then investigate the random process in first approximation as stationary, using Formula (10.89).

In this case, however, to determine the regular component $\tilde{x}$ we can no longer use the algebraic equation (10.85), but must turn to the differential equation (10.83).

The graphic solution described above does not work in this case and we must proceed differently. We first determine from (10.89) the dependence $\sigma_x(\tilde{x})$. For this purpose, in analogy with the graphical solution of (9.33), we break up (10.89) into two equations

$$\left.\begin{aligned} \sigma_x^2 &= \zeta, \\ hI_n(\tilde{x}, \sigma_x) &= \zeta. \end{aligned}\right\} \quad . \quad (10.92)$$

The first yields the parabola 1 (Fig. 10.14) and the second a series of curves 2 for different constant values of $\tilde{x}$. Transferring the ordinates from the points of intersection to the plane $\tilde{x}$, σ_x and marking for each of these the abscissas $\tilde{x}$ corresponding to Curves 2, we obtain the sought dependence $\sigma_x(\tilde{x})$ in the form of Curve 3 (Fig. 10.14).

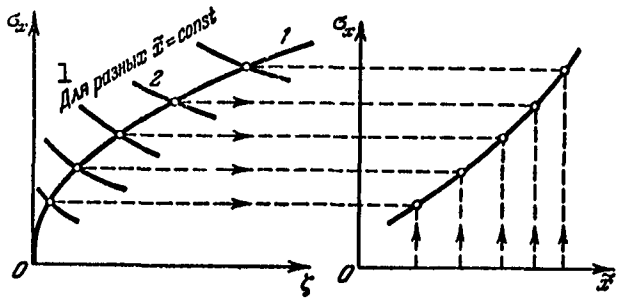


Fig. 10.14. 1) For different.

Substituting the dependence $\sigma_x(\tilde{x})$ obtained into the expression chosen for the specified nonlinearity from §10.2, namely

$$\tilde{F}(\tilde{x}, \sigma_x), \quad (10.93)$$

we eliminate from the latter the quantity σ_x and obtain a function of one variable

$$\tilde{F} = \Phi(\tilde{x}), \quad (10.94)$$

which, as in Chapter 5 and §9.2, can be called the bias function,* for here the mathematical expectations $\tilde{x}$ and $\tilde{F}$ represent the shifts (bias)

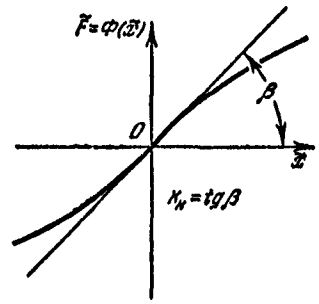


Fig. 10.15

of the random components.

Once the bias function (10.94) is determined, it can be substituted into (10.83):

$$Q(p)\tilde{x}+R(p)\Phi(\tilde{x})=S(p)\tilde{f}(t) \qquad (10.95)$$

from which we determine for the specified function $\tilde{f}(t)$ the regular component of the process $\tilde{x}(t)$ by solving the differential equation.

In most problems the bias function (10.94) will have the form of a smooth curve (Fig. 10.15) which within certain limits can be subjected to ordinary linearization

$$\tilde{F}=k_н\tilde{x}, \quad k_н=\left(\frac{d\Phi}{d\tilde{x}}\right)_{\tilde{x}=0}=\operatorname{tg}\beta. \qquad (10.96)$$

If the system is such that the linear part with transfer function

$$R(p)/Q(p)$$

does not pass the frequency spectrum corresponding to the fluctuations $f^{sl}(t)$ and defined by the spectral density $s_f(\omega)$, the determination of σ_x becomes much simpler, for we obtain from (10.87)

$$\sigma_x^2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{S(j\omega)}{Q(j\omega)}\right|^2 s_f(\omega)\,d\omega, \qquad (10.97)$$

i.e., σ_x will not depend on the form of the nonlinearity or on the value of $\tilde{x}$.

In this case we can determine k_n not by differentiating the bias function (10.94), but directly* from (10.93)

$$\tilde{F}=k_н\tilde{x}, \quad k_н=\left(\frac{d\tilde{F}}{d\tilde{x}}\right)_{x=0}. \qquad (10.98)$$

We obtain k_n here as a function of σ_x:

$$k_н=k_н(\sigma_x). \qquad (10.99)$$

It is then necessary to substitute the quantity σ_x, obtained from Formula (10.97), or simply take from Figs. 10.7b-10.12b the curve corresponding to the obtained value of σ_x. In this case the calculation of

the integral (10.97) is carried out by means of the ready-made formulas (10.22) and (10.23).

As a result of substitution of (10.96) or (10.98), the equation for the determination of the regular component (10.95) becomes linear

$$[Q(p)+k_{\text{н}}R(p)]\,\mathring{x}=S(p)\,\mathring{f}(t) \qquad (10.100)$$

and is solved as such on the basis of the ordinary characteristic equation

$$Q(p)+k_{\text{н}}R(p)=0. \qquad (10.101)$$

It is important to note, however, the following. According to Formulas (10.87) and (10.97) the value of σ_x depends on the spectral density of the interference $s_f(\omega)$. Consequently, the form of the bias function (10.94) and its slope k_n (Fig. 10.15), determined in terms of the quantity σ_x, also depend not only on the parameters of the system itself but also on the spectral density of the interference $s_f(\omega)$. But if k_n depends on $s_f(\omega)$, then, in accord with (10.100) and (10.101), all the static and dynamic properties and even the stability of the system relative to the useful signal will depend not only on the system parameters themselves, but also on the spectral density parameters of the external random noise. Consequently, a nonlinear system which is stable in the absence of noise may lose its properties at a definite noise level, i.e., it may go out of order as an automatic control system not because the system ceases to filter the useful signal, as is usually the case, but because the main control loop changes its dynamic properties with change in k_n, or may even become unstable.

Cases are possible when this phenomenon, which is specific of nonlinear systems, will occur for the system, which is calculated in the linear approximation, ceases to filter the useful signal. From this point of view, an account of the nonlinearities actually present in the automatic control system in the presence of high frequency (com-

pared with the useful signal) noise becomes exceedingly important for practical purposes. This is just as important as allowance for the influence of the vibrational sinusoidal noise considered above in §§9.2 and 9.6. The results of solving both these problems are analogous (see the example in §10.6 below).

It is obvious that the foregoing influence of noise, which is specific to nonlinear systems, can in some cases also improve the dynamic properties of the system.

The attractive aspect of the method described here is that an investigation of the transient properties of the processes, of all the frequency characteristics, and of other properties of the control system with respect to the useful (control) signal is carried out using any of the methods of ordinary linear theory of automatic control by means of Eq. (10.100). In spite of this linearization of the problem solution, all the specific nonlinear phenomena of practical importance are clearly displayed, owing to the above-described method of determining the coefficient k_n, which takes into account the fact that the superposition principle does not hold for nonlinear systems.

It is also important to bear in mind the following. By investigating by means of linear control theory, using Eq. (10.100), the variation of the static and dynamic properties of the system relative to the useful signal with variation of the structure and parameters of this system, one must without fail take into account also the change in the coefficient k_n itself, a change that follows from Expressions (10.99) and (10.97) or (10.87).

Third problem. If the general linearization of the bias function (10.96) or (10.98) is not desirable or impossible (because of the excessive curvature of this function), it becomes necessary the nonlinear equation (10.95) directly, for specified $\tilde{f}(t)$, using any of the

known methods for solving ordinary nonlinear equations. In this case the stability of the given nonlinear system can be determined, in particular, by the harmonic linearization method described in §2.7, after first carrying out the harmonic linearization bias function, namely

$$\left.\begin{aligned} \tilde{F}(\tilde{x}) &= q^{\text{ст}}(A,\ \sigma_x)\,\tilde{x}, \\ q^{\text{ст}}(A,\ \sigma_x) &= \frac{1}{\pi A}\int_0^{2\pi} \tilde{F}(A\sin\psi,\ \sigma_x)\sin\psi\, d\psi, \end{aligned}\right\} \qquad (10.102)$$

where $\tilde{F}(\tilde{x},\ \sigma_x)$ is taken, in accord with (10.93), in ready-made form for each nonlinearity from §10.2, followed by the substitution $\tilde{x} = A\sin\psi$, where A denotes the amplitude of the self-oscillations of the slowly varying component $\tilde{x}$, which arise in the system on the boundary of the stability region and outside this region. If necessary, we can investigate further also the self-oscillations in this nonlinear system with respect to the slowly varying regular component $\tilde{x}$, by determining their amplitude A and frequency Ω as functions of the system parameters, using any of the methods described in §2.3, in accordance with (10.95) and (10.102), from the harmonically linearized equation

$$[Q(p) + q^{\text{ст}}(A,\ \sigma_x)\,R(p)]\,\tilde{x} = 0. \qquad (10.103)$$

In this case the quantity σ_x which is contained in (10.102) and (10.103) is determined in a simplified method by means of Eq. (10.97), since it is obvious that when self-oscillations (with respect to the slowly varying component $\tilde{x}$) set in, the high-frequency fluctuations will not be passed by the linear part, which is defined by the transfer function R(p)/Q(p), owing to the presence of filtering properties in the system (properties needed to obtain self-oscillations in the form $\tilde{x} = A\sin\psi$).

We see, therefore, that inasmuch as σ_x is determined in terms of $s_f(\omega)$, the amplitude and frequency of the self-oscillations, determined from Eq. (10.103), as well as the very conditions of their oc-

currence in the nonlinear system, will depend essentially on the spectral density of the external fluctuations (random noise).

We give, furthermore, ready-made expressions for the harmonic linearization coefficient $q^{st}(A, \sigma_x)$, calculated by means of Formula (10.100) for several typical nonlinearities F(x), contained in the initial equation of the system (10.81). Let us note that a more detailed notation for the operation (10.100), taking into account Eq. (10.41), can be obtained in terms of the initial nonlinearity F(x) in the form

$$q^{ст}(A, \sigma_x) = \frac{1}{\pi A} \int_0^{2\pi} M[F(A \sin \psi + x^{сл})] \sin \psi \, d\psi$$

for a normal distribution of x^{sl} (10.42) and subject to the condition that the variation of A sin ψ is slow compared with the frequency spectrum of the random fluctuations x^{sl}.

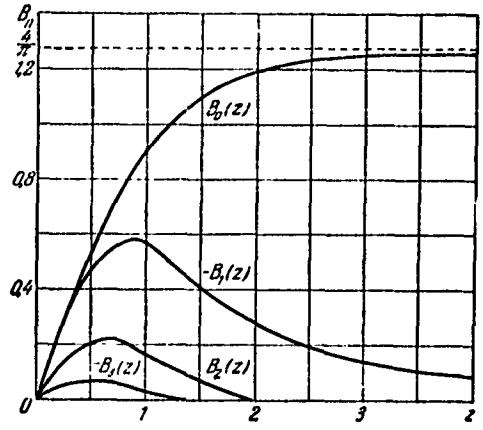


Fig. 10.16

The results of the calculations of the coefficient $q^{st}(A, \sigma_x)$, made by A.A. Pervozvanskiy [331], are as follows:

1. Ideal relay characteristic (Fig. 10.7a)

$$q^{ст}(A, \sigma_x) = \frac{c}{A} B_0(z), \qquad (10.104)$$

where $B_0(z)$ is graphically represented in Fig. 10.16 (the analytic expression is given below), with

$$z=\frac{A}{\sigma_x\sqrt{2}}. \qquad (10.105)$$

2. Single-valued relay characteristic with backlash zone (Fig. 10.8a)

$$q^{ст}(A,\ \sigma_x)=\frac{c}{A}\sum_{n=0}^{\infty}b_1^{2n}B_n(z), \qquad (10.106)$$

where

$$b_1=\frac{b}{\sigma_x\sqrt{2}}, \qquad (10.107)$$

and the functions $B_n(z)$ are also given in Fig. 10.16 (it is sufficient to take a finite sum up to n = 3). The plots of Fig. 10.16 are based on the formula

$$B_n(z)=\frac{4}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^{i+n}\,|2(i+n)|}{(2n)!\,(i!)^2\,(i+n)!\,(i+1)}\left(\frac{z}{2}\right)^{2i+1} \qquad (n=0,\ 1,\ 2,\ \ldots).$$

3. Loop-type relay characteristic of general form (Fig. 10.9a)

$$q^{ст}(A,\sigma_x)=\frac{c}{A}\sum_{n=0}^{\infty}\frac{1+m^{2n}}{2}b_1^{2n}B_n(z). \qquad (10.108)$$

4. Characteristic of the saturation type (Fig. 10.10a)

$$q^{ст}(A,\sigma_x)=\frac{c}{A}\sum_{n=0}^{\infty}\frac{b_1^{2n}}{2n+1}B_n(z). \qquad (10.109)$$

5. Characteristic with backlash zone (Fig. 10.11b)

$$q^{ст}(A,\ \sigma_x)=k-\frac{kb}{A}\sum_{n=0}^{\infty}\frac{b_1^{2n}}{2n+1}B_n(z) \qquad (10.110)$$

(in practically all cases it is sufficient to sum only up to n = 3).

6. Quadratic characteristic (Fig. 10.11c)

$$\left.\begin{aligned}F&=kx^2\operatorname{sign}x=k|x|x,\\ q^{ст}(A,\sigma_x)&=kAD(z),\end{aligned}\right\} \qquad (10.111)$$

where D(z) is shown graphically in Fig. 10.17. It is plotted from the formula

$$D(z)=\frac{1}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^{i+1}\,(2i)!}{(i!)^2\,(2i-1)(i+1)}\left(\frac{z}{2}\right)^{2i-1}.$$

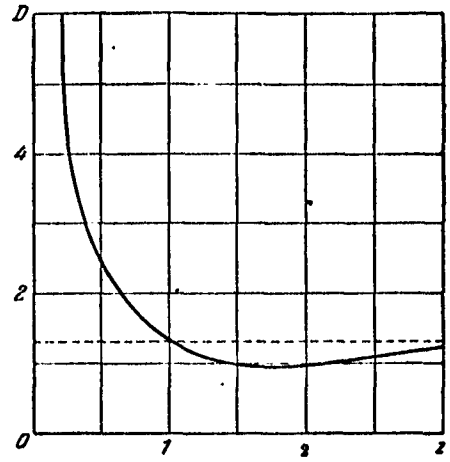


Fig. 10.17

7. Power-law characteristics with odd powers (Fig. 10.11c)

$$F=kx^3, \quad q^{\text{ст}}(A,\ \sigma_x)=\frac{3k}{4}(A^2+4\sigma_x^2); \tag{10.112}$$

$$F=kx^5, \quad q^{\text{ст}}(A,\ \sigma_x)=\frac{15k}{2}\left(\frac{1}{12}A^4+A^2\sigma_x^2+2\sigma_x^4\right); \tag{10.113}$$

finally, in general form

$$F=kx^{2n+1} \qquad (n=1,\ 2,\ 3,\ldots),$$

$$q^{\text{ст}}(A,\ \sigma_x)=\frac{k(2n+1)!}{2^n}\sum_{l=0}^{n}\frac{2^l A^{2(n-l)}\sigma_x^{2l}}{l!\,(n-l)!\,(n-l+1)!}. \tag{10.114}$$

Thus, after going through the above-described determination of the harmonic linearization coefficient of the nonlinearity in the presence of high-frequency external fluctuations (random noise), we can determine from Eq. (10.103) by means of the methods of Chapter 2 (illustrated with examples in Chapter 4) the stability boundary and the parameters of the self-oscillations of the nonlinear automatic system, but now as functions not only of the structure and parameters of the system, but also of the spectral density of the high-frequency fluctuations acting on the system.

We can also use Eq. (10.103) to investigate later on the quality of the transients in such a system by means of the methods of Chapter 7, and determine by the procedures described in §9.1 also the single-frequency forced oscillations, in accordance with (10.95) and (10.102),

using the equation

$$[Q(p)+q^{cr}(A_v,\ \sigma_x)R(p)]\tilde{x}=S(p)\tilde{f}(t),$$

where the regular component of the external signal $\tilde{f}(t)$ varies as

$$\tilde{f}(t)=B\sin\Omega_v t$$

with a frequency Ω_v which lies considerably below the spectrum of the frequencies of the random oscillations $f^{sl}(t)$. In this case, the expression for $q^{st}(A_v, \sigma_x)$, where A_v is the sought amplitude of the forced oscillations, is taken for each nonlinearity from the formulas written out above for $q^{st}(A, \sigma_x)$.

§10.4. Low-Frequency Random Processes

We consider here three other principally different problems in general form. They will be illustrated by means of examples in §§10.5 and 10.7.

Fourth problem. Let us return to the problem considered in §10.1. Assume that slowly varying random signals flow through a self-oscillating nonlinear system, operating in the forced oscillation mode. The frequency spectrum of the random signal is in this case, unlike in the third problem, considerably below the frequency of the self-oscillations or of the forced vibrations. The solution x(t) is resolved into two components, a slowly varying one x^0 and an oscillatory one x*.

We leave the start of the solution of the problem, i.e., Formulas (10.1)-(10.10) or (10.27)-(10.35), respectively, in exactly the same form as in §10.1. In addition, there we carried out ordinary linearization of the bias function and solved the ordinary problem of the flow of random signals through a purely linear closed loop automatic system. Now, however, we consider a case when ordinary linearization of the bias function is undesirable or impossible (owing to its strong curvature).

In this case we carry out statistic linearization of the bias

function, for it is necessary here to seek not for the regular slowly varying component $x^0(t)$, as in the third problem, but for the random slowly varying component.

In §10.1 we indicated two methods for calculating the equivalent gain q^{sl} of the random component. We shall use the second of these, since it is simpler. Then the result of the statistic linearization of the bias function (10.10) or (10.35) will be

$$F^0 = q^{c.a}(\sigma_{x^0})\, x^0, \tag{10.115}$$

where in accordance with (10.48), recognizing that the mathematical expectation of x^0 is zero, we have

$$q^{c.a} = \frac{1}{\sigma^2_{x^0}} \int_{-\infty}^{+\infty} \Phi(x^0)\, x^0 w(x^0)\, dx^0, \tag{10.116}$$

and the distribution w of the random quantity x^0 will be assumed normal. An example with a specific nonlinearity will be given in §10.5.

With the substitution (10.115), the statistically linearized equation (10.6) for the slowly varying random component $x^0(t)$ and for a random external signal f(t) will assume the form

$$[Q(p) + q^{c.a}(\sigma_{x^0})\, R(p)]\, x^0 = S(p) f(t). \tag{10.117}$$

We next apply the formulas of the linear theory of random processes, of the type (10.17) and (10.18), for the determination of the dispersion $\sigma^2_{x^0}$, which yields

$$\sigma^2_{x^0} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{S(j\omega)}{Q(j\omega) + q^{c.a}(\sigma_{x^0}) R(j\omega)} \right|^2 s_f(\omega)\, d\omega, \tag{10.118}$$

where $s_f(\omega)$ is the specified spectral density of the external random signal f(t). Using further Formulas (10.23) and (10.22), we represent Eq. (10.118) in the form

$$\sigma^2_{x^0} = hI_n(\sigma_{x^0}), \tag{10.119}$$

where h is a constant factor, which separates out upon integration. From this we determine the sought dispersion $\sigma^2_{x^0}$ or the mean square

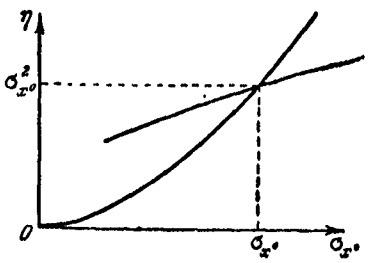


Fig. 10.18

value σ_{x^0} of the slowly varying random component x^0.

Equation (10.119) can be solved either graphically, by finding the point of intersection of the two curves (Fig. 10.18)

$$\eta = \sigma^2_{x^0} \quad \text{and} \quad \eta = hI_n(\sigma_{x^0}), \tag{10.120}$$

or analytically by successive approximation, using the solution obtained in §10.1 as the first approximation. Substituting the thus obtained σ_{x^0} in (10.119) we obtain a new value of $\sigma^2_{x^0}$ (second approximation), etc.

Once σ_{x^0} has been obtained, we calculate by means of the previous formulas (10.24) and (10.25) the mathematical expectation (mean value) of the self-oscillation amplitude $\tilde{A}$ or of the amplitude of the forced oscillations $\tilde{A}_v$, and the dispersion σ^2_A of the amplitude, thereby completing the solution of the problem.

First problem. In principle, an entirely different version of solving the same problem regarding the slowly varying random signals is possible. In the fourth problem the solution x(t) was separated into a slowly varying component x^0 and an oscillatory component x*, both of which were random. Another approach is possible, which was employed in the first three problems of the preceding section, where the solution x(t) was resolved into the regular and random components $\tilde{x}$ and x^{sl}, respectively. In this case, unlike in the third problem, we shall assume the regular component $\tilde{x}$ to be self-oscillating at a high frequency, and the random component $\tilde{x}^{sl}$ a slowly varying one, i.e., with a frequency spectrum that lies considerably below the self-oscillation frequency.

Then the equation of the nonlinear system

$$Q(p)x + R(p)F(x, px) = S(p)f(t) \tag{10.121}$$

for a slowly varying random external signal f(t) will break up as a result of statistical linearization of the nonlinearity (10.40) into the following two equations

$$Q(p)\tilde{x} + R(p)\tilde{F}(\tilde{x}, \sigma_x) = 0, \tag{10.122}$$

$$[Q(p) + R(p)q^{ст}(\tilde{x}, \sigma_x)]x^{сл} = S(p)f(t) \tag{10.123}$$

respectively, for the regular and random components.

Inasmuch as the regular component $\tilde{x}$ is assumed to be self-oscillating:

$$\tilde{x} = A \sin \Omega t, \tag{10.124}$$

it is necessary, in order to determine the amplitude A and the frequency Ω of the self-oscillations, to carry out in (10.122) harmonic linearization of the nonlinearity $\tilde{F}(\tilde{x}, \sigma_x)$, namely:

$$\left.\begin{aligned} \tilde{F} &= q^{ст}(A, \sigma_x)\tilde{x}, \\ q^{ст}(A, \sigma_x) &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{F}(A \sin \psi, \sigma_x) \sin \psi \, d\psi. \end{aligned}\right\} \tag{10.125}$$

As a result of this Eq. (10.122) assumes the form

$$[Q(p) + q^{ст}(A, \sigma_x)R(p)]\tilde{x} = 0. \tag{10.126}$$

We see that these equations coincide formally with the previous equations (10.102) and (10.103) which were used in the third problem (§10.3). We can therefore use here all the specific expressions for q^{st} (10.104)-(10.114) indicated above for the specific nonlinearities F(x). The essential difference from the preceding case will lie in the definition of the quantity σ_x, which enters into these formulas, in view of the different scope of the solved problem.

In the present case σ_x characterizes the slowly varying random component x^{sl}, defined by means of (10.123). The latter is complicated because the mathematical expectation $\tilde{x}$ contained in it is not constant, but varies sinusoidally (10.124). However, inasmuch as x^{sl} var-

ies slowly, the solution can be simplified by replacing the coefficient q^{sl} by means of a constant quantity over the period of the self-oscillations; this constant quantity is equal to the average value

$$q^{cp} = \frac{1}{2\pi}\int_0^{2\pi} q^{ca}(A\sin\psi,\ \sigma_x)\,d\psi. \qquad (10.127)$$

Then (10.123) will assume the form

$$[Q(p) + q^{cp}(A,\ \sigma_x)R(p)]\,x^{ca} = S(p)f(t). \qquad (10.128)$$

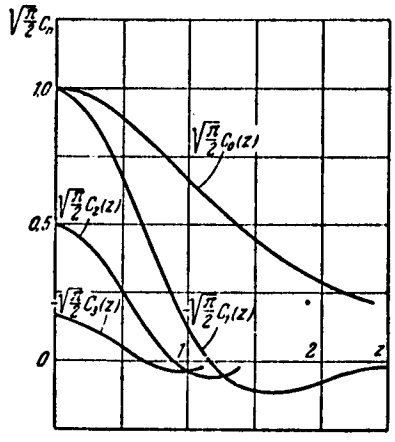


Fig. 10.19

Expressions for the coefficient $q^{sr}(A, \sigma_x)$ were derived [331] for standard nonlinearities F(x) by means of Formula (10.127) using the material of §10.2, where expressions for q^{sl} for the same nonlinearities are given. The second method of determining q^{sl} (10.48) was used here.

1. Ideal relay characteristic (Fig. 10.7a)

$$q^{cp}(A,\ \sigma_x) = \frac{c}{\sigma_x}C_0(z), \qquad (10.129)$$

where $C_0(z)$ is plotted in Fig. 10.19 (the analytic expression will be given below), with

$$z = \frac{A}{\sigma_x\sqrt{2}}.$$

2. Single-valued relay characteristic with backlash zone (Fig. 10.8a)

$$q^{cp}(A,\ \sigma_x) = \frac{c}{\sigma_x}\sum_{n=0}^{\infty} b_1^{2n}C_n(z), \qquad (10.130)$$

where

$$b_1 = \frac{b}{\sigma_x\sqrt{2}},$$

and the functions $C_n(z)$ are also given in Fig. 10.19 (it is sufficient

to take a finite sum up to n = 3). The plots of Fig. 10.19 are based on the formula

$$C_n(z) = \sqrt{\frac{2}{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^{i+n}[2(i+n)]!}{(2n)!\,(i!)^2\,(i+n)!} \left(\frac{z}{2}\right)^{2i} \qquad (n = 0, 1, 2, \ldots).$$

3. Loop-type relay characteristic of general form (Fig. 10.9a)

$$q^{\text{ср}}(A, \sigma_x) = \frac{c}{\sigma_x} \sum_{n=0}^{\infty} \frac{1+m^{2n}}{2} b_1^{2n} C_n(z). \qquad (10.131)$$

4. Saturation type characteristic (Fig. 10.10a)

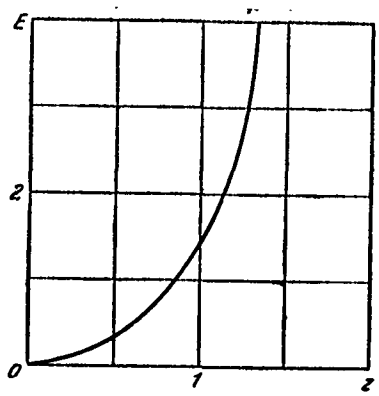


Fig. 10.20

$$q^{\text{ср}}(A, \sigma_x) = \frac{c}{\sigma_x} \sum_{n=0}^{\infty} \frac{b_1^{2n}}{2n+1} C_n(z). \qquad (10.132)$$

5. Characteristics with backlash zone (Fig. 10.11b)

$$q^{\text{ср}}(A, \sigma_x) = k - \frac{kb}{\sigma_x} \sum_{n=0}^{\infty} \frac{b_1^{2n}}{2n+1} C_n(z) \qquad (10.133)$$

(in practically all cases it is sufficient to sum only up to n = 3).

6. Quadratic characteristic (Fig. 10.11c)

$$\left.\begin{aligned} F &= kx^2 \operatorname{sign} x = k|x|x, \\ q^{\text{ср}}(A, \sigma_x) &= k\sigma_x E(z), \end{aligned}\right\} \qquad (10.134)$$

where E(z) is plotted in Fig. 10.20 on the basis of the formula

$$E(z) = z^2 \sqrt{\frac{2}{\pi}} \left[1 + 2z^2 - 2 \sum_{i=0}^{\infty} \frac{(-1)^i (2i+1)!}{i!\,[(i+2)!]^2} \left(\frac{z}{2}\right)^{2(i+1)}\right].$$

7. Power-law characteristics with odd powers (Fig. 10.11c)

$$F = kx^3, \quad q^{\text{ср}}(A, \sigma_x) = \frac{3k}{2}(A^2 + 2\sigma_x^2); \qquad (10.135)$$

$$F = kx^5, \; q^{\text{ср}}(A, \sigma_x) = \frac{15k}{2}\left(\frac{1}{4}A^4 + 2\sigma_x^2 A^2 + 2\sigma_x^4\right); \qquad (10.136)$$

in general form

$$F = kx^{2n+1} \qquad (n = 1, 2, 3, \ldots),$$

$$q^{\text{ср}}(A, \sigma_x) = \frac{k(2n+1)!}{2^n} \sum_{i=0}^{n} \frac{2^i A^{2(n-i)} \sigma_x^{2i}}{i!\,[(n-i)!]^2}. \qquad (10.137)$$

After writing out Eq. (10.128), a formula of the type (10.118) is used, namely

$$\sigma_x^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{S(j\omega)}{Q(j\omega)+q^{cp}(A,\sigma_x)R(j\omega)}\right| s_f(\omega)\,d\omega, \qquad (10.138)$$

from which, using (10.23), we obtain an equation for the determination of σ_x as a function of the self-oscillation amplitude A:

$$\sigma_x^2 = hI_n(A,\sigma_x). \qquad (10.139)$$

From this we can determine $\sigma_x(A)$ graphically by the same method as in Fig. 10.14, in which we replace the letter $\tilde{x}$ by A, and plot curve 1 by means of the equation

$$\sigma_x^2 = \zeta,$$

and curves 2 for the different constant values of A by means of the equations

$$hI_n(A,\sigma_x) = \zeta,$$

after which the ordinates of the points of intersection are transferred to the right-hand plot, where the abscissas are marked with the corresponding values of A.

The dependence $\sigma_x(A)$ obtained in this manner is then substituted in (10.126), after which any of the methods of §2.3 can be used to determine from this equation the amplitudes A and the frequency Ω of the self-oscillations. The solution of the problem is concluded by substituting this value of A into the previously obtained dependence $\sigma_x(A)$, which yields the sought mean square value σ_x of the slowly varying random component x^{sl}.

This constitutes the second version of the solution of the problem involving the flow of slowly varying random signals through a self-oscillating system. The same procedure can be employed to investigate the flow of slowly varying random signals through a nonlinear system operating in the forced oscillation mode (for example, vibration

smoothing of nonlinearities). In this case only Eqs. (10.21), (10.124), and (10.126) are replaced by the following

$$Q(p)x + R(p)F(x, px) = S(p)f(t) + S_2(p)B\sin\Omega_2 t, \qquad (10.140)$$

$$\tilde{x} = A_2 \sin(\Omega_2 t + \varphi),$$

$$[Q(p) + q^{cr}(A_2, \sigma_x)R(p)]\tilde{x} = S_2(p)B\sin\Omega_2 t. \qquad (10.141)$$

Formulas (10.125) and (10.127)-(10.139) remain the same as before, except that A is replaced in them by A_v. After determining $\sigma_x(A_v)$, Eq. (10.141) is solved by the method of Chapter 9.

Sixth problem. In those cases when one cannot count on having the frequency spectrum of the random component of the process much below or much above the self-oscillation frequency, the use of the statistical and harmonic linearization methods becomes difficult. Nevertheless, frequently it is useful to employ in these cases a device developed by A.A. Pervozvanskiy [331] on the basis of ideas by Rice [307] concerning the determination of the probability density of the sum of a sinusoidal signal and a random component.

Let, as before, the solution of the nonlinear equation (10.121) be sought in the form

$$x = \tilde{x} + x^{cl}, \quad \tilde{x} = A\sin\psi, \qquad (10.142)$$

where it is regarded as a whole as a single random process. Inasmuch as the choice of the phase ψ is arbitrary, it is considered to be a uniformly distributed quantity. The probability density $w_s(x)$ of the summary random process (10.142) is determined on the basis of probability theory by means of the formula

$$w_c(x) = \frac{1}{\pi}\int_0^{\pi} w(x - A\sin\psi)\,d\psi, \qquad (10.143)$$

where w is the normal distribution law (10.42), namely

$$w(x - A\sin\psi) = \frac{1}{\sigma_x\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x - A\sin\psi}{\sigma_x}\right)^2}. \qquad (10.144)$$

Statistical linearization of the nonlinearity F(x, px) is carried out here, unlike the preceding equation (10.40), for the entire process as a whole

$$F = q^{\text{эк}} x = q^{\text{эк}} (\tilde{x} + x^{\text{сл}}), \tag{10.145}$$

where the summary equivalent gain q^{ek} is determined from the condition that the mean square deviation of the substitute function from the original function be a minimum, i.e., from the condition

$$M[(F - q^{\text{эк}} x)^2] = \min,$$

from which we obtain for single-valued nonlinearities F(x)

$$q^{\text{эк}} = \frac{M[Fx]}{M[x^2]} = \frac{\int_{-\infty}^{+\infty} F(x)\, x w_c(x)\, dx}{\int_{-\infty}^{+\infty} x^2 w_c(x)\, dx}. \tag{10.146}$$

Substituting here (10.143) and reversing the order of integration in the numerator we obtain

$$J_1 = \frac{1}{\pi} \int_0^{\pi} d\psi \int_{-\infty}^{+\infty} F(x^{\text{сл}} + A \sin\psi)(x^{\text{сл}} + A \sin\psi) w(x)\, dx,$$

and in the denominator

$$J_2 = \frac{1}{\pi} \int_0^{\pi} d\psi \int_{-\infty}^{+\infty} (x^{\text{сл}} + A \sin\psi)^2 w(x)\, dx.$$

But from (10.125) and (10.41) we have

$$q^{\text{ст}}(A, \sigma_x) = \frac{2}{\pi A} \int_0^{\pi} \sin\psi\, d\psi \int_{-\infty}^{+\infty} F(x^{\text{сл}} + A \sin\psi) w(x)\, dx,$$

and from (10.127) and (10.48) we have

$$q^{\text{ср}}(A, \sigma_x) = \frac{1}{\pi \sigma_x^2} \int_0^{\pi} d\psi \int_{-\infty}^{+\infty} F(x^{\text{сл}} + A \sin\psi)\, x w(x)\, dx.$$

In addition, we use the expression

$$\int_{-\infty}^{+\infty} x^n w(x)\, dx = \begin{cases} (2l-1)!!\, \sigma_x^{2l} & (n = 2l), \\ 0 & (n = 2l-1) \end{cases} \qquad (l = 0, 1, 2, \ldots).$$

From the foregoing relations it follows that

$$J_1=\frac{1}{2}A^2q^{ст}(A,\sigma_x)+\sigma_x^2q^{ср}(A,\sigma_x),\quad J_2=\frac{1}{2}A^2+\sigma_x^2.$$

Therefore, Formula (10.146) assumes ultimately the form

$$q^{эк}=\frac{J_1}{J_2}=\frac{q^{ст}(A,\sigma_x)+\frac{2\sigma_x^2}{A^2}q^{ср}(A,\sigma_x)}{1+\frac{2\sigma_x^2}{A^2}}. \qquad (10.147)$$

Consequently, in order to obtain specific expressions for the coefficient q^{ek} for typical nonlinearities, we can use the ready-made expressions for q^{st} (10.104)-(10.114) and q^{sr} (10.129)-(10.137).

As a result, the dispersion $\sigma_x^2(A)$ of the random component x^{sl} will be determined in analogy with (10.138) by the formula

$$\sigma_x^2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{S(j\omega)}{Q(j\omega)+q^{эк}(A,\sigma_x)R(j\omega)}\right|^2 s_f(\omega)\,d\omega \qquad (10.148)$$

using the same graphic procedure, while the amplitude A and the frequency Ω of the self-oscillations are determined by the methods of §2.3 using the characteristic equation

$$Q(p)+q^{эк}(A,\sigma_x)R(p)=0, \qquad (10.149)$$

in which we substitute $\sigma_x(A)$ from (10.148). By determining A from it, we obtain ultimately also the quantity σ_x.

It must only be noted that these formulas do not provide the solution to the problem in the "resonant" case, i.e., it is necessary to impose the condition that the spectral density of the external signal $s_f(\omega)$ vanish at a frequency $\omega = \Omega$ equal to the self-oscillation frequency, otherwise the integral (10.148) may diverge since, in accordance with (10.149), its denominator vanishes when $\omega = \Omega$. The "resonant" case is treated separately in Reference [338].

§10.5. Example of Slowly Varying Random Processes in a Self-Oscillating System

Let us illustrate by means of specific systems and numbers the

methods used to investigate random processes in nonlinear systems, as developed in the present chapter.

In this section we consider the example of a magnetoelectric accelerometer. The instrument represents a closed-loop automatic system, operating in the vibration self-oscillation mode. Its diagram is shown in Fig. 10.21. This system was described earlier in §6.4, where calculations were given for its use to measure a constant acceleration, and, in accordance with §5.3, calculation can be carried out for a specified time variation of the acceleration. We now assume that the acceleration measured with this instrument* is a random function of the time; for example, this may be the lateral acceleration of an airplane, resulting from the turbulence in the atmosphere.

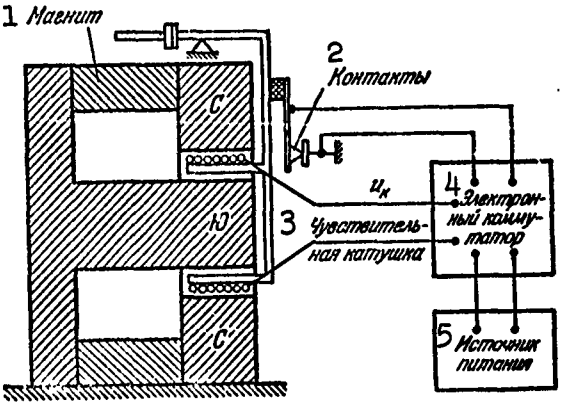


Fig. 10.21. 1) Magnet; 2) contacts; 3) sensitive coil; 4) electronic commutator; 5) power supply.

According to (6.54), the dynamics of the instrument system with the simplification $F_2(\varphi) = \varphi$ is described by the equation**

$$[T_1^2 T_k p^3 + (T_1^2 + T_1 T_k) p^2 + (T_1 + T_k + k_3 T_k + T_e) p + 1 + k_3] x + k_1 F(x) = k_3 (T_k p + 1) f(t), \qquad (10.150)$$

where $T_1 = 0.6$ sec is the damping constant of the sensitive coil, $T_2^2 = 0.007$ sec^2 the inertial constant of the sensitive coil, $T_k = 0.0001$ sec is its electric constant, $T_e = 0.004$ sec is its constant connected

with the counter emf, $k_1 = 0.7$ 1/a, $k_2 = 0.0715 \cdot 10^{-3}$ $g^{-1}cm^{-1}$, and $k_3 = 1.2 \cdot 10^{-3}$ sec^2/cm are transfer ratios, F(x) is a nonlinear characteristic of the ideal relay type:

$$F = \operatorname{sign} x, \tag{10.151}$$

f(t) is the measured lateral acceleration of the airplane, with

$$f(t) = 0{,}1\, v(t), \tag{10.152}$$

(v(t) is the velocity of the lateral wind). Based on the literature data [323], we shall assume the latter to be a stationary random function of the time, with a correlation function $r_v(\tau)$ experimentally obtained (curve 1 on Fig. 10.22). For $\tau \leq 15$ this curve is well approximated by the following curve (curve 2):

$$r_v(\tau) = \sigma_v^2 e^{-\alpha|\tau|} \cos \omega_1 \tau, \tag{10.153}$$

where

$$\sigma_v^2 = 0.01 \text{ [m/sec]}^2, \quad \alpha = 0.2 \text{ 1/sec}, \quad \omega_1 = 0.1 \text{ 1/sec}.$$

On the basis of (10.152) we can write for the correlation function of the measured acceleration

$$r_f(\tau) = \sigma_f^2 e^{-\alpha|\tau|} \cos \omega_1 \tau, \quad \sigma_f^2 = 10^{-4}. \tag{10.154}$$

From this we obtain by means of Formula (10.14) the spectral density of the external random signal (the measured lateral acceleration of the airplane)

$$s_f(\omega) = 2\alpha\sigma_f^2 \frac{\omega^2 + \gamma^2}{(\omega^2 + \gamma^2)^2 - 4\omega_1^2\omega^2}, \tag{10.155}$$

where

$$\alpha = 0{,}2, \quad \sigma_f^2 = 10^{-4}, \quad \omega_1 = 0{,}1, \quad \gamma^2 = \alpha^2 + \omega_1^2 = 0{,}05. \tag{10.156}$$

Let us first find the self-oscillations of the system in the absence of an external signal (f = 0)

$$x = A_c \sin \Omega t;$$

Using harmonic linearization of the nonlinearity (10.151), in accordance with Chapter 3 and Formula (10.150), we obtain the characteristic

equation

$$T_2^2 T_k p^3 + (T_2^2 + T_1 T_k) p^2 + (T_1 + T_k + k_2 T_k + T_c) p + 1 + k_2 + \frac{4k_1}{\pi A_c} = 0.$$

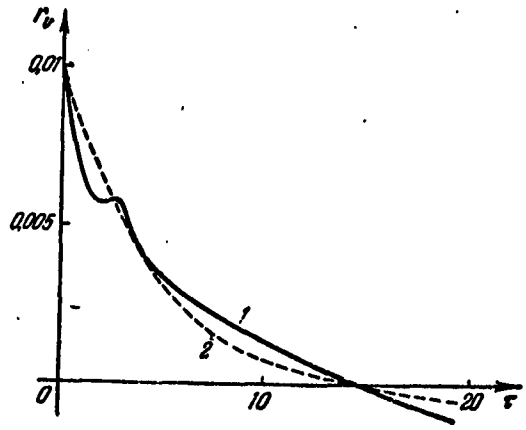


Fig. 10.22

Substituting $p = j\Omega_s$ and separating the real and imaginary parts, we have two equations

$$1 + k_2 + \frac{4k_1}{\pi A_c} - (T_2^2 + T_1 T_k)\Omega_c^2 = 0,$$

$$(T_1 + T_k + k_2 T_k + T_c)\Omega_c - T_2^2 T_k \Omega_c^3 = 0,$$

from which we obtain the frequency and amplitude of the self-oscillations in the absence of an external signal (f = 0):

$$\Omega_c = \sqrt{\frac{T_1 + T_k + k_2 T_k + T_c}{T_2^2 T_k}} = 929 \text{ 1/sec}, \qquad (10.157)$$

$$A_c = \frac{4k_1}{\pi} \frac{T_2^2 T_k}{(T_2^2 + T_1 T_k)(T_1 + T_k + k_2 T_k + T_c) - T_2^2 T_k (1 + k_2)} = 1{,}463 \cdot 10^{-4}. \qquad (10.158)$$

At such a self-oscillation frequency (10.157), the random signal described by the spectral density (10.155) can be fully assumed to be slowly varying. Therefore to determine the self-oscillations $x^* = A \sin \Omega t$ in the presence of a random signal in the present system, we make use of Eq. (10.7). The characteristic equation corresponding to it will be, on the basis of (10.150)

$$T_2^2 T_k p^3 + (T_2^2 + T_1 T_k) p^2 + (T_1 + T_k + k_2 T_k + T_c) p + 1 + k_2 + k_1 q(A, x^*) = 0,$$

where, in accord with §5.6, we have for the nonlinearity (10.151)

$$q=\frac{4}{\pi A}\sqrt{1-\left(\frac{x^0}{A}\right)^2}. \qquad (10.159)$$

Substituting $p = j\Omega$ in the characteristic equation and separating the real and imaginary parts we obtain

$$1+k_2+k_1q(A, x^0)-(T_2^2+T_1T_k)\Omega^2=0,$$
$$(T_1+T_k+k_2T_k+T_c)\Omega-T_2^2T_k\Omega^3=0,$$

from which we obtain the previous value of the frequency

$$\Omega=\Omega_c=\sqrt{\frac{T_1+T_k+k_2T_k+T_c}{T_2^2T_k}}=929\ \underline{1/\text{sec}},$$

and also, taking (10.158) into account, a new value for the amplitude, which depends on the slowly varying component x^0, namely

$$A=\frac{A_c}{\sqrt{2}}\sqrt{1+\sqrt{1-\left(\frac{2x^0}{A_c}\right)^2}}. \qquad (10.160)$$

It is obvious that the limits of the quantities contained here are

$$|x^0|\leqslant\frac{A_c}{2},\quad \frac{A_c}{\sqrt{2}}\leqslant A\leqslant A_c. \qquad (10.161)$$

To determine the random slowly varying component $x^0(t)$ it is necessary to use Eq. (10.6), which for this system, in accordance with (10.150), assumes the form

$$[T_2^2T_kp^3+(T_2^2+T_1T_k)p^2+(T_1+T_k+k_2T_k+T_c)p+1+k_2]x^0+$$
$$+k_1F^0=k_2(T_kp+1)f(t), \qquad (10.162)$$

where according to §5.6, for the specified nonlinearity (10.151) we have

$$F^0=\frac{2}{\pi}\arcsin\frac{x^0}{A}\qquad(|x^0|\leqslant A); \qquad (10.163)$$

when $x^0 < -A$ we get $F^0 = -1$, and when $x^0 > A$ we get $F^0 = +1$.

Substituting here expression A (from (10.160)), we obtain the bias function (the dependence of F^0 on x^0 and on the system parameters only)

$$F^0=\Phi(x^0)=\frac{2}{\pi}\arcsin\frac{x^0\sqrt{2}}{\sqrt{A_c^2+A_c}\sqrt{A_c^2-4(x^0)^2}}$$

or, putting $2x^0/A_s = \sin\beta$, we get $\Phi = \beta/\pi$, i.e.,

$$F^0 = \Phi(x^0) = \frac{1}{\pi}\arcsin\frac{2x^0}{A_c} \quad \left(|x^0| \leqslant \frac{A_c}{2}\right); \tag{10.164}$$

when $x^0 < -A_s/2$ we get $F^0 = -1$, and when $x^0 > A_s/2$ we get $F^0 = +1$, and the quantity A_s which is contained in (10.164) depends, in accord with (10.158), on the parameters of the given system. The bias function obtained is shown in Fig. 10.23. This is precisely the nonlinear characteristic, smoothed with the aid of self-oscillations, of the slowly varying component $x^0(t)$ of the process, in place of the pure relay characteristic (10.151).

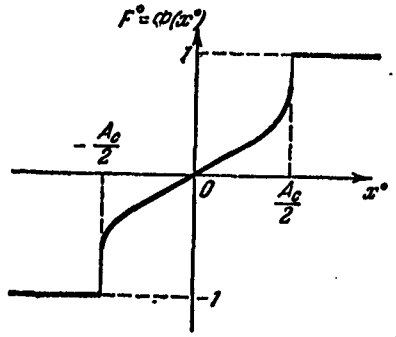


Fig. 10.23

Further solution of the problem involving the determination of $\sigma^2_{x^0}$ of the slowly varying random component $x^0(t)$, and also of the mathematical expectation $\tilde{A}$ and of the dispersion σ^2_A of the amplitude A of the oscillatory component x*, will be carried out in two versions: 1) by ordinary linearization of the bias function in accordance with §10.1, and 2) by statistical linearization of this function (fourth problem of §10.4). This problem was solved by A.A. Pervozvanskiy [334].

Ordinary linearization of the bias function (10.164) by means of Formula (10.11) for small deviations of x^0 near the origin yields

$$F^0 = k_н x^0, \quad k_н = \left(\frac{d\Phi}{dx^0}\right)_{x^0=0} = \frac{2}{\pi A_c} = 0{,}435 \cdot 10^4. \tag{10.165}$$

We verify incidentally, by means of the same example, the agreement between the results obtained from Formulas (10.11) and (10.12). For this purpose we calculate k_n in accordance with (10.12) by partially differentiating with respect to x^0 the expression for F^0 (10.163) directly, without transforming the latter into the bias function, We obtain

$$k_н = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0} = \left(\frac{2}{\pi A}\right)_{x^0=0} = \frac{2}{\pi A_c},$$

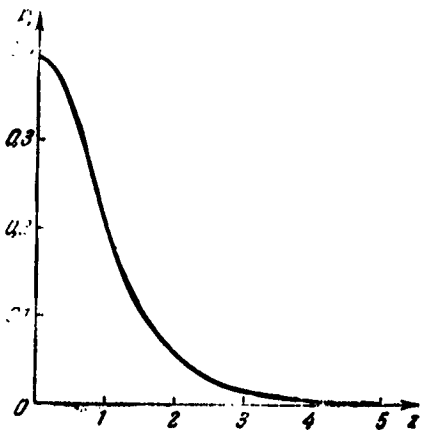


Fig. 10.24

which indeed coincides with (10.165).

On the other hand, statistical linearization of the bias function (10.164) by means of Formula (10.115) yields

$$F^0 = q^{c,1}(\sigma_{x^0})\, x^0, \qquad (10.166)$$

where, in accord with (10.116), (10.164), and (10.26) we have on the interval $|x^0| \leq A/2$

$$q^{c,1} = \frac{1}{\pi \sigma_{x^0}^2 \sqrt{2}} \int_{-\frac{A_c}{2}}^{\frac{A_c}{2}} x^0 \arcsin \frac{2x^0}{A_c} e^{-\frac{1}{2}\left(\frac{x^0}{\sigma_{x^0}}\right)^2} dx^0$$

or

$$q^{c,1}(\sigma_{x^0}) = \frac{8}{A_c \pi \sqrt{\pi}} z^3 P_1(z) \quad \left(|x^0| \leq \frac{A_c}{2}\right), \qquad (10.167)$$

where

$$\left.\begin{aligned} P_1(z) &= \int_0^1 y \arcsin y e^{-z^2 y^2} dy, \\ z &= \frac{A_c}{2\sigma_{x^0}\sqrt{2}}, \quad y = \frac{2x^0}{A_c}, \end{aligned}\right\} \qquad (10.168)$$

with the function $P_1(z)$ calculated beforehand and plotted (Fig. 10.24).

On the other hand, if one cannot assume that the quantity x^0 indicated in (10.167) is bounded, we obtain

$$q^{c,1}(\sigma_{x^0}) = \frac{8}{A_c \pi \sqrt{\pi}} z^3 P_1(z) + \frac{2}{\sigma_{x^0}^2 \sqrt{2\pi}} \int_{\frac{A_c}{2}}^{\infty} x^0 e^{-\frac{1}{2}\left(\frac{x^0}{\sigma_{x^0}}\right)^2} dx^0$$

or

$$q^{c,1}(\sigma_{x^0}) = \frac{4z}{A_c \sqrt{\pi}} \left[\frac{2z^2}{\pi} P_1(z) + e^{z^2}\right]. \qquad (10.169)$$

The dispersion of the slowly varying random component of the process $x^0(t)$, in accord with (10.18), (10.17), and (10.162), under the ordinary linearization (10.165), is determined by the formula

$$\sigma_{x0}^2 = h\frac{1}{2\pi}\times$$
$$\times\int_{-\infty}^{+\infty}\left|\frac{T_k j\omega+1}{-T_2^2T_k j\omega^3-(T_2^2+T_1T_k)\omega^2+(T_1+T_k+k_2T_k+T_e)\omega+1+k_2+k_1k_u}\right|^2\times$$
$$\times\frac{(\omega^2+\gamma^2)\,d\omega}{(\omega^2+\gamma^2)^2-4\omega_1^2\omega^2},$$

where

$$h = 2\alpha\sigma_f^2 k_2^2 = 0{,}576\cdot 10^{-10}. \qquad (10.170)$$

Noting that with allowance for (10.156) we have

$$(\omega^2+\gamma^2)^2-4\omega_1^2\omega^2 = |(j\omega)^2+2\alpha(j\omega)+\gamma^2|^2, \qquad (10.171)$$

we represent the entire denominator under the integral sign in the form

$$|H(j\omega)|^2 = |a_0(j\omega)^5+a_1(j\omega)^4+a_2(j\omega)^3+a_3(j\omega)^2+a_4(j\omega)+a_5|^2,$$

where

$$a_0 = T_2^2T_k = 0{,}7\cdot 10^{-6};\quad a_1 = T_2^2+T_1T_k+2\alpha T_2^2T_k = 0{,}706\cdot 10^{-2};$$
$$a_2 = T_1+T_k+k_2T_k+T_e+2\alpha(T_2^2+T_1T_k)+\gamma^2T_2^2T_k = 0{,}607;$$
$$a_3 = 2\alpha(T_1+T_k+k_2T_k+T_e)+\gamma^2(T_2^2+T_1T_k)+1+k_2+k_1k_u =$$
$$= 1{,}314+0{,}7\,k_u;$$
$$a_4 = \gamma^2(T_1+T_k+k_2T_k+T_e)+2\alpha(1+k_2+k_1k_u) = 0{,}459+0{,}28\,k_u;$$
$$a_5 = \gamma^2(1+k_2+k_1k_u) = 0{,}0536+0{,}035\,k_u.$$

The entire numerator under the integral sign, as called for by the procedure indicated in §10.1 for evaluating the integral, will be written in the form

$$Q(\omega) = |T_k j\omega+1|^2(\omega^2+\gamma^2) = T_k^2\omega^4-(1+T_k^2\gamma^2)\omega^2+\gamma^2 =$$
$$= b_0\omega^8+b_1\omega^6+b_2\omega^4+b_3\omega^2+b_4,$$

where

$$b_0 = b_1 = 0,\quad b_2 = T_k^2 = 10^{-8},\quad b_3 = 1+T_k^2\gamma^2\approx 1,\quad b_4 = \gamma^2 = 0{,}05.$$

Thus,

$$\sigma_{x0}^2 = h\,\frac{1}{2\pi}\int_0^\infty\frac{b_0\omega^8+\ldots+b_4}{|a_0(j\omega)^5+\ldots+a_5|^2}\,d\omega = hI_5. \qquad (10.172)$$

Using the ready-made formula (10.22) for I_5 and substituting the values of all the coefficients, we obtain an expression for the sought dispersion σ_{x0}^2 as a function of the slope k_n of the bias function in the form

$$\sigma_{x^0}^2 = \frac{0{,}295 \cdot 10^{-8}(k_n^2 - 0{,}869 \cdot 10^4 k_n \quad 0{,}148 \cdot 10^8)}{k_n^6 - 0{,}869 \cdot 10^4 k_n^5 - 0{,}432 \cdot 10^8 k_n^4 - 0{,}708 \cdot 10^8 k_n - 0{,}390 k_n^0}. \qquad (10.173)$$

Substituting the value of k_n (10.165) we obtain the dispersion

$$\sigma_{x^0}^2 = 0{,}156 \cdot 10^{-16}$$

and the mean square value

$$\sigma_{x^0} = 0{,}395 \cdot 10^{-8}. \qquad (10.174)$$

We see that the mean square value of x^0 is much smaller than the quantity $A_s/2 = 0.732 \cdot 10^{-4}$ involved in Fig. 10.23. Therefore, calculations carried out with the aid of the ordinary linearization (10.165) is fully justified in the present example.

On the other hand, if for other numerical data the value of σ_{x^0} will be larger, then the expression of the type of (10.173) obtained with the aid of the ordinary linearization can be taken as the first approximation, and subsequently corrected by using the statistical linearization (10.166). For this purpose, as follows from comparison of (10.17) with (10.118), it is merely necessary to substitute into the expression (10.173) for $\sigma_{x^0}^2$ the quantity $q^{sl}(\sigma_{x^0})$ in place of k_n, and use Formula (10.167) or (10.169). We then obtain for σ_{x^0} an equation of the form

$$\sigma_{x^0} = hI_s(\sigma_{x^0}),$$

which can be readily solved graphically (Fig. 10.18).

Finally, it is necessary to find the mathematical expectation (mean value) $\tilde{A}$ and the dispersion σ_A^2 of the amplitude of the oscillatory component. From Formulas (10.24) and (10.25) we have with allowance for (10.160)

$$\tilde{A} = \frac{A_c}{2\sigma_{x^0}\sqrt{\pi}} \int_{-\frac{A_c}{2}}^{\frac{A_c}{2}} \sqrt{1 + \sqrt{1 - \left(\frac{2x^0}{A_c}\right)^2}}\, e^{-\frac{1}{2}\left(\frac{x^0}{\sigma_{x^0}}\right)^2} dx^0,$$

$$\sigma_{\tilde{A}}^2 = \frac{1}{\sigma_{x^0}\sqrt{2\pi}} \int_{-\frac{A_c}{2}}^{+\frac{A_c}{2}} \left(\frac{A_c}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \left(\frac{2x^0}{A_c}\right)^2}} - \tilde{A} \right)^2 e^{-\frac{1}{2}\left(\frac{x^0}{\sigma_{x^0}}\right)^2} dx^0$$

or

$$\left.\begin{aligned} \tilde{A} &= \frac{4}{\sqrt{\pi}} \sigma_{x^0} z^2 P_2(z), \\ \sigma_{\tilde{A}}^2 &= \frac{3}{\sqrt{\pi}} \sigma_{x^0}^2 z^3 P_3(z) - \frac{8}{\pi} \sigma_{x^0}^2 z^2 \tilde{A} P_2(z) + \frac{2}{\sqrt{\pi}} z \tilde{A}^2 P_4(z), \end{aligned}\right\} \qquad (10.175)$$

where

$$P_2(z) = \int_0^1 \sqrt{1 + \sqrt{1 - y^2}}\, e^{-z^2 y^2} dy, \quad z = \frac{A_c}{2\sigma_{x^0}\sqrt{2}}, \quad y = \frac{2x^0}{A_c},$$

$$P_3(z) = \int_0^1 \left(1 + \sqrt{1 - y^2}\right) e^{-z^2 y^2} dy, \quad P_4(z) = \int_0^1 e^{-z^2 y^2} dy.$$

For relatively small values of $\underline{z}$ ($0 \leq z \leq 5$) the functions $P_2(z)$, $P_3(z)$, and $P_4(z)$ are plotted in Fig. 10.25.

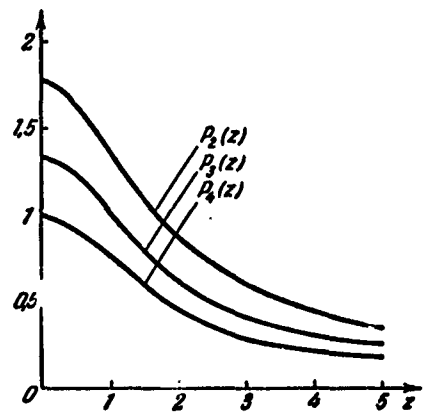


Fig. 10.25

In the numerical example considered, with allowance for (10.174), we have $z = 1.31 \cdot 10^4$, i.e., $\underline{z}$ greatly exceeds five, and $\underline{y}$ is small. In this case Formulas (10.175) can be simplified by assuming

$$\tilde{A} = \frac{2z}{\sqrt{\pi}} z^2 \int_0^\infty \sqrt{2}\, e^{-z^2 y^2} dy = 2\sqrt{2}\sigma_{x^0} z = A_c \qquad (10.176)$$

and furthermore

$$\sigma_{\tilde{A}}^2 = \frac{2z}{\sqrt{\pi}} \int_0^1 \frac{A_c^2}{2} \left(\sqrt{1 + \sqrt{1 - y^2}} - \sqrt{2}\right)^2 e^{-z^2 y^2} dy =$$

$$= \frac{8z^3}{\sqrt{\pi}} \sigma_{x^0}^2 \int_0^1 \left(\sqrt{1 + \sqrt{1 - y^2}} - \sqrt{2}\right)^2 e^{-z^2 y^2} dy.$$

For small values of $\underline{y}$ we can assume

$$\sqrt{1 + \sqrt{1 - y^2}} = \sqrt{2}\left(1 - \frac{y^2}{8}\right),$$

and the upper limit of integration can be replaced by infinity. Then

$$\sigma_{\dot{A}}^2 = \frac{8z^3}{\sqrt{\pi}}\sigma_{x^0}^2\int_0^\infty\left(-\frac{y^2\sqrt{2}}{8}\right)^2 e^{-z^2y^2}dy = \frac{z^3}{4\sqrt{\pi}}\sigma_{x^0}^2\int_0^\infty y^4 e^{-z^2y^2}dy.$$

But we know of the following expression for the gamma function:

$$\Gamma(u) = (z^2)^u\int_0^\infty t^{u-1}e^{-z^2t}dt.$$

In the integral under consideration $t = y^2$ and $u = 5/2$. Therefore,

$$\sigma_{\dot{A}}^2 = \frac{z^3}{4\sqrt{\pi}}\sigma_{x^0}^2\frac{1}{2z^5}\Gamma\left(\frac{5}{2}\right).$$

We also know the formula

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}\prod_{k=1}^{n}(2k-1),$$

hence

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(2+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}6.$$

We ultimately obtain

$$\sigma_{\dot{A}}^2 = \frac{3}{16z^5}\sigma_{x^0}^2. \qquad (10.177)$$

Thus, in our example, taking (10.174), (10.176), and (10.158) into account, we have

$$\sigma_{x^0} = 0{,}395\cdot10^{-8};\quad \tilde{A} = 1{,}463\cdot10^{-4};\quad \sigma_{\dot{A}} = 1{,}305\cdot10^{-12}.$$

As a result, all the sought probability characteristics of the solution $x = x^0 + x^*$, where $x^* = A\sin\Omega t$ have been determined, and, as previously found, $\Omega = 929\ \text{sec}^{-1}$.

§10.6. Example of the Effect of Noise on the Dynamics of a Nonlinear System

A nonlinear automatic control system (Fig. 10.26a) is acted upon by a random noise f(t), which has a high frequency compared with the slowly varying useful control signal in the given system. Passage of noise through the nonlinear element changes the gain of the useful signal in the latter (second problem of §10.3). It is required to estimate

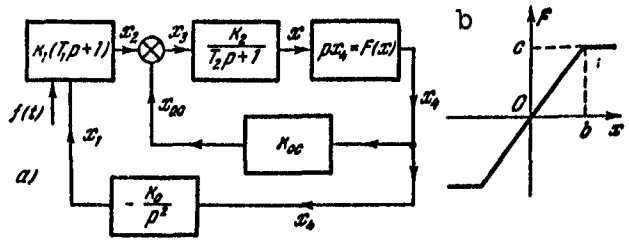


Fig. 10.26

the influence of this phenomenon on the dynamic properties of the given automatic control system relative to the useful signal [336].

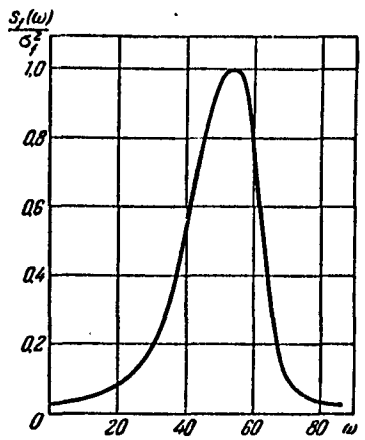


Fig. 10.27

The equation of the closed loop system (Fig. 10.26) as a whole will be

$$p^3(T_2p+1)x+(k_2k_{o.c}p^2+kk_0T_1p+kk_0)F(x)=kp^3(T_1p+1)f(t), \quad (10.178)$$

where $k = k_1k_2$, $F(x)$ is a specified nonlinearity (Fig. 10.26b). In addition, the following are specified: $k = 18$, $k_2 = 60$, $k_{o.s} = 0.03$, $k_0 = 0.5$, $T_1 = 0.5$, $T_2 = 0.02$, $c/b = 4$.

The noise has a normal distribution and is specified in terms of the spectral density (Fig. 10.27)

$$s_f(\omega)=\frac{2\beta\sigma_f^2}{(\omega_1^2-\alpha^2\omega^2)^2+\mu^2\omega^2}, \quad (10.179)$$

where $\alpha = 0.05$, $\beta = 1.35$, $\omega_1^2 = 7.5$, $\mu = 0.03$. By varying the noise dispersion σ_f^2, which characterizes the "noise level," we determine the dynamic qualities of the system as a function of the quantity σ_f.

Carrying out the statistical linearization (10.40), let us break down the system equation (10.178) into two equations, for the regular and random components, respectively:

$$p^3(T_2p+1)\tilde{x}+(k_2k_{o.c}p^3+kk_0T_1p+kk_0)\tilde{F}=0,$$
$$[p^3(T_2p+1)+(k_2k_{o.c}p^3+kk_0T_1p+kk_0)q^{c.s}]x^{c.s}=kp^3(T_1p+1)f(t). \quad (10.180)$$

Inasmuch as the transfer function of the linear part of the system is

$$W_s(p) = \frac{k_y k_{0.c} p^2 + k k_0 T_1 p + k k_0}{p^2 (T_2 p + 1)}$$

and for the previously assigned parameters this system does not pass in practice* the frequencies at which the spectral noise density (Fig. 10.27) has an appreciable value, according to (10.97) the dispersion of the noise at the output of the nonlinear element will be

$$\sigma_x^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{S(j\omega)}{Q(j\omega)} \right|^2 s_f(\omega)\, d\omega =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{k(T_1 j\omega + 1)}{T_2 j\omega + 1} \right|^2 \frac{2\beta\sigma_f^2}{(\omega_1^2 - \alpha^2\omega^2)^2 + \mu^2\omega^2}\, d\omega.$$

In order to transform this integral into the standard form (10.23), we first transform the denominator of the spectral density, namely

$$(\omega_1^2 - \alpha^2\omega^2)^2 + \mu^2\omega^2 = |\alpha^2 (j\omega)^2 + \mu j\omega + \omega_1^2|^2.$$

We then obtain

$$H(j\omega) = a_0 (j\omega)^2 + a_1 (j\omega)^2 + a_2 (j\omega) + a_3,$$

where

$$a_0 = \alpha^2 T_2, \qquad a_1 = \alpha^2 + \mu T_2,$$
$$a_2 = \omega_1^2 T_2 + \mu, \quad a_3 = \omega_1^2.$$

In the numerator we get

$$Q(\omega) = |T_1 j\omega + 1|^2 = b_0\omega^4 + b_1\omega^2 + b_2,$$

where

$$b_0 = 0,\ b_1 = T_1^2,\ b_2 = 1.$$

As a result we obtain

$$\sigma_x = k\sigma_f \sqrt{2\beta I_3}, \tag{10.181}$$

where, according to (10.22)

$$I_3 = \frac{\alpha^2 + \mu T_2 - T_1^2\omega_1^2}{2\mu\omega_1^2 (\alpha^2 + \mu T_2 + T_2^2\omega_1^2)}. \tag{10.182}$$

We now proceed to Eq. (10.180) for the regular component, i.e., for the useful signal. The function $\tilde{F}$ is determined in it by the plot of Fig. 10.10b as a function of $\tilde{x}_1 = \tilde{x}/b$ and $\sigma_1 = \sigma_x/b$. In the initial part, all the curves of this plot are nearly straight lines. We can

therefore carry out the ordinary linearization of these plots in the form

$$\tilde{F} = k_{\text{н}}\tilde{x}, \tag{10.183}$$

where k_n is the slope at the origin (Fig. 10.10b), which depends on the value of σ_1. For our problem we obtain

σ_1	0	0,1	0,3	0,6	1,0	2,0	3,0	5,0	10	∞
$k_{\text{н}}$	4,0	4,0	4,0	3,2	2,3	1,4	1,0	0,6	0,4	0

Physically, the quantity k_n is the gain of the useful signal in the nonlinear element in the presence of noise, and the presented table gives the dependence of this gain on the noise level, i.e., on its mean square value $\sigma_1 = \sigma_x/b$, at the input of the nonlinear element.

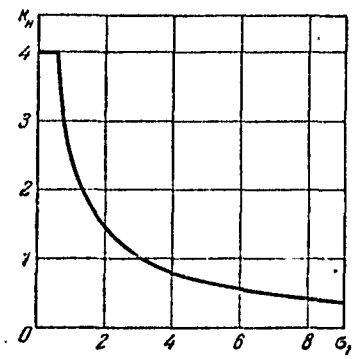


Fig. 10.28

We see that an increase in the noise level brings about an appreciable reduction in the gain of the useful signal in the nonlinear element, as illustrated graphically in Fig. 10.28. This is the principal distinguishing feature of a nonlinear system, which causes all the static and dynamic properties relative to the useful signal, including the stability, to be dependent on the noise level.

Let us find, for example, the dependence of the system stability on the noise level. For this purpose we write, in accordance with (10.180) and (10.183), the characteristic equation of the system:

$$T_2p^4 + p^3 + k_2k_{\text{о.с}}k_{\text{н}}p^2 + kk_0k_{\text{н}}T_1p + kk_0k_{\text{н}} = 0. \tag{10.184}$$

The Hurwitz criterion for the system stability assumes the form

$$k_{\text{н}} > \frac{1}{T_1(k_2k_{\text{о.с}} - kk_0T_1T_2)}. \tag{10.185}$$

For the parameters specified at the start of this section, this yields $k_n > 1.17$. This corresponds, in accord with Fig. 10.28, to a value

$$\sigma_1 = \frac{\sigma_x}{b} = 2{,}65.$$

But, according to (10.181)

$$\sigma_1 = \frac{\sigma_x}{b} \cdot \frac{\sigma_f}{b_f} \sqrt{2 I_3}, \tag{10.186}$$

where

$$b_f = \frac{b}{k};$$

this quantity is best used to express the mean square value of the external noise σ_f in relative units, recognizing that in accordance with Fig. 10.26 the dimensionalities of the variables f(t) and $\underline{x}$ are interrelated precisely through the coefficient $k = k_1 k_2$.

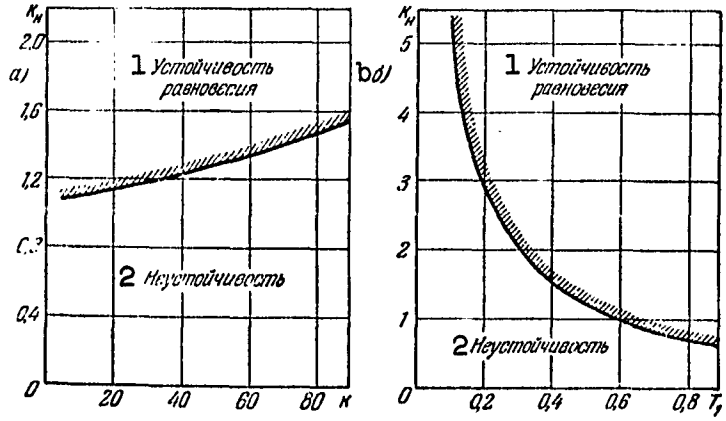


Fig. 10.29. 1) Stable equilibrium; 2) instability.

After calculating I_3 by means of Formula (10.182) we obtain from (10.186) for the system parameters given above

$$\frac{\sigma_f}{b_f} = 0{,}00437.$$

This means that only when the noise level does not exceed this value is our system stable. Beyond that it loses stability with respect to the useful signal.

Let us explain now the influence of the parameters $\underline{k}$ and T_1 on the system stability in the presence of noise. For this purpose we de-

termine first with the aid of Formula (10.185) the system stability limit on the planes of the parameters $\underline{k}$, k_n and T_1, k_n (Fig. 10.29a and b). On the stability limit we obtain for each value of k_n, using the curve of Fig. 10.28 (or the table given above), the value of σ_1 from which, in accord with (10.186), we obtain also the mean square value of the external noise, at which the system loses stability

$$\frac{\sigma_f}{b_f} = \frac{\sigma_1}{\sqrt{2\beta I_3}}. \qquad (10.187)$$

This enables us to replot the stability limits obtained on Fig. 10.29 into new coordinates, respectively:

$$k, \frac{\sigma_f}{b_f} \text{ and } T_1, \frac{\sigma_f}{b_f}$$

(Figs. 10.30a and b). It must be borne in mind here that in accord with (10.182) the quantity I_3 depends on the parameter T_1, and consequently the calculations by means of Formula (10.187) used to plot Fig. 10.30b must be carried out with account of the change of I_3 with changing T_1.

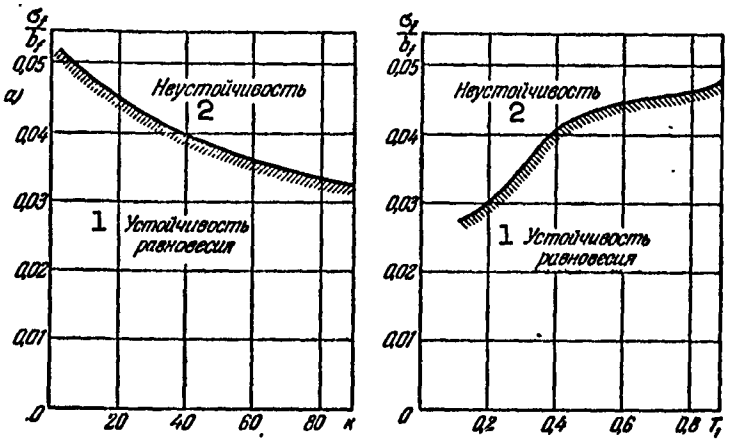


Fig. 10.30. 1) Stable equilibrium; 2) instability.

We see that the dangerous noise level drops with increasing parameter $\underline{k}$ and rises with increasing parameter T_1. This is quite natural, inasmuch as T_1 is, in accord with Fig. 10.26a, the coefficient of the intensity with which the derivative, which improves the stabilization

of the system, is introduced.

Analogously, using the linear equation that follows from (10.180) and (10.183)

$$[p^3(T_2p+1)+(k_2k_{0.c}p^2+kk_0T_1p+kk_0)k_n]\tilde{x}=\dot{0},$$

we can further investigate by means of linear automatic control theory also all other dynamic properties of the given nonlinear system with respect to the useful signal in the presence of noise, recognizing at all times, however, that the value of the coefficient k_n depends on the noise level σ_f, on the over-all structure of the system, and on some of its parameters.

§10.7. Other Examples of Investigation of Random Processes

Example 1. Let the dynamics of the automatic system (Fig. 10.31) be described by a nonlinear differential equation of first order [317]:

$$(Tp+1)x_1=F(x), \quad x=y+f(t)-x_1, \tag{10.188}$$

where F(x) is an ideal relay characteristic

$$F(x)=c\,\mathrm{sign}\,x, \tag{10.189}$$

with c = 0.1 and T = 0.2 specified; f(t) is a stationary random function of the time, with a spectral density

$$s_f(\omega)=\frac{8\pi\sigma_f^2}{4\pi^2+\omega^2}, \quad \sigma_f=0{,}177, \tag{10.190}$$

and with zero mathematical expectation ($\tilde{f}$ = 0). It is required to find the mathematical expectation x_1 and the mean square deviation σ_{x_1} of the output quantity x_1 (Fig. 10.31) in the steady state for y = const = = 0.2. This is the simplest example of the solution of the first problem of §10.3.

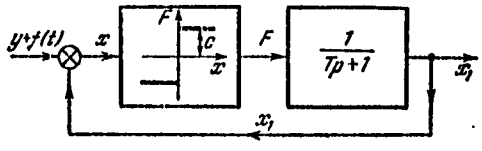


Fig. 10.31

Writing down the sought solution of the equations in (10.188) in the form

$$x_1 = \tilde{x}_1 + x_1^{c.s}, \quad x = \tilde{x} + x^{c.s} \tag{10.191}$$

and carrying out, in accordance with (10.40) statistical linearization of the nonlinearity, we obtain in place of (10.188) the following two systems of equations

$$(Tp+1)\tilde{x}_1 = \tilde{F}(\tilde{x}, \sigma_x), \qquad \tilde{x} = y - \tilde{x}_1, \tag{10.192}$$

$$(Tp+1)x_1^{c.s} = q^{c.s}(\tilde{x}, \sigma_x)x^{c.s}, \quad x^{c.s} = f(t) - x_1^{c.s}, \tag{10.193}$$

for the regular (i.e., for the mathematical expectations) and random components, respectively, with

$$q^{c.s}(\tilde{x}, \sigma_x) = \frac{e}{2\sigma_x}[\varphi^{(1)}(\tilde{x}, \sigma_x) + \varphi^{(2)}(\tilde{x}, \sigma_x)], \tag{10.194}$$

where $\varphi^{(1)}$ and $\varphi^{(2)}$ are determined for the specified nonlinearity (10.189) by means of the formulas (10.52). This corresponds to the arithmetic mean q^{sl} of the two values corresponding to the two methods of determining its value, namely (10.46) and (10.48). These functions are plotted in Fig. 10.7c. In the same place (Fig. 10.7b) is given a plot of the expression (10.49) for the function $\tilde{F}(\tilde{x}, \sigma_x)$ which is contained in Eq. (10.192).

From Eq. (10.192) we obtain for the investigated steady-state mode ($\tilde{x}_1 = \text{const}$, $p\tilde{x}_1 = 0$)

$$\tilde{x} = y - \tilde{F}(\tilde{x}, \sigma_x), \tag{10.195}$$

and from the condition (10.193) we get

$$[Tp+1+q^{c.s}(\tilde{x}, \sigma_x)]x^{c.s} = (Tp+1)f(t). \tag{10.196}$$

From the latter we arrive by means of Formula (10.87) with account of (10.190) at an equation for the determination of σ_x:

$$\sigma_x^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{Tj\omega+1}{Tj\omega+1+q^{c.s}(\tilde{x}, \sigma_x)}\right|^2 \frac{8\pi\sigma_f^2}{4\pi^2+\omega^2}d\omega,$$

or

$$\sigma_x^2 = h \cdot I_2(\tilde{x}, \sigma_x), \qquad (10.197)$$

where

$$h = 8\pi\sigma_f^2, \quad I_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{G(\omega)}{|H(j\omega)|^2}\, d\omega.$$

Noting that the numerator under the integral sign is

$$G(\omega) = T^2\omega^2 + 1,$$

and that the denominator contains the square of the modulus of the function

$$H(j\omega) = T(j\omega)^2 + [2\pi T + 1 + q^{c.s}(\tilde{x}, \sigma_x)]\, j\omega + 2\pi[1 + q^{c.s}(\tilde{x}, \sigma_x)],$$

we obtain in accord with (10.21) and (10.22)

$$I_2 = \frac{2\pi T[1 + q^{c.s}(\tilde{x}, \sigma_x)] + 1}{2[2\pi T + 1 + q^{c.s}(\tilde{x}, \sigma_x)]}. \qquad (10.198)$$

Solving the system of equations (10.195) and (10.197) graphically (Fig. 10.13) with allowance for the formulas (10.198) and (10.194) and for the plots of Fig. 10.7c, we get

$$\tilde{x} = 0{,}140; \quad \sigma_x = 0{,}160.$$

To solve the problem it was necessary to find $\tilde{x}_1$ and σ_{x_1}. From Formulas (10.192), (10.194) and the plot of Fig. 10.7c we get

$$\tilde{x}_1 = y - \tilde{x} = 0{,}060, \quad q^{c.s} = 0{,}394;$$

and from Eq. (10.193) we get

$$(Tp + 1 + q^{c.s})\, x_1^{c.s} = q^{c.s} f(t).$$

Therefore, in accord with (10.87), (10.190) and (10.21), (10.22), we obtain

$$\sigma_{\tilde{x}1}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{q^{c.s}}{Tj\omega + 1 + q^{c.s}}\right|^2 \frac{8\pi\sigma_f^2}{4\pi^2 + \omega^2}\, d\omega =$$

$$= \frac{(q^{c.s}\sigma_f)^2}{(1 + q^{c.s})(2\pi T + 1 + q^{c.s})} = 1{,}51 \cdot 10^{-4},$$

giving for the mean square deviation $\sigma_{x_1} = 0.0123$.

Example 2. Let us determine the self-oscillations and the conditions under which they exist in a servomechanism (Fig. 10.32), the dy-

namics of which is described by a third-order equation

$$(Tp+1)^3 px_1 = kF(x), \quad x = f(t) - x_1, \tag{10.199}$$

for a high-frequency random external signal f(t) with specified mean square value σ_f [331]. The nonlinearity is F(x) = sign x (Fig. 10.32, where c = 1). This corresponds to answering one of the questions in the third problem of §10.3.

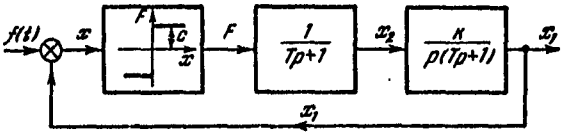


Fig. 10.32

It is easy to check (by any of the methods of §2.3) that in the absence of an external signal (f = 0) self-oscillations exist in the given system for any value of the gain k, and their frequency Ω_0 and amplitude A_0 are determined by the expressions

$$\Omega_0 = \frac{1}{T}, \quad A_0 = \frac{2}{\pi} kT. \tag{10.200}$$

In the presence of high-frequency fluctuations f(t), the self-oscillations in the system are determined by Eq. (10.103), which in the present example, in accord with (10.199), assumes the form

$$[(Tp+1)^3 p + kq^{ст}(A, \sigma_x)]\dot{x} = 0, \tag{10.201}$$

where, in accordance with (10.104), we have

$$q^{ст}(A, \sigma_x) = \frac{1}{A} B_0(z), \quad z = \frac{A}{\sigma_x \sqrt{2}}, \tag{10.202}$$

with the function $B_0(z)$ represented by the plot of Fig. 10.16. In the present example (Fig. 10.32) we have

$$\sigma_x = \sigma_f, \tag{10.203}$$

since the linear part of the system does not pass the high-frequency fluctuations. The same result is obtained also from Formula (10.97), where

$$\left|\frac{S(j\omega)}{Q(j\omega)}\right|^2 = 1,$$

inasmuch as it follows from (10.199) after the elimination of x_1 that

$$S(p)=Q(p)=(Tp+1)^3p.$$

Let us find the self-oscillations in the system in the presence of high-frequency external fluctuations. The characteristic equation for (10.201) with allowance for (10.203) will be

$$T^3p^3+2Tp^2+p+kq^{\text{ст}}(A,\sigma_f)=0.$$

Substituting in it $p = j\Omega$ and separating the real and imaginary parts we obtain

$$kq^{\text{ст}}(A,\sigma_f)-2T\Omega^2=0,$$
$$\Omega-T^3\Omega^3=0.$$

From the second equation we obtain the frequency of self-oscillations

$$\Omega = 1/T$$

(it coincides with the same frequency in the absence of self-oscillations), while the first equation yields

$$q^{\text{ст}}(A,\sigma_f)=\frac{2}{kT}, \qquad (10.204)$$

where, in accord with (10.202) and (10.203), we have

$$q^{\text{ст}}(A,\sigma_f)=q^{\text{ст}}(z)=\frac{B_0(z)}{z\sigma_f\sqrt{2}}, \quad z=\frac{A}{\sigma_f\sqrt{2}}. \qquad (10.205)$$

According to the formula for $B_n(z)$, given in §10.3, we have for $B_0(z)$ the expression

$$B_0(z)=\frac{2z}{\sqrt{\pi}}\left[1-\left(\frac{z}{2}\right)+72\left(\frac{z}{2}\right)^4-\ldots\right],$$

from which it follows that

$$B_0(z)\to\frac{2z}{\sqrt{\pi}} \quad \text{as} \quad z\to 0,$$

meaning that, in accordance with (10.204),

$$q^{\text{ст}}=\frac{1}{\sigma_f}\sqrt{\frac{2}{\pi}} \quad \text{for } z=0 \ (A=0).$$

The plot of $\sigma_f q^{\text{ст}}(A/\sigma_f)$, defined by Formula (10.205), is shown in Fig. 10.33, from which we see that

$$\frac{1}{\sigma_f}\sqrt{\frac{2}{\pi}} \geqslant q^{cr} \geqslant 0 \text{ for } 0 \leqslant A \leqslant \infty. \qquad (10.206)$$

From this, in accordance with (10.204), we obtain the condition for the existence of self-oscillations in a system acted upon by high-frequency external fluctuations

$$\frac{1}{\sigma_f}\sqrt{\frac{2}{\pi}} \geqslant \frac{2}{kT} \geqslant 0,$$

i.e., for the specified fluctuations, self-oscillations will take place in the system when the gain is

$$k \geqslant \frac{\sigma_f \sqrt{2\pi}}{T}, \qquad (10.207)$$

whereas in the absence of fluctuations the self-oscillations took place for any value of k. On the other hand, if the parameters of the system k and T are specified, then the condition (10.207) for the existence of self-oscillations can be written in the form

$$\sigma_f \leqslant \frac{kT}{\sqrt{2\pi}}, \qquad (10.208)$$

i.e., self-oscillations will take place in the system (with fluctuations superimposed on them) only when the level of the external random signal does not exceed a definite threshold value

$$\sigma_f^{пор} = \frac{kT}{\sqrt{2\pi}}, \qquad (10.209)$$

beyond which ($\sigma_f > \sigma_f^{por}$) the self-oscillations cease. This is similar to the condition for the locking of the system by forced oscillations, considered in Chapter 9.

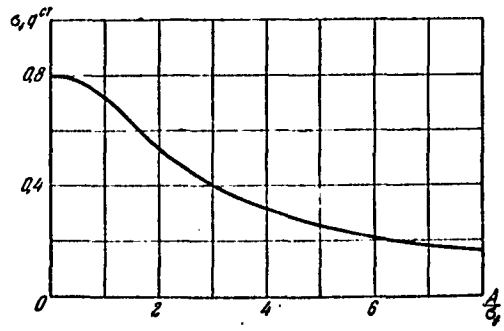


Fig. 10.33

Equation (10.204) enables us to determine readily with the aid of the plot of Fig. 10.33 the amplitude of the self-oscillations A for all specified values $\underline{k}$, T, and σ_f and the region of existence of self-oscillations.

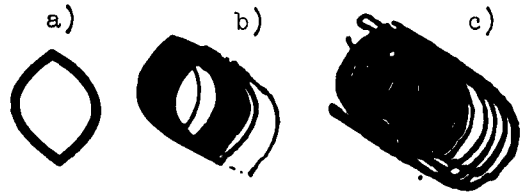


Fig. 10.34

Figure 10.34 shows oscillograms obtained by means of an electronic analog model for the system (10.199) at $k = 50\ \text{sec}^{-1}$, $T = 0.1$ sec, and $\sigma_f^{por} = 2.0$ using Formula (10.209). The following values were used in the experiment: a) $\sigma_f = 0$; b) $\sigma_f = 1.0$; c) $\sigma_f = 2.2$. From the corresponding oscillograms (Fig. 10.34) one can see clearly the deformation introduced into the limit cycle by the self-oscillations, and the cessation of self-oscillations at $\sigma_f > \sigma_f^{por}$, thus confirming the results of the calculations.

Example 3. Let us consider a vacuum tube oscillator [331] (Fig. 10.35). Let the external random signal be applied to the grid circuit of the oscillator. Neglecting the grid current, the oscillations are described by the equation

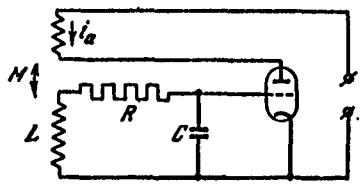


Fig. 10.35

$$(LCp^2 + RCp + 1)u - Mpi_a = f(t), \qquad (10.210)$$

where $\underline{u}$ is the grid voltage and i_a the plate current. The nonlinear characteristic of the tube is

$$i_a = b_1 u - \frac{1}{3} b_3 u^3. \qquad (10.211)$$

These equations can be written in the form

$$(p^2 - \varkappa p + \nu^2)u + \frac{1}{3}\gamma^2 pF(u) = \nu^2 f(t), \qquad (10.212)$$

where

$$F(u)=u^3,\quad \nu^2=\frac{1}{LC},\quad \varkappa=\frac{1}{LC}(Mb_1-RL),\quad \gamma^2=\frac{Mb_3}{LC}.$$

We recall that in the case of the sinusoidal external signal f = = B sin $\Omega_v t$ the condition for locking (cessation of self-oscillations) can be expressed in the form (see [321], page 298)

$$B>\frac{\sqrt{2\varkappa}\,|\nu^2-\Omega_B^2|}{\gamma\nu^2}. \tag{10.213}$$

We now consider the behavior of the generator for a random external signal f(t) in three cases: 1) high-frequency external fluctuations; 2) low-frequency random external signal; 3) the intermediate case, which pertains to the third, fifth, and sixth problems of §§10.3 and 10.4.

1. To determine the self-oscillations in the case of high-frequency external fluctuations f(t), we have in accord with (10.103) and (10.212) the characteristic equation

$$p^2+\left[\frac{1}{3}\gamma^2 q^{\text{ст}}(A,\sigma_u)-\varkappa\right]p+\nu^2=0, \tag{10.214}$$

from which we obtain for the self-oscillation frequency $\Omega = \nu$, and the amplitude is determined from the condition

$$q^{\text{ст}}(A,\sigma_u)=\frac{3\varkappa}{\gamma^2} \tag{10.215}$$

(condition for the presence of a pair of pure imaginary roots in the characteristic equation). But for the given nonlinearity $F(u) = u^3$ we have in accordance with (10.112)

$$q^{\text{ст}}=\frac{3}{4}A^2+3\sigma_u^2, \tag{10.216}$$

where the mean square of the voltage fluctuations σ_n is determined from Formula (10.97), which with allowance for (10.212) yields

$$\sigma_u^2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\nu^2}{-\omega^2-\varkappa j\omega+\nu^2}\right|^2 s_f(\omega)\,d\omega, \tag{10.217}$$

i.e., σ_u is independent of A. We therefore obtain from (10.215) and (10.216) the self-oscillation amplitude

$$A = 2\sqrt{\frac{\varkappa}{\gamma^2} - \sigma_u^2}. \qquad (10.218)$$

The condition for the cessation of the self-oscillations brought about by an external random signal will therefore be

$$\sigma_u > \frac{\sqrt{\varkappa}}{\gamma}. \qquad (10.219)$$



Fig. 10.36

In order to compare this condition for random fluctuations with the locking condition for a sinusoidal signal (10.213), let us attempt to derive from (10.219) the condition (10.213) as a particular case, using the fact that for a sinusoidal signal $f = B \sin \Omega_v t$ we can represent the spectral density $s_f(\omega)$ in the form of two instantaneous pulses at $\omega = \pm\Omega_v$ (Fig. 10.36), each having an area $\pi B^2/2$, something that can be written with the aid of a delta function in the form

$$s_f(\omega) = \frac{\pi}{2} B^2 [\delta(\omega - \Omega_u) + \delta(\omega + \Omega_u)]. \qquad (10.220)$$

Recognizing that for any function x(t) we get

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t - t_0)\,dt = x(t_0),$$

we obtain from Formula (10.217)

$$\sigma_u^2 = \frac{B^2}{2} \frac{\nu^4}{(\nu^2 - \Omega_u^2)^2 + \varkappa^2 \Omega_u^2} \approx \frac{B^2 \nu^4}{2\Omega_u^4},$$

since $\Omega_v >> \nu$ for high-frequency external vibrations.

Substituting this in the condition for the cessation of self-oscillations (10.219), we get

$$B > \frac{\sqrt{2\varkappa}}{\gamma} \frac{\Omega_u^2}{\nu^2},$$

which indeed agrees with (10.213) when $\Omega_v >> \nu$.

2. In the case of low-frequency (slowly varying) random external

signal we obtain, in accord with (10.126), our previous equation (10.214) for the determination of the self-oscillations, meaning also the previous relations (10.215) and (10.216). However, the mean squared value of the random component of the voltage σ_u is determined here in a different manner, namely, we obtain from (10.138) and (10.212)

$$\sigma_u^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\nu^2}{-\omega^2+\left[\frac{1}{3}\gamma^2 q^{cp}(A,\sigma_u)-\varkappa\right]j\omega+\nu^2}\right|^2 s_f(\omega)\,d\omega, \qquad (10.221)$$

where from Formula (10.135) we have for the specified nonlinearity $F(u) = u^3$

$$q^{cp} = \frac{3}{2}A^2 + 3\sigma_u^2. \qquad (10.222)$$

For a specified spectral density $s_f(\omega)$ we determine from this the function $\sigma_u(A)$ graphically, as was described following Formula (10.139). The obtained function $\sigma_u(A)$ is then substituted in (10.216), i.e.,

$$q^{cr}(A) = \frac{3}{4}A^2 + 3\sigma_u^2(A),$$

after which the equation for the determination of the amplitude of the self-oscillations (10.215) assumes the form

$$\frac{1}{4}A^2 + \sigma_u^2(A) = \frac{\varkappa}{\gamma^2}. \qquad (10.223)$$

From the same relations we obtain also the condition for the existence of self-oscillations in the case of a low-frequency (slowly varying) external random signal f(t), as the condition under which Eq. (10.223) gives a real positive value for the unknown A.

Let us attempt in this case, too, to derive as a particular case the condition for locking (10.213) for a sinusoidal signal $f = B \sin \Omega_v t$, using the expression for its spectral density (10.220). In this case we obtain from Formulas (10.221) and (10.222) when $\Omega_v \ll \nu$:

$$\sigma_u^2 = \frac{B^2\nu^4}{2(\nu^2-\Omega_u^2)^2+[\gamma^2(A^2+2\sigma_u^2)-2\varkappa]^2\Omega_u^2} \approx \frac{B^2}{2}.$$

Substituting this in (10.223) we obtain the condition for the cessation

of the self-oscillations

$$B > \frac{\sqrt{2\varkappa}}{\gamma},$$

which coincides with (10.213) for $\Omega_v \ll \nu$.

3. Let us assume now that the external random signal is such that one cannot count on having its spectrum much higher or much lower than the self-oscillation frequency. Then to determine the self-oscillations in the presence of an external random signal we make use of the characteristic equation (10.149) which in accord with (10.212) now assumes the form

$$p^2 + \left[\frac{1}{3}\gamma^2 q^{3\varkappa}(A, \sigma_x) - \varkappa\right] p + \nu^2 = 0, \tag{10.224}$$

where, in accord with (10.147), (10.216), and (10.222), we have

$$q^{3\varkappa}(A, \sigma_u) = \frac{\frac{3}{4}A^2 + 3\sigma_u^2 + \frac{2\sigma_u^2}{A^2}\left(\frac{3}{2}A^2 + 3\sigma_u^2\right)}{1 + \frac{2\sigma_u^2}{A^2}}. \tag{10.225}$$

From the condition for the presence of a pair of pure imaginary roots in the equation (10.224) we obtain an equation for the determination of the amplitude of the self-oscillations in the form

$$q^{3\varkappa}(A, \sigma_u) = \frac{3\varkappa}{\gamma^2} \tag{10.226}$$

or

$$\frac{A^4 + 8\sigma_u^2 A^2 + 8\sigma_u^4}{4(A^2 + 2\sigma_u^2)} = \frac{\varkappa}{\gamma^2}. \tag{10.227}$$

In this case the mean square value of the random voltage component σ_u is determined, in accordance with Formulas (10.148) and (10.212), in the form

$$\sigma_u^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\nu^2}{(j\omega)^2 + \left[\frac{1}{3}\gamma^2 q^{3\varkappa}(A, \sigma_u) - \varkappa\right] j\omega + \nu^2}\right|^2 s_f(\omega)\, d\omega.$$

Using Condition (10.226), we have

$$\sigma_u^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\nu^2}{(j\omega)^2 + \nu^2}\right|^2 s_f(\omega)\, d\omega. \tag{10.228}$$

Inasmuch as the mean square value σ_u determined for a specified external signal spectral density $s_f(\omega)$ is independent of A, we obtain from (10.227) the self-oscillation amplitude in the form

$$A^2 = \frac{2\varkappa}{\gamma^2} - 4\sigma_{\dot{u}}^2 \pm \sqrt{4\frac{\varkappa^2}{\gamma^2} - 8\sigma_{\dot{u}}^2\left(\frac{\varkappa}{\gamma^2} - \sigma_{\dot{u}}^2\right)}. \qquad (10.229)$$

From this we get for the condition for the cessation of self-oscillations in the presence of an external random signal

$$\sigma_{\dot{u}} > \frac{\sqrt{\varkappa}}{\gamma}. \qquad (10.230)$$

If we employ this in the particular case for a sinusoidal external signal $f = B \sin \Omega_v t$, we obtain from (10.228) and (10.220)

$$\sigma_{\dot{u}}^2 = \frac{B^2\nu^4}{2(\nu^2 - \Omega_v^2)^2}.$$

Substituting this into the condition for the cessation of self-oscillations (10.230), we obtain precisely the same condition (10.213), which we know from the theory of nonlinear forced oscillations.

The foregoing examples illustrate certain applications of the methods described in the present chapter.

By way of conclusion to the entire book as a whole, we must note that the harmonic linearization method developed in it (in conjunction with statistic linearization in the last chapter) makes it possible to solve many problems in the analysis and synthesis of nonlinear automatic systems. There is no doubt that additional new capabilities of the method will be found in the future, in which connection its further development is quite urgent both from the theoretical point of view and in applications to various engineering calculations for nonlinear automatic and other dynamic systems.

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[Footnotes]

875 The nonlinear characteristic for the slowly varying components of the process, smoothed with the aid of self-oscillations.

877 This is explained by the fact that the polynomial $H(j\omega)$, as seen from Formula (10.17), must contain the denominator of the transfer function of the closed-loop system, i.e., the left half of the characteristic equation of the given system, with the substitution $p = j\omega$.

879 The problem can, of course, be solved also in the case when the mathematical expectations of $\underline{f}$ and x^0 are not equal to zero.

881 The nonlinear characteristic for the slowly varying component of the process (Fig. 10.1), smoothed with the aid of forced oscillations.

893 All the general solutions of the problem, considered in §§10.3 and 10.4 as applied to nonlinear systems of the first class (10.81), can be generalized also to the other classes (see §1.2) with several nonlinearities, similar to what was done in the preceding chapters.

894 In all the problems we shall henceforth seek an approximate solution only for the variable $\underline{x}$ under the nonlinearity sign. Once this solution is found, it is always possible by using the suitable transfer functions to determine the approximate solution for other variables, too, as will be shown in Example 1 of §10.7.

896 In analogy with (10.10) and (10.35), this will be the nonlinear characteristic for the slowly varying component of the process, smoothed with the aid of random fluctuations.

897 Graphically this will be the slope of the curves on Figs. 10.7b-10.12b for the corresponding nonlinearities.

914 But with account of only one nonlinearity in place of two in §6.4.

914 Here φ has been replaced by $\underline{x}$.

925 This is determined by simple plotting of the amplitude frequency characteristic of the linear part of the system from its specified transfer function.

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[List of Transliterated Symbols]

875 н = n = nelineynyy = nonlinear

879 в = v = vynuzhdennyy = forced

884 сл = sl = sluchaynyy = random

900 ст = st = staticheskiy = static

908 ср = sr = sredniy = average

911 с = s = summarnyy = summary, resultant, over-all

912 эк = ek = ekvivalentnyy = equivalent

914 к = k = katushka = coil

915 с = s = sistema = system

924 о.с = o.s = obratnaya svyaz' = feedback

925 л = l = lineynyy = linear

934 пор = por = porogovoy = threshold

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