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APPLIED MATHEMATICS AND STATISTICS LABORATORIES

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A NEW DERIVATION OF THE EQUATIONS OF ALREADY-UNIFIED FIELD THEORY

BY

MENAHEM SCHIFFER AND RONALD ADLER

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A NEW DERIVATION OF THE EQUATIONS OF ALREADY-UNIFIED FIELD THEORY

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The algebraic Rainich conditions and the differential relations of Wheeler and Misner which form the basis of the already unified field theory are usually obtained from the Einstein-Maxwell system of equations by rather laborious tensor analysis. We here obtain these results by utilizing classical matrix theory and a special local coordinate system.

0. Introduction

The combined equations of classical vacuum electrodynamics and general relativity,

$$
(0.1) \quad \begin{aligned}
&\text{a)} \quad R_{\mu\nu} = f_{\mu}{}^{\alpha} f_{\alpha\nu} + \frac{1}{4} g_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta} = T_{\mu\nu} \\
&\text{b)} \quad f^{\mu\nu}{}_{\|\nu} = 0 \\
&\text{c)} \quad {*f}^{\mu\nu}{}_{\|\nu} = 0
\end{aligned}
$$

are usually referred to as the Einstein-Maxwell equations. These were investigated by Rainich in 1925 [1] and Wheeler and Misner in 1957 [2,3], who found that the system (1) is equivalent to the following set of purely geometric relations:

$$a)\quad R^{\mu}{}_{\mu} = 0$$

$$b)\quad R_{\mu\nu}\, R^{\nu}{}_{\alpha} = \frac{1}{4}\,(R_{\tau\beta}\, R^{\tau\beta})\; g_{\mu\alpha}$$

$$(0.2)\qquad c)\quad R_{oo} \geqq 0 \qquad \text{(system of real coordinates)}$$

$$d)\quad v_{\lambda|\tau} - v_{\tau|\lambda} = 0\;; \qquad v_{\lambda} = \frac{\sqrt{|g|}\;\epsilon_{\lambda\nu\beta\gamma}\, R^{\beta\mu||\nu}\, R_{\mu}{}^{\gamma}}{R_{\rho\kappa}\, R^{\rho\kappa}}$$

Subsequently, Wheeler, Misner and others used the system (2) as the basis of a geometrical-topological theory of gravitation and electromagnetism, i.e. the already-unified field theory [2,3]. It is our purpose to give an alternative derivation of (2) from (1) which is based on classical matrix theory and is considerably simpler than the original derivation.

1. Some properties of antisymmetric matrices

Let us consider a 4×4 antisymmetric matrix $f = -f^T$, with complex components. We will identify this matrix with the Minkowski electromagnetic field tensor in part 3. The characteristic polynomial $\phi(\lambda) = |\lambda I - f|$ of f is easily shown to be an even function of λ:

$$(1.1)\qquad \phi(\lambda) = |\lambda I - f| = |\lambda I + f^T| = |\lambda I + f| = (-1)^4\, |-\lambda I - f| = \phi(-\lambda)$$

Thus $\phi(\lambda) = \phi(-\lambda)$, and it follows that ϕ has the form

$$(1.2)\qquad \phi(\lambda) = \lambda^4 + a_2\, \lambda^2 + a_0$$

Furthermore, it is evident from (1.1) that if λ is an eigenvalue of f then $-\lambda$ is also. The eigenvalues, therefore, occur in two pairs: $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2$. It is easy to express the coefficients a_2 and a_0 in (1.2) in terms of these eigenvalues; since the eigenvalues are roots of $\phi(\lambda)$, we can write

$$(1.3) \qquad \phi(\lambda) = (\lambda-\lambda_1)(\lambda+\lambda_1)(\lambda-\lambda_2)(\lambda+\lambda_2) = \lambda^4 - (\lambda_1^2+\lambda_2^2)\,\lambda^2+\lambda_1^2\lambda_2^2$$

Thus the coefficients a_2 and a_0 are easily identified as

$$(1.4) \qquad a_2 = -\,(\lambda_1^2 + \lambda_2^2) \quad ; \quad a_0 = \lambda_1^2\lambda_2^2$$

The Cayley-Sylvester theorem tells us that f itself satisfies $\phi(f) = 0$, so we have from (1.2) and (1.4)

$$(1.5) \qquad f^4 + a_2\, f^2 + a_0 = f^4 - (\lambda_1^2 + \lambda_2^2)\, f^2 + \lambda_1^2\lambda_2^2 = 0$$

The preceding results allow us to construct an interesting symmetric matrix from f. Define

$$(1.6) \qquad T = f^2 + \frac{a_2}{2}\, I = f^2 - \frac{1}{2}\,(\lambda_1^2 + \lambda_2^2)\, I$$

This will be identified in part 3 with the energy-momentum tensor of the electromagnetic field. From the definition (1.6) and equation (1.5) it follows immediately that T^2 <u>is a multiple of the identity matrix</u>:

$$T^2 = \frac{1}{4}(\lambda_1^2 - \lambda_2^2)^2 I \tag{1.7}$$

We will always assume that $\lambda_1^2 \neq \lambda_2^2$, so that T^2 is not zero.

A second interesting property of T follows from a consideration of its Jordan canonical form for similarity. By Jordan's theorem any complex matrix is similar to a direct sum matrix as follows:

$$T = Q^{-1}\mathcal{T}Q = Q^{-1}\begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_N \end{pmatrix} Q \tag{1.8}$$

where the C_i have scalars τ_i on the diagonal and "1"s on the first superdiagonal:

$$C_i = \begin{pmatrix} \tau_i & 1 & 0 & 0 \cdots 0 \\ 0 & \tau_i & 1 & 0 \cdots 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \cdots \tau_i \end{pmatrix} \tag{1.9}$$

However, for the present case the C_i are severely limited by the requirement (1.7). Indeed, if one squares the equation (1.8) and compares with (1.7) it is evident that the C_i must all be 1×1 matrices and that the τ_i must have the values $\pm \frac{1}{2}(\lambda_1^2 - \lambda_2^2)$. Thus

$$(1.10) \qquad T = Q^{-1}\begin{pmatrix} \tau_1 & & & \\ & \tau_2 & & \\ & & \tau_3 & \\ & & & \tau_4 \end{pmatrix} Q$$

and the τ_i are clearly the eigenvalues of T.

The signs of the τ_i can be determined from the definition (1.6) and the fact that they are the eigenvalues of T:

$$(1.11) \qquad |\tau_i I - T| = |[\tau_i - \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2)]I - f^2| = 0$$

Thus $\tau_i - \frac{1}{2}(\lambda_1^2 + \lambda_2^2)$ is an eigenvalue of f^2 and must consequently be λ_1^2 or λ_2^2, each occurring twice. This gives

$$(1.12) \qquad \tau_1 = \tau_2 = -\tau_3 = -\tau_4 = \tfrac{1}{2}(\lambda_1^2 - \lambda_2^2) \equiv \tau$$

and upon substitution into (1.10)

$$(1.13) \qquad T = Q^{-1}\eta Q = Q^{-1}\begin{pmatrix} \tau I & 0 \\ 0 & -\tau I \end{pmatrix} Q \quad ; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus it is evident that T <u>has a null trace</u> and can be written as

$$(1.14) \qquad T = f^2 - \tfrac{1}{4}(\mathrm{Tr}\, f^2) I$$

In part 4 we will need the fact that the Q which appears in (1.13) may be chosen to be orthogonal: $Q^{-1} = Q^T$. The proof of this is straightforward. Since T is symmetric, (1.13) gives

$$T^T = Q^T \eta (Q^{-1})^T = T = Q^{-1} \eta Q \tag{1.15}$$

from which

$$\eta (Q Q^T) = (Q Q^T) \eta \tag{1.16}$$

If $Q Q^T$ is now written in terms of 2×2 submatrices as

$$Q Q^T = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} \tag{1.17}$$

then substitution into (1.16) reveals that $\gamma = \delta = 0$. Thus

$$Q Q^T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{1.18}$$

and the submatrices α and β are clearly symmetric and have nonzero determinants.

Next note that the choice of Q is somewhat arbitrary since it may be replaced by

$$\tilde{Q} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} Q \tag{1.19}$$

where R and S are arbitrary 2×2 nonsingular matrices. This is evident from the fact that

$$T = Q^{-1} \eta Q = \tilde{Q}^{-1} \eta \tilde{Q} \tag{1.20}$$

Furthermore, it is evident that

$$\tilde{Q}\tilde{Q}^T = \begin{pmatrix} R\alpha R^T & 0 \\ 0 & S\beta S^T \end{pmatrix} \tag{1.21}$$

Thus in order to complete the proof we need only show that R and S may be chosen so that $R\alpha R^T = S\beta S^T = I$, for then by (1.20) and (1.21) $\tilde{Q}$ will be the desired orthogonal matrix. This is an easy task; let

$$\alpha = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \; ; \quad R = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \tag{1.22}$$

and substitute into $R\alpha R^T = I$. This gives three equations in the four unknowns u, v, w, and z.

$$\begin{aligned} au^2 + 2buv + cv^2 &= 1 \\ aw^2 + 2bwz + cz^2 &= 1 \\ auw + buz + bwv + czv &= 0 \end{aligned} \tag{1.23}$$

If a is nonzero, a solution is

$$v = 0 \; ; \quad u = \frac{1}{\sqrt{a}} \; ; \quad z = \sqrt{\frac{a}{|\alpha|}} \; ; \quad w = \frac{-b}{\sqrt{a|\alpha|}} \tag{1.24}$$

The case of nonzero c is completely analogous. If both a and c, however, are zero we can use the following solution:

$$(1.24') \qquad z = 1 \; ; \quad u = \frac{1}{2b} \; ; \quad v = -1 \; ; \quad w = \frac{1}{2b}$$

This exhausts all cases, so we have shown that R does exist. In completely similar fashion there exists a nonsingular S such that $S\beta S^T = I$, so the proof of the statement is complete: the Q in (1.13) can be chosen to be orthogonal.

The canonical form (1.15) for T can be utilized to investigate the structure of its generating matrix f. Write f in the form

$$(1.25) \qquad f = Q^T \begin{pmatrix} K & L \\ M & N \end{pmatrix} Q$$

where Q is the same orthogonal matrix as in (1.13) and K, L, M, and N are to be determined. Since f is antisymmetric, both K and N are also antisymmetric. From the definition of T (1.6) and the canonical form (1.13), one finds easily that

$$(1.26) \qquad f^2 = Q^T \begin{pmatrix} \lambda_1^2 I & 0 \\ 0 & \lambda_2^2 I \end{pmatrix} Q$$

Using (1.25) for f and (1.26) for f^2 and the obvious identity $f^2 f - f f^2 = 0$, we obtain

$$(1.27) \qquad (\lambda_1^2 - \lambda_2^2) L = (\lambda_2^2 - \lambda_1^2) M = 0$$

Since λ_1^2 and λ_2^2 are assumed to be unequal, L and M are zero. Similarly, the identity $ff = f^2$ yields

$$K^2 = \lambda_1^2 I \;; \quad N^2 = \lambda_2^2 I \tag{1.28}$$

Since K and N are antisymmetric this tells us that

$$K = i\lambda_1 J \;; \quad N = i\lambda_2 J \;; \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.29}$$

The signs in (1.29) are arbitrary and have been chosen to be positive. This gives the following canonical form for f

$$f = Q^T \begin{pmatrix} i\lambda_1 J & 0 \\ 0 & i\lambda_2 J \end{pmatrix} Q \tag{1.30}$$

where Q is an orthogonal matrix.

2. A simplification of preceding results

Having obtained the canonical form of f in (1.30) we can put all the preceding results in very simple form. Define

$$p = Q^T \begin{pmatrix} iJ & 0 \\ 0 & 0 \end{pmatrix} Q \;; \quad q = Q^T \begin{pmatrix} 0 & 0 \\ 0 & iJ \end{pmatrix} Q \tag{2.1}$$

where Q is the same orthogonal matrix as in (1.30). Then one finds by an elementary calculation that

(2.2)
$$p^2 + q^2 = I \; ; \quad p^3 = p \; ; \quad q^3 = q$$
$$qp = pq = 0 \; ; \quad p^4 + q^4 = I \quad \text{etc.}$$

The canonical form (1.30) now is expressible as

(2.3)
$$f = \lambda_1 p + \lambda_2 q$$

From the definition of T and (2.2), we have

(2.4)
$$T = \frac{1}{2}(\lambda_1^2 - \lambda_2^2)(p^2 - q^2)$$

It immediately follows that T <u>is traceless</u> and that T^2 <u>is a multiple of the identity</u>

(2.5)
$$T^2 = \frac{1}{4}(\lambda_1^2 - \lambda_2^2)^2(p^4 + q^4) = \frac{1}{4}(\mathrm{Tr}\, T^2)\, I$$

These are the principal results of part 1, but now made quite transparent.

The degree of uniqueness in the relation of f to T is quite interesting and easily ascertained. If we construct a $\tilde{T}$ matrix from a different $\tilde{f}$ defined by

(2.6)
$$\tilde{f} = \tilde{\lambda}_1 p + \tilde{\lambda}_2 q \; ; \quad \tilde{\lambda}_1 = \sqrt{\lambda_1^2 - \lambda_2^2}\; \cosh \alpha$$
$$\tilde{\lambda}_2 = \sqrt{\lambda_1^2 - \lambda_2^2}\; \sinh \alpha$$

where α is an arbitrary parameter, then

$$\tilde{T} = \frac{1}{2}(\tilde{\lambda}_1^2 - \tilde{\lambda}_2^2)(p^2 - q^2) = \frac{1}{2}(\lambda_1^2 - \lambda_2^2)(p^2 - q^2) = T \tag{2.7}$$

that is, f and $\tilde{f}$ generate the same T independent of the choice of α. Note in particular that the choice

$$\cosh\alpha = \frac{\lambda_1}{\sqrt{\lambda_1^2 - \lambda_2^2}}; \quad \sinh\alpha = \frac{\lambda_2}{\sqrt{\lambda_1^2 - \lambda_2^2}} \tag{2.8}$$

clearly yields the original f, i.e. $\tilde{f} = f$. It is, therefore, clear that an entire one-parameter family of f matrices generates the same T matrix.

3. The algebraic Rainich conditions

We now wish to apply the results of parts 1 and 2 to the task of deriving the Rainich conditions (0.2a, b, c). For convenience we will work in a locally geodesic system, so that at some fixed world-point P the Christoffel symbols vanish and the ordinary and covariant derivatives of first order coincide. Such a locally geodesic system is only determined up to a linear transformation with constant coefficients. Thus, in order to use matrix theory, we may use a geodesic system in which the metric tensor $g_{\mu\nu}$ is the Kronecker $\delta_{\mu\nu}$. This allows us to ignore the distinction between contravariant and covariant indices and makes tensor algebra and matrix algebra the same. Two further features of this special system should be noted;

firstly, it is only unique up to an orthogonal transformation, which leaves $\delta_{\mu\nu}$ invariant. Secondly, in such a system both the coordinates x^{μ} and the Minkowski tensor $f_{\mu\nu}$ will in general be complex.

With the above choice of a coordinate system, we can write the energy-momentum tensor of the electromagnetic field (0.1a) in matrix notation as

$$T = f^2 - \frac{1}{4}(\mathrm{Tr}\, f^2)\, I \tag{3.1}$$

Comparison with (1.16) shows that T depends on f in precisely the same way as the T matrix which we investigated in part 1 depended on the antisymmetric matrix f. Thus all the results concerning T in part 1 are immediately applicable to the electromagnetic energy-momentum tensor. In particular, we can assert that T is traceless and T^2 is a multiple of I. By equation (0.1a) we can say the same about $R_{\mu\nu}$. That is, in tensor notation,

$$\text{a)}\ \ R^{\mu}{}_{\mu} = 0\ ; \qquad \text{b)}\ \ R_{\mu\nu}\, R^{\nu}{}_{\alpha} = \frac{1}{4}(R_{\tau\beta}\, R^{\tau\beta})\, g_{\mu\alpha} \tag{3.2}$$

Furthermore, since (3.2) is written in covariant form it is true at all world-points and in all coordinate systems.

It is possible to strengthen equation (3.2b) by demonstrating that, under reasonable physical assumptions (as explained below), the scalar $R_{\tau\beta}\, R^{\tau\beta}$ is positive and real. This is easily shown as follows. From (2.5) we know that $R_{\tau\beta}\, R^{\tau\beta}$ may be expressed in terms of the eigenvalues of $f_{\alpha\beta}$ as $(\lambda_1^2 - \lambda_2^2)^2$. This is a covariant statement.

Indeed, the eigenvalue equation of $f_{\alpha\beta}$ may be written covariantly as

$$f_{\alpha\beta}\,\xi^{\beta} = \lambda\xi_{\alpha} = \lambda g_{\alpha\beta}\,\xi^{\beta} \tag{3.3}$$

from which it is clear that the eigenvalue λ is a scalar. Equation (3.3) gives rise in the usual way to a covariant secular equation for λ:

$$|f_{\alpha\beta} - \lambda g_{\alpha\beta}| = 0 \tag{3.4}$$

What we wish to show now is that if we make the physically reasonable demand that $f_{\alpha\beta}$ __be real in a system of real coordinates then__ $(\lambda_1^2 - \lambda_2^2)^2$ __is a positive real scalar__.

A scalar can be calculated in any coordinate system so we will momentarily utilize a real tangent Lorentz system with

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{3.5}$$

We can write $f_{\alpha\beta}$, which is now assumed real, in the general form

$$f_{\alpha\beta} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \tag{3.6}$$

and for convenience we introduce the real three-tuple vectors $C = (a, b, c)$ and $D = (f, -e, d)$. The secular equation (3.4) with $g_{\alpha\beta}$ from (3.5) and $f_{\alpha\beta}$ from (3.6) is found by direct expansion to be

$$\lambda^4 - (C^2 - D^2)\,\lambda^2 - (C\cdot D)^2 = 0 \tag{3.7}$$

where we have used three-dimensional vector notation. This equation has the two solutions

$$\lambda^2 = \frac{C^2 - D^2 \pm \sqrt{(C^2 - D^2)^2 + 4(C\cdot D)^2}}{2} \tag{3.8}$$

from which we find immediately that

$$R_{\tau\beta}\, R^{\tau\beta} = (\lambda_1^2 - \lambda_2^2)^2 = (C^2 - D^2)^2 + 4(C\cdot D)^2 \tag{3.9}$$

By our standing assumption that $\lambda_1^2 \neq \lambda_2^2$ this is obviously a positive real number. Since it is also a scalar, we have shown that it is real and positive in general.

The above fact is interesting as a statement concerning (3.2b), but it is also important in the derivation of condition (2.b) as we shall see presently.

It is a well-known fact that in special relativity the component T_{oo} of the energy-momentum tensor is proportional to the energy density of the electromagnetic field and must, therefore, be a positive number. In order to preserve the same interpretation of T_{oo} in general relativity we must show that

$$R_{oo} \geq 0 \tag{3.10}$$

is a consistent covariant demand, at least in all real coordinate systems.

To demonstrate the consistency of this relation consider $R_\alpha{}^\beta$ as a linear transformation on an arbitrary vector v_β

$$w_\alpha = R_\alpha{}^\beta \, v_\beta \tag{3.11}$$

From (3.2b) it is evident that

$$w_\alpha \, w^\alpha = \frac{1}{4} \, (R_{\tau\beta} \, R^{\tau\beta}) \, v_\eta \, v^\eta \tag{3.12}$$

Since $R_{\tau\beta} \, R^{\tau\beta}$ has been shown to be positive, we see that w_α and v_α are both timelike, both spacelike, or both null. In particular, we can say that $R_\alpha{}^\beta$ carries the light cone into itself.

Next we consider R_{oo} in a new coordinate system

$$\tilde{R}_{oo} = \frac{\partial x^\mu}{\partial \tilde{x}^o} \frac{\partial x^\nu}{\partial \tilde{x}^o} \, R_{\mu\nu} \tag{3.13}$$

Let $\frac{\partial x^\mu}{\partial \tilde{x}^o}$ be v^μ and $\frac{\partial x^\nu}{\partial \tilde{x}^o} \, R_{\mu\nu}$ be w_μ as above, so that $\tilde{R}_{oo}$ can be written

$$\tilde{R}_{oo} = v^\mu \, w_\mu \tag{3.14}$$

Thus it is clear that the demand that $\tilde{R}_{oo}$ be positive definite is equivalent to the covariant requirement that $R_{\alpha\beta}$ carry a vector v_α into a vector w^α such that $v^\alpha w_\alpha \geq 0$. This can be interpreted physically by saying that $R_\alpha{}^\beta$ carries the <u>forward</u> light cone into itself and the <u>backward</u> light cone into itself. Thus, by the statement (3.10), one gives a sign sense to $R_{\mu\nu}$ in a covariant way for any real coordinate system.

The conditions (3.2) and (3.10) were first obtained by Rainich in 1925, using somewhat different matrix methods than we have used. They are usually referred to as the <u>algebraic Rainich conditions</u> and, indeed, are algebraic in the sense that they follow from a consideration of $f_{\mu\nu}$ and $T_{\mu\nu}$ at a single world-point P.

This completes the derivation of (0.2a, b, c) from (0.1), so only the differential relation (0.2d) remains to be considered. This differential relation will be our most important and interesting result. We have derived the algebraic conditions (0.2a, b, c) mainly for the sake of completeness and in order to lay a groundwork for the final relation (0.2d).

4. <u>The differential relations of Wheeler and Misner</u>

In order to obtain (1.2d) we need to take into account the fact that $f_{\mu\nu}$ obeys the Maxwell equations (0.1b,c). At the world-point P in our special system we can ignore the distinction between contravariant and covariant indices and between ordinary and covariant derivatives of first order. This allows us to rewrite (0.1b,c) as

$$(4.1) \qquad \text{a)} \quad f_{\mu\nu|\nu} = 0 \; ; \qquad \text{b)} \quad *f_{\mu\nu|\nu} = 0$$

where we have retained the Einstein summation convention without regard for index position.

Our aim now is to study how equations (4.1) reflect themselves in properties of the tensor $R_{\mu\nu}$. Observe first that, given an energy momemtum tensor $T_{\mu\nu}$ according to (2.6), one can only obtain $f_{\mu\nu}$ up to a parameter α which may depend on the point considered. Algebraically, the α field at different world-points could be completely incoherent, but we will find that the Maxwell equations (4.1) provide a differential system for α which makes it a determined point function. The integrability condition on the system will be the Wheeler-Misner relation which we seek.

Let $r = \sqrt{\lambda_1^2 - \lambda_2^2}$ and write $f_{\mu\nu}$ in the general form (2.6).

$$(4.2) \qquad f_{\mu\nu} = r \cosh \alpha \, p_{\mu\nu} + r \sinh \alpha \, g_{\mu\nu}$$

By inspection of (2.1) the dual of f is found to be

$$(4.3) \qquad *f_{\mu\nu} = r \cosh \alpha \, g_{\mu\nu} + r \sinh \alpha \, p_{\mu\nu}$$

Recall that our special coordinate system is still arbitrary to an orthogonal transformation. This allows us to assume without loss of generality that the matrix Q which appears in the definition of p and q is precisely the identity matrix I. Then p and q at the world-point P take the particularly simple forms

$$(4.4) \qquad \tilde{p} = \begin{pmatrix} iJ & 0 \\ 0 & 0 \end{pmatrix} ; \quad \tilde{q} = \begin{pmatrix} 0 & 0 \\ 0 & iJ \end{pmatrix}$$

It follows then from (2.4) that the energy-momentum tensor becomes

$$(4.5) \qquad \tilde{T} = \frac{1}{2} r^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and that the Maxwell equations at P are

$$(4.6) \quad \begin{aligned} &\text{a)} \quad f_{\mu\nu|\nu} = \tilde{p}_{\mu\nu} (r \cosh \alpha)_{|\nu} + \tilde{q}_{\mu\nu} (\sinh \alpha)_{|\nu} \\ &\qquad\qquad + r \cosh \alpha \, p_{\mu\nu|\nu} + r \sinh \alpha \, q_{\mu\nu|\nu} = 0 \\ &\text{b)} \quad {*f}_{\mu\nu|\nu} = q_{\mu\nu} (r \cosh \alpha)_{|\nu} + p_{\mu\nu} (r \sinh \alpha)_{|\nu} \\ &\qquad\qquad + (r \cosh \alpha) \, q_{\mu\nu|\nu} + (r \sinh \alpha) \, p_{\mu\nu|\nu} = 0 \end{aligned}$$

To simplify (4.6) define at the world-point P vectors Π_μ and K_μ and scalars A and B as follows:

$$(4.7) \quad \Pi_\mu = p_{\mu\nu|\nu} \; ; \quad K = q_{\mu\nu|\nu} \; ; \quad A = r \cosh \alpha \; ; \quad B = r \sinh \alpha$$

Then Maxwell's equations (4.6) reduce to

$$(4.8) \quad \begin{aligned} &\text{a)} \quad \tilde{p}_{\mu\nu} A_{|\nu} + \tilde{q}_{\mu\nu} B_{|\nu} + A \, \Pi_\mu + B \, K_\mu = 0 \\ &\text{b)} \quad \tilde{q}_{\mu\nu} A_{|\nu} + \tilde{p}_{\mu\nu} B_{|\nu} + A \, K_\mu + B \, \Pi_\mu = 0 \end{aligned}$$

Now multiply a) by $\tilde{p}$, b) by $\tilde{q}$, add the two results, and use (2.2) to obtain easily

$$(4.9) \quad A[\tilde{p}_{\mu\nu} \Pi_\nu + \tilde{q}_{\mu\nu} K_\nu + (\log r)_{|\mu}] + B[\tilde{q}_{\mu\nu} \Pi_\nu + \tilde{p}_{\mu\nu} K_\nu + \alpha_{|\mu}] = 0$$

Similarly, multiply a) by $\tilde{q}$, b) by $\tilde{p}$, and add to get

$$(4.10) \quad A[\tilde{q}_{\mu\nu} \Pi_\nu + \tilde{p}_{\mu\nu} K_\nu + \alpha_{|\mu}] + B[\tilde{p}_{\mu\nu} \Pi_\nu + \tilde{q}_{\mu\nu} K_\nu + (\log r)_{|\mu}] = 0$$

By the definition of A and B and the structure of these equations, it is clear that (4.9) and (4.10) are consistent only if the coefficients of A and B are zero. Thus we obtain

$$(4.11) \quad \begin{aligned} &\text{a)} \qquad \alpha_{|\mu} = -(\tilde{q}_{\mu\nu} \Pi_\nu + \tilde{p}_{\mu\nu} K_\nu) \\ &\text{b)} \qquad (\log r)_{|\mu} = -(\tilde{p}_{\mu\nu} \Pi_\nu + \tilde{q}_{\mu\nu} K_\nu) \end{aligned}$$

As we indicated above the α field is now seen to be subjected to a simple differential system.

The next step in our development is to calculate the vectors Π_μ and K_μ and to use the result to put (4.11) into a more informative and interesting form. In order to do this we first will calculate the derivatives of the matrices p and q. At the world-point P these matrices are precisely the $\tilde{p}$ and $\tilde{q}$ in (4.4), but as we move to a nearby world-point $P(\epsilon;\mu)$, displaced from P by a distance ϵ in the μ-th coordinate direction, these matrices will become $p^{(\mu)}$

and $q^{(\mu)}$ which differ from $\tilde{p}$ and $\tilde{q}$ by a small rotation in space-time corresponding to an orthogonal matrix $Q_{(\mu)}$. Let us represent this matrix $Q_{(\mu)}$ by a series

$$(4.12) \qquad Q_{(\mu)} = I + \epsilon\, C_{(\mu)} + \epsilon^2 D_{(\mu)} + \cdots$$

The orthogonality of $Q_{(\mu)}$ then implies that $C_{(\mu)}$ must be anti-symmetric, as is well known, and that its inverse is

$$(4.13) \qquad Q^T_{(\mu)} = Q^{-1}_{(\mu)} = I - \epsilon\, C_{(\mu)} + \cdots$$

It is now easy to calculate the derivatives of p and q at P in the μ-th direction. We have from (2.1), (4.12), and (4.13)

$$(4.14) \qquad p^{(\mu)} = Q^T_{(\mu)}\, \tilde{p}\, Q_{(\mu)} = \tilde{p} + \epsilon[\tilde{p}\, C_{(\mu)} - C_{(\mu)}\, \tilde{p}\,] + O(\epsilon^2)$$

Now subtract $\tilde{p}$ from both sides, divide by ϵ, and pass to the limit $\epsilon \to 0$ to obtain

$$(4.15) \qquad p_{|\mu} = \tilde{p}\, C_{(\mu)} - C_{(\mu)}\, \tilde{p}$$

and similarly for $q_{|\mu}$. If we write $C_{(\mu)}$ in terms of 2×2 sub-matrices as

$$(4.16) \qquad C_{(\mu)} = \begin{pmatrix} a_{(\mu)} & b_{(\mu)} \\ c_{(\mu)} & d_{(\mu)} \end{pmatrix}$$

then the antisymmetry of $C_{(\mu)}$ implies that $a^T_{(\mu)} = - a_{(\mu)}$, $d^T_{(\mu)} = - d_{(\mu)}$, and $b^T_{(\mu)} = - c_{(\mu)}$. Now substitute (4.16) and (4.4) into (4.15) to obtain an explicit form for $p_{|\mu}$

$$p_{|\mu} = i \begin{pmatrix} 0 & Jb_{(\mu)} \\ -c_{(\mu)}J & 0 \end{pmatrix} \tag{4.17}$$

Similarly, we find for $q_{|\mu}$

$$q_{|\mu} = i \begin{pmatrix} 0 & -b_{(\mu)}J \\ Jc_{(\mu)} & 0 \end{pmatrix} \tag{4.18}$$

We will later need the derivative $T_{|\mu}$ at P, so we will obtain it now while it is most convenient. Recall that T may be written as

$$T = \frac{1}{2} r^2(p^2 - q^2) \tag{4.19}$$

Using (4.4), (4.5), (4.17), and (4.18) it is then easily seen that

$$T_{|\mu} = (\log r)_{|\mu} \, 2 \tilde{T} + r^2 \begin{pmatrix} 0 & b_{(\mu)} \\ -c_{(\mu)} & 0 \end{pmatrix} \tag{4.20}$$

Observe that in the above expressions for $p_{|\mu}$, $q_{|\mu}$, and $T_{|\mu}$

only the 2×2 matrices $b_{(\mu)}$ and $c_{(\mu)} = - b^T_{(\mu)}$ occur. Let us write these explicitly as

$$(4.21) \qquad b_{(\mu)} = \begin{pmatrix} k_\mu & \ell_\mu \\ m_\mu & n_\mu \end{pmatrix} \; ; \quad c_{(\mu)} = - \begin{pmatrix} k_\mu & m_\mu \\ \ell_\mu & n_\mu \end{pmatrix}$$

and substitute into (4.17), (4.18), and (4.20) to get

$$\text{a)} \qquad p_{|\mu} = i \left(\begin{array}{cc|cc} 0 & 0 & m_\mu & n_\mu \\ 0 & 0 & -k_\mu & -\ell_\mu \\ \hline -m_\mu & k_\mu & 0 & 0 \\ -n_\mu & \ell_\mu & 0 & 0 \end{array}\right)$$

$$(4.22) \quad \text{b)} \qquad q_{|\mu} = i \left(\begin{array}{cc|cc} 0 & 0 & \ell_\mu & -k_\mu \\ 0 & 0 & n_\mu & -m_\mu \\ \hline -\ell_\mu & -n_\mu & 0 & 0 \\ k_\mu & m_\mu & 0 & 0 \end{array}\right)$$

$$\text{c)} \quad T_{|\mu} = (\log r)_{\mu} \, 2 \, \tilde{T} + r^2 \left(\begin{array}{cc|cc} 0 & 0 & k_\mu & \ell_\mu \\ 0 & 0 & m_\mu & n_\mu \\ \hline k_\mu & m_\mu & 0 & 0 \\ \ell_\mu & n_\mu & 0 & 0 \end{array}\right)$$

A straightforward calculation then yields

$$(4.23)\qquad \text{a)}\quad \Pi_\mu = p_{\mu\nu|\nu} = i \begin{pmatrix} m_3 + n_4 \\ -k_3 - \ell_4 \\ -m_1 + k_2 \\ -n_1 + \ell_2 \end{pmatrix}$$

$$\text{b)}\quad K_\mu = q_{\mu\nu|\nu} = i \begin{pmatrix} \ell_3 - k_4 \\ n_3 - m_4 \\ -\ell_1 - n_2 \\ k_1 + m_2 \end{pmatrix}$$

We have achieved an explicit form for K_μ and Π_μ in terms of the components of $C_{(\mu)}$. Let us substitute now the results (4.23) into (4.11a) and use (4.4) to get an explicit form for $\alpha_{|\mu}$

$$(4.24)\qquad \alpha_{|\mu} = \begin{pmatrix} n_3 - m_4 \\ k_4 - \ell_3 \\ \ell_2 - n_1 \\ m_1 - k_2 \end{pmatrix}$$

Observe an important fact at this point: the vector $\alpha_{|\mu}$ is composed of the same components k_μ, ℓ_μ, m_μ, and n_μ which occur in $T_{|\mu}$ and is, therefore, expressible in terms of $T_{\mu\nu}$ alone. Indeed, our only remaining problem is to express the correspondence of $\alpha_{|\mu}$ and $T_{|\mu}$ in covariant form. To do this consider the covariant vector

$$v_\lambda = \frac{\sqrt{|g|}\,\epsilon_{\lambda\nu\beta\gamma}\,T^{\beta\mu\|\nu}\,T_\mu{}^\gamma}{T_{\rho\kappa}\,T^{\rho\kappa}} \tag{4.25}$$

which we will now calculate at P in our special coordinate system. By (4.5) the denominator is simply r^4. The numerator takes the simple form

$$\sqrt{|g|}\,\epsilon_{\lambda\nu\beta\gamma}\,T^{\beta\mu\|\nu}\,T_\mu{}^\gamma = \epsilon_{\lambda\nu\beta\gamma}\,\tilde{T}_{\beta\mu|\nu}\,T_{\mu\gamma} \tag{4.26}$$

Using the explicit form (4.20) for $T_{\beta\mu|\nu}$ we have

$$\epsilon_{\lambda\nu\beta\gamma}\,T_{\beta\mu|\nu}\,\tilde{T}_{\mu\gamma} = 2(\log r)_{|\nu}\,\epsilon_{\lambda\nu\beta\gamma}\,\tilde{T}_{\beta\mu}\,\tilde{T}_{\mu\gamma} + r^2\,\epsilon_{\lambda\nu\beta\gamma}\begin{pmatrix} 0 & b_{(\mu)} \\ -c_{(\mu)} & 0 \end{pmatrix}_{\nu\beta}\tilde{T}_{\mu\gamma} \tag{4.27}$$

The Rainich condition (3.2b) and the Einstein equations (0.1a) tell us that $\tilde{T}_{\beta\mu}\,\tilde{T}_{\mu\gamma}$ is a multiple of the metric tensor and is, therefore, symmetric in β and γ; since $\epsilon_{\lambda\nu\beta\gamma}$ is totally antisymmetric the first term of (4.28) vanishes identically. A short calculation using (4.21) and (4.5) then reveals that

$$T_{\beta\mu|\nu}\,\tilde{T}_{\mu\gamma} = \frac{1}{2}r^4\left(\begin{array}{cc|cc} 0 & 0 & -k_\nu & -\ell_\nu \\ 0 & 0 & -m_\nu & -n_\nu \\ \hline k_\nu & m_\nu & 0 & 0 \\ \ell_\nu & n_\nu & 0 & 0 \end{array}\right)_{\beta\gamma} \tag{4.28}$$

and a slightly tedious but elementary calculation gives

$$(4.29) \qquad \nu_\lambda = \frac{\epsilon_{\lambda\nu\beta\gamma}\, T_{\beta\mu|\nu}\, \tilde{T}_{\mu\gamma}}{T_{\rho k}\, T^{\rho k}} = \begin{pmatrix} n_3 - m_4 \\ k_4 - \ell_3 \\ \ell_2 - n_1 \\ m_1 - k_2 \end{pmatrix}$$

Comparing this with (4.24) we see that

$$(4.30) \qquad \alpha_{|\lambda} = \nu_\lambda = \frac{\sqrt{|g|}\,\epsilon_{\lambda\nu\beta\gamma}\, T^{\beta\mu\|\nu}\, T^{\gamma}{}_{\mu}}{T_{\rho k}\, T^{\rho k}}$$

which is in tensor form and therefore valid in general. Using the Einstein equations (0.1a), we can also write this in geometric form as

$$(4.31) \qquad \alpha_{|\lambda} = \nu_\lambda = \frac{\sqrt{|g|}\;\epsilon_{\lambda\nu\beta\gamma}\, R^{\beta\mu\|\nu}\, R^{\gamma}{}_{\mu}}{R_{\rho k}\, R^{\rho k}}$$

The differential relations of Wheeler and Misner follow immediately from (4.31), for in order that (4.31) be an integrable relation we must have

$$(4.32) \qquad \nu_{\lambda|\tau} - \nu_{\tau|\lambda} = 0$$

Thus we have finally obtained (0.2d), the last basic relation of the already-unified field theory.

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