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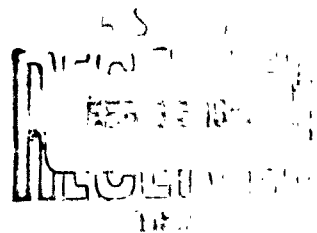
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SURVEILLANCE PROBLEMS: POISSON MODELS WITH NOISE

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SURVEILLANCE PROBLEMS: POISSON MODELS WITH NOISE

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0. Introduction.

The model discussed in [2]^{1/} is generalized by the introduction of noise. The generalization increases the domain of application as well as making the associated problems statistical (Bayes) rather than purely probabilistic as formerly.

A process, the production process, is either producing a continuous stream of goods or else is in a state of repair. A cycle consists of all of the events from the time that the production process leaves the repair state until it has gone through production and repair and is once again ready to leave the repair state. The variable, t , is used to measure time from the beginning of a cycle. It takes m time units to go through repair at a cost of K units per unit of time. When the production process is producing and is in state x , the income from production per unit of time is $i(x)$. When the production process leaves the repair state, it is in the 0 state, i.e., $x(0)=0$. It is assumed that $x(t)$ is a Poisson stochastic process with parameter Δ_x . In particular, when $x(t)$ changes, it is a unit increase, the number of changes per unit of time is Poisson distributed with parameter Δ_x , and the times between changes have independent exponential distributions with expectation $1/\Delta_x$.

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^{1/} The present paper is self contained as far as results. It is necessary to read [2] to obtain additional motivation as well as details for some of the proofs.

Observations are made on a process $y(t)=x(t)+z(t)$, where $z(t)$ is a Poisson process with parameter Δ_z . The x - and z - processes are assumed independent. Hence $y(t)$ is a Poisson process with parameter $\Delta_y=\Delta_x+\Delta_z$. (It will be useful to define $H=1/\Delta_y=(\Delta_x+\Delta_z)^{-1}$.) Thus we cannot observe the production process directly. The noise, $z(t)$, contaminates the observations. For example, if $y(t)$ is an instrument reading used to make inferences about $x(t)$, then $z(t)$ can be thought of as the accumulated calibration error. It is assumed $z(0)=0$ and hence $y(0)=0$, e.g., repairs include recalibration.

In Section 1, it is assumed that $y(t)$ is observed continuously. In Section 2, it is assumed that each observation of $y(t)$ costs L units; observation results are obtained instantly and an observation must be obtained exactly at the time that repairs begin. The primary purpose of the analysis is to obtain methods for finding qualitative properties of the optimal solutions.

The methods used are sufficiently general to allow minor variations in the model without requiring an entire redevelopment. For instance, dropping the restriction of looking at the production process immediately before beginning repairs would not involve difficult changes. The full strength of the assumption that the x - and z - processes are Poisson might not be required, but the weakening of this assumption certainly can not be made as completely as it was done when just dealing with $y(t)=x(t)$ [2, Section 3]. Specific choices of the income function $i(x)$, computation procedures, and numerical examples are not discussed. It is hoped that the general qualitative properties obtained for the solutions will help sufficiently in a particular problem so that detailed numerical analysis will not be necessary.

1. Continuous Surveillance.

A cycle will consist of T units of production time where T is determined by the rule used for placing the production process in the repair state.

T will be a random variable and can depend on the data obtained up to the time when production is stopped, i.e., T can depend on $y(\cdot)$ for $0 \leq t \leq T$. Since the cycles all begin in the same state, the choice of the rule should be the same for each cycle. If R is a specific rule, let $T(R)$ be the associated random time. Then the long run average income per unit of time, $I(R)$, is

$$I(R) = \frac{E\left[\int_0^{T(R)} i(x(t))dt\right] - mK}{ET(R) + m}.$$

The basic problems are to isolate the reasonable rules, to evaluate $I(R)$ for those rules, and to find properties of the best rule among the reasonable rules. When $y(t)=x(t)$, it was found that the reasonable rules were to select an integer w and to stop production as soon as $x(t)$ exceeded w, i.e., $T(w)$ was the smallest solution of $x(t)=w+1$. In this section, it is shown, even when $y(t)$ contains noise, that the reasonable rules involve the selection of an integer w and $T(w)$ is the smallest solution of $y(t)=w+1$. When $w=-1$, the production process will always be kept in repair. When $w=\infty$, the production process is never repaired. Thus for decision purposes the history of how $y(\cdot)$ arrived at $w+1$ is not relevant. In other words, in making inferences about $x(\cdot)$ at time t from observing $y(\cdot)$ up to time t, $y(t)$ is a sufficient statistic. The basic result is contained in:

Theorem 1.

$$P(x(T)=k|y(\cdot), 0 \leq t \leq T) = P(x(T)=k|y(T)).$$

Let $p = \Delta_x / (\Delta_x + \Delta_z) = \Delta_x / \Delta_y = H\Delta_x$, and $y(T)=n$. Then the following is well known.

Theorem 2.

$$P(x(T)=k|y(T)=n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(Using Theorems 1 and 2, one can say that $x(\cdot)$ for $t > T$ is a Poisson process with a random starting point selected from a binomial distribution.)

Proof of Theorem 1.

The proof is by induction on the value of $y(T)$. If $y(T) = 0$, then $x(T) = 0$ and the result is trivial. Now let t_i be the smallest solution of $y(T) = 1$ for $i=0,1,2,\dots$. Clearly,

$$P(x(T)=k|y(\cdot), 0 \leq t \leq T \text{ and } y(T)=1) = P(x(t_i)=k|y(\cdot), 0 \leq t \leq t_i).$$

Now proceed with the induction argument, using the following decomposition,

$$\begin{aligned} P(x(t_{i+1})=k|y(\cdot), 0 \leq t \leq t_{i+1}) \\ = P(x(t_i)=k \text{ and } x(t_{i+1})-x(t_i)=0|y(\cdot), 0 \leq t \leq t_i, \text{ and } t_{i+1}) \\ + P(x(t_i)=k-1 \text{ and } x(t_{i+1})-x(t_i)=1|y(\cdot), 0 \leq t \leq t_i, \text{ and } t_{i+1}). \end{aligned}$$

Now use the independence of the increments of the x - and y - processes, i.e., the future does not effect the past and the past does not effect the changes which will occur in the future. Then

$$\begin{aligned} P(x(t_{i+1})=k|y(\cdot), 0 \leq t \leq t_{i+1}) \\ = P(x(t_i)=k|y(\cdot), 0 \leq t \leq t_i)P(x(t_{i+1})-x(t_i)=0|t_i \text{ and } t_{i+1}) \\ + P(x(t_i)=k-1|y(\cdot), 0 \leq t \leq t_i)P(x(t_{i+1})-x(t_i)=1|t_i \text{ and } t_{i+1}). \end{aligned}$$

Using the induction hypothesis, one obtains

$$\begin{aligned} (1.1) \quad P(x(t_{i+1})=k|y(\cdot), 0 \leq t \leq t_{i+1}) \\ = P(x(t_i)=k|y(t_i)=1)P(x(t_{i+1})-x(t_i)=0|t_i \text{ and } t_{i+1}) \\ + P(x(t_i)=k-1|y(t_i)=1)P(x(t_{i+1})-x(t_i)=1|t_i \text{ and } t_{i+1}). \end{aligned}$$

At this point we require

$$(1.2) \quad 1-P(x(t_{i+1})-x(t_i)=0|t_i \text{ and } t_{i+1}) = P(x(t_{i+1})-x(t_i)=1|t_i \text{ and } t_{i+1}) = p.$$

Now using (1.2) and Theorem 2 in (1.1), one finds

$$P(x(T)=k|y(.), 0 \leq t \leq T \text{ and } y(T)=i+1) \\ = \binom{i}{k} p^k (1-p)^{i-k} (1-p) + \binom{i}{k-1} p^{k-1} (1-p)^{i-k+1} p = \binom{i+1}{k} p^k (1-p)^{i+1-k}.$$

This completes the proof of the theorem, since the desired probability depends on $y(.), 0 \leq t \leq T$, only through the value of $y(T)$.

The following theorem will not be used here but is easily proved at this point.

Theorem 3.

Let $y(t)=x(t)+z(t)$ where $x(t)$ and $z(t)$ are independent Wiener processes with variance parameters C_x and C_z . Then

$$P(x(T)=x|y(.), 0 \leq t \leq T)dx = P(x(T)=x|y(T))dx$$

and the conditional distribution of $x(T)$ given $y(T)$ is normal with mean $y(T)C_x/(C_x+C_z)$ and variance $TC_x C_z/(C_x+C_z)$.

Proof.

In Theorem 1, let $\Delta_x = \Delta C_x$ and $\Delta_z = \Delta C_z$, then the variables $[x(t)-t\Delta C_x]\Delta^{-\frac{1}{2}}$, $[z(t)-t\Delta C_z]\Delta^{-\frac{1}{2}}$, and $[y(t)-t\Delta(C_x+C_z)]\Delta^{-\frac{1}{2}}$ will have the properties of the variables in the present theorem as Δ tends to infinity.

Now consider the evaluation of $I(w)$. It is clear that

$$ET(w) = (w+1)H$$

and thus the denominator of $I(w)$ is of the form $(w+1)H+m$. The evaluation of the numerator will require more detail:

$$E\left[\int_0^{T(w)} 1(x(t))dt\right] = E\left[E\left(\int_0^{T(w)} 1(x(t))dt\right)|x(T(w)) \text{ and } T(w)\right].$$

Given the value of $x(T(w))$, the points of increase of $x(.)$ in the interval $0 \leq t \leq T(w)$ will be uniformly distributed, so that

$$\begin{aligned}
E\left[\int_0^{T(w)} i(x(t))dt\right] &= E T(w) \sum_{j=0}^{x(T(w-1))} i(j)/(1+x(T(w-1))) \\
&= E T(w) \sum_{a=0}^w \binom{w}{a} p^a (1-p)^{w-a} \left[\sum_{j=0}^a i(j)/(1+a) \right] \\
&= (w+1)H \sum_{a=0}^w \binom{w}{a} p^a (1-p)^{w-a} \left[\sum_{j=0}^a i(j)/(1+a) \right] \\
&= H \sum_{a=0}^w \binom{w+1}{a+1} p^a (1-p)^{w-a} \left[\sum_{j=0}^a i(j) \right]
\end{aligned}$$

Finally

$$(1.3) \quad I(w) = [H \sum_{a=0}^w \binom{w+1}{a+1} p^a (1-p)^{w-a} \left(\sum_{j=0}^a i(j) \right) - mK] / [(w+1)H+m].$$

The remaining problem is to find the best choice of w . The following theorems give bounds for $I(w)$.

Theorem 4.

$$I(w) \leq \underset{w}{\text{maximum}} [\Delta_x^{-1} \sum_{j=0}^w i(j) - mK] / [\Delta_x^{-1} (w+1) + m].$$

Proof.

The quantity being maximized on the right hand side in the statement of the theorem is $I(w)$ with $p=1$. This expression was obtained in [2] when there was no noise. (It should be easier to evaluate than $I(w)$, in that it does not involve a double summation.) The proof is made by noting: If it were possible to observe directly the process $x(t)$, then at no cost an artificial process $z(t)$ could be added to it. Hence, any strategy which is available when observing $x(t)+z(t)$ is also available when observing $x(t)$.

Theorem 5.

$$\underset{w}{\text{maximum}} I(w) \geq \underset{t}{\text{maximum}} I(t)$$

where $I(t)$ is the income corresponding to the rule of having each cycle of length exactly $t+m$. Furthermore

$$I(t) = [\Delta_x e^{-\Delta_x t} \sum_{a=1}^{\infty} \frac{(t\Delta_x)^a}{a!} (\sum_{j=0}^{a-1} i(j)) - mK] / [t+m].$$

Proof.

Since the best fixed time strategy can not be as good as the best strategy, one immediately obtains the main result of the theorem. The expression for $I(t)$ is a routine computation. When $i(x)$ is a low degree polynomial, it is not difficult to evaluate $I(t)$.

Below, the symbols on the left are defined by the symbols on the right:

$$p(a,w) = \binom{w}{a} p^a (1-p)^{w-a}$$

$$c(a) = \sum_{j=0}^a i(j) / (a+1)$$

$$b(w) = \sum_{a=0}^w p(a,w) c(a)$$

Notice

$$I(w) = [(w+1) H b(w) - mK] / [(w+1) H + m].$$

Lemma 1.

$p(a,w)$ considered as a density function in a with parameter w is Polya type ∞ and in particular

$$\sum_{a=0}^A p(a,w) > \sum_{a=0}^A p(a,w+1).$$

$p(a,w)$ considered as a density function in a with parameter p is Polya type ∞ and in particular

$$\sum_{a=0}^A p(a,w)$$

is for each A and w a decreasing function of p .

Proof.

See [1] for definitions and methods.

The following Lemma will help in the analysis of $I(w)$.

Lemma 2.

If $i(x)$ is non increasing then $c(a)$ and $b(w)$ are non increasing.

Also, $b(w)$, for fixed w , is a non increasing function of p .

Proof.

That $c(a)$ is non increasing is immediate. The proof for $b(w)$ is based on the following

$$b(w) = \sum_{a=1}^{\infty} (c(a-1) - c(a)) \left[\sum_{j=0}^{a-1} p(j, w) \right].$$

In this expression for $b(w)$ the coefficients of the inner summation, $c(a-1) - c(a)$, are non negative and the value of the inner summation is a non increasing function of w . Hence $b(w)$ is a non increasing function of w . On the other hand the inner summation is a non increasing function of p , so that $b(w)$ is a non increasing function of p .

2. Costly Surveillance.

A strategy in this case consists of the following: Select a time $T(0)$ at which the first inspection will be made; select a $W_0(T(0))$ such that if $y(T(0))$ is not in $W_0(T(0))$, begin repairs, but if $y(T(0))$ is in $W_0(T(0))$ then; select a $T(T(0), y(T(0)))$ and make the next observation at time $T(0) + T(T(0), y(T(0)))$: select a $W_1(T(0), y(T(0)), T(T(0), y(T(0))))$ such that if the observation at this time is not in W_1 , begin repairs, but if the observation is in W_1 then; select $T(T(0), y(T(0)))$

The class of possible strategies is large. The following theorem, however, simplifies the matter.

Theorem 6.

Let $t_0, t_1, \dots, t_r$ be an increasing sequence of numbers such that for $i=0, \dots, r$, one has that t_i depends on t_j and $y(t_j)$ for $j < i$ (t_{-1} and $y(t_{-1})$ can be arbitrary fixed numbers). Then

$$P(x(t_r)=k | y(t_i) \text{ and } t_i, \text{ for } i=1, \dots, r) = P(x(t_r)=k | y(t_r)).$$

Remark.

In interpreting this theorem in terms of the above discussion, let $t_0=T(0)$, $t_1=T(0) + T(T(0), y(T(0))), \dots$. The implication of the theorem is that in predicting the future history of $x(t)$ given this kind of past history for $y(t)$, the only relevant information is $y(t_r)$, i.e., the last observation. This implies that the class of reasonable strategies consists of the selection of a set W and a sequence of numbers $T(0), T(1), \dots, T(k)$. Then, if an observation is not in W , begin repairs, and if an observation equals k which is in W , wait $T(k)$ units of time before making the next observation.

Proof of Theorem 6.

The increment in $x(\cdot)$ in the interval (t_{i-1}, t_i) is binomially distributed with parameters p and $y(t_i) - y(t_{i-1})$. Also the increments are independently distributed. Hence the sum of the increments is binomially distributed with parameters p and $y(t_k)$. Which is the desired conclusion and yields the following theorems.

Theorem 7.

$$P(x(t_r)=k | y(t_i) \text{ and } t_i, \text{ for } i=1, \dots, r) = \binom{y(t_r)}{k} p^k (1-p)^{y(t_r)-k}.$$

Theorem 8.

Let $y(t)=x(t)+z(t)$ where $x(t)$ and $z(t)$ are independent Wiener processes with variance parameters C_x and C_z . Then

$$P(x(t_r)=x | y(t_i) \text{ and } t_i, \text{ for } i=1, \dots, r) dx = P(x(t_r)=x | y(t_r)) dx$$

and the conditional distribution of $x(t_r)$ given $y(t_r)$ is normal with mean $y(t_r)C_x/(C_x+C_z)$ and variance $t_r C_x C_z/(C_x+C_z)$.

[In the statements of Theorems 7 and 8, it is implicitly assumed that the conditions on t_0 of Theorem 6 are satisfied.]

Assume there exists a best strategy and let I^* be the maximum income per unit of time. Our objective is to bound I^* and to find qualitative properties of the best strategy. (It will be left as a conjecture that there is a best strategy. In fact, it will be assumed that there is a unique best strategy.)

If an observation has just been obtained^{2/} and found to be y , then let $F^*(y)$ be the expected income remaining in the cycle if the best strategy is followed. Let $T^*(y)$ be the expected time to complete the cycle if the best strategy is followed. Our interest will be centered on the function $F(y)=F^*(y)-T^*I^*$. An interpretation of $F(y)$ is the maximum expected income remaining in a cycle when the observation y is obtained and at each instant of time the rate of income is I^* less than in the original problem. It can be shown that $F(0)=0$. Also the following functional equation can be obtained:

$$(2.1) \quad F(y) = \underset{W, T(y)}{\text{maximum}} \left\{ \begin{array}{l} -m(K+I^*) - L, \\ E \left[\int_0^{T(y)} (i(x(t)) - I^*) dt | y(0)=y \right] - L \end{array} \right. \quad y \notin W$$

$$+ \sum_{j=0}^{\infty} e^{-T(y)\Delta_y} (T(y)\Delta_y)^j (j!)^{-1} F(y+j), \quad y \in W$$

The unknowns in this equation are $F(y)$, W , $T(y)$, and I^* . Clearly

$$(2.2) \quad F(y) \geq -m(K+I^*) - L.$$

Since $F(0)=0$, one obtains

$$(2.3) \quad I^* \geq -K-m^{-1}L.$$

^{2/} The cost, L , to obtain the observation is to be paid immediately after the observation is made.

Also, if $i(x)$ is non increasing then

$$(2.4) \quad F(y) < 0, \quad y=1,2,\dots$$

To find an upper bound on I^* , consider the impossible strategy of stopping the production process at the moment $y(t)$ leaves the optimal W and to do this with one look. Since $y(t)$ increases by unity, this would correspond to stopping as soon as $y(t) > w$ for the best choice of w . Then, the upper bound on I^* would be

$$(2.5) \quad \underset{0 \leq w}{\text{maximum}} [(w+1)Hb(w) - mK - L] / [(w+1)H + m].$$

A weaker upper bound could be obtained by using Theorem 4. A lower bound on I^* could be obtained by using the non optimal rule of having each cycle exactly of length $T+m$ and choose the best value of T . Thus, a lower bound for I^* would be

$$(2.6) \quad \underset{0 \leq T}{\text{maximum}} [E[\int_0^T i(x(t))dt] - L - mK] / [T+m].$$

In any case, I^* is an increasing function of p .

If $y \in W$ then it is desirable to continue production at least until $y(t)=y+1$. If we could progress from y to $y+1$ without paying for observations, there would be a savings. Hence

$$(2.7) \quad F(y) \leq F(y+1) + E[(\int_0^T i(x(t))dt) | y(0)=y] - HI^*$$

where T is the first time point of increase in $y(\cdot)$ after leaving y , i.e., T is the smallest solution of $y(t)=y+1$ given $y(0)=y$. The expected value of the integral in (2.7) can be expressed in the following form

$$H \sum_{j=0}^y \binom{y}{j} p^j (1-p)^{y-j} i(j).$$

The expected value of the integral in (2.1) can be expressed in the following forms:

$$\begin{aligned}
 (2.8) \quad E \left[\int_0^{T(y)} (i(x(t)) - I^*) dt \mid y(0) = y \right] \\
 = \sum_{a=0}^y \sum_{a'=0}^{\infty} \binom{y}{a} p^a (1-p)^{y-a} e^{-\Delta_x T(y)} (\Delta_x T(y))^{a'} (a'!)^{-1} \\
 \left\{ \sum_{j=0}^{a'} T(y) (i(a+j) - I^*) / (a'+1) \right\} \\
 = \Delta_x^{-1} \sum_{a=0}^y \sum_{a'=1}^{\infty} \binom{y}{a} p^a (1-p)^{y-a} e^{-\Delta_x T(y)} (\Delta_x T(y))^{a'} (a'!)^{-1} \\
 \left\{ \sum_{j=0}^{a'-1} (i(a+j) - I^*) \right\}.
 \end{aligned}$$

Now assume $i(x)$ is a non increasing function of x . Then

$$d(a) = \sum_{a'=0}^{\infty} e^{-\Delta_x t} (\Delta_x t)^{a'} (a'!)^{-1} \left\{ \sum_{j=0}^{a'} t (i(a+j) - I^*) / (a'+1) \right\}$$

is a non increasing function in a . Hence, for a fixed value of $T(y)$, the expression (2.8) is a non increasing function. This result implies:

Theorem 9.

If $i(x)$ is non increasing, then $F(y)$ is non increasing and W is either empty or of the form $(0, 1, \dots, w)$.

Proof.

If W is empty, i.e., the process is always in repair, there is nothing to prove. If $F(y)$ is non increasing, it is clear that W should be of the desired form. Hence the crucial result is to show that $F(y)$ is non increasing. The proof is by contradiction. Assume there exists a y' such that y' and $y'+1$ are in W and $F(y') < F(y'+1)$. Then consider the following non optimal

strategy: If during a cycle y' is observed behave as if $y'+1$ had been observed, i.e., for the rest of the cycle if y^* is observed use $T(y^*+1)$ and stop production as soon as a y^* is observed such that y^*+1 is in W . Now use the monotonicity property of (2.8) and the one to one probability mapping of paths through y' and $y'+1$. This yields the desired contradiction.

Theorem 10.

If $i(x)$ is non increasing and y is in W , then

$$E(i(x)|y) = \sum_{j=0}^y \binom{y}{j} p^j (1-p)^{y-j} i(j) \geq I^*.$$

If I^* is replaced by a lower bound, say I_1^* , then the largest value of y satisfying $E(i(x)|y) \geq I_1^*$ is an upper bound for w , defined in Theorem 9.

Theorem 11.

If $i(x)$ is non increasing then $T(y)$ is non increasing.

Proof.

The proof is similar to that of Proposition 3.14 of [2]. Define $g(y,t)$ as the expected income (reduced by I^* per unit of time) remaining in the cycle when the best strategy is followed, the process is now at y , and t units of time remain until the next observation. ($g(y,t)$ is defined only for y in W .) The main part of the proof is to show that the functions $g(y,t)$ have a unique maximum in t for each y . First, it will be shown that $g(w,t)$ has a unique maximum. After some computations, one can obtain:

$$g(w,t) = -L - m(K+I^*) + \sum_{a'=0}^{\infty} p(a',t)C(a')$$

where

$$p(a',t) = e^{-\Delta_x t} (\Delta_x t)^{a'} / (a')! ,$$

$$C(0) = F(w) + m(K+I^*)$$

$$C(a') = \sum_{j=0}^{a'-1} B(j), \text{ for } a'=1,2,\dots ,$$

and

$$B(j) = \Delta_x^{-1} \sum_{a=0}^w \binom{w}{a} p^a (1-p)^{w-a} [i(a+j) - I^*].$$

Now $p(a', t)$ is of Polya type 3 in t , so that if the sequence $C(a')$ has at most one maximum, then $g(w, t)$ has at most one maximum [1]. Because $i(x)$ is non increasing, it is clear that $B(j)$ is non increasing. Then for $a' \geq 1$, $C(a')$ is increasing, decreasing, or first increasing and then decreasing. Next, we must show

$$C(0) < C(1)$$

or

$$(2.9) \quad F(w) < \Delta_x^{-1} \sum_{a=0}^w \binom{w}{a} p^a (1-p)^{w-a} [i(a) - I^*] - m(K + I^*).$$

The quantity on the right hand side corresponds to using the super optimal strategy of stopping production as soon as a transition in the $x(\cdot)$ process occurs after $y(t)=w$ and this is done with no inspection cost. Hence inequality (2.9). Hence the entire $C(j)$ sequence has at most one maximum and $g(w, t)$ has at most one maximum. (If there is no maximum, either there is no production or there is no repair.)

The remainder of the proof easily follows form

$$(2.10) \quad g'(y, t) = e^{-\Delta_y t} [F(y+1) - F(y) + \sum_{a=0}^y \binom{y}{a} p^a (1-p)^{y-a} (i(a) - I^*) + \int_0^t e^{\Delta_y z} g(y+1, z) dz]$$

where $y < w$ and g' is the derivative with respect to t .

Theorem 10 can be improved, to yield:

$$(2.11) \quad E \left[\int_0^{T(y)} (i(x(t)) - I^*) dt \mid y(0)=y \right] \geq 0.$$

It can be shown that $T(y)$ is less than the unique root of this inequality and if I^* is replaced by a lower bound, then the root will be increased. The

bounds from (2.11) will be decreasing in y . ($i(x)$ is assumed non increasing.)
A lower bound can be found corresponding to Proposition (3.16) of [2].

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Appendix: Questions

- 1. When does $I(w)$, see equation (1.3), have a unique maximizing value when
considered as a function of w ?
- 2. When does the expression in Theorem 4 have a unique maximizing value when
considered as a function of w ?
- 3. When are the best choices of w in equation (1.3) and in Theorem 9 decreasing
functions of p for fixed H ?
- 4. When is $T(y)$ a decreasing function of p ?