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TWO SIMPLE DISTRIBUTION-FREE TESTS OF GOODNESS OF FIT

by

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Summary.

If X has the continuous cumulative distribution function F and $X_1, X_2, \cdots, X_n$ is a sample of X then each of the two statistics $\overline{F} = \frac{1}{n} \sum_{j=1}^{n} F(X_j)$ and, for n odd, $U'' =$ median of $[F(X_1), \cdots, F(X_n)]$ has a probability distribution independent of F. Tests of goodness of fit based on these statistics are proposed, some numerical tables are presented, and power and consistence of the tests are discussed.

1. Introduction.

1.1. Basic concepts.

Let Ω and Ω' be two families of cumulative distribution functions. A real-valued function

$$S(x_1, x_2, \cdots, x_n, G)$$

is called a <u>statistic in Ω with regard to Ω'</u> when for every $G \varepsilon \Omega$ and every $F \varepsilon \Omega'$ the following requirements are fulfilled: if $X_1, X_2, \cdots, X_n$ are identically independently distributed (i.i.d.) random variables with the c.d.f. F then

(i) $S(X_1, X_2, \cdots, X_n, G)$ is defined except for a set of probability zero in $(X_1, X_2, \cdots, X_n)$, and

(ii) $W = S(X_1, X_2, \cdots, X_n, G)$ has a probability distribution, which will be denoted by

$$P[S(X_1, X_2, \ldots, X_n, G) ; F] = P(W; F).$$

For example, consider $X_1, X_2, \ldots, X_n$ i.i.d. with probability density $N(a, \sigma^2)$, and let $\Omega = \Omega'$ be the family of all normal distributions. Then

$$W = S(X_1, \ldots, X_n, G) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i - E_G(X)\right)/\sigma_G(X)$$

is a statistic in Ω w.r.t. Ω'.

If $\Omega = \Omega'$ and the statistic $S(X_1, X_2, \ldots, X_n, G)$ has the property that the probability distribution $P[S(X_1, X_2, \ldots, X_n, G); G]$ is independent of G for $G \in \Omega$, then the statistic $S(X_1, X_2, \ldots, X_n, G)$ is called <u>distribution-free with regard to</u> Ω (d.f.w.r.t. Ω).

1.2. Statistics of structure (d).

A statistic $S(X_1, X_2, \ldots, X_n, G)$ is said to have <u>structure (d)</u> with respect to Ω if there exists a function $\Phi(u_1, u_2, \ldots, u_n)$ such that, for every $G \in \Omega$,

$$S(X_1, X_2, \ldots, X_n, G) = \Phi[G(X_1), G(X_2), \ldots, G(X_n)].$$

We shall from now on denote by Ω_2 the class of all continuous one-dimensional cumulative distribution functions. The following theorem will be repeatedly used.

1.2.1. Theorem.*) A statistic of structure (d) is distribution free w.r.t. Ω_2.

2. The $\mathbb{F}$ statistic.

2.1. Definitions and basic properties of the statistics $\mathbb{F}$ and U_n^*.

If $X_1, X_2, \ldots, X_n$ is a sample of a random variable X

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*) See [1]

which has the c.d.f. $F \varepsilon \Omega_2$, then $U_1 = F(X_1), \ldots, U_n = F(X_n)$ form a sample of the random variable U with uniform distribution on $(0,1)$. Since $E(U) = \frac{1}{2}$ and $\sigma^2(U) = \frac{1}{12}$, it follows that the <u>$\bar{F}$ statistic</u> defined by

$$(2.1.1) \qquad \bar{F} = \frac{1}{n}\sum_{i=1}^{n} F(X_i)$$

has the expectation and variance

$$(2.1.2) \qquad E(\bar{F}) = \frac{1}{2}, \quad \sigma^2(\bar{F}) = \frac{1}{12n}.$$

We standardize $\bar{F}$ and obtain the statistic

$$(2.1.3) \qquad U_n^* = \sqrt{12n}\,(\bar{F} - \tfrac{1}{2})$$

with expectation and variance

$$E(U_n^*) = 0, \quad \sigma^2(U_n^*) = 1.$$

Since U_n^* is of structure (d), it is by 1.2.1 a distribution free statistic w.r.t. Ω_2. Its probability distribution can be calculated exactly for small n by evaluating the convolution of n random variables with uniform distribution on $(0,1)$. For example, for $n = 2$ one has

$$(2.1.5) \qquad P\left\{\bar{F}_2 \le s\right\} = \begin{cases} 2s^2 & \text{for } 0 \le s \le \frac{1}{2} \\ 1 - 2(1-s)^2 & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

hence

$$(2.1.6) \quad P\left\{U_2^* \leq u\right\} = \begin{cases} \frac{1}{2}\left(1 + \frac{u}{\sqrt{6}}\right)^2 & \text{for } -\sqrt{6} \leq u \leq 0 \\ 1 - \frac{1}{2}\left(1 - \frac{u}{\sqrt{6}}\right)^2 & \text{for } 0 \leq u \leq \sqrt{6}. \end{cases}$$

According to the central limit theorem, U_n^* converges in distribution to $N(0,1)$ with $n \to \infty$, and numerical calculations have shown that this convergence is so fast that already for $n = 8$ there is practically no difference between

$$P\left\{U_n^* \leq u\right\} \quad \text{and} \quad \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{s^2}{2}} \, ds .$$

The statistic U_n^* offers, therefore, the practical advantage that, if X has the given distribution F, the probability distribution of U_n^* can be computed exactly for n small, say $n \leq 10$, and from then on the normal approximation can be used.

2.2. The $\mathbb{F}$ tests.

2.2.1. The one-sided test.

We consider the hypothesis

$$H : X \text{ has c.d.f. } F(x) \in \Omega_2$$

and the one-sided alternative

$$A : X \text{ has c.d.f. } A(x) \text{ such that } A(x) \geq F(x) \text{ for all } x$$
$$\text{and } A(\xi) \neq F(\xi) \text{ for some } \xi .$$

For given significance level α and sample size n [illegible]

$u_{n,\alpha}$ be so determined that

$$(2.2.1.1) \qquad P\left\{U_n^* > u_{n,\alpha} \mid F\right\} = \alpha ,$$

hence also

$$(2.2.1.2) \qquad P\left\{U_n^* < -u_{n,\alpha} \mid F\right\} = \alpha .$$

We then define as our region of rejection

$$(2.2.1.3) \qquad U_n^* < -u_{n,\alpha} .$$

In view of (2.2.1.2) this test has size α.

2.2.2 Tabulation of critical values $u_{n,\alpha}$.

Let X denote a random variable with uniform probability distribution on $(-1, +1)$, and let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be the mean of a sample of X. A table of the exact values of $P\left\{\bar{X}_n \leq s\right\}$ for s, proceeding by steps of .01, had been previously computed for $n = 2,3,4,5,6,$ and 10, and was available. Making use of this table and of the relationships

$$U = \frac{1}{2}(X + 1)$$

$$U_n^* = \sqrt{3n}\bar{X}_n$$

the equations

$$P\left\{U_n^* < -u_{n,\alpha}\right\} = P\left\{U_n^* > u_{n,\alpha}\right\} = P\left\{\bar{Y}_n > \frac{u_{n,\alpha}}{\sqrt{3n}}\right\} = \alpha$$

were solved by quadratic inverse interpolation for $\alpha = .05, .025, .01, .005$. The results are presented in Table I. The last row, $n = \infty$, contains values taken from the normalized normal distribution.

2.2.3. Lower bound for the power.

We assume that $F, G \in \Omega_2$, and that X has the c.d.f. G, so that F is the hypothesis and G the alternative. We assume furthermore that

$$G(x) \geq F(x)$$

(2.2.3.1)

$$G(\xi) = F(\xi) + \delta \quad \text{for a given } \xi ,$$

as indicated in Figure 1.

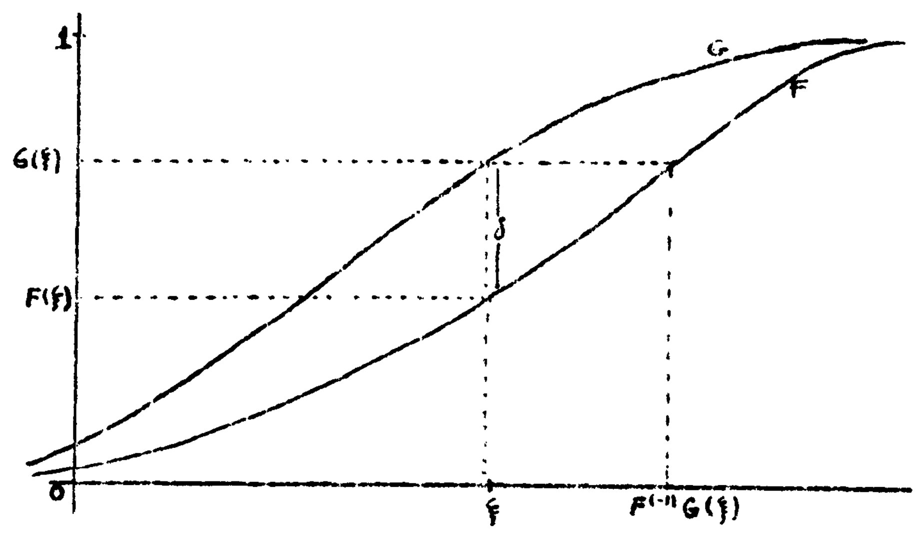

Figure 1

TABLE I

Values $u_{n,\alpha}$ such that

$$P\{U^*_{n,\alpha} < -u_{n,\alpha}\} = P\{U^*_{n,\alpha} > u_{n,\alpha}\} = \alpha$$

n \ α	.05	.025	.01	.005
2	1.67499	1.90641	2.10318	2.20454
3	1.66116	1.93737	2.21700	2.37861
4	1.65127	1.93969	2.25180	2.44472
5	1.65047	1.94253	2.26631	2.47462
6	1.64945	1.94572	2.27647	2.49883
10	1.64764	1.95161	2.29725	2.52692
∞	1.645	1.96	2.327	2.575

Setting $L_{\xi,\delta}(x) = \begin{cases} F(x) & \text{for } x < \xi \\ F(\xi) + \delta = G(\xi) & \text{for } \xi \leq x < F^{(-1)}(G(\xi)) \\ F(x) & \text{for } F^{(-1)}(G(\xi)) \leq x , \end{cases}$

we clearly have $G(x) \geq L_{\xi,\delta}(x)$ for all x, and conclude

$$P\left\{F(X) \leq s \mid G\right\} = P\left\{X \leq F^{(-1)}(s) \mid G\right\} = G[F^{(-1)}(s)] \geq$$

(2.2.3.2)

$$\geq L_{\xi,\delta}[F^{(-1)}(s)] = \begin{cases} s & \text{for } 0 \leq s < F(\xi) \\ F(\xi) + \delta = G(\xi) & \text{for } F(\xi) \leq s < G(\xi) \\ s & \text{for } G(\xi) \leq s \leq 1 , \end{cases}$$

We define, for $\gamma, \delta > 0$ and $\gamma + \delta < 1$,

$$A_{\gamma,\delta}(s) = \begin{cases} s & \text{for } 0 \leq s < \gamma \\ \gamma + \delta & \text{for } \gamma \leq s < \gamma + \delta \\ s & \text{for } \gamma + \delta \leq s \leq 1, \end{cases}$$

as indicated in Figure 2 and then rewrite (2.2.3.2) in the form

(2.2.3.3) $\quad P\left\{F(X) \leq s \mid G\right\} \geq A_{\gamma,\delta}(s)$, where $\gamma = F(\xi)$, $\gamma + \delta = G(\xi)$.

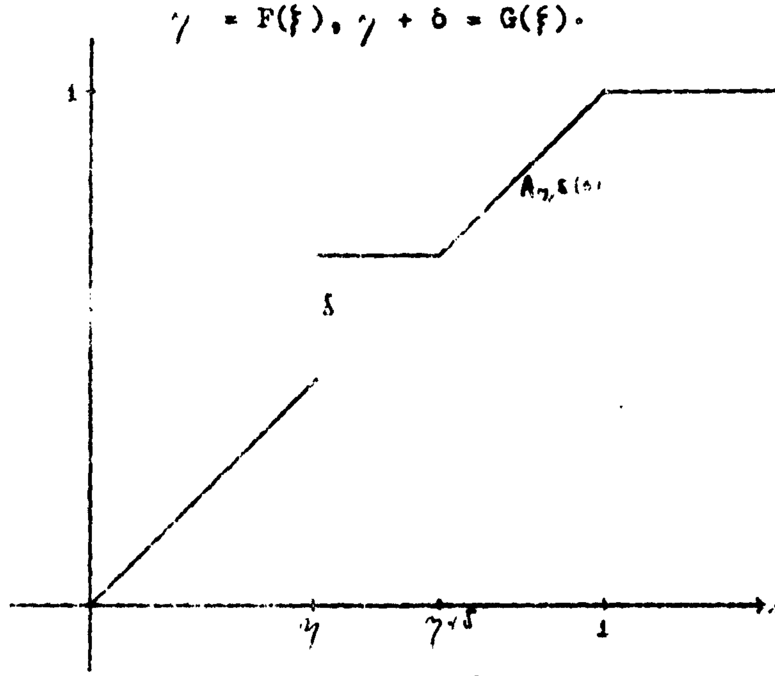

Figure 2

Since the test statistic U_n^* is monotone in the sense of Chapman [2] and has structure (d), it follows from a theorem in [2], p. 657, that the power of test (2.2.1.3) of hypothesis F against alternative G is greater than or equal to its power of F against alternative $L_{\xi,\delta}$, and that this latter power is a sharp lower bound for the power for given $\gamma = F(\xi)$ and $\gamma + \delta = G(\xi)$.

Without loss of generality we may now replace $F(x)$ by the uniform distribution $R(s)$ on $(0,1)$ and $L_{\xi,\delta}(x)$ by the distribution function $A_{\gamma,\delta}(s)$ on $(0,1)$. The power of our test of R against $A_{\gamma,\delta}$ will be the lower bound for its power for F against G such that $F(s) \leq G(s)$ for all real G, and $F(\xi) = \gamma$ and $G(\xi) = F(\xi) + \delta$ for some ξ.

While it is difficult to compute the exact power for finite sample size n of the F-test for hypothesis R against alternative $A_{\gamma,\delta}$, the asymptotic power for $n \to \infty$ can be easily computed in the following manner.

The expectation and variance of X under the alternative are

$$E(X;\ A_{\gamma,\delta}) = \tfrac{1}{2}(1 - \delta^2)$$

(2.2.3.4)

$$\sigma^2(X;\ A_{\gamma,\delta}) = \frac{1}{12} - \gamma\delta^2 + \frac{1}{2}\delta^2 - \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4.$$

The statistic U_n^* has therefore under the alternative expectation and variance

$$E(U_n^*; A_{\eta,\delta}) = -\sqrt{3n}\,\delta^2$$

(2.2.3.5)

$$\sigma^2(U_n^* ; A_{\eta,\delta}) = 1 - 12\eta\delta^2 + 6\delta^2 - 4\delta^3 - 3\delta^4 .$$

According to the central limit theorem U_n^* is asymptotically normal and in view of (2.2.3.5) we have

(2.2.3.6) $$\lim_{n\to\infty} P\left\{\frac{U_n^* + \sqrt{3n}\,\delta}{\sqrt{1-12\eta\delta^2+6\delta^2-4\delta^3-3\delta^4}} < s; A_{\eta,\delta}\right\} =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}}\,dt.$$

For $\delta = 0$ one has $A_{\eta,\delta} = R$, (2.2.3.6) becomes

$$\lim_{n\to\infty} P\left\{ U_n^* \leq s ; R\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt$$

and, for $u_{n,\alpha}$ determined by (2.2.1.2) one has

$$\alpha = \lim_{n\to\infty} P\left\{ U_n^* < -u_{n,\alpha}; R\right\} = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-u_{n,\alpha}} \epsilon^{-\frac{t^2}{2}} dt$$

so that $\lim_{n\to\infty} (-u_{n,\alpha}) = -z_\alpha$, with $$\frac{1}{\sqrt{2\pi}} \int_{z_\alpha}^{+\infty} e^{-\frac{t^2}{2}} dt = \alpha .$$

For $\delta > 0$ and $u_{n,\alpha}$ determined by (2.2.1.2) one obtains from (2.2.3.6)

$$\lim_{n\to\infty} P\{U_n^* < -u_{n,\alpha}; A_{\eta,\delta}\}$$

$$= \lim_{n\to\infty} P\left\{\frac{U_n^* + \sqrt{3n}\,\delta}{\sqrt{1-12\eta\delta^2+6\delta^2-4\delta^3-3\delta^4}} < \frac{-u_{n,\alpha} + \sqrt{3n}\,\delta}{\sqrt{1-12\eta\delta^2+6\delta^2-4\delta^3-3\delta^4}}; A_{\eta,\delta}\right\}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b_n} e^{-\frac{t^2}{2}} dt = 1$$

where
$$b_n = \frac{-u_{n,\alpha} + \sqrt{3n}\,\delta}{\sqrt{1-12\eta\delta^2+6\delta^2-4\delta^3-3\delta^4}}.$$

This expression for the asymptotic power of our test against $A_{\eta,\delta}$ shows that the test is consistent against every one-sided alternative.

The results of this and of the next section are not new. Equivalent statements have been obtained by Chapman [2, expressions (46), (47)], who reports that the test discussed here has been previously proposed by L. E. Moses.

2.2.4. Upper bound for the power.

By an argument similar to that in the preceding section, one can show that if the test (2.2.1.3) is applied to the hypothesis F and the alternative G in (2.2.3.1) its power is not greater than it is when that test is applied to hypothesis F and the alternative.

$$M_\delta(x) = \min\{F(x) + \delta, 1\}, \tag{2.2.4.1}$$

which ascribes the discrete probability δ to the value $-\infty$. Without loss of generality one may replace F and M_δ by the uniform distribution $R(s)$ on the unit interval and the distribution

$$B_\delta(s) = \begin{cases} 0 & \text{for } s < 0 \\ s + \delta & \text{for } 0 \leq s < 1 - \delta \\ 1 & \text{for } 1 - \delta \leq s \end{cases} \tag{2.2.4.2}$$

respectively, so that the power of our test for R against B_δ is the upper bound for its power for any F in Ω_2 against an alternative (2.2.3.1). Again, we derive an asymptotic expression for the power for R against B_δ, by computing

$$\begin{aligned} E(X;\, B_\delta) &= \tfrac{1}{2}(1 - \delta)^2 \\ \sigma^2(X;\, B_\delta) &= \tfrac{1}{12}(1 - \delta)^3(1 + 3\delta), \end{aligned} \tag{2.2.4.3}$$

hence

$$\begin{aligned} E(U_n^*;\, B_\delta) &= -\sqrt{3n}\,\delta(2 - \delta) \\ \sigma^2(U_n^*;\, B_\delta) &= (1 - \delta)^3(1 + 3\delta) \end{aligned} \tag{2.2.4.4}$$

and observing that U_n^* is, under alternative B_δ, asymptotically normal with expectation and variance (2.2.4.4). The power of the test (2.2.1.3) for R against B_δ is therefore for large n asymptotically

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c_n} e^{-\frac{s^2}{2}} \, ds$$

where

$$c_n = \frac{-u_n \sqrt{3n}\, \delta(2 - \delta)}{(1 - \delta)\sqrt{(1 - \delta)(1 + 3\delta)}} \quad .$$

2.2.5. The two-sided F-test.

We consider a hypothesis $F \varepsilon \Omega_2$ and an alternative $G \varepsilon \Omega_2$, and agree to reject F when

(2.2.5.1) $$|U_n^*| > u_{n,\frac{\alpha}{2}} \quad .$$

From (2.2.1.1) and (2.2.1.2) follows that (2.2.5.1) defines a test of size α.

To study the power of this test we consider the expectation and the variance of $F(x)$ under the alternative G

$$E_G(F) = \int_{-\infty}^{+\infty} F(x)\,dG(x)$$

$$\sigma_G^2(F) = \int_{-\infty}^{+\infty} F^2(x)\,dG(x) - E_G^2(F),$$

make the assumption

$$0 < \sigma_G^2(F) < \infty$$

and note that

$$E_F(F) = \frac{1}{2}, \quad \sigma_F^2(F) = \frac{1}{12}.$$

By virtue of the central limit theorem the random variable

$$\frac{\bar{F} - E_G(F)}{\sigma_G(F)} \sqrt{n}$$

tends in distribution to the normalized normal random variable.

In view of (2.1.3) and (2.2.5.1) we have

$$P\left\{|U_n^*| < u_{n,\frac{\alpha}{2}}; G\right\} = P\left\{\sqrt{12n}\,|\bar{F} - \tfrac{1}{2}| < u_{n,\frac{\alpha}{2}};\ G\right\} =$$

$$= P\left\{\sqrt{12n}\,|\bar{F} - E_G(F) - [\tfrac{1}{2} - E_G(F)]\,| < u_{n,\frac{\alpha}{2}};\ G\right\} =$$

$$= P\left\{\frac{\sqrt{12n}[\frac{1}{2} - E_G(F)] - u_{n,\frac{\alpha}{2}}}{\sqrt{12}\ \sigma_G(F)} < \frac{\bar{F} - E_G(F)}{\sigma_G(F)}\sqrt{n} < \frac{\sqrt{12n}[\frac{1}{2} - E_G(F)] + u_{n,\frac{\alpha}{2}}}{\sqrt{12}\ \sigma_G(F)};\ G\right\}$$

and this is asymptotically equal to

$$\frac{1}{\sqrt{2\pi}} \int_{a_n}^{b_n} e^{-\frac{s^2}{2}}\, ds$$

where

$$a_n = \frac{\sqrt{12n}[\frac{1}{2} - E_G(F)] - z_{\frac{\alpha}{2}}}{\sqrt{12}\ \sigma_G(F)}$$

(2.2.5.2)

$$b_n = \frac{\sqrt{12n}[\frac{1}{2} - E_G(F)] + z_{\frac{\alpha}{2}}}{\sqrt{12}\ \sigma_G(F)}$$

and $z_{\frac{\alpha}{2}}$ is such that

$$\frac{1}{\sqrt{2\pi}} \int_{z_{\frac{\alpha}{2}}}^{\infty} e^{-\frac{s^2}{2}} ds = \frac{\alpha}{2} .$$

From (2.2.5.2) one concludes that $P\{|U_n^*| < u_{n,\frac{\alpha}{2}};\ G\} \to 0$

with $n \to \infty$ for every alternative G such that

(2.2.5.3) $\qquad E_G(F) \neq \frac{1}{2}$,

so that the two-sided $\bar{F}$ test (2.2.5.1) is consistent against every alternative $G \in \Omega_2$ satisfying (2.2.5.3).

3. The median-F statistic.

3.1 Definitions and basic properties of the statistic M_{2m+1}.

Let $X_1, X_2, \ldots, X_{2m+1}$ be a sample of size $2m+1$ (an odd integer) of a random variable X which has c.d.f.

$F \varepsilon \Omega_2$, and $X_1' < X_2' < \cdots < X_{m+1}' < \cdots < X_{2m+1}'$ the corresponding ordered sample, so that X_{m+1}' is the sample-median. Then

$$U_1' = F(X_1'),\ U_2 = F(X_2'),\ \cdots,\ U_{m+1}' = F(X_{m+1}'),\ \cdots, U_{2m+1}' = F(X_{2m+1}')$$

is an ordered sample of a random variable U which has uniform distribution on (0, 1), and U_{m+1}' the median of that sample. It is well known that U_{m+1}' has the c.d.f.

$$(3.1.1)\quad P\{U_{m+1}' \leq s\} = \frac{(2m+1)!}{(m!)^2}\int_0^s u^m(1-u)^m du = \frac{B_s(m+1,m+1)}{B(m+1,m+1)}$$

where B and B_s are the beta-function and the incomplete beta-function, that therefore

$$(3.1.2)\qquad E(U_{m+1}') = \frac{1}{2},\quad \sigma^2(U_{m+1}') = \frac{1}{4(2m+1)}\ ,$$

and that the normalized random variable

$$(3.1.3)\qquad M_{2m+1} = 2\sqrt{2m+1}\,(U_{m\ 1}' - \tfrac{1}{2}) = 2\sqrt{2m+1}[F(X_{m+1}') = \tfrac{1}{2}]$$

converges in distribution to the normalized normal variable.

The statistic M_{2m+1} defined by (3.1.3) offers obvious advantages in practical use: 1) it is a statistic of structure (d), hence is distribution-free w.r.t. Ω_2, 2) if X has c.d.f. F then the probability distribution of M_{2m+1} can be computed exactly for m small from (3.1.3) and (3.1.1) by using available tables of the incomplete beta-function; [3] for m large the probability distribution of M_{2m+1} is approximated by the normalized normal distribution; [4] the statistic M_{2m+1} is easily computed for given sample $X_1, \dots, X_{2m+1}$ and given γ, since all one has to evaluate (or even only to know) is the sample median X'_{m+1} and the value $F(X'_{m+1})$.

3.2. The median-F tests.

3.2.1. The one-sided test.

To test the hypothesis H against the alternative A described in 2.2.1., we determine $\gamma_{m,\alpha}$ so that

$$(3.2.1.1) \qquad P\left\{M_{2m+1} < -\gamma_{m,\alpha}\right\} = P\left\{M_{2m+1} > \gamma_{m,\alpha}\right\} = \alpha$$

and reject H when

$$(3.2.1.2) \qquad M_{2m+1} < -\gamma_{m,\alpha} .$$

This test clearly has size α.

3.3.2. Tabulation of critical values $\gamma_{m,\alpha}$.

To obtain solutions $\gamma_{m,\alpha}$ of equation (3.2.1.1), this equation may be written in the form

$$P\left\{M_{2m+1} > \gamma_{m,\alpha}\right\} = P\left\{X'_{m+1} > \frac{\gamma_{m,\alpha}}{2\sqrt{2m+1}} - \frac{1}{2}\right\} = \alpha$$

which, under the hypothesis, and in view of (3.1.1) is equivalent with

$$(3.2.2.1) \qquad \frac{(2m+1)!}{(m!)^2} \int_0^{\frac{\gamma_{m,\alpha}}{2\sqrt{2m+1}} + \frac{1}{2}} u^m (1-u)^m du = 1-\alpha .$$

For given sample size $2m+1$ and significance level α

the value $\frac{\gamma_{m,\alpha}}{2\sqrt{2m+1}} + \frac{1}{2}$ can be obtained by inverse

interpolation from the Tables of the Incomplete Beta Function [4]. The values presented in Table II were calculated by using Lagrange's interpolation formula of degree 4 and solving the resulting equation by Newton's method.

TABLE II

Values $\gamma_{m,\alpha}$ such that

$$P\{M_{2m+1} < -\gamma_{m,\alpha}\} = P\{M_{2m+1} > \gamma_{m,\alpha}\} = \alpha$$

$2m+1$ \ α	.05	.025	.01	.005
19	1.56845	1.84257	2.14498	2.33986
29	1.59396	1.88142	2.20428	2.41629
39	1.60670	1.90094	2.23437	2.45539
49	1.61435	1.91268	2.25255	2.47899
59	1.61943	1.92054	2.26470	2.49486
69	1.62308	1.92615	2.27341	2.50633
79	1.62579	1.93036	2.27995	2.51480
89	1.62793	1.93363	2.28510	2.52140
99	1.62962	1.93626	2.28906	2.52677
∞	1.645	1.96	2.327	2.575

3.2.3. Consistence of the median-F test.

Let the hypothesis F and the alternative G both be in Ω_2, and let their medians be

(3.2.3.1) $$\mu_F = F^{(-1)}(\tfrac{1}{2}) \ , \ \mu_G = G^{(-1)}(\tfrac{1}{2}) \ .$$

The probability of rejection when test (3.2.1.2) is used is

$$P\left\{M_{2m+1} < -\gamma_{m,\alpha} | G\right\} = P\left\{2\sqrt{2m+1}\,[F(X'_{m+1}) - \tfrac{1}{2}] < -\gamma_{m,\alpha} | G\right\}$$

$$= P\left\{G(X'_{m+1}) < G[F^{(-1)}(\tfrac{1}{2} - \frac{\gamma_{m,\alpha}}{2\sqrt{2m+1}})] | G\right\} .$$

We define

$$\Upsilon_m = G[F^{(-1)}(\tfrac{1}{2} - \frac{\gamma_{m,\alpha}}{2\sqrt{2m+1}})]$$

(3.2.3.2)

$$\Upsilon = G[F^{(-1)}(\tfrac{1}{2})] = \lim_{m\to\infty} \Upsilon_m .$$

Since, under alternative G, the random variable $G(X)$ has uniform distribution on $(0, 1)$, the random variable $G(X'_{m+1})$ is the median of a sample of size $2m + 1$ of such a uniformly distributed random variable and by a well-known theorem (e.g. [3] p. 369) has asymptotically normal distribution with expectation $\frac{1}{2}$ and variance $\frac{1}{4(2m+1)}$. The probability of rejection is therefore asymptotically

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\Upsilon_m - \frac{1}{2}) 2\sqrt{2m+1}} e^{-s^2/2}\, ds .$$

If $\Upsilon > \frac{1}{2}$, we have $(\Upsilon_m - \frac{1}{2})\, 2\sqrt{2m+1} \to +\infty$ with $m \to \infty$, and since $\Upsilon > \frac{1}{2}$ is equivalent with $\mu_F > \mu_G$

we conclude that test (3.2.1.2) is consistent for every alternative G such that $\mu_G < \mu_F$.

3.2.4. Lower bound for the power

Using the assumptions and notations of 2.2.3 and noting that the statistic M_{2m+1} is monotone and has structure

(d), we conclude as in 2.2.3 that a sharp lower bound for the one-sided test defined by (3.2.1.2) for given η and δ is obtained by choosing as hypothesis the uniform c.d.f. $R(s)$ on $(0,1)$ and as alternative the c.d.f. $A_{\eta,\delta}(s)$ defined in 2.2.3. Again, the exact power in this case for finite sample size appears to be difficult to compute, but an asymptotic expression can be obtained as follows.

If the random variable V has the c.d.f. $A_{\eta,\delta}(s)$,

and $V_1', V_2', \ldots, V_{m+1}', \ldots, V_{2m+1}'$ is an ordered sample of V,

then the sample median V_{m+1}' has the c.d.f.

$$P\{V'_{m+1} \leq v | A_{\eta,\delta}\} = \begin{cases} \sum_{i=m+1}^{2m+1} \binom{2m+1}{i} v^i (1-v)^{2m+1-i} & \text{for } 0 \leq v < \eta \\ \sum_{i=m+1}^{2m+1} \binom{2m+1}{i} (\eta+\delta)^i (1-\eta-\delta)^{2m+1-i} & \text{for } \eta \leq v < \eta+\delta \\ \sum_{i=m+1}^{2m+1} \binom{2m+1}{i} v^i (1-v)^{2m+1-i} & \text{for } \eta+\delta \leq v < 1 . \end{cases}$$

Since now (3.1.3) becomes

$$M_{2m+1} = 2\sqrt{2m+1} \quad (V'_{m+1} - \tfrac{1}{2})$$

we obtain for the power of the test (3.2.1.2) the expression

$$P\{M_{2m+1} < -\gamma_{m,\alpha} | A_{\eta,\delta}\} = P\{V'_{m+1} < \tfrac{1}{2} - \frac{\gamma_{m,\alpha}}{2\sqrt{2m+1}} | A_{\eta,\delta}\}$$

If $\eta < \frac{1}{2} < \eta + \delta$, then this becomes

$$P\{M_{2m+1} < -\gamma_{m,\alpha} | A_{\eta,\delta}\} = \sum_{i=m+1}^{2m+1} \binom{2m+1}{i} (\eta+\delta)^i (1-\eta-\delta)^{2m+1-i}$$

and this, for $m \to \infty$, is asymptotically equal to

$$\frac{1}{\sqrt{2\pi}} \int_{d_m}^{\infty} e^{-\frac{s^2}{2}} ds \tag{3.2.4.1}$$

where

$$d_m = \frac{m+1 - (2m+1)(\eta + \delta)}{\sqrt{(2m+1)(\eta+\delta)(1-\eta-\delta)}} .$$

It is clear that for $m \to \infty$ the expression (3.2.4.1) tends to 1 when $\eta + \delta > \frac{1}{2}$ and to 0 when $\eta + \delta < \frac{1}{2}$.

3.2.5. The two-sided median-F test.

The test with the region of rejection defined by

$$(3.2.5.1) \qquad |M_{2m+1}| > \Upsilon_{m,\frac{\alpha}{2}}$$

clearly has size α , so that the values under headings .025 and .005 in Table II may be used to apply the two-sided test (3.2.5.1) on the .05 and .01 levels of significance. An argument analogous to that in section 3.2.3 shows that, for hypothesis F and alternative G both in Ω_2, this test is consistent if $\mu_F \neq \mu_G$.

References

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