

UNCLASSIFIED

AD_ 296 168

*Reproduced
by the*

**ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA**



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

296168

AFCRL-62-1078

CATALOGED BY ASTIA
AS AD No. _____

296 168

**THE EFFECT OF FINITE DIFFERENCES
ON THE GROWTH RATES OF UNSTABLE WAVES
IN A SIMPLE BAROCLINIC MODEL**

by

CHRISTOPHER A. RIEGEL
University of California, Los Angeles

Project 8628
Task 862803

Scientific Report No. 7

DYNAMICAL WEATHER PREDICTION PROJECT
Department of Meteorology

1 October 1962

The research reported in this document has been sponsored in part by the Geophysics Research Directorate of the Air Force Cambridge Research Laboratories, Office of Aerospace Research, United States Air Force, L. G. Hanscom Field, Bedford, Mass., under Contract No. AF 19(604)-6149.

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

**ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA**

Department of Defense contractors must be established for ASTIA services or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

All other persons and organizations should apply to the:

**U. S. DEPARTMENT OF COMMERCE
OFFICE OF TECHNICAL SERVICES
WASHINGTON 25, D.C.**

The Effect of Finite Differences
on the Growth Rates of Unstable Waves
in a Simple Baroclinic Model

by

Christopher A. Riegel

University of California, Los Angeles

Scientific Report No. 7

Dynamical Weather Prediction Project

Department of Meteorology

1 October 1962

Abstract

The distortion of the growth rates of unstable baroclinic waves is investigated using implicit finite differences to replace the linearized governing equations of a two-parameter model. A variety of space and time mesh sizes Δx and Δt are used, together with three values of the static stability parameter $\partial\theta/\partial p$. On the whole, for commonly used values of Δx and Δt , and for typical values of $\partial\theta/\partial p$, the introduction of finite differences appears to cause only a slight distortion of the growth rates over the major portion of the unstable spectrum, the long waves being less affected than the short waves. Effects due to Δt are particularly minor. The finite-difference growth rates depend very slightly on the zonal wind U , in contrast to the continuous case where no such dependence is found.

1. Introduction

In a recent paper Gates (1961) discussed the effect of finite-differences on baroclinic instability criteria for the case of a linearized two-parameter model, using an implicit difference scheme. His results show that, for a given vertical windshear, the spectrum of unstable waves is shifted slightly toward shorter wavelengths, leaving the minimum critical shear unaltered. The magnitude of the shift was found to be directly proportional to the space mesh, Δx , and inversely proportional to the static-stability. In this connection it might be interesting to know what effect, if any, the introduction of certain finite differences has on the growth rates of unstable baroclinic waves. It is the purpose of the present discussion to throw some light on this problem. The same atmospheric model, and the same differencing scheme as in the paper mentioned above, will be used.

2. Baroclinic difference equations

The particular baroclinic model employed is the thermotropic model, whose details are described elsewhere (Thompson and Gates, 1956), and whose linearized governing differential equations, with the usual assumption of y -independent flow, may be written as

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial \psi}{\partial t} \right) + U \frac{\partial^3 \psi}{\partial x^3} + \beta \frac{\partial \psi}{\partial x} + \frac{U^*}{4} \frac{\partial^3 \phi}{\partial x^3} = 0, \quad (1)$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial \phi}{\partial t} \right) + U \frac{\partial^3 \phi}{\partial x^3} + (\beta - \mu^2 U) \frac{\partial \phi}{\partial x} - \mu^2 \frac{\partial \phi}{\partial t} + U^* \frac{\partial^3 \psi}{\partial x^3} + \mu^2 U^* \frac{\partial \psi}{\partial x} = 0, \quad (2)$$

where ψ and ϕ are the streamfunctions of the vertically-integrated flow and the thermal wind, respectively. U is the zonal wind, and U^* is the shear or thermal wind, both assumed constant; β is the Rossby parameter, and μ^2 depends on the static stability, and is given by

$$\mu^2 = - \frac{4f^2\theta}{R p_0 T \partial\theta/\partial p},$$

where f is the Coriolis parameter, R the specific gas constant for dry air, $p_0 = 1000$ mb, T is the absolute temperature, and θ is the potential temperature. For the purpose of avoiding extraneous roots in the frequency equation below, and of using a numerically stable scheme, implicit differences are now introduced, and the appropriate finite-difference analogues of (1) and (2) are

$$\begin{aligned} & (\Delta x^2 \Delta t)^{-1} (\psi_{m+1, \tau+1} - 2\psi_{m, \tau+1} + \psi_{m-1, \tau+1} - \psi_{m+1, \tau} + 2\psi_{m, \tau} - \psi_{m-1, \tau}) \\ & + U(4\Delta x^2)^{-1} (\psi_{m+2, \tau} - 2\psi_{m+1, \tau} + 2\psi_{m-1, \tau} - \psi_{m-2, \tau} + \psi_{m+2, \tau+1} - 2\psi_{m+1, \tau+1} \\ & + 2\psi_{m-1, \tau+1} - \psi_{m-2, \tau+1}) + \beta(4\Delta x)^{-1} (\psi_{m+1, \tau} - \psi_{m-1, \tau} + \psi_{m+1, \tau+1} - \psi_{m-1, \tau+1}) \\ & + U^*(16\Delta x^2)^{-1} (\phi_{m+2, \tau} - 2\phi_{m+1, \tau} + 2\phi_{m-1, \tau} - \phi_{m-2, \tau} + \phi_{m+2, \tau+1} - 2\phi_{m+1, \tau+1} \\ & + 2\phi_{m-1, \tau+1} - \phi_{m-2, \tau+1}) = 0 \end{aligned} \quad (3)$$

and

$$\begin{aligned}
 & (\Delta x^2 \Delta t)^{-1} (\phi_{m+1, \tau+1} - 2\phi_{m, \tau+1} + \phi_{m-1, \tau+1} - \phi_{m+1, \tau} + 2\phi_{m, \tau} - \phi_{m-1, \tau}) \\
 & + U(4\Delta x^3)^{-1} (\phi_{m+2, \tau} - 2\phi_{m+1, \tau} + 2\phi_{m-1, \tau} - \phi_{m-2, \tau} + \phi_{m+2, \tau+1} - 2\phi_{m+1, \tau+1} \\
 & + 2\phi_{m-1, \tau+1} - \phi_{m-2, \tau+1}) + (4\Delta x)^{-1} (\beta - \mu^2 U) (\phi_{m+1, \tau} - \phi_{m-1, \tau} + \phi_{m+1, \tau+1} \\
 & - \phi_{m-1, \tau+1}) - \mu^2 (\Delta t)^{-1} (\phi_{m, \tau+1} - \phi_{m, \tau}) + U^* (4\Delta x^3)^{-1} (\psi_{m+2, \tau} - 2\psi_{m+1, \tau} \\
 & + 2\psi_{m-1, \tau} - \psi_{m-2, \tau} + \psi_{m+2, \tau+1} - 2\psi_{m+1, \tau+1} + 2\psi_{m-1, \tau+1} - \psi_{m-2, \tau+1}) \\
 & + \mu^2 U^* (4\Delta x)^{-1} (\psi_{m+1, \tau} - \psi_{m-1, \tau} + \psi_{m+1, \tau+1} - \psi_{m-1, \tau+1}) = 0,
 \end{aligned} \tag{4}$$

where $\psi_{m, \tau} = \psi(m\Delta x, \tau\Delta t)$, $\phi_{m, \tau} = \phi(m\Delta x, \tau\Delta t)$, and where Δx and Δt denote space and time increments, respectively.

In order to investigate the behavior of growth rates, it is convenient to assume that (3) and (4) have solutions of the form

$$\left. \begin{aligned}
 \psi_{m, \tau} &= \bar{\Psi} e^{i(m\alpha - n\tau\Delta t)} \\
 \phi_{m, \tau} &= \bar{\Phi} e^{i(m\alpha - n\tau\Delta t)}
 \end{aligned} \right\}, \tag{5}$$

where i is the imaginary unit, $\alpha = k\Delta x$, $n = kC_N$, $k = 2\pi/L$, and where L is the wavelength, and C_N is the phase speed of the finite-difference solution. The amplitudes $\bar{\Psi}$ and $\bar{\Phi}$ are assumed constant. Substituting (5) into (3) and (4), one finds readily enough that the system of equations for the

unknown amplitudes is

$$\left. \begin{aligned} & \left[\kappa^2 \left(\frac{1}{\lambda} \tan \frac{n \Delta t}{2} - \nu \right) + \beta \right] \Psi - \frac{\nu^*}{4} \kappa^2 \Phi = 0 \\ & \nu^* (\mu^2 - \kappa^2) \Psi + \left[(\mu^2 + \kappa^2) \left(\frac{1}{\lambda} \tan \frac{n \Delta t}{2} - \nu \right) + \beta \right] \Phi = 0 \end{aligned} \right\}, \quad (6)$$

where

$$\kappa^2 = k^2 \left(\frac{\sin \alpha/2}{\alpha/2} \right)^2, \quad (7)$$

$$\lambda = \frac{1}{2} k \Delta t \left(\frac{\sin \alpha}{\alpha} \right). \quad (8)$$

The condition that (6) should have non-trivial solutions requires that the determinant of the coefficients of Ψ and Φ must vanish. This leads to the frequency equation of the numerical solution,

$$C_N = \frac{2}{k \Delta t} \tan^{-1}(\lambda Z) \quad (9)$$

where

$$Z = \nu - \frac{1}{2} \left(\frac{\beta}{\kappa^2} + \frac{\beta}{\mu^2 + \kappa^2} \right) \pm \frac{1}{2} \left[\left(\frac{\beta}{\kappa^2} - \frac{\beta}{\mu^2 + \kappa^2} \right)^2 - (\nu^*)^2 \left(\frac{\mu^2 \kappa^2}{\mu^2 + \kappa^2} \right) \right]^{1/2}. \quad (10)$$

In comparison, one finds by the usual methods that the frequency equation for the differential system (1) and (2) is

$$C = \nu - \frac{1}{2} \left(\frac{\beta}{k^2} + \frac{\beta}{\mu^2 + k^2} \right) \pm \frac{1}{2} \left[\left(\frac{\beta}{k^2} - \frac{\beta}{\mu^2 + k^2} \right) - (\nu^*)^2 \left(\frac{\mu^2 - k^2}{\mu^2 + k^2} \right) \right]^{1/2}, \quad (11)$$

where C is the phase speed, and $k = 2\pi/L$ as above. Equation (9) expresses the well-known fact that the numerical phase speed C_N depends on the choice of Δx and Δt .

3. Growth rates in the difference case

Let C_N be complex with real part C_{N_r} , and imaginary part C_{N_i} , i.e., $C_N = C_{N_r} + iC_{N_i}$. Since (5) is a solution of the difference system (3) and (4), subject to condition (9), unstable baroclinic waves can exist in this system if C_N has a non-trivial imaginary part. If (5) is now written in the form

$$\left. \begin{aligned} \psi_{m,z} &= \Psi e^{k C_{N_i} z \Delta t} e^{i(ma - k C_{N_r} \tau \Delta t)} \\ \phi_{m,z} &= \Phi e^{k C_{N_i} z \Delta t} e^{i(ma - k C_{N_r} \tau \Delta t)} \end{aligned} \right\}, \quad (12)$$

it is clear that the waves will amplify provided $C_{N_i} > 0$. Now, the inverse tangent in (9) has real solutions for all real values of λZ , and C_N can be complex only if Z is complex, because λ is always real. For complex Z , let $Z = Z_r + iZ_i$ where, from (10)

$$Z_r = U - \frac{1}{2} \left(\frac{\beta}{\kappa^2} + \frac{\beta}{\mu^2 + \kappa^2} \right), \quad (13)$$

$$Z_i = \pm \frac{1}{2} \left[(U^2)^2 \left(\frac{\mu^2 - \kappa^2}{\mu^2 + \kappa^2} \right) - \left(\frac{\beta}{\kappa^2} - \frac{\beta}{\mu^2 + \kappa^2} \right) \right]^{1/2}. \quad (14)$$

Thus, the critical curve, that is, the curve which separates the region

of stability from that of instability, is defined by the requirement $Z_i = 0$. The analogous curve in the differential system is given by the condition

$$(U^*)^2 \left(\frac{\mu^2 - k^2}{\mu^2 + k^2} \right) - \left(\frac{\beta}{k^2} - \frac{\beta}{\mu^2 + k^2} \right) = 0,$$

which is the same as setting $Z_i = 0$, and dividing κ^2 by $(\frac{\sin \alpha/2}{\alpha/2})^2$. The cut-off wavelength L_c , such that all waves for which $L \leq L_c$ are stable, the minimum shear U_m^* , such that all waves for which $U^* \leq U_m^*$ are stable, and the wavelength L_m corresponding to U_m^* are all obtainable from (14). Expressions for L_c , U_m^* , and L_m have already been given elsewhere (Gates, 1961) and it need only be mentioned here that the method used in the present paper gives the same results for these parameters.

To discuss the effect of finite differences on the growth rate, it is necessary to find the imaginary part of the phase speed (9). Thus, letting Z be complex, one finds after some manipulation that (See Appendix),

$$C_{N_r} = \frac{1}{k \Delta t} \tan^{-1} \left[\frac{2 \lambda \hat{z}_r}{1 - (\lambda \hat{z}_r)^2 - (\lambda \hat{z}_i)^2} \right], \quad (15)$$

$$C_{N_i} = \frac{1}{2 k \Delta t} \ln \left[\frac{(\lambda \hat{z}_r)^2 + (1 + \lambda \hat{z}_i)^2}{(\lambda \hat{z}_r)^2 + (1 - \lambda \hat{z}_i)^2} \right]. \quad (16)$$

The form of (16) indicates that the positive square root should be taken in (14) for amplifying waves ($C_{N_i} > 0$). Similarly, $C_{N_i} < 0$, and the waves decay if the negative root is taken in (14). For finite, non-zero Δx and

At the waves are neutral ($C_{N_1} = 0$) when $Z_1 = 0$, and when $\lambda = 0$. The case $Z_1 = 0$ is the one mentioned above. The case $\lambda = 0$ occurs whenever Δx is any integer multiple of $L/2$. This means, at least in the linear case, that the harmonic components whose wavelengths are below the resolution limit of the grid remain neutral when an implicit difference scheme is used.

Various expressions for the growth rate are possible; but in the finite-difference case, a convenient measure of this quantity is the doubling time. This is the time required by the wave amplitude to double in magnitude, and can be written as

$$t_{D_N} = \frac{\ln 2}{* C_{N_i}}, \quad (17)$$

where C_{N_1} is given by (16). The distortion of the doubling time by implicit finite differences is shown in figs. 1-9 for selected values of Δx , Δt , and the static stability $\partial\theta/\partial p$. It is evident from the figures that the distortion is slight for commonly used values of Δx and Δt , and for a typical value of $\partial\theta/\partial p$ (-6 deg/100 mb). These figures also show that the distortion is more serious for large values of Δx , short waves, and small static stabilities, in agreement with the behavior of the shift of baroclinic instability criteria found by Gates (1961). In general, the growth rate of the unstable shorter waves is increased, and that of the longer waves decreased, except in the case of neutral static stability ($\partial\theta/\partial p = 0$), where the growth rate is always decreased. Another effect of the finite differences is that, for a given shear U^* , the wavelength of the most rapidly growing wave is slightly lowered. This shift is more pronounced for the smaller shears, larger space mesh sizes, and small static stabilities. For a given static stability, the maximum value of

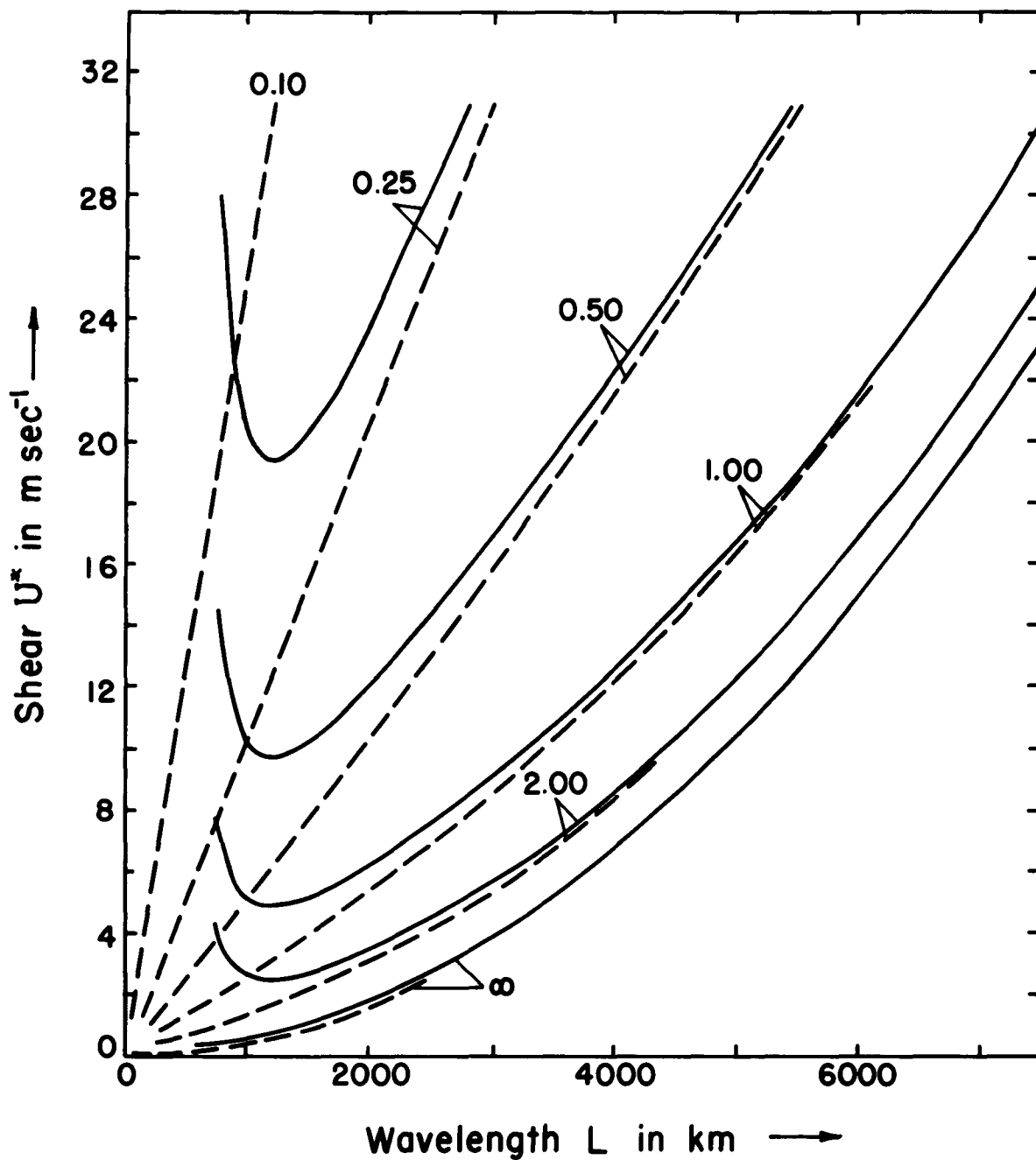


Figure 1. Doubling time in days. The solid lines are for the finite-difference case, the dashed lines for the continuous case. The latitude is 45 deg N. Here the static stability parameter $\partial\theta/\partial p = 0$, $\Delta x = 300$ km,

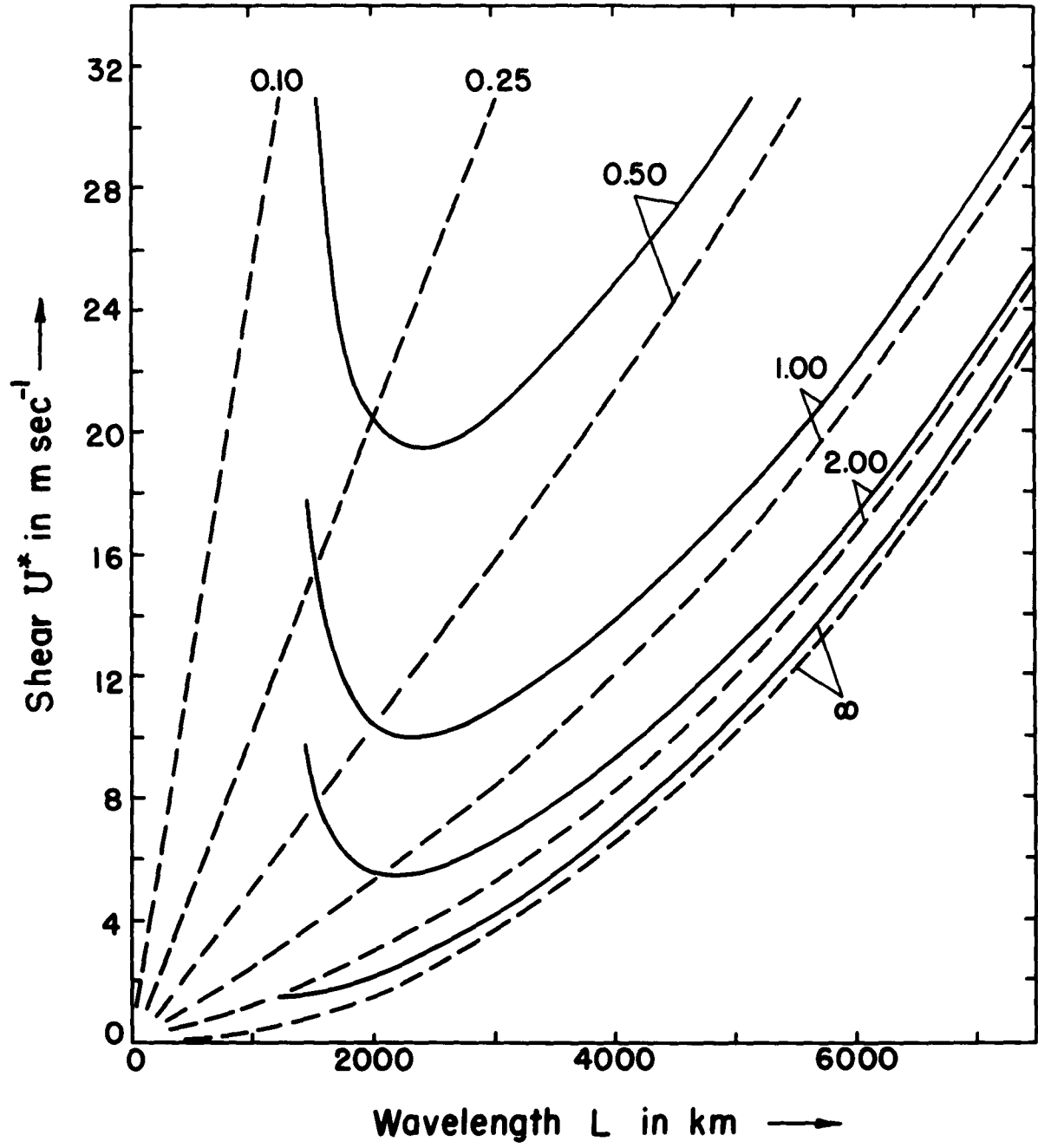


Figure 2. Same as fig. 1, but with $\Delta x = 600 \text{ km}$.

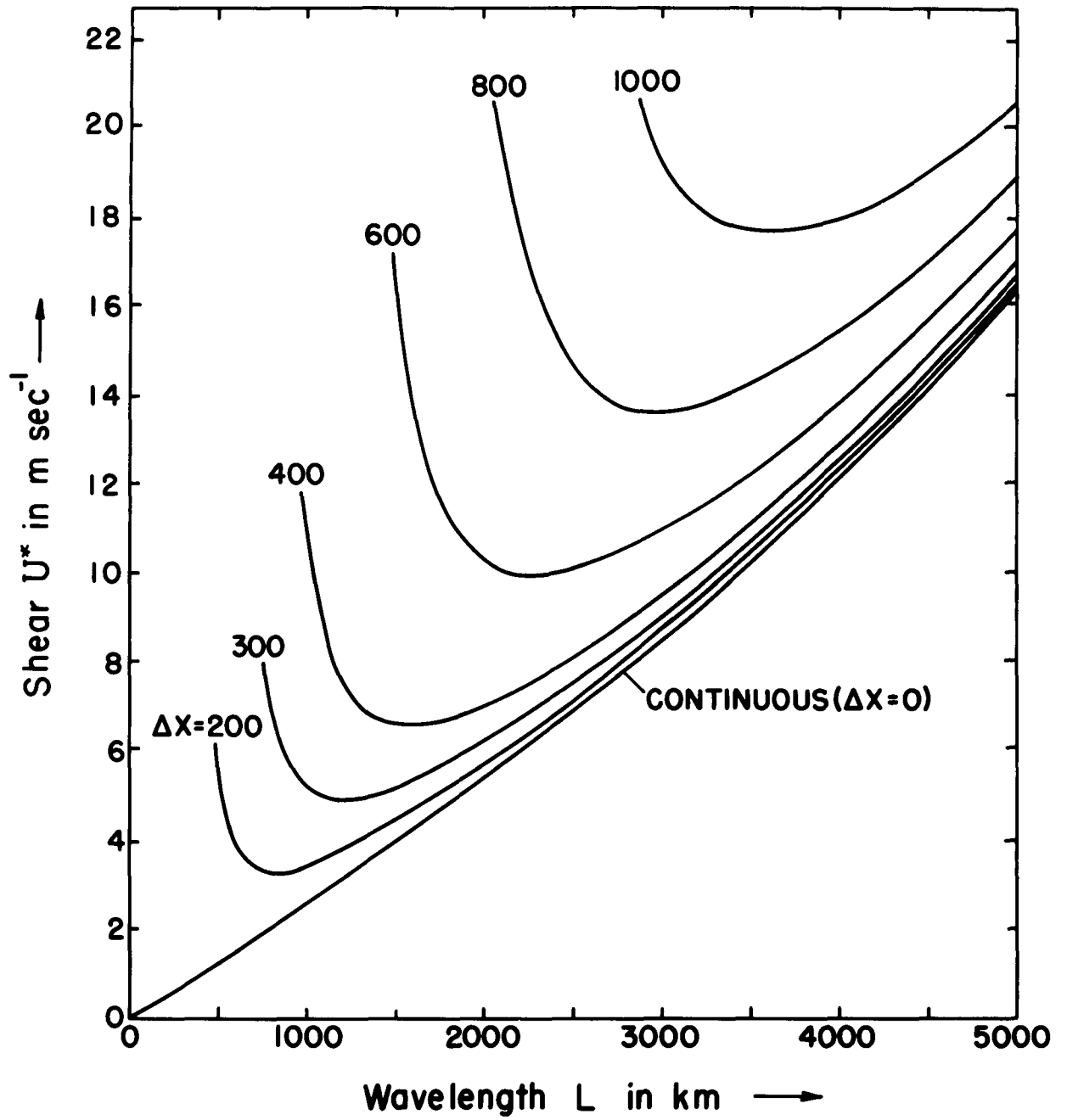


Figure 3. The effect of the space mesh Δx on a doubling time of exactly 1 day. The latitude is 45 deg N, $\partial\theta/\partial p = 0$, and $\Delta t = 1$ hr.

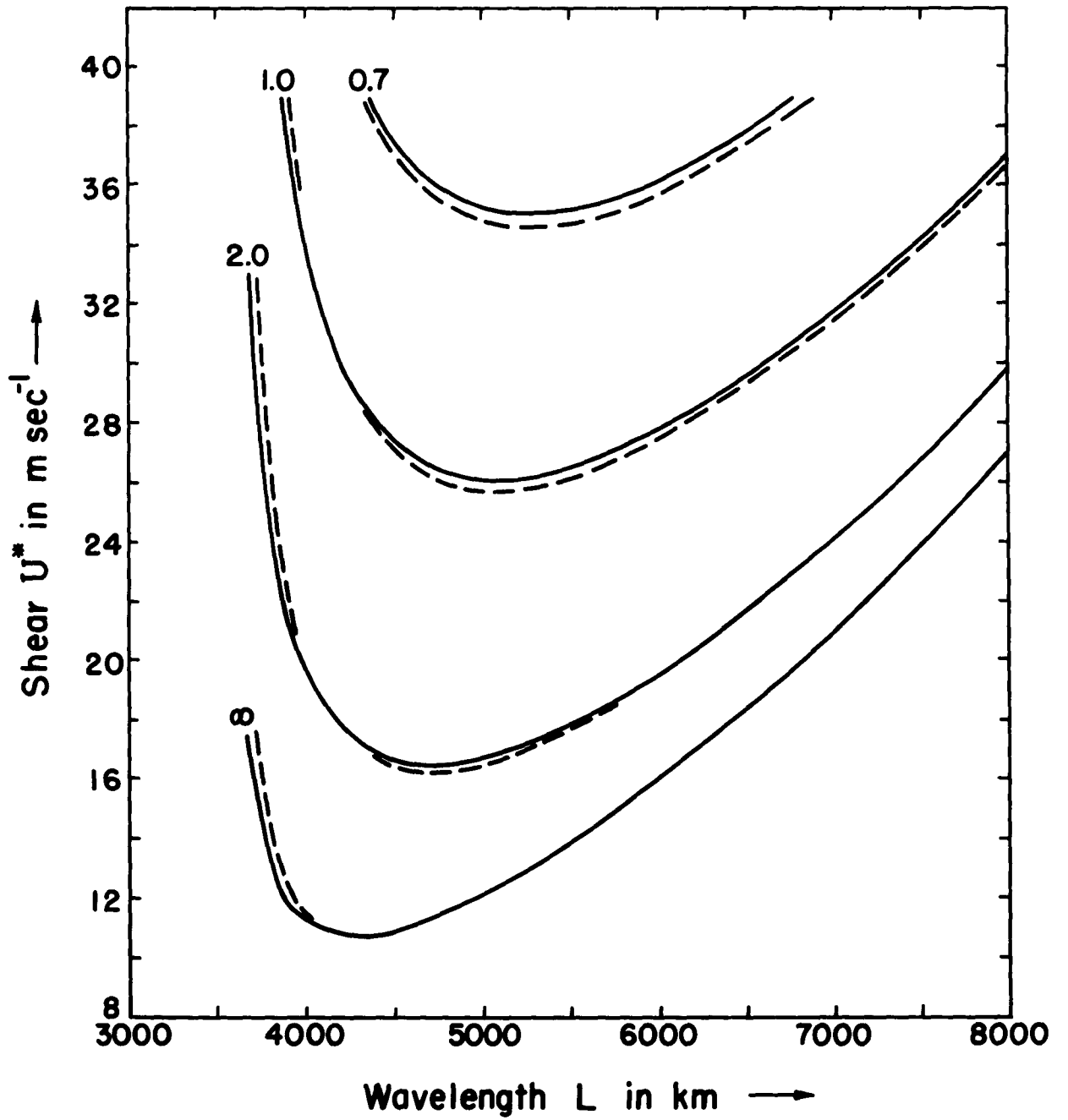


Figure 4. Same as fig. 1, but with $\partial\theta/\partial p = -6 \text{ deg (100 mb)}^{-1}$.

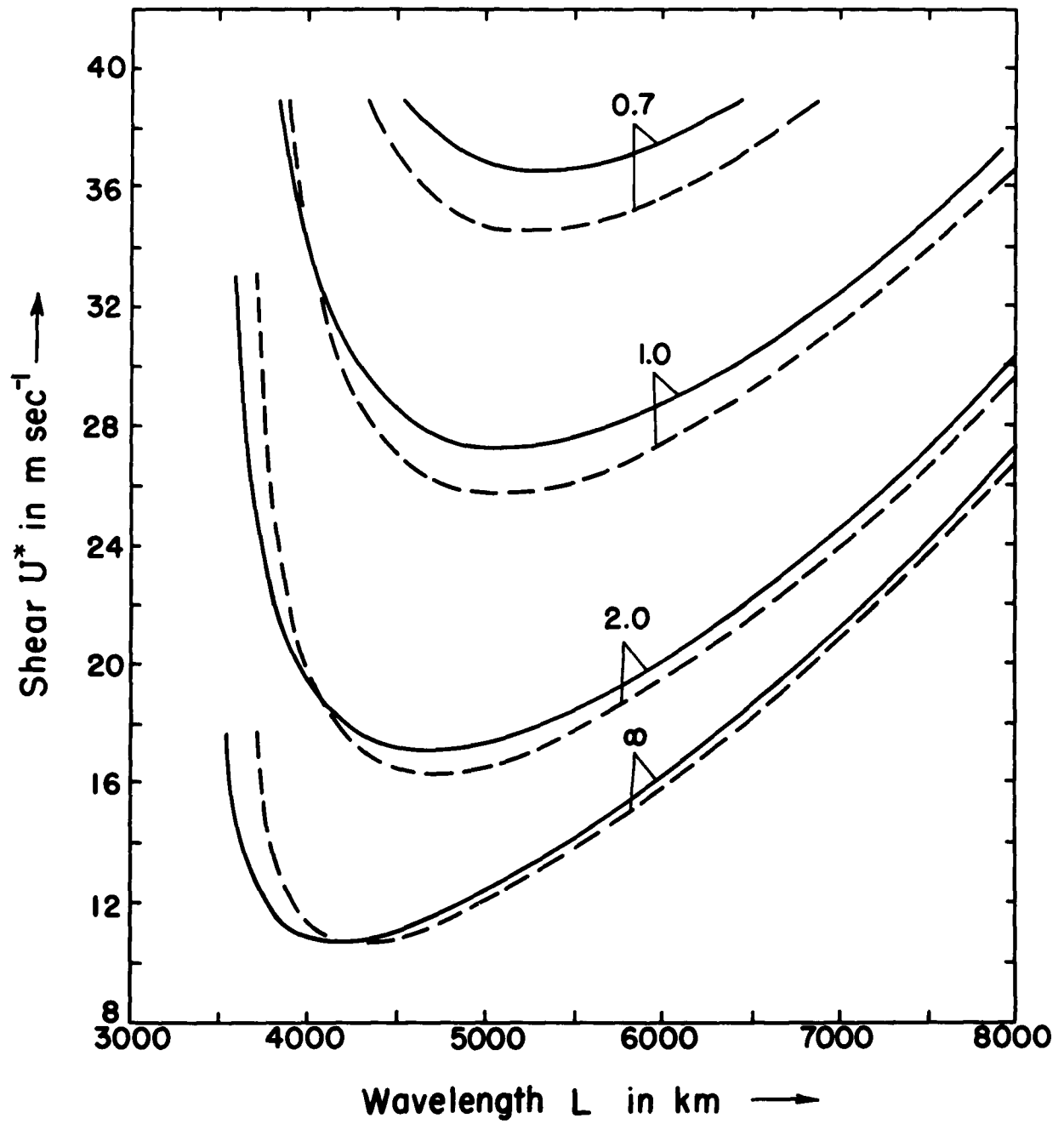


Figure 5. Same as fig. 2, but with $\partial\theta/\partial p = -6\ deg\ (100\ mb)^{-1}$.

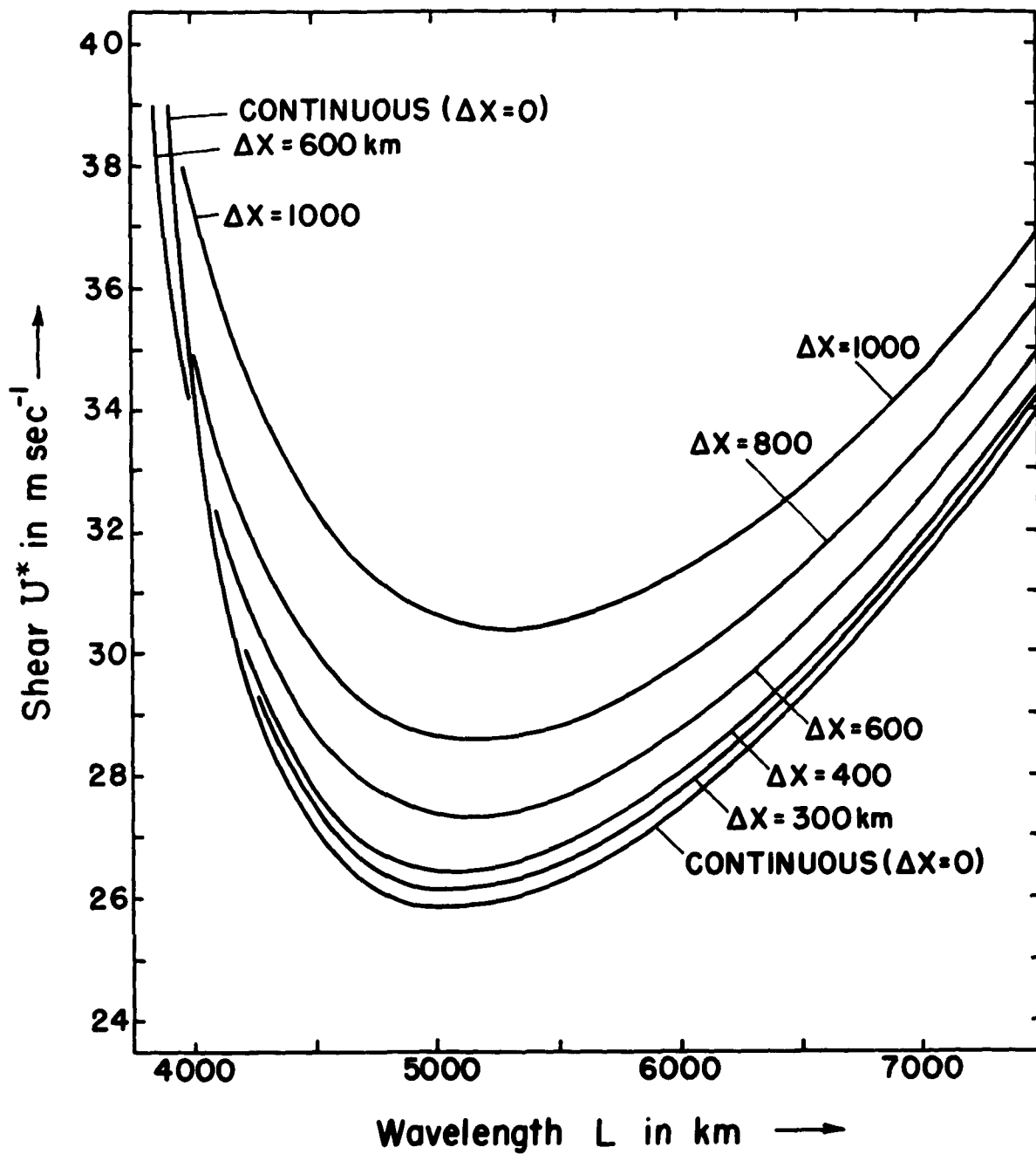


Figure 6. Same as fig. 3, but with $\partial\theta/\partial p = -6 \text{ deg (100 mb)}^{-1}$.

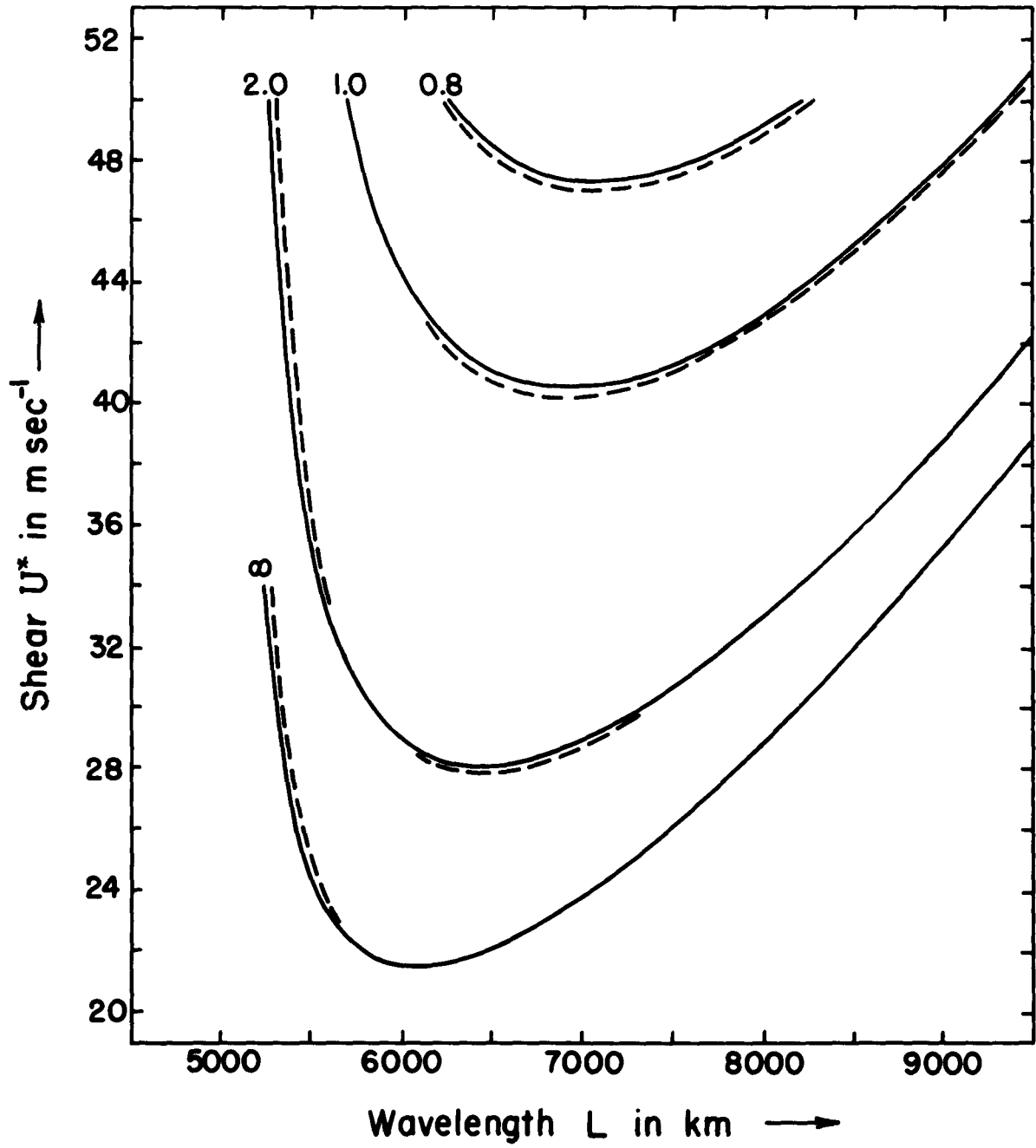


Figure 7. Same as fig. 1, but with $\partial\theta/\partial p = -12 \text{ deg (100 mb)}^{-1}$.

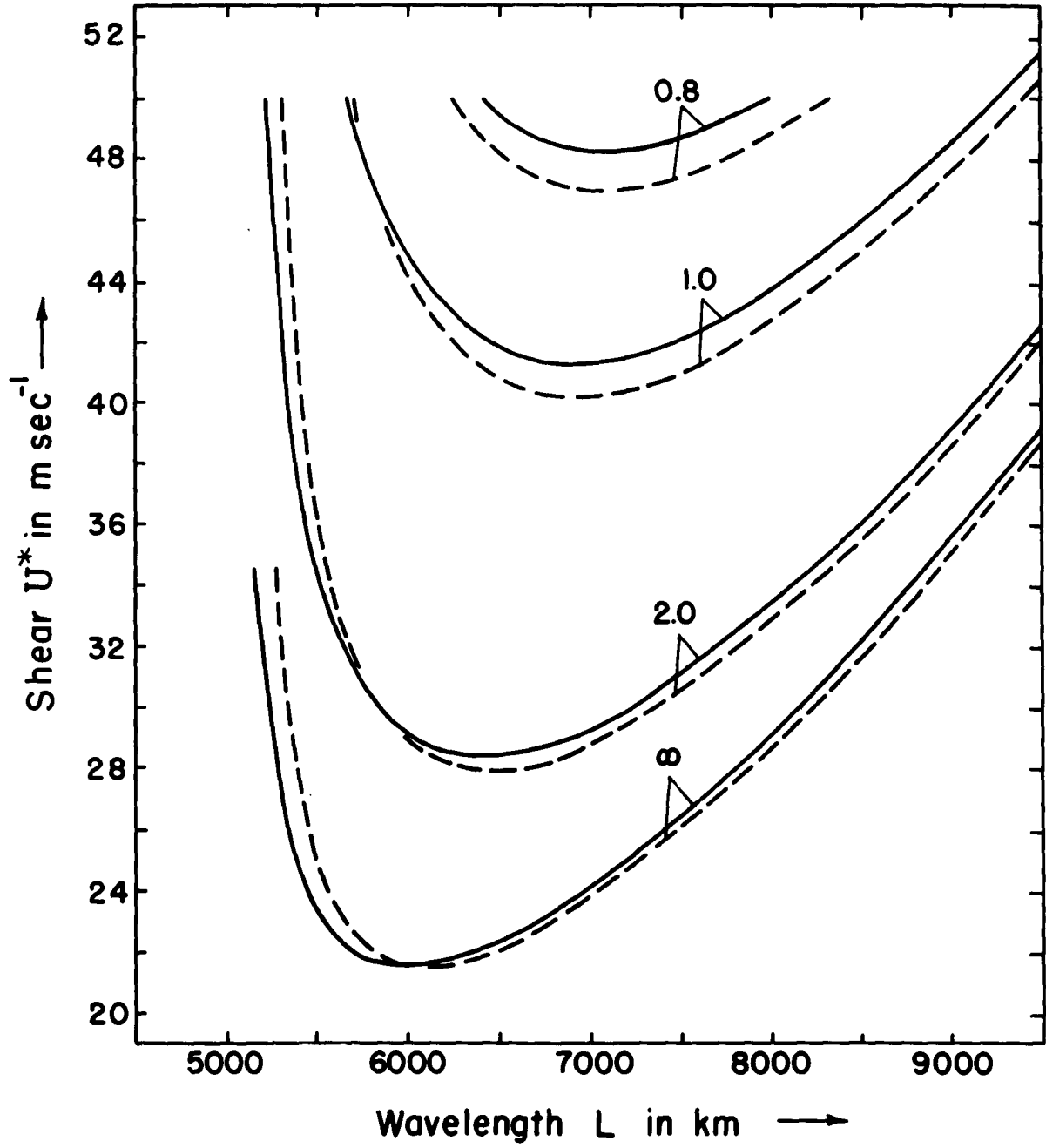


Figure 8. Same as fig. 2, but with $\partial\theta/\partial p = -12 \text{ deg (100 mb)}^{-1}$.

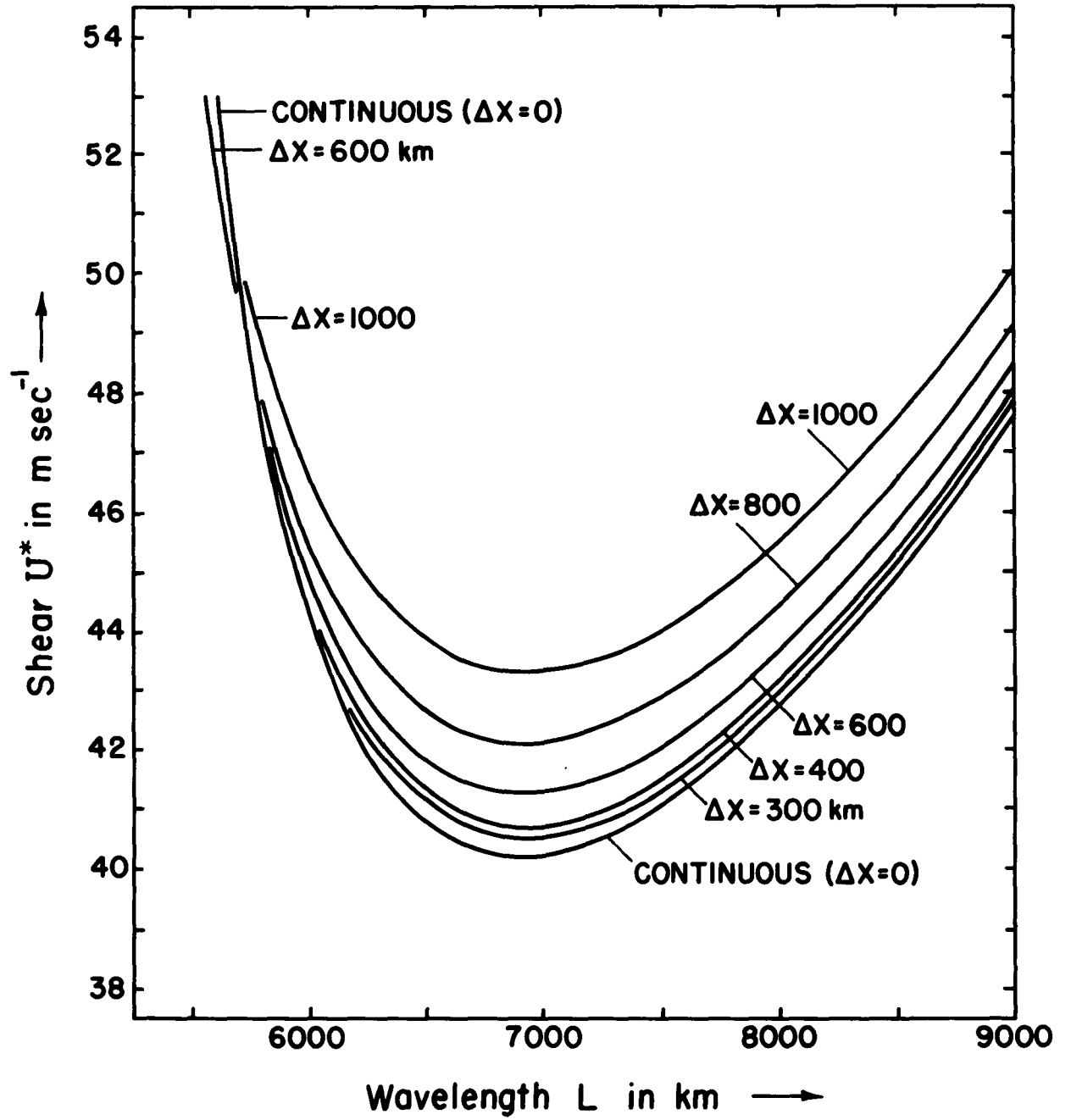


Figure 9. Same as fig. 3, but with $\partial\theta/\partial p = -12 \text{ deg (100 mb)}^{-1}$.

this shift is equal to the displacement of the minimum point on the critical curve, corresponding to a given Δx , and decreases with increasing shear. Typical maximum shifts are 35 km for $\Delta x = 300$ km, and 146 km for $\Delta x = 600$ km, when $\partial\theta/\partial p = -6$ deg. $(100 \text{ mb})^{-1}$.

It may be seen from fig. 5, for example, that the distance between the solid and the dashed curves does not in itself indicate how serious the distortion is. Therefore, the percentage change in the growth rate of the finite-difference case, as compared to the continuous case, would provide a better measure of the distortion. Now, let ξ be an arbitrary quantity, and let ξ' differ from ξ by an amount $\Delta\xi$. Then the percentage change of ξ' relative to ξ may be written as

$$\Delta\xi(\%) = 100 \left(\frac{\xi' - \xi}{\xi} \right).$$

This expression has been used in the construction of figs. 10-12. The implication of fig. 10 is that the percentage changes due to the introduction of a fairly large space mesh Δx are probably not serious, except possibly in the immediate vicinity of the critical curve. The changes due to a mesh size of 300 km amount to about 1/3 to 1/2 of those shown in the figure. Smaller (larger) static stabilities $\partial\theta/\partial p$ increase (reduce) those changes proportionately. It is well-known that effects due to the time mesh Δt are, in general, smaller than those due to the space mesh Δx when equations (1) and (2) are replaced by finite differences. This is also borne out by figs. 11 and 12, which indicate the relative insensitivity of the doubling times due to the changes in Δt alone. Another feature shown in figs. 10-12 is that effects of Δx and Δt on the growth rates are virtually independent of the vertical shear U^* over large portions of the instability region.

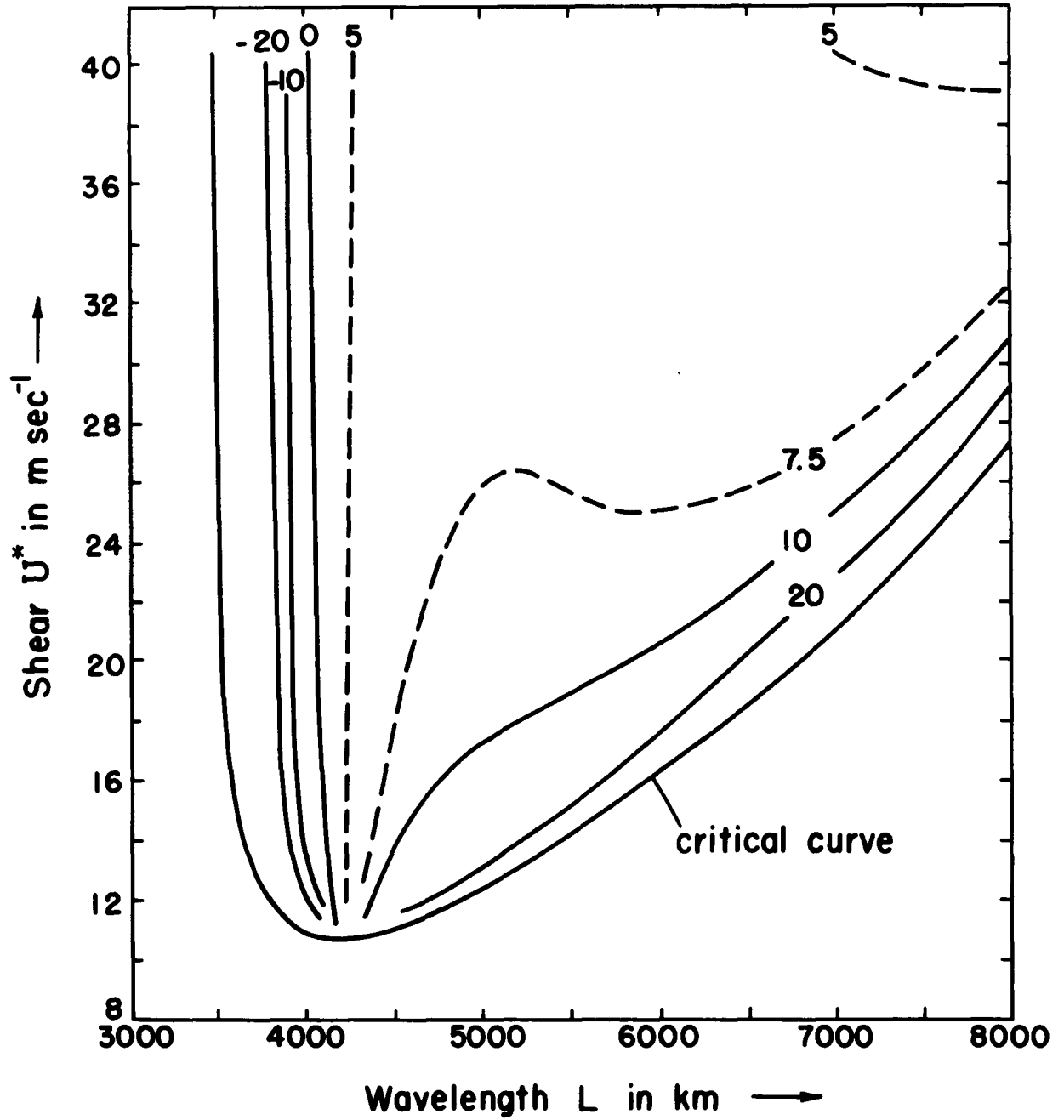


Figure 10. Percentage change in doubling time relative to the continuous case, for $\Delta x = 600\ km$, $\Delta t = 1\ hr$, and $\partial\theta/\partial p = -6\ deg\ (100\ mb)^{-1}$. The units are per cent, the latitude is $45\ deg\ N$, and the critical curve is that for $\Delta x = 600\ km$.

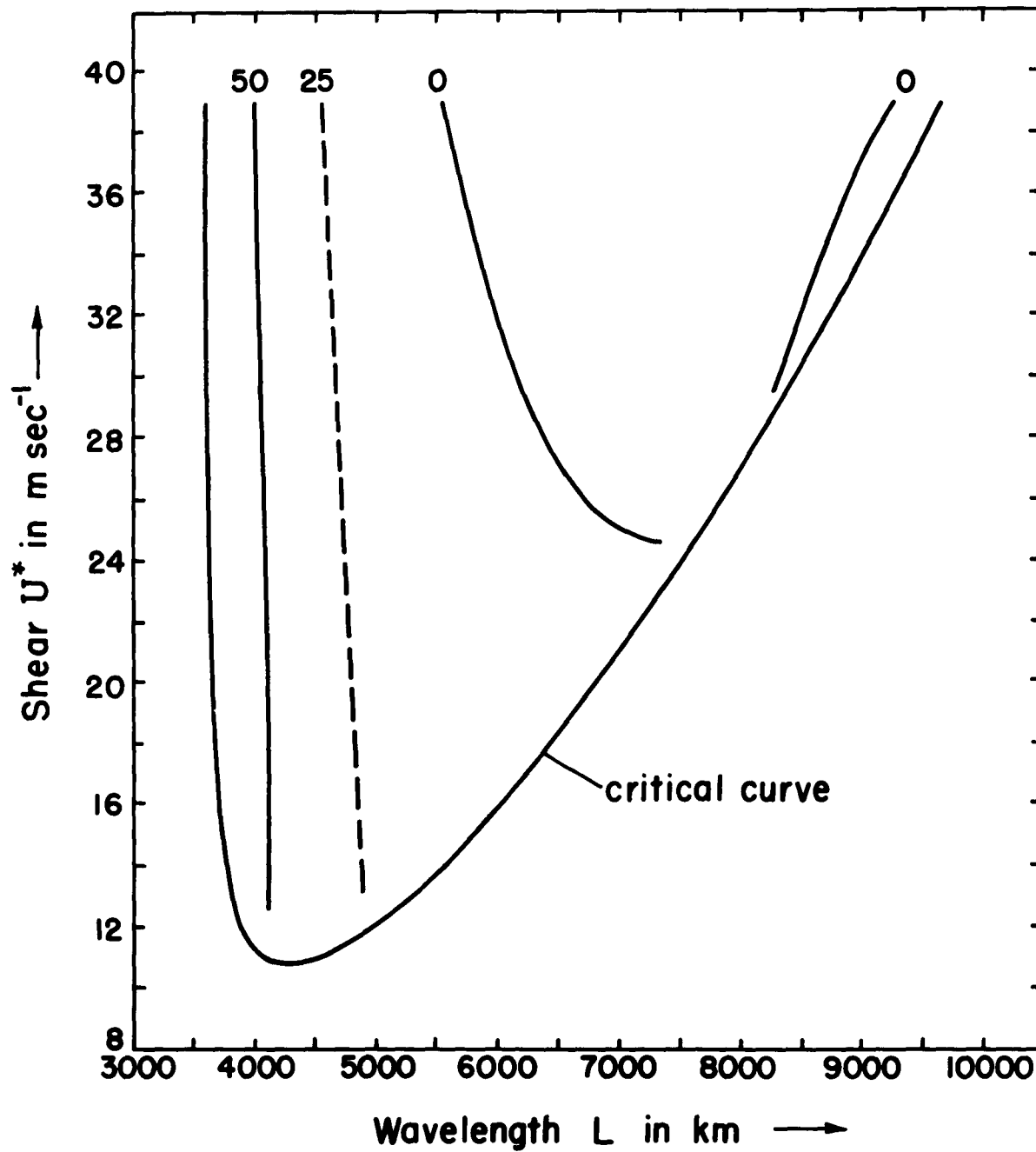


Figure 11. Percentage change in doubling time due to increasing Δt from $\frac{1}{2}$ hr to 1 hr. Here the latitude is 45 deg N, $\partial\theta/\partial p = -6 \text{ deg } (100 \text{ mb})^{-1}$, $\Delta x = 300 \text{ km}$, and the units are 10^{-3} per cent. The critical curve is that for $\Delta x = 300 \text{ km}$.

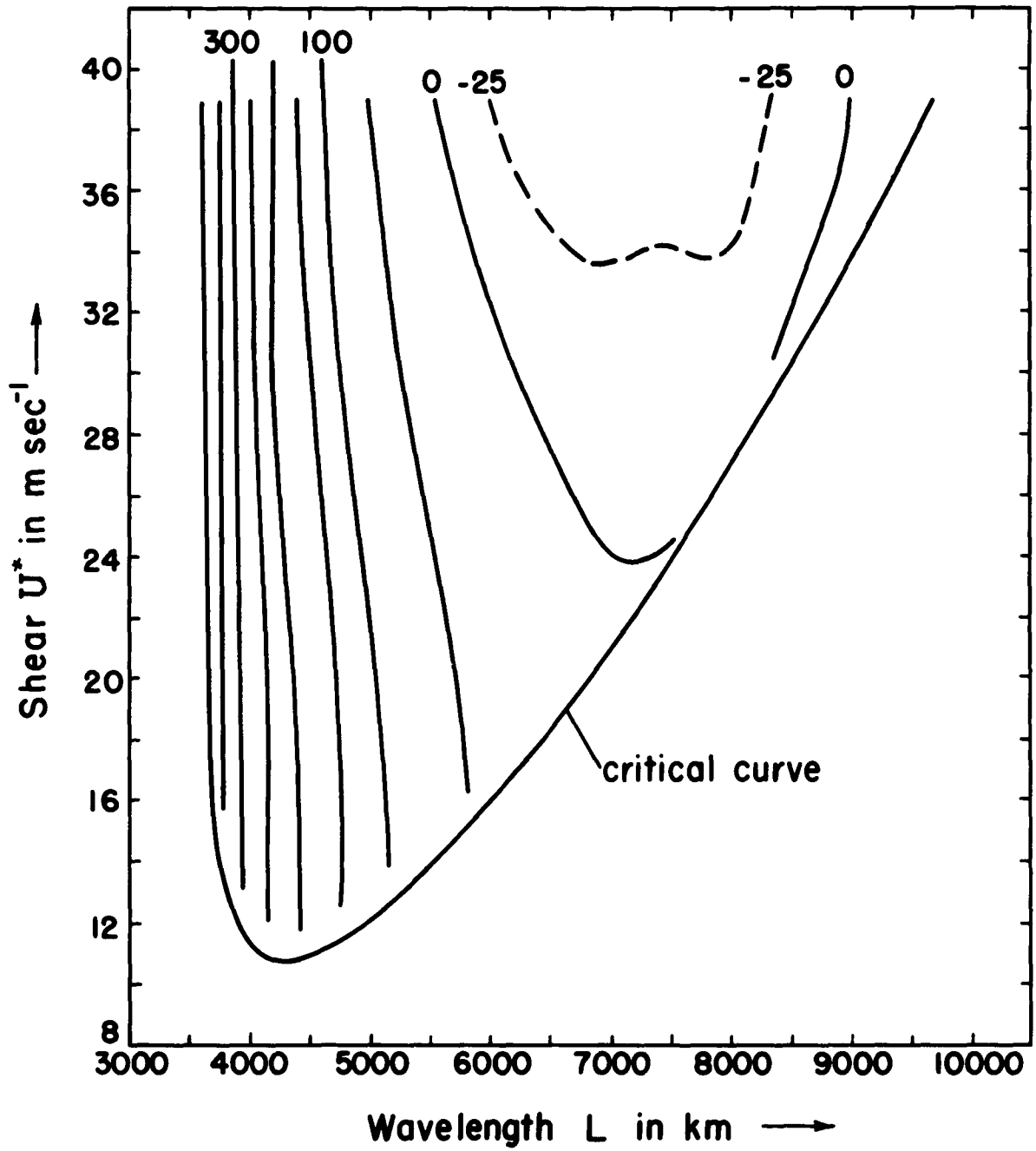


Figure 12. Same as fig. 11, but for an increase in Δt from $\frac{1}{2}$ hr to 2 hr.

In contrast to the differential system, the doubling time in the finite-difference case depends on the undisturbed zonal flow U due to the presence of Z_r in (16). This dependence, however, is slight. The percentage changes in the growth rates due to increasing U from 15 to 30 m sec^{-1} are generally less than $\frac{1}{2}$ per cent for $\partial\theta/\partial p < 0$, and less than 2 per cent in the neutral case, assuming $\Delta x = 300$ km, $\Delta t = 1$ hr. Here the effect of increasing Δx is to reduce the changes brought about by increasing the speed of the horizontal mean flow.

4. Conclusions

The introduction of implicit finite differences into the governing equations of a linearized two-parameter model causes only slight distortions of the growth rates of baroclinically unstable waves, except in the immediate vicinity of the critical curve. For the commonly used space and time mesh sizes, $\Delta x = 300$ km, $\Delta t = 1$ hr, the percentage change in growth rate, compared to that of the differential system, amounts to only about 2 per cent when $\partial\theta/\partial p = -6 \text{ deg (100 mb)}^{-1}$, and the wavelength $L = 4500$ km. In general, the distortion is greater for smaller static stabilities, shorter waves and larger Δx . It is very insensitive to changes in Δt , and depends only slightly on the undisturbed zonal flow. With the exception of the neutrally stable case, the growth rate of shorter waves is increased, and that of the longer waves slightly decreased when an implicit scheme is introduced, as compared to the continuous case. Finally, the percentage changes in the growth rates due to changes in Δx , Δt , and U are virtually independent of the vertical shear U^* .

Acknowledgement

I would like to thank Professor W. Lawrence Gates for his constructive comments and for making it possible to perform the calculations on the UCLA Computing Facility's IBM 7090.

APPENDIX

The expressions for the real and imaginary parts of the numerical phase speed C_N , equations (15) and (16), can be obtained from (9) using the relationship

$$\tan^{-1} w = \frac{i}{2} \log \left(\frac{1-iw}{1+iw} \right) \quad (18)$$

(Churchill, 1960), where $w = u + iv$. Let

$$1-iw = R_1 e^{i\omega_1},$$

$$1+iw = R_2 e^{i\omega_2},$$

where

$$R_i = [(1 \pm v)^2 + u^2]^{1/2},$$

$$\omega_i = \mp \tan^{-1} \left(\frac{u}{1 \pm v} \right),$$

and where the upper sign corresponds to subscript 1, and the lower sign to subscript 2. Identity (18) now becomes

$$\tan^{-1} w = \frac{1}{2} (\omega_2 - \omega_1) + \frac{i}{2} \ln(R_1/R_2). \quad (19)$$

Here $-\pi < (\omega_1 - \omega_2) < \pi$, and $\ln(R_1/R_2)$ is the natural logarithm of the real positive number R_1/R_2 . Writing now

$$C_N = C_{N_r} + i C_{N_i} = \frac{2}{k \Delta t} \tan^{-1} (\lambda Z_1),$$

setting $w = \lambda Z$, $u = \lambda Z_r$, $v = \lambda Z_i$, where λ is given by (8), and Z_r and Z_i by (13) and (14) respectively, expressions (15) and (16) follow directly from (19).

It can also be shown that C_N , C_{N_r} and C_{N_i} tend to their corresponding counterparts of the continuous system when Δx and Δt tend to zero simultaneously.

REFERENCES

1. Churchill, R.V., 1960: Complex Variables and Applications, New York, McGraw-Hill, 297 pp.
2. Gates, W.L., 1961: The distortion of baroclinic instability criteria by finite differences. J. Meteor., 18, 697-700.
3. Thompson, P.D., and W.L. Gates, 1956: A test of numerical prediction methods based on the barotropic and two-parameter baroclinic models. J. Meteor., 13, 127-141.