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THE EFFECT OF CLOSE COLLISIONS ON THE TWO-BODY DISTRIBUTION FUNCTION IN A PLASMA

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Abstract

This investigation is concerned with the development of a two-body distribution function in a plasma for use in a kinetic equation for the one-body distribution function. The kinetic equation is obtained for a uniform plasma for those circumstances in which the time dependence of higher-order distribution functions can be assumed to occur within a functional dependence on the one-particle distribution function. The conditions of validity for this functional-dependence assumption are discussed. The resulting interaction term is new in the sense that it contains no divergent integrals requiring cutoffs, and it may be considered accurate to first order in $(e^2/kT\lambda_D)$. The interaction term is composed of two parts. The first is a Boltzmann collision integral with a Debye-shielded interaction. The second term is due to the deviation of the shielding cloud from a Debye shield and is the Fokker-Planck form, the coefficients of which are finite and well-behaved. Because of its form, with a convergent collision integral and convergent Fokker-Planck coefficients, the solution may be considered as a joining of the previous solutions to this problem.

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I. THE NATURE OF THE PROBLEM

1.1 INTRODUCTION

In the description of the evolution of a fluid, the interplay of the properties of the interaction and the density determine whether the fluid is a liquid, gas or plasma. If at the interparticle distance the motions of the particles are strongly affected by the forces that are due to other particles, we consider the fluid a liquid. Forces that are weak at the interparticle distance will give rise to fluids described as gases or plasmas. If we consider forces that fall off at large distances to be $r^{-\alpha}$, we may distinguish gases and plasmas as $\alpha > 2$ and $\alpha \leqslant 2$, respectively. For gases the effect of the volume element in a summation over particles is not enough to counteract the $r^{-\alpha}$-decay of the force. The summation will receive its greatest contribution from near particles. In a gas of this type, for a density that is sufficiently low that particles interact only infrequently, we are led to the concept of a two-body collision and a Boltzmann gas. For $\alpha \leqslant 2$, the r^2-increase of the volume element is sufficient to increase the effect of the more distant particles. In particular, for Coulomb interactions in a plasma the volume element just offsets the r^{-2}-decay, and the concept of a collision becomes vague, since many particles interact at once.

Kirkwood and Poirier[1] show that for a plasma in equilibrium the effect of the Coulomb potential is modified by the screening of other particles and an effective potential is established. The effective potential can be well approximated by the Debye potential, $\frac{e}{r} e^{-r/\lambda_D}$. The Debye length, $\lambda_D = \sqrt{\frac{kT}{4\pi ne^2}}$ for n particles per unit volume, becomes a new range of interaction. As we shall see (sec. 1.2), plasmas of interest will be of such a density that they will have many particles within a radius λ_D; $n\lambda_D^3$ is a large number. We are still faced with many particles interacting at once.

Because the information that is eventually desired about a gas will not depend on the detailed motion of particular particles, but rather on an average over many particles, it is convenient to introduce distribution functions. Thus we introduce the distribution functions

$$F_1(x_1;t),\ F_2(x_1,x_2;t),\ \ldots,\ F_s(x_1,\ldots x_s;t),$$

where the subscript denotes the number of particles in the argument; t, the time; and x_i, the 6-dimensional vector $\{\vec{p}_i, \vec{q}_i\}$. We shall assume that the F_s are invariant under an interchange of particles, so that the particular x_i appearing in the arguments are simply labels. The F_s may be defined as follows: The function $\frac{1}{V^s} F_s(x_1,\ldots x_s;t)\, dx_1 \ldots dx_s$ is the probability at time t that the particles 1 ... s will be found, respectively, at $x_1, \ldots x_s$ within $dx_1 \ldots dx_s$. We shall take the relation between the distribution functions to be

$$F_s(x_1,\ldots x_s;t) = \frac{1}{V^{k-s}} \int F_k(x_1 \ldots x_k, t)\, dx_k \ldots dx_{s+1} \quad k \geqslant s,$$

where the volume V that is available to the gas is inserted in both cases to allow a smooth passage to infinite volume. For each dx_i, the integration is over all momentum space and the volume V.

There is one more concept that will be of interest to us. If the motion of the s particles can be considered statistically independent, we have

$$F_s(x_1, \dots x_s) = \prod_{i=1}^{s} F_1(x_i). \tag{1}$$

In this report we shall suppress the time variable t when it is not important to the relation considered. We shall refer to motion for which the property (1) holds as uncorrelated motion. Motion for which (1) does not hold will be referred to as correlated motion.

The distribution functions have final interpretation when used to evaluate the average or expectation value of some mechanical property A_s that depends on s particles. We interpret this property as

$$\langle A_s \rangle = \frac{1}{V^s} \int A_s(x_1, \dots x_s)\; F_s(x_1, \dots x_s)\; dx_1 \dots dx_s, \tag{2}$$

where we have used the fact that the distribution functions are normalized as $\frac{1}{V}\int F_1(x_1)\, dx_1 = 1$. Obviously, we are primarily interested in s = 1 and s = 2.

Upon introducing the distribution functions, we are interested in their equations of motion rather than those of individual particles, although the two are closely related. One of the first successes in this direction was Boltzmann's equation for the one-particle distribution function for dilute gas of short-range potential.

$$\frac{\partial F_1(x_1)}{\partial t} + \vec{V}_1 \cdot \frac{\partial F_1(x_1)}{\partial \vec{q}_1}$$

$$= n \int_0^{2\pi} \int_0^{\infty} \int_{(p_2)} |\overrightarrow{V_2 - V_1}| \{F_1(\vec{p}_1^{\,*}, \vec{q}_1)\; F_1(\vec{p}_2^{\,*}, \vec{q}_2) - F_1(\vec{p}_1, \vec{q}_2)\; F_1(\vec{p}_2, \vec{q}_2)\}\, d\vec{p}_2 a\, da\, d\phi. \tag{3}$$

Here, a is the collision impact parameter and $\vec{p}_1^{\,*}$ and $\vec{p}_2^{\,*}$ are the momenta that the particles must have had before the collision, given that their coordinates are now x_1 and x_2. We shall consider no external force.

It may be recalled that Boltzmann's derivation depended upon a long free path between relatively quick collisions. As an approach to the problem of plasmas, this equation with a modified Coulomb potential was used by Spitzer and Härm.[2] The collision integral with a straight r^{-2}-force diverges at long distances. Using the known fact that at equilibrium the effective interaction is the Debye potential, they cut the integral off at λ_D.

Another approach was originally proposed by Vlasov[3] and solved in detail by Landau.[4] This emphasized the Coulomb nature of the interactions by considering the force on a charge to be given by the gradient of a potential whose source is $\int F_1(x_1)\, dp_1$. The equation for F_1 becomes

$$\frac{\partial F_1(x_1)}{\partial t} + \vec{V}_1 \cdot \frac{\partial F_1(x_1)}{\partial \vec{q}_1} = n \frac{e^2}{m} \frac{\partial F_1(x_1)}{\partial \vec{V}_1} \cdot \frac{\partial}{\partial \vec{q}_1} \int \frac{F_1(x_2)\, dx_2}{|\overrightarrow{q_1 - q_2}|}. \tag{4}$$

As we shall see later, this equation is a first step but does not include or account for particle correlations.

A third approach to the problem may be made through a Fokker-Planck type of equation for F_1,[5]

$$\frac{\partial F_1(x_1)}{\partial t} + V_1 \cdot \frac{\partial F_1(x_1)}{\partial \vec{q}_1} = \frac{\partial}{\partial \vec{V}} \cdot \left(\overrightarrow{D^o} F_1(x_1) + \overleftrightarrow{B^o} \cdot \frac{\partial F_1(x_1)}{\partial \vec{V}_1} \right), \tag{5}$$

where $\overrightarrow{D^o}$ and $\overleftrightarrow{B^o}$ are functionals of F_1. The form of this equation is derived by assuming that the particles undergo a large number of small deflections – a condition violated by Coulomb particles undergoing a close collision with large momentum transfer.

Before the Fokker-Planck equation can be of use, we must obtain the coefficients from considerations of the interactions. One method, described by Allis,[6] is to expand the Boltzmann collision integral in terms of small deflections and obtain the Fokker-Planck form of equation. This, however, necessitates using the collision integral in the region in which its accuracy is most suspect – long-range interactions or grazing collisions. Another method described later[7-11] uses the two-body distribution function under the assumption that the two bodies never get close together or experience an interaction that is strong compared with their kinetic energy. As discussed by Balescu,[7] the coefficients of this equation have great intuitive appeal. However, integrals in them diverge at short distances because of a violation of the initial assumption; in this report we shall resolve this divergence.

One more fact should be noticed about these three equations. On the right-hand side of the Boltzmann, Vlasov, and Fokker-Planck equations we have functionals of F_1 only. The future of F_1 is determined by its present value, but not obviously; we might, for example, find that the equations for F_1, F_2, and so forth, are all interrelated in a set of simultaneous equations. As it is, the equation for F_2 and the higher-order distribution functions must, in some sense, be trivial in the time variable so that they can be solved immediately with the result of a single equation for F_1. This occurrence will be discussed in great detail in section 2.1.

The purpose of this report is to derive a kinetic equation for F_1. We shall derive this equation by starting with the general Liouville equation for the plasma and then examining the circumstances under which a kinetic equation for F_1 can be assumed to exist. For these circumstances we shall obtain an equation for F_1 to first order in the small parameter $(e^2/kT\lambda_D)$. This will be an improvement upon the attempts mentioned above in the sense that it will contain no divergences in those terms corresponding to interactions with other particles, that is, the right-hand sides of (3)-(5).

1.2 STATEMENT OF THE PROBLEM

Several authors[8-10] have used the Liouville equation for the distribution function in the phase space of 6-N dimensions as a starting point for the discussion of the evolution of a gas of N particles in a volume V. This distribution function, $D_N(x_1, \ldots x_s;t)$, is assumed to be symmetric under the interchange of any pair of particles. The Liouville equation is then

$$\frac{\partial D_N}{\partial t} = \left[H_N^o + \sum_{i<j}^{N} \psi_{ij} ; D_N \right]. \tag{6}$$

Here, the brackets are Poisson brackets,

$$[F, G] \equiv \sum_{i=1}^{N} \left\{ \frac{\partial F}{\partial \vec{q}_i} \cdot \frac{\partial G}{\partial \vec{p}_i} - \frac{\partial F}{\partial \vec{p}_i} \cdot \frac{\partial G}{\partial \vec{q}_i} \right\}, \tag{7}$$

and the kinetic and potential energies are

$$H_N^o \equiv \sum_{i=1}^{N} \frac{p_i^2}{2m}$$

and

$$\psi_{ij} \equiv \frac{e^2}{|\vec{q}_i - \vec{q}_j|}.$$

For our purposes throughout this report we shall assume only Coulomb interactions and identical electrons of mass m and charge e imbedded in a uniform background of opposite charge. The net charge will be taken to be neutral. The points to be investigated may be studied with this idealized model, without the complication of different particles.

We define the reduced distribution function for s particles

$$F_s(x_1, \ldots x_s;t) \equiv \frac{1}{V^{N-s}} \int D_N \, dx_{s+1} \ldots dx_N, \tag{8}$$

where the factor V^{N-s} is introduced to allow a transition to infinite volume. We shall also have use for the following identities:

$$\frac{1}{V^{N-s}} \int \frac{\partial D_N}{\partial t} dx_{s+1} \ldots dx_N \equiv \frac{\partial F_s}{\partial t},$$

$$\frac{1}{V^{N-s}} \int \left[H_N^o ; D_N \right] dx_{s+1} \ldots dx_N \equiv \left[H_s^o ; F_s \right],$$

$$\frac{1}{V^{N-s}} \int \left[\sum_{i<j}^{N} \psi_{ij} ; D_N \right] dx_{s+1} \ldots dx_N \tag{9}$$

$$\equiv \left[\sum_{i<j}^{s} \psi_{ij} ; F_s \right] + \frac{1}{V^{N-s}} \sum_{j=s+1}^{N} \sum_{i=1}^{s} \int [\psi_{ij} ; D_N] \, dx_{s+1} \ldots dx_N .$$

By using the symmetry of D_N under interchange of particles, the sum over j consists of identical terms so that the last identity may be written

$$\frac{1}{V^{N-s}} \int \left[\sum_{i<j}^{N} \psi_{ij} ; D_N \right] dx_{s+1} \ldots dx_N$$

$$\equiv \left[\sum_{i<j}^{s} \psi_{ij} ; F_s \right] + \frac{N-s}{V} \int \left[\sum_{i=1}^{s} \psi_{is+1} ; F_{s+1} \right] dx_{s+1} . \tag{10}$$

Integration of (6) over $dx_{s+1} \ldots dx_N$ and use of (9) produce

$$\frac{\partial F_s}{\partial t} = \left[H_s^o ; F_s \right] + \left[\sum_{i<j}^{s} \psi_{ij} ; F_s \right] + \frac{N-s}{V} \int \left[\sum_{i=1}^{s} \psi_{is+1} ; F_{s+1} \right] dx_{s+1} . \tag{11}$$

Throughout this report the argument of the function F_s is suppressed when the meaning is clear.

For fixed s we may pass to the limit of infinite volume and infinite number so that $\frac{N}{V}$ remains constant. This replaces $\frac{N-s}{V}$ with n. For spacially nonuniform plasmas, n is not a density in quite the usual sense, but is the limit $\frac{N}{V}$.

The introduction of n gives (11) its final form. The meanings of its terms are clear. The left-hand side and the first two terms on the right-hand side constitute the Liouville equation including interactions for the s particles under consideration. The integral term represents the contribution to the rate of change of F_s which is due to the interactions with the rest of the particles.

In order to approach the analysis of the equations (11), we shall estimate the size of the terms. It will be found that under certain circumstances one term is small and thus gives rise to the possibility of a perturbation expansion. We assume a plasma that is near enough to equilibrium that we may define a shielding distance $\lambda_D = \sqrt{kT/4\pi ne^2}$, a plasma frequency $\omega_p = \sqrt{4\pi ne^2/m}$, and a characteristic velocity $\bar{v} = \sqrt{kT/m}$. By using these as units, the sizes of the various integrals and derivatives can be estimated. The change of coordinates $\tau = \omega_p t$, $\rho = r/\lambda_D$, and $u = v/\bar{v}$ in (11) will make, for the average particle, each of the integrals and derivatives of order one. The size of each term will be given by the following coefficients

$$\omega_p, \quad \bar{v}/\lambda_D, \quad e^2/m\bar{v}b^2, \quad ne^2\lambda_D/m\bar{v}$$

which are the ratios

$$1, \quad 1, \quad 1/n\lambda_D b^2, \quad 1.$$

Here, b is the distance between the two particles under consideration. The last three terms are the three sums in (11), and the coefficients above simply represent their magnitudes. In particular, in the third term there will be a different b for the separation of each pair of particles; the term for each pair is subject to the analysis given below.

A requirement that the third term be small implies that $b \gg \sqrt{e^2\lambda_D/kT}$. The solution of (11) has been discussed by several authors[8-10] for the situations that satisfy this requirement for all pairs. The process has been to assume that all of the s-particle interaction terms are small and to assign to them an expansion parameter g that is later set to one. The terms designated by g^r will be of order $(e^2/kT\lambda_D)^r$. Then F_s is expanded in g as a perturbation expansion. This results in zero and first order in g.

$$\frac{\partial F_s^o}{\partial t} = \left[H_s^o;F_s^o\right] + n \int \left[\sum_{i=1}^{s} \psi_{is+1};F_{s+1}^o\right] dx_{s+1} \tag{12}$$

and

$$\frac{\partial F_s^1}{\partial t} = \left[H_s^o;F_s^1\right] + \left[\sum_{i<j}^{s} \psi_{ij};F_s^o\right] + n \int \left[\sum_{i=1}^{s} \psi_{is+1};F_{s+1}^1\right] dx_{s+1}. \tag{13}$$

The derivation and solution of these equations will be discussed; of more interest to us now is the general form of the solution. We shall see that for a uniform plasma F_s^o and F_s^1 may be reduced to

$$\begin{aligned} F_s^o &= \prod_{i=1}^{s} F_1(\vec{p}_i) \\ F_s^1 &= \sum_{i<j}^{s} \prod_{k\neq i,j}^{s} F_1(\vec{p}_k)\, F_2^1(\vec{p}_i, \vec{p}_j, \overline{\vec{q}_i - \vec{q}_j}). \end{aligned} \tag{14}$$

Here, F_2^1 is the solution of the equation

$$\frac{\partial F_2^1(x_1, x_2)}{\partial t} = \left[H_2^o; F_2^1(x_1, x_2)\right] + [\psi_{12}; F_1(\vec{p}_1)\, F_1(\vec{p}_2)]$$

$$+ n \int \left\{ \left[\psi_{13}; F_1(\vec{p}_1)\, F_2^1(x_2, x_3)\right] + \left[\psi_{23}; F_1(\vec{p}_2)\, F_2^1(x_1, x_3)\right] \right\} dx_3, \tag{15}$$

and to first order in g

$$\frac{\partial F_1(\vec{p}_1)}{\partial t} = n \int \left[\psi_{12}; F_2^1(x_1, x_2)\right] dx_2. \tag{16}$$

Equation 15 has been solved under the adiabatic hypothesis defined in section 2.1. When this solution is substituted in (16), the final equation becomes

$$\frac{\partial F_1(p_1)}{\partial t} = n \int \left[\psi_{12}; F_2^1(x_1, x_2)\right] dx_2 \cong n \int \left[\psi_{12}; f_2^1(x_1, x_2)\right] dx_2$$

$$= \frac{16\pi^3 n e^4}{m(2\pi)^3} \int\int d\vec{p}_2 d\vec{k}\, \vec{k} \cdot \frac{\partial}{\partial \vec{V}_1} \frac{\delta(\vec{k}\cdot(\overrightarrow{V_1 - V_2}))}{\left| k^2 + \omega_p^2 \int_{+ie} \frac{\vec{k} \cdot \frac{\partial F_1(\vec{p}_2)}{\partial \vec{p}_2}}{k \cdot (\overrightarrow{V_1 - V_2})} d\vec{p}_2 \right|^2} \vec{k}$$

$$\cdot \left(\frac{\partial}{\partial \vec{V}_1} - \frac{\partial}{\partial \vec{V}_2} \right) F_1(\vec{p}_1)\, F_1(\vec{p}_2), \tag{17}$$

where, under the adiabatic hypothesis,

$$F_2^1(x_1, x_2) \equiv f_2^1(x_1, x_2). \tag{18}$$

For definiteness, we reserve a special symbol for this function and refer to it as the large-separation solution.

In (17) $\vec{k}$ is the Fourier-transform variable corresponding to $(\overrightarrow{r_1 - r_2})$. This integration diverges logarithmically for large values of $|\vec{k}|$ corresponding to small values of $|\overrightarrow{r_1 - r_2}|$. As mentioned previously, this divergence occurs because we assumed that ψ_{12} is small in the derivation of (15). The divergence occurs in a region that violates this assumption. To remove the divergence, one should allow for the possible mutual approach of particles 1 and 2 in (11).

In the classical gas of electrons considered, one never encounters the problem of two particles close together because of the mutual repulsions. This fact should be borne out in the solution of the set of equations (11). If discrete positive charges had been included, real problems might have occurred because of the attractive potential. The occurrence of bound states and the effects of very fast electrons would require analysis that would go far beyond the techniques employed here. In this investigation we shall study the effect of close collisions on the two-body distribution function and shall thereby remove divergence in the classical problem of Coulomb repulsions.

In the past this divergence has been handled by cutting off the integration in (17) at kT/e^2, the value corresponding to the distance of average closest approach. Since the dependence on the cutoff is logarithmic, the final results are not expected to be much in error from a numerical point of view. However, it is of interest to see what happens to the distribution function for close collisions and to see how accurate the method of cutting off may be.

In order to proceed let us return to the arguments that led to the assignments of orders of magnitude to the various terms in (11). We found that the requirement that the pair interaction term be small implied that the s particles are mutually separated by distances $b \gg \sqrt{e^2\lambda_D/kT}$. Let us now imagine a set of concentric spheres, of possible separation of two particles, with radii e^2/kT, $\sqrt{e^2\lambda_D/kT}$, and λ_D. The radii are in the constant ratio $\sqrt{kT\lambda_D/e^2} = \sqrt{4\pi n\lambda_D^3}$. For a wide range of plasmas the quantity $4\pi n\lambda_D^3$ is much larger than one. For example, if $kT = 100$ ev and $n = 10^{16}$ cm^{-3}, then $4\pi n\lambda_D^3 \sim 10^5$. These spheres are quite distinct, and they are useful in visualizing the process of interaction.

Let us label the spheres I, II, and III in order of increasing radii. The volume inside sphere I may be considered forbidden to the particles because e^2/kT is the distance of closest approach for the particle of average energy kT. Particles are allowed between spheres I and II; but in this region the potential energy is larger than the average kinetic energy, and thus the pair interaction term may not be considered small. The solution may be considered correct from sphere II outward, and an evaluation of (17) shows that the result is exponentially cut off outside λ_D, in agreement with the Debye theory. In order to correctly handle the integral occurring in the equation for F_1, we must consider the possibility of particles occurring between spheres I and II.

Consider the number $n(e^2\lambda_D/kT)^{3/2}$, the probable number of particles inside sphere II. In terms of $4\pi n\lambda_D^3$ this number is $(n\lambda_D^3)^{-1/2}$, which is small. We may argue from a strictly probabilistic point of view that it would be correspondingly even more unlikely that more particles should be inside this sphere. We are led to the concept of a "close collision," one in which two particles experience a short time interaction within sphere II. In line with the foregoing argument, we shall assume that the close collisions are binary and shall ignore the possibility of three particles occurring within this short range.

The analogy with the Boltzmann gas should be mentioned. For the Boltzmann gas we consider free particles undergoing binary collisions. For the Coulomb case we realize that the particles interact over a long range, but we use the fact that the strong interactions occur only in binary types of events. In both cases those collisions that cause a large change in momentum are assumed to be binary.

We shall carry out the solution to (11) under the assumption that the interaction potential of one pair of particles, 1 and 2, is not necessarily small, while all other pairs are assumed to be small. The Hamiltonian to zero order for s particles, including 1 and 2, will be $H_s^o + \psi_{12}$. Sets of particles not including 1 and 2 will be assumed to be outside a range corresponding to a close collision; their zero-order Hamiltonian will be H_s^o, and their solutions will be assumed to be (14)-(17). Whereas earlier treatments have assumed a gas of electrons experiencing entirely grazing collisions, we assume that one pair really collides with no limitations. This assumption is not as restricted as it sounds, since we are in reality saying that there are many mutually separated pairs in close collision.

This entire procedure will not get rid of all divergences, since in the equation for the two-body distribution function we encounter terms of the form

$$\int [\psi_{13}; F_3(x_1, x_2, x_3)]\, dx_3. \tag{19}$$

By the above-given procedure we correctly allow for ψ_{12}, but not the approach of 1 and 3 — the combination would entail a three-body collision. In section 3.2 it will be seen that this divergence can be circumvented in a plausible way. However, it is reasonable to expect that, if we kept this integral and merely cut it off, the dependence on the cut-off in the final equation for F_1 would be much weaker than the logarithmic dependence found in (17) because we have carried the problem to one more step of accuracy.

Section 2.1 will be devoted to a discussion of the methods, operators, and notation to be used in solving these equations. Section 2.2 will include, as an example, some discussion of the equations for the large-separation solution. In section 2.3 we shall modify the ideas of section 2.2 so that they will be applicable to the present problem. Section III will be a discussion of the actual problem, and Section IV will be a discussion of the results. Some of the material contained in section 2.1 is taken from a book by Bogoliubov[12] but is included here since it is not usually used.

II. THE METHOD OF APPROACH

2.1 THE ADIABATIC HYPOTHESIS AND ITS IMPLICATIONS

In order to study the methods to be used in the solution, introduce the Hamiltonian for the kinetic energies and mutual interactions of s particles.

$$H_s = H_s^o + \sum_{i<j}^{s} \psi_{ij}.$$

For simplicity, we assume that there is no external field. The generalization is conceptual immediately, since these are one-particle processes. However, the resulting particle trajectories are very difficult to solve. One would not expect that the evolutions of correlations are much affected by the presence of weak external fields; thus this model is useful intuitively for the more difficult case. The strong field, in which the external field exerts more force than most of the interactions, would be complicated. But for this simple case even the most simple equation, the Vlasov equation (4), is not understood, since its nonlinear character becomes important.

For the whole system the Liouville equation,

$$\frac{\partial D_N}{\partial t} = [H_N;D_N], \tag{20}$$

has as a formal solution

$$D_N(x_1, \dots x_N;t) = S_{-t}^N \; D_N(x_1, \dots x_N;0). \tag{21}$$

Here, the operator S_{-t}^N operates on the particle coordinates $x_1 \dots x_N$ and projects them backward in time t seconds on the basis of the paths given by their Hamiltonian; that is, D_N flows like an ideal fluid in phase space.

Note the identity for an arbitrary ϕ:

$$\frac{\partial}{\partial t} S_{-t}^N \phi(x_1, \dots x_N, t) = \left[H_N, S_{-t}^N \phi(x_1, \dots x_N, t)\right] + S_{-t}^N \frac{\partial}{\partial t} \phi(x_1, \dots x_N, t). \tag{22}$$

This follows from the definition of S_{-t}^N and the fact that

$$\frac{\partial}{\partial t} S_{-\tau}^N \phi = \left(\frac{\partial S_{-t}^N}{\partial t}\right) \phi + S_{-t}^N \frac{\partial \phi}{\partial t}. \tag{23}$$

Another property that we shall need in this investigation is the solution of equations of the form

$$\frac{\partial}{\partial t} \phi(x_1, \dots x_N;t) = [H_N;\phi(x_1, \dots x_N;t)] + f(x_1, \dots x_N;t). \tag{24}$$

Let $\phi(x_1, \dots x_N;t) \equiv S_{-t}^N \chi(x_1, \dots x_N;t)$; then, using (23) and (24), we obtain

$$S_{-t}^{N} \frac{\partial}{\partial t} \chi(x_1, \ldots x_N;t) = f(x_1, x_2 \ldots x_N, t).$$

Multiplying by S_t^N and integrating, we obtain

$$\chi(x_1, \ldots x_N;t) - \chi(x_1, \ldots x_N;0) = \int_0^t S_\tau^N f(x_1, \ldots x_N;\tau)\, d\tau. \tag{25}$$

In (25) we use the relation

$$S_t^N S_{-t_o}^N = S_{t-t_o}^N \xrightarrow[t - t_o \to 0]{} 1,$$

and consider it the inversion property of the operators. Finally, multiplying by S_{-t}^N and resubstituting ϕ, we have

$$\phi(x_1, \ldots x_N;t) = S_{-t}^N \phi(x_1, \ldots x_N;0) + \int_0^t d\tau\, S_{-(t-\tau)}^N f(x_1, \ldots x_N;\tau). \tag{26}$$

In (26) the first term on the right-hand side is the contribution of flow in phase space and the second is the effect of the source.

Now let us examine the exact equation for F_1,

$$\frac{\partial F_1(x_1)}{\partial t} = [H_1^o;F_1(x_1)] + n \int [\psi_{12};F_2(x_1, x_2)]\, dx_2. \tag{27}$$

This is obtained by integrating the Liouville equation and is equivalent to it. To use F_2, we must solve the equation for F_2, which involves knowing F_3, and so forth. The advantage of this chain over the original Liouville equation is that, if we can break the chain in a physically sensible way, we can obtain a closed set of equations and know the precise approximations made in departing from the full Liouville equation. The problem is to perform the break in a manner that will balance the physical and computational reasonability.

Now look at (27) in the light of the fact that, as mentioned in section 1.1, the various forms of kinetic equations for F_1 have one thing in common – they can generally be written

$$\frac{\partial F_1(x_1)}{\partial t} = A_1(x_1;F_1). \tag{28}$$

Here, $A_1(x_1;F_1)$ is a functional of F_1. The importance of this is that the entire time dependence of the right-hand side of (28) lies inside F_1 and depends only upon the current value of F_1. This fact is implied by having a kinetic equation for F_1: that its present value is sufficient to predict its future.

If (27) is to be of the form of (28), then F_2 must be such that

$$F_2(x_1, x_2;t) = F_2(x_1, x_2;F_1). \tag{29}$$

If (29) is to hold, it must be true for all $s \geq 2$ that

$$F_s(x_1 \ldots x_s;t) = F_s(x_1 \ldots x_s;F_1). \tag{30}$$

Bogoliubov has shown[12] that for a Boltzmann gas this is a very good assumption in that any initial F_s that violates (30) will relax to the form (30) in a collision time that is very short compared with the characteristic time of change of F_1. Thus it is safe to assume that F_s is of the form (30).

This whole argument breaks down for a plasma, particularly one that is not spacially uniform. In this case the collision time is of the order $\lambda_D/\bar{v} = \sqrt{m/4\pi ne^2} = \omega_p^{-1}$, which is the characteristic time of change of F_1 for a nonuniform plasma. However, to get a kinetic equation of the form (28), one is forced to take (30) as an assumption and to look for those solutions satisfying this form which will provide the most general equation of the form (28).

We begin our investigation with a generalization of the foregoing equations. We shall look for a pair of equations which is made up of (27) and an equation of the form

$$\frac{\partial F_2(x_1, x_2)}{\partial t} = A_2(x_1, x_2;F_1, F_2), \tag{31}$$

where the time dependence of A_2 resides within a functional dependence on F_1 and F_2. Since we know from (11) that

$$\frac{\partial F_2}{\partial t} = \left[H_2^o+\psi_{12};F_2\right] + n \int [\psi_{13}+\psi_{23};F_3]\, dx_3, \tag{32}$$

we may say in analogy with (29) that

$$F_3(x_1, x_2, x_3;t) = F_3(x_1, x_2, x_3;F_1, F_2), \tag{33}$$

and, therefore, that

$$F_s(x_1, \ldots x_s;t) = F_s(x_1 \ldots x_s;F_1, F_2). \tag{34}$$

We shall refer to these functional-dependence assumptions as the "adiabatic hypotheses" in the following sense. For example, in (34), we assume that F_s for $s > 2$ relaxes very rapidly to a form depending only on the instantaneous values of F_1 and F_2. This assumption is analogous to the adiabatic approximation to the time-dependent perturbation theory in quantum mechanics.

This procedure serves the following purpose. We assume that a kinetic equation for F_2 exists, (31). Equation 31 implies certain limitations upon the time dependence of F_3, (33). We shall find the solution for F_3 which satisfies these limitations and use this function in the integral of (32) to obtain a general kinetic equation for F_2. We then have a method of investigating directly the possibilities and limitations of making another restriction on F_2, that is, that it is of the form $F_2(x_1, x_2;F_1)$. We shall make this restriction and obtain a solution for F_2 in this form. Since the additional assumption that was made to obtain (17) from (15) is (29), this procedure will yield a result that is directly equivalent to (17) except that the divergence will no longer exist.

A look at the equations to be solved will show the origin of the adiabatic hypothesis and the ensuing statements. If the interaction ψ_{12} is of order g, a condition that we shall call Case I, we can write

$$\frac{\partial F_s}{\partial t} = \left[H_s^o; F_s\right] + g\left[\sum_{i<j}^{s} \psi_{ij}; F_s\right] + n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}\right] dx_{s+1}. \tag{35}$$

When particles 1 and 2 are allowed to collide, ψ_{12} is not small and must be included in the zero-order Hamiltonian. In this form we have Case II,

$$\frac{\partial F_s}{\partial t} = \left[H_s^o + \psi_{12}; F_s\right] + g\left[{\sum_{i<j}^{s}}' \psi_{ij}; F_s\right] + n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}\right] dx_{s+1}, \tag{36}$$

where the $\sum'$ denotes the omission of the pair $\{12\}$.

We shall see that if F_s is expanded in powers of g,

$$F_s = F_s^o + g\,F_s^1 + g^2\,F_s^2 + \ldots, \tag{37}$$

and a perturbation expansion is carried out, then the zero solutions are

CASE I:

$$F_s^o = \prod_{i=1}^{s} F_1(x_i) \tag{38a}$$

and

CASE II:

$$F_s^o = F_2(x_1, x_2) \prod_{i=3}^{s} F_1(x_i). \tag{38b}$$

In the perturbation expansions of (35) and (36), F_s^o will serve as the source term for F_s^1 and we obtain

CASE I:

$$\frac{\partial F_s^1}{\partial t} = \left[H_s^o; F_s^1\right] + \left[\sum_{i<j}^{s} \psi_{ij}; F_s^o\right] + n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}^1\right] dx_{s+1} \tag{39a}$$

and

CASE II:

$$\frac{\partial F_s^1}{\partial t} = \left[H_s^o + \psi_{12}; F_s^1\right] + \left[{\sum_{i<j}^{s}}' \psi_{ij}; F_s^o\right] + n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}^1\right] dx_{s+1}. \tag{39b}$$

With the adiabatic hypothesis applied to (39) we can study the time dependence of F_s^1 while holding F_s^o constant in the source. If, upon solution, F_s^1 is found to vary as rapidly as F_s^o, then the hypothesis is invalid and another method is required.

The basis for the belief that for a uniform plasma the hypothesis is valid follows from the equation for F_1 in which we expand and use (38)

$$\frac{\partial F_1(x_1)}{\partial t} = \left[H_1^o;F_1(x_1)\right] + n \int [\psi_{12};F_1(x_1)F_1(x_2)]\, dx_2$$

$$+ g\, n \int \left[\psi_{12};F_2^1(x_1, x_2)\right] dx_2 + g^2 \dots . \tag{40}$$

The first integral is the effect of a potential U whose source is F_1,

$$\nabla^2 U = -4\pi n e^2 \int F_1(x)\, d\vec{p}. \tag{41}$$

For the neutral uniform plasma, U and the Poisson bracket of F_1 with H_1^o are zero. The rate of change of F_1 is of order g; thus in equations like (39) we can hold F_s^o constant to the same order in g.

In this report we assume for Case I and Case II that the F_s depends functionally upon the functions appearing in F_s^o. In section 2.2 we work out Case I by assuming a dependence on F_1. This solution will serve as an introduction to the methods because this problem is closely related to the more general one. Section 2.3 and Section III will be concerned with Case II.

2.2 THE LARGE-SEPARATION SOLUTION

In this section we shall discuss the solution to those equations in which all two-particle interactions are assumed to be small, which were referred to as Case I in section 2.1. We take (30) to be our form of the adiabatic hypothesis.

To carry out the solution, we must be able to handle terms of the form

$$\frac{\partial}{\partial t} F_s(x_1, x_2 \dots x_3;F_1),$$

for which we know that the derivative will operate only on the F_1, since that is the only place in which t appears. For $s \geq 2$ expand F_s in a power series in g, where g is the expansion parameter described in section 1.2, to obtain (37). In (30) the expansion of F_2 will have the effect

$$\frac{\partial F_1(x_1)}{\partial t} = \left[H_1^o;F_1(x_1)\right] + n \int \left[\psi_{12};F_2^o(x_1, x_2;F_1) + g\, F_2^1(x_1, x_2;F_2) + \dots\right] dx_2, \tag{42}$$

and we shall consider this an expansion of $\partial F_1/\partial t$ in powers of g. For convenience of notation we write (42)

$$\frac{\partial F_1(x_1)}{\partial t} = A_1^o(x_1;F_1) + g\, A_1^1(x_1;F_1) + \dots \tag{43}$$

and define the A_1^r by comparison with (43):

$$A_1^o(x_1;F_1) = \left[H_1^o;F_1(x_1)\right] + n\int\left[\psi_{12};F_2^o(x_1,x_2;F_1)\right]dx_2,$$

$$A_1^1(x_1;F_1) = n\int\left[\psi_{12};F_2^1(x_1,x_2;F_1)\right]dx_2, \tag{44}$$

$$\vdots$$

Let $\chi(x_1,\ldots x_s;F_1)$ be any expression whose time dependence is completely defined by F_1. For infinitesimal variations in F_1,

$$\delta\chi(x_1,x_2\ldots x_s;F_1) = \phi(x_1,\ldots x_s;F_1,\delta F_1),$$

where ϕ is some new functional that will be linear in δF_1. From this relationship and (44), we have

$$\frac{\partial}{\partial t}\chi(x_1,\ldots x_s;F_1) = \phi\left(x_1,\ldots x_s;F_1\,\frac{\partial F_1}{\partial t}\right)$$

$$= \phi\left(x_1,\ldots x_s;F_1,A^o\right) + g\phi\left(x_1,\ldots x_s;F_1,A_1^1\right) + g^2\ldots$$

or

$$\frac{\partial\chi}{\partial t} = D_o\chi + gD_1\chi + g^2\ \ldots\ . \tag{45}$$

Here, D_r denotes an operator that differentiates with respect to t (by operating on F_1) and then replaces $\frac{\partial F_1}{\partial t}$ with A_1^r.

We wish to apply these definitions to the equation for F_s, Case I,

$$\frac{\partial F_s}{\partial t} = \left[H_s^o;F_s\right] + g\left[\sum_{i<j}^{s}\psi_{ij};F_s\right] + n\int\left[\sum_{i=1}^{s}\psi_{is+1};F_{s+1}\right]dx_{s+1}. \tag{46}$$

In (46) expand $\partial F_s/\partial t$ as in (45), and expand F_s as in (37) to obtain

$$\left(D_o+gD_1+g^2\ldots\right)\left(F_s^o+gF_s^1+g^2F_s^2+\ldots\right) = \left[H_s^o + g\sum_{i<j}^{s}\psi_{ij};F_s^o + gF_s^1 + g^2F_s^2\ \ldots\right]$$

$$+ n\int\left[\sum_{i=1}^{s}\psi_{is+1};F_{s+1}^o + gF_{s+1}^1 + g^2F_{s+1}^2\ \ldots\right]dx_{s+1}.$$

Using the first two powers of g, we obtain the following two equations:

$$D_o F_s^o = \left[H_s^o ; F_s^o\right] + n \int \left[\sum_{i=1}^{s} \psi_{is+1} ; F_{s+1}^o\right] dx_{s+1} \tag{47}$$

and

$$D_o F_s^1 + D_1 F_s^o = \left[H_s^o ; F_s^1\right] + \left[\sum_{i<j}^{s} \psi_{ij} ; F_s^o\right] + n \int \left[\sum_{i=1}^{s} \psi_{is+1} ; F_{s+1}^1\right] dx_{s+1} . \tag{48}$$

Since in the limit of infinite separation of all s particles the motions of the particles must become uncorrelated, we impose the following boundary conditions:

$$\begin{aligned} F_s^o &\xrightarrow[\text{all } \overline{q_i - q_j} \to \infty]{} \prod_{i=1}^{s} F_1(x_i) \\ F_s^i &\xrightarrow[\text{all } \overline{q_i - q_j} \to \infty]{} 0 \qquad i \geqslant 1 . \end{aligned} \tag{49}$$

Here, F_s^i goes to zero because F_s^o is assumed to have all of the boundary contribution.

By direct substitution and the use of the definition of the D_r operators in terms of A_1^r, we see that the solution of (47) which satisfies (38) is

$$F_s^o(x_1, \ldots x_s ; F_1) = \prod_{i=1}^{s} F_1(x_i) . \tag{50}$$

In solving (48) we use the fact that, by definition of D_1,

$$\begin{aligned} D_1 F_s^o = D_1 \prod_{i=1}^{s} F_1(x_i) &= \sum_{i=1}^{s} \prod_{\substack{j=1 \\ \neq i}}^{s} F_1(x_j)\, A_1^1(x_i ; F_1) \\ &= n \sum_{i=1}^{s} \prod_{\substack{j=1 \\ \neq i}}^{s} F_1(x_j) \int \left[\psi_{is+1} ; F_2^1(x_i, x_{s+1} ; F_1)\right] dx_{s+1} . \end{aligned} \tag{51}$$

Substitution of (51) in (48) yields

$$\begin{aligned} D_o F_s^1 = \left[H_s^o ; F_s^1\right] &+ \left[\sum_{i<j}^{s} \psi_{ij} ; F_s^o\right] + n \int \left[\sum_{i=1}^{s} \psi_{is+1} ; F_{s+1}^1\right] dx_{s+1} \\ &- n \sum_{i=1}^{s} \prod_{\substack{j=1 \\ \neq i}}^{s} F_1(x_j) \int \left[\psi_{is+1} ; F_2^1(x_i, x_{s+1})\right] dx_{s+1} . \end{aligned} \tag{52}$$

We see that by straightforward substitution and application of (44) and (50), the solution of (52) which is consistent with the boundary condition (49) is

$$F_s^1 = \sum_{1 \le i < j \le s} \prod_{\substack{k=1 \\ \ne i, j}}^{s} F_1(x_k)\, F_2^1(x_i, x_j; F_1).$$

The final equation for $F_2^1(x_1, x_2; F_1)$ is

$$\begin{aligned} D_o F_2^1(x_1, x_2; F_1) &= \left[H_2^o; F_2^1(x_1, x_2; F_1)\right] + [\psi_{12}; F_1(x_1)\, F_1(x_2)] \\ &+ n \int \left\{\left[\psi_{13}; F_1(x_1)\, F_2^1(x_2, x_3; F_1)\right] + \left[\psi_{23}; F_1(x_2)\, F_2^1(x_1, x_3; F_1)\right]\right\} dx_3 \\ &+ n \int \left[\psi_{13} + \psi_{23}; F_2^1(x_1, x_2; F_1)\, F_1(x_3)\right] dx_3. \end{aligned} \tag{53}$$

For uniform plasmas, this equation has been solved by Lenard[10] and Dupree.[11] The latter's solution is carried out in the Appendix by using methods to be introduced in Section III. This solution results in the expression (17), which was solved under the adiabatic hypothesis. Thus in line with (18), the solution to (53) will be denoted $f_2^1(x_1, x_2; F_1)$.

2.3 COLLISION PROBLEM ASSUMPTIONS

In this section we discuss the equations referred to as Case II, (36), in which ψ_{12} is not assumed to be of order g and the expansion parameter has been inserted before those terms taken to be small. (See section 1.2.)

By selecting particles 1 and 2 as the particular particles whose close approach will be allowed, we temporarily destroy the interchangeability of particles in F_s. Thus when we make an adiabatic hypothesis for F_s (Eq. 34) and take $\partial F_s/\partial t$, we must interpret the effect of $\partial F_1/\partial t$ and $\partial F_2/\partial t$ differently, according to their arguments. Since in (36) we assumed that all ψ_{ij} are small for i and j which are not equal to 1 and 2, we expect the correlations between these pairs to be the same as those studied in section 2.2 because the correlation of a pair of particles is generated by the past history of their mutual force. Therefore, for these i and j, we take $\partial F_1(x_i)/\partial t$ and $\partial F_2(x_i, x_j)/\partial t$ to be given by the results of section 2.2.

In (10) we obtained the factor $\frac{N-s}{V}$ before the integral by summing over identical particles. If in this new interpretation we are careful to sum over only the identical ones, we will have as an equation for $F_1(x_i)$

$$\begin{aligned} \frac{\partial F_1(x_i)}{\partial t} &= \left[H_1^o; F_1(x_i)\right] + \frac{N-3}{V} \int [\psi_{is+1}; F_2(x_i, x_{s+1})]\, dx_{s+1} \\ &+ \frac{1}{V} \int [\psi_{i1}; F_2(x_i, x_1)]\, dx_1 + \frac{1}{V} \int [\psi_{i2}; F_2(x_i, x_2)]\, dx_2. \end{aligned}$$

However, the last two integrals have small effect for two reasons. First, in the limit of large N, their contribution will become negligible. Second, we know that outside a range corresponding to a close collision all F_2 are the same, so that to this extent they can be included in the first integral. We have said the ψ_{i1} or ψ_{i2} must be small, just as ψ_{is+1} is small; thus the contribution must be essentially the same. A similar argument holds for the evolution of $F_2(x_1, x_2)$.

This distinction may be summarized in the following way. When we write an equation for F_s, the Hamiltonian will contain ψ_{12} to zero order in g only if the set s contains both 1 and 2. We assume that, if this zero-order Hamiltonian does not contain ψ_{12}, the evolution of F_s is the same as that discussed in section 2.2. This assumption is connected with the binary collision assumption. We do not complicate the picture by including close collisions of mutually separated pairs and we explicitly exclude 3-body collisions.

With this in mind we are able to interpret differentiations of the form $\frac{\partial}{\partial t} F_s(x_1, \ldots x_s; F_1, F_2)$. When the derivative operates on $F_1(x_i)$ or $F_2(x_i, x_j)$, we apply A_1^r defined in section 2.2. When the derivative operates on $F_2(x_1, x_2)$ we must define new A_2. By comparing (31) with the equation for $F_2(x_1, x_2)$,

$$\frac{\partial F_2(x_1, x_2)}{\partial t} = \left[H_2^o + \psi_{12}; F_2(x_1, x_2)\right] + n \int [\psi_{13} + \psi_{23}; F_3(x_1, x_2, x_3; F_1, F_2)]\, dx_3, \tag{54}$$

expanding in (54)

$$F_3 = F_3^o + g F_3^1 + g^2 F_3^2 + \ldots$$

and expanding in (31)

$$A_2(x_1, x_2; F_1, F_2) = A_2^o(x_1, x_2; F_1, F_2) + g A_2^1(x_1, x_2; F_1, F_2) + g^2 \ldots,$$

we obtain

$$\begin{aligned} A_2^o(x_1, x_2; F_1, F_2) &= \left[H_2^o + \psi_{12}; F_2(x_1, x_2)\right] \\ &\quad + n \int \left[\psi_{13} + \psi_{23}; F_3^o(x_1, x_2, x_3; F_1, F_2)\right] dx_3 \\ A_2^1(x_1, x_i; F_1, F_2) &= n \int \left[\psi_{13} + \psi_{23}; F_3^1(x_1, x_2, x_3; F_1, F_2)\right] dx_3 \\ &\ \ \vdots \end{aligned} \tag{55}$$

III. COLLISION PROBLEM SOLUTION

In section 3.1 we shall use the results of section 2.3 in an equation for F_3^1. In section 3.2 we shall solve this equation and use it in a solution of the equation for F_2.

3.1 THE EQUATION FOR F_3^1

In order to derive an equation for $F_3^1(x_1, x_2, x_3; F_1, F_2)$, we start with Eq. 36 in which we wish to obtain an expansion for F_3 which is analogous to (37). In (36) we no longer require that ψ_{12} be of order g, and we assume that its time dependence occurs through a functional dependence on F_1 and F_2. Expand $\partial/\partial t$ as in (45), F_s as in (37), and F_{s+1}; then

$$(D_o + gD_1 + \ldots)\left(F_s^o + gF_s^1 + \ldots\right) = \left[H_s^o + \psi_{12}; F_s^o + gF_s^1 + \ldots\right] + g\left[\sum_{i<j}^{s\,'} \psi_{ij}; F_s^o + gF_s^1 + \ldots\right]$$
$$+ n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}^o + gF_{s+1}^1 + \ldots\right] dx_{s+1}.$$

Equating the first two powers of g, we obtain

$$D_o F_s^o = \left[H_s^o + \psi_{12}; F_s^o\right] + n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}^o\right] dx_{s+1} \tag{56}$$

and

$$D_o F_s^1 + D_1 F_s^o = \left[H_s^o + \psi_{12}; F_s^1\right] + \left[\sum_{i<j}^{s} \psi_{ij}; F_s^o\right] + n\int\left[\sum_{i=1}^{s} \psi_{is+1}; F_{s+1}^1\right] dx_{s+1}. \tag{57}$$

Since in the limit of infinite separation of all s particles except 1 and 2 the motions must become uncorrelated, we impose the following boundary conditions

$$\begin{aligned} &F_s^o(x_1, \ldots x_s; F_1, F_2) \xrightarrow[\substack{\text{all } \overline{q_j - q_i} \to \infty \\ \text{except } 12}]{} F_2(x_1, x_2)\prod_{i=3}^{s} F_1(x_i) \\ &F_s^i(x_1, \ldots x_s; F_1, F_2) \xrightarrow[\substack{\text{all } \overline{q_i - q_j} \to \infty \\ \text{except } 12}]{} 0 \qquad i > 1. \end{aligned} \tag{58}$$

The boundary condition for $F_2(x_1, x_2)$ will not be introduced here because it is not needed, since F_s^o has $F_2(x_1, x_2)$ in it. The boundary conditions are used only to show the asymptotic form of F_s^o.

The solution to (56) which matches these boundary conditions is

$$F_s^o(x_1,\ldots x_s;F_1,F_2) = F_2(x_1,x_2)\prod_{i=3}^{s} F_1(x_i).$$

Here, we have used the facts that

$$D_oF_2(x_1,x_2) = A_2^o(x_1,x_2;F_1,F_2)$$

and

$$D_oF_1(x_i) = A_1^o(x_i;F_1).$$

To solve (57), we must know the function $D_1F_s^o$. Using the section 2.2 prescription for D_1, we have

$$D_1F_s^o = \prod_{i=3}^{s} F_1(x_i)\, D_1F_2(x_1,x_2) + F_2(x_1,x_2)\sum_{i=3}^{s}\prod_{\substack{j=3\\ \neq i}}^{s} F_1(x_j)\, D_1F_1(x_i)$$

$$= n\prod_{i=3}^{s} F_1(x_i)\int\left[\psi_{1s+1}+\psi_{2s+1};F_3^1(x_1,x_2,x_{s+1};F_1,F_2)\right]dx_{s+1}$$

$$+ nF_2(x_1,x_2)\sum_{i=3}^{s}\prod_{\substack{j=3\\ \neq i}}^{s} F_1(x_j)\int\left[\psi_{is+1};f_2^1(x_i,x_{s+1};F_1)\right]dx_{s+1}. \qquad (59)$$

The implications of the discussion in section 2.3 are used. Since the i^{th} particle is assumed not to interact closely with the other particles, the function $f_2^1(x_i,x_{s+1})$ in (59) is the large-separation function of section 2.3.

Using (59) in (57) we arrive at the following expression for $D_oF_s^1$:

$$D_oF_s^1(x_1,\ldots x_s;F_1,F_2) = \left[H_s^o+\psi_{12};F_s^1(x_1,\ldots x_s;F_1,F_2)\right] + \left[\sum_{i<j}^{s}\psi_{ij};F_s^o(x_1,\ldots x_s;F_1,F_2)\right]$$

$$+ n\int\left[\sum_{i=1}^{s}\psi_{is+1};F_{s+1}^1(x_1\ldots x_s+1;F_1,F_2)\right]dx_{s+1}$$

$$- n\prod_{i=3}^{s} F_1(x_i)\int\left[\psi_{1s+1}+\psi_{2s+1};F_3^1(x_1,x_2,x_{s+1};F_1,F_2)\right]dx_{s+1}$$

$$- nF_2(x_1,x_2)\sum_{i=3}^{s}\prod_{\substack{j=3\\ \neq i}}^{s} F_1(x_j)\int\left[\psi_{is+1};f_2^1(x_i,x_{s+1})\right]dx_{s+1}. \qquad (60)$$

By direct substitution the solution to (60) is

$$F_s^1 = F_2(x_1, x_2) \sum_{3\leq i<j\leq s} \prod_{\substack{k=3\\ \neq i,j}}^{s} F_1(x_k)\, f_2^1(x_i, x_j)$$

$$+ \sum_{i=3}^{s} \prod_{\substack{k=3\\ \neq i}}^{s} F_1(x_k)\, F_3^1(x_1, x_2, x_i; F_1, F_2).$$

Here, F_3^1 is the solution to

$$\begin{aligned} D_o F_3^1(x_1, x_2, x_3; F_1, F_2) = & \left[H_3^o + \psi_{12}; F_3^1(x_1, x_2, x_3; F_1, F_2)\right] \\ & + n \int \left[\psi_{14} + \psi_{24} + \psi_{34}; F_1(x_4) F_3^1(x_1, x_2, x_3; F_1, F_2)\right] dx_4 \\ & + n \int \left[\psi_{34}; F_1(x_3) F_3^1(x_1, x_2, x_4; F_1, F_2)\right] dx_3 \\ & + [\psi_{13} + \psi_{23}; F_2(x_1, x_2) F_1(x_3)] \\ & + n \int \left[\psi_{14} + \psi_{24}; F_2(x_1, x_2) f_2^1(x_3, x_4; F_1)\right] dx_4. \end{aligned} \tag{61}$$

The first integral is zero for a uniform plasma, since it is the effect of the potential U in (41).

For convenience we put (61) into another form and thus give an analytical meaning to D_o.

Since for the uniform plasma the integral terms in $A_2^o(x_1, x_2)$ and $A_1^o(x_3)$ are zero, we have

$$\begin{aligned} D_o F_2(x_1, x_2) &= \left[H_2^o + \psi_{12}; F_2(x_1, x_2)\right] \\ D_o F_1(x_3) &= \left[H_1^o; F_1(x_3)\right]. \end{aligned} \tag{62}$$

Now introduce the operator $S_{-\tau}^1$ that projects particle 3 backward τ seconds along a path given by the free-particle Hamiltonian. For any function $\phi(x_3)$,

$$\frac{\partial}{\partial \tau} S_{-\tau}^1 \phi(x_3) = \left[H_1^o; S_{-\tau}^1 \phi(x_3)\right]. \tag{63}$$

Likewise introduce the operator $S_{-\tau}^2$ that projects particles 1 and 2 backward along the paths given by their Hamiltonian including interactions. In operating on the coordinates of a colliding pair at x_1 and x_2, this operator will produce the coordinates $x_1(-\tau)$ and $x_2(-\tau)$. In analogy with (63), we have

$$\frac{\partial}{\partial \tau} S_{-\tau}^2 \phi(x_1, x_2) = \left[H_2; S_{-\tau}^2 \phi(x_1, x_2)\right]. \tag{64}$$

Using $F_1(x_3)$ and $F_2(x_1, x_2)$ in (63) and (64), we have

$$\frac{\partial}{\partial\tau} S^1_{-\tau} F_1(x_3) = \left[H^o_1; S^1_{-\tau} F_1(x_3)\right]$$
$$\frac{\partial}{\partial\tau} S^2_{-\tau} F_2(x_1, x_2) = \left[H^o_2 + \psi_{12}; S^2_{-\tau} F_2(x_1, x_2)\right]. \tag{65}$$

Comparing (62) and (65), we have the identity

$$D_o F^1_3\left(x_1, x_2, x_3; S^1_{-\tau} F_1, S^2_{-\tau} F_2\right) = \frac{\partial}{\partial\tau} F^1_3\left(x_1, x_2, x_3; S^1_{-\tau} F_1, S^2_{-\tau} F_2\right); \tag{66}$$

that is, if in the functional dependence we use $S^1_{-\tau} F_1$ and $S^2_{-\tau} F_2$ instead of F_1 and F_2, the D_o operator can be replaced with $\partial/\partial\tau$ because the effect is the same. By using this relationship, (61) becomes

$$\begin{aligned}\frac{\partial}{\partial\tau} F^1_3\left(x_1, x_2, x_3; S^1_{-\tau} F_1, S^2_{-\tau} F_2\right) = &\left[H^o_3 + \psi_{12}; F^1_3\left(x_1, x_2, x_3; S^1_{-\tau} F_1, S^2_{-\tau} F_2\right)\right] \\ &+ n \int \left[\psi_{34}; S^1_{-\tau} F_1(x_3) F^1_3\left(x_1, x_2, x_4; S^1_{-\tau} F_1, S^2_{-\tau} F_2\right)\right] dx_4 \\ &+ \left[\psi_{13} + \psi_{23}; S^2_{-\tau} F_2(x_1, x_2) S^1_{-\tau} F_1(x_3)\right] \\ &+ n \int \left[\psi_{14} + \psi_{24}; S^2_{-\tau} F_2(x_1, x_2) f^1_2\left(x_3, x_4; S^1_{-\tau} F_1\right)\right] dx_4,\end{aligned} \tag{67}$$

where we have replaced F_1 by $S^1_{-\tau} F_1$ and F_2 by $S^2_{-\tau} F_2$.

Note that in (67) τ is not the time variable t. Here, t occurs as a parameter inside F_1 and F_2, and τ is a dummy variable introduced to give analytical meaning to D_o. Equation 67 holds for any value of τ; we shall pick the value of τ that is most convenient to us. Because of the adiabatic hypothesis and the implications of it, t lost its position as a variable and became a parameter in the equation for F_s. The t-dependence has become a functional dependence on F_1 and F_2.

In order to make the functional substitution clear, (67) is more general than is necessary. Since we have assumed that F_1 has no spacial dependence, we can use

$$S^1_{-\tau} F_1(\vec{p}_3) = F_1(\vec{p}_3) \tag{68}$$

throughout this report.

3.2 SOLUTION OF THE EQUATION FOR F^1_3

We shall now solve the equation for F^1_3 for use in the equation for F_2 without expanding F_2 in g; however, we must expand F_2 in order to obtain its solution. Some of these operator techniques were developed by Dupree.[11]

To proceed, go back to (79) and use (80), to obtain

$$\frac{\partial}{\partial\tau} F_3^1\left(x_1, x_2, x_3; F_1, S_{-\tau}^2 F_2\right) = \left[H_3^0 + \psi_{12}; F_3^1\left(x_1, x_2, x_3; F_1, S_{-\tau}^2 F_2\right)\right]$$

$$+ n \int \left[\psi_{34}; F_1(\vec{p}_3) F_3^1\left(x_1, x_2, x_4; F_1, S_{-\tau}^2 F_2\right)\right] dx_4$$

$$+ \left[\psi_{13} + \psi_{23}; S_{-\tau}^2 F_2(x_1, x_2) F_1(\vec{p}_3)\right]$$

$$+ n \int \left[\psi_{14} + \psi_{24}; S_{-\tau}^2 F_2(x_1, x_2) f_2^1(x_3, x_4; F_1)\right] dx_4. \tag{69}$$

For the moment abbreviate

$$F_3^1\left(x_1, x_2, x_3; F_1, S_{-\tau}^2 F_2\right) \equiv F_3^1(\tau),$$

where the other dependences are understood. Using (22), we may then write (69)

$$\frac{\partial}{\partial\tau} S_{+\tau}^2 F_3^1(\tau) + L S_{+\tau}^2 F_3^1(\tau) = S_{+\tau}^2 \phi(\tau). \tag{70}$$

Here, the operator L acting on any function $\phi(x_3)$ is

$$L\phi(x_3) \equiv -\left[H_1^0; \phi(x_3)\right] - \frac{\partial F_1(\vec{p}_3)}{\partial \vec{p}_3} \cdot \frac{\partial}{\partial \vec{q}_3} \int \psi_{34} \phi(x_4)\, dx_4, \tag{71}$$

and $\phi(\tau)$ is the source term made up of the last two terms of (69). Note that S_τ^2 and L commute since they operate on different coordinates. We call L the Landau operator since

$$\frac{\partial f(x_3, t)}{\partial t} + L f(x_3, t) = 0$$

is the equation involving L, which has been discussed in detail by Landau.[4]

Writing out the arguments, we obtain the formal solution of (70):

$$F_3^1\left(x_1, x_2, x_3; F_1, S_{-\tau}^2 F_2\right) = e^{-L\tau} S_{-\tau}^2 \int_0^\tau d\tau'\, e^{L\tau'} S_{\tau'}^2 \phi\left(x_1, x_2, x_3; F_1, S_{-\tau'}^2 F_2\right)$$

$$+ e^{-L\tau} S_{-\tau}^2 \phi(x_1, x_2, x_3; F_1, F_2), \tag{72}$$

where we use the fact that L does not depend on τ when F_1 has no $\vec{q}$-dependence.

In (72) let $F_2 \to S_{+\tau}^2 F_2$, use the inversion property of the $S_{-\tau}$ operator, and let $\tau' \to \tau - \tau'$, to obtain

$$F_3^1(x_1, x_2, x_3; F_1, F_2) = \int_0^\tau e^{-L\tau'} S_{-\tau'}^2 \phi\left(x_1, x_2, x_3; F_1, S_{\tau'}^2 F_2\right) d\tau'$$

$$+ e^{-L\tau} S_{-\tau}^2 \phi(x_1, x_2, x_3; F_1, F_2). \tag{73}$$

The left-hand side of (73) is independent of τ, and thus we are free to pick τ arbitrarily. We pick $\tau = \infty$ and thus remove the initial condition term because by (58) F_3^1 must go to zero for infinite separation of 1 and 2.

We then have a new understanding of the adiabatic hypothesis and the D_r expansion. We shall see that (73) implies that, to this order of g, the correlations are calculated by integrating along unperturbed (that is, zero-order) orbits while F_1 and F_2 are held constant. If F_1 and F_2 are known to change in times that are comparable to the time of build-up of correlations, for example in a nonuniform plasma, then this analysis is incorrect. The dummy variable τ gives us a way of studying this mathematically.

We must interpret the meaning of the operator $e^{-L\tau}$ in (73). For any $\phi(x_3)$, $e^{-L\tau}$ satisfies

$$\frac{\partial}{\partial\tau} e^{-L\tau} \phi(x_3) + L\, e^{-L\tau} \phi(x_3) \equiv 0. \tag{74}$$

For the moment let

$$h(x_3, \tau) \equiv e^{-L\tau} \phi(x_3)$$

for any $\phi(x_3)$, and let us represent by $\bar{\bar{h}}$ the Fourier-Laplace transform

$$\bar{\bar{h}}(\bar{p}_3, \bar{k}, \sigma) = \int_0^\infty d\tau\, e^{-\sigma\tau}\, \bar{h}(\bar{p}_3, \bar{k}, \tau) \tag{75}$$

where

$$\bar{h}(\bar{p}_3, \bar{k}, \tau) = \int d\bar{q}_3\, e^{i\bar{k}_o \bar{q}_3}\, h(\bar{p}_3, \bar{q}_3, \tau).$$

As shown by Landau,[4] (75) reduces (74) to

$$(\sigma - i\bar{k}\cdot\bar{v}_3)\, \bar{\bar{h}} + \frac{m\omega_p^{2i}}{k^2} \bar{k} \cdot \frac{\partial F_1(p_3)}{\partial \bar{p}_3} \int \bar{\bar{h}}\, d\bar{p}_3' = \bar{h}(\bar{p}_3, \bar{k}, 0) \tag{76}$$

or

$$\int \bar{\bar{h}}\, d\bar{p}_3 = \frac{\displaystyle\int \frac{\bar{h}(\bar{p}_3, \bar{k}, 0)}{\sigma - i\bar{k}\cdot\bar{v}_3}\, d\bar{p}_3}{\displaystyle 1 + m\frac{\omega_p^{2i}}{k^2} \int \frac{\bar{k}\cdot\dfrac{\partial F_1}{\partial \bar{p}_3'}}{\sigma - i\bar{k}\cdot\bar{v}_3'}\, d\bar{p}_3'}. \tag{77}$$

(We are actually interested only in the integral of $\bar{\bar{h}}$ over $d\bar{p}_3$.) Since Re $\sigma > 0$ in the definition of $\bar{\bar{h}}$ in (75), the integral in (77) is defined. To stay above the point $(-L\sigma)$, we take the same integral, with the $\bar{p}_3'$-integration deformed, to be the analytic continuation of the function Re $\sigma \leq 0$. All of this computation is identical to Landau's work.

Finally, inverting the transform, we obtain

$$\int h\, d\hat{p}_3 = \frac{1}{(2\pi)^4 i} \int_{-\infty i+\beta}^{\infty i+\beta} d\sigma\, e^{\sigma\tau} \int d\hat{k}\, e^{-i\hat{k}\cdot\hat{q}_3} \cdot \frac{\displaystyle\int \frac{\bar{h}(\hat{p}_3, \hat{k}, 0)}{\sigma - i\hat{k}\cdot\hat{v}_3}\, d\hat{p}_3}{1 + L + (\sigma)}, \tag{78}$$

where we define

$$L_{\pm}(\sigma) \equiv m \frac{i\omega_p^2}{k^2} \int_{\pm} \frac{\hat{k}\cdot \dfrac{\partial F_1(p_3')}{\partial \hat{p}_3'}}{\sigma - i\hat{k}\cdot\hat{v}_3}\, d\hat{p}_3. \tag{79}$$

The plus and minus signs indicate that the contour is to pass above or below the singularity.

Resorting to the definition of h, we have

$$\int e^{-L\tau}\, \phi(x_3)\, d\hat{p}_3 = \frac{1}{(2\pi)^4 i} \int_{-i\infty+\beta}^{i\infty+\beta} d\sigma\, e^{\sigma\tau} \int d\hat{k}\, e^{-i\hat{k}\cdot\hat{q}_3} \frac{\displaystyle\int \frac{\phi(\hat{k}, \hat{p}_3)}{\sigma - i\hat{k}\cdot\hat{v}_3'}\, d\hat{p}_3}{1 + L_+(\sigma)}.$$

This implies that

$$\int d\hat{p}_3\, F_3^1(x_1, x_2, x_3; F_1, F_2)$$

$$= \int d\hat{p}_3 \int_0^\infty d\tau\, e^{-L\tau}\, \phi\left(S_{-\tau}^2 x_1, x_2, x_3; F_1, F_2\right)$$

$$= \frac{1}{(2\pi)^4 i} \int_0^\infty d\tau \int_{-\infty i+\beta}^{\infty i+\beta} d\sigma\, e^{\sigma\tau} \int d\hat{k}\, e^{-i\hat{k}\cdot\hat{q}_3} \frac{\displaystyle\int \frac{\bar{\phi}\left(S_{-\tau}^2 x_1, x_2, \hat{k}, \hat{p}_3; F_1, F_2\right)}{\sigma - i\hat{k}\cdot\hat{v}_3}\, d\hat{p}_3}{1 + L_+(\sigma)} \tag{80}$$

where

$$\bar{\phi}\left(S_{-\tau}^2 x_1, x_2, \hat{k}, \hat{p}_3; F_1, F_2\right) = \int d\hat{q}_3\, e^{i\hat{k}\cdot\hat{q}_3}\, \phi\left(S_{-\tau}^2 x_1, x_2, x_3; F_1, F_2\right).$$

In (80) we have used the fact that

$$S_{-\tau}^2 \phi\left(x_1, x_2, x_3; F_1, S_\tau^2 F_2\right) = \phi\left(S_{-\tau}^2 x_1, x_2, x_3; F_1, F_2\right). \tag{81}$$

This follows, since $S_{-\tau}^2$ is defined to operate inside the function wherever x_1 and x_2 occur. In (81) we interpret the $S_{-\tau}^2$ as operating on those x_1 and x_2 lying outside the function $F_2(x_1, x_2)$.

To continue we must insert the form of $\overline{\Phi}$. Remember that we defined

$$\phi \equiv [\psi_{23}+\psi_{13}; F_2(x_1, x_2) F_1(\vec{p}_3)]$$

$$+ n \int \left[\psi_{14}+\psi_{24}; F_2(x_1, x_2) f_2^1(\vec{p}_3, \vec{p}_4, \overline{\vec{q}_2 - \vec{q}_4})\right] dx_4. \tag{82}$$

Using these definitions we obtain

$$\overline{\phi}(x_1, x_2, \vec{k}, \vec{p}_3; F_1, F_2)$$

$$= \frac{4\pi i e^2}{k^2} \vec{k} \cdot \left\{ \left(\vec{d}_{13} F_2(x_1, x_2)\ F_3(\vec{p}_3) + n \frac{\partial F_2(x_1, x_2)}{\partial \vec{p}_1} \int \bar{f}_2^1(\vec{p}_3, \vec{p}_4, \vec{k})\ d\vec{p}_4 \right) e^{i\vec{k}\cdot\vec{q}_1} + (1 \leftrightarrow 2) \right\}. \tag{83}$$

In (83) the notation $(1 \leftrightarrow 2)$ means that the previous term is repeated with 1 and 2 interchanged,

$$\vec{d}_{ij} \equiv \frac{\partial}{\partial \vec{p}_i} - \frac{\partial}{\partial \vec{p}_j},$$

and

$$\bar{f}_2^1(\vec{p}_3, \vec{p}_4, \vec{k}) \equiv \int d\vec{q}\ e^{i\vec{k}\cdot\vec{q}}\ f_2^1(\vec{p}_3, \vec{p}_4, \vec{q}).$$

If $\vec{q}_1(-\tau)$ is the position of particle 1 at $-\tau$, given that it was at $\vec{q}_1$ at $\tau = 0$, with the mechanism of the motion governed by the two-body collision, and similarly for $\vec{q}_2(-\tau)$, then (83) becomes

$$\overline{\Phi}\left(S_{-\tau}^2 x_1, x_2, \vec{k}, \vec{p}_3; F_1, F_2\right)$$

$$= \frac{4\pi i e^2}{k^2} k \cdot \left\{ \left(\vec{d}_{13} F_2(x_1, x_2)\ F_3(\vec{p}_3) + n \frac{\partial F_2(x_1, x_2)}{\partial \vec{p}_1} \int \bar{f}_2^1(\vec{p}_3, \vec{p}_4, \vec{k})\ d\vec{p}_4 \right) e^{i\vec{k}\cdot\vec{q}_1(-\tau)} + (1 \leftrightarrow 2) \right\}. \tag{84}$$

Let us now discuss the interpretation of some of the integrals appearing in (80). Note that the τ-dependence in $\overline{\Phi}$ is in the form of an exponential with an imaginary argument. Thus the τ-integration can proceed only if β is negative. Since the inversion in the σ-plane must go to the right of all singularities, β can be negative only if all of the zeros of the denominator lie to the left of the imaginary axis. For a wide range of $F_1(\vec{p}_3)$ this is true as long as $|k| > 0$.[13] Thus we exclude an infinitesimal region from the origin of $\vec{k}$. Under these restrictions we may proceed with the τ-integration, followed by the σ, then by the $\vec{k}$.

Those distribution functions that give zeros of $1 + L_+(\sigma)$ in the right-half plane present a difficult problem. For these situations the past is not damped out, and the current value of F_3^1 must depend on the initial conditions — a situation that is incompatible with the adiabatic hypothesis.[11] It would appear that the hypothesis is invalid for plasmas containing these instabilities. In order to proceed we shall limit ourselves to the more well behaved functions F_1. As Baccus shows,[13] these are all single-humped momentum distributions.

A physical understanding of why some distributions $F_1(\vec{p})$ give rise to damped solutions and others do not can be seen from the arguments used to explain Landau damping and plasma instabilities.[14] Because the shielding integral (the L operator) in (70) is the same as that considered by Landau[4] in his use of the Vlasov equation, we obtained Landau's characteristic denominator $1 + L_+(\sigma)$ and can use his analysis of the study of waves to study the growth of correlations.

For the $\vec{k}^{th}$ partial wave in the analysis of wave motion in a plasma, those particles traveling at the phase velocity of the wave will see a constant field. If

$$\vec{k} \cdot \frac{\partial}{\partial \vec{p}} F_1(\vec{p}) \Big|_{v=\sigma/ik}$$

is negative, more particles will be speeded up than are slowed down, and damping of the wave will result. If this derivative is positive, energy will be fed from the particles to the wave, and a growing wave will result.

In the problem considered, the plasma is homogeneous and the $\vec{k}$-analysis refers to the coordinates $\overline{q_1 - q_3}$ and $\overline{q_2 - q_3}$. Whereas for waves we had damped or increasing energy, in the problem considered we have decreasing or increasing correlation.

Using the form of ϕ in (69) and the meaning of the $S^2_{-\tau}$ operators, we have

$$\phi\left(S^2_{-\tau}x_1, x_2, x_3, F_1, F_2\right) = \left[S^2_{-\tau}(\psi_{13}+\psi_{23}); F_2(x_1, x_2)F_1(x_3)\right]$$

$$+ n \int \left[S^2_{-\tau}(\psi_{14}+\psi_{24}); F_2(x_1, x_2)f_2^1(x_3, x_4)\right] dx_4. \qquad (85)$$

Throughout this analysis we have assumed that we never have a three-body collision, that is, that the interactions of all pairs except 1 and 2 are of order g. This assumption implies that the distances $|\overrightarrow{q_1 - q_3}|$, $|\overrightarrow{q_2 - q_3}|$, $|\overrightarrow{q_1 - q_4}|$, and $|\overrightarrow{q_2 - q_4}|$ are large in the sense of the g expansion. The $S_{-\tau}$ operators in (85) have the effect of

$$S^2_{-\tau}\psi_{13}(\overrightarrow{q_1 - q_3}) = \psi_{13}(\overrightarrow{q_1(-\tau) - q_3}). \qquad (86)$$

The change in ψ_{13} will become important only when $|\overrightarrow{q_1(-\tau) - q_1}|$ becomes of the order of $|\overrightarrow{q_1 - q_3}|$. By assumption this change takes a long time, a time that is sufficient for 1 and 2 to be far apart. For example, we assume that $|\overrightarrow{q_2 - q_3}|$ and $|\overrightarrow{q_2 - q_3}| \geqslant \sqrt{e^2\lambda_D/kT}$. At this separation the potential ψ_{12} has the magnitude

$$\psi_{12}\left(\sqrt{\frac{e^2\lambda_D}{kT}}\right) \cong kT\sqrt{\frac{e^2}{kT\lambda_D}}.$$

which is much smaller than the average kinetic energy kT. Thus for those values of τ in (86) which have any effect, 1 and 2 can be considered to be interacting only weakly. In (86) we will specifically use the fact that as q_1 is projected backward in time, the difference $|\overline{q_1 - q_3}|$ is relatively insensitive to the change in $\vec{q}_1$ while $\vec{q}_1$ is in the immediate vicinity of $\vec{q}_2$. A similar argument holds for ψ_{23} by interchanging 1 and 2.

These statements can be made quantitative by introducing in (86) the operator $\mathring{S}^2_{-\tau}$ that projects 1 and 2 backward with zero interaction, that is, along straight lines and with constant velocity,

$$S^2_{-\tau}\psi_{13}(\overline{q_1 - q_3}) \equiv S^2_{-\tau}\mathring{S}^2_{\tau}\mathring{S}^2_{-\tau}\psi_{13}(\overline{q_1 - q_3}) = S^2_{-\tau}\mathring{S}^2_{\tau}\psi_{13}(\overline{q_1 - v_1\tau - q_3}). \tag{87}$$

As we shall see below, for large values of τ, the product $S^2_{-\tau}\mathring{S}^2_{\tau}$ becomes stationary, independent of τ. This fact, coupled with the preceding discussion, will be used to approximate (87).

To understand the foregoing assertions, consider the operation on the velocity of 1 or 2

$$\lim_{\tau \to \infty} S^2_{-\tau}\vec{v}_i .$$

This will approach a constant limit defined as

$$\lim_{\tau \to \infty} S^2_{-\tau}\vec{v}_i \equiv \vec{V}_i(x_1, x_2).$$

By its definition $\vec{V}_i(\vec{P}_i = m\vec{V}_i)$ is the velocity (momentum) that the i^{th} particle had in the distant past before undergoing the two-body collision, given that the colliding particles have current coordinates x_1 and x_2. For this reason we shall refer to it as the precollision velocity (momentum).

Equation 88 proceeds at such a rate that

$$\lim_{\tau \to \infty} \tau\left(\vec{V}_i - S^2_{-\tau}\vec{v}_i\right) \to 0.$$

This condition ensures the existence of the limit

$$\lim_{\tau \to \infty}\left(S^2_{-\tau}\vec{q}_i + \vec{V}_i\tau\right) = \lim_{\tau \to \infty}\left(\vec{q}_i + \int_0^{\tau}\left(\vec{V}_i - S^2_{-\tau'}\vec{v}_i\right)d\tau'\right) \to \vec{Q}_i$$

where

$$\vec{Q}_i(x_1, x_2) \equiv \vec{q}_i + \int_0^{\infty}\left(\vec{V}_i - S^2_{-\tau}\vec{v}_i\right)d\tau .$$

To find the position $Q_i(x_1, x_2)$, start at the point $\vec{q}_i$, project backward τ seconds along the trajectories of a two-body interaction, and project forward τ seconds with the constant precollision velocity $\vec{V}_i$. In the limit $\tau \to \infty$, this procedure will give the position that the i^{th} particle would have had with no collision interaction. For this reason we call $\vec{Q}_i$ the undeflected position.

Now consider for any $\phi(x_1, x_2)$

$$\lim_{\tau \to \infty} S^2_{-\tau} \mathring{S}^2_\tau \phi(\vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2)$$

$$= \lim_{\tau \to \infty} S^2_{-\tau} \phi(\vec{p}_1, \vec{p}_2, \overrightarrow{q_1 - v_1 \tau}, \overrightarrow{q_2 - v_2 \tau})$$

$$= \lim_{\tau \to \infty} \phi\left(S^2_{-\tau}\vec{p}_1, S^2_{-\tau}\vec{p}_2, q_1 + \int_0^\tau \left(S^2_{-\tau} v_1 - S^2_{-\tau'} v_1\right) d\tau', \; q_2 + \int_0^\tau \left(S^2_{-\tau}\vec{v}_2 - S^2_{-\tau'} v_2\right) d\tau'\right)$$

$$= \phi(\vec{P}_1, \vec{P}_2, \vec{Q}_1, \vec{Q}_2)$$

$$= \phi(X_1, X_2).$$

We define the 6-dimensional vector $X_i = \{P_i, Q_i\}$.

Notice that the transformation from (x_1, x_2) to (X_1, X_2) is a contact transformation, since it is obtained from a product of transformations governed by Hamiltonians. This property will be useful because of the invariance of the Poisson brackets with respect to a contact transformation.

To approximate the product $S^2_{-\tau} \mathring{S}^2_\tau$ on any function $\phi(x_1, x_2)$, we may expand the result around $\tau = \infty$:

$$S^2_{-\tau} \mathring{S}^2_\tau \phi(x_1, x_2) = \left\{1 + \left(S^2_{-\tau}\vec{p}_1 - \vec{P}_1\right) \cdot \frac{\partial}{\partial \vec{P}_1} + \left(S^2_{-\tau}\vec{p}_2 - \vec{P}_2\right) \cdot \frac{\partial}{\partial \vec{P}_2} + \left(S^2_{-\tau} \mathring{S}^2_\tau \vec{q}_1 - Q_1\right) \cdot \frac{\partial}{\partial \vec{Q}_1} + \left(S^2_{-\tau} \mathring{S}^2_\tau \vec{q}_2 - \vec{Q}_2\right) \cdot \frac{\partial}{\partial \vec{Q}_2}\right\} S^2_{-\infty} \mathring{S}^2_\infty \phi(x_1, x_2) + \text{higher-order terms.}$$

Application of this to $\psi_{13}(\overrightarrow{q_1 - v_1 \tau - q_3})$ yields

$$S^2_{-\tau} \mathring{S}^2_\tau \psi_{13}(\overrightarrow{q_1 - v_1 \tau - q_3}) \cong \psi_{13}(\overrightarrow{Q_1 - V_1 \tau - q_3}) + (S_{-\tau}\vec{q}_1 - (\overrightarrow{Q_1 - V_1 \tau})) \cdot \frac{\partial}{\partial Q_1} \psi_{13}(\overrightarrow{Q_1 - V_1 \tau - q_3}) + \dots .$$

The coefficient of the second term is, as a function of τ, the deflection of particle 1 along the path of its collision with 2. By using the specific form of ψ, the ratio of the second to the first term is of order

$$\frac{|S^2_{-\tau}\vec{q}_1 - (\overrightarrow{Q_1 - V_1 \tau})|}{|\vec{Q}_1 - \vec{V}_1 \tau - \vec{q}_3|} \leqslant \frac{|\overrightarrow{q_1 - Q_1}|}{|\overrightarrow{Q_1 - q_3}|}. \tag{89}$$

The quantity $|\overrightarrow{Q_1 - q_3}|$ is large for the same reason that $|\overrightarrow{q_1 - q_3}|$ is large, and it is of the same order of magnitude. The ratio is then small for all situations except those in which 1 and 2 are widely separated but have had a collision in the past (that is, $|\overrightarrow{q_1 - Q_1}|$

is large enough to be in the range $|\overline{Q_1 - q_3}|$). For 1 and 2 close together and 1 and 2 experiencing a close collision, the ratio (89) is small.

We shall ignore the exception noted above and assume that the ratio is small for all situations of interest. We shall see that those situations for which (89) is not small have negligible contribution to the problems of our interest.

These approximations are in the source term of F_3^1. If we were to consider corrections, they would occur as new (small) source terms and be additive.

We shall then use the fact that the approximation

$$S^2_{-\tau}\psi_{13}(\overline{q_1 - q_3}) \cong \psi_{13}(\hat{Q}_1 - \hat{V}_1\tau - \hat{q}_3) \tag{90}$$

is a valid outcome of the g expansion; and similarly for $S^2_{-\tau}\psi_{23}$. Since in the integration over dx_4 in the source term ϕ we assumed that $|\overline{q_1 - q_4}|$ and $|\overline{q_2 - q_4}|$ are large (that is, we cut off the integral), we can make similar expansions for $S^2_{-\tau}\psi_{14}$ and $S^2_{-\tau}\psi_{24}$.

This step represents the real departure from the analysis when all particle interactions are assumed small. Rather than the approximation (90), these treatments have implicitly used the approximation

$$S^2_{-\tau}\dot{S}^2_{-\tau} \cong 1$$

or

$$S^2_{-\tau}\psi_{13}(\overline{q_1 - q_3}) \cong \psi_{13}(\overline{q_1 - v_1\tau - q_3}). \tag{91}$$

By the discussion above we see that these approximations are valid only when ψ_{12} is weak at present and in the past. Use of (91) and the following procedure would lead directly to the large-separation solution.

We may then use these approximations in (84). Since we effectively make the transformation $\hat{q}_i \to \hat{Q}_i$ inside ψ, to retain the form of the Poisson brackets we must transform $\partial/\partial\hat{p}_i \to \partial/\partial\hat{P}_i$. With these changes, (84) becomes

$$\bar{\phi}\left(S^2_{-\tau}x_1, x_2, \hat{k}, \hat{p}_3; F_1, F_2\right) \cong \frac{4\pi i e^2}{k^2}\hat{k}\cdot\left\{\left(\hat{D}_{13}F_2(x_1, x_2)\, F_1(\hat{p}_3) + n\frac{\partial F_2(x_1, x_2)}{\partial\hat{P}_1}\int \bar{f}_2^1(\hat{p}_3, \hat{p}_4, \hat{k})\, d\hat{p}_4\right) e^{i\hat{k}\cdot(\hat{Q} - \hat{V}_1\tau)} + (1 \leftrightarrow 2)\right\} \tag{92}$$

where $\hat{D}_{13} \equiv \frac{\partial}{\partial\hat{P}_1} - \frac{\partial}{\partial\hat{p}_3}$ and $\hat{D}_{23} \equiv \frac{\partial}{\partial\hat{P}_2} - \frac{\partial}{\partial\hat{p}_3}$.

Using (92) in (80), we obtain another expression for F_3^1,

$$\int F_3^1(x_1, x_2, x_3; F_1, F_2)\, d\hat{p}_3 = \frac{4\pi e^2}{(2\pi)^4}\int_0^\infty d\tau \int_{-i\infty+\beta}^{i\infty+\beta} d\sigma\, e^{\sigma\tau}\int \frac{d\hat{k}\, e^{-i\hat{k}\cdot\hat{q}_3}}{k^2}\, \frac{1}{1 + L_+(\sigma)}\int \frac{d\hat{p}_3}{\sigma - i\hat{k}\cdot\hat{v}_3}$$

$$\vec{k}\cdot\left\{\left(\vec{D}_{13}F_2(x_1,x_2)\,F_1(\vec{p}_3) + n\,\frac{\partial F_2(x_1,x_2)}{\partial\vec{P}_1}\int \bar{f}_2^1(\vec{p}_3,\vec{p}_4,\vec{k})\,d\vec{p}_4\right) e^{i\vec{k}\cdot(\overrightarrow{Q_1-V_1\tau})} + (1\leftrightarrow 2)\right\}. \tag{93}$$

If $\beta < 0$, we may carry out the τ-integration, which will bring down $(\sigma-i\vec{k}\cdot\vec{V}_1)^{-1}$ and $(\sigma-i\vec{k}\cdot\vec{V}_2)^{-1}$ in the two terms. Note that the approximation (90) was made to make the τ-integration possible. For those functions $F_1(\vec{p}_3)$ for which $1 + L_+(\sigma)$ has no poles in the right-half plane, the σ-integration can be closed in the right-half plane and will only enclose poles at $i\vec{k}\cdot\vec{V}_1$ and $i\vec{k}\cdot\vec{V}_2$. Carrying out these two steps, we obtain

$$\int F_3^1(x_1,x_2,x_3;F_1,F_2)\,d\vec{p}_3$$

$$= \frac{4\pi e^2 i}{(2\pi)^3}\int \frac{d\vec{k}\,e^{-i\vec{k}\cdot\vec{q}_3}}{k^2}\,\vec{k}\cdot\left\{\int_+ \frac{d\vec{p}_3}{\vec{k}\cdot(\overrightarrow{V_1-v_3})}\left(\vec{D}_{13}F_2(x_1,x_2)\,F_1(\vec{p}_3)\right.\right.$$

$$\left.\left. + n\,\frac{\partial F_2(x_1,x_2)}{\partial\vec{P}_1}\int \bar{f}_2^1(\vec{p}_3,\vec{p}_4,\vec{k})\,d\vec{p}_4\right)\frac{e^{i\vec{k}\cdot\vec{Q}_1}}{1+L_+(i\vec{k}\cdot\vec{V}_1)} + (1\leftrightarrow 2)\right\}. \tag{94}$$

The $\int_+$ means that the $\vec{p}_3$-integration is to stay above the poles, since $(-i\sigma)$ was to be below the axis.

We wish to use this result in an integral of the form

$$n\int\left[\psi_{13}(\overrightarrow{q_1-q_3})+\psi_{23}(\overrightarrow{q_2-q_3});F_3^1(x_1,x_2,x_3;F_1,F_2)\right]dx_3,$$

where we are faced with the integration of a coordinate that is assumed to be far removed from $\vec{q}_1$ and $\vec{q}_2$. Accordingly, we may approximate under the integral

$$n\int\left[\psi_{13}(\overrightarrow{Q_1-q_3})+\psi_{23}(\overrightarrow{Q_2-q_3});F_3^1(x_1,x_2,x_3;F_1,F_2)\right]dx_3. \tag{95}$$

This approximation can be analyzed exactly as before, that is,

$$\psi_{13}(\overrightarrow{Q_1-q_3}) \cong \psi_{13}(\overrightarrow{q_1-q_3}) + (\overrightarrow{Q_1-q_1})\cdot\frac{\partial}{\partial\vec{q}_1}\psi_{13}(\overrightarrow{q_1-q_3}) + \ldots\,.$$

The error term is of order

$$\frac{|\overrightarrow{Q_1-q_1}|}{|\overrightarrow{q_1-q_3}|},$$

which has the same characteristics as (89).

Putting (94) into (95) yields the interaction integral in the equation for F_2:

$$\frac{\partial F_2(x_1,x_2)}{\partial t} = \left[H_2^o+\psi_{12};F_2(x_1,x_2)\right] - g\,\frac{n(4\pi e^2)^2\, i}{(2\pi)^3}\int \frac{d\vec{k}}{k^2}\vec{k}\cdot$$

$$\left\{\frac{\partial}{\partial \vec{P}_1}\,\frac{1}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\int_+ \frac{d\vec{p}_3}{\vec{k}\cdot(\overline{V_1-v_3})}\,k\cdot\left(\vec{D}_{13}F_2(x_1,x_2)\,F_1(\vec{p}_3)\right.\right.$$

$$\left. + n\,\frac{\partial F_2(x_1,x_2)}{\partial \vec{P}_1}\int \bar{f}_2^1(\vec{p}_3,\vec{p}_4,\vec{k})\,d\vec{p}_4\right)$$

$$+ e^{i\vec{k}\cdot(\overline{Q_2-Q_1})}\left[\frac{\partial}{\partial \vec{P}_2}\,\frac{1}{(1+L_+(i\vec{k}\cdot\vec{V}_1))}\int_+ \frac{d\vec{p}_3}{\vec{k}\cdot(\overline{V_1-v_3})}\,\vec{k}\cdot\left(\vec{D}_{13}F_2(x_1,x_2)\,F_1(\vec{p}_3)\right.\right.$$

$$\left.\left.\left. + n\,\frac{\partial F_2(x_1,x_2)}{\partial \vec{P}_1}\int \bar{f}_2^1(\vec{p}_3,\vec{p}_4,\vec{k})\,d\vec{p}_4\right)\right] + (1\leftrightarrow 2)\right\}. \tag{96}$$

This is an equation for $\partial F_2/\partial t$ to first order in g. It is complicated but will yield some interesting insights. The integral terms can be simplified somewhat by examining the equation for f_2^1.

Let us rewrite (53) for a uniform plasma by using $F_2^1(x_3,x_1;F_1) \to f_2^1(x_3,x_1)$ and the fact that f_2^1 depends only on the difference of spacial coordinates $\vec{q} = \overline{q_3 - q_1}$; we obtain

$$D_o f_2^1(\vec{p}_3,\vec{p}_1,\vec{q};F_1) + \frac{(\overline{p_3-p_1})}{m}\cdot\frac{\partial}{\partial \vec{q}}\,f_2^1(\vec{p}_3,\vec{p}_1,\vec{q};F_1)$$

$$+ n\,\frac{\partial F_1(\vec{p}_1)}{\partial \vec{p}_1}\cdot\frac{\partial}{\partial \vec{q}}\int d\vec{p}_5\,d\vec{q}^{\,1}\,\psi(\overline{q-q^1})\,f_2^1(\vec{p}_3,\vec{p}_5,\vec{q}^{\,1};F_1)$$

$$- n\,\frac{\partial F_1(\vec{p}_2)}{\partial \vec{p}_3}\cdot\frac{\partial}{\partial \vec{q}}\int d\vec{p}_5\,d\vec{q}^{\,1}\,\psi(\overline{q-q^1})\,f_2^1(\vec{p}_5,\vec{p}_1,\vec{q}^{\,1})$$

$$= \frac{\partial}{\partial \vec{q}}\,\psi(\vec{q})\cdot d_{31}F_1(\vec{p}_3)\,F_1(\vec{p}_1). \tag{97}$$

Again we introduce $f_2^1(\vec{p}_3,\vec{p}_1,\vec{q};S_{-\tau}^1F_1)$ and use the fact that $D_o \to \partial/\partial\tau$. We are again free to pick the value of τ, since it is a dummy variable. However, since τ appears in the unknown function, its value must be selected carefully. We shall pick $\tau \to \infty$ but do so in the Laplace-transform space by letting the transform variable $\sigma \to 0$. This will also set $\partial f_2^1/\partial\tau$ equal to zero, as it must since $D_oF_1 = 0$. This technique can be used because of the fact that for

$$\bar{g}(\sigma) = \int_0^\infty g(\tau)\,e^{-\sigma\tau}\,d\tau, \tag{98}$$

$$\lim_{\sigma\to 0}\sigma\bar{g}(\sigma) = g(0),$$

and

$$f_2^1\left(\vec{p}_3,\vec{p}_1,\vec{q};S^1_{-\tau}F_1\right) \xrightarrow[\tau\to 0]{} f_2^1(\vec{p}_3,\vec{p}_1,\vec{q};F_1).$$

The initial condition term will also disappear in the limit $\sigma \to 0$.

Putting (98) into (97) and carrying out a Fourier transform in space as in (75), we have

$$\lim_{\epsilon\to 0}\;(\epsilon - i\vec{k}\cdot(\overline{v_3 - v_1}))\,\bar{f}_2^1(\vec{p}_3,\vec{p}_1,\vec{k}) + \frac{4\pi n\, e^2\, i}{k^2}\vec{k}\cdot\frac{\partial F_1(\vec{p}_3)}{\partial\vec{p}_3}\int d\vec{p}_5 \bar{f}_2^1(\vec{p}_5,\vec{p}_1,\vec{k})$$

$$-\frac{4\pi n\, e^2\, i}{k^2}\vec{k}\cdot\frac{\partial F_1(\vec{p}_1)}{\partial\vec{p}_1}\int d\vec{p}_5 \bar{f}_2^1(\vec{p}_3,\vec{p}_5,\vec{k})$$

$$= -\frac{4\pi\, e^2\, i}{k^2}\vec{k}\cdot d_{31}F_1(\vec{p}_3)\,F_1(\vec{p}_1). \tag{99}$$

Dividing by $(\epsilon - i\vec{k}\cdot(\overline{v_3 - v_1}))$ and then integrating over $\vec{p}_3$ yields

$$\int \bar{f}_2^1(\vec{p}_3,\vec{p}_1,\vec{k})\,d\vec{p}_3 = \lim_{\epsilon\to 0}\frac{4\pi\, e^2}{k^2}\left\{ n\int\frac{\vec{k}\cdot\dfrac{\partial F_1(\vec{p}_3)}{\partial\vec{p}_3}}{\vec{k}\cdot(\overline{v_3 - v_1}) + i\epsilon}\,d\vec{p}_3\int d\vec{p}_5\bar{f}_2^1(\vec{p}_5,\vec{p}_1,\vec{k})\right.$$

$$\left. - n\vec{k}\cdot\frac{\partial F_1(\vec{p}_1)}{\partial\vec{p}_1}\int\frac{d\vec{p}_5\,d\vec{p}_3\bar{f}_2^1(\vec{p}_3,\vec{p}_5,\vec{k})}{\vec{k}\cdot(\vec{v}_3 - \vec{v}_1) + i\epsilon} + \int\frac{\vec{k}\cdot\vec{d}_{31}F_1(\vec{p}_3)\,F_1(\vec{p}_1)}{\vec{k}\cdot(\overline{v_3 - v_1}) + i\epsilon}\,d\vec{p}_3\right\}.$$

Rearranging terms and taking the limit $\epsilon \to 0$, we have

$$\frac{m\omega_p^2}{k^2}\vec{k}\cdot\frac{\partial F_1(\vec{p}_1)}{\partial\vec{p}_1}\int_+\frac{\left(F_1(\vec{p}_3) + n\int f_2^1(\vec{p}_3,\vec{p}_5,\vec{k})\,d\vec{p}_5\right)dp_3}{\vec{k}\cdot(\overline{v_1 - v_3})}$$

$$= n\int\bar{f}_2^1(\vec{p}_3,\vec{p}_1,\vec{k})\,d\vec{p}_3 + \frac{F_1(\vec{p}_1)\,L_+(i\vec{k}\cdot\vec{v}_1)}{1 + L_+(i\vec{k}\cdot\vec{v}_1)}. \tag{100}$$

We may use (100), together with $\vec{v}_1 \to \vec{V}_1$ and $\vec{v}_2 \to \vec{V}_2$, in (96) to obtain

$$\frac{\partial F_2(x_1,x_2)}{\partial t} = \left[H_2^0 + \psi_{12};F_2(x_1,x_2)\right] - g\,\frac{4\pi\, e^2\, i}{(2\pi)^3}\int\frac{d\vec{k}}{k^2}\vec{k}\cdot\left\{\frac{\partial}{\partial\vec{P}_1}\left[\left(n\int\bar{f}_2^1(\vec{p}_3,\vec{P}_1,\vec{k})\,d\vec{p}_3\right.\right.\right.$$

$$+\frac{F_1(\vec{P}_1)\,L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\Bigg)\frac{1}{\vec{k}\cdot\dfrac{\partial F_1(\vec{P}_1)}{\partial\vec{P}_1}}\,\vec{k}\cdot\frac{\partial F_2(x_1,x_2)}{\partial\vec{P}_1}$$

$$-\frac{F_2(x_1,x_2)\,L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\Bigg]+e^{ik\cdot(Q_2-Q_1)}\Bigg[\Bigg\{n\int f_2^1(\vec{p}_3,\vec{P}_1,\vec{k})\,d\vec{p}_3$$

$$+\frac{F_1(\vec{P}_1)\,L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\Bigg\}\frac{1}{\vec{k}\cdot\dfrac{\partial F_1(\vec{P}_1)}{\partial\vec{P}_1}}\,\frac{\partial}{\partial\vec{P}_2}\left(\vec{k}\cdot\frac{\partial}{\partial\vec{P}_1}F_2(x_1,x_2)\right)$$

$$-\frac{L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\,\frac{\partial F_2(x_1,x_2)}{\partial\vec{P}_2}\Bigg]+(1\longleftrightarrow 2)\Bigg\}. \tag{101}$$

This form of the equation for F_2 is still very complex; however, with it we can easily obtain results to be compared with (17). Obtaining these results will entail making an adiabatic hypothesis for F_2 in the sense that we now assume that the time dependence of F_2 resides within a functional dependence of F_1.

In line with this hypothesis, we expand

$$F_2(x_1,x_2;F_1)=F_2^o(x_1,x_2;F_1)+gF_2^1(x_1,x_2;F_1)+\dots$$

and introduce D_r operators that replace $\partial F_1/\partial t$ with A_1^r in (96) and then we equate equal powers of g, for the first two powers, to obtain

$$D_oF_2^o(x_1,x_2;F_1)=\left[H_2^o+\psi_{12};F_2^o(x_1,x_2;F_1)\right] \tag{102}$$

and

$$D_1F_2^o(x_1,x_2;F_1)+D_oF_2^1(x_1,x_2;F_1)=\left[H_2^o+\psi_{12};F_2^1(x_1,x_2;F_1)\right]$$

$$-\frac{4\pi e^2 i}{(2\pi)^3}\int\frac{d\vec{k}}{k^2}\,\vec{k}\cdot\Bigg\{\frac{\partial}{\partial\vec{P}_1}\Bigg[\Bigg\{n\int \bar{f}_2^1(\vec{p}_3,\vec{P}_1,\vec{k})\,d\vec{p}_3$$

$$+\frac{F_1(\vec{P}_1)\,L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\Bigg\}\frac{1}{\vec{k}\cdot\dfrac{\partial F_1(\vec{P}_1)}{\partial\vec{P}_1}}\,\vec{k}\cdot\frac{\partial F_2^o\left(x_1,x_2^2\right)}{\partial\vec{P}_1}$$

$$-\frac{F_2^o(x_1,x_2)\,L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\Bigg]+e^{i\vec{k}\cdot(\overrightarrow{Q_2-Q_1})}\Bigg[\Bigg\{n\int \bar{f}_2^1(\vec{p}_3,\vec{P}_1,\vec{k})\,d\vec{p}_3$$

$$+\frac{F_1(\vec{P}_1)\,L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\Bigg\}\frac{1}{k\cdot\frac{\partial F_1(\vec{P}_1)}{\partial\vec{P}_1}}\frac{\partial}{\partial\vec{P}_2}\left(\vec{k}\cdot\frac{\partial}{\partial\vec{P}_1}F_2^o(x_1,x_2)\right)$$

$$-\frac{L_+(i\vec{k}\cdot\vec{V}_1)}{1+L_+(i\vec{k}\cdot\vec{V}_1)}\frac{\partial F_2^o(x_1,x_2)}{\partial\vec{P}_2}\Bigg]+(1\longleftrightarrow 2)\Bigg\}. \tag{103}$$

Let us first discuss the solution to (102). By the same procedure that we used in section 3.1 we can replace the D_o operator and obtain

$$\frac{\partial}{\partial t}F_2^o(x_1,x_2;S_{-\tau}^1F_1)=\left[H_2^o+\psi_{12};F_2^o(x_1,x_2;S_{-\tau}^1F_1)\right].$$

By (26) the solution to this is

$$F_2^o(x_1,x_2;S_{-\tau}^1F_1)=S_{-\tau}^2F_2^o(x_1,x_2;F_1)$$

or, letting $F_1\to S_{+\tau}^1F_1$ in the functional dependence, we have the solution

$$F_2^o(x_1,x_2;F_1)=S_{-\tau}^2F_2^o(x_1,x_2;S_\tau^1F_1). \tag{104}$$

Since the left-hand side is independent of τ, we may again pick the value of τ arbitrarily. We pick $\tau\to\infty$ and use the boundary condition that is similar to (49)

$$F_2^o(x_1,x_2;F_1)\xrightarrow[|q_1-q_2|\to\infty]{}F_1(x_1)\,F_1(x_2). \tag{105}$$

Using these, we obtain

$$\begin{aligned}F_2^o(x_1,x_2;F_1)&=\lim_{\tau\to\infty}S_{-\tau}^2\mathring{S}_\tau^2F_1(x_1)F_1(x_2)\\&=F_1(X_1)\,F_1(X_2).\end{aligned} \tag{106}$$

For the homogeneous plasma

$$F_2^o(x_1,x_2;F_1)=F_1(\vec{P}_1)\,F_1(\vec{P}_2).$$

By the definition of the D_1 operator we have

$$\begin{aligned}D_1F_2^o&=D_1F_1(\vec{P}_1)\,F_1(\vec{P}_2)\\&=nF_1(\vec{P}_2)\int\left[\psi_{13}(\vec{Q}_1-\vec{q}_3);f_2^1(\vec{P}_1,\vec{p}_3,\overrightarrow{Q_1-q_3})\right]dx_3\\&\quad+nF_1(\vec{P}_1)\int\left[\psi_{23}(\overrightarrow{Q_2-q_3});f_2^1(\vec{P}_2,\vec{p}_3,\overrightarrow{Q_2-q_3})\right]dx_3\end{aligned}$$

$$= -\frac{4\pi e^2 i}{(2\pi)^3} F_1(P_2) \int \frac{d\vec{k}}{k^2} \vec{k} \cdot \frac{\partial}{\partial \vec{P}_1} \int \bar{f}_2^1(\vec{P}_1, \vec{p}_3, \vec{k}) \, dp_3 + (1 \longleftrightarrow 2). \tag{107}$$

Upon inserting F_2^o into (103), we see that (107) just cancels the two integral terms that have no $(\overrightarrow{Q_1 - Q_2})$-dependence. This cancellation is the plausible way of avoiding the secondary divergence, mentioned with (19), which results from an integration over $\vec{q}_3$. The subtraction of $D_1 F_2^o(x_1, x_2)$ involves the difference of two integrals that are to be cut off. This removes the dependence on the cutoff distance in the equation for $D_o F_2^1$.

Thus (103) becomes

$$\begin{aligned} D_o F_2^1(x_1, x_2; F_1) = &\left[H_2^o + \psi_{12};\ F_2^1(x_1, x_2; F_1) \right] \\ &- n \frac{4\pi e^2}{(2\pi)^3} i \, \frac{\partial F_1(\vec{P}_2)}{\partial \vec{P}_2} \cdot \int \frac{d\vec{k}}{k^2} \vec{k} e^{i\vec{k} \cdot (\overrightarrow{Q_2 - Q_1})} \int \bar{f}_2^1(\vec{p}_3, \vec{P}_1, \vec{k}) \, d\vec{p}_3 \\ &- n \frac{4\pi e^2}{(2\pi)^3} i \, \frac{\partial F_1(\vec{P}_1)}{\partial \vec{P}_1} \cdot \int \frac{d\vec{k}}{k^2} \vec{k} e^{i\vec{k} \cdot (\overrightarrow{Q_1 - Q_2})} \int \bar{f}_2^1(\vec{p}_3, \vec{P}_2, \vec{k}) \, d\vec{p}_3 \\ = &\left[\frac{P_1^2 + P_2^2}{2m};\ F_2^1(x_1, x_2; F_1) \right] \\ &+ n \int \left[\psi_{13}(\overrightarrow{Q_1 - q_3}); F_1(\vec{P}_1) f_2^1(\vec{P}_2, \vec{p}_3, \overrightarrow{Q_2 - q_3}) \right] dx_3 \\ &+ n \int \left[\psi_{23}(\overrightarrow{Q_2 - q_3}); F_1(\vec{P}_2) f_2^1(\vec{P}_1, \vec{p}_3, \overrightarrow{Q_1 - q_3}) \right] dx_3, \end{aligned} \tag{108}$$

where, in the latter form, we have written the integrals in $\vec{q}$-space and used the fact that

$$H_2^o + \psi_{12} = \frac{P_1^2 + P_2}{2m}.$$

Now add the term $\left[\psi_{12}(\overrightarrow{Q_1 - Q_2}); F_1(\vec{P}_1) \, F_1(\vec{P}_2) \right]$ to and subtract it from the right-hand side of (108). By comparing the results with (97) and using the invariance of the Poisson brackets with respect to a contact transformation, we see that F_2^1 can be split into two functions

$$F_2^1(x_1, x_2; F_1) = f_2^1(\vec{P}_1, \vec{P}_2; \overrightarrow{Q_1 - Q_2}) + h_2(x_1, x_2; F_1), \tag{109}$$

where h_2 is the solution of

$$D_o h_2(x_1, x_2; F_1) = \left[H_2^o + \psi_{12}; h_2(x_1, x_2; F_1) \right] - \left[\psi_{12}(\overrightarrow{Q_1 - Q_2}); F_1(\vec{P}_1) \, F_1(\vec{P}_2) \right]. \tag{110}$$

By the familiar use of the D_o operators we can see that the solution to (110) is

$$h_2(x_1, x_2; F_1) = -\left[\int_0^\infty d\tau\ \psi_{12}(\overline{Q_1 - Q_2} - (\overline{V_1 - V_2})\tau); F_1(\hat{P}_1)\ F_1(\hat{P}_2)\right]. \tag{111}$$

Combining (106), (109), and (111), we obtain a final form for F_2 to first order in g under the adiabatic hypothesis.

$$F_2(x_1, x_2; F_1) = F_1(\hat{P}_1)\ F_1(\hat{P}_2)$$
$$+ f_2^1(\hat{P}_1, \hat{P}_2, \overline{Q_1 - Q_2}) - \left[\int_0^\infty d\tau\ \psi_{12}(\overline{Q_1 - Q_2} - (\overline{V_1 - V_2})\tau); F_1(\hat{P}_1)\ F_1(\hat{P}_2)\right]. \tag{112}$$

In Section IV we shall see that use of this function results in no divergence in the equation for F_1.

We shall look at (101) in its more general form, as a kinetic equation for F_2. As such it is exact in ψ_{12} and accurate to first order in all other interactions. If any collision term in the equation for F_1 is going to be used, it should be obtained from (101). For example, the expansion in g and the adiabatic hypothesis led to (112) and will be used in such a collision term.

We can also use (101) to study the limitations of making the adiabatic hypothesis for F_2. If in the solution we had expanded in g but not made the hypothesis, we would have had as a zero-order equation

$$\frac{\partial F_2^o}{\partial t} = \left[H_2^o + \psi_{12}; F_2^o\right].$$

This is exactly the form of equation that Bogoliubov[12] considers in showing that, for short-range forces, the adiabatic hypothesis is valid. He shows that, for forces of range r_o, $F_2^o(x_1, x_2; t)$ would relax to the form $F_2^o(x_1, x_2; F_1)$ in a time of the order $r_o/\bar{v}$. If for the long-range interaction we interpret a collision as described in section 1.2, this would be a time of order

$$\sqrt{\frac{\lambda_D e^2}{kT}}\ \frac{1}{\bar{v}} = \omega_p^{-1}\ \sqrt{\frac{e^2}{kT\lambda_D}} \ll \omega_p^{-1},$$

where ω_p^{-1} is the characteristic time of change of F_1. We can restate this by saying that an adiabatic hypothesis for F_2^o is forced on us by the boundary conditions. To zero order in g, the only force producing correlated motion is the two-body force, and the time that is such that this force is not of order g is small compared with the time of change of F_1. (These statements for F_2^o can be carried over immediately to the inhomogeneous case; but we shall not carry this out since the transition is not as simple for F_2^1.)

If we use the value of F_2^o thus obtained in the equation to first order, we have

$$\frac{\partial F_2^1(x_1,x_2)}{\partial t} = \left[H_2^o + \psi_{12}; F_2^1(x_1,x_2) \right]$$

$$+ n \int \left[\psi_{13}(\overrightarrow{Q_1 - q_3}); F_1(\vec{P}_1)\, f_2^1(\vec{P}_2, \vec{p}_3, \overrightarrow{Q_2 - q_3}) \right] dx_3$$

$$+ n \int \left[\psi_{23}(\overrightarrow{Q_2 - q_3}); F_1(\vec{P}_2)\, f_2^1(\vec{P}_1, \vec{p}_3, \overrightarrow{Q_1 - q_3}) \right] dx_3. \qquad (113)$$

Again an adiabatic hypothesis is attractive, since the whole source term is a functional of F_1. If we call this source term $\phi(x_1, x_2; t)$, we can write the solution to (113) as

$$F_2^1(x_1, x_2; t) = S^2_{-(t-t_o)}\, F_2^1(x_1, x_2; t_o)$$

$$+ \int_{t_o}^{t} S^2_{t'-t}\, \phi(x_1, x_2; t')\, dt' \xrightarrow[t_o \to -\infty]{} \int_0^\infty dt'\, S^2_{-t'}\, \phi(x_1, x_2; t-t').$$

Here, we use the boundary condition that F_2^1 is zero at infinite separation, ϕ goes to zero for $|\overrightarrow{q_1 - q_2}| \gtrsim \lambda_D$, and the t'-integration will back the particles off to this distance in a time $\sim \lambda_D/\bar{v}$. If ϕ has not changed in this time, we hold it constant throughout and say that

$$F_2^1(x_1, x_2; t) = \int_0^\infty dt'\, S^2_{-t'}\, \phi(x_1, x_2; t). \qquad (114)$$

Since t in ϕ occurs only inside F_1, (114) is equivalent to the adiabatic hypothesis. The argument connected with (40) indicates that for a uniform plasma F_1 changes sufficiently slowly that these approximations in the integral of (114) are valid.

From this discussion we see that the adiabatic hypothesis depends on the order of g. To zero order it is well founded. To first order two problems enter. The first is the necessity of assuming a slow variation of F_1, as mentioned above. The second is that the hypothesis for $F_2^1(x_1, x_2)$ depends upon the hypothesis for $F_2^1(x_1, x_i)$ where $i \neq 2$. (The fact that the source term in (113) is a functional of F_1 depends on this.) Dupree[11] has shown that the hypothesis for $F_2^1(x_1, x_i)$ depends also on the slow variation of F_1. It simply shows that the hypotheses for the various F_2^1 are interrelated and cannot be considered separately. The first-order effects in a plasma include shielding, which is a cooperative phenomenon.

In this section we discuss the result obtained in the previous section and see that it removes the divergence in (17), as desired. Section 4.1 will be devoted to a discussion of the equilibrium case as an example to increase the understanding of the results. Section 4.2 will contain a discussion of the more general case.

4.1 THE EQUILIBRIUM CASE

At equilibrium the momentum distributions are Maxwellian. Using the fact that $\vec{p}$ and $\vec{P}$ are related by the conservation of energy,

$$\frac{P_1^2 + P_2^2}{2m} = \frac{p_1^2 + p_2^2}{2m} + \frac{e^2}{|\overrightarrow{q_1 - q_2}|},$$

we may write the first term of (112) as

$$F_1^M(P_1)\, F_1^M(P_2) = F_1^M(p_1)\, F_1^M(p_2) \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right], \tag{115}$$

where $F_1^M(p)$ is the Maxwellian distribution.

Other authors[9, 10, 12] have shown that for equilibrium the large-separation solution reduces to

$$f_2^1(\vec{p}_1, \vec{p}_2, \overrightarrow{q_1-q_2}) = -\frac{e^2}{kT|\overrightarrow{q_1-q_2}|} \exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right] F_1^M(p_1)\, F_1^M(p_2). \tag{116}$$

In terms of the undeflected variables, and using (115), we obtain

$$f_2^1(\vec{P}_1, \vec{P}_2, \overrightarrow{Q_1-Q_2}) = -\frac{e^2}{kT|\overrightarrow{Q_1-Q_2}|} \exp\left[-\frac{|\overrightarrow{Q_1-Q_2}|}{\lambda_D}\right] F_1^M(p_1)\, F_1^M(p_2) \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right]. \tag{117}$$

For equilibrium the final term in (112) may be evaluated as follows:

$$\left[\int_0^\infty d\tau\, \psi_{12}(\overrightarrow{Q_1-Q_2}-(\overrightarrow{V_1-V_2})\tau); F_1^M(P_1)\, F_1^M(P_2)\right]$$

$$= -\frac{1}{kT}\frac{\partial}{\partial \vec{Q}_1}\int_0^\infty d\tau\, \psi_{12}(\overrightarrow{Q_1-Q_2}-(\overrightarrow{V_1-V_2})\tau)\cdot(\overrightarrow{V_1-V_2})\, F_1^M(P_1)\, F_1^M(P_2)$$

$$= \frac{1}{kT}\int_0^\infty d\tau\, \frac{\partial}{\partial\tau}\psi_{12}(\overrightarrow{Q_1-Q_2}-(\overrightarrow{V_1-V_2})\tau)\, F_1^M(P_1)\, F_1^M(P_2)$$

$$= -\frac{1}{kT|\overrightarrow{Q_1-Q_2}|} F_1^M(p_1) F_1^M(p_2) \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right]. \tag{118}$$

The results (115), (117), and (118) give, for the equilibrium function $F_2^e(x_1, x_2)$,

$$F_2^e(x_1, x_2) = F_1^M(p_1) F_1^M(p_2) \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right] \cdot \left\{1 + \frac{e^2}{kT|\overrightarrow{Q_1-Q_2}|}\left(1 - \exp\left[-\frac{|Q_1-Q_2|}{\lambda_D}\right]\right)\right\}. \tag{119}$$

Those regions for which $|\overrightarrow{Q_1-Q_2}|$ differs from $|\overrightarrow{q_1-q_2}|$ are those for which $|\overrightarrow{Q_1-Q_2}| \ll \lambda_D$ and for which, therefore, the last term in parentheses is small. To the same order of $e^2/kT\lambda_D$ we may replace $|\overrightarrow{Q_1-Q_2}|$ with $|\overrightarrow{q_1-q_2}|$ in (119). This substitution is consistent with our assumptions concerning the insensitivity of F_3^1 and F_2^1 to the difference between $\overleftarrow{Q}$ and $\overleftarrow{q}$.

We could approach the equilibrium case from a straightforward application of statistical mechanics. It has been shown[1] that the effective potential field of a charge imbedded in a fluid of charged particles is approximately the Debye potential

$$\frac{e^2 e^{-r/\lambda_D}}{r}.$$

If we were to assume that this is the potential energy between two charges, we would write the two-body distribution function as

$$F_2^e(x_1, x_2) = F_1^M(p_1) F_1^M(p_2) \exp\left[-\frac{e^2 \exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right]}{kT|\overrightarrow{q_1-q_2}|}\right].$$

If $|\overrightarrow{q_1-q_2}| \gg e^2/kT$, we can expand the exponential and obtain the result corresponding to the large-separation solution, (116),

$$F_2^e(x_1, x_2) \cong F_1^M(p_1) F_1^M(p_2)\left\{1 - \frac{e^2}{kT|\overrightarrow{q_1-q_2}|} \exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right]\right\}. \tag{120}$$

If we do not wish to assume that $|\overrightarrow{q_1-q_2}| \gg e^2/kT$, we can write F_2^e as

$$F_2^e(x_1,x_2) = F_1^M(p_1) F_1^M(p_2) \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right] \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\left(\exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right] -1\right)\right]$$

and expand the exponential of

$$\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\left(\exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right] - 1\right).$$

This function has its maximum at $|\overrightarrow{q_1-q_2}| = 0$, the value at which it equals $e^2/kT\lambda_D$, our small expansion parameter. Under this expansion we have

$$F_2^e(x_1, x_2) \cong F_1^M(p_1)\, F_1^M(p_2) \exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right] \cdot \left\{1 - \frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\left(\exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right] - 1\right)\right\}. \tag{121}$$

Equation 121 is (119) when we replace $|\overrightarrow{Q_1-Q_2}|$ with $|\overrightarrow{q_1-q_2}|$, and is the equilibrium function that we obtained by allowing for possible small values of $|\overrightarrow{q_1-q_2}|$. Both (119) and (121) result from expansions in the weakness of the forces giving rise to shielding.

If we regroup (121) as

$$F_2^e = F_1^M(p_1)\, F_1^M(p_2)\left\{\exp\left[-\frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right]\left(1 + \frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\right) - \frac{e^2}{kT|\overrightarrow{q_1-q_2}|}\exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right]\right\}$$

and expand for $|\overrightarrow{q_1-q_2}| \gg e^2/kT$, we obtain (120) to first order in $e^2/kT|\overrightarrow{q_1-q_2}|$. The effect of the additional function $h_2(x_1, x_2)$ of section 3.2 in the transition to large separation is to cancel the first-order effect in the expansion of F_2^o. We shall see this in the more general case in section 4.2.

Another use for $F_2(x_1, x_2)$ will be in the equation

$$\frac{\partial F_1(x_1)}{\partial t} = \left[H_1^o; F_1(x_1)\right] + n \int \left[\psi_{12}; F_2(x_1, x_2)\right] dx_2. \tag{122}$$

The use of $F_2(x_1, x_2)$ in the interaction term of (122) involves the removal of the distinction of particles 1 and 2, which was discussed at the beginning of section 2.3. The coordinate x_2 in (122) is simply a dummy variable in the integration over a function that is considered correct for all values of $(\overrightarrow{q_1-q_2})$.

We can use the expression (121) for F_2^e in order to estimate the contribution of various terms in F_2 to the integral in (122). We know that the angular integrals over F_2^e will give zero contribution, but we are justified in examining the $\left(|\overrightarrow{q_2-q_1}|\right)$-dependence for this case. In order to have convergence in the $\left(|\overrightarrow{q_2-q_1}|\right)$-integration, it is necessary to subtract the effect of the uniform background (also zero on the angular integration). Thus we shall examine the $\left(|\overrightarrow{q_2-q_1}|\right)$-dependence of the integral

$$\int \left[\psi_{12}; F_2^e(x_1, x_2) - F_1^M(p_1)\, F_1^M(p_2)\right] dx_2. \tag{123}$$

Our motivation for this is as follows. In the nonequilibrium case the angular integrations of (123) will not give zero. However, we can expect the $\left(|\overrightarrow{q_2-q_1}|\right)$-dependence to be roughly the same. If we can solve the equilibrium case, we can gain some insight into the nonequilibrium case.

Using (121) for F_2^e in (123), we can write the $|\overrightarrow{q_2-q_1}|$ part of the dx_2-integration as

$$\int_0^\infty dr\left[\exp\left[-\frac{e^2}{kTr}\right]\left(1+\frac{e^2}{kTr}\right)-1-\frac{e^2}{kTr}\exp-\left[\frac{e^2}{kTr}-\frac{r}{\lambda_D}\right]\right]. \tag{124}$$

Let us integrate the last term,[15]

$$\frac{e^2}{kT}\int_0^\infty \frac{dr}{r}\exp\left[-\frac{e^2}{kTr}-\frac{r}{\lambda_D}\right]=\frac{2e^2}{kT}K_o\left(2\sqrt{\frac{e^2}{kT\lambda_D}}\right).$$

We can expand the modified Bessel function[16] for small argument and obtain

$$\frac{e^2}{kT}\int_0^\infty \frac{dr}{r}\exp\left[-\frac{e^2}{kTr}-\frac{r}{\lambda_D}\right]\cong-\frac{e^2}{kT}\left(\ln\gamma\frac{e^2}{kT\lambda_D}+0\left(\frac{e^2}{kT\lambda_D}\ln\frac{e^2}{kT\lambda_D}\right)\right), \tag{125}$$

where $\gamma = e^C$, and C is Euler's number, 0.577.

Use of the function (121) has resulted in a convergent integral. Had we followed other authors and used (116), the integral would have diverged at small distances. Cutting it off at the distance of closest approach, e^2/kT, yields

$$\frac{e^2}{kT}\int_{e^2/kT}^\infty \frac{dr}{r}e^{-r/\lambda_D}\cong-\frac{e^2}{kT}\left(\ln\gamma\frac{e^2}{kT\lambda_D}+0\left(\frac{e^2}{kT\lambda_D}\ln\frac{e^2}{kT\lambda_D}\right)\right).$$

This result agrees with the more exact treatment to the same order of $e^2/kT\lambda_D$.

This surprising accuracy of the cutoff procedure occurs because, to a function that varies as λ_D, the function $\exp(-e^2/kTr)$ cuts off very sharply. Because this result was obtained only from the $\left(|\overrightarrow{q_2-q_1}|\right)$-dependence of the integral of (123) and did not rely on the symmetry of the equilibrium function, we can expect similar results for any non-equilibrium function $f_2^1(\bar{P}_1, \bar{P}_2, \overrightarrow{q_1-q_2})$. We therefore expect that we can still retain accuracy to the same order in g if we set

$$f_2^1(\bar{P}_1, \bar{P}_2, \overrightarrow{Q_1-Q_2}) \Longrightarrow f_2^1(\bar{p}_1, \bar{p}_2, \overrightarrow{q_1-q_2}) \tag{126}$$

and cut off the integration at e^2/kT.

The new results obtained by considerations of the close approach are in the first two terms of the integral in (124),

$$\int_0^\infty dr\left[\exp\left[-\frac{e^2}{kTr}\right]\left(1+\frac{e^2}{kTr}\right)-1\right]=-\frac{e^2}{kT}. \tag{127}$$

This is of the same order of g as (125), and thus we obtain a non-negligible contribution as a result of the close collisions. The generalization of (127) to the nonequilibrium case will not be as simple as the result for f_2^1, (126). The generalization of (127) depends very intimately on the nature of the functions for $|\overrightarrow{q_2-q_1}| \sim e^2/kT$, and will be investigated in the next section.

4.2 THE NONEQUILIBRIUM CASE

The discussion relating to the general case is necessarily more complex, since we must deal with general expressions rather than evaluate integrals of known functions. We shall see that to first order in g, the entire effect of the close collisions can be put into one term that is a generalization of the Boltzmann collision integral. To this will be added another accounting for the velocity dependence of the collective interaction.

Consider (99) for $\bar{f}_2^1(\bar{P}_1, \bar{P}_2, \bar{k})$,

$$\bar{f}_2^1(\bar{P}_1, \bar{P}_2, \bar{k}) = -\frac{4\pi i e^2}{k^2}\frac{\bar{k}\cdot\bar{D}_{12}F_1(\bar{P}_1)\,F_1(\bar{P}_2)}{(\epsilon - i\bar{k}\cdot(\overline{V_1 - V_2}))}$$

$$-\frac{4\pi n e^2 i}{k^2(\epsilon - i\bar{k}\cdot(\overline{V_1 - V_2}))}\left\{\bar{k}\cdot\frac{\partial F_1(\bar{P}_1)}{\partial\bar{P}_1}\int d\bar{p}_3\,\bar{f}_2^1(\bar{p}_3, \bar{P}, \bar{k})\right.$$

$$\left.-\bar{k}\cdot\frac{\partial F_1(\bar{P}_2)}{\partial\bar{P}_2}\int d\bar{p}_3\,\bar{f}_2^1(\bar{P}_1, \bar{p}_3, \bar{k})\right\}. \tag{128}$$

In (128) we define

$$\bar{f}_2^1(\bar{P}_1, \bar{P}_2, \bar{k}) = \int d^3\bar{Q}\, e^{i\bar{k}\cdot\bar{Q}}\, f_2^{1'}(\bar{P}_1, \bar{P}_2, \bar{Q}) \tag{129}$$

and

$$-\frac{4\pi i e^2}{k^2}\frac{\bar{k}\cdot\bar{D}_{12}F_1(\bar{P}_1)\,F_1(\bar{P}_2)}{(\epsilon - i\bar{k}\cdot(\overline{V_1 - V_2}))} = \int d\bar{Q}_{12}\, e^{i\bar{k}\cdot\bar{Q}_{12}}\left[\int_0^\infty d\tau\,\psi_{12}(\bar{Q}_{12} - (\overline{V_1 - V_2})\tau); F_1(\bar{P}_1)\,F_1(\bar{P}_2)\right]. \tag{130}$$

From (130) we see that small distances, $|\overline{Q_1 - Q_2}| \ll \lambda_D$, correspond to large wave numbers, $k \gg \lambda_D^{-1}$, in the region of chief contribution of the transform. For large k, (129) can be expanded in inverse powers of k by using the first term of (129) as the first approximation in an iteration expansion,

$$n\bar{f}_2^1(\bar{P}_1, \bar{P}_2, \bar{k}) + \frac{4\pi n e^2}{k^2}\, i\,\frac{\bar{k}\cdot\bar{D}_{12}F_1(\bar{P}_1)\,F_1(\bar{P}_2)}{(\epsilon - i\bar{k}\cdot(\overline{V_1 - V_2}))}$$

$$\cong \frac{\left(\frac{4\pi n e^2}{k^2}\right)^2 i}{(\epsilon - i\bar{k}\cdot(\overline{V_1 - V_2}))}\left\{\bar{k}\cdot\frac{\partial F_1(\bar{P}_1)}{\partial\bar{P}_1}\int_+ d\bar{p}_3\,\frac{\bar{k}\cdot\bar{D}_{32}F_1(\bar{p}_3)\,F_1(\bar{P}_2)}{\bar{k}\cdot(\overline{v_3 - V_2})}\right.$$

$$- \tilde{k} \cdot \frac{\partial F_1(\tilde{P}_2)}{\partial \tilde{P}_2} \int_{-} d\tilde{p}_3 \frac{\tilde{k} \cdot \tilde{D}_{13} F_1(\tilde{P}_1) F_1(\tilde{P}_2)}{\tilde{k} \cdot (\overline{V_1 - v_3})} \Bigg\} + 0\left(\frac{1}{\lambda_D^6 k^6}\right) . \tag{131}$$

Successive terms fall off as k^{-2n}.

For $k > \sqrt{kT/e^2\lambda_D}$ the right-hand side of (131) is of order $(e^2/kT\lambda_D)^2$, which is of second-order smallness in g. As was pointed out in section 1.2, the difference between $\tilde{P}$ and $\tilde{p}$ and $\tilde{Q}$ and $\tilde{q}$ becomes important only in the region $|\overrightarrow{q_1 - q_2}| < \sqrt{e^2\lambda_D/kT}$. This fact implies that on the left-hand side of (131) we can replace $\tilde{Q}$ with $\tilde{q}$ and $\tilde{P}$ with $\tilde{p}$ and know that in those regions in which this replacement is not valid the function (131) is very small. Thus in $\tilde{q}$-space

$$f_2^1(\tilde{P}_1, \tilde{P}_2, \overrightarrow{Q_1 - Q_2}) - \left[\int_0^\infty d\tau\, \psi_{12}(\overrightarrow{Q_1 - Q_2} - (\overrightarrow{V_1 - V_2})\tau); F_1(\tilde{P}_1)\, F_1(\tilde{P}_2)\right]$$

$$\longrightarrow f_2^1(\tilde{p}_1, \tilde{p}_2, \overrightarrow{q_1 - q_2}) - \left[\int_0^\infty d\tau\, \psi(\overrightarrow{q_1 - q_2} - (\overrightarrow{v_1 - v_2})\tau); F_1(\tilde{p}_1)\, F_1(\tilde{p}_2)\right], \tag{132}$$

where the error incurred will be of order $(e^2/kT\lambda_D)^2$. (The case of $|\overrightarrow{Q_1 - Q_2}| \sim \sqrt{e^2\lambda_D/kT}$ will be mentioned later.)

The final use for $F_2(x_1, x_2)$ will be in the interaction term (I.T.) of the equation for F_1,

$$\frac{\partial F_1(x_1)}{\partial t} - \left[H_1^o; F_1(x_1)\right] \equiv \text{I. T.}$$

$$= n \int \Bigg[\psi_{12}; \Bigg\{F_1(\tilde{P}_1)\, F_1(\tilde{P}_2) + f_2^1(\tilde{p}_1, \tilde{p}_2, \overrightarrow{q_1 - q_2})$$

$$- \left[\int_0^\infty d\tau\, \psi_{12}(\overrightarrow{q_1 - q_2} - (\overrightarrow{v_1 - v_2})\tau); F_1(\tilde{p}_1)\, F_1(\tilde{p}_2)\right]\Bigg\}\Bigg]\, dx_2, \tag{133}$$

where we have proved that there is no divergence for $|\overrightarrow{q_1 - q_2}| \to 0$.

The third term of (133) appears to diverge for $|\overrightarrow{q_1 - q_2}| \to \infty$, but we can easily show that this divergence is cancelled by the first term. Consider the equation for $F_1(\tilde{P}_1)\, F_1(\tilde{P}_2)$ in the region of small ψ_{12}, that is, large $|\overrightarrow{q_1 - q_2}|$,

$$D_o F_2^o(x_1, x_2) = \left[H_2^o; F_2^o(x_1, x_2)\right] + \left[\psi_{12}; F_2^o(x_1, x_2)\right].$$

Using the usual solution methods and (26), we may obtain an expansion for small ψ_{12},

$$F_1(\tilde{P}_1)\, F_1(\tilde{P}_2) = F_1(\tilde{p}_1)\, F_1(\tilde{p}_2)$$

$$+ \left[\int_0^\infty \psi_{12}(\overrightarrow{q_1 - q_2} - (\overrightarrow{v_1 - v_2})\tau)\, d\tau; F_1(\tilde{p}_1)\, F_1(\tilde{p}_2)\right]$$

$$+\int_0^\infty \left[\psi_{12}(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau);\left[\int_\tau^\infty d\tau' \,\psi_{12}(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau');F_1(\vec{p}_1)\,F_1(\vec{p}_2)\right]\right] d\tau$$

$$+\ \ldots . \tag{134}$$

Using this expansion in (133) in the region of large $|\overrightarrow{q_1-q_2}|$, we see that the second term of (134) cancels the divergent term of (133). The higher powers of ψ_{12} in (134) will converge in the integral of (133) for $|\overrightarrow{q_1-q_2}| \to \infty$.

We see that (133), although a sufficient result, is in an inconvenient form because the final term in the interaction term cancels the divergence in f_2^1 as $|\overrightarrow{q_1-q_2}| \to 0$ and the divergence in F_2^o as $|\overrightarrow{q_1-q_2}| \to \infty$. We shall obtain another form that is equivalent to (133) but that is more convenient in the kinetic equation for F_1.

In the Appendix we derive the result (17) corresponding to the f_2^1 term in the interaction term of (133),

$$\frac{16\pi^2 ne^4}{(2\pi)^3}\iint \frac{d\vec{k}d\vec{p}_2}{k^4}\vec{k}\cdot\frac{\partial}{\partial\vec{p}_1}\frac{\delta(\vec{k}\cdot(\overrightarrow{v_1-v_2}))}{|1+L_+(i\vec{k}\cdot\vec{v}_1)|^2}\vec{k}\cdot\vec{d}_{12}\,F_1(\vec{p}_1)\,F_1(\vec{p}_2). \tag{135}$$

The corresponding expression for the last term in (133) is obtained by expressing the integral in $\vec{k}$-space and is given by

$$\frac{16\pi^2 ne^4}{(2\pi)^3}\iint \frac{d\vec{k}d\vec{p}_2}{k^4}\vec{k}\cdot\frac{\partial}{\partial\vec{p}_1}\delta(\vec{k}\cdot(\overrightarrow{v_2-v_1}))\,\vec{k}\cdot\vec{d}_{12}\,F_1(\vec{p}_1)\,F_1(\vec{p}_2). \tag{136}$$

(Using the fact that $L_+(i\vec{k}\cdot\vec{v}) \xrightarrow[k\to\infty]{} 0$, we can easily see that the difference between (135) and (136) converges for large $|\vec{k}|$ in agreement with (132).)

In the interaction term of (133) add and subtract the term

$$\frac{16\pi^2 ne^4}{(2\pi)^3}\iint \frac{d\vec{k}d\vec{p}_2}{\left(k^2+k_D^2\right)^2}\,\vec{k}\cdot\frac{\partial}{\partial\vec{p}_1}\,\delta(\vec{k}\cdot(\overrightarrow{v_1-v_2}))\,\vec{k}\cdot\vec{d}_{12}\,F_1(\vec{p}_1)\,F_1(\vec{p}_2)$$

$$= n\int\left[\psi_{12}^D;\left[\int_0^\infty d\tau\,\psi_{12}^D(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau);\,F_1(\vec{p}_1)\,F_1(\vec{p}_2)\right]\right] dx_2, \tag{137}$$

where

$$k_D \equiv \lambda_D^{-1}$$

and

$$\psi_{12}^D \equiv \frac{e^2}{|\overrightarrow{q_1-q_2}|}\exp\left[-\frac{|\overrightarrow{q_1-q_2}|}{\lambda_D}\right].$$

The difference between (135) and (137) also converges for $|\vec{k}| \to \infty$.

With these additions, the interaction term becomes

$$\text{I.T.} = n\int\left[\psi_{12};\, F_1(\vec{P}_1)\, F_1(\vec{P}_2) - \left[\int_0^\infty \psi_{12}(\overline{q_1-q_2}-(\overline{v_1-v_2})\tau);\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)\right]\right] dx_2$$

$$+\, n\int\left[\psi_{12}^D;\, \left[\int_0^\infty d\tau\, \psi_{12}^D(\overline{q_1-q_2}-(\overline{v_1-v_2})\tau);\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)\right]\right] dx_2$$

$$+\, n\int\left\{\left[\psi_{12};\, f_2^1(\vec{p}_1, \vec{p}_2, \overline{q_1-q_2})\right]\right.$$

$$\left.-\left[\psi_{12}^D;\, \left[\int_0^\infty d\tau\, \psi_{12}^D(\overline{q_1-q_2}-(\overline{v_1-v_2})\tau);\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)\right]\right]\right\} dx_2. \tag{138}$$

In order to combine the first three terms of (138), consider the equation for $F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right)$ where

$$\vec{P}_1^D = \lim_{\tau\to\infty} S_{-\tau}^D \vec{p}_1,$$

$$\vec{P}_2^D = \lim_{\tau\to\infty} S_{-\tau}^D \vec{p}_2, \tag{139}$$

and

$$\frac{\partial}{\partial\tau} S_{-\tau}^D \phi(x_1, x_2) = \left[H_2^o + \psi_{12}^D; S_{-\tau}^D \phi(x_1, x_2)\right]$$

for any $\phi(x_1, x_2)$. Here, $S_{-\tau}^D$ is the operator that projects the particles 1 and 2 back along trajectories corresponding to a Debye-shielded collision.

By the familiar means, we have

$$D_o F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right) = \left[H_2^o+\psi_{12}^D; F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right)\right]$$

$$= \left[H_2^o+\psi_{12}; F_1\left(P_1^D\right) F_1\left(P_2^D\right)\right]$$

$$+ \left[\psi_{12}^D-\psi_{12}; F_1\left(P_1^D\right) F_1\left(P_2^D\right)\right]. \tag{140}$$

Since

$$|\psi_{12}^D - \psi_{12}| \le e^2/\lambda_D,$$

we can expand $F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right)$, using (26), in terms corresponding to powers of $(e^2/kT\lambda_D)$,

$$F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right) \cong F_1(\vec{P}_1)\, F_1(\vec{P}_2) + \int_0^\infty d\tau\, S_{-\tau}^2\left[\psi_{12}^D-\psi_{12}; F_1(\vec{P}_1)\, F_1(\vec{P}_2)\right]. \tag{141}$$

The difference $\psi_{12}^D - \psi_{12}$ is very small until the τ-integration backs the particles to a separation, $\sim \lambda_D$, so that we obtain in the integral of (141) only the asymptotic velocities,

$$\left[\int_0^\infty d\tau \left\{\psi_{12}^D(\overrightarrow{Q_1-Q_2}-(\overrightarrow{V_1-V_2})\tau)-\psi_{12}(\overrightarrow{Q_1-Q_2}-(\overrightarrow{V_1-V_2})\tau)\right\};\ F_1(\vec{P}_1)\ F_1(\vec{P}_2)\right]. \tag{142}$$

As in (132), this difference is small unless $|\overrightarrow{q_1-q_2}| \gtrsim \lambda_D$, in which case $\vec{Q} \to \vec{q}$ and $\vec{P} \to \vec{p}$ $\left(|\overrightarrow{Q_1-Q_2}| \sim \lambda_D \text{ will be discussed later}\right)$. Thus (141) becomes

$$F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right) \cong F_1\left(\vec{P}_1\right) F_1\left(\vec{P}_2\right) + \left[\int_0^\infty d\tau \left\{\psi_{12}^D(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau)-\psi_{12}(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau)\right\};\ F_1(\vec{p}_1)\ F_1(\vec{p}_2)\right]. \tag{143}$$

Using (143), we can obtain

$$\begin{aligned}\int \left[\psi_{12};\ F_1(\vec{P}_1)\ F_1(\vec{P}_2)\right] dx_2 \cong \int \Bigg\{ & \left[\psi_{12}^D;\ F_1\left(\vec{P}_1^D\right)\ F_1\left(\vec{P}_2^D\right)\right] \\ & + \left[\psi_{12};\ \left[\int_0^\infty d\tau\ \psi_{12}(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau);\ F_1(\vec{p}_1)\ F_1(\vec{p}_2)\right]\right] \\ & - \left[\psi_{12}^D;\ \left[\int_0^\infty d\tau\ \psi_{12}^D(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau);\ F_1(\vec{p}_1)\ F_1(\vec{p}_2)\right]\right]\Bigg\} dx_2,\end{aligned} \tag{144}$$

where the equality holds to first order in $(e^2/kT\lambda_D)$. Using (144) in (138), we have

$$\begin{aligned}\text{I. T.} = n \int & \left[\psi_{12}^D; F_1\left(\vec{P}_1^D\right)\ F_1\left(\vec{P}_2^D\right)\right] dx_2 \\ & + \int \Bigg\{\left[\psi_{12}; f_2^1(\vec{p}_1, \vec{p}_2, \overrightarrow{q_1-q_2})\right] \\ & - \left[\psi_{12}^D;\ \left[\int_0^\infty d\tau\ \psi_{12}^D(\overrightarrow{q_1-q_2}-(\overrightarrow{v_1-v_2})\tau);\ F_1(\vec{p}_1)\ F_1(\vec{p}_2)\right]\right]\Bigg\} dx_2.\end{aligned} \tag{145}$$

We may see from the following considerations that the first term of (145) is a Boltzmann collision integral with a Debye-shielded interaction. The method used is due to Bogoliubov.[12]

From the definition of $\vec{P}^D$

$$\left[H_2^o+\psi_{12}^D; F_1\left(\vec{P}_1^D\right)\ F_1\left(\vec{P}_2^D\right)\right] = 0.$$

Thus we set

$$\int \left[\psi_{12}^D; F_1\left(\vec{P}_1^D\right) F_1\left(\vec{P}_2^D\right)\right] dx_2 = \iint \frac{(\overrightarrow{p_2-p_1})}{m} \cdot \frac{\partial}{\partial \vec{q}_2}\ F_1\left(\vec{P}_1^D\right)\ F_1\left(\vec{P}_2^D\right)\ d\vec{p}_2 d\vec{q}_2 \tag{146}$$

and recognize that $\vec{P}_i^D$ depends on $\vec{q}_2$ only through $\vec{q} = \overrightarrow{q_2 - q_1}$. For the $\vec{q}$-integration set up a cylindrical coordinate system with the positive z-axis parallel to $\overrightarrow{v_2 - v_1}$. Denote the radius and polar angle by a and ϕ, respectively. Then

$$\int \frac{(\overrightarrow{p_2 - p_1})}{m} \cdot \frac{\partial}{\partial \vec{q}} F_1\left(P_1^D\right) F_1\left(P_2^D\right) d\vec{q}$$

$$= \left|\frac{\overrightarrow{p_2 - p_1}}{m}\right| \int_0^{2\pi} d\phi \int_0^{\infty} da\ a \int_{-\infty}^{\infty} d\xi\ \frac{\partial}{\partial \xi} F_1\left(P_1^D\right) F_1\left(P_2^D\right). \tag{147}$$

From the two-body problem under consideration, $\vec{P}_1^D(x_1, x_2)$ and $\vec{P}_2^D(x_1, x_2)$ are the precollision momenta that will yield the state x_1, x_2. From this definition

$$\begin{aligned} P_1^D(x_1, x_2)\Big|_{\xi=-\infty} &= \vec{p}_1 \qquad & P_2^D(x_1, x_2)\Big|_{\xi=-\infty} &= \vec{p}_2 \\ P_1^D(x_1, x_2)\Big|_{\xi=+\infty} &= \vec{p}_1^{\,*} \qquad & P_2^D(x_1, x_2)\Big|_{\xi=+\infty} &= \vec{p}_2^{\,*} \end{aligned} \tag{148}$$

where $\vec{p}_1^{\,*}$ and $\vec{p}_2^{\,*}$ are those momenta that, as precollision momenta, will give $\vec{p}_1$ and $\vec{p}_2$ as a final momentum. For coordinates x_1 and x_2 which are such that an interaction is in progress, $\vec{P}_i^D$ and $\vec{p}_i^{\,*}$ are not equal. Only when the particles are separated by a distance that is much greater than λ_D do $\vec{P}_i^D$ and $\vec{p}_i^{\,*}$ become equal.

Using (146)-(148) in (145), we have

$$n \int \left[\psi_{12}^D ; F_1\left(P_1^D\right) F_1\left(P_2^D\right)\right] dx_2$$

$$= n \int_0^{2\pi} \int_0^{\infty} \int_{(\vec{p}_2)} \left|\frac{\overrightarrow{p_2 - p_1}}{m}\right| \left(F_1\left(\vec{p}_1^{\,*}\right) F_1\left(\vec{p}_2^{\,*}\right) - F_1(\vec{p}_1) F_1(\vec{p}_2)\right) d\vec{p}_2\ a da\ d\phi, \tag{149}$$

which is a form of the Boltzmann collision integral with the Debye-shielded interaction.

Using (135), (137), (145), and (149) in (133), we obtain

$$\frac{\partial F_1(x_1)}{\partial t} = \left[H_1^o ; F_1(x_1)\right]$$

$$+ \frac{n}{m} \int_0^{2\pi} \int_0^{\infty} \int_{(\vec{p}_2)} |\overrightarrow{p_2 - p_1}| \left(F_1\left(\vec{p}_1^{\,*}\right) F_1\left(\vec{p}_2^{\,*}\right) - F_1(\vec{p}_1) F_1(\vec{p}_2)\right) d\vec{p}_2\ a da\ d\phi$$

$$+ \frac{16\pi^2 n e^4}{(2\pi)^3} \int d\vec{k}\ d\vec{p}_2\ \vec{k} \cdot \frac{\partial}{\partial \vec{p}_1} \left\{ \frac{1}{k^4 |1 + L_+(i\vec{k} \cdot \vec{v}_1)|^2} \right.$$

$$-\left.\frac{1}{(k^2+k_D)^2}\right\} \delta(\vec{k}\cdot(\overrightarrow{v_2-v_1}))\, \vec{k}\cdot \vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2). \tag{150}$$

The collision integral is the same as that used by Allis[6] and by Spitzer and Härm[2] insofar as they replace the Coulomb field with a Debye field in adapting the Boltzmann theory to plasmas. It is convenient that the entire effect of close collisions is contained in this term. This is consistent with the concept of the close collision and its analogy with the Boltzmann gas discussed in section 1. 1.

The integral over $\vec{k}$ in (150) is the contribution to the interaction term which is due to the deviation of the shielding cloud from the Debye shield. This is caused by the nature of F_1 and the velocity $\vec{v}_1$, as can be seen from the fact that

$$\lim_{v_1\to 0} L_+(i\vec{k}\cdot\vec{v}_1) = \lim_{v_1\to 0} \frac{m\omega_p^2}{k^2} \int_+ \frac{\vec{k}\cdot\frac{\partial F_1(\vec{p}_2)}{\partial \vec{p}_2}}{\vec{k}\cdot(\overrightarrow{v_1-v_2})}\, d\vec{p}_2$$

$$\longrightarrow -\frac{k_D^2}{k^2}(kT)\int_+ \frac{\frac{\partial F_1(u)}{\partial u}\,du}{u}. \tag{151}$$

For a distribution that is spherically symmetric the integral is real. For a Maxwellian distribution (151) reduces to k_D^2/k^2, which makes the $\vec{k}$ integral of (150) zero.

If the distribution $F_1(p)$ is not Maxwellian, the contribution of the integral over $\vec{k}$ in (150) is of the Fokker-Planck form. If we define D^o and B^o in (5) we have

$$\overrightarrow{D^o} \equiv -\frac{16\pi^2 ne^4}{m(2\pi)^3}\iint \frac{d\vec{p}_2\, d\vec{k}\,\vec{k}}{k^4}\left(\frac{1}{|1+L_+(i\vec{k}\cdot\vec{v}_1)|^2} - \frac{1}{\left(1+\frac{k_D^2}{k^2}\right)^2}\right)\delta(\vec{k}\cdot(\overrightarrow{v_2-v_1}))\,\vec{k}\cdot\frac{\partial F_1(p_2)}{\partial \vec{p}_2}$$

$$\overleftrightarrow{B^o} \equiv \frac{16\pi^2 ne^4}{m^2(2\pi)^3}\iint \frac{d\vec{p}_2\, d\vec{k}\,\vec{k}\,\vec{k}}{k^4}\left(\frac{1}{|1+L_+(i\vec{k}\cdot\vec{v}_1)|^2} - \frac{1}{\left(1+\frac{k_D^2}{k^2}\right)^2}\right)\delta(\vec{k}\cdot(\overrightarrow{v_2-v_1}))\,F_1(\vec{p}_2). \tag{152}$$

Armed with solution (150), we are in a position to justify the assumptions (90), (95), (132), and (142). We have shown explicitly that these assumptions are good for $|\overrightarrow{Q_1-Q_2}| \ll \lambda_D$.

We can now show that the error introduced by these assumptions for $|\overrightarrow{Q_1-Q_2}| \sim \lambda_D$ is negligible. We are interested in an integral of $F_2(x_1, x_2)$ over $\vec{p}_2$ and $\vec{q}_2$. In this integration consider a value of $(\overrightarrow{q_2-q_1})$ which is such that $|\overrightarrow{Q_1-Q_2}| \sim \lambda_D$ (which also means that $|\overrightarrow{q_1-q_2}| \sim \lambda_D$). The number of particles at this separation which have undergone a close collision in the past is given by the number of particles that would undergo a close collision in an inverse collision starting from the same spacial coordinates with the velocities reversed. The probability for this is given by the solid angle that is available for close collisions and that is given by A/R^2, where A is the cross section for close collision and R is the initial separation of the particles. For particles separated by λ_D, this ratio is $(e^2\lambda_D/kT)/\lambda_D^2 = e^2/kT\lambda_D$. Since nothing radical happens for these particles, for example, we do not predict a divergent result for their contribution, we may say that the approximations (90), (95), (132), and (142) are valid for all those situations that yield a significant contribution to the kinetic equation for F_1.

As a final step we shall evaluate the $|\vec{k}|$-integration for $\overleftrightarrow{B}^o$ and $\overrightarrow{D}^o$ in (152), which will leave these functions expressed in angular integrals. The $|\vec{k}|$-integration can be performed in the general case as follows. Define

$$L'_{\pm}(i\vec{k}\cdot\vec{v}_1) \equiv k^2 L_{\pm}(i\vec{k}\cdot\vec{v}_1) \tag{153}$$

and note that $L'_{\pm}$ does not depend on $|\vec{k}|$, only on the direction of $\vec{k}/k$. We define

$$W(\theta,\phi,\vec{v}_1) \equiv \int_0^\infty dk k^3 \left(\frac{1}{|k^2+L'_+(i\vec{k}\cdot\vec{v}_1)|^2} - \frac{1}{\left(k^2+k_D^2\right)^2}\right) \tag{154}$$

where θ and ϕ are the polar and azimuthal angles of $\vec{k}$ in $\vec{k}$-space. W converges at the upper limit as a result of the subtraction of the two terms, and the integral in (154) can be evaluated by standard means to yield

$$W(\theta,\phi,\vec{v}_1) = \mathrm{Re}\left[\frac{L'_+ - L'_+ \log' \dfrac{L'_+}{k_D^2}}{L'_+ - L'_-}\right]. \tag{155}$$

In terms of W, the functions $\overleftrightarrow{B}^o$ and $\overrightarrow{D}^o$ can be written

$$\overrightarrow{D}^o = -\frac{16\pi^2 ne^4}{m(2\pi)^3}\int d\vec{p}_2 \int_0^{2\pi} d\phi \int_0^\pi \sin\theta \, d\theta \, \frac{\vec{k}}{k} W(\theta,\phi,\vec{v}_1)\, \delta(\vec{k}\cdot(\vec{v}_2-\vec{v}_1))\, \vec{k}\cdot\frac{\partial F_1(\vec{p}_2)}{\partial \vec{p}_2}$$

$$\overleftrightarrow{B}^o = \frac{16\pi^2 ne^4}{m^2(2\pi)^3}\int d\vec{p}_2 \int_0^{2\pi} d\phi \int_0^\pi \sin\theta \, d\theta \left(\frac{\vec{k}\vec{k}}{k}\right) W(\theta,\phi,\vec{v}_1)\, \delta(\vec{k}\cdot(\vec{v}_2-\vec{v}_1))\, F_1(\vec{p}_2). \tag{156}$$

For a spherically symmetric distribution $F_1(p)$, it is convenient to pick the z-axis of the $\vec{k}$-integration along the direction of $\vec{v}_1$. For this case the azimuthal integration yields 2π, and $\overleftrightarrow{B}^o$ and $\overrightarrow{D}^o$ are expressed as one-dimensional integrals. The final evaluation of these integrals must, of course, be made with a particular $F_1(\vec{p})$.

In the use of (150)-(156) the following should be noted. Although $\overleftrightarrow{B}^o$ and $\overrightarrow{D}^o$ are in the form of Fokker-Planck coefficients, they do not represent the complete contribution to the dynamical friction and the diffusion in velocity space. One method of treating the Boltzmann collision integral is to expand the integral in a series of powers of deflections in velocity.[2, 6] The coefficients of the first two powers of this expansion will give additional contribution to the dynamical friction and the diffusion in velocity space. The contributions (136) are due to the particular shape of the shielding cloud and happen to be in the form of Fokker-Planck coefficients.

V. SUMMARY

Equation (150) is the result that was sought. As a kinetic equation for $F_1(\vec{p})$, it may be considered accurate to first order in $(e^2/kT\lambda_D)$. The use of the adiabatic hypothesis has yielded the most general kinetic equation for F_1; and insofar as the hypothesis holds, this equation can be said to exist.

The form (150) is appealing because, although it is a new result, it contains terms corresponding to those of various earlier treatments, the Boltzmann equation (3) and the Fokker-Planck equation (5). (There is no Vlasov type of contribution for the uniform plasma.) Equation 3 was correct for close collisions and in error for grazing collisions; and (5), the opposite. Equation 150 combines these two solutions and forms a bridge between them.

Discussions of the solution of (150) will not be carried out here, since the various terms in (150) correspond to earlier treatments. Spitzer and Harm[2] and Allis[6] have discussed the Boltzmann collision integral with an assumed Debye-shielded interaction. Several authors[8-10] have obtained the term containing the integral over f_2^1 in (150). This result was (17) mentioned in section 1.2. The divergence in the $\vec{k}$-integration is cancelled by the last term of (150), although the other properties remain the same.

The solution of (150) is still complicated and must be subjected to other approximations. However, within the limitations mentioned above, (150) may be taken to be the equation of evolution of F_1, and these approximations of solution may be used and analyzed in their own right. The ability to separate the approximations used in deriving the equation from the approximations used in its solution aids greatly in understanding the accuracy, meaning, and limitations of the analysis.

APPENDIX

We shall carry out a derivation of the large-separation solution f_2^1 which is due to Dupree.[11] We shall then show that Dupree's result can be put into the form obtained by Lenard.[10]

Dupree considers (97) for f_1^2 with the substitutions $F_1 \to S_{-\tau}^1 F_1$ and $D_o \to \frac{\partial}{\partial t}$ as in (66). We write $\vec{q} = \overrightarrow{q_1 - q_2}$ and obtain

$$\frac{\partial}{\partial t} f_2^1 (\vec{p}_1, \vec{p}_2, \vec{q}; S_{-\tau}^1 F_1) + \frac{(\overrightarrow{p_1 - p_2})}{m} \cdot \frac{\partial}{\partial \vec{q}} f_2^1 (p_1, p_2, \vec{q}; S_{-\tau}^1 F_1)$$

$$- n \frac{\partial F_1(\vec{p}_1)}{\partial \vec{p}_1} \cdot \frac{\partial}{\partial \vec{q}} \int d\vec{p}_3 \, d\vec{q}' \, \psi(\overrightarrow{q - q'}) \, f_2^1 (\vec{p}_3, \vec{p}_2, \vec{q}'; S_{-\tau}^1 F_1)$$

$$+ n \frac{\partial F_1(\vec{p}_2)}{\partial \vec{p}_2} \cdot \frac{\partial}{\partial \vec{q}} \int d\vec{p}_3 \, d\vec{q}' \, \psi(\overrightarrow{q - q'}) \, f_2^1 (\vec{p}_1, \vec{p}_3, q'; S_{-\tau}^1 F_1)$$

$$= \frac{\partial}{\partial \vec{q}} \psi(\vec{q}) \cdot \vec{d}_{12} \, F_1(\vec{p}_1) \, F_1(\vec{p}_2). \tag{157}$$

This can be written

$$\left(\frac{\partial}{\partial t} + L_1 + L_2\right) f(\tau) = \frac{\partial}{\partial \vec{q}} \psi(\vec{q}) \cdot \vec{d}_{12} \, F_1(\vec{p}_1) \, F_1(\vec{p}_2) \tag{158}$$

where L_1 and L_2 are Landau operators defined in (71) and operate on coordinate 1 and coordinate 2, respectively. Note that L_1 and L_2 commute.

The formal solution to (158) is

$$f_2^1 (\vec{p}_1, \vec{p}_2, \vec{q}) = \int_0^\infty d\tau' \, e^{-(L_1 + L_2)\tau'} \frac{\partial}{\partial \vec{q}} \psi(\vec{q}) \cdot \vec{d}_{12} \, F_1(\vec{p}_1) \, F_1(\vec{p}_2)$$

where we have taken the limit $\tau \to \infty$ in agreement with the discussion preceding (98).

It is easier to evaluate

$$\int d\vec{p}_2 \, f_2^1 (\vec{p}_1, \vec{p}_2, \vec{q})$$

and to then compute f_2^1 from (157). Using (78) and (79) for the result of the operator L_i, we have

$$\int \bar{f}_1^2 (\vec{p}_1, \vec{p}_2, \vec{k})\, d\vec{p}_2 = -\frac{1}{(2\pi)^2} \int_0^\infty d\tau \iint_{-\infty i+\beta}^{\infty i+\beta} d\sigma_1 d\sigma_2 \frac{e^{(\sigma_1+\sigma_2)\tau}}{1 + L_-(-\sigma_2)} \cdot \int_- \frac{d\vec{p}_2}{\sigma_2 + i\vec{k}\cdot\vec{v}_1}$$

$$\left\{ \frac{\dfrac{m\omega_p^2}{k^2}}{1 + L_+(\sigma_1)} \frac{\vec{k}\cdot\dfrac{\partial F_1(\vec{p}_1)}{\partial \vec{p}_1}}{\sigma_1 - i\vec{k}\cdot\vec{v}_1} \int_+ \frac{d\vec{p}_1}{\sigma_1 - i\vec{k}\cdot\vec{v}_1} + \frac{i}{\sigma_1 - i\vec{k}\cdot\vec{v}_1} \right\}$$

$$\cdot \frac{4\pi c^2}{k^2} \vec{k} \cdot \vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2), \tag{159}$$

where

$$\bar{f}_2^1 (\vec{p}_1, \vec{p}_2, \vec{k}) = \int d\vec{q}\, e^{i\vec{k}\cdot\vec{q}}\, f_2^1 (\vec{p}_1, \vec{p}_2, \vec{q}).$$

Equation 159 is the form obtained by Dupree. The $\int_+ dp$ and L_+ are defined in (78) and (79). The $\int_- dp$ and L_- are the same integrals with the path of integration going under the poles.

As mentioned in connection with (80), the σ-integration can be moved just to the left of the imaginary axis and the τ-integration carried out. This brings down a $(\sigma_1+\sigma_2)^{-1}$ if we can assume Re σ_2 < Re σ_1 (if not, carry through the following operation with σ_1 and σ_2 interchanged). For the same reasons and conditions that the zeros of $1 + L_+(\sigma)$ are in the left-half plane, the zeros of $1 + L_-(-\sigma)$ are also in the left-half plane. Therefore, we close the σ_2 in the right-half plane and have only the contribution at $\sigma_2 = -\sigma_1$. This yields

$$\int f_2^1 (\vec{p}_1, \vec{p}_2, \vec{k})\, d\vec{p}_2 = \frac{2c^2}{k^2} \int_{-\infty i+\beta}^{\infty i+\beta} d\sigma \frac{1}{1 + L_-(\sigma)} \cdot \int_- \frac{d\vec{p}_2}{\sigma - i\vec{k}\cdot\vec{v}_2}$$

$$\cdot \left\{ \frac{\dfrac{m\omega_p^2}{k^2} i\vec{k}\cdot\dfrac{\partial F_1(\vec{p}_1)}{\partial \vec{p}_1}}{[1 + L_+(\sigma)]\,[\sigma - i\vec{k}\cdot\vec{v}_1]} \int_+ \frac{d\vec{p}_1}{\sigma - i\vec{k}\cdot\vec{v}_1} - \frac{1}{\sigma - i\vec{k}\cdot\vec{v}_1} \right\}$$

$$\vec{k} \cdot \vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)\,. \tag{160}$$

By algebraic manipulations (160) can be put into the form

$$\int f_2^1(\vec{p}_1, \vec{p}_2, \vec{k}) = -\frac{2c^2}{\pi k^2} \int_{-\infty i+\beta}^{\infty i+\beta} \frac{d\sigma}{\sigma - i\vec{k}\cdot\vec{v}_1} \frac{1}{(1+L_+(\sigma))(1+L_-(\sigma))} \int_- \frac{d\vec{p}_2}{\sigma - i\vec{k}\cdot\vec{v}_2}$$

$$\cdot \left\{ 1 + i\frac{m\omega_p^2}{k^2} \vec{k}\cdot \frac{\partial F_1(\vec{p}_2)}{\partial \vec{p}_2} \int_+ \frac{d\vec{p}_2}{\sigma - i\vec{k}\cdot\vec{v}_2} \right\} \vec{k}\cdot\vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2). \tag{161}$$

In the limit of very small β

$$\frac{1}{m}\int_\pm \frac{d\vec{p}_2}{\sigma - i\vec{k}\cdot\vec{v}_2} = \frac{1}{m} ⨍_\sigma \frac{d\vec{p}_2}{\sigma - i\vec{k}\cdot\vec{v}_2} \pm \pi \int \delta(\vec{k}\cdot\vec{v}_2 + i\sigma) \frac{d\vec{p}_2}{m}$$

$$\int_{-\infty i+\beta}^{\infty i+\beta} \frac{d\sigma}{\sigma - ik\cdot v_1} = ⨍ \frac{d\sigma}{\sigma - ik\cdot v_1} + \pi i \int d\sigma\, \delta(\sigma - i\vec{k}\cdot\vec{v}_1). \tag{162}$$

Using (162) in (161) with further algebra, we obtain

$$\int \bar{f}_2^1(\vec{p}_1, \vec{p}_2, \vec{k})\, d\vec{p}_2 = -\frac{2c^2}{k^2} \int_{-\infty i+\beta}^{\infty i+\beta} \frac{d\sigma}{\sigma - i\vec{k}\cdot\vec{v}_1} \frac{1}{(1+L_+(\sigma))(1+L_-(\sigma))}$$

$$\Bigg[(1+L_-(\sigma)) \int_+ \frac{d\vec{p}_2}{\sigma - i\vec{k}\cdot\vec{v}_2} \cdot \vec{k}\cdot\vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)$$

$$- 2\pi \int d\vec{p}_2\, \delta(\vec{k}\cdot\vec{v}_2 + i\sigma)\, \vec{k}\cdot\vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2) \Bigg].$$

The first integral may be closed in the right-half plane with zero contribution because there are no poles in this plane. (By reflection the zeros of $1 + L_-(\sigma)$ are in the right-half plane because the zeros of $1 + L_-(-\sigma)$ are in the left-half plane.)

We obtain, as a final result,

$$\int \bar{f}_2^1(\vec{p}_1, \vec{p}_2, \vec{k})\, d\vec{p}_2$$

$$= \frac{4\pi c^2}{k^2} \int_{-\infty i+\beta}^{\infty i+\beta} \frac{d\sigma}{\sigma - i\vec{k}\cdot\vec{v}_1} \frac{1}{(1+L_+(\sigma))(1+L_-(\sigma))}$$

$$\int d\vec{p}_2\, \delta(\vec{k}\cdot\vec{v}_2 + i\sigma)\, \vec{k}\cdot d_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)$$

$$= \frac{4\pi c^2}{k^2} \oint \frac{d\sigma}{\sigma - i\vec{k}\cdot\vec{v}_1} \frac{1}{|1+L_+(\sigma)|^2} \int d\vec{p}_2\, \delta(\vec{k}\cdot\vec{v}_2 + i\sigma)\, \vec{k}\cdot\vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)$$

$$+ i\frac{4\pi c^2}{k^2} \frac{1}{|1+L_+(ik\cdot v_1)|^2} \int d\vec{p}_2\, \delta(\vec{k}\cdot(\vec{v}_2 - \vec{v}_1))\, \vec{k}\cdot\vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)\,, \qquad (163)$$

where we use (162) and the fact that $(1+L_+(\sigma))(1+L_-(\sigma)) = |1+L_+(\sigma)|^2$ for σ that is pure imaginary.

Fortunately the principal value term in (163) is even in $\vec{k}$ while the second term is odd. Thus in the integral

$$n \int [\psi_{12};\, f_2^1(\vec{p}_1, \vec{p}_2, \overline{q_1 - q_2})]\, d\vec{p}_2 d\vec{q}_2 = -\frac{m\omega_p^2 i}{(2\pi)^3} \int \frac{d\vec{k}}{k^2} \vec{k}\cdot\frac{\partial}{\partial\vec{p}_1} \int \bar{f}_2^1\, (\vec{p}_1, \vec{p}_2, \vec{k})\, d\vec{p}_2 \qquad (164)$$

we need only the second term.

Introducing (163) into (164), we obtain (17) for the evolution of a spacially homogeneous F_1,

$$\frac{\partial F_1(\vec{p}_1)}{\partial t} = \frac{16\pi^3 n c^4}{(2\pi)^3} \iint \frac{d\vec{k}\, d\vec{p}_2}{k^4} \vec{k}\cdot\frac{\partial}{\partial\vec{p}_1} \frac{\delta(\vec{k}\cdot(\vec{v}_2 - \vec{v}_1))}{|1+L_+(i\vec{k}\cdot\vec{v}_1)|^2} \vec{k}\cdot\vec{d}_{12}\, F_1(\vec{p}_1)\, F_1(\vec{p}_2)\,.$$

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References

1. J. Kirkwood and J. Poirier, J. Phys. Chem. 58, 591 (1954).
2. L. Spitzer, Jr. and R. Härm, Phys. Rev. 89, 977 (1953).
3. A. Vlasov, J. Exptl. Theoret. Phys. (U.S.S.R.) 8, 291 (1938).
4. L. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).
5. S. Chandrasekhar, Revs. Modern Phys. 15, 1 (1943).
6. W. P. Allis, Motions of Ions and Electrons, Handbuch der Physik, Vol. 21, edited by S. Flügge (Springer Verlag, Berlin, 1956), pp. 384-444; Technical Report 299, Research Laboratory of Electronics, M.I.T., June 13, 1956.
7. R. Balescu, Phys. Fluids 3, 52 (1960).
8. C. Tchen, Phys. Rev. 114, 394 (1959).
9. N. Rostocker and M. Rosenbluth, Phys. Fluids 3, 1 (1960).
10. A. Lenard, Ann. Phys. 3, 390 (1960).
11. T. Dupree, Phys. Fluids 4, 696 (1961).
12. N. Bogobiubov, Problems of Dynamic Theory in Statistical Physics (Leningrad, 1946); Translation, L. Venters, AEC-tr-3852.
13. G. Baccus, J. Math. Phys. 1, 178 (1960).
14. J. Drummond, Plasma Physics (McGraw-Hill Book Company, Inc., New York, 1961), Chapter 1.
15. Bateman Manuscript Project, California Institute of Technology, Tables of Integral Transforms, Vol. 1 (McGraw-Hill Book Company, Inc., New York, 1954), Chapter 7.
16. Bateman Manuscript Project, California Institute of Technology, Higher Transcendental Functions, Vol. 2 (McGraw-Hill Book Company, Inc., New York, 1954).