

UNCLASSIFIED

AD 296 495

*Reproduced
by the*

**ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA**



UNCLASSIFIED

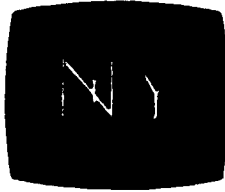
NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-2-4

IMM-NYU 306
JANUARY 1963

CATALOGED BY ASTIA

AS AD No. 29 6495



NEW YORK UNIVERSITY
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES

On the Combination of Independent Two Sample Tests of a General Class

M. L. PURI

296 495

PREPARED UNDER
CONTRACT NO. NONR-285(38)
OFFICE OF NAVAL RESEARCH

ERRATA

Title - ON THE COMBINATION OF INDEPENDENT TWO SAMPLE TESTS OF A
GENERAL CLASS

Author - M. L. Puri

Report - IMM-NYU 306

CORRECTIONS:

Page 3: Line 9

From: $y_i n_i$

To: y_{in_i}

Page 4: Line 10

From: $V_1^{(i)}$

To: $V_1^{(i)}$

Page 5: Line 11 (Assumption 2)

From:

$$\int_{I_{N_1}} [J_{N_1}^{(H_{N_1})} - J(H_{N_1})] dS_{m_1}^{(1)}(x) = o_p(1/N_1^{1/2})^3; i=1, \dots, k$$

To:

$$\int_{I_{N_1}} [J_{N_1}^{(H_{N_1})} - J(H_{N_1})] dS_{m_1}^{(1)}(x) = o_p \left(\frac{1}{\sqrt{N_1}} \right); i=1, \dots, k$$

Page 5: Line 13 (Assumption 3)

From: $J_{N_1}(1) = o(\sqrt{N_1})$

To: $J_{N_1}(1) = o(\sqrt{N_1})$

New York University
Courant Institute of Mathematical Sciences

On the Combination of Independent Two
Sample Tests of a General Class¹

M. L. Puri

This report represents results obtained at the University of California, Berkeley under Contract Nonr-222(43) and at the Courant Institute of Mathematical Sciences under the sponsorship of the Office of Naval Research, Contract No. Nonr-285(38). Reproduction in whole or in part is permitted for any purpose of the United States Government.

1. Summary

Let $X_{11}, \dots, X_{1m_1}$ and $Y_{11}, \dots, Y_{1n_1}$; $i=1, \dots, k$ be k pairs of samples of mutually independent observations from continuous distribution functions $F_i(x)$ and $G_i(y)$ respectively; $i=1, \dots, k$. Then for testing the hypothesis $F_i = G_i$; $i=1, \dots, k$; test statistics of

the form (i) $T = \sum_{i=1}^k c_i t_i$ and (ii) $Q = \sum_{i=1}^k c_i Q_i$ are considered.

Here c_i are the weights which may depend upon the sample sizes, t_i student's t statistic for testing the equality of means between two normal populations with the same variance corresponding to the i^{th} pair of samples and Q_i is the Chernoff-Savage Statistic² (1958) for the i^{th} pair of samples. Under suitable assumptions, the weights c_i which maximize the local asymptotic powers of the tests (i) and (ii) are obtained. These results are specialized to (a) Pitman's shift alternatives, (b) Lehmann's distribution free alternatives and (c) contaminated alternatives. Finally, the asymptotic efficiencies of Q test relative to some of its parametric as well as non-parametric competitors against the above mentioned alternatives are discussed.

2. Introduction

It frequently happens that several independent test statistics are available for testing the same null hypothesis. These may have arisen from several sets of independent samples which cannot be combined perhaps because they are reported by different investigators or because they have not all been gathered under the same conditions. In such situations, it is often considered reasonable

to combine the various results into a single measure on which an objective judgment of the evidence as a whole can be based. One measure is advanced by Fisher (1932). He proposed as a test statistic the product of the tail errors of the individual tests. It turns out that -2 times the logarithm of this product has a chi-square distribution with $2k$ degrees of freedom when the null hypothesis is true, k being the number of tests. For detailed discussion about Fisher's method, the reader is referred to the paper of Wallis (1942). General discussion of combining independent tests can also be found in Birnbaum (1954) and Pearson (1938). Recently, an interesting technique was advanced by Ph. van Elteren (1960). He analyses a class of tests based on linear

combinations $\sum_{i=1}^k c_i W_i$ of test statistics $W_1, \dots, W_k$ of k independent two sample Wilcoxon tests. He considers in particular two special linear combinations, when (i) $c_i = c/m_i n_i$ and (ii) $c_i = c/(m_i + n_i + 1)$ where c is a positive real number and m_i, n_i are the sample sizes of the i^{th} set and shows that the test (i) has a region of consistency independent of sample sizes and the test (ii) has asymptotically the maximum power. In this paper, we consider a similar problem in a more general frame work which includes as a special case the problem considered by Ph. van Elteren (1960), mentioned above. Precisely, we consider the following problem.

3. Problem

Let $\underline{X}_i, \underline{Y}_i$; $i=1, \dots, k$ be k pairs of independent stochastic variables about whose cumulative distribution functions, nothing

$$H_1(x) = \lambda_1 F_1(x) + (1-\lambda_1) G_1(x).$$

Let $Z_{N,j}^{(1)} = 1$, if the j^{th} smallest observation in the combined sample of the i^{th} pair comes from X_1 and otherwise let $Z_{N,j}^{(1)} = 0$. Then the Chernoff-Savage statistic (1958) for the i^{th} pair of samples is

$$(4.1) \quad Q_i = \frac{1}{m_i} \sum_{j=1}^{N_1} E_{N,j}^{(1)} Z_{N,j}^{(1)}$$

where the $E_{N,j}^{(1)}$ are given numbers. Note that Wilcoxon's statistic for the i^{th} pair of samples is obtained from (4.1) by letting $E_{N,j}^{(1)} = j/N_1$ and the normal score statistic for the corresponding samples by letting $E_{N,j}^{(1)} = E(V_j^{(1)})$ where $V_1^{(1)} < \dots < V_{N_1}^{(1)}$ is an ordered sample of size N_1 from a Standard normal distribution.

Following Chernoff-Savage (1958), we shall use the following equivalent form of Q_i :

$$(4.2) \quad Q_i = \int_{-\infty}^{+\infty} J_N(H_{N_1}(x)) dS_{m_i}^{(1)}(x); \quad i=1, \dots, k,$$

where $E_{N,j}^{(1)} = J_N(j/N_1)$.

While J_N need be defined only at $1/N_1, \dots, N_1/N_1$ but may have its domain of definition extended to $(0,1]$ by letting J_N be constant on $(j/N_1, (j+1)/N_1]$.

In this paper, we consider the statistics of the form

$$(4.3) \quad Q = \sum_{i=1}^k c_i Q_i$$

where the c 's are real positive numbers and may depend upon the sample sizes.

We may test the hypothesis $H_0: F_1(x) = G_1(x); i=1, \dots, k$ by means of the critical region $Q \geq Q_\alpha$ where Q_α is given by

$$(4.4) \quad P_{H_0}(Q \geq Q_\alpha) = \alpha$$

where α is the level of significance. If the distribution of Q under H_0 is symmetric with respect to the origin, then the corresponding left-sided test will have a critical region: $Q \leq -Q_\alpha$ and the two sided test will have a critical region $|Q| \geq Q_{\alpha/2}$.

5. General Properties of the distribution of Q

In what follows, we make the following assumptions:

(1) $J(H) = \lim_{N \rightarrow \infty} J_N(H)$ exists for $0 < H < 1$ and is not constant.

(2) $\int_{I_{N_1}} [J_N(H_{N_1}) - J(H_{N_1})] dS_{m_1}^{(1)}(x) = O_p(1/N_1^{1/2})^3; i=1, \dots, k$

where $I_{N_1} = \{x: 0 < H_{N_1}(x) < 1\}$.

(3) $J_{N_1}(1) = O(\sqrt{N_1})$

(4) $|J^{(r)}(H)| = \left| \frac{d^r J}{dH^r} \right| \leq K [H(1-H)]^{-r-(1/2)+\delta}$ for $r = 0, 1, 2$

and for some $\delta > 0$ and some K .

Then, the application of Chernoff-Savage theorem (1958) yields

$$(5.1) \quad \lim_{N_1 \rightarrow \infty} P\left(\frac{Q_1 - \mu_1(\theta)}{\sigma_1(\theta)} \leq x\right) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

where

$$(5.2) \quad \mu_1(\theta) = \int_{-\infty}^{+\infty} J[H_1(x)] dF_1(x)$$

and

$$\begin{aligned}
(5.3) \quad N_1 \sigma_1^2(\theta) = 2(1-\lambda_1) & \left\{ \iint_{-\infty < x < y < \infty} G_1(x)[1-G_1(y)] J[H_1(x)] J[H_1(y)] \right. \\
& dF_1(x) dF_1(y) \\
& + \left. \left(\frac{1-\lambda_1}{\lambda_1} \right) \iint_{-\infty < x < y < \infty} F_1(x)[1-F_1(y)] J[H_1(x)] J[H_1(y)] \right. \\
& dG_1(x) dG_1(y) \left. \right\}
\end{aligned}$$

provided $\sigma_1(\theta) \neq 0$.

Thus

$$(5.4) \quad \mu(0) = E_{H_0}(Q) = \sum_{i=1}^k c_i a_i$$

where

$$(5.5) \quad a_i = \int_{-\infty}^{+\infty} J[F_1(x)] dF_1(x)$$

$$(5.6) \quad \sigma^2(0) = \text{var}_{H_0}(Q) = \sum_{i=1}^k c_i^2 \frac{n_i}{m_i N_1} A^2$$

where

$$(5.7) \quad A^2 = \int_0^1 J^2(x) dx - \left(\int_0^1 J(x) dx \right)^2$$

$$(5.8) \quad \mu(\theta) = E(Q) = \sum_{i=1}^k c_i \mu_i(\theta)$$

$$(5.9) \quad \sigma^2(\theta) = \text{var}(Q) = \sum_{i=1}^k c_i^2 \sigma_i^2(\theta)$$

where $\mu_i(\theta)$ and $\sigma_i^2(\theta)$ are given by (5.2) and (5.3) respectively.

By the Central Limit Theorem, the distribution of Q will be approximately normal.

It follows that the critical value Q_α is approximately equal to

$$(5.10) \quad Q_\alpha = \mu(0) + \lambda_\alpha A \sqrt{\sum_{i=1}^k (c_i^2 n_i) / m_i N_1}$$

where

$$(5.11) \quad \int_{\lambda_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \alpha$$

and the power of the Q test with respect to a given set of distribution functions $F_1(x)$ and $G_1(x)$ is approximately equal to

$$(5.12) \quad \beta_\alpha(\theta) = 1 - \Phi\left[\lambda_\alpha - \frac{\mu(\theta) - \mu(0)}{\sigma(0)} \frac{\sigma(0)}{\sigma(\theta)}\right]$$

where $\Phi(x)$ is the standard normal distribution function.

6. Locally Best Q Test

From this section onward, we assume that m_1 , n_1 and k are non-decreasing functions of a natural number n that tends to infinity. The dependence on n is indicated when necessary, by writing $m_1(n)$, $n_1(n)$, $k(n)$, $\mu^{(n)}(\theta)$, etc. We shall consider the following two special cases:

Case 1: $m_1(n)$ and $n_1(n)$ tend to infinity as n tends to infinity but $\frac{m_1(n)}{n}$ and $\frac{n_1(n)}{n}$ remain bounded away from zero, $k(n) = k$ for each n .

Case 2: $m_1(n)$ and $n_1(n)$ remain constants and $k(n)$ tends to infinity as n tends to infinity. For simplicity sake, we assume that $m_1(n) = m_1$, $n_1(n) = n_1$ and $k(n) = n$.

Furthermore, we make the following assumption:

Assumption 6.1:

For sufficiently large n ,

$$\sqrt{n}[J\{H_1(x;n)\} - J\{F_1(x;n)\}] / A$$

remains bounded as n tends to infinity. Then we prove the following

Theorem 6.1.

For each index n , assume the validity of the case 1 and assumption 6.1. Then the Q test with

$$(6.1) \quad c_1(n) = c \frac{d_1(n)m_1(n)N_1(n)}{n_1(n)}$$

where

$$(6.2) \quad d_1(n) = \int_{-\infty}^{+\infty} [J\{H_1(x;n)\} - J\{F_1(x;n)\}] dF_1(x;n)$$

and c is an arbitrary positive constant, has for $n \rightarrow \infty$, asymptotically the largest power against all alternatives for which $d_1(n)$ are positive.

Proof. We shall first prove that

$$\frac{\sigma^{(n)}(0)}{\sigma^{(n)}(\theta)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

For this it suffices to show that $\sigma^{(n)}(\theta)$ is continuous at $\theta = 0$, uniformly in n . Consider first, the first integral on the right hand side of (5.3) and let it be denoted by $A_1(\theta)$. Thus

$$(6.3) \quad A_1(\theta) = \iint_{0 < x < y < \infty} G_1(x)[1-G_1(y)]J[H_1(x)]J[H_1(y)] \\ dF_1(x)dF_1(y).$$

Setting $F_1(x) = u$ and $F_1(y) = v$, we rewrite (6.3) as

$$(6.4) \quad A_1(\theta) = \iint_{0 < u < v < 1} G_1^*(u)[1-G_1^*(v)]J[H_1^*(u)]J[H_1^*(v)]dudv$$

where $G_1^*(u) = G_1[F_1^{-1}(u)]$ and $H_1^*(u) = \lambda_1 u + (1-\lambda_1)G_1^*(u)$.

It is clear that integrand is continuous at $\theta = 0$ for almost all u and v .

Furthermore, since

$$G_1^*(u) \leq \frac{1}{1-\lambda_1} H_1^*(u)$$

$$1-G_1^*(v) \leq \frac{1}{1-\lambda_1} [1-H_1^*(v)]$$

$$|\dot{J}[H_1^*(u)]| \leq K[H_1^*(u) \{1-H_1^*(u)\}]^{-3/2+\delta}$$

we have, from (6.4)

$$\begin{aligned} (6.5) \quad & |G_1^*(u)[1-G_1^*(v)]\dot{J}[H_1^*(u)]\dot{J}[H_1^*(v)]| \\ & \leq K \frac{1}{(1-\lambda_1)^2} [H_1^*(u)]^{-1/2+\delta} [1-H_1^*(v)]^{-1/2+\delta} [H_1^*(v)]^{-3/2+\delta} \\ & \quad [1-H_1^*(u)]^{-3/2+\delta} . \end{aligned}$$

We may assume $\delta < 1/2$, without loss of generality.

Then, from (6.5)

$$\begin{aligned} & |G_1^*(u)[1-G_1^*(v)]\dot{J}[H_1^*(u)]\dot{J}[H_1^*(v)]| \\ & \leq K \frac{1}{(1-\lambda_1)^2} \lambda_1^{-4+4\delta} u^{-1/2+\delta} (1-v)^{-1/2+\delta} v^{-3/2+\delta} (1-u)^{-3/2+\delta} \end{aligned}$$

Hence by Cramér ([1957], p. 67) $A_1(\theta)$ is continuous at $\theta = 0$.

Similarly the second integral on the right side of (5.3) is continuous at $\theta = 0$. Hence $\sigma_1^{2(n)}(\theta)$ and so a fortiori $\sigma^{2(n)}(\theta)$ is continuous at $\theta = 0$.

Next, because of the assumption (6.1)

$$\frac{\mu^{(n)}(\theta) - \mu^{(n)}(0)}{\sigma^{(n)}(0)} = o(1),$$

Hence the power of the Q test can be approximated by

$$(6.6) \quad 1 - \Phi(\lambda_\alpha - \frac{\mu^{(n)}(\theta) - \mu^{(n)}(0)}{\sigma^{(n)}(0)}).$$

This is maximum, when

$$(6.7) \quad \frac{\mu^{(n)}(\theta) - \mu^{(n)}(0)}{\sigma^{(n)}(0)} = \frac{\sum_{i=1}^k c_i(n) \left[\int_{-\infty}^{+\infty} [J \{H_i(x;n)\} - J \{F_i(x;n)\}] dF_i(x;n) \right]}{A \sqrt{\sum_{i=1}^k [c_i^2(n) n_i(n)] / m_i(n) N_i(n)}}$$

is maximum, which is so when $c_i(n)$ is as defined in (6.1)

This completes the proof of the theorem.

7. Computation of $d_1(n)$

The computation of $d_1(n)$ highly depends upon the sequences of alternatives, we have in mind. In subsequent analysis, we shall concern ourselves with three sequences of admissible alternative hypotheses viz. H_n^P , H_n^L and H_n^C . The hypothesis H_n^P specifies that for each $i=1, \dots, k$; $G_i(x) = F(x + \gamma_i + \frac{\xi}{\sqrt{n}})$, the hypothesis H_n^L specifies that for each $i=1, \dots, k$; $G_i(x) = [F(x + \gamma_i)] \sqrt{1 - \frac{\xi}{\sqrt{n}}}$ and the hypothesis H_n^C specifies that for each $i=1, \dots, k$; $\underline{x}_i$ has the distribution function $F(x + \gamma_i)$ and $\underline{y}_i$ has the distribution function $(1 - \frac{\xi}{\sqrt{n}})F(x + \gamma_i) + \frac{\xi}{\sqrt{n}}G(x + \gamma_i)$; where γ_i is a real number, ξ is a finite positive constant independent of n , and $F_i(x) = F(x + \gamma_i)$. Alternatives of the form H_n^P were introduced by Pitman, those of the form H_n^L by Lehmann (1953) in order to study the non-parametric procedures

when the alternatives themselves are given in a non-parametric form. For an extensive study of Lehmann's alternatives, the reader is referred to an interesting paper of Savage (1956). Alternatives of the form H_n^C are referred to as contaminated alternatives, which have been considered by Hodges and Lehmann (1956) among others.

We shall, therefore, compute $d_1(n)$ and hence $c_1(n)$ for the above mentioned classes of alternatives. We shall make use of a lemma due to Hodges and Lehmann (1961) and the reader is referred to this reference regarding it. A consequence of this lemma in a form appropriate for our purpose, may be stated as follows:

Lemma 7.1. (Hodges-Lehmann).

If

(i) F is continuous cumulative distribution function
differentiable in each of the open intervals $(-\infty, a_1)$, $(a_1, a_2), \dots$,
 (a_{s-1}, a_s) , (a_s, ∞) and the derivative of F is bounded in each of
these intervals and either

(ii) for the alternative H_n^P or H_n^C the function $\frac{dJ[F(x)]}{dx}$ is bounded
as $x \rightarrow \pm\infty$, or

(ii') for the alternatives H_n^L , the function $F(x) \log F(x) \frac{dJ[F(x)]}{dx}$
is bounded as $x \rightarrow \pm\infty$, then

(7.1) $\sqrt{n} d_1(n) \sim \xi(1-\lambda_1) \int \frac{dJ[F(x)]}{dx} dF(x)$, in case the hypothesis
 H_n^P is valid,

(7.2) $\sqrt{n} d_1(n) \sim \xi(1-\lambda_1) \int -F(x) \log F(x) \frac{dJ[F(x)]}{dx} dx$ in case the
hypothesis H_n^L is valid, and

(7.3) $\sqrt{n} d_1(n) \sim \xi(1-\lambda_1) \int [G(x)-F(x)] \frac{dJ[F(x)]}{dF(x)} dF(x)$ in case the
hypothesis H_n^C is valid.

The proof of this lemma follows by the method used in section 3 and 4 of Hodges-Lehmann (1961).

In order to save space the details are omitted.

Now the quantities $\xi \int \frac{dJ[F(x)]}{dx} dF(x)$, $\xi \int -F(x) \log F(x) \frac{dJ[F(x)]}{dx}$ and $\int [G(x)-F(x)] \frac{dJ[F(x)]}{dF(x)} dF(x)$ being constants, can be absorbed into the constant c of (6.1), with the result that we have $c_1(n) = cm_1(n)$. Thus the material discussed in this section coupled with the one discussed in the previous section yields the following

Theorem 7.1

For each index n , assume the validity of the hypotheses H_n^P or H_n^L or H_n^C and the assumptions of lemma 7.1. Then for the case 1, the Q test with weights $c_1(n) = cm_1(n)$, where c is an arbitrary positive constant, has asymptotically the maximum power.

In what follows, we shall denote the locally best Wilcoxon form of Q -test by the symbol Q_W and we shall call it locally best Q_W test. Thus

$$Q_W = \sum_{i=1}^k cm_1(n) Q_{W_1}$$

where Q_{W_1} is obtained from (4.1) by letting $E_{N,j}^{(1)} = j/N_1$.

8. Relation between Elteren's W test and locally best Q_W test

Let $X_{1,r}$ and $Y_{1,s}$ denote the r^{th} and the s^{th} observations of $\underline{X}_1$ and $\underline{Y}_1$ respectively; $r=1, \dots, m_1$; $s=1, \dots, n_1$.

Denote

$$\text{sgn}(X_{1,r} - Y_{1,s}) = \begin{cases} -1, & \text{if } X_{1,r} - Y_{1,s} < 0 \\ 0, & \text{if } X_{1,r} - Y_{1,s} = 0 \\ +1, & \text{if } X_{1,r} - Y_{1,s} > 0 \end{cases}$$

then the Elteren's locally best W test [cf. Elteren, Ph. van (1960)] for case I as well as case II is defined as

$$(8.1) \quad W = c \sum_{i=1}^k \frac{W_i}{m_i(n_i+1)}$$

where

$$(8.2) \quad W_i = \sum_{r=1}^{m_i} \sum_{s=1}^{n_i} \text{sgn}(X_{1,r} - Y_{1,s})$$

which is equivalent to the Wilcoxon's statistic [cf. Wilcoxon (1945)] for the i^{th} pair of samples.

It is easy to check that

$$(8.3) \quad W_i = 2m_i N_i Q_{W_i} - m_i(N_i+1)$$

so that

$$(8.4) \quad W = 2c \sum_{i=1}^k \frac{m_i N_i}{m_i + n_i + 1} Q_{W_i} - c \sum_{i=1}^k m_i.$$

Hence, asymptotically, the following linear relation exists between the W statistic and Q_W statistic.

$$(8.5) \quad W = 2c Q_W - c \sum_{i=1}^k m_i.$$

In our subsequent analysis, we shall use the following expressions connected with the Elteren's W test:

$$(8.6) \quad \mu(\theta) = E(W) = 2c \sum_{i=1}^k \frac{m_i n_i}{m_i + n_i + 1} \int_{-\infty}^{+\infty} [G_i(x) - F_i(x)] dF_i(x)$$

$$(8.7) \quad \sigma^2(0) = \text{var}_{H_0}(W) = \frac{1}{3} c^2 \sum_{i=1}^k \frac{m_i n_i}{m_i + n_i + 1}.$$

9. On the Combination of Independent Two-Sample tests based on Student's t-statistic.

Let $F_1(x)$ and $G_1(x)$ be normal distribution functions with the same variance σ^2 . Then the student's t-test for the 1th pair of samples, is based on the statistic

$$(9.1) \quad t_1 = \frac{(\bar{X}_1 - \bar{Y}_1) / \sqrt{\frac{1}{m_1} + \frac{1}{n_1}} \sigma}{\sqrt{\left[\sum_{j=1}^{m_1} (X_{1j} - \bar{X}_1)^2 + \sum_{k=1}^{n_1} (Y_{1k} - \bar{Y}_1)^2 \right] / (m_1 + n_1 - 2) \sigma^2}}$$

where

$$(9.2) \quad \bar{X}_1 = \sum_{j=1}^{m_1} X_{1j} / m_1 \quad \text{and} \quad \bar{Y}_1 = \sum_{k=1}^{n_1} Y_{1k} / n_1.$$

But since the denominator of t_1 tends to one in probability, therefore an asymptotically equivalent statistic is

$$(9.3) \quad t'_1 = (\bar{X}_1 - \bar{Y}_1) / \sqrt{\frac{1}{m_1} + \frac{1}{n_1}} \sigma$$

which has normal distribution. Now proceeding as in sections 6 and 7, we conclude

Theorem 9.1.

For each index n , assume the validity of the hypotheses H_n^P, H_n^L or H_n^C . Then the t-test with weights $c_1(n) = c \sqrt{\frac{m_1(n)n_1(n)}{N_1(n)\sigma^2}}$, has for $n \rightarrow \infty$, asymptotically the largest power. (c is an arbitrary positive constant).

We may note that

(a) Under H_n^P ,

$$(9.4) \quad \mu_t(\theta) = E(t) = \sum_{i=1}^k c \frac{m_1(n)n_1(n)}{N_1(n)\sigma^2} \frac{\xi_i}{\sqrt{n}}$$

(b) Under H_n^L ,

$$(9.5) \quad \mu_t(\theta) = E(t) = - \sum_{i=1}^k c \frac{m_1(n)n_1(n)}{N_1(n)\sigma^2} \frac{\xi_i}{\sqrt{n}} \int x(1+\log F(x))dF(x)$$

(c) Under H_n^C ,

$$(9.6) \quad \mu_t(\theta) = E(t) = \sum_{i=1}^k c \frac{m_1(n)n_1(n)}{N_1(n)\sigma^2} \frac{\xi_i}{\sqrt{n}} \int x d(G(x)-F(x))$$

and

$$(9.7) \quad \sigma_t^2(\theta) = \text{var}(t) = \sum_{i=1}^k c^2 \frac{m_1(n)n_1(n)}{N_1(n)\sigma^2}$$

under H_n^P , H_n^L and H_n^C .

10. The asymptotic relative efficiencies of the tests

Briefly, the idea of the asymptotic relative efficiency is the following:

Suppose that for testing the hypothesis H_0 against H_n , two tests T and T^* require N and N^* observations to achieve the same power β at the level of significance α . Then the asymptotic efficiency of T with respect to T^* is defined as

$$N^*/N = e_{T, T^*}(\alpha, \beta, H_0, \{H_n\}).$$

We shall be interested in studying the asymptotic efficiency of (i) the Elteren's W test relative to an arbitrary Q test against (a) Pitman's shift alternatives and (b) Lehmann's distribution free alternatives, and (ii) the Elteren's W test relative to the locally best t test against (a) and (iii) arbitrary Q test relative to the locally best t test against contaminated alternatives.

The asymptotic relative efficiency of the Elteren's W test relative to an arbitrary Q test is stated in the following

Theorem 10.1(a)

If

(i) for all i, $\lim_{n \rightarrow \infty} \frac{m_i(n)}{n} = r_i$ and $\lim_{n \rightarrow \infty} \frac{n_i(n)}{n} = s_i$ exist and are positive,

(ii) the distribution function F is such that

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-\infty}^{+\infty} [J \left\{ \lambda_1 F_1(x) + (1-\lambda_1) F_1\left(x + \frac{\xi}{\sqrt{n}}\right) \right\} - J \left\{ F_1(x) \right\}] dF_1(x) / A$$

exists,

(iii) the hypothesis of lemma 7.1 are assumed,

then

the asymptotic relative efficiency of the Elteren's W test relative to an arbitrary Q test for testing the hypothesis H_0 against H_n^P is

$$(10.1) \quad e_{W,Q}^P = 12A^2 \left(\frac{\int_{-\infty}^{+\infty} f^2(x) dx}{\int_{-\infty}^{+\infty} \frac{dJ[F(x)]}{dx} dF(x)} \right)^2$$

where f is the density of F.

Proof.

Let n index the sample size for the Elteren's W test and n* the corresponding index for the Q test. Furthermore, let the level of significance be fixed at α and the limiting power at β . Then the W and Q tests will have the same limiting power, if

$$\begin{aligned}
& 2 \frac{\sum_{i=1}^k \frac{m_1(n)n_1(n)}{m_1(n)+n_1(n)+1} \int_{-\infty}^{+\infty} [F(x+\gamma_1+\xi/\sqrt{n}) - F(x+\gamma_1)] dF(x+\gamma_1)}{\sqrt{\frac{1}{2} \sum_{i=1}^k \frac{m_1(n)n_1(n)}{m_1(n)+n_1(n)+1}}} \\
&= \frac{\sum_{i=1}^k m_1(n^*) \left[\int_{-\infty}^{+\infty} \left\{ J \left\{ \frac{m_1(n^*)}{N_1(n^*)} F(x+\gamma_1) + \frac{n_1(n^*)}{N_1(n^*)} F(x+\gamma_1+\xi^*/\sqrt{n^*}) \right\} \right. \right. \\
&\quad \left. \left. - J \left\{ F(x+\gamma_1) \right\} \right\} dF(x+\gamma_1) \right]}{A \sqrt{\sum_{i=1}^k \frac{m_1(n^*)n_1(n^*)}{N_1(n^*)}}}
\end{aligned}$$

i.e. if

$$(10.2) \quad 2\sqrt{3} \sqrt{\sum_{i=1}^k \frac{r_1 s_1}{(r_1+s_1)}} \xi \int_{-\infty}^{+\infty} f(x) dF(x) = \frac{1}{A} \sqrt{\sum_{i=1}^k \frac{r_1 s_1}{r_1+s_1}} \xi^* \int_{-\infty}^{+\infty} \frac{dJ[F(x)] dF(x)}{dx}$$

and the same alternatives, if $\xi/\sqrt{n} = \xi^*/\sqrt{n^*}$.

Substituting $\xi^* = \xi \sqrt{n^*/n}$ in (10.2) yields the desired result.

It may be remarked that (10.1) agrees with the result found by Chernoff-Savage (1958), Hodges-Lehmann (1961) for the two-sample problem and Puri (1962) for the c-sample problem. Hence the efficiency results of this paper as well as those mentioned above apply directly to the present problem.

Proofs of theorems 10.1(b) to 10.1(d) are similar to those of theorem 10.1(a) and are therefore omitted.

Theorem 10.1(b).

If

(i) for all i , $\lim_{n \rightarrow \infty} \frac{m_i(n)}{n} = r_i$ and $\lim_{n \rightarrow \infty} \frac{n_i(n)}{n} = s_i$ exist and are positive,

(ii) the distribution function F is such that

$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-\infty}^{+\infty} [J \{ \lambda_1 F_1(x) + (1-\lambda_1)[F_1(x)]^{1-\xi/\sqrt{n}} \} - J \{ F_1(x) \}] dF_1(x)/A$
exists,

(iii) the hypotheses of lemma 7.1 are satisfied,
then,

the asymptotic relative efficiency of the Elteren's W test relative to an arbitrary Q test for testing the hypothesis H_0 against H_n^L is

$$(10.3) \quad e_{W,Q}^L = \frac{3}{4} A^2 \left(\frac{1}{\int_{-\infty}^{+\infty} F(x) \log F(x) \frac{dJ}{dF(x)} \{F(x)\} dF(x)} \right)^2.$$

In particular when $J = \Phi^{-1}$, where Φ is the cumulative normal distribution function,

$$(10.4) \quad e_{W,Q}^L(F(x)) = \frac{3}{4} \left(\frac{1}{\int_{-\infty}^{+\infty} x \log \Phi(x) d\Phi(x)} \right)^2$$

= .927 by numerical evaluation.

We may remark that (10.4) agrees with the result found by Puri (1962) for the c-sample problem.

Theorem 10.1 (c).

If

(i) for all i , $\lim_{n \rightarrow \infty} \frac{m_i(n)}{n} = r_i$ and $\lim_{n \rightarrow \infty} \frac{n_i(n)}{n} = s_i$ exist and are positive,

(ii) the distribution function F and G are such that

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-\infty}^{+\infty} [J \{ \lambda_1 F(x+\gamma_1) + (1-\lambda_1) \{ (1-\theta)F(x+\gamma_1) + \theta G(x+\gamma_1) \} \} - J \{ F(x+\gamma_1) \}] dF(x+\gamma_1) / A$$

exists,

(iii) the hypotheses of lemma 7.1 are assumed,

then,

the asymptotic relative efficiency of an arbitrary Q test relative to locally best T test for testing the hypothesis H_0 against H_n^c is

$$(10.5) \quad e_{Q,T}^c(F,G) = \frac{\sigma^2}{A^2} \left(\frac{\int_{-\infty}^{+\infty} [F(x)-G(x)] \frac{dJ \{ F(x) \}}{dF(x)} dF(x)}{\int_{-\infty}^{+\infty} [F(x)-G(x)] dx} \right)^2$$

If, in particular, $J(u) = u$, then Q test becomes Elteren's W test and we have

$$(10.6) \quad e_{W,T}^c = 12 \sigma^2 \left(\frac{\int_{-\infty}^{+\infty} [F(x)-G(x)] dF(x)}{\int_{-\infty}^{+\infty} [F(x)-G(x)] dx} \right)^2$$

We may remark that the result (10.6) agrees with the result found by Hodges, Lehmann (1956) for the two sample problem. Hence their general comments regarding the merits of the performance of Wilcoxon test relative to t test against contaminated alternatives may be carried along the present situation.

Theorem 10.1(d).

If

(i) for all 1, $\lim_{n \rightarrow \infty} \frac{m_1(n)}{n} = r_1$ and $\lim_{n \rightarrow \infty} \frac{n_1(n)}{n} = s_1$ exist and are positive.

(ii) the hypotheses of lemma 7.1 are satisfied,

then,

the asymptotic relative efficiency of the Elteren's W test relative to the locally best T test for testing the hypothesis H_0 against H_n^L is

$$(10.7) \quad e_{W,T}^L(F(x)) = \frac{3}{4} \sigma^2 \frac{1}{\left(\int_{-\infty}^{+\infty} x[1+\log F(x)]dF(x) \right)^2}$$

which agrees with the result obtained by the author (1962) for the c-sample problem.

Similarly, it can be shown that

$$(10.8) \quad e_{W,T}^P(F(x)) = 12 \sigma^2 \left(\int_{-\infty}^{+\infty} f^2(x)dx \right)^2$$

which is known to be the asymptotic efficiency of the two sample Wilcoxon test relative to the student's t test. Hodges and Lehmann (1956) have shown that always $e_{W,t}^P(F(x)) \geq 0.864$. In case $F(x)$ is normal distribution function, this is $3/\pi$.

We now consider the case II. Let $m_1(n) = m_1$, $n_1(n) = n_1$ and suppose that the number, say v , of pair of samples tend to infinity in such a way that the limits:

$$L = \lim_{v \rightarrow \infty} \frac{1}{v} \sum_{i=1}^v \frac{m_1 n_1}{m_1 + n_1 + 1}$$

and

$$M = \lim_{v \rightarrow \infty} \frac{1}{v} \sum_{i=1}^v \frac{m_1 n_1}{m_1 + n_1}$$

exist. Then subject to the conditions that underlying distributions satisfy some general regularity conditions, it can be shown that

$$(10.9) \quad e_{W,T}^P(F(x)) = \frac{12 \sigma^2 L \left(\int_{-\infty}^{+\infty} f^2(x) dx \right)^2}{M} .$$

As a special case, suppose that N repetitions of k blocks are needed for Elteren's locally best W test and N^* for the locally best T test. Then we shall have

$$(10.10) \quad e_{W,T}^P(F(x)) = \frac{12 \sigma^2 \sum_{i=1}^k \frac{m_i n_i}{m_i + n_i + 1} \left(\int_{-\infty}^{+\infty} f^2(x) dx \right)^2}{\sum_{i=1}^k \frac{m_i n_i}{m_i + n_i}} .$$

In particular, where $m_1 = n_1 = 1$

$$(10.11) \quad e_{W,T}^P(F(x)) = 8 \sigma^2 \left(\int_{-\infty}^{+\infty} f^2(x) dx \right)^2$$

which is the asymptotic efficiency of the sign test relative to the student's test, a quantity which is usually expressed as

$$(10.12) \quad e_{s,t}^P(F(x)) = 4 \sigma^2 f^2(0)$$

see in this connection Hodges-Lehmann (1956) and Noether (1958).

It is interesting to note that the asymptotic relative efficiency (10.10) depends on the number of blocks as well as their sizes. When the sample sizes are equal from block to block, say $m_1 = n_1 = m$, then the asymptotic efficiency (10.10) depends only on the block size $2m$. In the special case where $F(x)$ is normal distribution function $\Phi(x)$, we have

$$(10.13) \quad e_{W,T}^P(F(x)) = \frac{3}{\pi} \frac{2m}{2m+1}$$

some values of this expression are tabulated below:

$$(10.14) \quad \begin{array}{cccccccccccccc} m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots & \infty \\ e_{W,T}^P(F) & .637 & .764 & .818 & .849 & .868 & .881 & .891 & .898 & .904 & .909 & \dots & .955 \end{array}$$

In conclusion, we may mention that the results given here are valid for large number of replications.

Acknowledgment

I wish to express my gratitude to Professor Erich L. Lehmann for proposing this investigation and for his constant advice and suggestions while it was in progress.

I also wish to express my appreciation to Doctor Ph. van Elteren for many useful suggestions.

Footnotes

1. This paper was prepared with the support of the Office of Naval Research (Nonr - 222(43) and Nonr - 285(38)). Reproduction in whole or in part is permitted for the purpose of the United States Government.
2. Chernoff and Savage use the symbol T_N instead of Q_1 .
3. If $\{x_n\}$ is a sequence of random variables and $\{r_n\}$ a sequence of positive numbers, we write $x_n = o_p(r_n)$, if x_n/r_n tends to zero in probability, or equivalently, if, for each $\varepsilon > 0$, $P_n\{|x_n|/r_n \leq \varepsilon\} \rightarrow 1$ as $n \rightarrow \infty$.

Resumé

Sur la combinaison de testes indépendantes d'une classe générale pour deux échantillons.

Dans cet article, on analyse des testes qui sont basées sur des combinaisons linéaires de k statistiques indépendantes pour deux échantillons. On compare deux classes de ces testes, où les statistiques employées sont d'une part du type de Chernoff et Savage [1] et de l'autre part du type de "Student". Sous certaines conditions, on obtient les coefficients de ces combinaisons linéaires qui donnent les plus grandes puissances asymptotiques locales de ces testes. Ces résultats sont particularisés au cas où les hypothèses alternatives sont les hypothèses (non paramétriques) de Pitman ou de Lehmann ou des moyennes pondérées souvent appelées "distributions contaminées". Enfin on discute les efficacités asymptotiques du teste Q relatives à quelques-uns de ses compétiteurs paramétriques ainsi que compétiteurs non-paramétriques en relation aux alternatives mentionnées ci-devant.

References

- [1] Birnbaum, Allan (1954). Combining independent tests of significance. *J. Amer. Statist. Assoc.* 49, 559-574.
- [2] Capon, Jack (1961). Asymptotic efficiency of certain locally most powerful rank tests. *Ann. Math. Statist.* 32, 88-100.
- [3] Chernoff, H. and Savage, I.R. (1958). Asymptotic normality and efficiency of certain non-parametric test statistics. *Ann. Math. Statist.* 29, 972-994.
- [4] Cramér, Harold (1957). *Mathematical Methods of Statistics*. Princeton University Press.
- [5] Elteren, Ph. van (1960). On the combination of independent two sample tests of Wilcoxon. *Bull. Inst. Internat. Statist.* 37, 1-13.
- [6] Fisher, R.A. (1932). *Statistical methods for research workers*. Oliver and Boyd, Edinburgh and London; (4th ed.).
- [7] Hodges, J.L., Jr., and Lehmann, Erich L. (1956). The efficiency of some non-parametric competitors of the t-test. *Ann. Math. Statist.* 27, 324-335.
- [8] Hodges, J.L., Jr., and Lehmann, Erich L. (1961). Comparison of the normal scores and Wilcoxon tests. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* 1, 307-317.
- [9] Lehmann, Erich L. (1953). The power of rank tests. *Ann. Math. Statist.* 24, 23-43.
- [10] Lehmann, Erich L. (1959). *Testing Statistical Hypotheses*. John Wiley and Sons, Inc.
- [11] Noether, G.E. (1958). The efficiency of some distribution free tests. *Statistica Neerlandica* 12, 63-72.
- [12] Pearson, E.S. (1938). The probability of integral transformation for testing goodness of fit and combining independent tests of significance. *Biometrika* 30, 134-148.

- [13] Puri, M.L. (1962). Asymptotic efficiency of a class of c-sample tests. Submitted to the Ann. Math. Statist.
- [14] Savage, I.R. (1956). Contribution to the theory of rank order statistics: Two sample case. Ann. Math. Statist. 27, 590-615.
- [15] Wallis, W.A. (1942). Compounding probabilities from independent significance tests. Econometrika 10, 229-248.
- [16] Wilcoxon, F. (1945). Individual comparison by ranking methods. Biometrics 1, 80-82.

Approved Distribution List for Unclassified Technical Reports

Head, Statistics Branch Office of Naval Research Washington 25, D.C. (2)	Professor Herman Chernoff Applied Math. and Stat. Lab. Stanford University Stanford, California (1)
Commanding Officer Office of Naval Research Branch Office, Navy 100 Fleet Post Office New York, New York (2)	Professor W.G. Cochran Department of Statistics Harvard University Cambridge, Massachusetts (1)
ASTIA Document Service Center Arlington Hall Station Arlington 12, Virginia (10)	Professor Benjamin Epstein Applied Math. and Stat. Lab. Stanford University Stanford, California (1)
Technical Information Officer Naval Research Laboratory Washington 25, D.C. (6)	Professor W. Hirsch Inst. of Math. Sciences New York University New York 3, New York (1)
Institute for Defense Analyses Communications Research Division von Neumann Hall Princeton, New Jersey (1)	Dr. Paul G. Hoel Department of Mathematics University of California Los Angeles 24, California (1)
Bureau of Supplies and Accounts Code OW Department of the Navy Washington 25, D.C. (1)	Professor Harold Hotelling Associate Director Institute of Statistics University of North Carolina Chapel Hill, North Carolina (1)
Professor T.W. Anderson Department of Mathematical Statistics Columbia University New York 27, New York (1)	Professor L. Murwicz School of Business Administration University of Minnesota Minneapolis, Minnesota (1)
Professor E.W. Birnbaum Laboratory of Statistical Research Department of Mathematics University of Washington Seattle 5, Washington (1)	Professor Leo Katz Department of Statistics Michigan State University East Lansing, Michigan (1)
Professor A.H. Bowker Applied Mathematics and Statistical Lab. Stanford University Stanford, California (1)	Professor Oscar Kempthorne Statistics Laboratory Iowa State College Ames, Iowa (1)
Professor Ralph A. Bradley Department of Stat. and Statistical Lab. Virginia Polytechnic Institute Blacksburg, Virginia (1)	Dr. Carl F Kossack I.B.M. Corp. Data Processing 2601 Main St. Houston 2, Texas (1)

Approved Distribution List (Con't.)

Professor Gerald J. Lieberman
Applied Mathematics and
Statistics Laboratory
Stanford University
Stanford, California (1)

Professor William G. Madow
Stanford Research Institute
Menlo Park, California (1)

Professor J. Neyman
Department of Statistics
University of California
Berkeley 4, California (1)

Professor Herbert Robbins
Mathematical Statistics Department
Columbia University
New York 27, New York (1)

Professor Murray Rosenblatt
Department of Mathematics
Brown University
Providence, R.I.

Professor L.J. Savage
Statistical Research Laboratory
Chicago University
Chicago 37, Illinois (1)

Dr. Herbert Solomon
Department of Statistics
Stanford University
Stanford, California (1)

Professor Frank Spitzer
Department of Mathematics
Cornell University
Ithaca, New York (1)

Professor S.S. Wilks
Department of Mathematics
Princeton University
Princeton, New Jersey (1)

Professor Evan J. Williams
Institute of Statistics
State College Section
North Carolina State College
Raleigh, North Carolina (1)

Professor J. Wolfowitz
Department of Mathematics
Cornell University
Ithaca, New York (1)

Professor W.H. Kruskal
Department of Statistics
The University of Chicago
Chicago 37, Illinois (1)