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Operations Analysis Working Paper Nr 4

MACHINE SELECTION STUDY

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January 1960

Operations Analysis Office
Directorate of Plans and Programs
Headquarters, Air Materiel Command

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MACHINE SELECTION STUDY

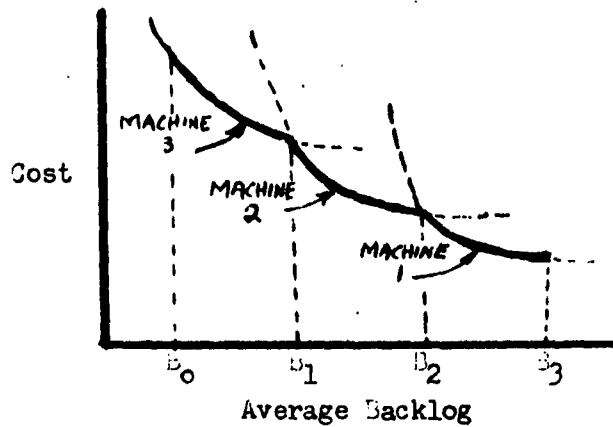
INTRODUCTION

↘ The selection of a machine from a class of available machines is usually a question of some interest and complexity. One finds available equipment with various service rates, reliabilities and various levels of maintainability. Assuming that purchase price or rental varies across the class of machines it is usually desired to select the ^{best} machine, measured in some sense, from those available to perform a specific task. ~~In this paper we will consider and develop~~ some measures of performance ^{are considered} for a very restricted class of machines. [≡] One might expect the following measures of performance to be important: ↗

- a. Average work backlog.
- b. Average work backlog when machine is out of commission.
- c. In commission rate.
- d. Machine utilization when in commission.
- e. Fraction of time during which work backlog exceeds a specified level.

If one could predict these measures of performance, given machine reliability, maintainability, service rate and arrival rate of work for service, it follows that the effectiveness of a particular machine might be assessed and choice exercised. To illustrate suppose one chose average work backlog as a measure of machine effectiveness. Then one might choose that machine which generated an acceptable backlog at least overall cost, i.e., rental and maintenance costs. Suppose three machines were available to perform a specified task and suppose the rate of flow of work into the installation were known. One might find that the following chart represented the average backlog versus total cost for a fixed arrival rate for the three machines.

Figure 1.01 - Cost vs Average Backlog



Therefore, if a backlog between $[B_0, B_1]$ is acceptable then machine 3 is the best choice. If a larger backlog between $[B_1, B_2]$ is acceptable then machine 2 is best. If a very large backlog is acceptable between $[B_2, B_3]$ then machine 1 is best. As one might expect, no one machine of the three is best for the whole range of acceptable backlogs. Further figure 1 assumes constant average arrival rate and would undergo alteration for a different rate, in which case a different machine selection might be involved for the various acceptable backlogs.

Clearly average backlog is only one of many measures of effectiveness that might be selected, but whatever the choice it is usually possible to construct a chart comparable to figure 1 and make a "best" selection.

The paper which follows deals only with the characterization of candidate equipments in terms of the measures of performance listed on Page 1. The technique for making an optimal choice will appear in a subsequent paper. Further, the paper does not provide a fully developed procedure which is ready for implementation and use by operating personnel. Rather, it is an exploratory study which

deals with a hypothetical machine and process which are subject to severe restrictions. The study is presented with the following purposes in mind:

- a. To demonstrate the feasibility of the approach.
- b. To demonstrate how, although technical in its conception and development, it can be used by non-technical personnel.
- c. To indicate the course which has to be taken to relax the restrictions in order that the approach may become more general, realistic and usable.

The paper attempts to introduce the non-mathematical reader to the fundamental elements of probability theory which form the basic structure of the technique. It is believed that this will form a desirable background for those personnel who will be required to evaluate the approach and ultimately decide whether or not to use it.

For those who do not choose to follow the mathematical rationale, the final portion of the study includes a description and sample of the nomographs and tables which would be used by operating personnel. These would be the end-products of the technique, the tools which would permit operating personnel to predict the operation of the various candidate equipments. These appear on pages 21, 25, and 29, respectively..

1. MODEL

We make (perhaps unrealistic) restrictions on the type of machine and process modeled, in the interest of indicating a type of approach that might be taken in more general cases; in short we consider this an exploratory study. We make the following restrictions:

- 1.01 a. Machine capable of performing single functions (e.g., printer only prints.)
- b. Machine processes only one unit at a time (no parallel processing).
- c. Machine functionally disconnected from all other machines.

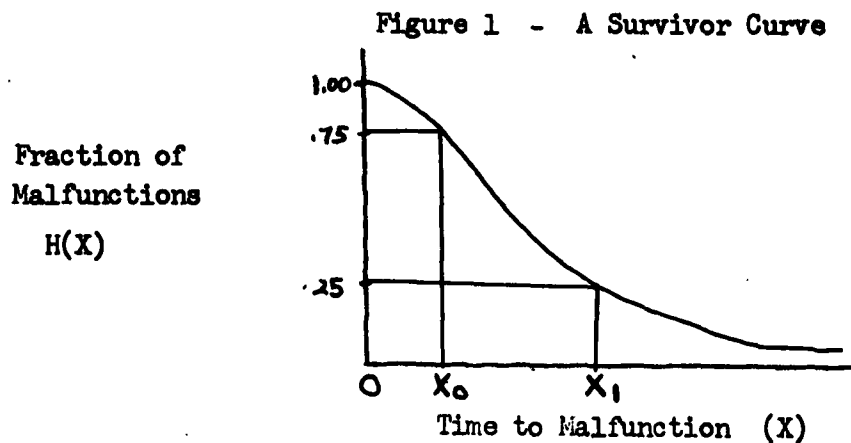
- d. Machine service time much less than one hour and constant.
- e. Machine malfunctions only during service and unit in service not completed.
- f. Arrival of units requiring service is independent of machine status.
- g. Arrivals non-lumpy i.e., units of work do not arrive in groups.
- h. Order of service is first come-first served.

We may summarize by saying we are considering machines capable of one function, rapid in performance, that process units sequentially and independently of other machines. Further the work arrives in units rather than batches and the machine processes work out of backlog without priority. The machine only malfunctions during operation and work in service at time of malfunctions is not completed.

2. NATURE OF TIME DELAY DISTRIBUTIONS CHOSEN

We assume that the times between malfunction, between repair, and between arrival of units in backlog are random variables. We mean by this, for example, that one cannot predict exact time to malfunction, but may specify the percentage of malfunctions for which the time to malfunction will exceed a particular value.

For example in Figure 1 we have a particular survivor curve for a machine.



The survivor curve indicates for a machine, the fraction of malfunctions for which the time to malfunction exceeds a specific time. This curve can be represented by the expression, $H(X)$, where X expresses time in some chosen unit.

We thus see that the fraction of malfunctions for which the time to malfunctions exceeds X_0 is .75. Mathematically, this can be written as

$$2.01 \quad P[X \geq X_0] = H(X_0) = .75$$

This expression states that the probability that the time to malfunction (X) is greater than or equal to X_0 is .75; i.e., $H(X)$, at the point X_0 , is .75. Notice as we choose a larger time X_1 that the fraction of malfunctions occurring with time exceeding X_1 , decreases. Thus if $X_1 \geq X_0$ then

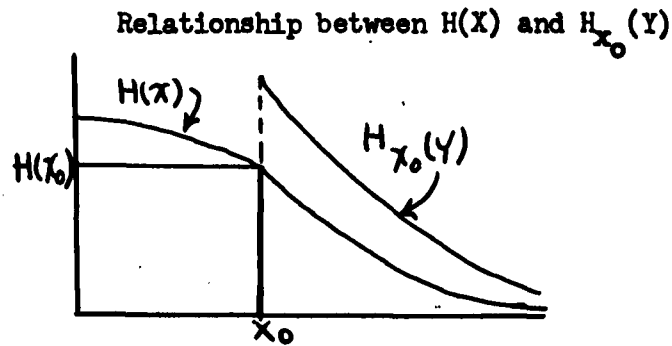
$$2.02 \quad H(X_1) = P[X \geq X_1] \leq P[X \geq X_0] = H(X_0).$$

When $X = 0$, $H(0) = 1$, since all malfunctions must occur subsequent to that time. Now suppose that one observed that the machine had not malfunction at time X_0 . Thus the machine is no longer a so-called "zero age" machine but has accumulated operating time X_0 since its last malfunction and is "age" X_0 . Since we know that the machine must malfunction at some age greater than or equal to X_0 it follows that $H(X_0)$ must be 1, (analogous to $H(0)$ as described earlier) if it is to properly specify the survivor distribution for an " X_0 aged" machine. If one defines a new function $H_{X_0}(Y)$ which gives the fraction of time a machine will not fail in the interval of time X_0 to $X_0 + Y$, given that it has survived to age X_0 , then it can be shown that the new function will be related to the old function, $H(X)$, by the formula.

$$2.03 \quad H_{X_0}(Y) = \frac{H(X_0 + Y)}{H(X_0)}, \quad Y \geq 0$$

In Figure 2.01 we have charted $H(X)$ from Figure 1.01 and $H_{x_0}(Y)$.

Figure 2.01

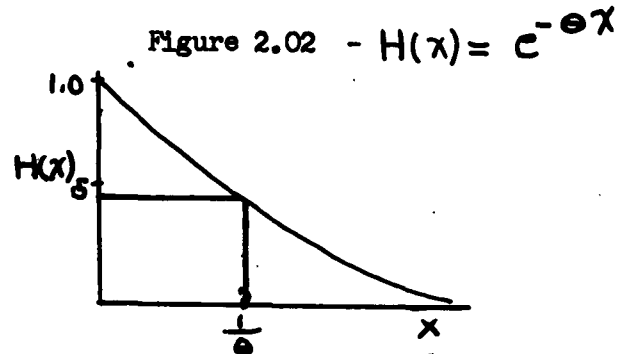


Thus from 2.03 we see that $H_{x_0}(Y)$ is obtained from the $H(X)$ curve by obtaining its value at $x_0 + Y$ and dividing by $H(x_0)$. The important point is: The survivor function for a machine of any age may be computed if one has the survivor function for a "zero aged machine".

The survivor function for this study is assumed to have the form

2.04
$$H(X) = e^{-\theta X} ; \theta > 0, X \geq 0.$$

We have in Figure 2.02 a graph of this function



It can be shown that for this survivor function the average time to malfunction is

2.05
$$E(X) = \frac{1}{\theta}$$

We see from Figure 2.02 that for a zero aged machine, with survivor distribution given by 2.04, 45% of all failures will occur at an age equal to or exceeding the mean time to failure. This result is independent of the value for the mean failure time.

Now suppose an item were known to have age X_0 and had expression 2.04 for a survivor function. Then we see that the survivor function for the item, given that it is age X_0 , is

$$2.06 \quad H_{X_0}(Y) = \frac{H(X_0+Y)}{H(X_0)} = \frac{e^{-\theta(X_0+Y)}}{e^{-\theta X_0}} = e^{-\theta Y}; Y \geq 0, \theta > 0.$$

Notice that $H_{X_0}(Y)$ is exactly the same function as $H(X)$. This means the survivor function, if of form 2.04, implies that the accrual of age on a device does not effect its chance for future survival. In other words the likelihood that an equipment will fail after the elapse of a time Y is not related to the age of the item. It turns out that the only survivor distribution that can have this property is the one specified in 2.04.

We make the assumption in this paper that malfunction time, repair time, and work arrival time have survivor distributions of the form

$$2.07 \quad H(t) = e^{-\alpha t}; E(X) = \frac{1}{\alpha}; t \geq 0, \alpha > 0.$$

$$2.08 \quad H(r) = e^{-\theta r}; E(r) = \frac{1}{\theta}; r \geq 0, \theta > 0.$$

$$2.09 \quad H(a) = e^{-\beta a}; E(a) = \frac{1}{\beta}; a \geq 0, \beta > 0.$$

2.10 $E(t)$ denotes average malfunction time,

2.11 $E(r)$ denotes average repair time,

2.12 $E(a)$ denotes average work arrival time,

2.13 t denotes malfunction time,

2.14 r denotes repair time,

2.15 a denotes arrival time.

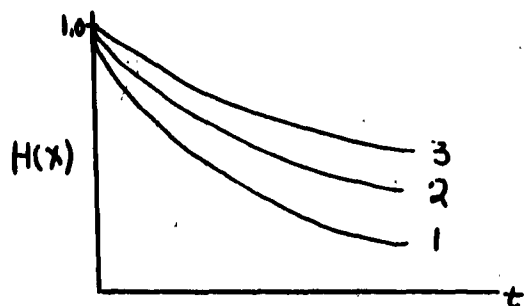
This assumption may be relaxed if actual data for machines indicate a different survivor distribution for repair and malfunction times, or if items arrive at the machines with a different survivor function. A paper given by R. M. Lewis, IBM, at a conference of the IRE, 1959, titled "A Study on the Collection and Analysis of Maintainability Data," indicated that the exponential distribution characterized repair times adequately for a computing system. Therefore, unless evidence to the contrary can be offered we shall maintain the above assumption for repair time. Exponential distributions have also been used extensively in reliability studies; hence, its use for the malfunction distribution appears reasonable. While the exponential distribution may be unrealistic for arrivals, its impact may not be too severe for heavy workloads. In any event, the effect of relaxing this assumption will be investigated in a subsequent paper.

We now discuss the relationship between $E(t)$ and $H(T)$. To do this we graph in Figure 2.04 three survivor curves for the three different mean times. Suppose

2.16
$$E_3(t) > E_2(t) > E_1(t).$$

The curves labeled 1, 2, and 3 are associated with $E_1(t)$, $E_2(t)$ and $E_3(t)$. Further we see that these curves intersect in only one point, mainly where $t = 0$ and $H(0) = 1$.

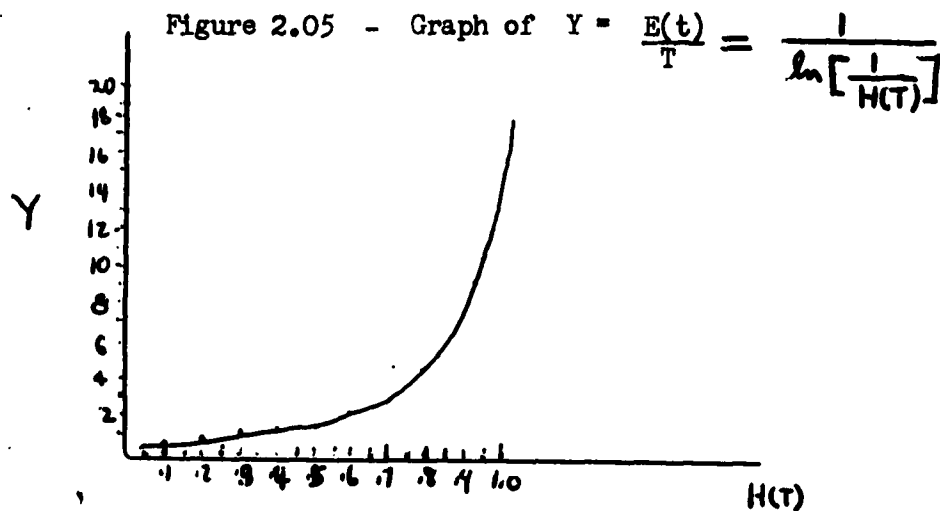
Figure 2.04 - Three Survivor Curves



This suggests that for the particular curve chosen the particular $H(t)$ is uniquely specified by $E(t)$. This is indeed the case. However, this suggests another fact; that if one knew the fraction of malfunctions that a machine survived for a specified time, T , then this would specify the mean time to failure, $E(t)$. This is in fact true and we have

$$2.17 \quad \frac{E(t)}{T} = \frac{1}{\ln\left[\frac{1}{H(T)}\right]}, \quad T > 0.$$

This equation permits the specification of $E(t)$ if one knows for a specific time T , the fraction of malfunctions the machine accrues for age larger than T , i.e., $H(T)$. Since 2.17 is a bit inconvenient we have Figure 2.05 which is a graph for this relationship.



Thus if $H(T) = .7$ and if T is 5 hours then $E(t)$ is 13.89 hours. In other words if the mean time to failure for a machine is 13.89 hours then it will operate without malfunction for at least 5 hours for 70% of all malfunctions. Of course, this result holds only if the survivor curve for the machine is

$$2.18 \quad H(T) = e^{-\theta T}, T \geq 0, \theta > 0 \text{ as assumed in (2.07).}$$

Figure 2.05 can be used to estimate $E(t)$ from empirical data. In order to do this one must be able to determine a value for $H(T)$ at some point of the distribution. To do this, the following procedure can be followed:

- a. Specify a time, T .
- b. For a large number of malfunctions, N , count the number that occur when the machine has accumulated time t or less at malfunction, where $t < T$. Call this number, n .

c. Then $H(T) = \frac{N - n}{N}$. Using Figure 2.05, one then obtains $\frac{E(t)}{T}$, or Y . From this, it is immediately obvious that $E(t) = YT$.

In the next section we describe the operation of the machine and process, and it is necessary to speak of the occurrence of a machine malfunction in one machine service interval or, in short, malfunction during one service time. Further, we often speak of repair completion during one machine service interval or repair completion during one service time. Finally we refer to a unit work arrival during one machine service interval; that is, an arrival during one machine service time.

Specifically we will determine the probability that a machine will malfunction in one service time if it has been operating for time X_0 since the last malfunction.

We have defined in 2.03 the survivor distribution for a "non-zero age machine", i.e., $H_{x_0}(Y)$. We denote a machine service time by

$$2.19 \quad \lambda = \text{one machine service time, } \lambda > 0.$$

Then clearly the fraction of malfunctions for which a machine aged X_0 survives at least one service time is $H_{X_0}(\lambda)$. Therefore, the probability that the machine does malfunction in one service time, given that it has been operating for time X_0 since the last malfunction, is

$$2.20 \quad P[0 \leq x \leq \lambda | \text{AGE} = X_0] = 1 - H_{X_0}(\lambda).$$

This expression states that the probability of the machine malfunctioning in one service time, given that it is of age, X_0 , at the start of the service period is $1 - H_{X_0}(\lambda)$. Now for the specific survivor function used in this study, we have from 2.06 and 2.07 that

$$2.21 \quad P[0 \leq x \leq \lambda | \text{AGE} = X_0] = 1 - e^{-\alpha \lambda}; \quad \lambda > 0.$$

This result states the very important fact that the probability that the machine malfunctions in a service interval is independent of the age of the machine.

In other words, if the survivor distribution is as specified in 2.07 then the chance of malfunction in a service interval for the machine is not affected by the operating time accumulated since last repair. It can be shown that if the product $\alpha\lambda$ is small then (2.20) is approximately equal to $\alpha\lambda$. Written mathematically, we have:

2.22 $P[0 \leq X \leq \lambda | AGE = X_0] = 1 - e^{-\alpha \lambda} \cong \alpha \lambda$ (where \cong means approximately equal). The error committed by this approximation is less than or equal to

$$2.23 \quad \text{Error} \leq \frac{(\alpha \lambda)^2}{2}.$$

Now it is important to note that the mean time to malfunction was specified in 2.07 as

2.24 $E(\lambda) = \frac{1}{\alpha}$. Therefore the result in 2.22 may be written in an interesting form, namely

$$2.25 \quad P_1 = P[0 \leq X \leq \lambda | AGE = X_0] = 1 - e^{-\alpha \lambda} \cong \alpha \lambda = \frac{\lambda}{E(X)}.$$

Thus the probability that the machine will malfunction in one service interval, given that it has operated for a time X_0 , is approximately the service time divided by the mean time to failure for the machine. If the mean failure time for the machine is large relative to the service time, this is a very good approximation.

We let

$$2.26 \quad P_1 = 1 - e^{-\alpha \lambda} \cong \alpha \lambda = \frac{\lambda}{E(t)}$$

$$2.27 \quad P_2 = 1 - e^{-\theta \lambda} \cong \theta \lambda = \frac{\lambda}{E(r)}$$

$$2.28 \quad \mu = 1 - e^{-\beta \lambda} \cong \beta \lambda = \frac{\lambda}{E(a)}.$$

Thus P_1 , P_2 and μ represent the probability that the machine will malfunction in one service time; the probability that the machine repair will be completed in one service time; the probability that one unit of work will arrive in one service time, respectively and these are only a function of average failure time, average repair time, average inter-arrival time and machine service time.

We are now in the position for

3. DESCRIPTION OF PROCESS.

a. Notation employed.

(1) To adequately describe the process by which the model was derived, it is necessary to define several of the basic terms used. The remaining terms will be defined later in the discussion. For the purpose of this model, " i " is used to indicate the status of the machine - i.e., in or out of commission. If $i = 1$, the machine is in commission and conversely if $i = 0$, the machine is out of commission.

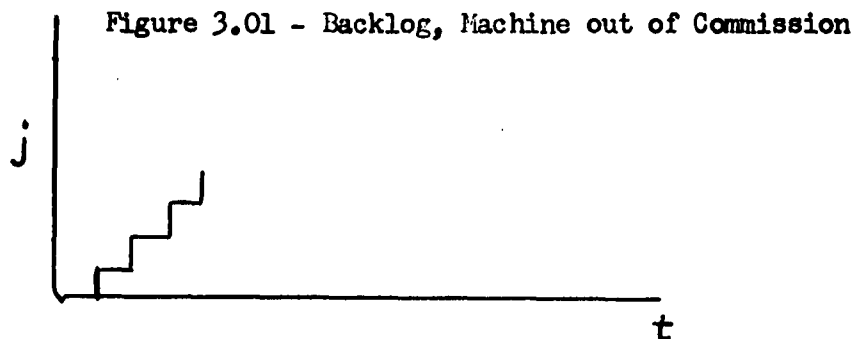
(2) " j " is used to indicate the magnitude of the backlog at a particular point in time.

(3) " t " is used to indicate time.

b. Verbal description.

(1) In a previous section of this report we have specified that average rate at which work arrives is constant over time. We have also said that not more than one unit of work can arrive during a particular service time and the machine can only service one unit at a time. Further, the machine can go out of commission only during a servicing and if this happens, the unit in service is not completed.

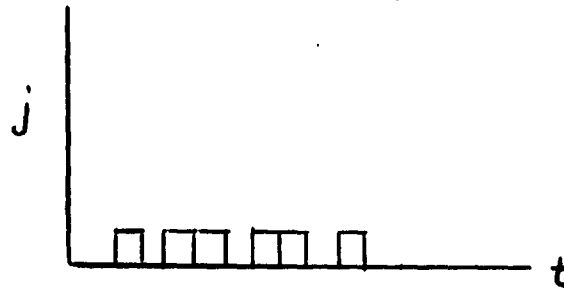
Since we have made the above restrictions we can now look at specific cases of backlog growth and decay. We first investigate the case where $i = 0$ (Machine out of commission). If we plot the backlog j against time in this case we get:



The backlog cannot decrease since there is no servicing possible.

We now investigate the case when $i = 1$ and the backlog is initially zero.

Figure 3.02 - Backlog, Machine in Commission



Since we have assumed that not more than one unit can arrive for servicing during a service time, the backlog with the machine in commission cannot exceed 1. If the backlog does exceed 1, the machine must then be in the state previously described when $i = 0$ (machine goes out of commission). The backlog will therefore grow and decay fluctuating between 0 and 1 as long as the machine remains in commission.

(2) Suppose we now investigate all the possible combinations of status changes as time advances one unit. We let "t" be a particular instant in time, "t+1" one service time later, $[j,1]$ a backlog condition with the machine in commission, $[j,0]$ a backlog condition with machine out of commission. "a" represents an arrival, "n" represents no arrival, "m" represents a malfunction, "n" represents no malfunction, "r" represents a repair, and finally "n" represents no repair. Therefore the following table can be constructed:

Figure 3.03 - Event Table

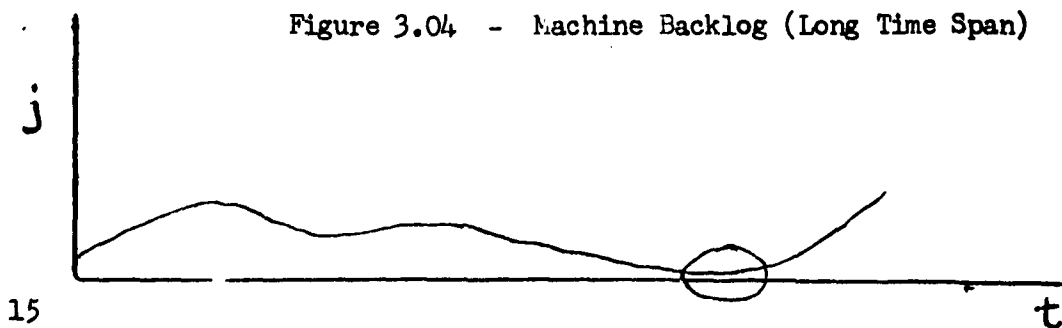
| t/t+1 | $[j,0]$ | $[j,1]$ | $[j-1,0]$ | $[j-1,1]$ | $[j+1,0]$ | $[j+1,1]$ |
|---------|------------------------------|-------------------|-----------|------------------------------|-------------------|-----------|
| $[j,1]$ | a m | a n | 0 | a n | a m | 0 |
| $[j,0]$ | a n | a r | 0 | 0 | a n | a r |

Let us now look at a few examples from this table. Suppose we are initially in the status $[j,1]$ (j items in backlog, machine in commission). To move to a $[j,0]$ condition in the next interval of time, we would have had to have no arrival and a malfunction. This would allow the backlog to remain constant but the status of the machine would go from 1 to 0. To go into status $[j-1,0]$ would be impossible (represented by 0) because we have said that if the machine malfunctions (goes from 1 to 0 status) the unit being serviced will not be completed and therefore the backlog can not decrease. To move into status $[j+1,0]$ from $[j,1]$, we need both an arrival and a malfunction. Since the unit being serviced is not completed, the arrival increases the backlog by one and the malfunction changes the status of the machine from 1 to 0.

Suppose we are now in the status $[j,0]$ (j items in backlog, machine out of commission). To move into $[j-1,0]$ or $[j-1,1]$ is impossible because with the machine out of commission, no servicing is possible and the backlog can therefore not decrease.

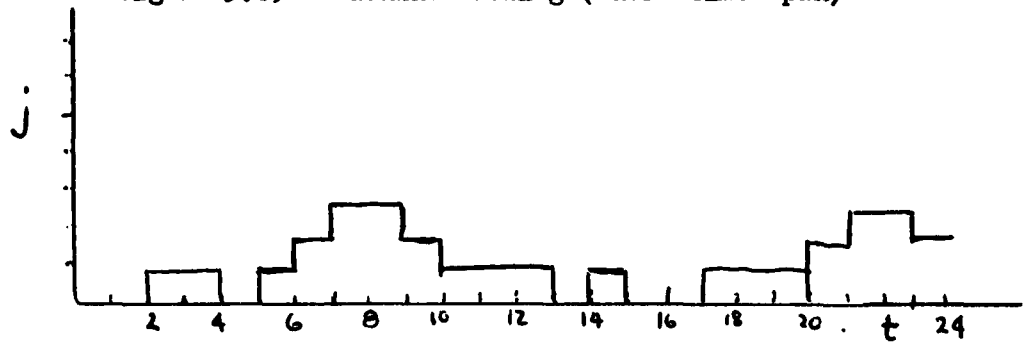
The remaining cells in the table can be determined in the same manner and the completed table will thus represent all possible combinations given the restrictions placed on the model.

(3) Now that we have considered all possible states into which the machine might move in one unit of time, we may graphically represent a long range status sequence for the particular machine in question. This would be done in much the same manner as discussed above but the time variable would be allowed to increase over a long period of time. Hence:



If we now investigate the small circled section of this graph for instance, we can examine in more detail the backlog fluctuations in this area.

Figure 3.05 - Machine Backlog (Short Time Span)



In this chart, each interval of t represents one service time. According to the restrictions placed on the process, there must be a unit in backlog at the beginning of a service interval in order for a service to take place during that interval.

If we look at the interval from $t = 0$ to 1 we see that the backlog was 0 at the beginning of the interval so there is no service performed and since the backlog is still zero at the end of the interval, there was no arrival. The interval from 1 to 2 also has no service but the backlog increases from 0 to 1, hence there was an arrival.

If we investigate the interval from 5 to 6 we see that the backlog has increased by one indicating that the machine went out of commission (since the backlog at the beginning of the interval > 0). Looking at interval 7 to 8, we see that such a condition could exist in one of two ways - either we had a service and an arrival or we had no service and no arrival. If we consider the former, the machine goes back into commission at point 7 and if we consider the latter, at point 8. We will assume for this discussion that the former is true.

We also see that the machine again goes out of commission at point 19 and comes back in commission at either 21 or 22. We will assume it comes back at 21.

This process can be continued for an indefinite time and a similar pattern

to the above could be expected.

By looking at the long term process (Figure 3.04), it is now possible to define certain terms for the process.

(4) Definition of terms

If we allow the intervals in which the machine is in commission (0-5, 7-19, etc.) to be represented by $t_1, t_2, \text{ etc.}, \text{ to } t_n$ and let the total time elapsed be represented by T we can define the fraction of time that the machine is in commission, $P(1)$, to be

$$3.01 \quad P(1) = \frac{t_1 + t_2 + t_3 + \dots + t_n}{T} = \frac{\sum t_i}{T} .$$

Conversely by letting the intervals in which the machine is out of commission be represented by a_1, a_2, \dots, a_n we may define the fraction of time that the machine is out of commission as $P(0)$ so that

$$3.02 \quad P(0) = \frac{a_1 + a_2 + a_3 + \dots + a_n}{T} = \frac{\sum a_i}{T} .$$

Now by letting $k_1 \dots k_n$ measure the time that backlog is greater than zero in each interval t_1 through t_n (when the machine is in commission) we establish a definition of utilization rate, $P[j \geq 1/1]$, so that

$$3.03 \quad P[j \geq 1/1] = \frac{k_1 + k_2 + k_3 + \dots + k_n}{t_1 + t_2 + t_3 + \dots + t_n} = \frac{\sum k_i}{\sum t_i}$$

(5) Process Averages

Having computed the foregoing fractions, it is now possible to compute the average backlogs for the process when the machine is in or out of commission, and we are also able to compute average backlog, over both conditions, which one might call the unconditional average backlog.

If we let b_1, b_2, \dots, b_n represent average backlog in intervals when the machine is out of commission we can compute the average backlog for all intervals when the machine is out of commission, $E[j/0]$, so that

$$3.04 \quad E[j/0] = \frac{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}{a_1 + a_2 + a_3 + \dots + a_n} = \frac{\sum a_i b_i}{\sum a_i}.$$

Conversely if we let $c_1, c_2, c_3, \dots, c_n$ be the average backlog in intervals when the machine is in commission we can compute the average backlog for all intervals in which the machine is in commission, $E[j/1]$, so that

$$3.05 \quad E[j/1] = \frac{c_1 t_1 + c_2 t_2 + \dots + c_n t_n}{t_1 + t_2 + t_3 + \dots + t_n} = \frac{\sum c_i t_i}{\sum t_i}.$$

Combining the two above formulae, we are able to compute the unconditional backlog, $E[j]$, so that

$$3.06 \quad E[j] = \frac{E[j/0] \sum a_i + E[j/1] \sum t_j}{\sum a_i + \sum t_j}.$$

However

$$3.07 \quad \sum a_i + \sum t_j = T \quad \text{since the machine is either in or out of commission, not both.}$$

Further

$$\left. \begin{aligned} P(0) &= \frac{\sum a_i}{T} \\ P(1) &= \frac{\sum t_j}{T} \end{aligned} \right\} \text{by 3.01 and 3.02.}$$

Therefore

$$3.08 \quad E[j] = E[j|0]P(0) + E[j|1]P(1)$$

In the next section we discuss $E[j/1]$, $P(0)$ and $P(1)$ as a function of machine and process characteristics, i.e., the average failure time, average time between work arrival, average repair time and machine service time. These equations were obtained by establishing a system of difference equations which describe mathematically the process described above and by obtaining the generating function for the process from these difference equations. Once we obtain the generating function the various averages and percentages defined above may be easily obtained. We have placed the system of difference equations and generating function for the process in the appendixes to the report.

4. RESULTS OF ANALYSIS. In Section 3 we defined several terms for the process. In this section we specify the relation between these terms and the parameters employed for characterizing the machine and the process and we assume that the process has operated for a long enough span of time so that the various measures developed have become statistically stable. We present first the precise relationships, then simple approximations to them, provided certain assumptions are made and, finally, graphic and tabular means for determining the various relationships.

a. Out of Commission Rate: We define the out of commission rate to be the fraction of total time that the machine is in reparable status. It can be shown that if restrictions 1.01 are in force for the machine and process that the out of commission rate, $P(0)$, is

$$4.01 \quad P(0) = \frac{\mu P_1}{P_2(1-P_1)} \quad \text{where } \mu, P_1 \text{ and } P_2 \text{ are as defined by } \\ 2.26, 2.27 \text{ and } 2.28.$$

A simple approximation for this result may be obtained if the mean malfunction time, mean interarrival time and mean repair times are large in relation to the machine service time. It can be shown that

$$4.02 \quad P(0) \cong \frac{\lambda}{E(a)} \left[\frac{E(r)}{E(t) - \lambda} \right], \text{ if}$$

$$E(r) \gg \lambda$$

$$E(t) \gg \lambda$$

$$E(a) \gg \lambda .$$

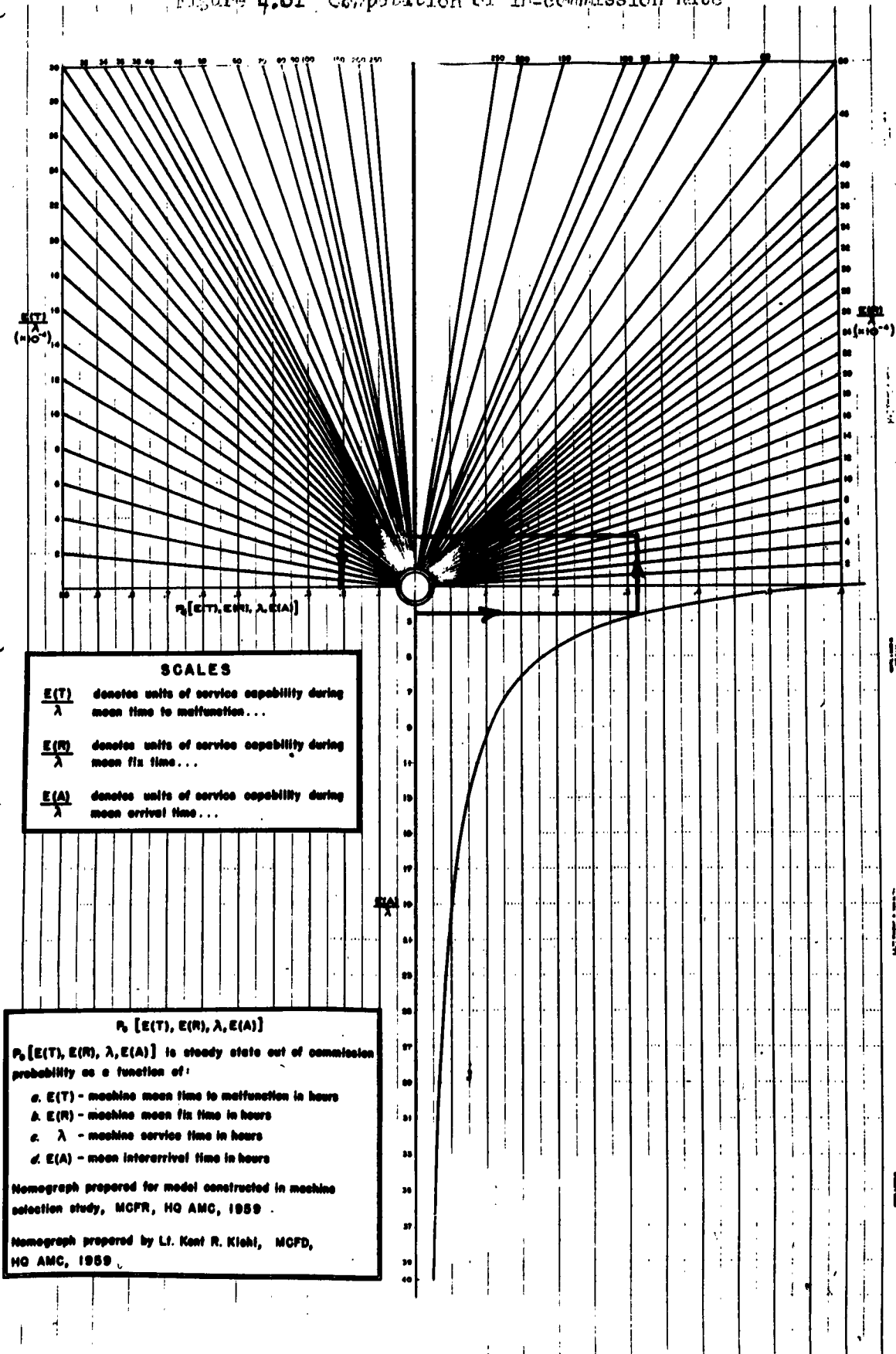
Since the mean malfunction time is much larger than a service interval

$$4.03 \quad P(0) \cong \frac{\lambda}{E(a)} \frac{E(r)}{E(t)} .$$

We therefore see that if mean repair time, mean malfunction time and mean interarrival time are all large relative to a machine service time, that the fraction of time the machine will be out of commission is approximately the product of the ratio of service time to average interarrival time and mean repair time to mean malfunction time. If this is not the case the computation for $P(0)$ is facilitated by employing the chart in Figure 4.01. Figure 4.01 is a nomograph which permits the determination of $P(0)$ with virtually no computation. It would be the tool actually used in the implementation of the technique for estimation of the in-commission rate of a candidate equipment.

In order to use the nomograph, one starts with the scale in the center of the lower half of the chart. To find the point of entry on the scale, take the mean time between arrivals, expressed in hours, and divide it by the service time per unit, also expressed in hours. Find the point on the scale corresponding to this ratio and draw a line horizontally to the right until it reaches the curve. At the point of contact, draw a vertical line until it intersects

Figure 4.01 Computation of In-commission Rate



one of the lines in the upper right hand quadrant. The particular line to be chosen is determined by the ratio of the mean time to repair, divided by the service time, each expressed in hours, and further divided by 10,000. If the ratio obtained does not coincide identically with one of the lines, visual interpolation can be performed. From the point of intersection, draw a line horizontally to the left until it intersects one of the lines in the upper left quadrant of the chart. This line is determined by the ratio of the expected time between malfunctions to the service time, in hours, and further divided by 10,000. Again interpolation can be visually achieved if the ratio obtained is not identical to the value shown at the left of each line. At the point of intersection, draw a vertical line downward until it hits the horizontal scale in the center of the chart to the left of the origin. On this scale can be read the probability that the machine is out of commission. Subtracting this value from 1 (one) gives the in-commission rate.

As an example, assume a machine has the following characteristics:

Service time: 200 lines per minute

Workload: Average of 40,000 lines per 8 hour day

Reliability: Average of 12 hours between malfunction

Maintainability: Average of 7 hours to fix

To use the nomograph, first note that 200 lines per minute is 12,000 lines per hour. Average service time per unit is therefore $1/12000$ hours or .0000833 hours. A workload of 40000 lines per 8 hour day is an average of 5000 lines per hour. Average time between arrivals is $\frac{1}{.5000}$ hours, or .0002 hours. The ratio of arrival time to service time, i.e., $\frac{E(a)}{\lambda}$ is equal to 2.4. This is the figure to be used in entering the chart on the center, lower scale. Choice of the proper line in the upper right quadrant is found by dividing the

maintainability figure (7 hours) by the service time as just computed (.0000833) and again by 10,000. This figure is 8.4. The appropriate line in the upper left hand quadrant is found by dividing the reliability figure, i.e., 12 hours, by the service time (.0000833) and again by 10,000, i.e., 14.4. Use of the nomograph then shows that the machine will be out of commission 20% of the time. The in-commission rate is therefore, 80%.

The machine utilization may be computed by

$$4.04 \quad P[J \geq 1 | 1] \cong \frac{\mu P_2}{P_2(1 - P_1) - \mu P_1} \quad \text{where}$$

μ, P_1, P_2 are defined as before. This result may be substantially simplified if the mean failure time and mean repair time are large relative to the service time for the machine.

In fact we have

$$4.05 \quad P[J \geq 1 | 1] \cong \frac{\mu}{1 - \frac{\lambda}{E(t)} - \mu \frac{E(r)}{E(t)}} \quad \text{if}$$

$E(t) \gg \lambda$ and $E(r) \gg \lambda$. Further if $E(t) \gg E(r)$ and $E(r) \gg \lambda$ then we have

$$4.06 \quad P[J \geq 1 | 1] \cong \frac{\mu}{1 - \mu \frac{E(r)}{E(t)}} \quad \text{and}$$

$$4.07 \quad P[J \geq 1 | 1] \cong \mu .$$

Finally if the mean interarrival time, $E(a)$, is much greater than the service time, λ , we have

$$4.08 \quad P[J \geq 1 | 1] \cong \frac{\lambda}{E(a)}$$

Thus we see that simple approximations for $P\{j > i\}$ are quite feasible if the mean failure time, mean repair time and mean interarrival time for the process are all large relative to machine service time. In fact if mean failure time is larger than mean repair time and if mean arrival time is larger than service time the utilization of the machine, when in commission, is very nearly the ratio of service time to mean interarrival time.

On the other hand, rather than perform these computations we have in Figure 4.02 a chart which permits computation of $P\{j > i\}$ graphically. This nomograph is the second tool which would be used in an operational environment and would permit evaluation of the utilization rate with virtually no computation. It is used in much the same fashion as the preceding one. The center scale on the lower half is entered at the same point as in the first nomograph. As before, a horizontal line is drawn until it intersects the curve. The line is then extended vertically until it intersects the appropriate line in the upper right hand quadrant of the chart. This line would have a value identical to the line used in the left hand quadrant of the first nomograph. Because of the nature of the relationship, however, only 4 lines are shown in the present nomograph with values of 1, 10, 100 and over 100 respectively. This is because the computation is relatively sensitive to values of $\frac{E(t)}{\lambda}$ between 1 and 10; much less sensitive to values between 10 and 100; and virtually insensitive to the range of numbers beginning at 100 and increasing without limit. As before, interpolation between the lines will probably be necessary. From the point of intersection (or from the interpolated point of intersection), a horizontal line is drawn to the left until it intersects the appropriate line in the upper left hand quadrant. This line is determined by the results obtained from using the first nomograph, namely the in-commission rate. From the point of intersection, a vertical line drawn downwards will intersect the scale at the utilization rate.

Using the same hypothetical example for a candidate equipment as we did in the first nomograph, we enter the scale in the center of the lower half of the chart at 2.4. We proceed to the right until we intersect the curve. At the point of intersection we move upwards until we hit an interpolated value of 14.4. We then move left to the line with value of .80, the in-commission rate. This was the value we derived from the first nomograph. Going down again, we read 43% as the utilization rate.

Thus we have approximations and exact functions specifying the in-commission and utilization rates as a function of mean malfunction, repair and arrival times for the process. We turn our attention to the average backlog of work given that the machine is out of commission, i.e., $E[j|0]$,

$$4.09 \quad E[j|0] = 1 + \frac{P(0)}{P_0(1)} \frac{1-M}{P_1 + P_2} \quad ,$$

where

$$4.10 \quad P_0(1) = 1 - \frac{M}{1-P_1} - P(0).$$

A slightly more convenient form is

$$4.11 \quad E[j|0] = 1 + \frac{M(1+P_1P_2) - M^2}{P_2(1-P_1) - M(P_1+P_2)} \quad \text{where}$$

P_1, P_2 and M are defined above.

If $E(t)$ and $E(r)$ are much larger than λ we have

$$4.12 \quad E[j|0] \cong 1 + \frac{E(t)}{\lambda} \frac{E(r)}{\lambda} \frac{M(1-M)}{\frac{E(t)}{\lambda}(1-M) - \frac{E(r)}{\lambda}M}$$

The function in 4.09 has been tabled in the $E[j|0]$ table and we illustrate their use by the following.

Example: Using the hypothetical piece of equipment as we did in the nomographs, we calculate ρ_1 , ρ_2 and M by using the approximations of 2.26, 2.27 and 2.28. We thus have

$$4.13 \quad \rho_1 \cong \frac{\lambda}{E(t)} = \frac{.0000833}{12} = .00000694 = .694 \times 10^{-5}$$

$$4.14 \quad \rho_2 \cong \frac{\lambda}{E(r)} = \frac{.0000833}{7} = .0000119 = .119 \times 10^{-4}$$

$$4.15 \quad M \cong \frac{\lambda}{E(a)} = \frac{.0000833}{.0002} = .417$$

The table is divided into pages, each page representing, for a particular value of ρ_2 , the average backlogs which would be expected for ranges of ρ_1 , and M . Thus, one first finds the proper page by matching the ρ_2 computed as in 4.14 with the ρ_2 at the top of each page of the table and selecting the page with the closest ρ_2 . One enters the table by choosing the closest ρ_1 and M to the ones computed as in 4.13 and 4.15, respectively. In the table shown as Figure 4.03 each entry consists of two numbers. The first number is seven and the second two digits. We notice that the first seven digit number has a decimal point to the left of the first digit. Notice further that the second two digit number has either a minus sign or no sign following it. This mode of representing a number is a form of floating point representation and is read as follows.

Rule: (a) If the sign of the second two digit number is positive, move the decimal point of the first seven digit number that many places to the right.

Example: .2540078 04 means 2540.078

(b) If the sign of the second two digit number is negative, move the decimal that many places left.

Example: .2500000 01- means .02500000

We notice that there is a number in the table that is negative, namely 1010101-01. This number does not represent backlog since backlog may not be negative. The number means that if one attempts operation of a process with the parameters indicated by this number that the process will be unstable and no steady state solution exists. Therefore we employ 1010101 -01 to indicate process parameters which create instability and thus little or no control. If one attempts operation of a machine at the parameter levels indicated, and if the assumptions governing the study hold, chaos will result.

We notice in Figure 4.03 that for

$$P_2 = .9999999 \text{ 05-} \cong .0000199$$

$$\mu = .4249998 \text{ 00} \cong .4170000$$

$$P_1 = .5999999 \text{ 05-} \cong .0000069$$

$$\text{that } E(j/0) = .7636958 \text{ 05 i.e.}$$

$$E(j/0) = 76369.58 \text{ items.}$$

This figure is the average backlog if machine is out of commission. Notice if the machine service rate is 12,000 lines per hour and if the machine goes back into commission with no new work arriving about $\frac{76370}{12,000} = 6.4$ hours will be required, on the average, to clean up the backlog of work on hand. We are now able to give the unconditional average backlog.

It can be shown that the unconditional average backlog is

$$4.16 \quad E(j) = \frac{\mu(P_1 + P_2)}{P_2(1 - P_1)} E[j|0] - \frac{\mu}{P_1 + P_2} .$$

If $E(r)$ and $E(t)$ are much larger than a service time then

$$4.17 \quad E(j) \cong \mu \frac{E(r) + E(t)}{E(t) - \lambda} E[j|0] - \frac{\mu}{\lambda} \frac{E(t)E(r)}{E(t) + E(r)}$$

E [J / 0] TABLE
MACHINE SELECTION STUDY

| p[2] = .999999 05- | | | | | | | | | | | |
|--------------------|------|----------|-----|-----------|-----|-----------|-----|-----------|-----|-----------|-----|
| u | p[1] | .999999 | 05- | .110000 | 04- | .160000 | 04- | .210000 | 04- | .260000 | 04- |
| .250000 | 01- | .2540078 | 04 | .2573589 | 04 | .2607996 | 04 | .2643336 | 04 | .2679648 | 04 |
| .4999999 | 01- | .5164077 | 04 | .5308327 | 04 | .5460870 | 04 | .5622441 | 04 | .5795866 | 04 |
| .7499999 | 01- | .7884576 | 04 | .8233528 | 04 | .8619183 | 04 | .9040334 | 04 | .9504762 | 04 |
| .9999999 | 01- | .1071536 | 05 | .1139336 | 05 | .1216342 | 05 | .1304487 | 05 | .1406407 | 05 |
| .1250000 | 00 | .1367298 | 05 | .1483173 | 05 | .1620509 | 05 | .1785873 | 05 | .1988830 | 05 |
| .1500000 | 00 | .1677744 | 05 | .1861443 | 05 | .2090318 | 05 | .2383371 | 05 | .2771996 | 05 |
| .1750000 | 00 | .2003323 | 05 | .2282748 | 05 | .2649260 | 05 | .3153982 | 05 | .3902400 | 05 |
| .2000000 | 00 | .2353062 | 05 | .2758773 | 05 | .3333344 | 05 | .4210838 | 05 | .5714915 | 05 |
| .2250000 | 00 | .2724735 | 05 | .3305855 | 05 | .4202068 | 05 | .5764960 | 05 | .9178982 | 05 |
| .2499999 | 00 | .3125131 | 05 | .3947539 | 05 | .5357486 | 05 | .8334206 | 05 | .1873496 | 06 |
| .2749999 | 00 | .3360405 | 05 | .4719156 | 05 | .6996104 | 05 | .1331896 | 06 | .1998925 | 07 |
| .2999999 | 00 | .4038607 | 05 | .5675943 | 05 | .9346244 | 05 | .3000904 | 06 | .1010101- | 01 |
| .3249999 | 00 | .4570468 | 05 | .6909785 | 05 | .1415477 | 06 | .1010101- | 01 | .1010101- | 01 |
| .3499999 | 00 | .5170624 | 05 | .8585358 | 05 | .2528233 | 06 | .1010101- | 01 | .1010101- | 01 |
| .3749998 | 00 | .5839361 | 05 | .1103008 | 06 | .9380963 | 06 | .1010101- | 01 | .1010101- | 01 |
| .3999998 | 00 | .6666876 | 05 | .1500112 | 06 | .1010101- | 01 | .1010101- | 01 | .1010101- | 01 |
| .4249998 | 00 | .7636928 | 05 | .2273493 | 06 | .1010101- | 01 | .1010101- | 01 | .1010101- | 01 |
| .4499998 | 00 | .8839371 | 05 | .4300898 | 06 | .1010101- | 01 | .1010101- | 01 | .1010101- | 01 |
| .4749997 | 00 | .1039098 | 06 | .1001851 | 08 | .1010101- | 01 | .1010101- | 01 | .1010101- | 01 |
| .4999997 | 00 | .1230047 | 06 | .1010101- | 01 | .1010101- | 01 | .1010101- | 01 | .1010101- | 01 |

and

$$4.18 \quad E(j) \cong M \frac{E(r) + E(t)}{E(t)} E[j|0] - \frac{M}{\lambda} \frac{E(t)E(r)}{E(t) + E(r)}$$

Finally the average backlog when machine is in commission is

$$4.19 \quad E[j|1] = \frac{E(j) - E[j|0]P(0)}{P(1)}$$

We have developed in-commission and utilization rates for the machine.

The backlogs for the process under various machine statuses as a function of average malfunction, repair and interarrival time and machine service time have also been developed. Various approximations have been given and nomographs and tables developed for these functions.

5. CONCLUSION. It is thus our contention that a machine and process may be characterized mathematically and that choice among alternative machines may be quantitatively, as well as qualitatively, exercised. The feasibility of making this characterization is clearly related to the generality one insists must be incorporated in that representation. Hence, we have shown that if one restricts a process severely it is rather easy to obtain results which permit explicit computation of process measures as a function of process parameter. The question of validity in the sense that such a characterization might permit prediction for real life processes is another, and not altogether unimportant, matter. It would seem prudent that if one were interested in "application" for such a model its predictive ability should be investigated relative to a real life process.

Considering for a moment the list of restrictions enumerated in 1.01 it seems, on the basis of one's intuition, that the most severe restriction in the list is "arrivals be non-lumpy". Experience of the briefest kind indicates otherwise; but fortunately, this restriction is easy to remove. The

restriction to a single machine would seem equally unrealistic and effort might be expended in relaxing it. It is believed that partial relaxation of this assumption can also be obtained without too much difficulty. The relaxation of these two assumptions may well make the technique workable in the sense that it provides reasonable descriptions of real world activity. As stated above, this could and should be tested against real data.

It should be relatively clear that any group of restrictions in the list may be relaxed but usually only at the expense of complexity. However, if one considers the benefits of extreme generalization worth the cost then simulation may be adopted and exceedingly general processes studied.

It is felt that consideration of machine rental and service rate, together with a few qualitative observations, are not adequate in characterizing the process, as is the common practice at the present time. Indeed the machine rental and service rate assume positions of importance only after one specifies the acceptable average backlogs and even then the mean time to malfunction and mean repair time are perhaps equally important, if not centrally important. This is clearly implied by the backlog equations in 4.12, 4.16 and 4.19.

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