AFCRL-62-794

University of Washington
College of Engineering
Department of Electrical Engineering

UNIDIRECTIONAL WAVES IN ANISOTROPIC MEDIA AND THE RESOLUTION OF THE THERMODYNAMIC PARADOX

By
AKIRA ISHIMARU

Contract No. AF19(604)-4098
Project No. 5635
Task No. 563503

TECHNICAL REPORT NO. 69

SEPTEMBER 1962

Prepared for

Electronics Research Directorate
Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Mass.

AFCRL-62-794

UNIDIRECTIONAL WAVES IN ANISOTROPIC MEDIA AND THE RESOLUTION OF THE THERMODYNAMIC PARADOX

by

Akira Ishimaru

DEPARTMENT OF ELECTRICAL ENGINEERING
College of Engineering
UNIVERSITY OF WASHINGTON

Contract No. AF19(604)-4098

Project 5635

Task 563503

TECHNICAL REPORT NO. 69

September 1962

The research reported in this document has been sponsored by the Electronics Research Directorate of the Air Force Cambridge Research Laboratories, Office of Aerospace Research, United States Air Force, Bedford, Massachusetts. The publication of this report does not necessarily constitute approval by the Air Force of the findings or conclusions contained herein.

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

Department of Defense contractors must be established for ASTIA services or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

All other persons and organizations should apply to the:

U.S. DEPARTMENT OF COMMERCE
OFFICE OF TECHNICAL SERVICES
WASHINGTON 25, D.C.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

This paper presents the resolution of so-called thermodynamic paradox, which was first pointed out by Lax and Button. It is shown that the Bresler's resolution is insufficient, and there can exist a single unidirectional mode in a lossless medium. The Poynting theorem is investigated for a discontinuity in the one-way system, and the solution is shown to exhibit a marked discontinuity depending on whether conductivity is zero or approaches zero.

It is shown that Maxwell's equations with completely lossless medium which leads to thermodynamic paradox is in fact "Improperly-posed problem," which does not correspond to physical reality.

1. Introduction

The first one-way transmission system was proposed by Lord Rayleigh [1901] [Hogan, 1956] using the Faraday rotation in an optical system. Since then, through the work of McMillian [1946], Gamo [1959] and others [Carlin, 1954], it is well-established from general energy considerations that the one-way transmission system must include resistive elements.

In 1955, however, Lax and Button [1955, 1956] pointed out the possibility of existence of one-way propagating mode in a lossless ferrite loaded waveguide, thus apparently violating the basic laws of thermodynamics. In an attempt to resolve this so-called "thermodynamic paradox", it was argued that the power flow in reverse direction takes place via cut-off modes [Kales, 1956]. That this does not provide a satisfactory resolution was shown by Bresler [1960] and Seidel [1959]. At present, there are two schools of thought on the resolution of this paradox. One approach [Seidel, 1957] is that a lossless ferrite medium possesses an "intrinsic loss" based on a consideration of the atomic model from which the ferrite properties are deduced. The other approach was advanced by Bresler [1960], in which he rejected the intrinsic loss approach, and on the basis of the lossless permeability diadic, he showed that while a single unidirectional mode is obtained for the ferrite slab placed at the guide wall, if a gap of width d between the ferrite and the wall were considered, the different secular equation results, and in the limit $d \to 0$, there are always an even number of propagating modes, half of which are the forward waves and the other half the backward waves. Thus, Bresler stated that this clears the way for the resolution of the thermodynamic paradox.

This paper first shows that the Bresler's resolution is not valid for the general case of a single unidirectional propagating mode. There exists a unidirectional mode in a lossless medium.

In order to resolve this thermodynamic paradox, we first note that there is no thermodynamic difficulties involved in an infinite lossless system, and that the difficulty occurs only when there is some discontinuities in the waveguide, which include input and output terminations.

Next, the Poynting theorem is studied in detail, in particular, its integral form for the case of one-way system terminated with a lossless short. It is shown that the solution and the power relation show a marked discontinuity whether conductivity is zero or approaches zero.

From this discontinuous behavior, it is shown that the Maxwell's equations with conductivity approaching zero is "Properly-posed problem" which corresponds to physical reality and which satisfies three basic requirements. On the other hand, the Maxwell's equations with zero conductivity is "Improperly-posed problem" which does not correspond to physical reality and satisfies only two of the three basic requirements.

Thus, the thermodynamic paradox is resolved by stating that only "Improperly-posed problem" presents any thermodynamic difficulties, and if the problem is properly-posed, there is no thermodynamic difficulty.

2. Bresler's Modes and the Existence of a Single Unidirection Mode

In this section, it is shown that the discontinuous behavior of the solution discussed by Bresler is not valid for the general case of a single unidirectional mode, and a single unidirectional mode can exist in a lossless medium.

Instead of the ferrite loaded waveguide, a simpler model shown in Fig. 1 is chosen to illustrate the point.

Consider a semi-infinite lossless ferrite region $y > d$ which is bounded by an air gap $d > y > 0$ and an electric wall at $y = 0$. The dc magnetic field is directed in z direction and it is assumed that the geometry and the field quantities are independent of z coordinate. The permeability tensor is given by

$$\bar{\bar{\mu}} = \begin{pmatrix} \mu & jK & 0 \\ -jK & \mu & 0 \\ 0 & 0 & \mu_z \end{pmatrix} \quad (1)$$

where

$$\mu = \mu_o \left(1 + \frac{\omega_M \omega_o}{\omega_o^2 - \omega^2}\right) \quad (2)$$

$$K = \mu_o \frac{\omega \omega_M}{\omega_o^2 - \omega^2}$$

$\omega_M = -\gamma\mu_o M_o$ saturation magnetization frequency

$\omega_o = -\gamma\mu_o H_i$ gyromagnetic resonance frequency

The differential equation for E_z in ferrite is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_f^2\right) E_z = 0 \tag{3}$$

where

$$k_f^2 = \omega^2 \varepsilon_f \frac{\mu^2 - K^2}{\mu}$$

H_x is given in terms of E_z.

$$H_x = -\frac{1}{j\omega\mu_T}\left(\frac{\partial}{\partial y} E_z + j \frac{K}{\mu} \frac{\partial}{\partial x} E_z\right) \tag{4}$$

where

$$\mu_T = \frac{\mu^2 - K^2}{\mu}$$

Now, matching the tangential electric field E_z and the magnetic field H_x at the boundaries, we obtain a secular equation

$$-jk_{fy} + \frac{K}{\mu} k_x = \frac{\mu_T}{\mu_o} \frac{k_y}{\tan k_y d} \tag{5}$$

where k_x is a propagation constant in x direction, and k_y and k_{fy} are propagation constant in y direction in air and in ferrite respectively. For comparison, let us consider the same problem with a magnetic wall at $y = 0$. Then, the secular equation is

$$-jk_{fy} + \frac{K}{\mu} K_x = \frac{\mu_T}{\mu_o} k_y \tan k_y d \tag{6}$$

First, we notice from (5) that when $d = 0$, the right hand side becomes infinite, and there is no solution. However, the solution of (6) exhibits a continuous behavior irrespective of whether $d = 0$ or $d \to 0$.

The same situation takes place when a lossless plasma is bounded by an air gap and an electric wall. It can be shown that in a frequency range

$$\sqrt{\omega_p^2 + \omega_c^2} < \omega < \frac{1}{2}\left[\sqrt{4\omega_p^2 + \omega_c^2} + \omega_c\right] \qquad (7)$$

where ω_p is plasma frequency and ω_c is cyclotron frequency, a unidirectional surface wave can propagate in a lossless opaque plasma [Ishimaru, 1961 and 1962]. In this range the dielectric constant is negative, and the plasma is opaque, thus, there is no radiation power. In such a case, the only power flow is by means of a single unidirectional surface wave mode without being accompanied by any backward waves.

It is important to note that this single unidirectional wave mode is a surface wave type mode, which means that the field decays exponentially in transverse direction. As will be shown later, this behavior is important in the discussion of the thermodynamic paradox.

3. Poynting Theorem and Thermodynamic Paradox

In the previous section, it was shown that a unidirectional surface wave can exist in a lossless medium.

We note, first of all, that no thermodynamic paradox occurs for an infinite uniform (or endless) lossless one way system. For example, an infinite lossless waveguide which carries the one-way mode does not violate any thermodynamic law, because an energy is being transferred from minus infinity to plus infinity and there is no source or load which may be cooled or heated. The same can be said about a ring type waveguide which propagates a one-way mode.

From the above considerations, we note that the so-called thermodynamic paradox occurs only when there is some discontinuities in the waveguide. The energy consideration which lead to McMillian's conclusion of resistive elements in the one-way system was based on the fact that the system is terminated at both input and output ends where the voltage or current can be specified. This is only possible when there is some discontinuities in the system.

Since there must be discontinuity in the waveguide in order for the thermodynamic difficulty to occur, let us consider a typical problem of terminating the one-way system by a lossless short circuit. This is shown in Fig. 2. In order to investigate this problem, it is necessary to examine the Poynting theorem.

From the Maxwell's equations, we get

$$\nabla \cdot E \times H^{*} = j\omega \left[E \cdot D^{*} - H^{*} \cdot B \right]$$

Taking the volume integral of the both sides of the equation, we get

$$\int_S E \times H^{*} \cdot dS = \int_V j\omega \left[E \cdot D^{*} - H^{*} \cdot B \right] dV \qquad (8)$$

provided that such an integral exists.

It is significant to note that this well known integral form is valid only when the integration can be performed to yield a finite value.

Let us first recognize that for the case of lossless medium, $\bar{\bar{\mu}}$ and $\bar{\bar{\epsilon}}$ are Hermitian, and thus the integrand in the right side of

(8) is pure imaginary as is well known. We note first the apparent difficulty associated with this problem. From the integral form of Poynting theorem, provided that this form is valid, the left side of (8) has a real part representing the real power carried by the one-way mode. The right hand side, however, has the pure imaginary integrand, and it appears that the integral may also be pure imaginary. Thus, the real power carried by the one-way mode must be equal to the imaginary power, which is contradictory.

To investigate this difficulty more closely, let us consider a simple example.

Let us consider an opaque plasma bounded by an electric wall as shown in Fig. 3. The one-way surface wave mode propagates towards the right. Suppose that the magnetic wall is placed as shown, which does not support the propagating mode. Thus, the real power carried by the one-way mode is being stopped by a lossless wall.

First, consider the behavior of the field components near this corner. H_z satisfies the wave equation.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_p^{\,2}\right) H_z = 0, \quad k_p^{\,2} = \omega^2 \mu \varepsilon_T \tag{9}$$

The boundary condition on the electric wall is

$$\left(\frac{\partial}{\partial y} + j\lambda \frac{\partial}{\partial x}\right) H_z = 0, \quad \lambda = \frac{g}{\varepsilon} \tag{10}$$

The boundary condition on magnetic wall is

$$H_z = 0 \tag{11}$$

The dielectric tensor of the plasma is given by

$$\bar{\bar{\varepsilon}} = \begin{pmatrix} \varepsilon & jq & 0 \\ -jq & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \qquad (12)$$

where

$$\varepsilon = 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2} \qquad (13)$$

$$q = \pm \frac{\omega_c \omega_p^2}{\omega(\omega_c^2 - \omega^2)} \qquad (14)$$

$$\varepsilon_z = 1 - \frac{\omega_p^2}{\omega^2} \qquad (15)$$

$$\varepsilon_T = \frac{\varepsilon^2 - q^2}{\varepsilon}$$

The plus sign in (14) is when the dc magnetic field is directed in +z direction, and the minus sign is for -z direction.

We note that in a frequency range given by (7),

$$\varepsilon_T < 0 \left|\frac{q}{\varepsilon}\right| > 1 \qquad (16)$$

H_z can be represented by

$$H_z = \sum_i a_{\nu_i} Z_{\nu_i} (k_p \rho) \sin \nu_i \phi \qquad (17)$$

where Z_{ν_i} is a Bessel function of order ν_i and a_{ν_i} is constant.

Consider small distance from the corner.

$$|k_p \rho| \ll 1$$

Then, the first dominant term of H_z is

$$H_z = \rho^{\nu} \sin \nu\phi, \; \nu \neq 0 \tag{18}$$

From the boundary condition,

$$\tan \nu\phi_o = j \frac{1}{\lambda} \tag{19}$$

Since $|\lambda| > 1$, there is at least one root of (19) which is pure imaginary. This imaginary root represents the exponential decay in transverse direction and this corresponds to the one-way mode. The other field components are

$$E_\rho = \frac{1}{j\omega\varepsilon_T} \rho^{\nu-1} \nu(\cos \nu\phi + j\lambda \sin \nu\phi)$$

$$E_\phi = \frac{1}{j\omega\varepsilon_T} \rho^{\nu-1} \nu(j\lambda \cos \nu\phi - \sin \nu\phi)$$

$$D_\rho = \frac{1}{j\omega} \rho^{\nu-1} \nu \cos \nu\phi \tag{20}$$

$$D_\phi = -\frac{1}{j\omega} \rho^{\nu-1} \nu \sin \nu\phi$$

Let us calculate the right hand integral of (8).

$$\int_V j\omega \, E \cdot D^* \, dV$$

$$= \frac{1}{j\omega\varepsilon_T} \int_o^\rho \nu(\nu + \nu^*) \rho^{\nu+\nu^*-1} d\rho \int_o^{\phi_o} (j\lambda \cos \nu\phi$$

$$- \sin \nu\phi) \sin \nu^*\phi \, d\phi \tag{21}$$

First, we note that if the medium and the boundaries are completely lossless, ν is pure imaginary, and thus, $\nu + \nu^* = 0$.

However, the integrand contains $\rho^{\nu+\nu^*-1}$ which becomes ρ^{-1}. Thus, this integrand is indeterminate, and moreover this integral is not defined because the behavior of the integrand at $\rho = 0$ cannot be definitely known.

However, if the integration is performed before the medium is allowed to become lossless, then

$$\int_V j\omega\, E \cdot D^* \, dV$$

$$= \frac{1}{j\omega\varepsilon_T} \rho^{\nu+\nu^*} \frac{\nu}{2}\left[\frac{\sin \nu_r\phi_o}{\nu_r}(j\lambda \sin \nu_r\phi_o + \cos \nu_r\phi_o)\right.$$

$$\left. + \frac{\sinh \nu_i\phi_o}{\nu_i}(\lambda \sinh \nu_i\phi_o - \cosh \nu_i\phi_o)\right] \qquad (22)$$

provided that $\nu + \nu^* > 0$, and $\nu = \nu_r + j\nu_i$. As the medium becomes lossless, $\nu_r \to 0$.

Thus

$$\int_V j\omega\, E \cdot D^* \, dV = \frac{\nu_i}{2\omega\varepsilon_T}\phi_o \qquad (23)$$

The left side of the integral of (8) is

$$\int_S E \times H^* \cdot dS$$

$$= \frac{1}{j\omega\varepsilon_r} \rho^{\nu+\nu^*} \frac{\nu}{2}\left[\frac{\sin \nu_r\phi_o}{\nu_r}(j\lambda \sin \nu_r\phi_o + \cos \nu_r\phi_o)\right.$$

$$\left. + \frac{\sinh \nu_i\phi_o}{\nu_i}(\lambda \sinh \nu_i\phi_o - \cosh \nu_i\phi_o)\right] \qquad (24)$$

irrespective of whether v is real, imaginary or complex. The above results can be summarized as follows:

The power relation represented by the integral form of the Poynting theorem shows a marked discontinuous behavior depending on whether the conductivity of dielectric medium and the resistivity of conductor is zero or approaches zero, i.e., $\sigma = 0$, $1/\sigma = 0$ or $\sigma \to 0$, $1/\sigma \to 0$.

In terms of the integrals in (8), these two cases correspond to the integration being performed <u>after</u> or <u>before</u> σ(and $1/\sigma$) is allowed to become zero.

The left side of (8)

$$\int_S E \times H^* \cdot dS$$

yields the real power, irrespective of whether integration is performed before or after the limit $\sigma \to 0$.

$$\lim_{\sigma \to 0} \int_S E \times H^* \cdot dS = \int_S \lim_{\sigma \to 0} \left[E \times H^* \right] \cdot dS$$

$$= \frac{v_1 \phi_o}{2\omega\epsilon_T} \text{ real} \qquad (25)$$

However, the right side of (8) is not defined if the limit is taken before the integration, and this is the case which yields the thermodynamic difficulty. If the limit is taken after the integration, this yields the same result as (25) and there is no difficulty associated with the power. It may be interesting to note that even though the integrand of the right side of (8) is pure imaginary for lossless medium, when the integration is performed before the limit $\sigma \to 0$ is taken, then the limit is purely real representing the real power dissipation.

$$\lim_{\sigma \to 0} \int_v j\omega(E \cdot D^* - H^* \cdot B)\, dv = \frac{v_1 \phi_0}{2\omega\epsilon_T} \text{ real} \qquad (26)$$

$$\int_v \lim_{\sigma \to 0} j\omega\, (E \cdot D^* - H^* \cdot B)\, dv = \text{Not defined} \qquad (27)$$

4. Resolution of Thermodynamic Paradox

In the preceding section, it was shown that the solution of Maxwell's equations show a remarkable discontinuity at $\sigma = 0$ for dielectric and $1/\sigma = 0$ for conductor. It was shown that if $\sigma = 0$ and $1/\sigma = 0$, then the integral form of Poynting theorem is not valid, and, therefore, the power relations cannot be meaningfully discussed. Also, this case leads to so-called thermodynamic paradox. On the other hand, if σ and $1/\sigma$ are allowed to approach zero, then the Poynting theorem is valid and there is no thermodynamic difficulty.

Our problem is to investigate why one solution of Maxwell's equations, namely the case $\sigma = 0$, $1/\sigma = 0$, yields such difficulties, while the other solution, $\sigma \to 0$ and $1/\sigma \to 0$, offers no difficulty at all.

We may be tempted to say that the latter is because of its intrinsic loss as was proposed by Seidel. However, we do not wish to rely on the intrinsic loss idea. We wish to resolve this difficulty within the framework of Maxwell's equation itself, as was advocated by Bresler. In other words, we wish to choose one of the two solutions, purely on the basis of mathematical problems, without employing the idea of atomic model or the intrinsic loss.

In order to do this, it is first necessary to investigate what must be the requirements of the mathematical problems if this problem is to correspond to physical reality. These requirements are explored by Courant [1962]. In general, there are three requirements which a mathematical problem should satisfy:

They are:

(1) The solution must exists.

(2) The solution must be uniquely determined.

(3) The solution should depend continuously on the data.

The last requirement is most important for our discussion. This requirement is necessary if this mathematical problem is to describe natural phenomena. The data such as time, space coordinate, angle, dielectric constant, permeability, conductivity, etc., can only be given within a certain margin, and these data cannot be measured without a certain amount of error. Therefore, for a small variation of the data, the variation of the solution must be also small if this problem is to describe physical phenomena.

The mathematical problem which satisfies these three requirements is called "Properly-posed problem", while the problem whose solution exists and is uniquely determined, but does not continuously depend on the data, is called "Improperly-posed problem".

From this consideration, it is now possible to clearly resolve the thermodynamic paradox.

We note that the solution shows a sharp discontinuity at

$\sigma = 0$ and $1/\sigma = 0$. Thus, the Maxwell's equation for purely lossless medium constitutes "Improperly-posed problem", which simply does not correspond to physical reality. This is why this lossless case leads to so-called "thermodynamic paradox", which does not exist in reality.

On the other hand, the Maxwell's equation with medium whose conductivity approaches zero, constitutes "Properly-posed problem", and indeed there is no thermodynamic difficulties involved.

5. Some Related Problems

It may be interesting to note that the idea of "Improperly-posed problem" may be applicable to other physical problems.

For example, the edge condition in a usual sense [Meins, 1955], which states that the energy in any small volume containing the edge should be finite, is meaningless in the problem discussed in the previous section. Instead, the condition should be stated as

$$\lim_{\sigma \to 0} \int_V (E \cdot D^* + H^* \cdot B)\, dV = \text{finite}. \qquad (28)$$

Felsen [1959] observed that, in the case of wedge with linearly varying impedance, the edge condition is apparently violated unless a small loss is introduced. This is in fact the same situation, and the above edge condition should be employed.

Another situation is Fourier transform applied to physical problems. In order for Fourier transform to be valid, it is necessary to assume that the wave number k has a small imaginary part. This is also an example of "Properly-posed problem".

REFERENCES

1. Bresler, A. D., "On the TE_{no} Modes of a Ferrite Slab Loaded Rectangular Waveguide and the Associated Thermodynamic Paradox, IRE Trans., MTT-8, 81 January 1960.

2. Button, K. J. and B. Lax, "Theory of Ferrites in Rectangular Waveguides", IRE Trans. PGAP, Vol 4, 531-537, July 1956.

3. Carlin, H. J., "Principles of Gyrator Networks", Symp. on Modern Advances in Microwave Techniques, P.I.B., 1954.

4. Courant, R. and D. Hilbert, "Methods of Mathematical Physics", Partial Differential Equations, Interscience Publishers, Vol II, 227-229, 1962.

5. Felsen, L. B., "Electromagnetic Properties of Wedge and Cone Surfaces with a Linearly Varying Surface Impedance", IRE Trans. PGAP, Vol 7, 231-243, Special Supplement, December 1959.

6. Gamo, Hideya, "On Passive One-way Systems", IRE Trans. on Circuit Theory, Vol Ct-6, 283-298, Special Supplement, May 1959.

7. Heins, A. and S. Silver, "The Edge and Conditions and Field Representation Theorems in the Theory of Electromagnetic Diffraction", Proc. Camb. Phil. Soc., Vol 51, 149-161, 1955.

8. Hogan, C. L., "The Elements of Nonreciprocal Microwave Devices", Proc. IRE, 1345-1368, October 1956.

9. Ishimaru, A., "Unidirectional Surface Wave Along a Perfectly Conducting Plane in a Plasma in the Presence of a DC Magnetic Field", presented at the URSI-IRE meeting, Austin, Texas, October 1961.

10. Ishimaru, A., "The Effect of a Unidirectional Surface Wave Along a Perfectly Conducting Plane on the Radiation from a Plasma Sheath", presented at the Second Symposium on the Plasma Sheath, Boston, Mass., April 10-12, 1962.

11. Kales, M. L., "Topics in Guided Wave Propagation in Magnetized Ferrites", Proc. IRE, Vol 44, 1403-1409, October 1956.

12. Lax, B. and K. J. Button, "New Ferrite Mode Configurations and Their Applications", J.A.P., 1186, September 1955.

13. McMillian, E. M. "Violation of the Reciprocity Theorem in Linear Passive Electromechanical Systems", J. Acoust. Soc. Amer., Vol 18, 344, 1946.

14. Rayleigh, Lord, "On the Magnetic Rotation of Light and the Second Law of Thermodynamics", Nature, Vol 64, 577, 1901.

15. Seidel, H., "Ferrite Slabs in Transverse Electric Modes Waveguide", J.A.P., Vol 28, 218-226, February 1957.

16. Seidel, H. and R. C. Fletcher, "Gyromagnetic Modes in Waveguide Partially Loaded with Ferrite", BSTJ, 1427-1456, November 1959.

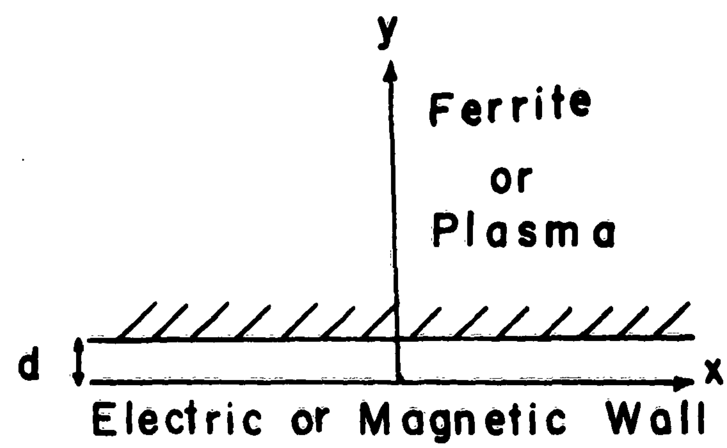

Figure 1 Unidirectional wave in anisotropic medium bounded by an electric or magnetic wall.

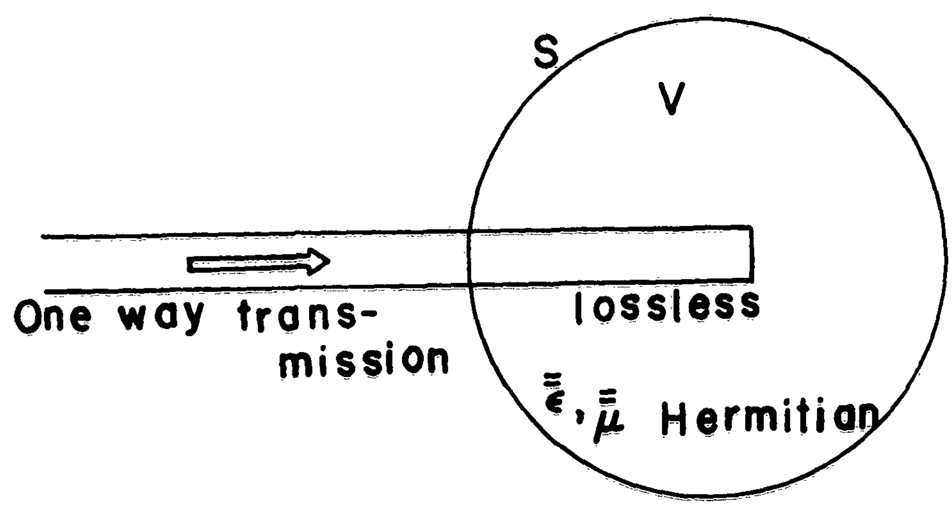

Figure 2 Poynting theorem applied to a one-way system terminated by a lossless short.

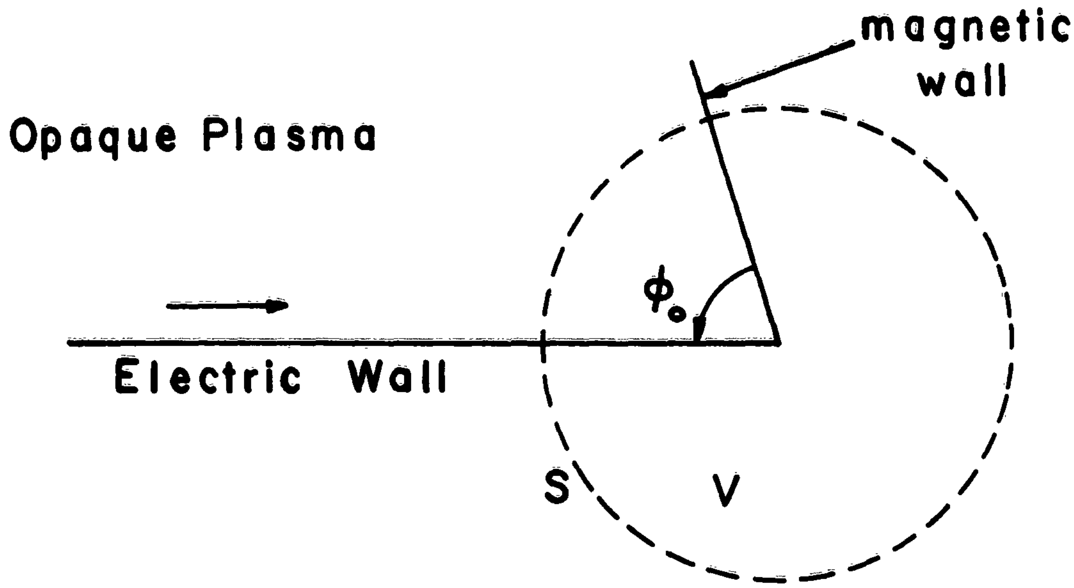

Figure 3 Unidirectional wave along an electric wall terminated by a magnetic wall.

TECHNICAL REPORTS PUBLISHED ON CONTRACT AF 19(604)4096

TR No.	Author	Title	Date
21	G. Held	The Impedance of a Slot in a Waveguide	1 May 1956
25	G. Held G. Hasserjian	An Experimental Analysis of the Surface and Tip Excitation of a Cone by a Slot	March 1958
26	A. Ishimaru G. Held	Radiation Pattern Synthesis with Sources Located on a Conical Surface AFCRC-TN-58-177	April 1958
29	G. Held A. Ishimaru	Optimum Radiation Pattern Synthesis for Circular Aperture	29 Sept 1958
32	J. W. Blundell G. Held A. Ishimaru	Synthesis of Rectangular Apertures AFCRC-TN-59-147	March 1959
33	A. Ishimaru G. Held	Analysis & Synthesis of Radiation Pattern for the Travelling Wave Type Sources on a Circular Aperture AFCRC-TN-59-170	March 1959
40	G. D. Bernard A. Ishimaru	Closed Form Analysis for the Radiation Pattern of the Modulated Antenna AFCRL-TN-60-1143	November 1960
45	R. B. Kieburtz A. Ishimaru G. Held	The Variational Method for Evaluation of Scattering of Electromagnetic Waves by Obstacles AFCRL-TN-60-961	August 1960
46	A. Ishimaru	Sinusoidally Modulated Wave Guides for Traveling-Wave Antennas AFCRL-TN-60-1117	September 1960
47	R. B. Kieburtz A. Ishimaru G. Held	The Variational Method for Evaluation of Scattering Electromagnetic Waves by Obstacles AFCRL 100	January 1961
48	R. B. Kieburtz A. Ishimaru G. Held	Diffraction by a Periodically Apertured Conducting Screen AFCRL 101	January 1961
50	G. Hasserjian A. Ishimaru G. Held	Currents Induced on the Surface of a Conducting Circular Cylinder by a Slot AFCRL 551	September 1961
51	L. J. Vanden-Bos G. Held	Propagation in Bounded Anistropic Media AFCRL 718	October 1961
52	A. Ishimaru R. G. Bosse	Azimuth Radiation Pattern Synthesis for Circular Aperture Antennas AFCRL 768	August 1961

TR No.	Author	Title	Date
53	G. D. Bernard A. Ishimaru	Tables of the Anger and Lommel-Weber Functions AFCRL 796	1962
54	A. Ishimaru Hang-Sheng Tuan	Frequency Scanning Antennas AFCRL 711	April 1961
57	A. Ishimaru C. Hsieh	Frequency Scanning of Slow Wave Antennas AFCRL 797	August 1961
58	A. Ishimaru F. R. Beich	Pattern Synthesis with a Flush-Mounted Leaky-Wave Antenna on a Conducting Circular Cylinder AFCRL 798	September 1961
63	A. Ishimaru	Theory of Unequally Spaced Arrays AFCRL-62-325	January 1962
64	A. Ishimaru	The Effect of the Radiation from a Plasma Sheath of a Unidirectional Surface Wave Along a Perfectly Conducting Plane AFCRL-62-355	April 1962