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SENSITIVITY AND MODAL RESPONSE FOR SINGLE-LOOP AND MULTILOOP SYSTEMS

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FOREWORD

This report represents one phase of an analytical investigation of handling qualities for multiloop airframe—human pilot systems. The research reported was sponsored by the Flight Control Laboratory, Aeronautical Systems Division, Air Force Systems Command, as part of Project No. 8219, Task No. 821905. It was conducted at Systems Technology, Inc., under Contract No. AF 33(616)-8024. The ASD project engineer was Mr. R. J. Wasicko.

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ABSTRACT

Two related facets of feedback system analysis are considered—the calculation of closed-loop response and the determination of the effects on closed-loop behavior of variations in open-loop parameters. The connection between the modal response coefficients (partial fraction expansion coefficients of the closed-loop transfer function), the sensitivity of the closed-loop poles to variations in the open-loop gains, poles, and zeros, and classical sensitivity are developed. A comprehensive summary is given of methods for determining modal response coefficients and sensitivities from open- or closed-loop transfer function representation. Response formulas for periodic or power series inputs are derived in terms of the modal response coefficients, with conventional error coefficients as a special case. The initial developments are for single-loop systems with first-order closed-loop poles; these are extended to cover multiple-order closed-loop poles, and generalized to multiloop systems. Examples are given using $j\omega$, ξ, σ, and shifted Bode plots, root locus, and analytical transfer function representations.

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PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.

FOR THE COMMANDER:

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SYMBOLS

a	Real part of z or p
a_k	Coefficient of s^{n-k} in numerator of G(s) when G(s) is in root-locus form
a_k	Amplitude of cos $\omega_k t$ input
A	Amplitude of G(s)
$A(s)$	Bode-form numerator of G(s)
A_{asy}	Closed-loop Bode amplitude asymptote
A_h	Root-locus gain of H_δ
A_N	Derivative $\partial^N \log A/\partial(\log \mu)^N$ in db/(dec)^N
A_θ	Root-locus gain of θ_δ
b	Imaginary part of z or p
b_k	Coefficient of s^{m+n-k} in denominator of G(s) when G(s) is in root-locus form
b_k	Amplitude of sin $\omega_k t$ input
$B(s)$	Bode-form denominator of G(s)
$c(t)$	System output
c_k	$\sqrt{a_k^2 + b_k^2}$
d	Distance imaginary axis is shifted to the left
d_M	Coefficient of $t^M/M!$ input
$e(t)$	Error, r(t) - c(t)
E	Error coefficient
F_1	Numerator of closed-loop transfer function
F_2	Denominator term of closed-loop transfer function (denominator is $1 + F_2$)
$G(s)$	Open-loop transfer function
G_a	Controller transfer function

G_f	Feedback transfer function
G_k	Transfer function of kth element in multiloop system
G_{rc}	Closed-loop output/input transfer function
G_{re}	Closed-loop error/input transfer function
h	Altitude
h	Index
H_δ	Altitude/elevator transfer function
i	Index
I(t)	System response to unit step input
j	Index
j	$\sqrt{-1}$
k	Index
k	Number of free s's in denominator of G(s)
K	Bode gain of G(s)
L	Number of transfer functions in multiloop system
$\mathcal{L}^{-1}$	Inverse Laplace transform
m	Number of poles of G(s) minus number of zeros
M	Index for power series inputs
n	Number of zeros of G(s)
N	Order of closed-loop pole
N	Noise
p	Negative of open-loop pole
q	Negative of closed-loop pole
Q	Modal-response coefficient
$Q^*(s)$	See Eq 8, 98, and 166
r(t)	System input

R	Radius of curvature of ξ plot
R	Response coefficient
s	Laplace transform variable
S_{λ}^{i}	Sensitivity of q_i to λ
$S_{x}^{G_{rc}}$	Classical sensitivity, $\frac{d \ln G_{rc}}{d \ln x}$
t	Time
$1/T_h$	See Eq 194
$1/T_{\theta}$	See Eq 194
u(t)	Unit step at t = 0
w(t)	System weighting function (response to unit impulse input)
Y	Controlled-element transfer function
Y_h	Altimeter transfer function
Y_{δ}	Elevator-servo transfer function
Y_{θ}	Equalization of pitch attitude feedback
z	Negative of open-loop zero
α	Numerator polynomial of G(s)
β	Denominator polynomial of G(s)
δ	Elevator deflection
$\delta(t)$	Unit impulse at t = 0
δ_j^i	Delta function ($\delta_j^i = 0$ for $i \neq j$, $\delta_j^i = 1$ for $i = j$)
Δ_{IL}	Denominator of $G_2G_3/(1 + G_2G_3)$ (multiloop example)
ζ	Damping ratio
ζ_s	Damping ratio of shifted roots
θ	Phase of G/κ
θ_{δ}	Pitch attitude/elevator transfer function

κ	Root-locus gain of G(s)
κ_{CL}	Root-locus gain of closed-loop transfer function
λ_q	Number of closed-loop poles of smaller magnitude than q_i
λ_u	Number of right-half-plane zeros and unstable closed-loop poles of larger magnitude than q_i, plus one if $\kappa/(1 + \kappa\delta_m^0) < 0$
λ_z	Number of open-loop zeros of smaller magnitude than q_i
μ	Magnitude of s
ξ	Real part of s divided by magnitude of s
σ	Real part of s
τ	Dummy variable of integration
φ	Phase of G(s)
φ_N	Derivative $\partial^N\varphi/\partial(\log \mu)^N$ in deg/(dec)^N
φ_{ki}	$\tan^{-1} \omega_k/q_i$
ψ_k	$\tan^{-1} b_k/a_k$
ω	Natural frequency
ω_k	Frequency of kth input term
ω_s	Natural frequency of shifted roots

SECTION I

INTRODUCTION

A. GENERAL LINEAR SYSTEM ANALYSIS PROBLEM

1. The Analysis Problem Defined

In this report detailed consideration is given to two major facets of feedback system analysis—the effects of component variations on system transfer functions, and the connection between transfer function characteristics and system time responses. For constant-coefficient linear systems typified by the single-loop system shown in Fig. 1, the system analysis problem shall be considered here to consist of five essential steps:

a. Delineation of nominal open-loop system characteristics. This step ordinarily starts with differential equations describing the nominal controlled-element characteristics and one or more controller possibilities. The step is concluded when one or more nominal open-loop transfer functions, G(s), are available, in factored form, for further analysis.

b. Determination of nominal closed-loop transfer functions, $G_{rc}(s)$ and $G_{re}(s)$, from the open-loop transfer function(s).

c. Calculation of closed-loop system time responses for pertinent inputs.

d. Determination of the changes in G(s) resulting from the expected variations in the controller and controlled-element characteristics.

e. Consideration of the effects of open-loop system variations on closed-loop behavior.

The major topics covered in this report are concerned with steps c and e, although desirable techniques to accomplish steps c and e are somewhat dependent on the methods used in steps a and b.

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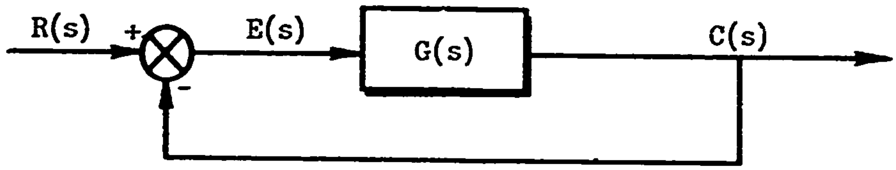

Open-Loop Transfer Function:

$$G(s) = \frac{C(s)}{E(s)} = \kappa \frac{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_n}{s^{m+n} + b_1 s^{m+n-1} + b_2 s^{m+n-2} + \cdots b_{m+n}} = \kappa \frac{\sum_{j=0}^{n} a_j s^{n-j}}{\sum_{i=0}^{m+n} b_i s^{m+n-i}}$$

$$= \kappa \frac{\alpha(s)}{\beta(s)} = \kappa \frac{\prod_{j=1}^{n} (s + z_j)}{\prod_{i=1}^{m+n} (s + p_i)}$$

$$= \frac{KA(s)}{s^k B(s)} = \frac{K \prod_{j=1}^{n} \left(\frac{s}{z_j} + 1\right)}{s^k \prod_{i=1}^{m+n-k} \left(\frac{s}{p_i} + 1\right)}$$

Output/Input Transfer Function:

$$G_{rc}(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\kappa\alpha(s)}{\beta(s) + \kappa\alpha(s)} = \frac{\kappa \prod_{j=1}^{n} (s + z_j)}{(1 + \kappa\delta_m^0) \prod_{i=1}^{m+n} (s + q_i)}$$

$$\delta_j^i = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Error/Input Transfer Function:

$$G_{re}(s) = \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{\beta(s)}{\beta(s) + \kappa\alpha(s)} = \frac{\prod_{i=1}^{m+n} (s + p_i)}{(1 + \kappa\delta_m^0) \prod_{i=1}^{m+n} (s + q_i)}$$

Figure 1. Single-Loop Linear Feedback System and Basic Notation

Delineating the open-loop characteristics in terms of a transfer function G(s) is extremely simple for time-invariant linear systems because transform methods can be used to convert the system differential equations to algebraic equations. This conversion permits the intermediate steps in an analysis sequence (e.g., reduction of simultaneous equations using Cramer's rule, transfer function development and manipulation) to be carried out using algebraic forms. Most such forms are rational polynomials, and the remainder can usually be approximated by this type of function. Thus, with the possible exception of polynomial factoring (which also enters step b), the delineation of open-loop transfer functions in the form indicated in Fig. 1 is basically elementary, as is the extension to similar open-loop transfer functions in multiloop systems.

The second step in the analysis sequence, i.e., given G(s), find $G_{rc}(s)$ and/or $G_{re}(s)$, is the central problem of feedback systems analysis. Trivial as it may seem, amounting only to finding roots of $1 + G(s) = 0$ when G(s) is given, a great deal of effort has been devoted to finding better methods for performing this operation. An eclectic view of this step, for single-loop systems, is presented in Ref. 1. Various representations of the open-loop transfer function (such as $G(j\omega)$, $G(\pm\sigma)$, and $G(\xi,\mu)$ Bode plots and pole-zero locations) are used there as part of a unified combination of methods to find closed-loop transfer functions in factored form. The techniques of Ref. 1 have also been extended to multiloop systems in Ref. 2.

2. System Time Response Behavior

When closed-loop transfer functions are available in factored form step c is, in principle, an elementary inversion process from the transform to the time domain. The major operation involved can again be algebraic, e.g., the system response to a unit impulse input requires the resolution of the transfer function $G_{rc}(s)$ into a partial fraction series. This algebraic operation is then followed with term-by-term inversion to the time domain. Thus, using the $G_{rc}(s)$ form shown in Fig. 1 (for $m \geq 1$),

$$G_{rc}(s) = \frac{K \prod_{j=1}^{n} (s + z_j)}{\prod_{i=1}^{m+n} (s + q_i)}$$

The first step is to resolve $G_{rc}(s)$ into partial fractions which, assuming that there are no repeated closed-loop poles, results in

$$G_{rc}(s) = \sum_{i=1}^{m+n} \frac{Q_i}{s + q_i}$$

Then, inverse transforming,

$$\mathcal{L}^{-1}\left[G_{rc}(s)\right] = \sum_{i=1}^{m+n} Q_i e^{-q_i t}$$

gives the system time response to the unit impulse input.

Standard procedures for finding the modal response coefficients—the Q_i's above—are routine but tedious. Techniques for finding the modal response coefficients from the open-loop transfer function G(s) are available but not too well known. So, the analyst often stops when $G_{rc}(s)$ is known in factored form, secure in the knowledge that the time variation portions (the $e^{-q_i t}$ terms) of the system response are defined by the q_i's appearing explicitly in the closed-loop transfer functions, and that the relative magnitudes of these time variations (the Q_i's) are indicated there implicitly. Much of this report is devoted to developments and methods which provide the modal response coefficients directly from the open- and/or closed-loop transfer function representations used in the second step of the general analysis procedure. Using these methods, the process of finding modal response coefficients (and system time responses) becomes a mere adjunct to the second step rather than a separate, often neglected, stage in analysis. Further, by extending the techniques to multiloop systems, the third step in the general analysis problem gains the same status of completeness as the first two already possess.

3. System Sensitivity to Parameter Variations

One reason for the fifth step (effects of component variations on system characteristics) is obvious—the system assumed in an analytical study will never precisely match the actual physical system, so it is important to know the effects of possible variations. A second, more subtle reason is that a general knowledge of the effects of parameter changes can be used as a guide to system modifications which would improve the over-all performance. For both of these reasons measures of closed-loop system sensitivity to open-loop system parameter variations are an integral part of the analysis problem.

Notions about system sensitivity were in the forefront when the feedback concept was initially developed. This was natural, even unavoidable, since feedback systems possess the "fundamental physical property that the effects of variation in the forward loop, whether they are taken as changes in $G(s)$ or as departures from strict linearity or from freedom from extraneous noise, are reduced by the factor $1 + G$ in comparison with the effects which would be observed in a non-feedback system."* Accordingly, sensitivity measures were indispensable to any rational discussion of feedback systems, and a useful, classical definition of sensitivity was made one of the two mathematical definitions of feedback (Ref. 3). Except for a minor modification (Ref. 4) this definition, by Bode (Ref. 3), remained unchanged for over a decade. Perhaps this static nature, surrounded by dynamic growth in most other areas of feedback systems engineering, made the concept fade—in any event, whole cadres of neophyte systems engineers were trained with precious little exposure to sensitivity concepts. In recent years classic sensitivity has become a more popular subject in automatic control (Ref. 4 and 5). Finally, the emphasis on pole-zero specifications for system characteristics gave rise to new conceptions of sensitivity, with associated new measures. These, called here "gain," "(open-loop) pole," and "(open-loop) zero" sensitivities, were evolved to account for changes in the position of closed-loop poles due to shifts or changes in open-loop gain, poles, or zeros. A number of thesis studies and research efforts, such as those reported in Ref. 6 - 13, have gone a long way toward bringing this subject to a logical conclusion.

In the simplest terms, sensitivity relationships connect open-loop differential variations with closed-loop pole differential shifts. Thus, the differential

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*Paraphrased quotation from Ref. 3, p. 44.

shift of a closed-loop pole is given by

$$dq_i = S_K^i \frac{dK}{K} + \sum_{j=1}^{n} S_{z_j}^i dz_j + \sum_{j=1}^{m+n} S_{p_j}^i dp_j$$

where the S^i factors are first-order sensitivities and the differentials dz_j, dp_j, and dK may be interpreted as incremental changes in the open-loop parameters.* The gain sensitivity, S_K^i, is basic, since the pole and zero sensitivities can be related to it, and since it is a component of the classical sensitivity.

When the sensitivity factors are known, they may be combined with the results of the fourth step in the general analysis sequence (estimates for K, z_j, and p_j uncertainties) to provide tolerances on the closed-loop roots. Further, a knowledge of the S^i values, and of their connection with system parameters, can lead to system changes which minimize the system variations to element deviations.

Sensitivities can also be very useful in the synthesis problem. Once a trial controller has been formed, the sensitivities permit the designer to estimate what changes he should make in the controller to obtain a more desirable set of nominal closed-loop poles. This approach is particularly beneficial for multiloop systems where the effects of inner-loop parameters on the final closed-loop poles are frequently obscure.

Sensitivities also have great value when a digital computer is used as the prime means of control analysis. By extending the program to compute sensitivities, the designer can use a manual or automatic iteration scheme to "home in" on the best controller rather than doing a costly parametric investigation of all the controller parameters.

On the surface, steps c and e of the analysis problem appear to be separate and distinct. However, there is a strong tie between the two—the gain

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*Note that in a strict mathematical sense the closed-loop poles are $-q_i$, the open-loop zeros are $-z_j$, and the open-loop poles are $-p_j$. The notation used in this report (Fig. 1) has been chosen to agree with a standard servoanalysis convention which has transfer function factors as $(s + z_j)$ and $(s + p_j)$.

sensitivities being, in fact, equal to the modal response coefficients for single-loop systems with single-order poles. This close connection provides additional methods for the calculation of modal response coefficients and gain sensitivities from the conventional open- and closed-loop system representations. These methods can also be extended to multiloop situations where sensitivity concepts should ultimately be very helpful in reducing to well-understood terms the many puzzling interactions present.

B. OUTLINE OF THE REPORT

The remainder of this report is organized into four sections (II through V). The basic relationships for modal response coefficients and sensitivities for single-loop systems are developed in Section II. Several methods for evaluating the modal response coefficients and sensitivities are derived in Section III. Section IV extends the developments to multiloop systems, and Section V summarizes the important results.

The developments of Section II begin with the modal response coefficients for single-order closed-loop poles. The connections between the modal response coefficients and the closed-loop responses for simple inputs are developed. Then the sensitivities and their relationship to the modal response coefficients and classical sensitivity are derived. The final article contains the modifications and extensions to the above which are necessary for systems with multiple-order closed-loop poles.

The evaluation methods presented in Section III are grouped as requiring

1. Direct calculation
2. Root locus plots
3. Open-loop Bode or ξ plots
4. Closed-loop Bode asymptotes

The numerical results obtained in an illustrative example are presented to indicate the accuracies obtainable with each technique. Details of the computations are given in the Appendix.

The extensions for multiloop systems, which are given in Section IV, generally parallel the developments in Section II. Section IV concludes with an example for a flight control system with pitch angle and altitude feedbacks to elevator.

The final section, Section V, summarizes the major identities and relationships developed in this report. The results have been put in a series of tables to provide a ready reference.

SECTION II

MODAL RESPONSE COEFFICIENTS AND SENSITIVITIES

A. MODAL RESPONSE COEFFICIENTS AND SYSTEM RESPONSES

For the single-loop system shown in Fig. 1 the feedback system analysis process of interest here can be presumed to begin with a given nominal open-loop transfer function G(s) in factored form. The results of the next stage of analysis are the closed-loop transfer functions $G_{rc}(s)$ and $G_{re}(s)$, which can also be in factored form. The closed-loop system has but one input, r(t), and two observable outputs, c(t) and e(t), so these two closed-loop transfer functions together constitute a complete mathematical model for the system. In essence the two open-loop elements of Fig. 2 replace the single open-loop transfer function G(s) plus the feedback connection (Fig. 1) as a system model.

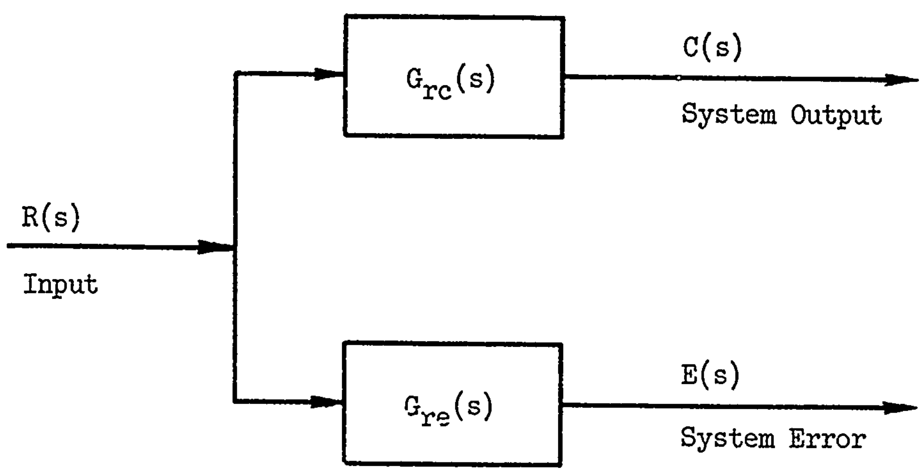


Figure 2. Open-Loop Elements for Single-Loop Feedback System

The closed-loop transfer functions can be combined with transforms of input functions to compute output time responses using routine inversion procedures. The system's natural modes are explicitly displayed by the q_i's of the transfer function denominator factors, but the relative magnitudes of these natural modes are indicated only implicitly. A basic set of these magnitudes, or modal response coefficients, are explicit features of the weighting functions, which represent the system time responses to a unit impulse input. Thus, a complete physical picture of system response characteristics is most easily achieved when both the transfer functions and weighting functions (or their constituent modal response coefficients) are known. To form a complete analysis structure, it should be possible to find the coefficients from either the open-loop or the closed-loop transfer functions, i.e., they should be explicitly defined by the elements contained in either Fig. 1 or 2.

In the present article modal response coefficients will be related to open- and closed-loop transfer functions for systems containing only first-order closed-loop poles. Several useful connections between the m+n modal response coefficients will also be developed. The coefficients will then be used as elements in system weighting functions to show their connection with system responses to various simple time function inputs. Subsequent articles in this section will relate the coefficients to sensitivity measures, and will generalize the developments to systems containing multiple-order poles. The actual determination of the coefficients from the several forms of transfer function representation (e.g., Bode and pole-zero plots) will be deferred to the next section.

1. Modal Response Coefficients

The system input-output weighting function is the output, $c(t)$, response when the input, $r(t)$, is a unit impulse. Since the Laplace transform of a unit impulse is unity, this weighting function, $w(t)$, is simply the inverse Laplace transform of the closed-loop system transfer function $G_{rc}(s)$,

$$w(t) = \mathcal{L}^{-1}\left[G_{rc}(s)\right] \tag{1}$$

If the number of poles of $G_{rc}(s)$ is greater than the number of zeros $(m \geq 1)$ and the closed-loop poles, $-q_i$, are all first order, then a partial fraction expansion of $G_{rc}(s)$ results in

$$\begin{aligned} w(t) &= \mathcal{L}^{-1}\left[\sum_{i=1}^{m+n} \frac{Q_i}{s+q_i}\right] \\ &= \sum_{i=1}^{m+n} Q_i e^{-q_i t} \end{aligned} \tag{2}$$

Q_i is the ith modal response coefficient. It may be evaluated from

$$Q_i = \left[(s + q_i)G_{rc}(s)\right]_{s=-q_i} \tag{3}$$

If m = 0, the bracketed quantity of Eq 2 has the constant term $\kappa/1 + \kappa)$ added, and the corresponding time response contains an additional term $\kappa\delta(t)/(1 + \kappa)$. For real systems $m \geq 1$ always, if higher frequency characteristics are taken into account; but low frequency approximations, for which m = 0, are often used.

As an aside from the main argument, it is worth noting that splitting $G_{rc}(s)$ into partial fractions, as in Eq 2, is equivalent to replacing the (m + n)th order differential equation for w(t) by m + n first-order differential equations of the form

$$\dot{w}_i(t) + q_i w_i(t) = Q_i\delta(t), \; i = 1, 2, \cdots, m + n \tag{4}$$

where $w_i(0) = 0$ and the total weighting function is

$$w(t) = w_1(t) + w_2(t) + \cdots + w_i(t) + \cdots w_{m+n}(t) \tag{5}$$

Thus the total weighting function is equivalent to the summed responses of m + n first-order systems excited by an impulse input, as shown in Fig. 3. The modal response coefficients can also be thought of as initial conditions on unity-gain elemental systems (the Q_i's in Fig. 3 replaced by 1's) which, with no other excitation (no impulse input to the system), results in a system output equal to the weighting function. For this interpretation the elemental differential equations would be

$$\dot{w}_i(t) + q_i w_i(t) = 0, \; i = 1, 2, \cdots, m + n \tag{6}$$

where $$w_i(0) = Q_i$$

The modal response coefficients can also be evaluated directly from the open-loop transfer function. In terms of G(s), Eq 3 can be written as

$$Q_i = \left[\frac{(s + q_i)G(s)}{1 + G(s)}\right]_{s=-q_i} \tag{7}$$

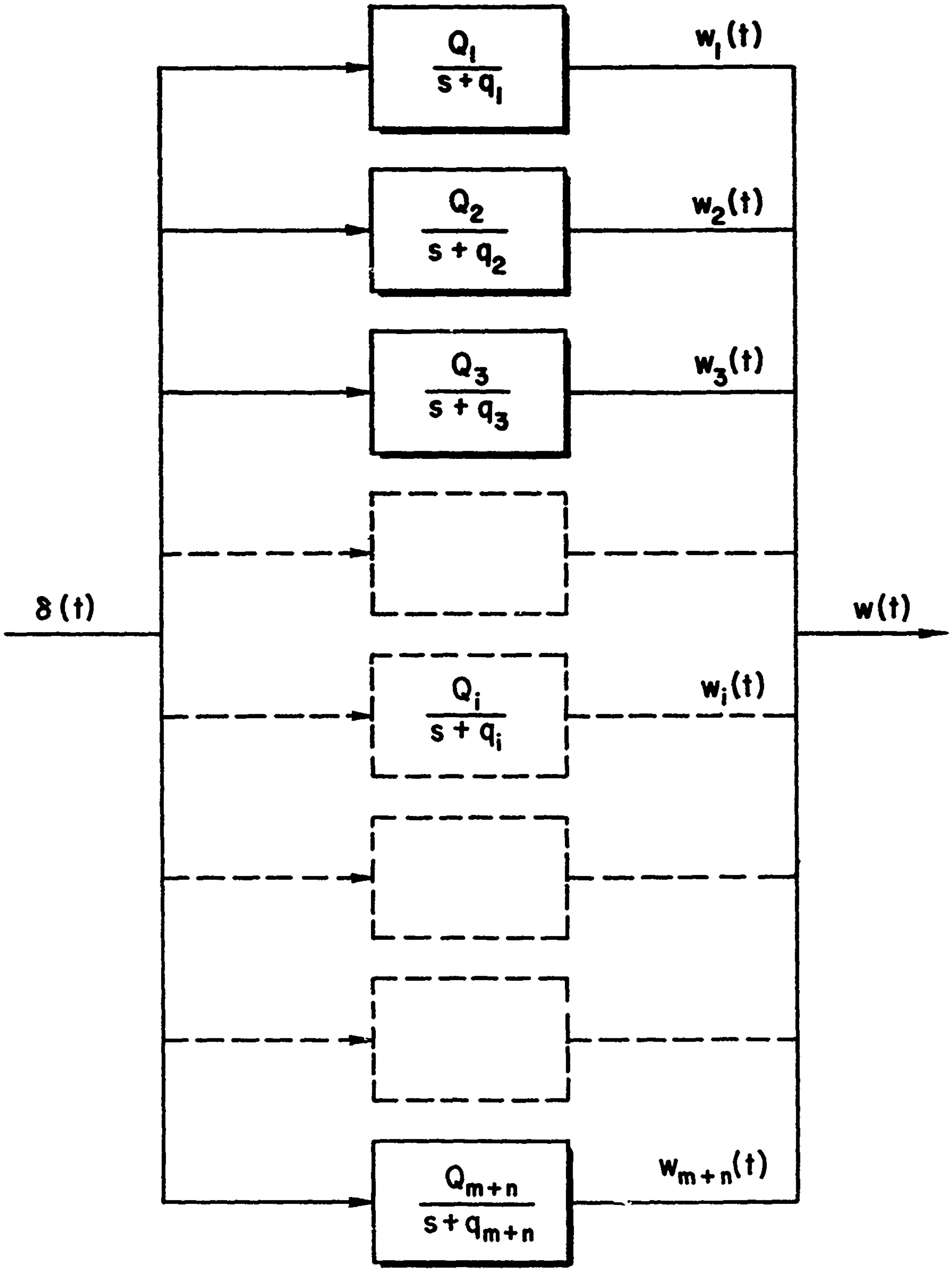


Figure 3. Elemental First-Order Systems Corresponding to Components of the Weighting Function

At this point it is convenient to introduce a new variable, $Q_i^*(s)$, which is defined as

$$Q_i^*(s) = \frac{(s + q_i)G(s)}{1 + G(s)} \tag{8}$$

and from Eq 7

$$Q_i^*(-q_i) = Q_i \tag{9}$$

Rewriting Eq 8 as

$$\left[1 + G(s)\right] Q_i^*(s) = (s + q_i)G(s)$$

and differentiating with respect to s, gives

$$\left[1 + G(s)\right] \frac{dQ_i^*(s)}{ds} + Q_i^*(s) \frac{dG(s)}{ds} = G(s) + (s + q_i) \frac{dG(s)}{ds} \tag{10}$$

Evaluating Eq 10 at $s = -q_i$ and recalling that $G(-q_i) = -1$,

$$Q_i = Q_i^*(-q_i) = \left[\frac{-1}{dG(s)/ds}\right]_{s=-q_i} \tag{11}$$

Besides providing a means for evaluating the Q_i's directly from the open-loop transfer function, Eq 11 will subsequently be used to show that a simple relationship exists between the modal response coefficients and the gain sensitivities defined in Article B of this section.

The modal response coefficients for the input-output weighting function, w(t), are closely connected with similar coefficients for the input-error weighting function, $w_{re}(t)$. In general the error is given by

$$e(t) = r(t) - c(t) \tag{12}$$

so the error response to a unit impulse input will be

$$
\begin{aligned}
w_{re}(t) &= \delta(t) - w(t) \\
&= \begin{cases} \delta(t) - \sum_{i=1}^{m+n} Q_i e^{-q_i t}, & m \geq 1 \\ \frac{1}{1 + \kappa} \delta(t) - \sum_{i=1}^{m+n} Q_i e^{-q_i t}, & m = 0 \end{cases}
\end{aligned}
\tag{13}
$$

The modal response coefficients for the input-output weighting function are thus seen to be the negatives of similar coefficients for the input-error weighting function.

Certain simple combinations of the modal response coefficients have properties which are occasionally useful. These are most conveniently developed by matching coefficients in expansions based on the partial fraction form and the closed-loop system expressions given as part of Fig. 1. In root-locus form, when $m \geq 1$,

$$
G_{rc}(s) = \frac{\kappa\alpha(s)}{\beta(s) + \kappa\alpha(s)} = \frac{\kappa \prod_{j=1}^{n} (s + z_j)}{\prod_{i=1}^{m+n} (s + q_i)}
\tag{14}
$$

$G_{rc}(s)$, when $m \geq 1$, is also equal to

$$
G_{rc}(s) = \sum_{i=1}^{m+n} \frac{Q_i}{s + q_i}
\tag{15}
$$

Equating Eq 14 and 15, and multiplying both sides by the product of the denominator factors,

$$
\begin{aligned}
&\sum_{i=1}^{m+n} \frac{Q_i}{s + q_i} \prod_{j=1}^{m+n} (s + q_j) \\
&= \sum_{i=1}^{m+n} Q_i \left[s^{m+n-1} + \left(\sum_{j=1}^{m+n} q_j - q_i \right) s^{m+n-2} + \cdots \right] \\
&= \kappa\alpha(s)
\end{aligned}
\tag{16}
$$

The sum of the negatives of the closed-loop poles is (Ref. 1), for $m \geq 1$,

$$\sum_{j=1}^{m+n} q_j = b_1 + K\delta_m^1 \tag{17}$$

where b_1 is the sum of the negatives of the open-loop poles $\left(b_1 = \sum_{j=1}^{m+n} p_j\right)$ and δ_m^1 is the Kronecker delta ($\delta_j^i = 0$ for $i \neq j$, $\delta_j^i = 1$ for $i = j$). Equation 16 then becomes

$$s^{m+n-1} \sum_{i=1}^{m+n} Q_i + s^{m+n-2}\left[(b_1 + K\delta_m^1) \sum_{i=1}^{m+n} Q_i - \sum_{i=1}^{m+n} Q_i q_i\right] + \cdots = Ks^n + Ka_1 s^{n-1} + \cdots \tag{18}$$

By matching coefficients,

$$\sum_{i=1}^{m+n} Q_i = K\delta_m^1 \tag{19}$$

$$(b_1 + K\delta_m^1) \sum_{i=1}^{m+n} Q_i - \sum_{i=1}^{m+n} Q_i q_i = K\delta_m^2 + Ka_1\delta_m^1 \tag{20}$$

Inserting Eq 19 into Eq 20

$$\sum_{i=1}^{m+n} Q_i q_i = K(b_1 - a_1 + K)\delta_m^1 - K\delta_m^2 \tag{21}$$

The results of Eq 19 and 21 are summarized below.

ITEM \ m	1	2	>2
$\sum_{i=1}^{m+n} Q_i$	K	0	0
$\sum_{i=1}^{m+n} Q_i q_i$	$K(b_1 - a_1 + K)$	$-K$	0

The Bode form for $G_{rc}(s)$ can also be used to establish similar results. The simplest of these occurs when s is allowed to approach zero, i.e., for $m \geq 1$,

$$\lim_{s \to 0} \sum_{i=1}^{m+n} \frac{Q_i}{s + q_i} = \lim_{s \to 0} \frac{KA(s)}{s^k B(s) + KA(s)} \tag{22}$$

or

$$\sum_{i=1}^{m+n} \frac{Q_i}{q_i} = \begin{cases} 1 & , \; k \geq 1 \\ \dfrac{K}{1 + K} & , \; k = 0 \\ 0 & , \; k \leq -1 \end{cases} \tag{23}$$

Equations 19, 21, and 23 can be used to calculate one or two of the Q_i's if all the others are known, or as checks on computation.

2. System Responses to Simple Inputs

Besides their central role in the system's weighting functions, the modal response coefficients also appear in simple ways in the system response to other elementary inputs. To illustrate this feature the outputs resulting from steps, periodic inputs, and inputs composed of power series in time will now be examined.

a. Response to Step Inputs. Next to the weighting function the step response is the most common transient response model used in systems analysis. The indicial response, or response to a unit step input, is just the integral of the weighting function,

$$\begin{aligned} I(t) &= \int_0^t w(\tau) d\tau \\ &= \mathcal{L}^{-1}\left[\frac{1}{s} G_{rc}(s)\right] \\ &= \mathcal{L}^{-1}\left[\frac{Q_o}{s} + \sum_{i=1}^{m+n} \frac{Q_{i_s}}{s + q_i}\right] \; , \quad q_i \neq 0 \end{aligned} \tag{24}$$

where the Q_{i_s}'s are the partial fraction coefficients for the step response, $m \geq 1$, and the q_i's are all first order.

The Q_{i_s}'s are related to the modal response coefficients, Q_i's, by

$$Q_{i_s} = \left[\frac{1}{s}(s + q_i)G_{rc}(s)\right]_{s=-q_i} = -\frac{Q_i}{q_i} \tag{25}$$

The amplitude of the step component in the system output, Q_o, is

$$Q_o = \left[G_{rc}(s)\right]_{s=0} = \begin{cases} 1 & , \ k \geq 1 \\ \dfrac{K}{1 + K} & , \ k = 0 \\ 0 & , \ k \leq -1 \end{cases} \tag{26}$$

or, comparing Eq 26 with Eq 23,

$$Q_o = \left[G_{rc}(s)\right]_{s=0} = \sum_{i=1}^{m+n} \frac{Q_i}{q_i} \tag{27}$$

Q_o is the magnitude of the output's final value when the input is a unit step and the system is stable (real part of $q_i > 0$). Combining Eq 24, 25, and 27,

$$I(t) = \sum_{i=1}^{m+n} \frac{Q_i}{q_i}\left(1 - e^{-q_i t}\right) \quad , \quad q_i \neq 0 \tag{28}$$

b. Response to Periodic Inputs. Any periodic input may be represented by a Fourier series of sine and cosine terms at various frequencies. Because the systems considered here are linear, the output is the sum of the outputs due to each term in the Fourier series. Consequently, only one sine and one cosine input need to be considered. To modify the periodic input (which is defined from $-\infty \leq t \leq +\infty$ from a Fourier series standpoint) to a periodic-like input which is zero until $t = 0$, the elemental periodic input will be multiplied by a unit step, $u(t)$. The input is then

$$r(t) = \left[a_k \cos \omega_k t + b_k \sin \omega_k t\right] u(t)$$
$$= \left[c_k \cos (\omega_k t - \psi_k)\right] u(t) \qquad (29)$$

where $c_k = \sqrt{a_k^2 + b_k^2}$, $\psi_k = \tan^{-1} \frac{b_k}{a_k}$.

The response for an arbitrary input is related to the impulse response by the convolution integral, i.e.,

$$c(t) = \int_0^t r(t - \tau) w(\tau) d\tau$$
$$= \int_0^t r(\tau) w(t - \tau) d\tau \qquad (30)$$

With the weighting function expressed in terms of the modal response coefficients (Eq 2) and the input given by Eq 29, the output becomes, using Eq 30,

$$c(t) = \sum_{i=1}^{m+n} \frac{Q_i (b_k \omega_k - a_k q_i)}{q_i^2 + \omega_k^2} e^{-q_i t}$$
$$+ \sum_{i=1}^{m+n} \frac{Q_i}{\sqrt{q_i^2 + \omega_k^2}} \left[a_k \cos (\omega_k t - \varphi_{ki}) + b_k \sin (\omega_k t - \varphi_{ki})\right] \qquad (31)$$

where $\varphi_{ki} = \tan^{-1} \omega_k / q_i$. In terms of c_k and ψ_k the steady-state portion of Eq 31 (the transient first term is unchanged) is

$$c_{ss}(t) = c_k \sum_{i=1}^{m+n} \frac{Q_i}{\sqrt{q_i^2 + \omega_k^2}} \cos (\omega_k t - \psi_k - \varphi_{ki}) \qquad (32)$$

The ith terms in the responses of Eq 31 and 32 are just the response of the ith elemental first-order system shown in Fig. 3 to the periodic-like input given by Eq 29. Considering only the steady-state portion, each elemental first-order system scales the input amplitude by the amplitude ratio $Q_i / \sqrt{q_i^2 + \omega_k^2}$, and contributes a phase lag φ_{ki}.

Equation 31 or 32 can also be used to prove the well-known fact that the amplitude ratio and phase of the steady-state response to a sinusoidal input of frequency ω_k is the amplitude and phase of $G_{rc}(j\omega_k)$. Thus, in more conventional form,

$$c_{ss}(t) = c_k |G_{rc}(j\omega_k)| \cos\left[\omega_k t - \psi_k + \measuredangle G_{rc}(j\omega_k)\right] \tag{33}$$

where

$$|G_{rc}(j\omega_k)| \cos \measuredangle G_{rc}(j\omega_k) = \sum_{i=1}^{m+n} \frac{Q_i}{\sqrt{q_i^2 + \omega_k^2}} \cos \varphi_{ki} = \sum_{i=1}^{m+n} \frac{Q_i q_i}{q_i^2 + \omega_k^2}$$

and

$$|G_{rc}(j\omega_k)| \sin \measuredangle G_{rc}(j\omega_k) = -\sum_{i=1}^{m+n} \frac{Q_i}{\sqrt{q_i^2 + \omega_k^2}} \sin \varphi_{ki} = -\sum_{i=1}^{m+n} \frac{Q_i \omega_k}{q_i^2 + \omega_k^2}$$

For a periodic-like input containing all the terms of a Fourier series, i.e.,

$$r(t) = \sum_{k=0}^{\infty} \left[a_k \cos \omega_k t + b_k \sin \omega_k t\right] u(t) \tag{34}$$

the response will be, by superposition,

$$c(t) = \sum_{k=0}^{\infty} \sum_{i=1}^{m+n} \left\{ \frac{Q_i (b_k \omega_k - a_k q_i)}{q_i^2 + \omega_k^2} e^{-q_i t} + \frac{Q_i}{\sqrt{q_i^2 + \omega_k^2}} \left[a_k \cos(\omega_k t - \varphi_{ki}) + b_k \sin(\omega_k t - \varphi_{ki})\right] \right\} \tag{35}$$

c. <u>Response to Power Series Inputs</u>. For inputs which can be represented by a power series in time, a procedure similar to that above can be used. It is again only necessary to consider the response to one term. For an input of $(d_M/M!)t^M$, the response is (assuming M is a non-negative integer and that the input is zero for $t < 0$)

$$c(t) = d_M \sum_{i=1}^{m+n} \frac{Q_i}{(-q_i)^{M+1}} \left[e^{-q_i t} - \sum_{j=0}^{M} \frac{(-q_i t)^j}{j!} \right] , \quad q_i \neq 0 \qquad (36)$$

Thus, for an input of

$$\begin{aligned} r(t) &= \sum_M \frac{d_M t^M}{M!} , \quad t \geq 0 \\ &= 0 , \quad t < 0 \end{aligned} \qquad (37)$$

the response is

$$c(t) = \sum_{i=1}^{m+n} \sum_M \frac{Q_i d_M}{(-q_i)^{M+1}} \left[e^{-q_i t} - \sum_{j=0}^{M} \frac{(-q_i t)^j}{j!} \right] , \quad q_i \neq 0 \qquad (38)$$

For example, the response to a unit step ($M = 0$) is

$$I(t) = \sum_{i=1}^{m+n} \frac{Q_i}{(-q_i)} \left(e^{-q_i t} - 1 \right) , \quad q_i \neq 0 \qquad (39)$$

which corresponds to the previous results (Eq 28). For a unit ramp input,

$$\left[c(t)\right]_{\text{Unit ramp}} = \sum_{i=1}^{m+n} \frac{Q_i}{q_i^2} \left(e^{-q_i t} - 1 + q_i t \right) , \quad q_i \neq 0 \qquad (40)$$

The modal response coefficients can be related to conventional output and error coefficients by using only the portion of Eq 38 which contains the power series terms in the output, i.e., by ignoring the $e^{-q_i t}$ transient terms. Then, for a power series input

$$r(t) = d_0 + d_1 t + \frac{d_2}{2!} t^2 + \frac{d_3}{3!} t^3 + \cdots , \quad t \geq 0 \qquad (41)$$

the output, less transient terms, will be

$$[c(t)]_{\text{Power series only}} = -\sum_{i=1}^{m+n} \sum_{M=0}^{\infty} \frac{Q_i d_M}{(-q_i)^{M+1}} \sum_{j=0}^{M} \frac{(-q_i t)^j}{j!} \tag{42}$$

By changing the summation index j to k = M - j and rearranging the order of the summations, Eq 42 becomes

$$[c(t)]_{\text{Power series only}} = -\sum_{k=0}^{\infty} \sum_{i=1}^{m+n} \frac{Q_i}{(-q_i)^{1+k}} \sum_{M=k}^{\infty} \frac{d_M t^{M-k}}{(M-k)!} \tag{43}$$

From Eq 41 it can be seen that

$$\sum_{M=k}^{\infty} \frac{d_M t^{M-k}}{(M-k)!} = \frac{d^k r(t)}{dt^k} \tag{44}$$

Combining Eq 43 and 44,

$$\begin{aligned} [c(t)]_{\text{Power series only}} &= -\sum_{k=0}^{\infty} \frac{d^k r(t)}{dt^k} \sum_{i=1}^{m+n} \frac{Q_i}{(-q_i)^{1+k}} \\ &= r(t) \sum_{i=1}^{m+n} \frac{Q_i}{q_i} - \dot{r}(t) \sum_{i=1}^{m+n} \frac{Q_i}{q_i^2} + \ddot{r}(t) \sum_{i=1}^{m+n} \frac{Q_i}{q_i^3} + \cdots \\ &= R_0 r(t) + R_1 \dot{r}(t) + R_2 \ddot{r}(t) + \cdots \end{aligned} \tag{45}$$

The R's in Eq 45 are output response coefficients which are occasionally useful for special purposes. Far more common are the error coefficients. The power series portion of the error response to a power series input is

$$\begin{aligned} [e(t)]_{\text{Power series only}} &= r(t) - [c(t)]_{\text{Power series only}} \\ &= E_0 r(t) + E_1 \dot{r}(t) + E_2 \ddot{r}(t) + \cdots \end{aligned} \tag{46}$$

Using the results of Eq 45, the error coefficients are seen to be

$$
\begin{aligned}
E_0 &= 1 - R_0 = 1 - \sum_{i=1}^{m+n} \frac{Q_i}{q_i} \\
E_1 &= -R_1 = \sum_{i=1}^{m+n} \frac{Q_i}{q_i^2} \\
E_2 &= -R_2 = -\sum_{i=1}^{m+n} \frac{Q_i}{q_i^3} \\
&\vdots \\
E_k &= -R_k = (-1)^{k+1} \sum_{i=1}^{m+n} \frac{Q_i}{q_i^{k+1}} \\
&\vdots
\end{aligned}
\qquad (47)
$$

Because of the many ways in which modal response coefficients can be obtained these formulas are especially handy for the calculation of error coefficients.

B. SENSITIVITY

The modal response coefficients discussed above are the last step in the analysis problem for linear feedback systems <u>if</u> the open-loop transfer function, G(s), is considered to be an exactly specified quantity. Unfortunately, the poles, zeros, and gain of G(s) ordinarily vary somewhat about nominal values which are themselves inexact. Thus, analyses are not really complete without some consideration of the effects of parameter variations. This purpose is served by the sensitivity factors to be discussed in this article.

As noted in the Introduction, sensitivity measures have been a major concern in feedback systems since the early days of feedback amplifiers. The pioneering investigators defined a sensitivity function as one of the essential features in a mathematical definition of feedback. This definition, which relates to the over-all sensitivity of closed-loop transfer functions to variations in the open-loop transfer function, we shall call "classical sensitivity." More recently attention has been centered on the effects upon closed-loop poles of variations in open-loop gain, pole, and zero locations. The sensitivities of this nature we shall call "gain sensitivity," "pole sensitivity," and "zero sensitivity," respectively. These sensitivities are especially important in servosystems because most such systems exhibit only one or two dominant closed-loop modes. The variations of these modes with open-loop characteristics are therefore of most concern, and the pole, zero, and gain sensitivities are of most value in assessing the effects of the variations. So, the sensitivity functions for closed-loop root variations will be the major topic below, with their relation to classical sensitivity being a short aside. Again the developments will presume first-order closed-loop poles, with generalization to the multiple-order situation being deferred to the last article of the section.

1. Gain, Pole, and Zero Sensitivities

The purpose of this article is to derive the expressions for the first-order sensitivity of the closed-loop poles to variations in the open-loop gain, κ, the open-loop poles, $-p_j$, and the open-loop zeros, $- {}_j$. With all these parameters being considered as variable, the open-loop transfer function, G, must be considered as a function of all of them, i.e.,

$$G = G(s,\kappa,z_j,p_j) \tag{48}$$

Then the total differential of G is

$$dG = \frac{\partial G}{\partial s}\, ds + \frac{\partial G}{\partial \kappa}\, d\kappa + \sum_{j=1}^{n} \frac{\partial G}{\partial z_j}\, dz_j + \sum_{j=1}^{m+n} \frac{\partial G}{\partial p_j}\, dp_j \tag{49}$$

Because the closed-loop poles, $-q_i$, are defined by the equation

$$\Big[1 + G(s)\Big]_{s=-q_i} = 0 \tag{50}$$

the total differential of Eq 48 must be zero for $s = -q_i$. Setting $dG = 0$ and $s = -q_i$ in Eq 49, and rearranging terms gives

$$dq_i = \left[\frac{1}{\partial G/\partial s}\right]_{s=-q_i} \left\{ \left(\frac{\partial G}{\partial \kappa}\right)_{s=-q_i} d\kappa + \sum_{j=1}^{n} \left(\frac{\partial G}{\partial z_j}\right)_{s=-q_i} dz_j + \sum_{j=1}^{m+n} \left(\frac{\partial G}{\partial p_j}\right)_{s=-q_i} dp_j \right\} \quad (51)$$

The variation in the negative of the closed-loop pole, dq_i, can also be expressed in another way by noting that q_i depends only on κ, the z_j's, and the p_j's, i.e.,

$$q_i = q_i(\kappa, z_j, p_j)$$

and writing another total differential as

$$\begin{aligned} dq_i &= \kappa \frac{\partial q_i}{\partial \kappa} \frac{d\kappa}{\kappa} + \sum_{j=1}^{n} \frac{\partial q_i}{\partial z_j} dz_j + \sum_{j=1}^{m+n} \frac{\partial q_i}{\partial p_j} dp_j \\ &= S_{\kappa}^{i} \frac{d\kappa}{\kappa} + \sum_{j=1}^{n} S_{z_j}^{i} dz_j + \sum_{j=1}^{m+n} S_{p_j}^{i} dp_j \end{aligned} \quad (52)$$

Here the factors $\kappa(\partial q_i/\partial \kappa)$, $\partial q_i/\partial z_j$, and $\partial q_i/\partial p_j$ are given a special symbol, S. These are the first-order sensitivity factors. The subscript and superscript notation indicates that a differential increment in the open-loop quantity, defined by the subscript, results in a differential increment of the ith closed-loop root equal to the sensitivity factor times the open-loop parametric variation.

Equating like coefficients in Eq 51 and 52 gives

$$\begin{aligned} S_{\kappa}^{i} &= \frac{\partial q_i}{\partial \kappa/\kappa} = \kappa\left(\frac{\partial G/\partial \kappa}{\partial G/\partial s}\right)_{s=-q_i} \\ S_{z_j}^{i} &= \frac{\partial q_i}{\partial z_j} = \left(\frac{\partial G/\partial z_j}{\partial G/\partial s}\right)_{s=-q_i} \\ S_{p_j}^{i} &= \frac{\partial q_i}{\partial p_j} = \left(\frac{\partial G/\partial p_j}{\partial G/\partial s}\right)_{s=-q_i} \end{aligned} \quad (53)$$

Note that the gain sensitivity is based on a fractional change in κ, while the pole and zero sensitivities are based on absolute shifts of p_j and z_j. These definitions were selected because they provide some simplifications in subsequent relationships.

If the open-loop transfer function, G, is written in root-locus form,

$$G = K \frac{\prod_{j=1}^{n} (s + z_j)}{\prod_{j=1}^{m+n} (s + p_j)} \tag{54}$$

the sensitivities are (remembering that at $s = -q_i$, $G = -1$)

$$S_K^i = \frac{-1}{\left(\frac{\partial G}{\partial s}\right)_{s=-q_i}}$$

$$S_{z_j}^i = \frac{S_K^i}{z_j - q_i} \tag{55}$$

$$S_{p_j}^i = \frac{S_K^i}{q_i - p_j}$$

Examination and further interpretation of Eq 55 reveal four very interesting properties of the sensitivity factors:

a. The gain sensitivity is a factor in each of the sensitivity terms. Thus, Eq 52 becomes

$$dq_i = S_K^i \left[\frac{dK}{K} + \sum_{j=1}^{n} \frac{dz_j}{z_j - q_i} + \sum_{j=1}^{m+n} \frac{dp_j}{q_i - p_j} \right] \tag{56}$$

b. The gain sensitivity is equal to the modal response coefficient. This is easily seen by comparing the expression for the gain sensitivity with Eq 11, i.e.,

$$Q_i = \left[\frac{-1}{dG(s)/ds} \right]_{s=-q_i} = S_K^i \tag{57}$$

This equality is very useful since all of the properties previously derived for the modal response cofficients are applicable to the gain sensitivity. Using this correspondence, other formulas for the gain sensitivity include

$$S_K^i = \Big[(s + q_i)G_{rc}(s)\Big]_{s=-q_i} \tag{58a}$$

$$= \left[\frac{(s + q_i)G(s)}{1 + G(s)}\right]_{s=-q_i} \tag{58b}$$

$$= \frac{K \prod_{j=1}^{n} (-q_i + z_j)}{(1 + K\delta_m^0) \prod_{\substack{j=1 \\ j \neq i}}^{m+n} (-q_i + q_j)} \tag{58c}$$

$$= \frac{- \prod_{j=1}^{m+n} (-q_i + p_j)}{(1 + K\delta_m^0) \prod_{\substack{j=1 \\ j \neq i}}^{m+n} (-q_i + q_j)} \tag{58d}$$

Another formula for S_K^i, which could have been derived previously for Q_i, is a direct result of Eq 57:

$$S_K^i = Q_i = \left[\frac{-1}{dG(s)/ds}\right]_{s=-q_i}$$

$$= \frac{1}{\left[\sum_{j=1}^{n} \frac{1}{z_j - q_i} + \sum_{j=1}^{m+n} \frac{1}{q_i - p_j}\right]} \tag{59}$$

c. Various gain sensitivity combinations have the same simple forms as the modal response coefficients. These follow from the equality of Eq 57 and the previous results of Eq 19, 21, and 23, i.e., for $m \geq 1$

$$\sum_{i=1}^{m+n} S_K^i = K\delta_m^1 \tag{60a}$$

$$\sum_{i=1}^{m+n} S_K^i q_i = K\Big[b_1 - a_1 + K\Big]\delta_m^1 - K\delta_m^2 \tag{60b}$$

$$\sum_{i=1}^{m+n} \frac{S_K^i}{q_i} = \begin{cases} 1 & , \ k \geq 1 \\ \dfrac{K}{1 + K} & , \ k = 0 \\ 0 & , \ k \leq -1 \end{cases} \tag{60c}$$

d. The sum of all the zero and pole sensitivities for each closed-loop pole must equal one. This follows directly from Eq 55 and 59. Thus,

$$\sum_{j=1}^{n} S^{i}_{z_j} + \sum_{j=1}^{m+n} S^{i}_{p_j} = 1 \qquad (61)$$

This initially surprising result is easily explained by recalling a root-locus plot. If all the open-loop zeros and poles are moved the same amount, all the closed-loop poles will be moved by that amount.

The gain sensitivity can be interpreted in two ways which are physically enlightening. First, S^{i}_{κ} is a measure of the slope of a conventional root locus. As root loci are ordinarily plotted for fixed open-loop poles and zeros, the only variable along the loci is gain. Thus the plot gives the zeros of

$$1 + G(s,\kappa) = 0 \qquad (62)$$

Taking the total derivative

$$dG = \frac{\partial G}{\partial s} ds + \frac{\partial G}{\partial \kappa} d\kappa \qquad (63)$$

Along the locus dG(s) = 0, so

$$ds = -\left[\frac{\kappa \frac{\partial G}{\partial \kappa}}{\frac{\partial G}{\partial s}}\right]_{s=-q_i} \frac{d\kappa}{\kappa}$$

$$= -S^{i}_{\kappa}\left(\frac{d\kappa}{\kappa}\right) \qquad (64)$$

$d\kappa/\kappa$ is a real number, so the direction of ds along the locus for positive $d\kappa/\kappa$ will be given by $-S^{i}_{\kappa}$ which, in general, will be complex. The minus sign appears because S^{i}_{κ} is the sensitivity of q_i which is the negative of the ith closed-loop pole.

The second interpretation of S^{i}_{κ} derives from the same argument as that used above by adding an additional step. The derivative of ln G(s) is

$$\frac{d \ln G(s)}{ds} = \frac{1}{G(s)} \frac{dG(s)}{ds} \qquad (65)$$

which, along the root locus, becomes

$$\frac{d \ln G(s)}{ds} = \frac{1}{S_K^i} \tag{66}$$

or

$$S_K^i = \frac{ds}{d \ln G(s)}$$

$$= \frac{ds}{dG/G} \tag{67}$$

Thus the gain sensitivity is a measure of the shift in a closed-loop pole due to a fractional change in the open-loop transfer function. Because only K changes along a conventional root locus, Eq 67 is really no different from Eq 64, except for the introduction of the logarithmic form.

2. Limiting Behavior and Special Cases

The magnitudes of gain sensitivities can cover the entire range of values from minus to plus infinity. Yet, intuitive notions of "sensitivity" as a general concept in closed-loop systems make part of this range appear unreasonable. One part of the problem is a direct consequence of the sensitivity definition, while another is associated with its first-order approximation nature. A better understanding of both facets can be gained by an examination of limiting cases.

In general, closed-loop poles depart from open-loop poles for low values of gain, and proceed to either open-loop zeros or unbounded values as the open-loop gain becomes very large. The gain sensitivity, as given by Eq 59, is

$$S_K^i = \frac{1}{\sum_{j=1}^{n} \frac{1}{z_j - q_i} + \sum_{j=1}^{m+n} \frac{1}{q_i - p_j}} \tag{59}$$

As K approaches zero, the closed-loop root $-q_i$ approaches the open-loop pole from which it derives, i.e., $q_i \to p_i$. Then the term $1/(q_i - p_i)$ in Eq 59 is dominant, so

$$S_K^i\Big]_{K \to 0} \to \frac{1}{\left(\frac{1}{q_i - p_i}\right)} \to 0 \tag{68}$$

Similarly, as K becomes very large, n of the closed-loop poles approach open-loop zeros. If the ith closed-loop pole is one of these, and it approaches the jth open-loop zero so that $q_i \to z_j$, then

$$S_K^i\Big]_{K \to \infty} \to \frac{1}{\left(\frac{1}{z_j - q_i}\right)} \to 0 \tag{69}$$

Finally, m of the closed-loop poles have no zeros to go to, and hence become very large relative to the p_j and z_j. The sensitivity for these poles is

$$S_K^i\Big]_{q_i \gg z_j, p_j} \to \frac{1}{\sum_{j=1}^{m} \frac{1}{q_i}} = \frac{q_i}{m} \tag{70}$$

When the gain is sufficiently large for the open-loop zero db line to intersect the high frequency asymptote, the open-loop transfer function is approximately

$$G(s) \doteq \frac{K}{s^m}$$

so that q_i will be

$$q_i \doteq -\sqrt[m]{-K} \tag{71}$$

Thus, the sensitivity of the unbounded pole will be

$$S_K^i\Big]_{q_i \gg z_j, p_j} \to -\frac{\sqrt[m]{-K}}{m} \tag{72}$$

Equation 72 indicates that the sensitivity increases as the mth root of K as gain is increased, although for finite gains the sensitivity is always finite.

Another circumstance in which the sensitivity can become very large is revealed by Eq 58d. Here it is apparent that the gain sensitivity for a closed-loop pole becomes very large as that pole nears another closed-loop pole. Indeed, as q_i becomes equal to q_j, indicating a branch point on the root locus, the gain sensitivity goes to infinity. This is to be expected since the sensitivity factors defined thus far have not considered multiple-order, closed-loop roots. As long as the gain is finite, an infinite gain sensitivity always indicates multiple-order, closed-loop poles.

A special situation of considerable interest can occur when a closed-loop root lies between an open-loop pole and zero which are much closer to each other than to all other open-loop poles and zeros. This is the so-called dipole case. The sensitivity for the bounded closed-loop pole will be, approximately,

$$S_K^i \doteq \frac{1}{\dfrac{1}{z_i - q_i} + \dfrac{1}{q_i - p_i}}$$

$$\doteq \frac{(z_i - q_i)(q_i - p_i)}{z_i - p_i} \tag{73}$$

The maximum value of S_K^i will occur when $q_i = (z_i + p_i)/2$, for which S_K^i becomes

$$\left. S_K^i \right]_{max} \doteq \frac{1}{4}(z_i - p_i) \tag{74}$$

3. Sensitivity Functions for Alternate Transfer Function Forms

The above equations for gain sensitivity are the same whether G is written in root-locus or Bode form, i.e., $S_K^i = S_{\text{K}}^i$. In fact, the equations are still valid if some of the terms in G are in root-locus form and some are in Bode form. It is, however, necessary to modify the open-loop pole and zero sensitivities for terms which are written in Bode form. From Eq 53 it can be shown that for zeros and poles which appear in G in Bode form, i.e., $\left[(s/z_j) + 1\right]$ or $\left[(s/p_j) + 1\right]$, the sensitivities are

$$S^i_{z_j} = \frac{q_i S^i_K}{z_j(z_j - q_i)}$$

$$S^i_{p_j} = \frac{q_i S^i_K}{p_j(q_i - p_j)} \tag{75}$$

Frequently open-loop zeros or poles will occur as complex-conjugate pairs, and variations in the system will change both zeros or poles. Consequently it becomes desirable to introduce sensitivities for the parameters which define a complex pair of zeros or poles. For example, consider a complex pair of zeros, z_1 and z_2, which are defined by their frequency, ω, and damping ratio, ζ, i.e.,

$$z_1 = \zeta\omega + j\omega\sqrt{1 - \zeta^2}$$

$$z_2 = \zeta\omega - j\omega\sqrt{1 - \zeta^2} \tag{76}$$

For this situation it is useful to define frequency and damping ratio sensitivities as

$$S^i_\omega = \frac{\partial q_i}{\partial \omega} = \frac{\partial q_i}{\partial z_1}\frac{dz_1}{d\omega} + \frac{\partial q_i}{\partial z_2}\frac{dz_2}{d\omega}$$

$$S^i_\zeta = \frac{\partial q_i}{\partial \zeta} = \frac{\partial q_i}{\partial z_1}\frac{dz_1}{d\zeta} + \frac{\partial q_i}{\partial z_2}\frac{dz_2}{d\zeta} \tag{77}$$

It is easily shown from Eq 55, 76, and 77 that if the term appears in G in root-locus form $(s^2 + 2\zeta\omega s + \omega^2)$, the frequency and damping-ratio sensitivities for a complex pair of zeros or poles are

$$S^i_\omega = \frac{\pm 2(\omega - \zeta q_i)S^i_K}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

$$S^i_\zeta = \frac{\mp 2\omega q_i S^i_K}{q_i^2 - 2\zeta\omega q_i + \omega^2} \tag{78}$$

where the upper sign is to be used for zeros and the lower one for poles.

If the term is written in Bode form $\left[(s^2/\omega^2) + (2\zeta s/\omega) + 1\right]$, the sensitivities are

$$S^i_\omega = \frac{\pm 2q_i}{\omega} \frac{(\zeta\omega - q_i)S^i_K}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

(79)

$$S^i_\zeta = \frac{\mp 2q_i\omega S^i_K}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

For some cases it may be more convenient to define a complex pair in terms of their real and imaginary parts, i.e.,

$$z_1 = a + jb$$

$$z_2 = a - jb$$

In this case, with the term in root-locus form $\left[s^2 + 2as + (a^2 + b^2)\right]$, the sensitivities are

$$S^i_a = \frac{\partial q_i}{\partial a} = \frac{\pm 2(a - q_i)S^i_K}{q_i^2 - 2aq_i + a^2 + b^2}$$

(80)

$$S^i_b = \frac{\partial q_i}{\partial b} = \frac{\pm 2bS^i_K}{q_i^2 - 2aq_i + a^2 + b^2}$$

If the term is in Bode form $\left[s^2/(a^2 + b^2) + 2as/(a^2 + b^2) + 1\right]$, the sensitivities are

$$S^i_a = \left[\frac{\pm 2q_i}{a^2 + b^2}\right]\left[\frac{(a^2 - b^2 - aq_i)S^i_K}{q_i^2 - 2aq_i + a^2 + b^2}\right]$$

(81)

$$S^i_b = \left[\frac{\pm 2q_i b}{a^2 + b^2}\right]\left[\frac{(2a - q_i)S^i_K}{q_i^2 - 2aq_i + a^2 + b^2}\right]$$

4. Connection Between Classical and Pole, Zero, and Gain Sensitivities

As pointed out by Bode, the usual conception of feedback involves two distinct ideas. The first is the more obvious and common—that feedback implies a loop transmission or return of some measure of output quantities to earlier stages of the system. The second notion is that of a reduction in the effects of variations in the elements of the forward loop.

To point up these old and well-known simple ideas, consider the system shown in Fig. 4. Here G_a is the main controller, G_f the feedback element, Y the

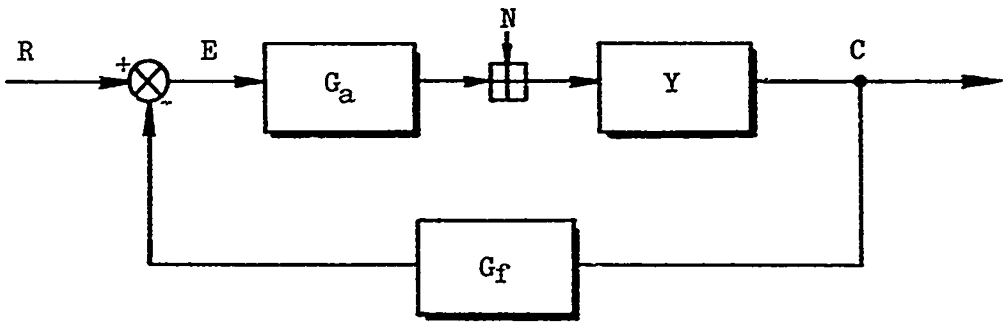


Figure 4. Generalized Single-Loop System

controlled-element, and N is a lumped source of unwanted signals. The system output, C(s), is

$$C(s) = \frac{G_a Y}{1 + G_a G_f Y} R + \frac{Y}{1 + G_a G_f Y} N$$

$$= G_{rc}\left[R + \frac{1}{G_a} N\right]$$

$$= G_{re}\left[G_a YR + YN\right] \tag{82}$$

where

$$G_{rc} = \frac{G_a Y}{1 + G_a G_f Y} = \frac{1}{G_f}\left[\frac{G}{1 + G}\right], \quad G = G_a G_f Y$$

$$G_{re} = \frac{1}{1 + G_a G_f Y} = \frac{1}{1 + G}$$

The presence of the loop transmission, i.e., the existence of the feedback element G_f, can lead to practical advantages when feedback is dominant. Thus, when $|G| \gg 1$, the output approximates

$$C(s) \doteq \frac{1}{G_f}\left[R + \frac{1}{G_a} N\right] \tag{83}$$

The relationship between output and input is made substantially independent of the controlled-element, and the effect of unwanted signals on the output is materially reduced (G_a being generally very large to make possible the inequality $|G| \gg 1$). As an open-loop control, the output would be

$$C(s) = G_a YR + YN \tag{84}$$

The difference due to the closing of the loop is just the error/input transfer function G_{re}. Thus G_{re} (the classical "return difference") is the fundamental measure of the improvement introduced by the loop transmission.

To illustrate the decreased sensitivity effects, consider a differential change in the closed-loop transfer function G_{rc}. For regions in the s-plane where G_{rc} and G are analytic,

$$dG_{rc} = \frac{1}{(1 + G)^2}\left[YdG_a + G_a dY - (G_a Y)^2 dG_f\right] \tag{85}$$

or, in terms of fractional changes,

$$\frac{dG_{rc}}{G_{rc}} = \frac{1}{1 + G}\left[\frac{dG_a}{G_a} + \frac{dY}{Y} - G\,\frac{dG_f}{G_f}\right]$$

$$= G_{re}\left[\frac{dG_a}{G_a} + \frac{dY}{Y} - G\,\frac{dG_f}{G_f}\right] \tag{86}$$

Again, when the feedback effect is dominant,

$$\frac{dG_{rc}}{G_{rc}} \doteq \frac{1}{G}\left[\frac{dG_a}{G_a} + \frac{dY}{Y}\right] - \frac{dG_f}{G_f} \tag{87}$$

The beneficial consequences of large values for the open-loop transfer function, G, in decreasing the effect on the over-all transfer function of forward-loop controller and controlled-element variations is apparent from Eq 87. The open-loop variations dG_a/G_a and dY/Y are decreased by the factor $1/G$. Note, however, that variations in the feedback elements are reflected directly into variations of the closed-loop transfer function G_{rc}. In general, the beneficial reduction is determined by the factor $1/(1 + G)$—again the error transfer function is a fundamental measure, this time of sensitivity.

Define, now, an over-all system sensitivity as

$$S_x^{G_{rc}} = \frac{dG_{rc}/G_{rc}}{dx/x} = \frac{d \ln G_{rc}}{d \ln x} = \frac{d \ln G_{rc}}{dx/x} \tag{88}$$

where x is some parameter or element in the open-loop system. If x is taken to be the forward-loop controller transfer function, G_a, the controlled-element transfer function, Y, or the forward-loop gain, κ_F, it is apparent that

$$S_{G_a}^{G_{rc}} = S_Y^{G_{rc}} = S_{\kappa_F}^{G_{rc}} = \frac{1}{1 + G} = G_{re} \tag{89}$$

This definition of sensitivity may be considered to be classical (Ref. 3 and 4), although Bode's original definition amounted to the inverse of that defined here. Examination of the role played by the classical sensitivity in Eq 82, 86, and 89 indicates the overwhelming importance of classical sensitivity as a fundamental quantity in feedback systems.

Classical sensitivity can also be used in expressions involving open-loop gain, pole, and zero changes. To illustrate this, return to the simplified block diagram of Fig. 1 for which

$$S_G^{G_{rc}} = \frac{dG_{rc}/G_{rc}}{dG/G} = \frac{1}{1 + G} = G_{re} \tag{90}$$

The fractional change in the open-loop transfer function is given by

$$\frac{dG}{G} = \frac{dK}{K} + \sum_{i=1}^{n} \frac{dz_i}{s + z_i} - \sum_{i=1}^{m+n} \frac{dp_i}{s + p_i} \tag{91}$$

so the fractional change in the closed-loop system characteristics is just

$$\frac{dG_{rc}}{G_{rc}} = S_G^{G_{rc}} \left\{ \frac{dK}{K} + \sum_{i=1}^{n} \frac{dz_i}{s + z_i} - \sum_{i=1}^{m+n} \frac{dp_i}{s + p_i} \right\} \tag{92}$$

The classical sensitivity is a function of s, and can range over extreme values. In the frequency domain ($s = j\omega$) its limiting cases occur when $|G| \gg 1$, which usually occurs for low frequencies, and $|G| \ll 1$, which always occurs for high frequencies ($m \geq 1$), viz:

$$|S_G^{G_{rc}}(j\omega)| = \begin{cases} \dfrac{1}{|G(j\omega)|} , & |G(j\omega)| \gg 1 \\ 1 , & |G(j\omega)| \ll 1 \end{cases} \tag{93}$$

Therefore, on a classical sensitivity basis a "good" system is one with lots of feedback—$|G(j\omega)| \gg 1$ over a broad frequency range—hardly a revelation of earth-shattering originality!

The classical and gain sensitivities are related in a very simple way. The relationship is most easily developed by noting that classical sensitivity corresponds to the input-error transfer function which, in turn, can be written in terms of the modal response coefficients,

$$S_G^{G_{rc}} = G_{re} = 1 - \sum_{i=1}^{m+n} \frac{Q_i}{(s + q_i)} , \quad m \geq 1 \tag{94}$$

Since the gain sensitivities are equal to the modal response coefficients,

$$S_G^{G_{rc}} = 1 - \sum_{i=1}^{m+n} \frac{S_K^i}{(s + q_i)} , \quad m \geq 1 \tag{95}$$

Equation 95 expresses the classical sensitivity as a weighted sum of gain sensitivities. This could be carried further formally be introducing pole and zero sensitivities to replace the one (see Eq 61), but this seems hardly worthwhile. Equation 95 also indicates that the gain sensitivities are the negatives of the residues, evaluated at the closed-loop poles, of the classical sensitivity (Ref. 10 and 12).

C. EFFECTS OF Nth-ORDER CLOSED-LOOP POLES

The developments of Articles A and B are restricted to cases for first-order closed-loop poles. This article considers the additional complexity introduced by the existence of Nth-order closed-loop poles. With the exception of N = 2, Nth-order closed-loop poles seldom occur in practice. The developments of this article are, therefore, somewhat academic, although required for completeness. This article is basically independent of the main body of the report, so the casual reader may skip it without seriously detracting from his understanding of other material. The procedure in this article will be to retrace most of the prior developments, noting what additions or modifications are required to generalize the results in the case of an Nth-order closed-loop pole.

If $-q_i$ is an Nth-order pole, the partial fraction expansion of the impulse response or weighting function, Eq 2, will contain the terms

$$\mathcal{L}^{-1}\left[\frac{Q_{i_1}}{s + q_i} + \frac{Q_{i_2}}{(s + q_i)^2} + \frac{Q_{i_3}}{(s + q_i)^3} + \cdots + \frac{Q_{i_N}}{(s + q_i)^N}\right]$$

$$= e^{-q_i t}\left[Q_{i_1} + Q_{i_2}t + \frac{Q_{i_3}t^2}{2!} + \cdots + \frac{Q_{i_N}t^{N-1}}{(N - 1)!}\right] \qquad (96)$$

where the N modal response coefficients are evaluated by

$$Q_{i_k} = \frac{1}{(N - k)!}\left\{\frac{d^{N-k}}{ds^{N-k}}\left[\frac{(s + q_i)^N G(s)}{1 + G(s)}\right]\right\}_{s=-q_i} \qquad (97)$$

There is a useful relationship between Q_{i_N} and a modified gain sensitivity, which will be developed shortly. To show the relationship, it is convenient to introduce the variable

$$Q^*_{i_N}(s) = \frac{(s + q_i)^N G(s)}{1 + G(s)} \qquad (98)$$

where

$$Q^*_{i_N}(-q_i) = Q_{i_N}$$

Rewriting Eq 98 as

$$\left[1 + G(s)\right] Q^*_{i_N}(s) = (s + q_i)^N G(s) \qquad (99)$$

and differentiating with respect to s, gives

$$\left[1 + G(s)\right] \frac{dQ^*_{i_N}(s)}{ds} + Q^*_{i_N}(s) \frac{dG(s)}{ds} = N(s + q_i)^{N-1} G(s) + (s + q_i)^N \frac{dG(s)}{ds} \qquad (100)$$

Evaluating Eq 100 at $s = -q_i$,

$$Q_{i_N} \left[\frac{dG(s)}{ds}\right]_{s=-q_i} = 0 \qquad (101)$$

therefore

$$\left[\frac{dG(s)}{ds}\right]_{s=-q_i} = 0 \qquad (102)$$

Repeated differentiation of Eq 100 shows that

$$\left[\frac{d^k G(s)}{ds^k}\right]_{s=-q_i} = 0 \quad , \quad 1 \le k \le N-1 \qquad (103)$$

and

$$Q_{i_N} = \frac{-N!}{\left[\dfrac{d^N G(s)}{ds^N}\right]_{s=-q_i}} \qquad (104)$$

Equation 103 expresses the interesting property that when $-q_i$ is an Nth-order pole, the first (N - 1) derivatives of G(s) with respect to s are zero at $s = -q_i$. Equation 104 is the general expression for Q_{i_N} and reduces to Eq 11 for N = 1. This equation will be used to show the relationship of Q_{i_N} to the modified gain sensitivity.

A relationship between the N modal response coefficients can be derived from the above by considering the functions

$$Q^*_{i_k}(s) = \frac{1}{(N-k)!} \frac{d^{N-k}}{ds^{N-k}} \left[\frac{(s+q_i)^N G(s)}{1 + G(s)} \right] \tag{105}$$

where $$Q^*_{i_k}(-q_i) = Q_{i_k}$$

Using an alternative expression for the closed-loop transfer function (Fig. 1), Eq 105 can be written

$$Q^*_{i_k}(s) = \frac{1}{(N-k)!} \frac{d^{N-k}}{ds^{N-k}} \left[\frac{\kappa \prod_{j=1}^{n} (s+z_j)}{(1+\kappa\delta_m^0) \prod_{\substack{j=1 \\ j\neq i}}^{m+n} (s+q_j)} \right] \tag{106}$$

where $\prod_{\substack{j=1 \\ j\neq i}}^{m+n} (s+q_j)$ = product of s minus each closed-loop pole except the Nth-order one, $-q_i$ (there are m+n-N terms in the product)

After differentiating once, Eq 106 becomes

$$Q^*_{i_k}(s) = \frac{1}{(N-k)!} \frac{d^{N-k-1}}{ds^{N-k-1}} \left\{ \frac{\kappa \prod_{j=1}^{n} (s+z_j)}{(1+\kappa\delta_m^0) \prod_{\substack{j=1 \\ j\neq i}}^{m+n} (s+q_j)} \left[\sum_{j=1}^{n} \left(\frac{1}{s+z_j} \right) - \sum_{\substack{j=1 \\ j\neq i}}^{m+n} \left(\frac{1}{s+q_j} \right) \right] \right\} \tag{107}$$

Combining Eq 107 with the formula for the derivative of a product,

$$\frac{d^k}{ds^k}\left[f_1(s)f_2(s)\right] = \sum_{h=0}^{k} \frac{k!}{h!\,(k-h)!}\,\frac{d^h f_1}{ds^h}\,\frac{d^{k-h} f_2}{ds^{k-h}} \tag{108}$$

gives

$$Q^*_{i_k}(s) = \frac{1}{N-k}\sum_{h=0}^{N-k-1}\frac{1}{h!\,(N-k-1-h)!}\,\frac{d^{N-k-1-h}}{ds^{N-k-1-h}}\left[\frac{\kappa\prod_{j=1}^{n}(s+z_j)}{(1+\kappa\delta_m^0)\prod_{\substack{j=1\\ j\neq i}}^{m+n}(s+q_j)}\right]$$

$$\times\ \frac{d^h}{ds^h}\left[\sum_{j=1}^{n}\left(\frac{1}{s+z_j}\right) - \sum_{\substack{j=1\\ j\neq i}}^{m+n}\left(\frac{1}{s+q_j}\right)\right] \tag{109}$$

Combining Eq 106 and 109,

$$Q^*_{i_k}(s) = \frac{1}{N-k}\sum_{h=0}^{N-k-1}\frac{Q^*_{i_{k+1+h}}(s)}{h!}\,\frac{d^h}{ds^h}\left[\sum_{j=1}^{n}\left(\frac{1}{s+z_j}\right) - \sum_{\substack{j=1\\ j\neq i}}^{m+n}\left(\frac{1}{s+q_j}\right)\right]$$

$$= \frac{1}{N-k}\sum_{h=0}^{N-k-1}(-1)^h Q^*_{i_{k+1+h}}(s)\left[\sum_{j=1}^{n}\left(\frac{1}{s+z_j}\right)^{h+1} - \sum_{\substack{j=1\\ j\neq i}}^{m+n}\left(\frac{1}{s+q_j}\right)^{h+1}\right] \tag{110}$$

Finally,

$$Q_{i_k} = \frac{1}{N-k}\sum_{h=0}^{N-k-1}(-1)^h Q_{i_{k+1+h}}\left[\sum_{j=1}^{n}\left(\frac{1}{z_j-q_i}\right)^{h+1} - \sum_{\substack{j=1\\ j\neq i}}^{m+n}\left(\frac{1}{q_j-q_i}\right)^{h+1}\right] \tag{111}$$

Once Q_{i_N} has been evaluated by one of the methods to be presented in Section III, Eq 111 can be used to calculate the remaining $(N - 1)$ coefficients for the Nth-order pole, $-q_i$. Note that the coefficients must be computed in the sequence

$Q_{i_{N-1}}, Q_{i_{N-2}}, Q_{i_{N-3}} \cdots Q_{i_1}$. A somewhat similar relationship is given in Ref. 14.

Although the general expression, Eq 111, is rather complicated, for the most common case of N = 2 it reduces to simply

$$Q_{i_1} = Q_{i_2}\left[\sum_{j=1}^{n}\left(\frac{1}{z_j - q_i}\right) - \sum_{\substack{j=1 \\ j\neq i}}^{m+n}\left(\frac{1}{q_j - q_i}\right)\right] \tag{112}$$

For Nth-order closed-loop poles, the previous expression for the sum of the modal response coefficients, Eq 19, is readily modified to

$$\sum_i Q_i + \sum_i Q_{i_1} = \kappa\delta_m^1 \quad , \quad m \geq 1 \tag{113}$$

where the summations include the Q_i's for all first-order poles and the Q_{i_1}'s for all higher order poles. The validity of Eq 113 can be shown very easily from the partial fraction expansion of $G_{rc}(s)$. Equation 113 is obtained by multiplying $G_{rc}(s)$ and its partial fraction expansion by s and letting $s \rightarrow \infty$.

For Nth-order poles the system responses are considerably more complicated than those given previously for N = 1. For example, if $-q_i$ is a second-order pole the system response to an input of sin ωt contains the terms

$$\frac{Q_{i_1}}{q_i^2 + \omega^2}\left[\omega e^{-q_i t} + q_i \sin \omega t - \omega \cos \omega t\right]$$

$$+ \frac{Q_{i_2}}{(q_i^2 + \omega^2)^2}\left\{\left[2q_i + (q_i^2 + \omega^2)t\right]\omega e^{-q_i t}\right.$$

$$\left. + \left[(q_i^2 - \omega^2) \sin \omega t - 2q_i\omega \cos \omega t\right]\right\}$$

For an input of cos ωt, the response contains the terms

$$\frac{Q_{i_1}}{q_i^2 + \omega^2}\left[-q_i e^{-q_i t} + q_i \cos \omega t + \omega \sin \omega t\right]$$

$$+ \frac{Q_{i_2}}{(q_i^2 + \omega^2)^2}\left\{\left[\omega^2 - q_i^2 - (q_i^2 + \omega^2) q_i t\right] e^{-q_i t}\right.$$

$$\left. + (q_i^2 - \omega^2) \cos \omega t + 2 q_i \omega \sin \omega t\right\}$$

A general expression for the response of an Nth-order pole to a cos ωt or sin ωt input can be derived by considering cos ωt and sin ωt to be the real and imaginary parts of $e^{j\omega t}$, i.e.,

$$\begin{aligned} \cos \omega t &= \mathrm{Re}\,(e^{j\omega t}) \\ \sin \omega t &= \mathrm{Im}\,(e^{j\omega t}) \end{aligned} \tag{114}$$

Using this technique we find that response of an Nth-order pole, $-q_i$, to a cos ωt input is

$$\sum_{k=1}^{N} Q_{i_k} \,\mathrm{Re}\left[\frac{e^{j\omega t} - e^{-q_i t} \sum_{h=0}^{k-1} \frac{(j\omega + q_i)^h \, t^h}{h!}}{(j\omega + q_i)^k}\right]$$

For an input of sin ωt, the response is

$$\sum_{k=1}^{N} Q_{i_k} \,\mathrm{Im}\left[\frac{e^{j\omega t} - e^{-q_i t} \sum_{h=0}^{k-1} \frac{(j\omega + q_i)^h \, t^h}{h!}}{(j\omega + q_i)^k}\right]$$

It is important to note that in evaluating the real and imaginary parts of the above expressions q_i must be treated as a purely real quantity even if it is actually complex.

The responses for power series inputs are equally as complicated. For example, if $-q_i$ is a second-order pole other than zero, the response to a unit step will contain the terms

$$\frac{Q_{i_1}}{q_i}\left(1 - e^{-q_i t}\right) + \frac{Q_{i_2}}{q_i^2}\left[1 - e^{-q_i t}(1 + q_i t)\right]$$

For a unit ramp, the terms are

$$\frac{Q_{i_1}}{q_i^2}\left(e^{-q_i t} - 1 + q_i t\right) + \frac{Q_{i_2}}{q_i^3}\left[q_i t - 2 + e^{-q_i t}(q_i t + 2)\right]$$

The general expression for the response of an Nth-order pole to an input of $t^M/M!$ can be derived by taking a partial fraction expansion of the product of the Laplace transform of the input and the q_i terms of the transfer function. After a considerable amount of manipulation, the inverse transform gives a response of

$$(-1)^M \sum_{k=1}^{N} \frac{Q_{i_k}}{(q_i)^{M+k}} \left\{ \frac{1}{(k-1)!} \sum_{j=0}^{M} \frac{(M+k-j-1)!\ (-q_i t)^j}{(M-j)!\ j!} \right.$$

$$\left. - \frac{e^{-q_i t}}{M!} \sum_{j=0}^{k-1} \frac{(M+k-j-1)!\ (q_i t)^j}{(k-j-1)!\ j!} \right\}$$

From the previous development it is obvious that the sensitivity definitions must be modified for Nth-order closed-loop poles. With the earlier definitions, the gain sensitivity is equal to $-(\partial G/\partial s)^{-1}_{s=-q_i}$, but from Eq 105

$$\left(\frac{\partial G}{\partial s}\right)_{s=-q_i} = 0 \quad , \quad N \geq 2 \tag{115}$$

or the original gain sensitivity becomes infinite. To modify the sensitivity definitions for Nth-order poles, we expand the total differential of G to include higher order terms, i.e.,

$$(dG)_{s=-q_i} = 0 = \Bigg[\left(ds \frac{\partial}{\partial s} + d\kappa \frac{\partial}{\partial \kappa} + \sum_{j=1}^{n} dz_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m+n} dp_j \frac{\partial}{\partial p_j}\right)G$$

$$+ \frac{1}{2!}\left(ds \frac{\partial}{\partial s} + d\kappa \frac{\partial}{\partial \kappa} + \sum_{j=1}^{n} dz_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m+n} dp_j \frac{\partial}{\partial p_j}\right)^2 G$$

$$+ \frac{1}{3!}\left(ds \frac{\partial}{\partial s} + d\kappa \frac{\partial}{\partial \kappa} + \sum_{j=1}^{n} dz_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m+n} dp_j \frac{\partial}{\partial p_j}\right)^3 G + \cdots \Bigg]_{s=-q_i} \tag{116}$$

where $$\left(ds \frac{\partial}{\partial s} + d\kappa \frac{\partial}{\partial \kappa}\right)^j G = (ds)^j \frac{\partial^j G}{\partial s^j} + j(ds)^{j-1} d\kappa \frac{\partial^j G}{\partial s^{j-1} \partial \kappa}$$

$$+ \frac{j(j-1)}{2!} (ds)^{j-2} (d\kappa)^2 \frac{\partial^j G}{\partial s^{j-2} \partial \kappa^2} + \cdots + (d\kappa)^j \frac{\partial^j G}{\partial \kappa^j}$$

Retaining only the lowest order terms for each parameter and remembering that the first $(N - 1)$ derivatives of G with respect to s are zero at $s = -q_i$, gives

$$dq_i = \left[\frac{(-1)^{N+1} N! \left(\frac{\partial G}{\partial \kappa} d\kappa + \sum_{j=1}^{n} \frac{\partial G}{\partial z_j} dz_j + \sum_{j=1}^{m+n} \frac{\partial G}{\partial p_j} dp_j\right)}{\frac{\partial^N G}{\partial s^N}}\right]^{1/N}_{s=-q_i} \tag{117}$$

Equation 117 suggests using modified sensitivity definitions such that

$$dq_i = \left[S^i_\kappa \frac{d\kappa}{\kappa} + \sum_{j=1}^{n} S^i_{z_j} dz_j + \sum_{j=1}^{m+n} S^i_{p_j} dp_j\right]^{1/N} \tag{118}$$

From Eq 117 and 118 it can be seen that if G is in root-locus, Bode, or mixed form,

$$S_{\kappa}^{i} = S_{K}^{i} = \frac{(-1)^{N} N!}{\left(\frac{\partial^{N} G}{\partial s^{N}}\right)_{s=-q_i}} \tag{119}$$

For zeros or poles which are written in root-locus form, the sensitivities are

$$S_{z_j}^{i} = \frac{S_{\kappa}^{i}}{z_j - q_i}$$

$$S_{p_j}^{i} = \frac{S_{\kappa}^{i}}{q_i - p_j} \tag{120}$$

For zeros or poles which are written in Bode form, the sensitivities are

$$S_{z_j}^{i} = \frac{q_i S_{\kappa}^{i}}{z_j(z_j - q_i)}$$

$$S_{p_j}^{i} = \frac{q_i S_{\kappa}^{i}}{p_j(q_i - p_j)} \tag{121}$$

Note that the relationship of the zero and pole sensitivities to the gain sensitivity is exactly the same as that for first-order closed-loop poles; Eq 120 is identical to Eq 55, and Eq 121 is identical to Eq 75. It is also important to note that from Eq 104 and 119

$$S_{\kappa}^{i} = (-1)^{N-1} Q_{i_N} \tag{122}$$

As pointed out by Ur in a similar development (Ref. 13), Eq 118 reflects the well-known characteristics of a branch point on a root-locus plot. The incoming branches, which at their junction represent an Nth-order closed-loop pole, are evenly spaced and separated from each other by $2\pi/N$. The outgoing branches are also separated from each other by $2\pi/N$ and are midway between the incoming branches.

SECTION III

MODAL RESPONSE COEFFICIENT AND GAIN SENSITIVITY EVALUATION FOR SINGLE-LOOP SYSTEMS

Modal response coefficients and gain sensitivities can be obtained in a wide variety of ways from either open- or closed-loop transfer function representations. The representations most commonly available in a typical problem are:

Open-loop	Closed-loop
$G(s)$ in factored form	
Pole-zero plot	Root loci, without complete sets of compatible roots
$G(j\omega)$ Bode plot $G(\pm\sigma)$ Bode plot (also serves as a plot of closed-loop real roots versus gain)	Closed-loop $\|G_{rc}\|_{db}$ and $\|G_{re}\|_{db}$ asymptotic plots (when decomposition method is applicable)

Correlating the information appearing on all these common representations serves to supplement the information available on any one. For example, a closed-loop $|G_{rc}|_{db}$ asymptotic plot, found using the $G(\pm\sigma)$ and $G(j\omega)$ open-loop Bode plots and the decomposition method, yields a set of roots which can be noted on the root locus. Similarly, one branch of the root locus complete with gains might provide sufficient information to allow a decomposition process to proceed using the Bode representation. In any event, for most cases in practice the above forms will be sufficient to provide complete information about gains and closed-loop poles. For the unusual circumstance where this is not true, the following additional representations may be necessary:

Open-loop	Closed-loop
$G(s)$ in a polynomial form	Root loci, with complete sets of compatible roots
$G(\xi,\mu)$ Bode plots	Root loci as functions of gain with ξ as parameter

These various forms of transfer function representation supply the raw data from which modal response coefficients and sensitivities are to be found.

The following discussion of modal response coefficient and gain sensitivity evaluation is organized along traditional lines, with an individual article for each general form of transfer function representation. The articles appear under the headings:

A. Direct Calculation
B. Root-Locus Methods
C. Methods Using Open-Loop Bode and ξ Plots
D. Method Using Closed-Loop Bode Asymptotes

Ordinarily different formulas for modal response coefficients or gain sensitivities are most appropriate to a particular form of representation. However, because of the supplementary character of the several transfer function representations available, certain formulas work well with several representations. Consequently some interplay at the detailed formula level is inevitable. In practice this is even more prevalent, since each of the transfer function representations available in a given problem is likely to be most suitable for computation of a particular modal response coefficient. Thus, in a practical problem one might get several coefficients from $G(j\omega)$ and $G(\pm\sigma)$ Bode forms, several other from a root locus, and the last one or two by direct calculation.

All the developments of this section are carried out for the general case of an Nth-order closed-loop pole, but at a convenient step the results for the specialized case of a first-order pole are often given.

To furnish a concrete example, all of the methods developed in this section have been applied to a simple third-order system. The details of the computations are given in the Appendix. The example covers the use of the techniques for closed-loop poles which are real and first-order, complex and first-order, and real and second-order. The results, summarized in Article E, are also used to illustrate the relative accuracy of the various methods.

A. DIRECT CALCULATION

Of the many methods presented in this report for the evaluation of modal response coefficients and gain sensitivities, only the two given in this article can readily yield values to any given degree of precision. All the other techniques involve a graphical construction or are based on certain approximations.

Either of the direct-calculation methods could be added to a digital computer program for loop-closing calculation to provide the modal response coefficients and sensitivities.

In Section II the modal response coefficients and gain sensitivities were given by, among other formulas,

$$Q_{i_N} = (-1)^{N-1} S_K^i = \frac{-N!}{\left(\frac{\partial^N G}{\partial s^N}\right)_{s=-q_i}} \tag{123}$$

Evaluation of Eq 123 is particularly convenient when the numerator and denominator of G are known polynomials in s. Expressing G as

$$G = \frac{K\alpha}{\beta} \tag{124}$$

gives

$$\frac{\partial G}{\partial s} = G\left(\frac{1}{\alpha}\frac{\partial \alpha}{\partial s} - \frac{1}{\beta}\frac{\partial \beta}{\partial s}\right) \tag{125}$$

and

$$\frac{\partial^2 G}{\partial s^2} = G\left[\frac{1}{\alpha}\frac{\partial^2 \alpha}{\partial s^2} - \frac{1}{\beta}\frac{\partial^2 \beta}{\partial s^2}\right] - G\left[\left(\frac{1}{\alpha}\frac{\partial \alpha}{\partial s}\right)^2 - \left(\frac{1}{\beta}\frac{\partial \beta}{\partial s}\right)^2\right] + \frac{\partial G}{\partial s}\left[\frac{1}{\alpha}\frac{\partial \alpha}{\partial s} - \frac{1}{\beta}\frac{\partial \beta}{\partial s}\right] \tag{126}$$

From Eq 123 and 125 it is obvious that for N = 1

$$Q_i = S_K^i = \left[\frac{K\alpha}{K\frac{\partial \alpha}{\partial s} + \frac{\partial \beta}{\partial s}}\right]_{s=-q_i}$$

$$= -\left[\frac{\beta}{K\frac{\partial \alpha}{\partial s} + \frac{\partial \beta}{\partial s}}\right]_{s=-q_i} \tag{127}$$

If N is greater than one, higher derivatives of G must be considered, such as $\partial^2 G/\partial s^2$ (Eq 126). Repeated differentiation of Eq 126 combined with the requirement that the first (N - 1) derivatives of G with respect to s vanish at $s = -q_i$, shows that

$$\left(\frac{1}{\alpha}\frac{\partial^k\alpha}{\partial s^k}\right)_{s=-q_i} = \left(\frac{1}{\beta}\frac{\partial^k\beta}{\partial s^k}\right)_{s=-q_i} \quad , \quad k < N \tag{128}$$

Consequently,

$$\begin{aligned}\left(\frac{\partial^N G}{\partial s^N}\right)_{s=-q_i} &= \left[G\left(\frac{1}{\alpha}\frac{\partial^N\alpha}{\partial s^N} - \frac{1}{\beta}\frac{\partial^N\beta}{\partial s^N}\right)\right]_{s=-q_i} \\ &= \left(\frac{\kappa\dfrac{\partial^N\alpha}{\partial s^N} \div \dfrac{\partial^N\beta}{\partial s^N}}{\beta}\right)_{s=-q_i} \\ &= -\left(\frac{\kappa\dfrac{\partial^N\alpha}{\partial s^N} + \dfrac{\partial^N\beta}{\partial s^N}}{\kappa\alpha}\right)_{s=-q_i}\end{aligned} \tag{129}$$

and

$$\begin{aligned}Q_{i_N} = (-1)^{N-1}S_\kappa^i &= -N!\left(\frac{\beta}{\kappa\dfrac{\partial^N\alpha}{\partial s^N} + \dfrac{\partial^N\beta}{\partial s^N}}\right)_{s=-q_i} \\ &= N!\left(\frac{\kappa\alpha}{\kappa\dfrac{\partial^N\alpha}{\partial s^N} + \dfrac{\partial^N\beta}{\partial s^N}}\right)_{s=-q_i}\end{aligned} \tag{130}$$

The second direct-calculation method derives from an alternate expression for G, i.e.,

$$G = \frac{\kappa \prod_{j=1}^{n} (s + z_j)}{\prod_{j=1}^{m+n} (s + p_j)} \tag{131}$$

Then

$$\frac{\partial G}{\partial s} = G\left[\sum_{j=1}^{n} \frac{1}{s + z_j} - \sum_{j=1}^{m+n} \frac{1}{s + p_j}\right] \tag{132}$$

and

$$\frac{\partial^2 G}{\partial s^2} = -G\left[\sum_{j=1}^{n}\left(\frac{1}{s + z_j}\right)^2 - \sum_{j=1}^{m+n}\left(\frac{1}{s + p_j}\right)^2\right] + \frac{\partial G}{\partial s}\left[\sum_{j=1}^{n} \frac{1}{s + z_j} - \sum_{j=1}^{m+n} \frac{1}{s + p_j}\right] \tag{133}$$

For $N = 1$, Eq 123 and 132 show that

$$Q_i = S_K^i = \frac{1}{\left[\sum_{j=1}^{n} \frac{1}{z_j - q_i} - \sum_{j=1}^{m+n} \frac{1}{p_j - q_i}\right]} \tag{134}$$

which is identical to Eq 59.

For N greater than one we must again consider higher derivatives of G, such as Eq 133. Repeated differentiation combined with the vanishing of the first $(N - 1)$ derivatives shows that

$$\sum_{j=1}^{n}\left(\frac{1}{z_j - q_i}\right)^k = \sum_{j=1}^{m+n}\left(\frac{1}{p_j - q_i}\right)^k , \quad k < N \tag{135}$$

Consequently,

$$\left(\frac{\partial^N G}{\partial s^N}\right)_{s=-q_i} = (-1)^N (N-1)!\left[\sum_{j=1}^{n}\left(\frac{1}{z_j - q_i}\right)^N - \sum_{j=1}^{m+n}\left(\frac{1}{p_j - q_i}\right)^N\right] \tag{136}$$

and

$$Q_{i_N} = (-1)^{N-1} S_K^i = \frac{(-1)^{N-1}\, N}{\sum_{j=1}^{n}\left(\frac{1}{z_j - q_i}\right)^N - \sum_{j=1}^{m+n}\left(\frac{1}{p_j - q_i}\right)^N} \tag{137}$$

The computation of Q_{i_N} or S_K^i by Eq 137 can easily be set up on a standardized work sheet and would probably be easier to use in a digital computer program than Eq 130. It is also simple to show that Eq 137 is valid if any or all of the terms of G are written in Bode form.

B. ROOT-LOCUS METHODS

Three methods of using a root-locus plot to estimate the modal response coefficients and gain sensitivities are described in this article. The first two are approximate techniques, while the accuracy of the third is limited only by the exactness of the plot and the required measurements.

The first method requires the locations of the closed-loop pole for two slightly different gains, such as the segment of a locus shown in Fig. 5a. Using finite increments as approximations to differential changes, Eq 118 and 122 give

$$S_K^i = (-1)^{N-1} Q_{i_N} = \frac{K(\Delta q_i)^N}{\Delta K} \qquad (138)$$

Equation 138 can also be utilized if the change in the closed-loop pole is obtained by some other technique, such as from ξ plots and root decomposition.

The second perturbation method is obtained by considering K to be a complex quantity. The normal root locus is then a graph of the closed-loop pole locations for K real. Now consider a small perturbation in the phase of K. The closed-loop pole must then be perturbed a small distance normal to the conventional root locus, and the phase perturbation of (G/K) must be minus that of K, see Fig. 5b. Consequently, for perturbations normal to the root locus, $\Delta K/K = -j\Delta\theta$, and Eq 138 becomes

$$S_K^i = (-1)^{N-1} Q_{i_N} \doteq \frac{j(\Delta q_i)^N}{\Delta \theta} \qquad (139)$$

where $\Delta\theta$ = phase change, in rad, of G/K

Note that for N greater than one the locus is not a single curve at $s = -q_i$, but is the junction of N incoming and N outgoing branches. In this case the Δq_i of

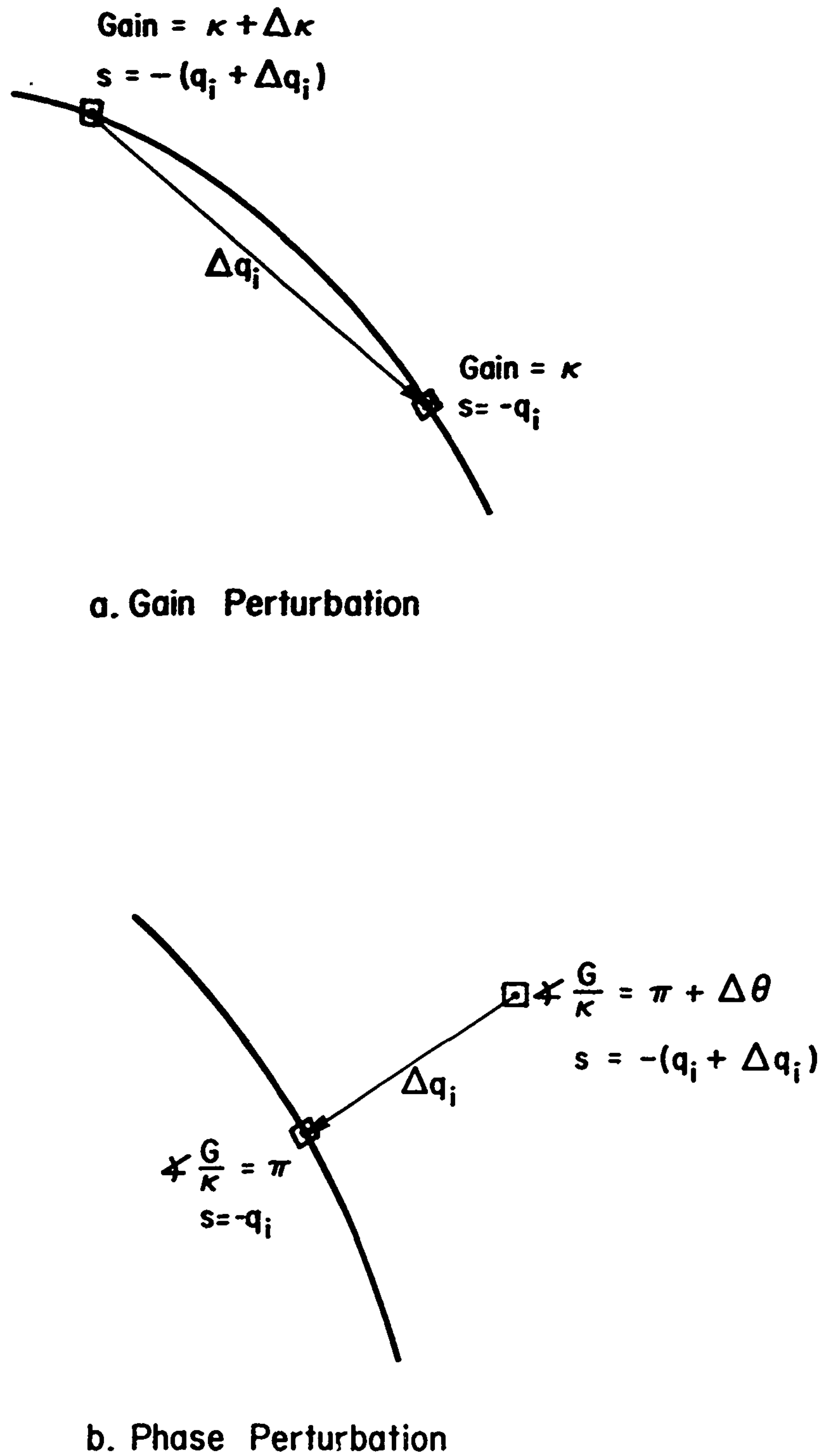


Figure 5. Root-Locus Perturbation Methods

Eq 139 should be picked midway between any incoming branch and an adjacent outgoing branch.

The third method, which will be called the vector method, has the advantage that its accuracy is limited only by the precision of the graph and the necessary measurements. Of the graphical techniques described in this section, this one should generally be the most accurate. Its primary disadvantage is that it requires a complete set of compatible closed-loop poles, i.e., all the closed-loop poles for a particular value of gain, and knowledge of the value of that gain.

From Eq 97 and 122,

$$S_{K}^{i} = (-1)^{N-1} Q_{i_N} = (-1)^{N-1} \left[\frac{(s + q_i)^{N_G}}{1 + G} \right]_{s=-q_i}$$

$$= \frac{(-1)^{N-1} K \prod_{j=1}^{n} (z_j - q_i)}{(1 + K\delta_m^0) \prod_{\substack{j=1 \\ j \neq i}}^{m+n} (q_j - q_i)} \qquad (140)$$

The right side of Eq 140 is a constant times a function which, with zeros at the open-loop zeros and poles at all the closed-loop poles except $-q_i$, is evaluated at $s = -q_i$. Thus, Eq 140 can be quickly evaluated using the normal graphical root-locus techniques with a Spirule or similar device. The principal quantities for a sample system are shown in Fig. 6.

C. METHODS USING OPEN-LOOP BODE AND ξ PLOTS

The derivative $\partial^N G/\partial s^N$, and consequently the modal response coefficient and gain sensitivity, can be determined from open-loop Bode or ξ plots. A ξ plot is a plot of the amplitude and phase of G as a function of μ for constant ξ, where

$$s = \left(\xi + j\sqrt{1 - \xi^2}\right)\mu \qquad (141)$$

The conventional Bode diagram is then a special case of the ξ plot ($\xi = 0$). Because ξ plots, particularly for $\xi = \pm 1$, are valuable analytical tools for

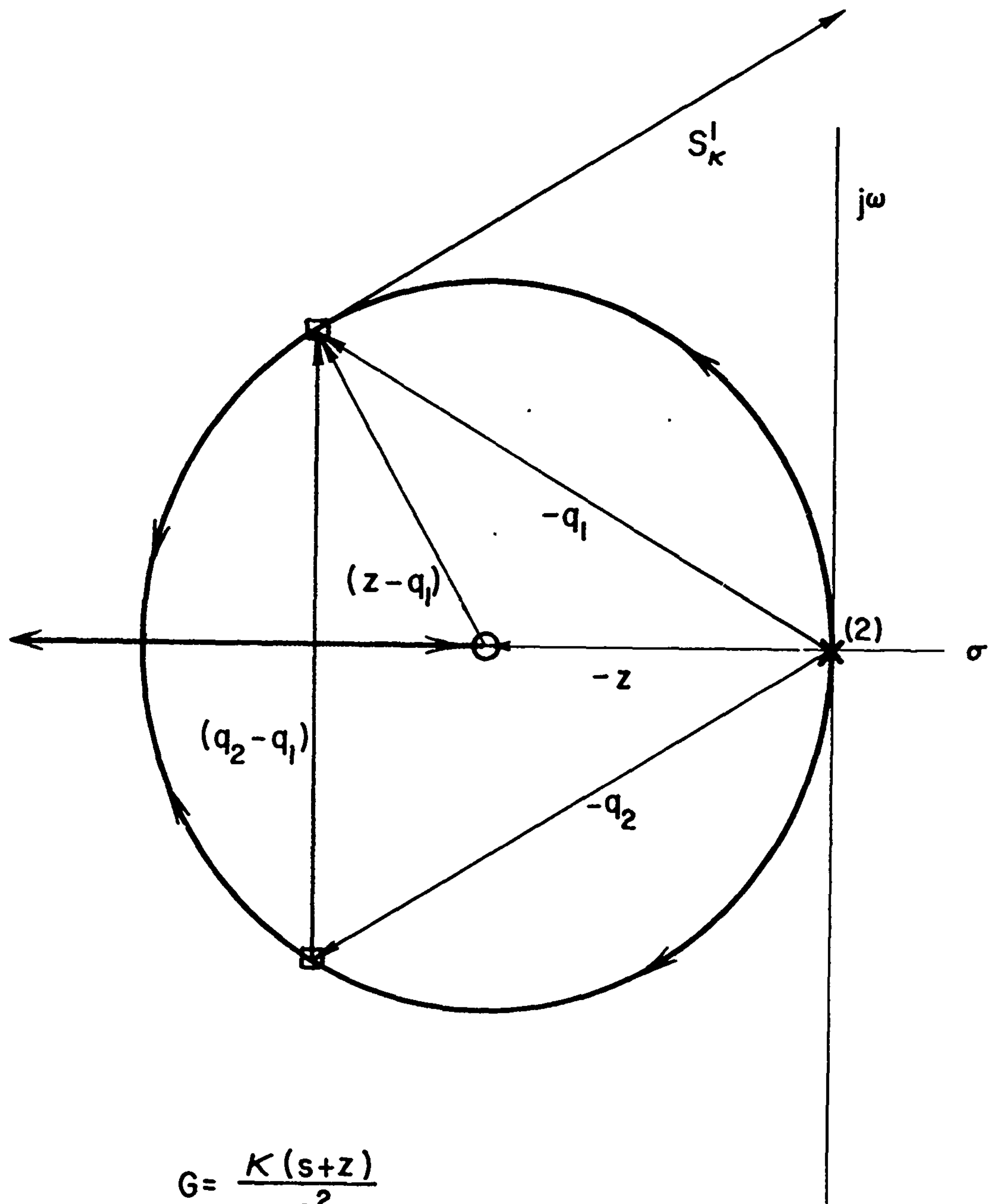


Figure 6. Vector Method

determining the closed-loop poles, it is highly desirable to be able to determine modal response coefficients and gain sensitivities from the ξ plots.

With ξ held constant,

$$\frac{\partial G}{\partial s} = G\frac{\partial \ln G}{\partial s} = G\frac{d \ln \mu}{ds}\frac{\partial \ln G}{\partial \ln \mu} \tag{142}$$

and from Eq 141,

$$\frac{d \ln \mu}{ds} = \frac{1}{s} \tag{143}$$

Equation 142 can now be written

$$\frac{\partial G}{\partial s} = \frac{G}{s}\frac{\partial \ln G}{\partial \ln \mu} \tag{144}$$

The second derivative is then

$$\frac{\partial^2 G}{\partial s^2} = \frac{G}{s^2}\frac{\partial^2 \ln G}{\partial(\ln \mu)^2} + \frac{\partial \ln G}{\partial \ln \mu}\frac{\partial}{\partial s}\left(\frac{G}{s}\right) \tag{145}$$

Repeated differentiation combined with the vanishing of the first (N - 1) derivatives at $s = -q_i$ gives

$$\left(\frac{\partial^N G}{\partial s^N}\right)_{s=-q_i} = \frac{-1}{(-q_i)^N}\left[\frac{\partial^N \ln G}{\partial(\ln \mu)^N}\right]_{s=-q_i} \tag{146}$$

Representing G in terms of its amplitude and phase, i.e.,

$$G = Ae^{j\varphi} \tag{147}$$

allows us to rewrite Eq 146 as

$$\left(\frac{\partial^N G}{\partial s^N}\right)_{s=-q_i} = \frac{-1}{(-q_i)^N}\left[\frac{\partial^N \ln A}{\partial(\ln \mu)^N} + \frac{j\partial^N \varphi}{\partial(\ln \mu)^N}\right]_{s=-q_i}$$

$$= \frac{-1}{(-q_i)^N}\left[\frac{1}{(2.3026)^{N-1}}\,\frac{\partial^N \log A}{\partial(\log \mu)^N} + \frac{j}{(2.3026)^N}\,\frac{\partial^N \varphi}{\partial(\log \mu)^N}\right]_{s=-q_i}$$

$$= \frac{-1}{(-q_i)^N(2.3026)^{N-1}}\left[\frac{A_N}{20} + \frac{j\varphi_N}{131.93}\right]_{s=-q_i} \qquad (148)$$

where

$$A_N = \text{the derivative } \frac{\partial^N \log A}{\partial(\log \mu)^N} \text{ in } \frac{\text{db}}{(\text{decade})^N}$$

$$\varphi_N = \text{the derivative } \frac{\partial^N \varphi}{\partial(\log \mu)^N} \text{ in } \frac{\text{deg}}{(\text{decade})^N}$$

Therefore for $N = 1$,

$$S_K^i = Q_i = \frac{-q_i}{\left[\frac{A_1}{20} + \frac{j\varphi_1}{131.93}\right]_{s=-q_i}} \qquad (149)$$

and in general

$$S_K^i = (-1)^{N-1} Q_{i_N} = \frac{-N!(2.3026)^{N-1}(q_i)^N}{\left[\frac{A_N}{20} + \frac{j\varphi_N}{131.93}\right]_{s=-q_i}} \qquad (150)$$

Thus, for a first-order pole the modal response coefficient and gain sensitivity can be determined from the slopes of the amplitude and phase curves of a ξ plot at $s = -q_i$. If q_i is purely imaginary, a conventional Bode diagram is used; if q_i is real, a Siggy ($\xi = \pm 1$) diagram is used and the φ derivative is zero. For a complex pole, a ξ plot or the shifted Bode diagram method, to be described shortly, must be used.

For multiple-order poles the higher derivatives, A_N and φ_N, must be evaluated. This might be done by:

1. Measuring the slopes, A_1 and φ_1, from a ξ plot
2. Plotting these slopes versus log μ
3. Measuring A_2 and φ_2
4. Plotting A_2 and φ_2 versus log μ
5. Repeating the process until A_N and φ_N are found

An alternate method for second-order poles is to measure the radius of curvature of the ξ plots. The radius of curvature of a curve defined by $y = y(x)$ at a point where $dy/dx = 0$ ($\partial G/\partial s$ is zero at $s = -q_i$) is given by

$$\frac{1}{R} = \frac{d^2y}{dx^2} \tag{151}$$

The radius of curvature can be used to determine A_2 and φ_2, but care must be exercised to properly account for the vertical and horizontal scales of the plot. A simple method of computing A_2 or φ_2 is to:

1. Measure the radius of curvature (at $s = -q_i$) in the units of the vertical scale, i.e., db or deg
2. Divide the vertical scale by the horizontal scale, e.g., (db/in.)/(dec/in.) or (deg/in.)/(dec/in.)
3. Square the ratio of step 2 and divide by the radius of curvature measured in step 1

An example of both methods for a second-order pole is given in the Appendix.

An alternative to the use of ξ plots for complex roots is to use shifted Bode diagrams. The basic idea is easily understood if one remembers that the shape of a root locus is independent of the location of the origin of the coordinates. Consequently, the imaginary axis may be shifted an arbitrary amount without changing the locus or the gain sensitivity. If the axis were translated so that it passed directly through a complex pair of closed-loop poles, the gain sensitivity for either of these two poles could be obtained from a Bode diagram and

Eq 150. Since Bode templates are readily available, and the Bode diagram is a well-known analytical tool, the shifted Bode diagram may, at times, be easier to use than the ξ plot.

In constructing the Bode diagram the real parts of all the poles and zeros must be translated by the proper increment. For poles or zeros on the real axis this involves only an addition or subtraction. For complex pairs the shift will change both the natural frequency, ω, and damping ratio, ζ, of the roots. It is easily shown that if the imaginary axis is shifted to the left (in the direction of the negative real axis) a distance $d = \mathrm{Re}(q_i)$, the natural frequency, ω_s, and damping ratio, ζ_s, of shifted poles or zeros are

$$\begin{aligned} \omega_s &= \sqrt{\omega^2 - 2\zeta\omega d + d^2} \\ \zeta_s &= \frac{\zeta\omega - d}{\sqrt{\omega^2 - 2\zeta\omega d + d^2}} = \frac{\zeta\omega - d}{\omega_s} \end{aligned} \tag{152}$$

For ζ_s equal to zero, i.e., $d = \zeta\omega$,

$$(\omega_s)_{\zeta_s=0} = \omega\sqrt{1 - \zeta^2} \tag{153}$$

A third method using ξ plots is especially appropriate when the decomposition loop-closure technique is applicable, although it also applies in general. The basic idea is to find two sets of closed-loop break points for neighboring open-loop gains. From these all of the Δq_i due to the gain change can be estimated. Because $\Delta K/K = \Delta\kappa/\kappa$, Eq 138 can be rewritten

$$S_{\kappa}^{i} = (-1)^{N-1} Q_{i_N} \doteq \frac{K(\Delta q_i)^N}{\Delta K} \tag{154}$$

This technique is the Bode diagram version of the gain perturbation technique discussed in connection with root-locus methods. Unfortunately, the method is an approximation, i.e., differentials are replaced by increments, and may have substantial errors if appreciable gain changes are used to develope the basic Δq_i data.

D. METHOD USING CLOSED-LOOP BODE ASYMPTOTES

The amplitude asymptotes of the closed-loop Bode can frequently be readily obtained by the methods of Ref. 1. These asymptotes can be used to estimate the modal response coefficients and gain sensitivities. This is only an approximate technique, and for accurate results the closed-loop pole must be widely separated in frequency from all other closed-loop poles (except its complex conjugate) and from all the open-loop zeros.

From previous developments we have

$$G_{rc} = \frac{\kappa}{(1 + \kappa\delta_m^0)} \frac{\prod_{k=1}^{n} (s + z_k)}{\prod_{k=1}^{m+n} (s + q_k)} \tag{155}$$

and

$$Q_{i_N} = (-1)^{N-1} S_\kappa^i = \frac{\kappa \prod_{k=1}^{n} (z_k - q_i)}{(1 + \kappa\delta_m^0) \prod_{\substack{k=1 \\ k \neq i}}^{m+n} (q_k - q_i)} \tag{156}$$

If $-q_i$ is a real root, the products of Eq 156 can be approximated by

$$\begin{aligned} \prod_{k=1}^{n} (z_k - q_i) &\doteq (-q_i)^{\lambda_z} \prod_{k}^{n-\lambda_z} (z_k) \\ \prod_{\substack{k=1 \\ k \neq i}}^{m+n} (q_k - q_i) &\doteq (-q_i)^{\lambda_q} \prod_{k}^{(m+n-N-\lambda_q)} (q_k) \end{aligned} \tag{157}$$

where

λ_z = number of zeros of smaller magnitude than q_i

$\prod_{k}^{n-\lambda_z} (z_k)$ = product of the $(n-\lambda_z)$ zeros of larger magnitude than q_i

λ_q = number of closed-loop poles of smaller magnitude than q_i

$\prod_{k}^{(m+n-N-\lambda_q)} (q_k)$ = product of the $(m+n-N-\lambda_q)$ closed-loop poles of larger magnitude than q_i

Combining Eq 156 and 157,

$$Q_{i_N} = (-1)^{N-1} S^i_\kappa \doteq \frac{\kappa(-q_i)^{\lambda_z - \lambda_q} \prod_k^{n-\lambda_z} (z_k)}{(1 + \kappa\delta_m^0) \prod_k^{(m+n-N-\lambda_q)} (q_k)} \tag{158}$$

From Eq 155 it can be seen that the amplitude asymptote, A_{asy}, of the closed-loop G_{rc} Bode diagram at a frequency equal to $|q_i|$ is

$$A_{asy} = \left|\frac{\kappa}{1 + \kappa\delta_m^0}\right| \frac{|q_i|^{\lambda_z} \prod_k^{n-\lambda_z} |z_k|}{|q_i|^{\lambda_q + N} \prod_k^{(m+n-N-\lambda_q)} |q_k|} \tag{159}$$

Note that several terms in Eq 158 and 159 are identical except for the absolute value sign. Consequently these equations can be combined to give

$$Q_{i_N} = (-1)^{N-1} S^i_\kappa \doteq \begin{cases} (-1)^{\lambda_z + \lambda_q + \lambda_u} (q_i)^N A_{asy} & , \; q_i \text{ real and } > 0 \\ (-1)^{\lambda_u} (-q_i)^N A_{asy} & , \; q_i \text{ real and } < 0 \end{cases} \tag{160}$$

where λ_u = number of closed-loop, non-minimum-phase zeros and poles of larger magnitude than q_i, plus one if $\kappa/(1 + \kappa\delta_m^0) < 0$

Note that because complex roots always occur in pairs, the powers of (-1) in Eq 160, $(\lambda_z + \lambda_q + \lambda_u)$ or λ_u, can be considered as the number of open-loop zeros and closed-loop poles which, in the s-plane, lie on the real axis and to the right of $-q_i$, plus one if $\kappa/(1 + \kappa\delta_m^0) < 0$.

If $-q_i$ is a complex pole, Eq 158 must be modified slightly because the complex conjugate pole has the same magnitude as q_i. Then Eq 158 should be written

$$Q_{i_N} = (-1)^{N-1} S_K^i \doteq \frac{K(-q_i)^{\lambda_z - \lambda_q} \prod_k^{n-\lambda_z} (z_k)}{(1 + K\delta_m^0)[-2j \text{ Im } (q_i)]^N \prod_k^{(m+n-2N-\lambda_q)} (q_k)} \qquad (161)$$

Combining Eq 159 and 161,

$$Q_{i_N} = (-1)^{N-1} S_K^i \doteq (-1)^{\lambda_u} \left(\frac{-q_i}{|q_i|}\right)^{\lambda_z - \lambda_q} \frac{|q_i|^{2N} A_{asy}}{[-2j \text{ Im } (q_i)]^N} \qquad (162)$$

The technique outlined above, although highly approximate, is especially easy to apply. Its principal merit is the quick, rough-cut view afforded from visual inspection of closed-loop asymptotic Bode plots. The major effect of a dipole (closely spaced pole-zero pair, q_i, z_h) can be taken into better account by modifying Eq 158 and 160 slightly. For this one modal response coefficient or sensitivity the factor $(z_h - q_i)$ should be used in the numerator of Eq 158 in place of one of the $-q_i$ or z_k terms; it replaces a $-q_i$ term if $|z_h| < |q_i|$ and replaces a z_k term if $|z_h| > |q_i|$. Consequently the effects of the dipole can be included by multiplying Eq 160 by $[1 - (z_h/q_i)]$ if $|z_h| < |q_i|$ or by $[1 - (q_i/z_h)]$ if $|z_h| > |q_i|$.

Further refinement of the technique is possible along these same lines. This is seldom warranted because the quick-view advantage of the method is lost and, even when refined, the technique cannot compete, on an accuracy basis, with the others discussed.

E. EXAMPLE

To illustrate the techniques described in the previous portions of this section, all of the methods have been used to compute the gain sensitivities and modal response coefficients for a simple example. The details of the calculations are given in the Appendix and the results are summarized here.

The example is a system with an open-loop transfer function given by

$$G(s) = \frac{K}{s(s+1)(s+5)} \qquad (163)$$

A root-locus plot for this system is shown in Fig. 7. The calculations were carried out for two different values of gain, κ. The high-gain case was selected to illustrate the procedures for complex closed-loop poles, and the low-gain case illustrates the procedures for second-order closed-loop poles. In addition, the actual shifts in the closed-loop poles due to the gain change are compared with the values predicted by both the high-gain and low-gain sensitivities.

For the high-gain case, the gain was selected to produce a complex pair of closed-loop poles with a damping ratio of $\sqrt{2}/2$. For this situation:

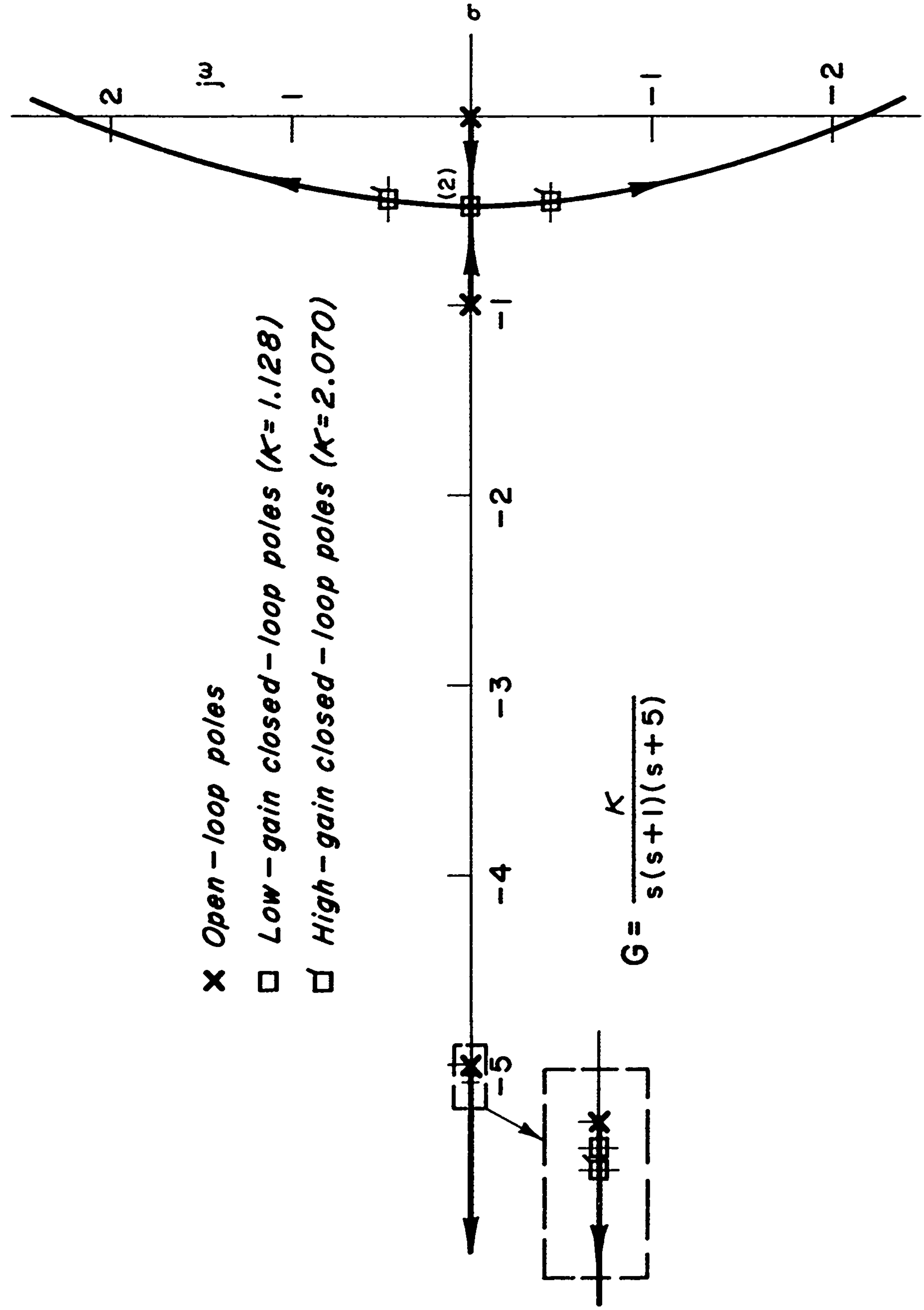
$$\kappa = 2.070$$
$$q_1 = 5.099$$
$$q_2 = 0.450(1 - j)$$
$$q_3 = 0.450(1 + j)$$

For the low-gain case, the gain was selected to produce a second-order closed-loop pole. For this condition:

$$\kappa = 1.128$$
$$q_1 = 5.055$$
$$q_2 = 0.472 \text{ (second-order pole)}$$

The values of gain sensitivity which were computed by each method are listed in Table I. The values of S_{κ}^{3} for the high-gain case are not listed because they are the complex conjugate of S_{κ}^{2}. No attempt will be made to form any general conclusions on the relative accuracies of the various methods except to note that the values obtained by the root-locus vector method, which would be expected to be the most accurate graphical technique for this example, agreed very well with the exact values obtained by the direct-calculation methods.

To get some indication of the accuracy of predicting the changes in closed-loop poles from the gain sensitivity, we can compare the actual shifts between the high- and low-gain cases with the changes predicted by the gain sensitivities. This comparison is summarized in Table II. Considering that the ratio of the two gains is more than 1.8, the agreement is generally quite good. The one relatively poor case is the shift in q_2 predicted by the high-gain sensitivity. The predicted change in the imaginary part of q_2 is only half of the actual shift.



Open-loop poles
Low-gain closed-loop poles (K = 1.128)
High-gain closed-loop poles (K = 2.070)
G = K / s(s+1)(s+5)
σ
jω
2
1
-1
-2
(2)
-1
-2
-3
-4
-5

TABLE I

COMPUTED GAIN SENSITIVITIES FOR SINGLE-LOOP EXAMPLE

CLASS	METHOD	HIGH GAIN		LOW GAIN	
		S_K^1	S_K^2	S_K^1	S_K^2
Direct calculation	Numerator and denominator derivatives	0.0949	-0.0474 - j0.490	0.0537	-0.246
	Summation of terms	0.0949	-0.0474 - j0.490	0.0537	-0.246
Root locus	Gain perturbation	0.083	-0.045 - j0.353	0.0526	-0.242 + j0.0237
	Phase perturbation	0.121	-0.0477 - j0.454	0.090	-0.245
	Vector	0.0952	-0.0470 - j0.488	0.0537	-0.246
Open-loop Bode and ξ plots	ξ successive slopes	0.0963	-0.0341 - j0.490	0.0539	-0.281
	ξ radius of curvature	NA	NA	NA	-0.247
	Shifted Bode	NA	-0.0431 - j0.466	NA	NA
	Gain perturbation	0.10	-0.05 - j0.345	0.03	-0.245
Closed-loop Bode	Amplitude asymptotes	0.0796	0 - j0.451	0.0441	-0.223

TABLE II

COMPARISON OF ACTUAL AND PREDICTED CHANGES IN CLOSED-LOOP POLES

	Δq_1	Δq_2
Actual	0.044	0.022 + j0.450
Predicted by high-gain sensitivity	0.043	0.022 + j0.223
Predicted by low-gain sensitivity	0.045	j0.455

SECTION IV

MULTILOOP SYSTEMS

A. MODAL RESPONSE COEFFICIENTS

The modal response coefficients for a multiloop system can be derived in a manner quite similar to that used in Section II for single-loop systems. The familiar expression

$$Q_{i_k} = \frac{1}{(N-k)!}\left\{\frac{d^{N-k}}{ds^{N-k}}\left[(s+q_i)^N G_{rc}\right]\right\}_{s=-q_i} \tag{164}$$

also applies to multiloop systems, but G_{rc} is now a more complicated function. In general, G_{rc} can be written as

$$G_{rc} = \frac{F_1}{1+F_2} \tag{165}$$

where F_1 and F_2 are sums of terms which contain the transfer functions of the various individual elements in the system.

As in Section II, define

$$Q^*_{i_N}(s) \equiv (s+q_i)^N G_{rc} = \frac{(s+q_i)^N F_1}{1+F_2} \tag{166}$$

or

$$(1+F_2)Q^*_{i_N}(s) = (s+q_i)^N F_1 \tag{167}$$

Repeated differentiation of Eq 167 shows that

$$\left[\frac{\partial^k F_2}{\partial s^k}\right]_{s=-q_i} = 0,\ 1 \le k \le N-1 \tag{168}$$

and $$Q_{i_N} = Q^*_{i_N}(-q_i) = \left(\frac{N!\ F_1}{\partial^N F_2/\partial s^N}\right)_{s=-q_i} \tag{169}$$

A relationship for the summation for the modal response coefficients can also be derived in a manner analogous to that of Section II. If there are no multiple-order closed-loop poles, and if the number of closed-loop poles is greater than the number of closed-loop zeros, we can write

$$G_{rc} = \sum_i \frac{Q_i}{s + q_i} \tag{170}$$

Multiplying both sides by s, and letting s approach infinity, gives

$$\sum_i Q_i = \begin{cases} \kappa_{CL} & \text{if number of closed-loop poles = 1 + number of closed-loop zeros} \\ 0 & \text{if number of closed-loop poles} \geq 2 + \text{number of closed-loop zeros} \end{cases} \tag{171}$$

where κ_{CL} is the closed-loop root-locus gain of the system. This is directly analogous to the result obtained for the single-loop case.

When there are Nth-order closed-loop poles, a result analogous to that for single-loop systems is

$$\sum_i Q_i + \sum_i Q_{i_1} = \begin{cases} \kappa_{CL} & \text{if number of closed-loop poles = 1 + number of closed-loop zeros} \\ 0 & \text{if number of closed-loop poles} \geq 2 + \text{number of closed-loop zeros} \end{cases} \tag{172}$$

B. SENSITIVITIES

For multiloop systems the closed-loop poles are functions of the gains, as well as the pole and zero locations, of a number of transfer functions. It is then desirable to know the sensitivity of a closed-loop pole to variations in the gain, poles, and zeros of any of the transfer functions. To derive these sensitivities, consider the characteristic equation of a multiloop system,

$$\left[1 + (F_2)\right]_{s=-q_i} = 0 \tag{173}$$

where F_2 is a summation of terms, each of which is the product of various transfer functions. To keep the discussion completely general, the transfer function of any individual element in the system will be denoted by G_k. Naturally the functional relationship between F_2 (or F_1) and any individual transfer function depends on the manner in which the various elements are connected. This functional relationship can be determined from a block diagram or signal flow diagram of the system.

Extending the notation to the gains, poles, and zeros, κ_k is the root-locus gain of the kth transfer function, and $(-z_{k_j})$ and $(-p_{k_j})$ are the jth zero and pole of the kth transfer function. The total differential of F_2 can be written (analogous to Eq 116 and 117)

$$(dF_2)_{s=-q_i} = 0 = \frac{1}{N!}\left(\frac{\partial^N F_2}{\partial s^N}\right)_{s=-q_i} (-dq_i)^N$$
$$+ \left[\sum_{k=1}^{L} \frac{\partial F_2}{\partial G_k}\left(\frac{\partial G_k}{\partial \kappa_k}\, d\kappa_k + \sum_{j=1}^{n_k} \frac{\partial G_k}{\partial z_{k_j}}\, dz_{k_j} + \sum_{j=1}^{m_k+n_k} \frac{\partial G_k}{\partial p_{k_j}}\, dp_{k_j}\right)\right]_{s=-q_i} \tag{174}$$

or

$$dq_i = \left[\frac{(-1)^{N+1} N!}{\partial^N F_2/\partial s^N} \sum_{k=1}^{L} \frac{\partial F_2}{\partial G_k}\left(\frac{\partial G_k}{\partial \kappa_k}\, d\kappa_k + \sum_{j=1}^{n_k} \frac{\partial G_k}{\partial z_{k_j}}\, dz_{k_j} + \sum_{j=1}^{m_k+n_k} \frac{\partial G_k}{\partial p_{k_j}}\, dp_{k_j}\right)\right]^{1/N}_{s=-q_i} \tag{175}$$

where n_k = number of zeros of kth transfer function, G_k

m_k+n_k = number of poles of kth transfer function

L = number of transfer functions

Note that as per Eq 168, the first (N - 1) derivatives of F_2 with respect to s have been set equal to zero.

As in the single-loop case, we will define the sensitivities so that

$$dq_i = \left[\sum_{k=1}^{L}\left(S^i_{\kappa_k}\frac{d\kappa_k}{\kappa_k} + \sum_{j=1}^{n_k} S^i_{z_{k_j}} dz_{k_j} + \sum_{j=1}^{m_k+n_k} S^i_{p_{k_j}} dp_{k_j}\right)\right]^{1/N} \tag{176}$$

Equating like coefficients of Eq 175 and 176 gives

$$S^i_{\kappa_k} = \left[\frac{(-1)^{N+1} N!}{\partial^N F_2/\partial s^N}\frac{\partial F_2}{\partial G_k}\frac{\partial G_k}{\partial \kappa_k}\kappa_k\right]_{s=-q_i} \tag{177}$$

$$\frac{S^i_{\kappa_j}}{S^i_{\kappa_k}} = \left[\frac{\kappa_j \dfrac{\partial F_2}{\partial G_j}\dfrac{\partial G_j}{\partial \kappa_j}}{\kappa_k \dfrac{\partial F_2}{\partial G_k}\dfrac{\partial G_k}{\partial \kappa_k}}\right]_{s=-q_i} \tag{178}$$

$$\frac{S^i_{z_{k_j}}}{S^i_{\kappa_k}} = \left[\frac{\dfrac{\partial G_k}{\partial z_{k_j}}}{\kappa_k \dfrac{\partial G_k}{\partial \kappa_k}}\right]_{s=-q_i} \tag{179}$$

$$\frac{S^i_{p_{k_j}}}{S^i_{\kappa_k}} = \left[\frac{\dfrac{\partial G_k}{\partial p_{k_j}}}{\kappa_k \dfrac{\partial G_k}{\partial \kappa_k}}\right]_{s=-q_i} \tag{180}$$

If the transfer functions are written in root-locus, Bode, or mixed form,

$$\kappa_k \frac{\partial G_k}{\partial \kappa_k} = K_k \frac{\partial G_k}{\partial K_k} = G_k \tag{181}$$

Consequently,

$$S^i_{\kappa_k} = S^i_{K_k} = \left[\frac{(-1)^{N+1} \; N! \; G_k \dfrac{\partial F_2}{\partial G_k}}{\dfrac{\partial^N F_2}{\partial s^N}} \right]_{s=-q_i} \qquad (182)$$

$$\frac{S^i_{\kappa_j}}{S^i_{\kappa_k}} = \left[\frac{G_j \dfrac{\partial F_2}{\partial G_j}}{G_k \dfrac{\partial F_2}{\partial G_k}} \right]_{s=-q_i} \qquad (183)$$

For zeros or poles which are written in root-locus form,

$$\frac{S^i_{z_{k_j}}}{S^i_{\kappa_k}} = \frac{1}{z_{k_j} - q_i} \qquad (184)$$

$$\frac{S^i_{p_{k_j}}}{S^i_{\kappa_k}} = \frac{1}{q_i - p_{k_j}} \qquad (185)$$

Likewise, for zeros or poles which are written in Bode form,

$$\frac{S^i_{z_{k_j}}}{S^i_{\kappa_k}} = \frac{q_i}{z_{k_j}(z_{k_j} - q_i)} \qquad (186)$$

$$\frac{S^i_{p_{k_j}}}{S^i_{\kappa_k}} = \frac{q_i}{p_{k_j}(q_i - p_{k_j})} \qquad (187)$$

Two important features of the above sensitivity ratios are worth mentioning. First, because the terms of F_2 will normally be products of transfer functions to the first power, $G_k(\partial F_2/\partial G_k)$ will simply be the sum of all the terms of F_2 which contain G_k. Thus, the ratio of the gain sensitivities for κ_j and κ_k will be the ratio of the terms of F_2 in which G_j and G_k appear, evaluated at $s = -q_i$. Second, the ratios of zero or pole sensitivities to the gain sensitivity of the same transfer function are identical to those for the single-loop case. Consequently, the previously derived (Eq 78 through 81) sensitivity relationships in terms of the damping ratio and frequency or the real and imaginary parts of an open-loop root also apply to the multiloop case.

Because the normal procedure for evaluating the closed-loop poles of a multiloop system is by means of a series of loop closures, the above equations can be utilized to form a relatively simple procedure for evaluating the sensitivities:

1. Calculate one of the gain sensitivities by applying one of the methods of Section III to the final loop closure
2. Compute the other gain sensitivities from Eq 183
3. Compute zero and pole sensitivities from Eq 78, 79, 80, 81, 184, 185, 186, or 187

In many cases step 2 of the above will be the greatest source of computational difficulty. No simple standard procedures can be established for this operation, but there are several ideas which may be helpful. One of these is to make use of the fact that at $s = -q_i$, $F_2 = -1$. For example, if F_2 were

$$F_2 = G_1G_2 + G_1G_3G_4 + G_1G_2G_3 + G_3G_4$$

then

$$\frac{S^i_{\kappa_2}}{S^i_{\kappa_1}} = \left(\frac{G_1G_2 + G_1G_2G_3}{G_1G_2 + G_1G_3G_4 + G_1G_2G_3}\right)_{s=-q_i} = -\left[\frac{G_1G_2(1 + G_3)}{1 + G_3G_4}\right]_{s=-q_i}$$

$$\frac{S^i_{\kappa_3}}{S^i_{\kappa_1}} = \left(\frac{G_1G_3G_4 + G_1G_2G_3 + G_3G_4}{G_1G_2 + G_1G_3G_4 + G_1G_2G_3}\right)_{s=-q_i} = \left(\frac{1 + G_1G_2}{1 + G_3G_4}\right)_{s = -q_i}$$

$$\frac{S^i_{\kappa_4}}{S^i_{\kappa_1}} = \left(\frac{G_1G_3G_4 + G_3G_4}{G_1G_2 + G_1G_3G_4 + G_1G_2G_3}\right)_{s=-q_i} = -\left[\frac{G_3G_4(1 + G_1)}{1 + G_3G_4}\right]_{s=-q_i}$$

The sensitivity ratios could be computed directly by setting $s = -q_i$ in the proper equations, but for complex poles graphical methods may be simpler. The products of transfer functions could be evaluated by constructing ξ plots or root-locus-like plots. From the known zeros and poles of the transfer functions, a ξ plot could be constructed for ξ equal to minus the damping ratio of q_i. The construction would only have to be accurate for the frequency equal to $|q_i|$. The amplitude and phase of the product of transfer functions can also be evaluated by plotting zeros and poles on an s-plane graph, and measuring the amplitude and phase at $s = -q_i$. Once the products have been evaluated, the summations with unity, which are sometimes necessary, could be done numerically, from a Nichols chart, or graphically (preferably on polar graph paper).

As in the single-loop case, a useful check on the sensitivity calculations is that the sum of all the zero and pole sensitivities is one if the transfer functions are entirely in root-locus form and if N equals one ($-q_i$ is a first-order pole). This can be shown from Eq 182, 184, and 185.

$$\sum_{k=1}^{L}\left(\sum_{j=1}^{n_k} S^i_{z_{k_j}} + \sum_{j=1}^{m_k+n_k} S^i_{p_{k_j}}\right) = \left[\frac{1}{\partial F_2/\partial s}\sum_{k=1}^{L} G_k \frac{\partial F_2}{\partial G_k}\left(\sum_{j=1}^{n_k}\frac{1}{z_{k_j} - q_i} + \sum_{j=1}^{m_k+n_k}\frac{1}{q_i - p_{k_j}}\right)\right]_{s=-q_i} \tag{188}$$

The derivative $\partial F_2/\partial s$ can be written as

$$\left(\frac{\partial F_2}{\partial s}\right)_{s=-q_i} = \sum_{k=1}^{L}\left(\frac{\partial F_2}{\partial G_k}\frac{\partial G_k}{\partial s}\right)_{s=-q_i}$$

$$= \sum_{k=1}^{L}\left(G_k\frac{\partial F_2}{\partial G_k}\right)_{s=-q_i}\left(\sum_{j=1}^{n_k}\frac{1}{z_{k_j} - q_i} + \sum_{j=1}^{m_k+n_k}\frac{1}{q_i - p_{k_j}}\right) \tag{189}$$

Consequently,

$$\sum_{k=1}^{L}\left(\sum_{j=1}^{n_k} S^i_{z_{k_j}} + \sum_{j=1}^{m_k+n_k} S^i_{p_{k_j}}\right) = 1 \tag{190}$$

The relationship between the gain sensitivity and the modal response coefficient can be seen directly from Eq 169 and 182.

$$Q_{i_N} = (-1)^{N-1} \left(\frac{F_1}{G_k(\partial F_2/\partial G_k)} \right)_{s=-q_i} S^{i}_{K_k} \qquad (191)$$

For a multiple input-output system there will be a set of modal response coefficients for each input-output combination. All the coefficients can be evaluated by means of Eq 191 if the proper numerator, F_1, for each input-output pair is used.

C. EXAMPLE

The altitude control system of Ref. 2 will be used as an example. In that system, pitch angle, θ, and altitude, h, are fed back to the elevator, Fig. 8.

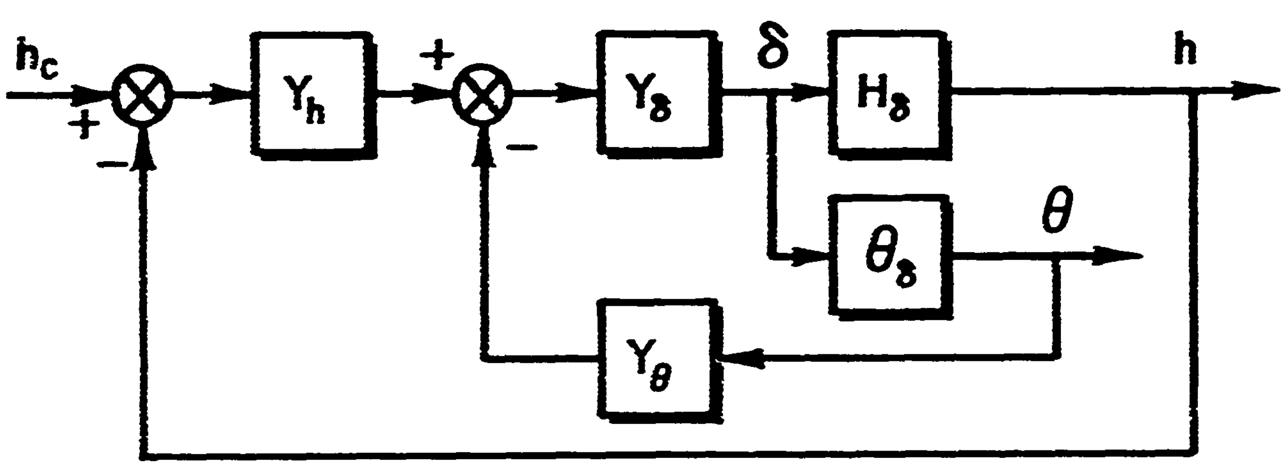


Figure 8. Multiloop Example

In Fig. 8, θ_δ and H_δ are the airframe pitch angle and altitude transfer functions, Y_θ and Y_h are the equalization and sensor dynamics, and Y_δ is the elevator servo dynamics. From Fig. 8 it can be seen that the closed-loop transfer function is

$$G_{rc} = \frac{h}{h_c} = \frac{Y_h H_\delta Y_\delta}{1 + Y_h H_\delta Y_\delta + Y_\theta \theta_\delta Y_\delta} \qquad (192)$$

For convenience, the notation

$$
\begin{aligned}
G_1 &= Y_h H_\delta \\
G_2 &= Y_\delta \\
G_3 &= Y_\theta \theta_\delta
\end{aligned}
\tag{193}
$$

will be used. The form of the transfer functions and numerical values which were used are:

$$
G_1 = \frac{\kappa_a A_h\left(s + \frac{1}{T_{h_1}}\right)\left(s + \frac{1}{T_{h_2}}\right)\left(s - \frac{1}{T_{h_3}}\right)}{(s + \omega_a)s(s^2 + 2\zeta_p\omega_p s + \omega_p^2)(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2)}
$$

$$
G_2 = \frac{\kappa_m}{s^2 + 2\zeta_m\omega_m s + \omega_m^2} \tag{194}
$$

$$
G_3 = \kappa_e A_\theta \frac{\left(s + \omega_L\right)\left(s + \frac{1}{T_{\theta_1}}\right)\left(s + \frac{1}{T_{\theta_2}}\right)}{(s^2 + 2\zeta_p\omega_p s + \omega_p^2)(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2)}
$$

where

$\kappa_a \kappa_m = 127.5$ | $\zeta_m = 0.7$

$A_h = -69.8$ | $\omega_m = 50$

$1/T_{h_1} = 0.0064$ | $\kappa_e \kappa_m = 2700$

$1/T_{h_2} = 1/T_{h_3} = 19.2$ | $A_\theta = 26$

$\omega_a = 15$ | $\omega_L = 2.4$

$\zeta_p = 0.0714$ | $1/T_{\theta_1} = 0.0098$

$\omega_p = 0.063$ | $1/T_{\theta_2} = 1.372$

$\zeta_{sp} = 0.493$

$\omega_{sp} = 4.27$

The characteristic equation of this system is

$$\left(1 + G_1G_2 + G_2G_3\right)_{s=-q_i} = 0 \qquad (195)$$

The technique for determining the closed-loop poles which was used in Ref. 2 was to rewrite the characteristic equation as

$$\left(1 + \frac{G_1G_2}{1 + G_2G_3}\right)_{s=-q_i} = 0 \qquad (196)$$

Equation 196 can also be written as

$$\left[1 + \kappa_a\kappa_m A_h \frac{\left(s + \frac{1}{T_{h_1}}\right)\left(s + \frac{1}{T_{h_2}}\right)\left(s - \frac{1}{T_{h_3}}\right)}{s(s + \omega_a)\Delta_{IL}}\right]_{s=-q_i} = 0 \qquad (197)$$

where Δ_{IL} contains the poles from the $1 + G_2G_3$ inner-loop closure, i.e.,

$$\Delta_{IL} = (s^2 + 2\zeta_m\omega_m s + \omega_m^2)(s^2 + 2\zeta_p\omega_p s + \omega_p^2)(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2) + \kappa_m\kappa_e A_\theta(s + \omega_L)\left(s + \frac{1}{T_{\theta_1}}\right)\left(s + \frac{1}{T_{\theta_2}}\right) \qquad (198)$$

$$= (s + 0.011)(s + 1.05)(s + 3.5)\left[s^2 + 2(0.32)(35)s + (35)^2\right](s + 47.5)$$

When the final closure is made, Eq 197, it is found that the dominant closed-loop poles are a complex pair at approximately s = -0.287 ± j0.869 (ω = 0.915, ζ = 0.314).* The sensitivities for this system will only be determined for these dominant poles. Actually, only the sensitivities for the pole, -0.287 + j0.869, will be computed. The sensitivities for the conjugate root are the complex conjugates of the computed sensitivities.

---

*These values for the closed-loop poles are slightly different from those given in Ref. 2. The values given above were obtained by using a more accurate approximation to the closure than was done in Ref. 2.

The first step is to determine one of the gain sensitivities. The simplest one to do is the one for K_1 (the gain of G_1) using the closure defined by Eq 197. Any one of the techniques of Section III could be used to evaluate $S^i_{K_1}$, but because all of the other sensitivities will be computed from $S^i_{K_1}$, we will use the exact analytical expression

$$S^i_{K_1} = \left[\sum_k \frac{1}{z_k - q_i} - \sum_k \frac{1}{p_k - q_i}\right]^{-1} \tag{199}$$

where $-p_k$ and $-z_k$ are the poles and zeros of $G_1G_2/(1 + G_2G_3)$, see Eq 196 and 197. This computation gives

$$\begin{aligned} S^i_{K_1} &= -(0.175 + j0.492) \\ &= -0.521 \exp\left(j \frac{70.4}{57.3}\right) \end{aligned} \tag{200}$$

For the sake of brevity, we will adopt the notation

$$Ae^{j\varphi} \equiv A \measuredangle \varphi \tag{201}$$

and express $S^i_{K_1}$ as

$$S^i_{K_1} = 0.521 \measuredangle 250.4 \text{ deg} \tag{202}$$

Note that because $F_1 = G_1G_2 = G_1(\partial F_2/\partial G_1)$, $S^i_{K_1}$ is equal to the modal response coefficient for h/h_c.

As the next step, $S^i_{K_2}$ and $S^i_{K_3}$ will be computed. From the sensitivity ratios of Eq 183,

$$\frac{S^i_{K_2}}{S^i_{K_1}} = \left(\frac{G_1G_2 + G_2G_3}{G_1G_2}\right)_{s=q_i} = \frac{-1}{(G_1G_2)_{s=-q_i}} \tag{203}$$

$$\frac{S^i_{K_3}}{S^i_{K_1}} = \left(\frac{G_2G_3}{G_1G_2}\right)_{s=-q_i} = \left(\frac{G_3}{G_1}\right)_{s=-q_i} \tag{204}$$

By plotting the poles and zeros of G_1 and G_2 on an s-plane plot, $(G_1G_2/K_1K_2)_{s=-q_i}$ can be measured with a Spirule or similar device. Multiplying by the gains gives

$$\frac{S^i_{K_2}}{S^i_{K_1}} = 0.160 \measuredangle 53.3 \text{ deg} \tag{205}$$

Therefore,

$$\begin{aligned} S^i_{K_2} &= (0.521)(0.160) \measuredangle (250.4 + 53.3) \text{ deg} \\ &= 0.0835 \measuredangle 303.7 \text{ deg} = 0.0464 - j0.0694 \end{aligned} \tag{206}$$

Similarly, by plotting the poles and zeros of G_3 and $1/G_1$,

$$\frac{S^i_{K_3}}{S^i_{K_1}} = \left(\frac{G_3}{G_1}\right)_{s=-q_i} = 0.965 \measuredangle 172.4 \text{ deg}$$

$$S^i_{K_3} = 0.504 \measuredangle 62.8 \text{ deg} = 0.230 + j0.446 \tag{207}$$

From Eq 203 and 204 it can be seen that $S^i_{K_1} + S^i_{K_3} - S^i_{K_2}$ should be zero. Using the computed values gives

$$S^i_{K_1} + S^i_{K_3} - S^i_{K_2} = 0.009 + j0.023$$

Considering the graphical techniques used, the agreement is quite good.

Now the pole and zero sensitivities can be computed from the simple sensitivity ratios of Eq 184 and 185. The results are summarized in Table III. It was noted earlier in this section that the sum of all the pole and zero sensitivities of each closed-loop pole should equal one. The actual sum of the sensitivities of Table III is 1.009 + j0.029. This agreement is excellent considering the inaccuracy of the closed-loop pole, the graphical techniques used in the calculations, and the fact that the numerical computation was all done by slide rule.

TABLE III

ZERO AND POLE SENSITIVITIES FOR MULTILOOP EXAMPLE

TRANSFER FUNCTION	p (z)	SENSITIVITY
1	0	0.568 ∠ -37.9 deg = 0.448 - j0.348
	15	0.0353 ∠ 67.0 = 0.014 + j0.033
	0.0045 + j0.0628	0.535 ∠ -36.4 = 0.431 - j0.317
	0.0045 - j0.0628	0.609 ∠ -38.8 = 0.475 - j0.382
	2.10 + j3.71	0.1058 ∠ 2.0 = 0.106 + j0.004
	2.10 - j3.71	0.1547 ∠ 127.9 = -0.095 + j0.122
	(0.0064)	0.571 ∠ 142.5 = -0.452 + j0.347
	(19.2)	0.0275 ∠ -112.2 = -0.010 - j0.025
	(-19.2)	0.0267 ∠ 73.0 = 0.008 + j0.026
2	35.0 + j35.7	0.00165 ∠ 77.2 = 0.0004 + j0.0016
	35.0 - j35.7	0.00170 ∠ 168.8 = -0.0017 + j0.0003
3	0.0045 + j0.0628	0.516 ∠ 136.0 = -0.371 + j0.358
	0.0045 - j0.0628	0.590 ∠ 133.6 = -0.406 + j0.426
	2.10 + j3.71	0.1023 ∠ 174.4 = -0.102 + j0.010
	2.10 - j3.71	0.1492 ∠ 300.3 = 0.075 - j0.129
	(2.4)	0.221 ∠ 40.4 = 0.168 + j0.143
	(0.0098)	0.552 ∠ 315.1 = 0.391 - j0.389
	(1.372)	0.361 ∠ 24.1 = 0.330 + j0.148

Note that the phugoid (ω_p) and short-period (ω_{sp}) poles appear in both G_1 and G_3. Consequently, the total sensitivities of these poles is the sum of their sensitivities when considered as poles of G_1 and of G_3. It is also desirable to convert the sensitivities for the complex-pair poles to damping ratio and frequency sensitivities per Eq 78. The final sensitivities of all the variable parameters are summarized in Table IV. The gain sensitivities are separated from the pole and zero sensitivities as a reminder that gain sensitivities are based upon fractional variations rather than absolute variations.

TABLE IV

MULTILOOP EXAMPLE SUMMARY

$q = 0.287 - j0.869$

GAIN SENSITIVITIES		
Parameter	Description	Sensitivity
K_a	Gain of altitude feedback	-0.175 - j0.492
K_e	Gain of pitch-angle feedback	0.230 + j0.446
K_m	Gain of elevator servo	0.046 - j0.069
A_h	Gain of H_δ	-0.175 - j0.492
A_θ	Gain of θ_δ	0.230 + j0.446

POLE AND ZERO SENSITIVITIES			
Parameter	Description	Nominal Value	Sensitivity
ω_a	Altimeter lag	15.0	0.014 + j0.033
ω_L	Lead in pitch-angle feedback	2.4	0.168 + j0.143
ζ_m	Damping ratio of elevator servo	0.7	-0.0019 - j0.0024
ω_m	Frequency of elevator servo	50.0	-0.0018 + j0.0028
$1/T_{h_1}$	First zero of H_δ	0.0064	-0.452 + j0.347
$1/T_{h_2}$	Second zero of H_δ	19.2	-0.010 - j0.025
$1/T_{h_3}$	Third (unstable) zero of H_δ	19.2	-0.008 - j0.026
$1/T_{\theta_1}$	First zero of θ_δ	0.0098	0.391 - j0.389
$1/T_{\theta_2}$	Second zero of θ_δ	1.372	0.330 + j0.148
ζ_p	Damping ratio of phugoid mode	0.0714	0.0084 + j0.0053
ω_p	Frequency of phugoid mode	0.063	0.0121 - j0.0045
ζ_{sp}	Damping ratio of short-period mode	0.493	-0.018 - j0.028
ω_{sp}	Frequency of short-period mode	4.27	-0.025 + j0.025

An examination of Table IV reveals a number of interesting bits of information:

1. The gain sensitivity of κ_a and A_h is nearly equal to the negative of that of κ_e and A_θ

2. The gain sensitivity of κ_m is much less than that of the other gains

3. On the basis of the same percentage change in parameters, the dominant mode is most sensitive to ω_a, ω_L, $1/T_{\theta_2}$, $1/T_{h_2}$, and $1/T_{h_3}$

We now know which parameters are the most important to the dominant mode of the system, and can estimate the variations in the dominant roots which could be achieved by changing the gains. By combining the above results with "approximate transfer functions" or an equivalent technique, which can relate the zeros and poles of the aircraft transfer function to the stability derivatives, we can estimate the changes in the dominant roots due to variations in the flight conditions and due to uncertainties in the inertial and aerodynamic characteristics of the vehicle.

SECTION V

SUMMARY

A good many derivations and relationships are scattered throughout this report. To facilitate the use of the material contained herein, the important results have been summarized in a number of tables which are contained in this section.

Table V lists the relationships between the modal response coefficients and the system responses for various inputs. Part A contains the relationships for $N = 1$, and Part B contains the general relationships.

Table VI lists a number of useful identities. Part A lists the identities for $N = 1$, and Part B lists the general identities.

Table VII contains the sensitivity ratios. Part A is the ratios for zeros or poles which are in root-locus form. Part B is the ratios for zeros or poles in Bode form.

Table VIII summarizes all the methods for computing gain sensitivities which were developed in Section III. The table lists the pertinent equations along with some of the advantages and disadvantages of each method.

TABLE V-A

RESPONSE RELATIONSHIPS

$N = 1$

Response to unit step input

$$\sum_i \frac{Q_i}{q_i}\left(1 - e^{-q_i t}\right) \quad , \quad q_i \neq 0$$

Response to power series input, $\sum_M \frac{d_M t^M}{M!}$

$$\sum_M d_M \sum_i \frac{Q_i}{(-q_i)^{M+1}}\left[e^{-q_i t} - \sum_{j=0}^{M} \frac{(-q_i t)^j}{j!}\right] \quad , \quad q_i \neq 0$$

Response to periodic input,* $\sum_k (a_k \cos \omega_k t + b_k \sin \omega_k t)$

$$\sum_k \sum_i \frac{Q_i}{q_i^2 + \omega_k^2}\left[(b_k\omega_k - a_k q_i)e^{-q_i t} + (a_k q_i - b_k\omega_k) \cos \omega_k t + (a_k\omega_k + b_k q_i) \sin \omega_k t\right] \quad , \quad q_i \neq \pm j\omega_k$$

*An alternate response expression, which may be useful when q_i is real, is given by Eq 35.

TABLE V-B

RESPONSE RELATIONSHIPS

GENERAL

Response of ith mode to unit step input (N = 2)

$$\frac{Q_{i_1}}{q_i}\left(1 - e^{-q_i t}\right) + \frac{Q_{i_2}}{q_i^2}\left[1 - e^{-q_i t}(1+q_i t)\right] \quad , \quad q_i \neq 0$$

Response of ith mode to power series input, $\sum_M \frac{d_M \, t^M}{M!}$

$$\sum_M (-1)^M d_M \sum_{k=1}^{N} \frac{Q_{i_k}}{(q_i)^{M+k}} \left[\frac{1}{(k-1)!} \sum_{j=0}^{M} \frac{(M+k-j-1)! \, (-q_i t)^j}{(M-j)! \, j!} - \frac{e^{-q_i t}}{M!} \sum_{j=0}^{k-1} \frac{(M+k-j-1)! \, (q_i t)^j}{(k-j-1)! \, j!}\right] \quad , \quad q_i \neq 0$$

Response of ith mode to periodic input, cos ωt

$$\sum_{k=1}^{N} Q_{i_k} \, \mathrm{Re}^* \left[\frac{e^{j\omega t} - e^{-q_i t} \sum_{h=0}^{k-1} \frac{(j\omega+q_i)^h \, t^h}{h!}}{(j\omega+q_i)^k}\right] \quad , \quad q_i \neq \pm j\omega$$

Response of ith mode to periodic input, sin ωt

$$\sum_{k=1}^{N} Q_{i_k} \, \mathrm{Im}^* \left[\frac{e^{j\omega t} - e^{-q_i t} \sum_{h=0}^{k-1} \frac{(j\omega+q_i)^h \, t^h}{h!}}{(j\omega+q_i)^k}\right] \quad , \quad q_i \neq \pm j\omega$$

*In evaluating the real and imaginary parts of the indicated quantity, q_i must be treated as a purely real quantity.

TABLE VI-A

IDENTITIES

$N = 1$

$$S^i_{\kappa} = S^i_K$$

Single Loop with Unit Feedback

$$Q_i = S^i_{\kappa}$$

$$\sum_{i=1}^{m+n} Q_i = \begin{cases} \kappa, & m = 1 \\ 0, & m \geq 2 \end{cases}$$

$$\sum_{i=1}^{m+n} Q_i q_i = \begin{cases} \kappa(b_1 - a_1 + \kappa), & m = 1 \\ -\kappa, & m = 2 \\ 0, & m \geq 3 \end{cases}$$

$$\sum_{i=1}^{m+n} \frac{Q_i}{q_i} = \left\{ \begin{matrix} 1, & k \geq 1 \\ \dfrac{K}{1+K}, & k = 0 \\ 0, & k \leq -1 \end{matrix} \right\} m \geq 1$$

$$E_0 = 1 - \sum_{i=1}^{m+n} \frac{Q_i}{q_i}$$

$$E_j = (-1)^{j+1} \sum_{i=1}^{m+n} \frac{Q_i}{q_i^{\,j+1}}, \quad j \neq 0$$

$$\sum_{j=1}^{n} S^i_{z_j} + \sum_{j=1}^{m+n} S^i_{p_j} = 1, \quad \text{all zeros and poles in root-locus form}$$

Multiloop

$$Q_i = \left[\frac{F_1}{G_k(\partial F_2/\partial G_k)}\right]_{s=-q_i} S^i_{\kappa_k}$$

$$\sum_i Q_i = \begin{cases} \kappa_{CL}, & \text{number of closed-loop poles} = 1 + \text{number of closed-loop zeros} \\ 0, & \text{number of closed-loop poles} \geq 2 + \text{number of closed-loop zeros} \end{cases}$$

$$\sum_{k=1}^{L} \left(\sum_{j=1}^{n_k} S^i_{z_{k_j}} + \sum_{j=1}^{m_k+n_k} S^i_{p_{k_j}} \right) = 1, \quad \text{all zeros and poles in root-locus form}$$

TABLE VI-B

IDENTITIES

GENERAL

$$S^i_{\kappa} = S^i_K$$

$$Q_{i_1} = Q_{i_2}\left[\sum_{j=1}^{n}\left(\frac{1}{z_j - q_i}\right) - \sum_{\substack{j=1\\ j\neq i}}^{m+n}\left(\frac{1}{q_j - q_i}\right)\right] \quad , \quad N = 2$$

$$Q_{i_k} = \frac{1}{N - k}\sum_{h=0}^{N-k-1}(-1)^h\, Q_{i_{k+1+h}}\left[\sum_{j=1}^{n}\left(\frac{1}{z_j - q_i}\right)^{h+1} - \sum_{\substack{j=1\\ j\neq i}}^{m+n}\left(\frac{1}{q_j - q_i}\right)^{h+1}\right]$$

Single Loop with Unit Feedback

$$Q_{i_N} = (-1)^{N-1}\, S^i_{\kappa}$$

$$\sum_i Q_{i_1} = \begin{cases} \kappa \ , & m = 1 \\ 0 \ , & m \geq 2 \end{cases}$$

Multiloop

$$Q_{i_N} = (-1)^{N-1}\left[\frac{F_1}{G_k(\partial F_2/\partial G_k)}\right]_{s=-q_i} S^i_{\kappa_k}$$

$$\sum_i Q_{i_1} = \begin{cases} \kappa_{CL} \ , & \text{number of closed-loop poles = 1 + number of closed-loop zeros} \\ 0 \ , & \text{number of closed-loop poles} \geq \text{2 + number of closed-loop zeros} \end{cases}$$

TABLE VII-A

SENSITIVITY RATIOS

ZEROS OR POLES IN ROOT-LOCUS FORM

$$S^{i}_{z_{k_j}} = \frac{S^{i}_{K_k}}{z_{k_j} - q_i}$$

$$S^{i}_{p_{k_j}} = \frac{S^{i}_{K_k}}{q_i - p_{k_j}}$$

$$S^{i}_{\omega} = \frac{{}^{*}\pm 2(\omega - \zeta q_i) S^{i}_{K}}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

$$S^{i}_{\zeta} = \frac{{}^{*}\mp 2\omega q_i S^{i}_{K}}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

$$S^{i}_{a} = \frac{{}^{*}\pm 2(a - q_i) S^{i}_{K}}{q_i^2 - 2a q_i + a^2 + b^2}$$

$$S^{i}_{b} = \frac{{}^{*}\pm 2b S^{i}_{K}}{q_i^2 - 2a q_i + a^2 + b^2}$$

*Use upper sign for zeros, lower sign for poles.

TABLE VII-B

SENSITIVITY RATIOS

ZEROS OR POLES IN BODE FORM

$$S^i_{z_{k_j}} = \frac{q_i S^i_{K_k}}{z_{k_j}(z_{k_j} - q_i)}$$

$$S^i_{p_{k_j}} = \frac{q_i S^i_{K_k}}{p_{k_j}(q_i - p_{k_j})}$$

$$S^i_{\omega} = \frac{{}^{*}\pm 2q_i}{\omega} \; \frac{(\zeta\omega - q_i)S^i_K}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

$$S^i_{\zeta} = \frac{{}^{*}\mp 2q_i \omega S^i_K}{q_i^2 - 2\zeta\omega q_i + \omega^2}$$

$$S^i_a = \left[\frac{{}^{*}\pm 2q_i}{a^2 + b^2}\right]\left[\frac{(a^2 - b^2 - aq_i)S^i_K}{q_i - 2aq_i + a^2 + b^2}\right]$$

$$S^i_b = \left[\frac{{}^{*}\pm 2q_i b}{a^2 + b^2}\right]\left[\frac{(2a - q_i)S^i_K}{q_i^2 - 2aq_i + a^2 + b^2}\right]$$

*Use upper sign for zeros, lower sign for poles.

METHODS OF COMPUTING

CLASS	METHOD	EQUATIONS: N = 1	EQUATIONS: [illegible]
Direct calculation	Numerator and denominator derivatives	$S_\kappa^i = \left[\frac{\kappa\alpha}{\kappa\frac{\partial\alpha}{\partial s}+\frac{\partial\beta}{\partial s}}\right]_{s=-q_i}$ $= -\left[\frac{\beta}{\kappa\frac{\partial\alpha}{\partial s}+\frac{\partial\beta}{\partial s}}\right]_{s=-q_i}$	$S_\kappa^i = (-1)^{N-1}N!\left[\frac{\kappa\alpha}{\kappa\frac{\partial^N\alpha}{\partial s^N} \dots}\right.$ [illegible] $= (-1)^N N!\left[\frac{\beta}{\kappa\frac{\partial^N\alpha}{\partial s^N} + \dots}\right.$ [illegible]
	Summation of terms	$S_\kappa^i = \left[\sum_{j=1}^{n}\left(\frac{1}{z_j-q_i}\right) - \sum_{j=1}^{m+n}\left(\frac{1}{p_j-q_i}\right)\right]^{-1}$	$S_\kappa^i = N\left[\sum_{j=1}^{n}\left(\frac{1}{z_j-q_i}\right)^N - \dots\right.$ [illegible]
Root locus	Gain perturbation	$S_\kappa^i \doteq \frac{\kappa\Delta q_i}{\Delta\kappa}$	$S_\kappa^i \doteq \frac{\kappa(\Delta q_i)^N}{\Delta\kappa}$
	Phase perturbation	$S_\kappa^i \doteq \frac{j\Delta q_i}{\Delta\theta}$	$S_\kappa^i \doteq \frac{j(\Delta q_i)^N}{\Delta\theta}$
	Vector	$S_\kappa^i = \frac{\kappa\prod_{j=1}^{n}(z_j-q_i)}{(1+\kappa\delta_m^0)\prod_{j=1,\,j\neq i}^{m+n}(q_j-q_i)}$	$S_\kappa^i = \frac{(-1)^{N-1}\kappa\prod_{j=1}^{n}\dots}{(1+\kappa\delta_m^0)\prod_{j=1,\,j\neq i}^{m+n}\dots}$ [illegible]
Open-loop Bode and ξ plots	Successive slopes	$S_\kappa^i = \frac{-q_i}{\left[\frac{A_1}{20}+\frac{j\varphi_1}{131.93}\right]_{s=-q_i}}$	$S_\kappa^i = \frac{-N!(2.3026)^{N-1}(\dots}{\left[\frac{A_N}{20}+\frac{j\varphi_N}{131.93}\right]_{s=\dots}}$ [illegible]
	Radius of curvature	Not applicable	$\frac{d^2y}{dx^2} = \frac{1}{R}\left(\frac{\text{vertical sc}\dots}{\text{horizontal}\dots}\right)$ [illegible]
	Shifted Bode	$\omega_s = \sqrt{\omega^2 - 2\zeta\omega d + d^2}$ $\zeta_s = \frac{\zeta\omega - d}{\omega_s}$	$\omega_s = \sqrt{\omega^2 - 2\zeta\omega d \dots}$ [illegible] $\zeta_s = \frac{\zeta\omega - d}{\omega_s}$
	Gain perturbation	$S_\kappa^i \doteq \frac{K\Delta q_i}{\Delta K}$	$S_\kappa^i \doteq \frac{K(\Delta q_i)^N}{\Delta K}$
Closed-loop Bode	Amplitude asymptotes	$S_\kappa^i \doteq \begin{cases} (-1)^{\lambda_z+\lambda_q+\lambda_i} q_i A_{asy}, & q_i \text{ real and } >0 \\ (-1)^{1+\lambda_u} q_i A_{asy}, & q_i \text{ real and } <0 \\ (-1)^{1+\lambda_u}\left(\frac{-q_i}{\lvert q_i\rvert}\right)^{\lambda_z-\lambda_q}\frac{\lvert q_i\rvert^2 A_{asy}}{2j\,\mathrm{Im}(q_i)}, & q_i \text{ complex}\end{cases}$	$S_\kappa^i \doteq \begin{cases} (-1)^{N-1+\lambda_z+\dots} \\ (-1)^{1+\lambda_u}(q_i \dots \\ (-1)^{1+\lambda_u}\left(\frac{-q_i}{\lvert q_i\rvert}\right)\dots\end{cases}$ [illegible]

TABLE VIII

METHODS OF COMPUTING GAIN SENSITIVITIES

EQUATIONS		ACCURACY		
		EXACT		APPR…
N = 1	GENERAL	Numerical	Graphical	
$= \left[\frac{\kappa\alpha}{\kappa\frac{\partial\alpha}{\partial s}+\frac{\partial\beta}{\partial s}}\right]_{s=-q_i}$ $= -\left[\frac{\beta}{\kappa\frac{\partial\alpha}{\partial s}+\frac{\partial\beta}{\partial s}}\right]_{s=-q_i}$	$S_\kappa^i = (-1)^{N-1}N!\left[\frac{\kappa\alpha}{\kappa\frac{\partial^N\alpha}{\partial s^N}+\frac{\partial^N\beta}{\partial s^N}}\right]_{s=-q_i}$ $= (-1)^N N!\left[\frac{\beta}{\kappa\frac{\partial^N\alpha}{\partial s^N}+\frac{\partial^N\beta}{\partial s^N}}\right]_{s=-q_i}$	✓		
$= \left[\sum_{j=1}^{n}\left(\frac{1}{z_j-q_i}\right) - \sum_{j=1}^{m+n}\left(\frac{1}{p_j-q_i}\right)\right]^{-1}$	$S_\kappa^i = N\left[\sum_{j=1}^{n}\left(\frac{1}{z_j-q_i}\right)^N - \sum_{j=1}^{m+n}\left(\frac{1}{p_j-q_i}\right)^N\right]^{-1}$	✓		
$\doteq \frac{\kappa\Delta q_i}{\Delta\kappa}$	$S_\kappa^i \doteq \frac{\kappa(\Delta q_i)^N}{\Delta\kappa}$			✓
$\doteq \frac{j\Delta q_i}{\Delta\theta}$	$S_\kappa^i \doteq \frac{j(\Delta q_i)^N}{\Delta\theta}$			✓
$= \frac{\kappa\prod_{j=1}^{n}(z_j-q_i)}{(1+\kappa\delta_m^0)\prod_{j=1, j\neq i}^{m+n}(q_j-q_i)}$	$S_\kappa^i = \frac{(-1)^{N-1}\kappa\prod_{j=1}^{n}(z_j-q_i)}{(1+\kappa\delta_m^0)\prod_{j=1, j\neq i}^{m+n}(q_j-q_i)}$		✓	
$= \frac{-q_i}{\left[\frac{A_i}{20}+\frac{j\varphi_i}{131.93}\right]_{s=-q_i}}$	$S_\kappa^i = \frac{-N!(2.3026)^{N-1}(q_i)^N}{\left[\frac{A_N}{20}+\frac{j\varphi_N}{131.93}\right]_{s=-q_i}}$		✓	
applicable	$\frac{d^2y}{dx^2} = \frac{1}{R}\left(\frac{\text{vertical scale}}{\text{horizontal scale}}\right)^2$		✓	
$= \sqrt{\omega^2 - 2\zeta\omega d + d^2}$ $= \frac{\zeta\omega - d}{\omega_s}$	$\omega_s = \sqrt{\omega^2 - 2\zeta\omega d + d^2}$ $\zeta_s = \frac{\zeta\omega - d}{\omega_s}$		✓	
$\doteq \frac{K\Delta q_i}{\Delta K}$	$S_\kappa^i \doteq \frac{K(\Delta q_i)^N}{\Delta K}$			✓
$\doteq \begin{cases} (-1)^{\lambda_z+\lambda_q+\lambda_u} q_i A_{asy}, & q_i \text{ real and } >0 \\ (-1)^{1+\lambda_u} q_i A_{asy}, & q_i \text{ real and } <0 \\ (-1)^{1+\lambda_u}\left(\frac{-q_i}{\lvert q_i\rvert}\right)^{\lambda_z-\lambda_q}\frac{\lvert q_i\rvert^2 A_{asy}}{2j\,\mathrm{Im}(q_i)}, & q_i \text{ complex} \end{cases}$	$S_\kappa^i \doteq \begin{cases} (-1)^{N-1+\lambda_z+\lambda_q+\lambda_u}(q_i)^N A_{asy}, & q_i \text{ real and } >0 \\ (-1)^{1+\lambda_u}(q_i)^N A_{asy}, & q_i \text{ real and } <0 \\ (-1)^{1+\lambda_u}\left(\frac{-q_i}{\lvert q_i\rvert}\right)^{\lambda_z-\lambda_q}\frac{\lvert q_i\rvert^{2N} A_{asy}}{[2j\,\mathrm{Im}(q_i)]^N}, & q_i \text{ complex} \end{cases}$			✓

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VIII

GAIN SENSITIVITIES

	ACCURACY			REMARKS
	EXACT		APPROXIMATE	
GENERAL	Numerical	Graphical		
$\left[\ldots\right]_{s=-q_i}$ $\left[\ldots\right]_{s=-q_i}$	✓			1. Simplest numerical method for low-order systems. 2. Requires open-loop numerator and denominator polynomials.
$\left[\sum_{1}^{n}\left(\frac{1}{p_j - q_i}\right)^{N}\right]^{-1}$	✓			1. Simplest numerical method for high-order systems.
			✓	
			✓	1. Simplest root-locus method. $\Delta\theta$ easily obtained with Spirule.
$\frac{\ldots q_i)}{\ldots q_i)}$		✓		1. Ordinarily the most accurate graphical technique. 2. Requires complete set of compatible closed-loop poles.
		✓		1. Need only the portion of the Bode or ξ plots about $\lvert G\rvert = 1$. 2. Accurate measurements of slopes or radii of curvature are frequently difficult to obtain.
$)^2$		✓		
		✓		
			✓	1. Method is simple and speedy when decomposition technique is applicable. 2. Sensitivities for all closed-loop poles can be computed from one gain change when decomposition technique is applicable.
$\ldots)^N A_{asy}$, q_i real and >0 , q_i real and <0 $\frac{\lvert q_i\rvert^{2N} A_{asy}}{[2j\ \mathrm{Im}\ (q_i)]^N}$, q_i complex			✓	1. Method is ordinarily simple and provides quick overview when closed-loop Bode asymptotic plot is available. 2. The closed-loop pole must be widely separated in frequency from all zeros (except dipole zero, $-z_h$) and from all other closed-loop poles except its complex conjugate; to correct for dipole, multiply sensitivity by $\left(1 - \frac{z_h}{q_i}\right)$ if $\lvert z_h\rvert < \lvert q_i\rvert$ or $\left(1 - \frac{q_i}{z_h}\right)$ if $\lvert q_i\rvert < \lvert z_h\rvert$

(3)

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APPENDIX

SINGLE-LOOP EXAMPLE

The use of all the methods for computing gain sensitivities and modal response coefficients will be illustrated for a simple example. The sample system has an open-loop transfer function given by

$$G = \frac{K}{s(s+1)(s+5)} \tag{A-1}$$

The computations will be given for two different gains. The high-gain case illustrates the procedure for complex closed-loop poles, and the low-gain case is an example of the computations for a second-order closed-loop pole.

A. HIGH GAIN

For this case the gain is set so that a pair of complex closed-loop poles with damping ratio of $\sqrt{2}/2$ exists. For this situation the important parameters

$$K = 31\sqrt{26} - 156 = 2.070$$

$$q_1 = \sqrt{26} = 5.099$$

$$q_2 = \frac{1}{2}(6 - \sqrt{26})(1 - j) = 0.450(1 - j)$$

$$q_3 = \frac{1}{2}(6 - \sqrt{26})(1 + j) = 0.450(1 + j)$$

1. Direct Calculation

Using Eq 127 for the numerator and denominator derivatives method,

$$Q_i = S_K^i = \left(\frac{K\alpha}{K \frac{\partial \alpha}{\partial s} + \frac{\partial \beta}{\partial s}} \right)_{s=-q_i} \tag{A-2}$$

From Eq A-1,

$$\alpha = 1$$

$$\frac{\partial \alpha}{\partial s} = 0$$

$$\beta = s^3 + 6s^2 + 5s$$

$$\frac{\partial \beta}{\partial s} = 3s^2 + 12s + 5$$

Therefore,

$$Q_i = S_K^i = \frac{K}{3q_i^2 - 12q_i + 5} \tag{A-3}$$

Using the value for q_2 gives

$$Q_2 = S_K^2 = -\frac{-3276 + 701\sqrt{26} + j(8112 - 987\sqrt{26})}{6290}$$

$$= -0.0474 - j0.490$$

$$= 0.492 \measuredangle 264.47 \text{ deg}$$

where the abbreviated notation

$$A \measuredangle \varphi \equiv Ae^{j\varphi}$$

has been adopted.

Because q_3 is the complex conjugate of q_2, Q_3 and S_K^3 are the complex conjugates of Q_2 and S_K^2 or

$$Q_3 = S_K^3 = -\frac{-3276 + 701\sqrt{26} - j(8112 - 987\sqrt{26})}{6290}$$

$$= -0.0474 + j0.490$$

$$= 0.492 \measuredangle 95.53 \text{ deg}$$

Using the value of q_1 in Eq A-3,

$$S_K^1 = \frac{-3276 + 701\sqrt{26}}{3145} = 0.0949$$

As a check, recall that in Section II (Eq 19) it was shown that i' there are no multiple-order closed-loop poles and the number of system poles is greater than the number of zeros by two or more, then the sum of the modal response coefficients or gain sensitivities is zero. These conditions are met for this example, and we can see that the values do sum to zero.

The second method of direct calculation, called the summation of terms method, uses Eq 134, i.e.,

$$S_K^i = \frac{1}{\sum_{k=1}^{n} \frac{1}{z_k - q_i} - \sum_{k=1}^{m+n} \frac{1}{p_k - q_i}} \qquad \text{(A-4)}$$

These computations may be performed in the following manner:

p_k	$(p_k - q_2)$	$\frac{1}{p_k - q_2}$
0	-0.45049 + j0.45049	-1.1099 - j1.1099
1	0.54951 + j0.45049	1.0883 - j0.8922
5	4.54951 + j0.45049	0.2177 - j0.0216

$$\sum\left(\frac{1}{p_k - q_2}\right) = 0.1961 - j2.0237$$

$$S_K^2 = -0.0474 - j0.490 = 0.492 \angle 264.47 \text{ deg}$$

$$S_K^3 = -0.0474 + j0.490 = 0.492 \angle 95.53 \text{ deg}$$

and

p_k	$(p_k - q_1)$	$\frac{1}{p_k - q_1}$
0	-5.09902	-0.1961
1	-4.09902	-0.2440
5	-0.09902	-10.0990

$$\sum\left(\frac{1}{p_k - q_1}\right) = -10.5391$$

$$S_K^1 = 0.0949$$

Naturally the results agree with the first method because both methods are exact.

2. Root-Locus Methods

The root locus of the example system was shown in Fig. 7. To estimate the gain sensitivity from Eq 138, a position on the locus of -0.41 + j0.76 was chosen. The measured gain at that point was 3.89. This gives an estimate of

$$S_K^2 \doteq \frac{K\Delta q_2}{\Delta K} = 2.07 \frac{[0.41 - j0.76 + (-0.45 + j0.45)]}{3.89 - 2.07}$$

$$= -0.045 - j0.353 = 0.356 \angle 262.7 \text{ deg}$$

which has an amplitude error of 28 percent and an angular error of 2 deg.

For q_1 the point -5.42 was selected, and the gain was 10.06. Then

$$S_K^1 \doteq \frac{2.07(5.42 - 5.10)}{10.06 - 2.07} = 0.083$$

Despite the fact that the gain was increased to nearly 5 times its original value, the estimate is within 13 percent of the exact value.

To obtain estimates from Eq 139, perturbations normal to the locus were considered. For q_2 the point -0.64 + j0.47 was selected, and the measured phase change was -24 deg. Then

$$S_K^2 \doteq \frac{j\Delta q_2}{\Delta\theta} = \frac{j[0.64 - j0.47 + (-0.45 + j0.45)]}{-24/57.3}$$

$$= -0.0477 - j0.454 = 0.457 \angle 264 \text{ deg}$$

This estimate is fairly good; the amplitude error is 7 percent and the angular error is less than 1 deg.

For q_1 the point -5.10 + j0.10 was chosen. The phase change was 47.5 deg, so

$$S_K^1 \doteq - \frac{j(j0.10)(57.3)}{47.5} = 0.121$$

This estimate is in error by 27 percent.

The third means of determining sensitivity from the root-locus plot is by the vector method, Eq 140. For the example, this equation reduces to

$$S_K^1 = \frac{K}{(q_2 - q_1)(q_3 - q_1)}$$

$$S_K^2 = \frac{K}{(q_1 - q_2)(q_3 - q_2)}$$

The following were measured with a Spirule:

$$\frac{1}{(q_2 - q_1)(q_3 - q_1)} = 0.0459$$

$$\frac{1}{(q_1 - q_2)(q_3 - q_2)} = 0.237 \measuredangle -95.5 \text{ deg}$$

which gives

$$S_K^1 \doteq 0.0952$$

$$S_K^2 \doteq 0.490 \measuredangle 264.5 \text{ deg}$$

These values are extremely close to the exact values. Both amplitude errors are less than 1 percent, and the angular error of S_K^2 is less than 1 deg.

3. Methods Using Open-Loop Bode and ξ Plots

A ξ plot of the example for $\xi = -0.7$ is shown in Fig. A-1. A small error has been introduced into the solution from the use of the $\xi = -0.7$ templates because the actual value of ξ at q_2 is $-1/\sqrt{2} = -0.707$. The slopes which were measured from this figure are

$$A_1 = -17 \text{ db/dec}$$

$$\varphi_1 = -129 \text{ deg/dec}$$

Then from Eq 149

$$S_K^2 \doteq \frac{[-0.450 + j0.450]}{-\frac{17}{20} - j\frac{129}{132}}$$

$$\doteq -[0.0341 + j0.490] = 0.491 \angle 266 \text{ deg}$$

This is within 1 percent of the correct amplitude and 2 deg of the correct angle. The extremely small error in the amplitude must be considered fortuitous as the errors in measuring the slopes will normally be higher than this.

For q_1 we must use a $\xi = -1$ or Siggy plot, see Fig. A-1. The measured slope from this plot is -830 db/dec, but the root is too close to the open-loop pole to get an accurate value. A better estimate can be obtained by noting that the two low frequency roots contribute about -40 db/dec, and the contribution of the pole at -5 can be approximated by (Ref. 1, Table III-A)

$$-20\frac{5.10}{5.10 - 5} = -1020 \text{ db/dec}$$

Therefore, $$A_1 \doteq -1060 \text{ db/dec}$$

and $$S_K^1 = \frac{20(-5.10)}{-1060} = 0.0963$$

This is within 2 percent of the exact value.

Instead of using the ξ plot for S_K^2, we might use the shifted Bode plot. Shifting the imaginary axis 0.450 to the left so that q_2 and q_3 are on the axis puts the open-loop poles at 0.450, -0.550, and -4.550. The shifted diagram is shown in Fig. A-2. The measured slopes are

$$A_1 = -19.2 \text{ db/dec}$$

$$\varphi_1 = -11.75 \text{ deg/dec}$$

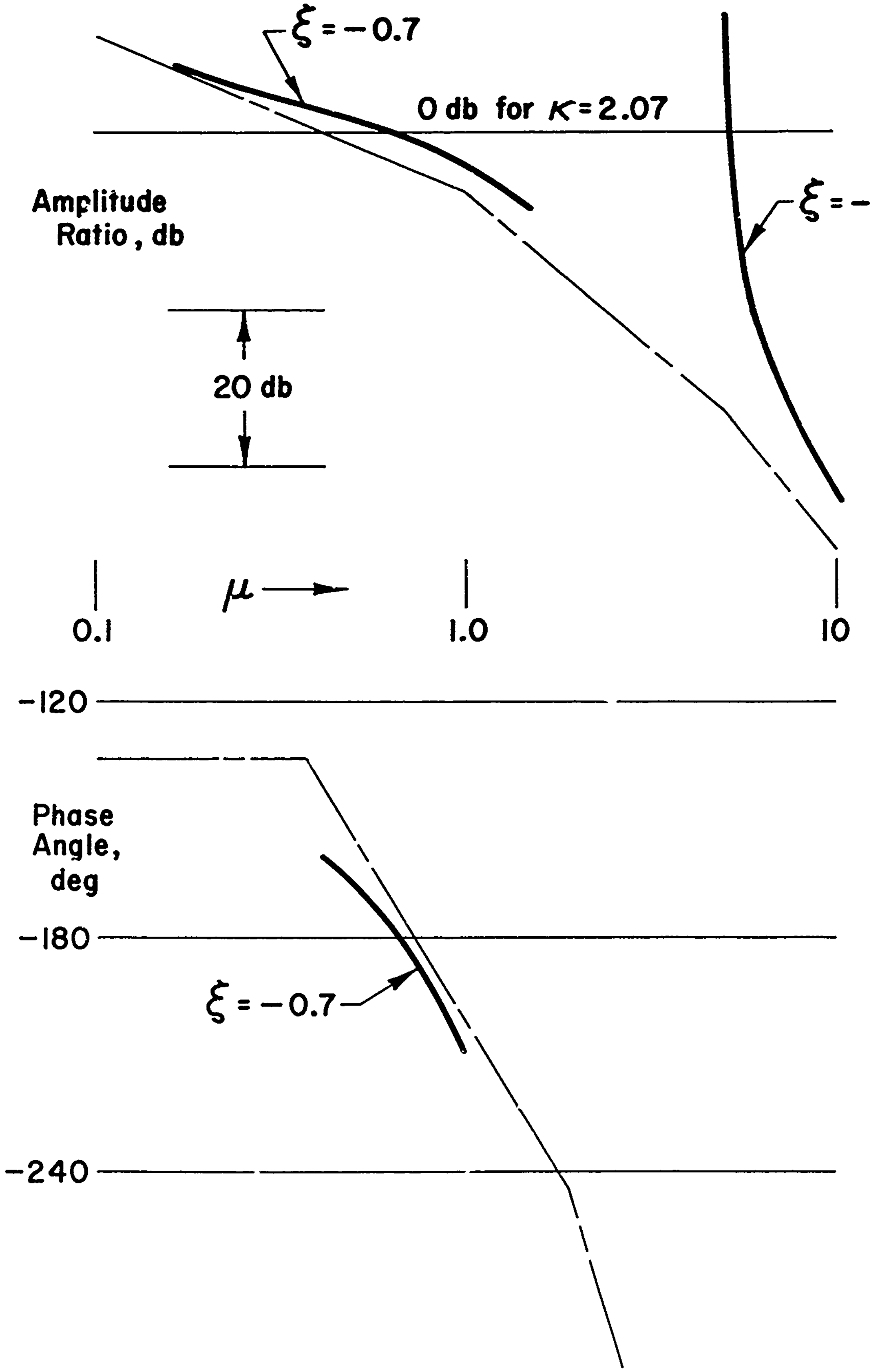

Figure A-1. High-Gain ξ Plot

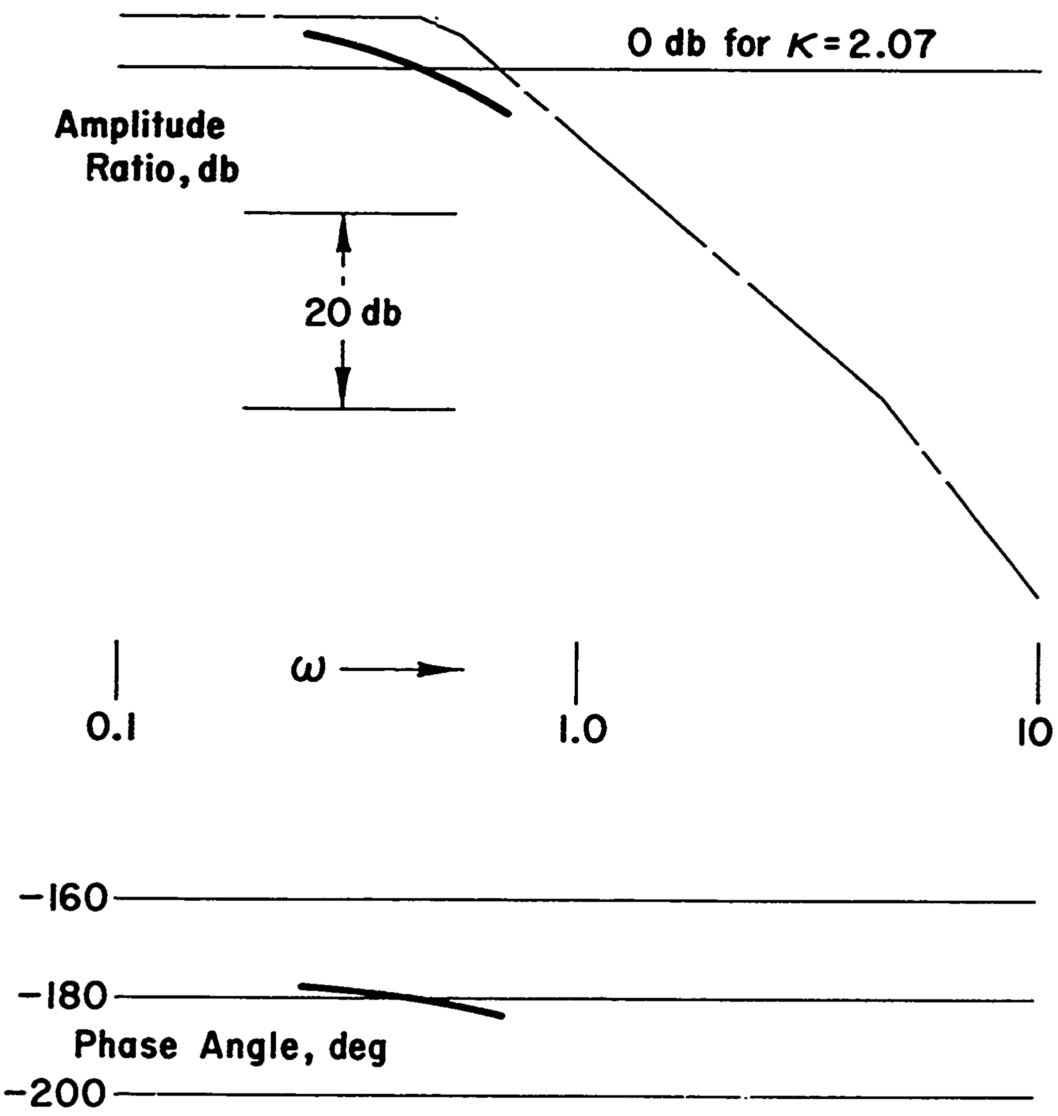

Figure A-2. High-Gain Shifted Bode Plot

Therefore,

$$S_K^2 \doteq \frac{j0.450}{-\frac{19.2}{20} - j\frac{11.75}{132}} = 0.467 \angle 264.7 \text{ deg}$$

This is an amplitude error of 5 percent and an angular error of less than 1 deg.

The Siggy of Fig. A-1 can also be used with the root decomposition technique to estimate Δq due to a gain change. For a 6-db increase in gain, $\Delta K/K = 1$, root decomposition gave

$$\Delta q_1 = 0.10$$

$$\Delta q_2 = -0.05 - j0.345$$

Therefore from Eq 154 $\quad S_K^1 \doteq 0.10$

which is in error by 5 percent, and

$$S_K^2 \doteq -0.05 - j0.345$$

$$\doteq 0.348 \angle 262 \text{ deg}$$

which is a magnitude error of 29 percent and an angle error of less than 3 deg.

4. Method Using the Closed-Loop Bode Asymptotes

The closed-loop Bode amplitude asymptote will be unity (0 db) for frequencies up to 0.637 and at that point it will break down at -40 db/dec. For q_2, then,

$$A_{asy} = 1$$

$$\lambda_z = \lambda_q = \lambda_u = 0$$

and from Eq 162

$$S_K^2 \doteq \frac{(0.637)^2}{2j(0.450)} = 0.451 \angle 270 \text{ deg}$$

This is an error of 8 percent in magnitude and 5.5 deg in direction.

For q_1 $\quad A_{asy} = \left(\frac{0.637}{5.10}\right)^2 = 0.0156$

$$\lambda_z = \lambda_u = 0 \ , \ \lambda_q = 2$$

From Eq 160 $\quad S^1_\kappa \doteq (5.10)(0.0156) = 0.0796$

This is an error of 16 percent.

B. LOW GAIN

We will now consider the case when the gain is set so that there is a second-order pole on the negative real axis. For this case the important parameters are:

$$\kappa = \frac{2}{9}\left(7\sqrt{21} - 27\right) = 1.128$$

$$q_1 = \frac{1}{3}\left(6 + 2\sqrt{21}\right) = 5.055$$

$$q_2 = \frac{1}{3}\left(6 - \sqrt{21}\right) = 0.472 \text{ (second-order pole)}$$

1. Direct Calculation

The numerator and denominator polynomials and their derivatives are

$$\alpha = 1$$

$$\frac{\partial \alpha}{\partial s} = \frac{\partial^2 \alpha}{\partial s^2} = 0$$

$$\beta = s^3 + 6s^2 + 5s$$

$$\frac{\partial \beta}{\partial s} = 3s^2 + 12s + 5$$

$$\frac{\partial^2 \beta}{\partial s^2} = 6s + 12$$

Then using Eq 130 for the numerator and denominator derivatives method,

$$S_K^2 = -2\left[\frac{K\alpha}{K(\partial^2\alpha/\partial s^2) + (\partial^2\beta/\partial s^2)}\right]_{s=-q_2} = \frac{-2K}{-6q_2 + 12}$$

$$= -\frac{2}{63}(49 - 9\sqrt{21}) = -0.246$$

$$S_K^1 = \left[\frac{K\alpha}{K(\partial\alpha/\partial s) + (\partial\beta/\partial s)}\right]_{s=-q_1} = \frac{K}{3q_1^2 - 12q_1 + 5}$$

$$= \frac{2}{189}(7\sqrt{21} - 27) = 0.0537$$

Note that for this case the evaluation of the gain sensitivities only determine two of the modal response coefficients,

$$Q_1 = S_K^1$$

$$Q_{2_2} = -S_K^2$$

To evaluate the third coefficient, Q_{2_1}, we must make additional calculations. For this case the easiest method is to use Eq 113, i.e.,

$$Q_1 + Q_{2_1} = 0$$

or

$$Q_{2_1} = -Q_1 = -S_K^1$$

The same result could be obtained from Eq 112, which for this case reduces to

$$Q_{2_1} = \frac{-Q_{2_2}}{q_1 - q_2} = \frac{-\frac{2}{63}(49 - 9\sqrt{21})}{\frac{1}{3}(6 + 2\sqrt{21}) - \frac{1}{3}(6 - \sqrt{21})}$$

$$= -\frac{2}{189}(7\sqrt{21} - 27) = -S_K^1$$

For the second method of direct calculation, called the summation of terms method, from Eq 137 we have

$$S_K^1 = \frac{-1}{\sum_{k=1}^{3}\left(\frac{1}{p_k - q_1}\right)}$$

$$S_K^2 = \frac{-2}{\sum_{k=1}^{3}\left(\frac{1}{p_k - q_2}\right)^2}$$

We can set up a computation procedure similar to that used for the high-gain case, i.e.,

p_k	$(p_k - q_2)$	$\left(\frac{1}{p_k - q_2}\right)^2$
0	-0.47247	4.4797
1	0.52753	3.5934
5	4.52753	0.0488

$$\sum\left(\frac{1}{p_k - q_2}\right)^2 = 8.1219$$

$$S_K^2 = -0.246$$

p_k	$(p_k - q_1)$	$\left(\frac{1}{p_k - q_1}\right)$
0	-5.05505	-0.198
1	-4.05505	-0.247
5	-0.05505	-18.165

$$\sum\left(\frac{1}{p_k - q_1}\right) = -18.610$$

$$S_K^1 = 0.0537$$

2. Root-Locus Methods

For the gain-perturbation method the gain can be increased to 2.070, for which the root was previously determined to be -0.450 + j0.450. Then from Eq 138

$$S_K^2 \doteq \frac{1.128[0.450 - j0.450 - 0.472]^2}{2.070 - 1.128}$$

$$\doteq -0.242 + j0.0237 = 0.243 \measuredangle 174.4 \text{ deg}$$

Even though the ratio of the gains is more than 1.8, the estimate is within 2 percent of the correct amplitude and 6 deg of the right direction.

Similarly for q_1,

$$S_K^1 \doteq \frac{1.128[5.099 - 5.055]}{2.070 - 1.128} = 0.0526$$

This is a 2-percent error.

Using the phase-perturbation method for q_2, we need a perturbation along a line midway between branches of the root locus. Selecting a perturbation of 0.25 at an angle of 45 deg gives a phase change of 14.6 deg. Therefore, from Eq 139

$$S_K^2 \doteq \frac{j57.3\left[-\frac{\sqrt{2}}{8}(1 + j)\right]^2}{45.5} = -0.245$$

This is an error of less than 1 percent.

For q_1 a perturbation of j0.1 was used. This gives a phase change of 63.7 deg. Then,

$$S_K^1 \doteq \frac{j57.3[-j0.1]}{63.7} = 0.090$$

This estimate has a large error, 68 percent, but this should not be surprising because the perturbation selected was nearly twice as large as the distance from q_1 to the open-loop pole at -5. Under these conditions, higher order effects are certain to be important.

The root-locus vector method, Eq 140, for this case reduces to simply

$$S_K^1 = \frac{K}{(q_2 - q_1)^2} = 0.0537$$

$$S_K^2 = \frac{-K}{q_1 - q_2} = -0.246$$

3. Methods Using Open-Loop Bode and ξ Plots

The ξ = -1 or Siggy plot for this case and the slope of the Siggy plot are shown in Fig. A-3.

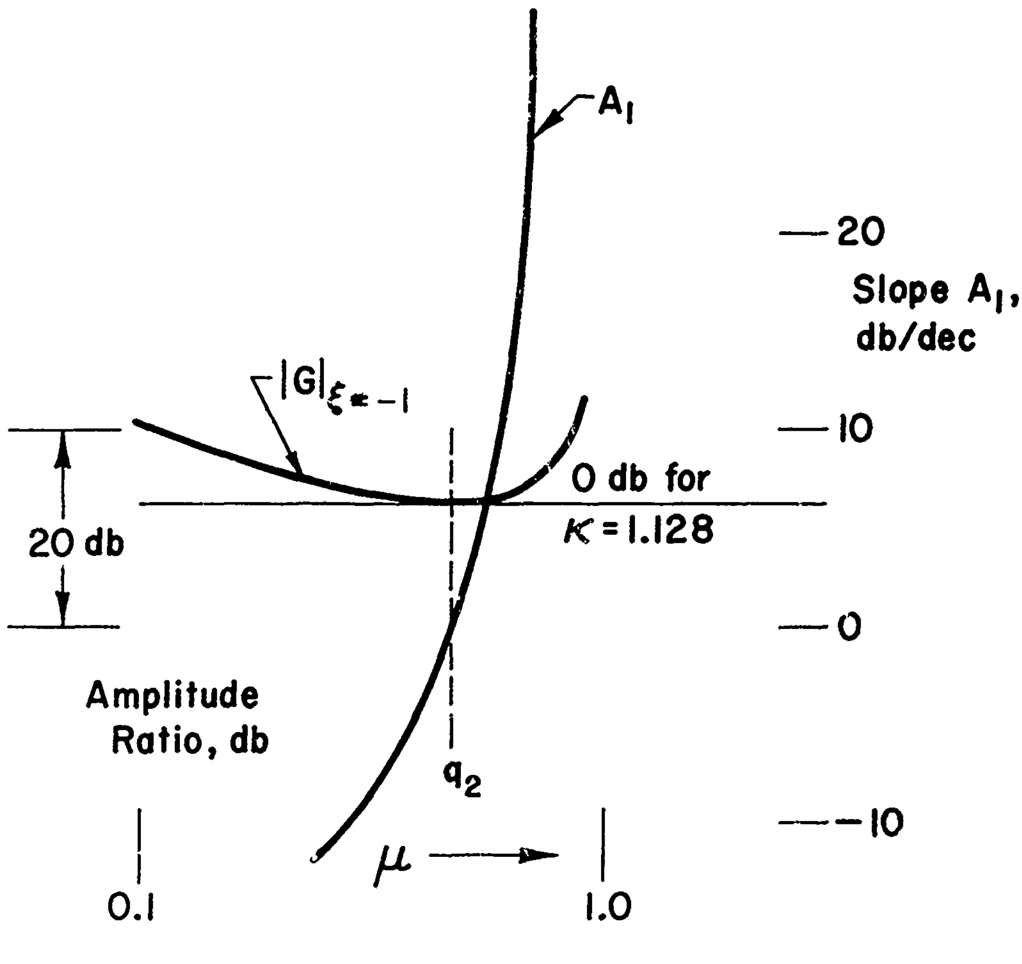

Figure A-3. Low-Gain ξ Plot

For q_2, Eq 150 becomes

$$S_K^2 = \frac{-2(2.30)(20)(q_2)^2}{(A_2)_{s=-q_2}}$$

Two methods of evaluating A_2 were used. The first was to measure the slope of the Siggy (this is A_1) and plot this slope versus frequency, see Fig. A-3. The slope of this curve is then A_2. The value measured from Fig. A-3 was 73 db/dec^2. This gives

$$S_K^2 \doteq -0.281$$

which is in error by 14 percent.

The second method was to estimate the radius of curvature of the Siggy at a frequency of 0.472. A radius of 30 db appeared to match the curvature of Fig. A-3. The vertical scale of the graph is 20 db/in., and the horizontal scale is 0.4 dec/in. Following the procedure outlined in Section III gives

$$A_2 \doteq \left(\frac{20 \text{ db/in.}}{0.4 \text{ dec/in.}}\right)^2 \frac{1}{30 \text{ db}} = 83.3 \text{ db/dec}^2$$

This gives a sensitivity of

$$S_K^2 \doteq -0.247$$

which is amazingly close to the exact value (less than 1 percent error). The surprising agreement is obviously fortuitous as the method is certainly not that accurate.

For q_1 we are again faced with the problem of being on an extremely steep portion of the Siggy. Rather than attempt to measure the slope, we will estimate it the same way we did for the high-gain case. The two low frequency roots contribute -40 db/dec and the root at -5 contributes

$$-20\left(\frac{5.055}{5.055 - 5}\right) = -1838 \text{ db/dec}$$

Therefore $$A_1 \doteq -1878 \text{ db/dec}$$

and $$S_K^1 \doteq \frac{-20(5.055)}{-1878} = 0.0539$$

which is within 1 percent.

For the gain-perturbation method a gain change of -6 db ($\Delta K/K = -1/2$) was selected. It can be seen from Fig. A-3 that this change splits the second-order pole into two first-order poles with

$$q = 0.13, 0.83$$

Thus,

$$\Delta q_2 = \begin{cases} 0.13 - 0.47 = -0.34 \\ \quad \text{or} \\ 0.83 - 0.47 = 0.36 \end{cases}$$

Either value of Δq_2 could be used in Eq 154, but because neither value has a theoretical advantage the average magnitude of the two was used. Therefore,

$$S_K^2 \doteq \frac{(0.35)^2}{-\frac{1}{2}} = -0.245$$

This is less than 1 percent error, although the method is obviously not generally that accurate.

For q_1 we make use of the fact that

$$\sum_i q_i = \sum_k p_k$$

so that with the gain change

$$q_1 = 6 - (0.13 + 0.83) = 5.04$$

Then

$$S_K^1 = \frac{5.04 - 5.055}{-\frac{1}{2}} = 0.03$$

This is an error of 44 percent.

4. Method Using the Closed-Loop Bode Asymptotes

The closed-loop Bode amplitude asymptote will be unity (0 db) for frequencies less than 0.472 and at that point it will break down at -40 db/dec. For q_2, then,

$$A_{asy} = 1$$

$$\lambda_z = \lambda_q = \lambda_u = 0$$

and from Eq 160 $$S_K^2 \doteq -(0.472)^2 = -0.223$$

This estimate is within 10 percent of the exact value.

For q_1

$$A_{asy} = \left(\frac{0.472}{5.055}\right)^2 = 0.00873$$

$$\lambda_z = \lambda_u = 0, \ \lambda_q = 2$$

and $$S_K^1 = (5.055)(0.00873) = 0.0441$$

which is an error of 18 percent.

Aeronautical Systems Division, Dir/Aeromechanics Flight Control Lab, Wright-Patterson AFB, Ohio. Rept Nr ASD-TDR-62-812. SENSITIVITY AND MODAL RESPONSE FOR SINGLE-LOOP AND MULTILOOP SYSTEMS. Final report, Jan 63, 105p. incl illus., tables 14 refs.

Unclassified Report

Two related facets of feedback system analysis are considered—the calculation of closed-loop response and the determination of the effects on closed-loop behavior of variations in open-loop parameters. The connection between the modal response coefficients (partial fraction expansion coefficients of the closed-loop transfer function), the sensitivity of the closed-loop poles to variations in the open-

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