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SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES

I. POLYHEDRAL GEODESIC STRIPS

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SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES[*]

I. POLYHEDRAL GEODESIC STRIPS

by

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I 1. Introduction. In this first note, we are concerned with an inequality between the area of a strip and the product of two lengths. A corresponding local inequality occurs in unpublished work of Aronszajn and Choquet, which is now quite old, and fairly well-known in its main features. We establish a polyhedral form of our inequality in the large, and we note that it necessitates modifying the local inequality by a constant factor, but we do not obtain the sharpest value for this factor. Our result extends without serious difficulty to relatively elementary curved surfaces, such as curved polyhedra and differentiable 2-manifolds with boundary, to which our methods apply virtually unaltered. For a more general extension, the apparatus would need modifying in a manner studied by K. H. Carlson

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[*] This series is devoted to questions, concerning simplicial complexes, which have important variational implications. We frame the questions in terms of simplicial complexes so as to give to them an elementary formulation. This does not mean that they possess an elementary solution, in fact we solve each question only in part, usually to the extent which suffices for the variational purpose from which it arose, although - as instanced by the delay in submitting some of the material - this is not a case of not having tried for more. The work has been sponsored by several agencies in the course of some years: Office of Ordnance Research, U. S. Army, Madison, Wisconsin, under Contract No. DA-11-022-ORD-1511, NSF: G 2987, NSF: AF 49(638)868, NSF: G 18909 and the Mathematics Research Center, U. S. Army, Madison, Wisconsin, under Contract No. DA-11-022-ORD-2059.

and by R. W. Rishel in their theses [1, 2]. The variational purpose of our inequality originates with [3]. Some connected questions are listed at the end of this note and in note II.

2. The strip inequality. For the purposes of this note, it is convenient to keep fixed, and to denote by Π, a special type of polydedron mod 2, on which our strip will be embedded, and we shall associate Π with a definite piecewise linear map $x(u, v)$ into Euclidean n-space, of a square of the (u, v) plane. We denote this square by S, and we suppose it subdivided in a definite way into triangles T; the perimeter of S is then subdivided correspondingly into segments, and we suppose that some of these are subject to identifications mod 2 in the usual manner, and induce corresponding identifications of points. Of $x(u, v)$ it will now be assumed that it is linear in each T and that it takes the same value at any two identified points. In other words, apart from fixing for convenience a particular representation $x(u, v)$, our polyhedron Π is a simplicial 2-complex of the special type which occurs in elementary topology mod 2, and is termed a polyhedral 2-dimensional pseudo-manifold by Seifert-Threlfall [4], except that here $x(u, v)$ need not be one to one.

In the sequel, the same symbol (u, v) will be used for a point of the plane, for a point of S (with the boundary identification-conventions), and for a point of Π; the latter will be distinguished by different notions of distance, and also of length and area. These last two are defined on Π by the usual parametric formulae, while the distance on Π of two points

(u', v') and (u'', v'') is taken to mean the minimal length of an arc of Π which joins the two points. With this notion of distance, the usual definition of length of a curve in a metric space reduces to that of length on Π.

A sum modulo 2 of a finite number of simple closed curves on our polyhedron Π will be termed a cycle on the polyhedron and homology of cycles will be modulo 2. The length of a cycle will be the minimum of the sum of lengths corresponding to its decompositions into such simple closed curves. Moreover a cycle Z_1 will be termed subcycle of a cycle Z_1 if the cycles Z_1 and $Z_2 - Z_1$ can be decomposed into finite systems of simple closed curves with no arcs in common. A cycle not homologous to a finite sum of shorter cycles will be termed a *geodesic cycle*, and its length a *characteristic period* of our polyhedron; in particular the *least characteristic period* will be the minimum of the lengths of cycles not homologous to 0; we agree to assign to it the value $+\infty$ if every cycle is homologous to 0.

By a *closed geodesic with the minimal short-cut* c, we shall mean a simple closed curve Γ on our polydedron for which c is the largest constant such that every arc γ of length $\leq c$, which has its extremities on Γ, separates Γ into two arcs one of which at least has length not greater than that of γ. In the case in which c is half the length of Γ, a value which can clearly never be exceeded, we term Γ an *absolute closed geodesic*. We observe that any geodesic cycle, and indeed any cycle, which is not the sum modulo 2 of two shorter cycles, it necessarily an absolute closed geodesic on our polyhedron: for it is evidently a simple closed curve; and any simple closed curve, other than an absolute closed geodesic, is the sum of two arcs γ' , γ' whose extremities can be joined by an arc γ shorter than both γ' and γ'', and is therefore the sum modulo 2 of the two cycles $\gamma + \gamma'$ and $\gamma + \gamma''$, both shorter than itself.

Given a closed geodesic Λ on a polyhedron Π , the set of the points of Π , each of which is distant $< t$ from a corresponding point of Λ , will be called "two-lane strip" of width $2t$ and central line Λ , or simply strip. It will be termed free strip (or free two-lane strip) if $2t$ does not exceed any of the numbers ω, c, d, where ω is the least characteristic period of Π , c the minimal short-cut of Λ and d is the distance, possibly ∞, of Λ to the boundary of Π . The length L of the central line Λ will be termed length of the strip. The main object of this note is to establish for any free two-lane strip of length L, area A, and width $2t$, the inequality

$$Lt \leq A \quad . \tag{I 2.1}$$

I 3. The proof. We denote by W_0 an arbitrarily chosen point of Λ , by $f(x)$ the distance on Π from W_0 to x, by W_ρ the set of points x for which $f(x) < \rho$, by E_ρ the set for which $f(x) = \rho$, and by $|W_\rho|$ and $|E_\rho|$ the area of W_ρ and the length of E_ρ, respectively. We remark that

$$|W_t| \geq \int_0^t |E_\rho| \, d\rho \quad . \tag{I 3.1}$$

(In fact this is true with equality.) It is sufficient by addition to establish the corresponding relation within each triangle of the subdivision of S , and to note that distances of points in a same triangle are Euclidean distances in n-space. By the triangle law of distance, it follows that $f(x)$ is locally

Lipschitzian, with constant one, in the interior of each triangle. So that (I 3.1) is a special case of the so-called Cavalieri inequality. (In fact since the latter is an equality when the underlying space is 2-dimensional and $f(x)$ is locally the sum of a constant and a Euclidean distance from a set of the form W_ρ, we have equality in each of the corresponding relations within the triangles of our subdivision, and so in (I 3.1) itself.)

This being so, let L_ρ be the length of the cycle bounding W_ρ. We assert that

$$|E_\rho| \geq L_\rho \geq 4\rho \quad (0 < \rho < t) \ . \tag{I 3.2}$$

The first of these two inequalities is obvious. We shall prove the second.

To this effect, let α be the arc of Λ of centre W_0 and length 2ρ. We remark that the open set W_ρ, which is a domain since every point in it can be joined in it to W_0, is separated by α: for otherwise there would be a path in $W_\rho - \alpha$ joining the two sides of α; this path would consist of points distant $< \rho$ from W_0 and since its extremities would be distant $< \rho$ respectively from the two sides of α at W_0, some point of the path would, by continuity, be distant $< \rho$ from both sides of α at W_0; so that the curves, on which the distance to this point is attained, would form together a path β of length $< \rho$ joining the two sides of α at W_0; since $2\rho < \omega$, the path β is homologous to 0 and so separates Λ, since Λ joins the two sides of β at W_0; but this is impossible, since the inequality $2\rho < c$ implies that β meets Λ at W_0 only. This contradiction shows that α separates W_ρ into two domains which contain respectively the two sides of α. The boundary of each of these two sub-domains contains a cycle of which α is an arc, and so contains a further arc joining the extremities of α. The two arcs so obtained, each have length $\geq 2\rho$, since $2\rho < c$ is the length of α, and they are non-overlapping since they are separated in a domain $W\rho'$ (where $\rho < \rho' < t$) by an arc of Λ. Since they both lie on the boundary of $W\rho'$ we deduce that $L_\rho \geq 4\rho$, as asserted.

From (I 3.1) and (I 3.2) it follows that

$$|W_t| \geq 2t^2 .$$

We now apply this inequality to N such sets W_t, corresponding to new positions of W_0, obtained by moving it along Λ through the distances $2t, 4t, \dots (2N-2)t$. We note that Λ can thus be covered an arbitrarily large number k of times by the sets W_t, except for an arc of Λ of length $< 2t$, which is covered $k-1$ times. The sets W_t so constructed will then all be in our strip, and they will cover no point of the strip more than k times. Hence

$$A \geq \frac{N}{k}(2t^2) = \frac{t}{k}(2tN) \geq \frac{t}{k}(kL - 2t) = tL - \frac{2}{k}t^2 ,$$

and (I 2.1) follows by making $k \to \infty$.

I 4. Additional remarks. (i) We have thus proved for $K = 1$, the inequality

$$A \geq K . Lt . \tag{I 4.1}$$

In the corresponding inequality for a rectangle or for a section of a circular cylinder the constant K becomes 2. This is also its value in the local inequality of Aronszajn and Choquet. (It must be born in mind that t is a half-width.) We shall show that, in general, however,

$$K \leq \frac{4}{\pi} \quad . \tag{I 4.2}$$

In fact it seems plausible that the best constant K for which (I 4.1) holds is actually $4/\pi$.

To establish (I 4.2), it is sufficient to observe that by polyhedral approximation (I 4.1) remains valid when Π is replaced by the surface of a sphere in 3-space, and Λ by a great circle. Taking the sphere to have radius unity, we find

$$A = 4\pi, \quad L = 2\pi, \quad t = \pi/2 ,$$

whence $K \leq A/(Lt) = 4/\pi$, as asserted.

(ii) There is a variant of (I 2.1) for a "one-lane" strip of area a, length L, and width t. This is virtually the same except that the area is halved, so that the inequality now reads

$$a \geq \frac{1}{2} Lt \quad . \tag{I 4.3}$$

and the "best" such inequality is probably $a \geq \frac{2}{\pi} Lt$. In deriving (I 4.3), we have to introduce a local right-hand side, possibly varying discontinuously along a given closed curve, and define in relation to it minimal right-hand and mixed short-cuts, and so forth. In the case in which Π is a polyhedral disc of area a, whose rim is an absolute closed geodesic of length L, our inequality would be

$$(I\ 4.4) \qquad a \geq L^2/8 \quad ,$$

and probably this could be strengthened to $a \geq L^2/(2\pi)$, an inequality which reduces to equality for a hemisphere. We observe that (I 4.4) can be deduced from (I 2.1) by a simple sticking together of two polyhedral discs to form a polyhedral sphere. It is sufficient to take these to be defined by maps $x(u, v) \pm \epsilon y(u, v)$ where $x(u, v)$ is the given map, $\epsilon > 0$ is suitably chosen, and $y(u, v)$ vanishes on the rim of the unit square and takes values in a plane orthogonal to the original n-space.

(iii) A converse question arises from the conjectured best possible form of (I 4.4). Given a simple closed polygon Λ, or a simple closed curve Λ, of length L, does there exist a polyhedral disc Π, or a topological disc Π, of area a, such that Λ is the rim of Π and constitutes on Π an absolute closed geodesic, when $a = L^2/(2\pi)$, or perhaps when a is arbitrarily close to $L^2/(2\pi)$?

We can establish, rather trivially, a result of this kind with a cruder constant: Given a simple closed polygon Λ, of length L, there exists a polyhedral disc Π, of area $a \leq \frac{5}{8} L^2$, such that Λ is both the rim of Π and an absolute closed geodesic on Π.

To this effect, let η denote a vector of length $L/4$ along an axis orthogonal to the n-space in which Λ is situated, and let Δ_0 be a polyhedral disc, with rim Λ, in the original n-space, such that Δ_0 has area

$\leq L^2/4$. The required polyhedral disc Π can then be defined very simply by a construction, which can be pictured intuitively as forming the union of a translation of Δ_0 by 2η with a cylinder of height 2η and base Λ.

(iv) We conclude with a crude estimate of certain characteristic periods of a closed polyhedron mod 2. Evidently (I 2.1) provides an estimate for the smallest of these, by taking $L = \omega$, $t = \frac{1}{4} L$, namely:

$$\omega^2 \leq 4A \tag{I 4.5}$$

where A is now the area of the given closed polyhedron Π. We denote by $\omega_1 = \omega$, ω_2, ..., ω_k the lengths on Π of absolute closed geodesics Γ_1, Γ_2, ..., Γ_k such that, for each $r > 1$, Γ_r is of minimal length in the class of cycles Γ, which are not homologous to zero mod Γ_1, Γ_2, ... Γ_{r-1}, and which have models in S which do not cross those of Γ_1, Γ_2, ..., Γ_{r-1}. Clearly

$$\omega_1 \leq \omega_2 \leq \ldots \leq \omega_k . \tag{I 4.6}$$

We shall, moreover, terminate the sequence at the value of $k = r$ for which no cycle Γ is both not homologous to zero mod Γ_1, Γ_2, ..., Γ_r and provided with a model in S which does not cross those of Γ_1, Γ_2, ..., Γ_r.

We wish to estimate crudelly the quantity ω_k. To this effect we derive from Π a polyhedron Π_1 of area $\leq 6A$ for which the sequence (I 4.6) is the same except for the omission of the first term. A simple induction coupled with (I 4.5) then shows that

$$\omega_k^2 \leq 4\{6^{k-1}A\} . \tag{I 4.7}$$

The construction of Π_1 may be described as cutting Π along Γ_1 and sticking, onto the two holes so formed, two lids consisting, in accordance with (iii), of two polyhedral discs of areas $\leq \frac{5}{8}\omega^2$ for each of which Γ_1 is an absolute closed geodesic. By (I 4.5) the area of Π_1 is then

$$\leq A + 2(\tfrac{5}{8}\omega^2) \leq A + 2(\tfrac{5}{8})(4A) = 6A.$$

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