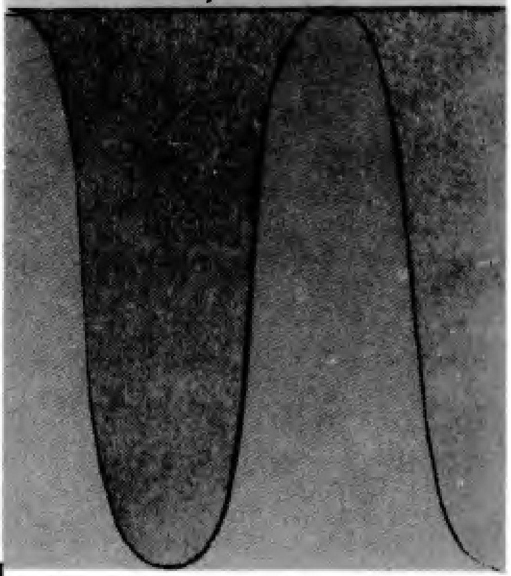
63-2-6

THE UNIVERSITY OF WISCONSIN

madison, wisconsin



UNITED STATES ARMY

MATHEMATICS RESEARCH CENTER

MATHEMATICS RESEARCH CENTER, UNITED STATES ARMY
THE UNIVERSITY OF WISCONSIN

Contract No.: DA-11-022-ORD-2059

SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES

V. THE RELATIVE AREA OF A KLEIN BOTTLE

L. C. Young

MRC Technical Summary Report #356
November 1962

Madison, Wisconsin

SOME EXTREMAL QUESTIONS FOR SIMPLICIAL COMPLEXES V

THE RELATIVE AREA OF A KLEIN BOTTLE

L. C. Young

V 1. Introduction. This final note of the series indicates how greatly we still lack information, not only as to some of the most basic facts in the higher Euclidean spaces, but also as to the question whether these facts, and even other much better known facts, are perhaps completely irrelevant in problems of analysis. This last question is further complicated by the partly Cartesian character of the underlying space in certain contexts: in fluid dynamics this space is not Euclidean, it is at best a Cartesian product of Euclidean, or locally Euclidean, spaces, corresponding to the highly dissimilar sets of variables, which specify the position, pressure, and thermo-dynamic or electro-magnetic state.

One striking, and certainly basic, fact in higher Euclidean spaces, is the familiar existence of a Klein bottle. It has had, so far, no very noticeable effect in the problems of classical analysis. We shall be concerned, however, with an example in least area, which exhibits in this respect a small but significant effect. We discuss its scope as far as our methods allow, and we indicate, as a challenge to the reader, the

---

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under the Contract No.: DA-11-022-ORD-2059.

very wide range of possibilities that are left open.

For the purposes of this note, surfaces and varieties, and in particular, polyhedra and polytopes, are oriented. We term Klein bottle, a K-dimensional variety V with boundary, such that

(a) V has a perimeter of the form mC, where C is a closed $(k-1)$- dimensional variety and m is an integer > 1;

(b) V is not expressible as a sum $V' + V''$, where V', V'' are varieties each bounded by non-negative integral multiples of C, and not identically zero.

We denote, for any variety V, by $|V|$ its extent, or area, and we term least area problem, that of the infimum of $|V|$ for varieties of a given perimeter. The relevance of Klein bottles in this problem depends on whether, for a perimeter of the form mC, the infimum is simply m times the one for the perimeter C.

We shall show that this is not the case, and in so doing we remain throughout in the domain of polytopes and polyhedra. Given a perimeter C, which we now suppose to consist of a closed $(k - 1)$-dimensional polytope in Euclidean n-space, where $1 < k < n - 1$, we denote, for each positive integer m, by a_m, the infimum of $|V|$ for all polytopes V with the perimeter mC. By the relative extent, or relative area, of a polytope V with this perimeter, we shall mean the ratio

$$\frac{|V|}{ma_1}.$$

We shall show that there exists a polytope of relative area < 1, or what amounts to the same, a polytopic perimeter C and an integer $m > 1$, for which $a_m < ma_1$. In fact, we shall simply set $m = 2$, and our example will consist of a closed $(k - 1)$-dimensional polytope C such that

$$a_2 < 2a_1. \tag{V 1.1}$$

We can apparently arrange, although we do not prove this, that, for any given $\epsilon > 0$, we should have, for a suitable C,

$$2a_1/a_2 > 1 + (1/\pi) - \epsilon. \tag{V 1.2}$$

In our example, it turns out further, that for all positive integers m, we have, at least approximately (as a certain δ tends to 0) and perhaps even exactly,

$$a_{2m} = ma_2 \quad , \quad a_{2m+1} = a_1 + ma_2 . \tag{V 1.3}$$

Before discussing our example, we indicate what possibilities are left open, and we prove two related theorems.

V 2. Discussion of prospects. The strict inequality sign in (V 1.1) is highly significant for variational theory. It implies, for instance, that the variational algorithm implicit in direct methods cannot solve what we have elsewhere [5 iii] termed "problem B", i.e., it cannot solve the least area problem for integral currents, as formulated by Federer and Fleming [2]. This is because a solution, subject to this algorithm, automatically provides a minimum in "problem A" of [5 iii], a problem which we now know to possess, on account of (V 1.1), a different solution to that of problem B.

However, a highly significant result may also be, in a sense, rather obvious; or again, it may be no more than a weak form of a much more precise result. A saleman's analogy suggests that this is perhaps the case of (V 1.1). We may think of C as a manufactured article, a_1 as its retail cost, and a_m/m as some sort of wholesale cost. (V 1.1) then says that it is sometimes a little cheaper to produce articles in pairs.

In this form, the proposition seems somewhat trite. Anyone familiar with Lewis Carrol's chapter on Woolgathering, where two eggs cost much less than one, would have suggested the possibility that, given $\epsilon > 0$, there exist an article C for which $a_2 < \epsilon a_1$. Indeed for many articles C, perhaps for most articles, one might at least expect the "cheaper by the dozen" possibility, that there exist an m for which

$$(V\ 2.1) \qquad \frac{a_m}{m} < \epsilon\, a_1 .$$

The appropriate m might depend on C as well as on ϵ.

On the other hand, the plausibility of the contingency just suggested in (V 2.1), as compared with the paradoxical nature of the one suggested by Lewis Carrol, depends entirely on the saleman's analogy, according to which the wholesale cost a_m/m should greatly decrease for large m. This analogy may be grossly misleading. In our example, the relations (V 1.3) show* that the least wholesale cost is attained for $m = 2$ and that nothing is there gained by taking m large. This may well be a rather general state of affairs. From the geometrical view, the case $m = 2$ is very different from the cases $m > 2$, in so far as an arc embedded in a surface, or a $(k - 1)$-variety in a k-variety, can only exceptionally have more than two banks. This brings us to a third, equally extreme, possibility, namely that (V 1.2) be best possible and that (V 1.3) be true of every polytopic perimeter C.

We have been unable so far to prove, or disprove, any of the three extreme possibilities just outlined. In this connection, it is of interest to note that the relations (V 1.3) are implied by an apparently much weaker statement. In the next section, we shall prove a theorem on subadditive functions (V 3.3), which includes the following:

---

* This conclusion applies even if the relations are only approximate.

(V 2.2) Reduction theorem. Let Γ be a class of polytopic perimeters such that $C \in \Gamma$ implies $2C \in \Gamma$ and $3C \in \Gamma$. Suppose further that, for each $C \in \Gamma$, there exists a positive integer $m^* = m^*(C)$ such that

$$a_{2m+1} = a_1 + ma_2 \quad \text{when} \quad m = m^* .$$

Then the relations (V 1.3) are valid for all positive integers m and all $C \in \Gamma$.

V 3. A remark on subadditive functions. For a given C the quantity a_m is a function $f(m)$ of the positive integer m, with the obvious property of subadditivity:

$$f\Big(\sum_{m \in E} m\Big) \leq \sum_{m \in E} f(m) \tag{V 3.1}$$

for every finite system E of positive integers. We recall that, as in (III 3) of [5vi], a system differs from a set in that repetitions of elements are allowed. In the sums in (V 3.1), a term is summed as often as its multiplicity indicates. Moreover, inclusion of systems is defined by the corresponding inequality between their multiplicities, and various other notions, familiar in set theory, can now be applied to systems.

The functions f occuring in this section will be restricted to have values which are positive reals or 0. When (V 3.1) reduces to equality for a system E, we say that f is additive on E. We have the following evident

(V 3.2) Additivity principle. The systems on which a subadditive f is additive are hereditary.

We pass on to another definition. Given a positive integer k, we shall say of a set M of functions of the positive integer m that it is a cone (mod k), if, whenever f belongs to M, so do also the functions f_ν given by $f_\nu(m) = f(\nu m)$ where $2 \leq \nu \leq k+1$. We shall establish the following:

(V 3.3) Theorem. Let k be a positive integer, let M be a cone (mod k) whose elements are subadditive functions of a positive integer, and suppose that, for each $f \in M$, there exists an integer $m^* = m^*(f)$ such that

$$f(km^* + 1) = m^* f(k) + f(1).$$

Then, for every $f \in M$ and all positive integers m, p,

$$f(km) = mf(k), \quad f(km+p) = mf(k) + f(p).$$

Proof. We may suppose, by (V 3.2), that $m^{*}(f)$ is the constant unity, and therefore that $f(k+1) = f(k) + f(1)$ for each $f \in M$. By applying this to the functions f_{ν}, defined as above, for $\nu = 1, k, k+1$, we deduce that

$$f\{(k+1)^{2}\} = f\{k(k+1)\} + f(k+1) = f\{k^{2}\} + 2f(k) + f(1),$$

and by an easy induction that

$$f\{(k+1)^{r}\} = f\{k^{r}\} + rf\{k^{r-1}\} + \ldots + rf(k) + f(1).$$

It follows by (V 3.2) that

$$f(kr) = rf(k) \quad , \quad f(kr+1) = rf(k) + f(1).$$

By applying this last relation to f_{ν}, for $\nu = 2, 3, \ldots, k-1$, and using the preceding one, we see that

$$f(kr\nu + \nu) = r\nu f(k) + f(\nu),$$

and therefore, by (V 3.2), for $m \leq r\nu$,

$$f(km + \nu) = mf(k) + f(\nu).$$

This last relation is thus valid for all positive m, and we deduce from it, by an easy induction, the corresponding relation, with ν replaced by an arbitrary positive integer p. This completes the proof.

V 4. Application of a theorem of Federer and Fleming. We denote by $\mathcal{K}_Q$ the class of $(k - 1)$-dimensional closed polytopes C, which are situated in some fixed cube Q and possess a $(k - 1)$-dimensional extent not exceeding some fixed constant b; when Q is replaced by the whole space, the corresponding class will be denoted by $\mathcal{K}$. Further, we write $C(f)$ for the integral on C of the integrand f, and $a(C)$ for the infimum in m, which is also the limit, of the ratio $a_m(C)/m$. Finally, we denote by $\mathcal{T}$ the class of infinitely differentiable linear $(k - 1)$-integrands, or exterior differential $(k - 1)$-forms. We shall prove:

(V 4.1) Theorem. Suppose $C_\nu \in \mathcal{K}_Q (\nu = 1, 2, \ldots)$ and $a(C_\nu) \to 0$ as $\nu \to \infty$. Then $a_1(C_\nu) \to 0$.

We shall deduce this from a theorem of Federer and Fleming [2, (7.1) p 489], slightly modified according to some earlier remarks of the same paper [(5.5), (5.6), pp 479,481]. A specialization of the theorem, thus modified, then states:

(V 4.2) Lemma. Suppose $C_\nu \in \mathcal{K}_Q (\nu = 1, 2, \ldots)$ and $C_\nu(f) \to 0$ as $\nu \to \infty$ for each $f \in \mathcal{T}$. Then there exists polytopes $V_\nu (\nu = 1, 2, \ldots)$ in Q, with the perimeters C_ν, such that $|V_\nu| \to 0$. In other words, we then have $a_1(C_\nu) \to 0$.

Proof of (V 4.1). We need only show that the relation $a(C_\nu) \to 0$ implies $C_\nu(f) \to 0$ for each $f \in \mathcal{T}$. This implication is a direct consequence of Stokes's theorem.

Theorem (V 4.1) can be strengthened in several ways: for instance we can replace the class $\mathcal{K}_Q$ by the class $\mathcal{K}$. However, none of these extensions appear to be deducible from the theorem of Federer and Fleming. We shall give a more precise result in the next section.

V 5. An isoperimetric inequality. We write $\mathcal{T}(C)$ for the ratio

$$\frac{|C|^k}{\kappa\{a(C)\}^{k-1}}$$

where κ is the appropriate isoperimetric constant, the $(k - 1)$-dimensional area of the $(k - 1)$-dimensional Euclidean sphere of radius k.

(V 5.1) Theorem. Let C be a $(k - 1)$-dimensional polytopic perimeter in n-space. Then for $\epsilon > 0$ we have an inequality of the form

$$a_1(C)/a(C) \le A\{\mathcal{T}(C)\}^\epsilon ,$$

where $A = A(\epsilon, k, n,)$ is independent of C.

(V 5.2) Corollary. Let C_ν ($\nu = 1, 2, \dots$) be $(k - 1)$-dimensional polytopic perimeters in n-space, subject, for some $\epsilon > 0$, to the relation $\lim_\nu a(C_\nu)|C_\nu|^\epsilon = 0$. Then $a_1(C_\nu) \to 0$.

To prove these results, it will be sufficient to establish, by an induction in k, the following statement, which is clearly valid for $k = 1$:

(V 5.3) Lemma. Let V be a k-dimensional polytope, bounded by a multiple mC of the closed $(k - 1)$-dimensional polytope C, and let $|V| = mM$, $|C|^k = N \cdot M^{k-1}$. Then, for $\epsilon > 0$, we have

$$a_1(C) \le AM \cdot N^{(k-1)\epsilon}$$

where $A = A(\epsilon, k, n)$ is independent of C, V.

Proof of (V 5.3). We choose a sufficiently large integer p. We write C_0, V_0, for C, V and we shall define further pairs C_ν, V_ν ($\nu = 1, 2, \dots p$); the latter will be defined by an inductive construction in terms of $C_{\nu-1}$, $V_{\nu-1}$ and of a grating which splits up n-space into cubes Δ, whose side h_ν we specify later.

We suppose already defined the closed polytope $C_{\nu-1}$, of dimension $k - 1$, and the polytope $V_{\nu-1}$, of dimension k; we suppose further that $V_{\nu-1}$ is bounded by the multiple $mC_{\nu-1}$ of $C_{\nu-1}$. We denote temporarily by C_Δ the intersection of $C_{\nu-1}$ with Δ and by H_Δ, Γ_Δ, the sections of $C_{\nu-1}$, $V_{\nu-1}$ by the frontier of Δ. We ensure that these sections lead to polytopes whose dimensions are $k - 2$, $k - 1$ respectively, by rotating our grating, if necessary, so that its axes are not orthogonal to any simplex occuring in $C_{\nu-1}$ or $V_{\nu-1}$. We then translate the grating, if necessary, to ensure further that

$$\begin{aligned} &\textstyle\sum_\Delta |H_\Delta| \leq 4n(h_\nu)^{-1} |C_{\nu-1}| , \\ (V\ 5.4) \qquad & \\ &\textstyle\sum_\Delta |\Gamma_\Delta| \leq 4n(h_\nu)^{-1} |V_{\nu-1}| . \end{aligned}$$

This is possible, as we shall verify, by a minor variant of a lemma in the theory of surfaces [5i].

In fact the lefthand sides in (V 5.4) may be written as sums of measures of sections by hyperplanes normal to the axes, each such term being taken twice. It is thus enough to verify for each axis a pair of inequalities

$$(V.\ 5.5) \qquad f \leq 2(h_\nu)^{-1} |C_{\nu-1}| , \quad g \leq 2(h_\nu)^{-1} |V_{\nu-1}|$$

where f, g are the $(k - 2)$ and $(k - 1)$ measures of the sections of $C_{\nu-1}$, $\dot{V}_{\nu-1}$ by the hyperplanes of the grating normal to the axis concerned. If we now denote by $f(t)$, $g(t)$ the measures thus obtained after a translation t of the grating along this axis, it is well known that the integrals from 0 to h_ν of $f(t)$, $g(t)$ cannot exceed $|C_{\nu-1}|$, $|V_{\nu-1}|$. The subsets of $(0 \leq t \leq h_\nu)$, in which $f(t)$ or $g(t)$ exceed the corresponding righthand sides in (V 5.5), thus have linear measures $< \frac{1}{2}h_\nu$, so that their complements have a non-empty intersection. Hence (V 5.5) must hold after a translation t, where t belongs to this intersection, and this justifies (V 5.4).

We orient H_Δ so as to bound C_Δ; and we orient Γ_Δ so that $\Gamma_\Delta + mC_\Delta$ bounds similarly the intersection of $V_{\nu-1}$ with Δ, and is therefore closed. This means that Γ_Δ has the perimeter $mH_\Delta{}^*$. (As usual here, an asterisk refers to the adjoint, i.e. indicates a reversal of orientation.) We can now construct a $(k - 1)$-dimensional polytope γ_Δ, of which we may suggest that it lies in Δ and that $|\gamma_\Delta| \leq |C_\Delta|$, such that γ_Δ has the perimeter H_Δ and is subject to an inequality of the form

$$|\gamma_\Delta| \leq B\,(m^{-1}|\Gamma_\Delta|)^{1-\alpha}|H_\Delta|^{\alpha+\beta};$$

the precise value of the exponents α, β does not matter here, they are non-negative and arbitrarily small, and B depends only on them and on k, n.

The construction is possible by our inductive hypothesis, according to which (V 5.3) holds when k is replaced by $k - 1$, although strictly the assertion then concerns the adjoints γ_Δ^* and H_Δ^* since Γ_Δ bounds the multiple mH_Δ^* of the latter; however the distinction is clearly immaterial in our inequality. It follows further, by Hölder's inequality and (V 5.4), that

$$\sum_\Delta |\gamma_\Delta| \le B(h_\nu)^{-1-\beta}(m^{-1}|V_{\nu-1}|)^{1-\alpha}|C_{\nu-1}|^{\alpha+\beta} \tag{V 5.6}$$

with now a different factor B subject to the same conditions.

We write, again temporarily, V_Δ for the cone with vertex at the centre of Δ and with perimeter $\Gamma_\Delta + m\gamma_\Delta$, and we define

$$C_\nu = \sum_\Delta \gamma_\Delta \quad , \quad V_\nu = \sum_\Delta V_\Delta \, .$$

We verify at once that V_ν has as perimeter $\sum_\Delta(\Gamma_\Delta + m\gamma_\Delta)$, and therefore has the perimeter $\sum_\Delta m\gamma_\Delta = mC_\nu$, since $\sum_\Delta \Gamma_\Delta$ is singular. The pairs C_ν, V_ν can thus be defined, in correspondence with the numbers h_ν, for $\nu = 1, 2, \ldots, p$.

This being so, we construct a k-dimensional polytope $S_\nu = \sum_\Delta S_\Delta$, where S_Δ is the cone with vertex at the centre of Δ and with perimeter $C_\Delta + \gamma_\Delta^*$. Then S_ν has the perimeter $C_{\nu-1} + C_\nu^*$, and since each cone S_Δ has its radii $\le \frac{1}{2}\sqrt{n}\,h_\nu$ and its perimeter of extent $\le 2|C_\nu|$, we find that

$$|S_\nu| \le k^{-1}\sqrt{n}\,h_\nu |C_{\nu-1}| \, .$$

It follows easily, by adding the relations

$$a_1(C_{\nu-1}) - a_1(C_\nu) \le |S_\nu| \qquad (\nu = 1, 2, \dots, p)$$

and the isoperimetric inequality

$$a_1(C_p) \le (\kappa^{-1}|C_p|^k)^{\frac{1}{k-1}} ,$$

that

$$\text{(V 5.7)} \quad \begin{cases} a_1(C) \le (\kappa^{-1}|C_p|^k)^{\frac{1}{k-1}} + \sum_{\nu=1}^{p} \frac{\sqrt{n}}{k} h_\nu |C_{\nu-1}| \\ \le \frac{k}{k-1} u_o + \sum_{\nu=1}^{p} u_\nu , \text{ say.} \end{cases}$$

We may suppose, by adding to some γ_Δ, if necessary, a small closed polytope, that each $|C_\nu| \ne 0$ in all the preceding constructions. Moreover since $|\gamma_\Delta| \le |C_\Delta|$, we observe that $|C_\nu| \le |C_{\nu-1}|$. We can then choose the numbers h_ν increasing with ν, so that in (V 5.7)

$$u_o = u_1 = \dots = u_p .$$

By replacing the sum by the appropriate multiple of the geometric mean, we find that

$$\text{(V 5.8)} \quad a_1(C) \le (p + \frac{k}{k-1})q$$

where

$$\text{(V 5.9)} \qquad q^{p+k/(k-1)} = \kappa^{-k/(k-1)} \left(\frac{\sqrt{n}}{k}\right)^p |C| \prod_{\nu=1}^{p} h_\nu |C_\nu| .$$

On the other hand, if we refer to the second part of (V 5.4), the definition of C_ν; and the definition of V_ν as the sum of cones of radii radii $\leq \frac{1}{2}\sqrt{n}\, h_\nu$ on the perimeters $\Gamma_\Delta + m\gamma_\Delta$, we find that $|V_\nu|$ cannot exceed the mean value, and therefore the greater, of the two quantities

$$4k^{-1} n\sqrt{n}\, |V_{\nu-1}| \qquad \text{and} \qquad mk^{-1}\sqrt{n}\, h_\nu |C_\nu| .$$

Since (V 5.6) provides the estimate

$$h_\nu |C_\nu| \leq B(m^{-1}|V_{\nu-1}|^{1-\alpha} |C_{\nu-1}|^{\alpha+\beta} (h_\nu)^{-\beta} ,$$

and so, in view of $|C_\nu| \leq |C_{\nu-1}|$, the estimate

$$(h_\nu |C_\nu|)^{1+\beta} \leq B(m^{-1}|V_{\nu-1}|)^{1-\alpha} |C_{\nu-1}|^{\alpha+2\beta} ,$$

we derive by induction an estimate of the type

$$\prod_{\nu=1}^{p} h_\nu |C_\nu| \leq AM^{p-\alpha} |C|^{\beta}$$

with new, but still arbitrarily small, exponents α, β, and with a coefficient A depending on p, k, n, α, β. Here, for dimensional reasons, we must have $k\alpha = (k - 1)\beta$, and from (V 5.8), (V 5.9) we can now derive the assertion of our lemma (V 5.3), by substituting suitable functions of ϵ for the quantities p, α, β, and by choosing for our new coefficient A the appropriate functions of ϵ, k, n.

V 6. Our main construction. We have now to construct a perimeter C subject to (V 1.1) or (V 1.2), and at least approximately to (V 1.3). We shall carry this out only for $k = 2$, since the general case can be derived by forming a Cartesian product with a polytopic $(k - 2)$-sphere. The lowest relevant value of n is then 4.

We shall exhibit C as a certain simple closed polygon, situated on a certain elementary polyhedral Klein bottle S. We therefore first define the polyhedron S, which we take, as usual, to be oriented. Our argument requires the oriented polyhedron S to be bounded by the multiple 2Θ of a certain simple closed polygon Θ, where Θ is further, the smallest contour on S which is homologous to Θ on S. We shall consider two special choices of this polyhedron S.

To this effect, we denote by M a polyhedral Mobius strip, so oriented that its boundary is augmented by twice a certain cross-section φ, where φ is a segment. We form the Cartesian product

$$Z = M \times I$$

of M with a segment I. The product Z is a 3-dimensional analogue, in 4-space, of a Mobius strip. We orient it correspondingly, so that its boundary is augmented by twice a certain cross-section of the form $\Phi = \varphi \times I$. We define S to be the polyhedron obtained by removing 2Φ from the elementary oriented boundary of the oriented polytope Z. The polyhedron S thus has the perimeter 2Θ, where Θ is the adjoint of the perimeter of the oriented rectangle Φ.

Our two choices of S will correspond to choices of M, I, and φ. In both cases we shall have an inequality of the type

$$\text{(V 6.1)} \qquad a_1(\Theta) > \tfrac{1}{2} c\,|S| - \eta$$

where η is a positive number at our disposal.

In our first example, the constant c has the value $1/\pi$. The example is intended to lead to (V 1.2), but we have not undertaken to prove this. We choose I very long, and we take for M a suitable polyhedral approximation to the Mobius strip shown in fig. 1. This strip is made up of three rectangles and of three sixty-degree circular sectors, which are situated in parallel planes and joined by the rectangles. The reader must imagine the diagram modified, by placing the sectors in three planes near to one another, so that their projections largely cover one another; in this way the rectangles joining them will have arbitrarily small

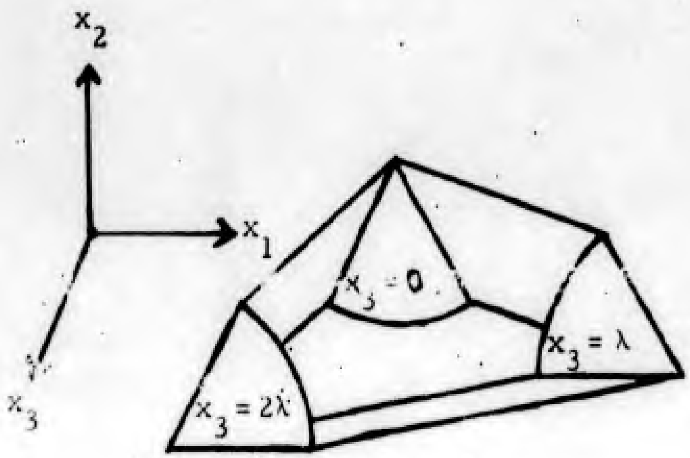


Figure 1

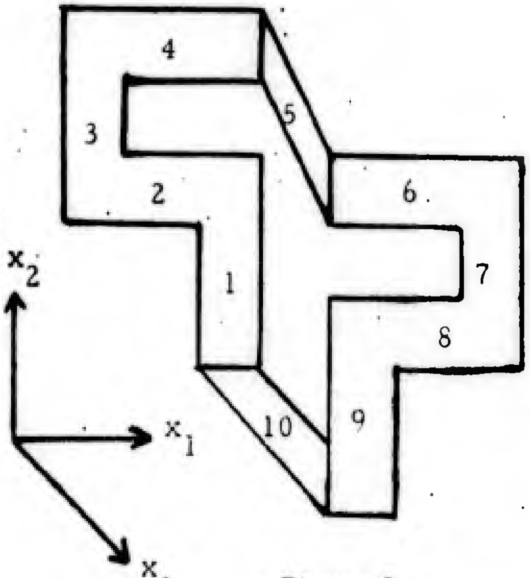


Figure 2

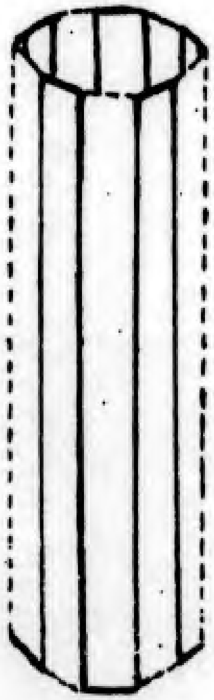


Figure 3

area, and we obtain the stated value for the constant c.

In our second example, the constant c is reduced to $1/21$. We choose M as shown in fig. 2, but with dimensions such that the various parts numbered do not interpenetrate when the cross-section is magnified five times. We choose I so that the cross-section of Z is a square.

In the rest of this paper we shall be concerned with this second choice of the Klein bottle S. The construction of the relevant C is similar in the two cases, but the first choice of S, which leads to the right constant in (V 1.2) — or at least to a better constant — requires much more ellaborate proofs.

We can thus visualize S as made up of ten tubes. Each tube is understood to be the portion cut off by two planes from a two-dimensional cylinder in a 3-space normal to one of the axes. The cross-section of each such cylinder is the perimeter of a square of fixed size. The tubes (1) to (4) and (6) to (9) are taken to lie in the 3-spaces $x_3 = 0$ and $x_3 = 1$, respectively, while the tubes (5) and (10) have generators parallel to the axis of x_3.

For definiteness, we take Θ to be the cross-section which bissects the tube (10), and S will be oriented accordingly and is bounded by 2Θ. We write $|\Theta| = 4p\delta$ where p is a large positive integer, and we denote by Θ', Θ'' two cross-sections of the tube (10), which are at a distance δ from one another, on the opposite sides of Θ. We denote by S' the polyhedron derived from S by removing the portion of the tube (10) between Θ' and Θ''. Thus S' consists of the tubes (1) to (9) together with two pieces of the tube (10); S' is the topological image of a tube T.

By a generator of S', which we picture simply by a generator of T, we shall mean the sum of eleven segments, consisting of generators of the respective tubes (1) to (9) and of the two portions in S' of the tube (10), which fit together to form a single eleven-sided polygonal arc. Similarly we define a generator of S, or one of Z.

On S' we now draw $4p$ of its generators, which are equidistant and include the edges. We denote by C a simple closed polygon consisting of these $4p$ generators, and of arcs of Θ' and Θ'' which connect them. In Figure 3, the contour C is pictured by its image on the tube T, where it is easier to visualize. We shall orient C to make it homologous on S to the cross-section Θ. It now depends, for fixed C, only on the choice of p, or what comes to the same, on that of δ.

We shall establish the following results, which clearly imply not only (VI.1), but also an approximate form of (VI.3) and an inequality similar to (VI.2) with the constant π replaced by 21:

$$(V6.2)\qquad \begin{aligned} a_{2m}(C) &= m|S| - O(\delta)\,, \\ a_{2m+1}(C) &= (m+\tfrac{1}{2})|S| + a_1(\Theta) - O(\delta)\,, \end{aligned}$$

where the symbols $O(\delta)$ denote non-negative quantities, of the same order of magnitude as δ when the latter tends to 0. The proofs of these results will occupy the remaining sections of this paper.

V 7. The separation of horizontal and vertical area. Since 2Θ bounds S, and Θ is homologous to C on S, we find easily that $2mC$ bounds a polyhedron of area $m|S|$, which is derived from mS by changing the orientation of certain parts of S. Thus

$$\text{(V 7.1)} \qquad a_{2m}(C) \leq m|S| .$$

Similarly, since C and the adjoint of Θ bound a part of S, and the complementary part is bounded by Θ and the adjoint of C, and one of the two parts has area $\leq \frac{1}{2}|S|$, we find that $a_1(C) - a_1(\Theta) \leq \frac{1}{2}|S|$. By combining this with (V 7.1) and with the subadditivity relation $a_{2m+1} \leq a_1 + a_{2m}$, we see that

$$\text{(V 7.2)} \qquad a_{2m+1}(C) \leq (m + \tfrac{1}{2})|S| + a_1(\Theta) .$$

It follows that to prove (V 6.2) it is enough to show that the area of every polyhedron V bounded by $2mC$, or by $(2m+1)C$, exceeds for suitable $O(\delta)$, the relevant right-hand side of (V 6.2). We shall reduce this to two further lemmas.

To this effect, we introduce additional notations. We fix the polyhedron V; and we fix also, in the case in which V has the perimeter $(2m+1)C$, a second polyhedron U, with the same perimeter as V, and with an area not exceeding that of V. The polyhedron U will be defined in the next section but one.

We shall use the letter i for an index, which may assume values $1, 2, \dots, 10$, and for which addition or subtraction of indices is understood modulo 10. We write θ for the number $\sqrt{a_1}(\Theta)$, i.e. for the length of the segment φ, and ρ_0, ρ_1 for the numbers $\theta + \frac{1}{2}\delta$, $2\theta + \frac{1}{2}\delta$. We suppose δ small enough to ensure that the tubes (1) to (10) do not penetrate one another after an enlargement of their cross-sections by the factor $5+\delta$. The symbol ρ will stand for a variable subject to $\frac{1}{2}\delta \leq \rho \leq \rho_1$.

The i-th of the tubes which constitute S is the portion of a 2-dimensional cylinder between two planes in its 3-space, and so between two 3-spaces of 4-space. We denote the latter by Π_{i-1} and Π_i; here Π_{i-1} denotes the 3-space, which bisects the right-angle between a pair of interesting generators of the $(i-1)$-th and the i-th tube, and which is normal to their plane. We denote by $[\Pi_i]$ the closed half-space, bounded by Π_i, which contains our i-tube, and by $Q_{i\rho}$ the set of the points of

$[\Pi_i] - [\Pi_{i-1}]$ whose distance from this tube is $\leq \rho$. Our convention concerning the symbol ρ ensures that the ten sets $Q_{i\rho}$ $(i = 1, 2, \dots, 10)$ are disjoint. We write Q_ρ for the union of these ten sets, i.e. for the set of the points distant $\leq \rho$ from the carrier of S. We write Q, Q_i for Q_ρ, $Q_{i\rho}$ when $\rho = \rho_1$; and S^+, S_i^+ for Q_ρ, $Q_{i\epsilon}$ when $\rho = \frac{1}{2}\delta$. We denote by C' the polygon C without those of its arcs which lie in Θ' or Θ''; and we write S_i^-, C_i^- for the parts of S', C' whose least distance from the complement of S_i^+ is $\frac{1}{2}\delta$. The sums in i of the parts S_i^-, C_i^- will be written simply S^-, C^-.

We denote by y_i, z_i, or for fixed i, by y, z, the pair of coordinates, taken from x_1, x_2, x_3, x_4, of which the former is constant in the i-th tube, and the latter has its axis parallel to the generators of that tube. We write Z_i for the intersection of Q_i with the carrier of Z; and Y_i for the maximal subset of Q_i with the same projection as Z_i orthogonal to y_i. We write Y for the union of the ten sets Y_i.

We shall speak of the direction of z_i as vertical. For sets and varieties situated in Q_i, this direction will be regarded as fixed. In accordance with [5 iv], we may then associate with every polyhedron, a vertical and a horizontal area of its intersection with Q_i; the sums with respect to i will be termed, respectively, the vertical and the horizontal area of the polyhedron in Q.

This being so, the relations (V 6.2) reduce to consequences of the following pair of statements:

(V 7.3) Lemma. Let V be a polyhedron with the perimeter mC. Then the vertical area of the part of V in S^+ is at least $\frac{1}{2}m|S^-| - O(\delta)$.

(V 7.4) Lemma. Let V be a polyhedron with the perimeter $(2m+1)C$, and let U be the associated polyhedron, with the same perimeter and with an area not exceeding that of V. Then the horizontal area of the part of U in $Y - S^+$ is at least $a_1(\Theta) - O(\delta)$.

To deduce the first relation (V 6.2) we now need only (V 7.3) with $2m$ in place of m; while, to deduce the second relation (V 6.2), we need only combine (V 7.4) with the result of substituting, in (V 7.3), $2m+1$ for m and U for V. For it is clear that an area is never less than the corresponding vertical, or horizontal, area. Moreover $|S^-| = |S| - O(\delta)$.

V 8. The vertical area. To prove lemma (V 7.3), it is enough, by addition, to show that, for each $i = 1, 2, \ldots, 10$, the vertical area of the part $V_i = V{\cdot}S^+_i$ is at least $\frac{1}{2}m|S^-_i| - O(\delta)$. To this effect, we observe that the vertical areas of V_i, S^-_i are expressible as integrals in z of the lengths v_i, s_i of their sections by $z = \text{const}$. Moreover, the horizontal area of S^-_i vanishes, so that its vertical area coincides with its area $|S^-_i|$. It will thus suffice to show that, for almost every z,

$$v_i \geq \tfrac{1}{2}m\, s_i - O(\delta),$$

and consequently, to verify the two relations

$$v_i \geq \tfrac{1}{2}m N\delta, \quad s_i \leq (N+2)\delta,$$

where N denotes the number $N(z)$ of points of the section $z = \text{const}$ of C^-_i. The second relation is verified at once, for every z. On the other hand, if E denotes the section $z = \text{const}$ of C^-_i, we remark that v_i is the total length of a system of arcs, with possible repetitions, whose ends include each point of E at least m times, but do not include any further points of S^+_i other than frontier points; arcs which begin and end at a same point are not regarded as having any end. Since the distance of each pair of points of E is $\geq \delta$, and the distance from each point of E to a frontier point of S^+_i is $\geq \frac{1}{2}\delta$, we find that $v_i \geq \frac{1}{2} m N \delta$, as required.

V 9. The reduction from V to U. We now suppose that V has the perimeter $(2m + 1)C$, where m is a positive integer or zero. We wish to construct a polyhedron U, with the same perimeter as V, and with an area not exceeding that of V, such that U is situated in Q. We shall distinguish two cases.

In the case in which

$$|V| \geq (m + \tfrac{1}{2})|S| + a_1(\Theta),$$

we define $U = U_1 + mU_2$, where U_2 is a polyhedron on S, derived by changes of orientation as in the proof of (V 7.1), so as to be bounded by $2C$ and of the same area as S; and where U_1 consists of the square Φ, bounded by Θ, together with a part of S, or of its adjoint S^*, constructed as in the proof of (V 7.2), so as to be of area $\leq \frac{1}{2}|S|$ and to be bounded by $C + \Theta^*$, where Θ^* is the adjoint of Θ. Evidently U then satisfies our requirements.

It remains to define U in the case in which

$$\text{(V 9.1)} \qquad |V| < (m + \tfrac{1}{2})\, |S| + a_1(\Theta)\,.$$

In this case, it follows from lemma (V 7.3), proved in the preceding section, that the part of V not in S^+ has area $< a_1(\Theta) + O(\delta)$. We denote by L_ρ the length of the section of V by the frontier of Q_ρ; and by M_ρ the area of the part of V outside Q_ρ. We recall the conventions of section V 7 in regard to ρ, ρ_0, ρ_1.

(V 9.2) Lemma. There exists a value ρ' subject to $\frac{1}{2}\delta < \rho' < \rho_0$, such that $M_{\rho'} \geq \frac{1}{4}(L_{\rho'})^2$.

Proof. The notion of partial area [5 iv], or the Cavalieri inequality in its various forms, provides the relation

$$M_\rho \geq \int_\rho^\infty L_{\rho'}\, d\rho' \ ;$$

also, we have initially, by (V 7.2) and (V 9.1),

$$\text{(V 9.3)} \qquad M_\rho \leq a_1(\Theta) \quad \text{for } \rho = \tfrac{1}{2}\delta\,.$$

The conclusion stated therefore follows by appealing to a familiar lemma of Cesari [1, p 272], or to the special case, with exponent $\frac{1}{2}$, of lemma (II 3.4) in [5 vi].

We are now in a position to define the polyhedron U in the case in which V is subject to (V 9.1). We choose ρ' as in the lemma just proved and we write U_o for the part of V in $Q_{\rho'}$. If now $L_{\rho'} = 0$, we set $U = U_o$. Otherwise we denote by C_ν ($\nu = 1, 2, \ldots, N$) the oriented closed curves of total length $L_{\rho'}$, which, together with $(2m+1)C$, bound U_o. We then define U as a suitable polyhedral approximation to the surface

$$\text{(V 9.4)} \qquad \sum_{\nu=o}^{N} U_\nu ,$$

where U_ν is an elementary surface subject to the following conditions for $\nu \neq o$:

U_ν is oriented and bounded by the adjoint of C_ν;
the support of U_ν is the convex hull of that of C_ν;
the area of U_ν is at most $|C_\nu|^2/(4\pi)$.

From this last condition and from lemma (V 9.2), we see that the sum of the areas $|U_\nu|$ for $\nu \neq o$ is less than $M_{\rho'}$ and hence that the surface (V 9.4) has smaller area than V. We observe further, by referring to lemma (V 9.2) and to the inequality (V 9.3) and remembering that M_ρ decreases as ρ increases, that

$$\tfrac{1}{2}L_{\rho'} \leq \sqrt{a_1}\,(\Theta) = \theta .$$

It follows that θ is not less than the half-length, and so the diameter of each C_ν, and therefore not less than the diameter of each U_ν $(\nu \neq 0)$. The surface (V 9.4) must therefore lie in Q_ρ for a value $\rho = \rho' + \theta < \rho_1$. It follows that we can determine the polyhedral approximation U with the perimeter $(2m+1)C$, so that its area is less than that of V and so that its support is in Q, as required.

V 10. The horizontal area. There only remains to prove lemma (V 7.4), for the polyhedron U just constructed. Given any generator g of Z, we denote, for $i = 1, 2, \ldots, 10$, by g_i the generator of Z_i which is its intersection with Z_i, and by $R_i = R_i(g)$ a rectangle, which is situated in the plane strip with the same projection, orthogonal to y_i, as g_i, and which consists, in that strip, of the points distant $\leq \rho_1$ from g_i.

We shall make use of the following:

(V 10.1) Lemma. Let W be an oriented polyhedron in Q with a perimeter Γ on S, where Γ is homologous on S to the cross-section Θ, and let g be a generator of Z not lying on S. Then W intersects at least one of the rectangles $R_i(g)$ $(i = 1, 2, \ldots, 10)$.

Proof. We shall suppose that W intersects none of the ten rectangles R_i, and we have to derive a contradiction. Without loss of generality, we shall use, repeatedly, the "procedure H" of altering W in its homology class in $Q - U_i R_i(g)$.

By H, we arrange first that $\Gamma = \Theta$. We then denote by K_i the intersection of Q_i with the 3-space Π_i, defined in section V 7; K_i is also the intersection of either with the closure of Q_{i+1}. The sets Q_i and K_i are convex. Further, K_i meets just two of our rectangles, namely R_i and R_{i+1}, and it meets their union in an arc γ, consisting of two segments at right-angles,which issue from a same point and which terminate on the frontier of K_i in its 3-space. We denote by σ the intersection of K_i with S.

Since σ links γ in K_i, we find that every contour in $K_i - \gamma$ is homologous to a multiple of σ. Since W does not meet R_i, R_{i+1}, its section by Π_i, which lies in K_i, does not meet γ. From these facts it follows that by adding to W ten pairs of adjoint polyhedra, situated in the K_i and not meeting the corresponding pairs of segments γ, we may suppose by H that W is a sum of ten polyhedra W_i $(i = 1, 2, \ldots, 10)$, where W_i lies in the closure of Q_i and does not meet R_i, and where the perimeter of each W_i lies on S.

This perimeter is then not altered by projecting W_i into the 3-space $y_i = \text{const}$ which contains Z_i. We may therefore suppose, by H, that each W_i lies in the 3-space containing Z_i. Further, since W_i does not meet the straight line through y_i, we find that in the set obtained by removing the line from the projection $y_i = \text{const}$ of Q_i, W_i is homologous to a polyhedron on the intersection of S with Q_i, that is one the i-th of the tubes (1) to (10).

We may thus suppose by H, that W is a polyhedron on S. This is impossible, since its perimeter Θ is not homologous to 0 on S.

We are now ready for the final stage:

Proof of (V 7.4). The horizontal area in question may be written $\sum_i |P_i|$, where P_i is the horizontal projection of the part of U in $Y_i - S^+_i$. We denote by G_i the set of generators g of Z not in S^+, such that $R_i(g)$ is met by U. We write E_i for the set of the intersections of these generators with a particular cross-section of Z. The measure $|E_i|$ does not depend on the choice of this cross-section, and we note that, if this cross-section is in Z_i, the set E_i becomes the carrier of the projection on $y_i = \text{const}$ of P_i. Hence $|P_i| \geq a_1(\Theta) - O(\delta)$. This last is the case, since by lemma (V 10.1) with $W = U$, the union of the G_i consists of all g in $Z - S^+$.

REFERENCES

[1] Cesari L. An existence theorem of the Calculus of Variations for integrals on parametric surfaces, Amer. J. of Math. 74(1952) 265-295.

[2] Federer H. and Fleming W.H. Normal and integral currents, Annals of Math. 72(1960) 458-520.

[3] Hardy G.H., Littlewood J.E., and Polya G. Inequalities

[4] Seifert H. and Threlfall W. Topologie

[5] Young L.C.

(i) A lemma in the theory of surfaces, J. London Math. Soc. 19 (1944) 209-212.

(ii) Generalized surfaces of finite type, Memoirs Amer. Math. Soc. no. 17 (1955).

(iii) A variational algorithm, Riv. di Mat. Univ. di Parma 5 (1954) 255-268.

(iv) Partial area I, II, III, ibidem 10 (1959) 103-113, 171-182, 257-263.

(v) Contours on generalized and extremal varieties, J. Math. and Mech. (in the press).

(vi) Some extremal questions for simplicial complexes I Polyhedral geodesic strips, II on the radius times periphery problem for area, III Problems of geometry and analysis in the higher Euclidean Spaces, IV the algebraic and the geometric resultant, an application of variational methods, MRC reports # 304, 307, 320, 322.