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The Measurement of Frequency with Scanning Spectrum Analyzers

by
W. R. Kincheloe, Jr.

October 1962

Technical Report No. 557-2

Prepared under
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United States Air Force
Griffiss Air Force Base, New York



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ERRATA

<u>Page</u>	<u>Position</u>	<u>Should read</u>
1	3rd line from bottom	... or, if the events are uniformly ...
5	6th line of footnotes	... freedom of a function [Refs. ...
6	line 7 of text	... $\phi(t)$; the <u>scan rate</u> $s(t)$ is ...
13	last term of Eq. (1.15)	... = $\frac{2 U_1 _{\max}^2 b_{1n}}{R}$...
16	Fig. 6	$u_n(t) \rightarrow \left[\dots \right]$
21	line 10	... for a given scan-rate or achievement ...
29	last line on page	... by addition of time delay ...
30	Eq. (3.6a)	$b_2 = \frac{\int_0^\infty A_2^2(\omega) d\omega}{2\pi A_{2m}^2} \quad (3.6a)$
32	Eq. (3.10), line 4	- $A_1(2\sigma_1 t) \sin \dots$
32	Eq. (3.10), line 9	- $A_1(2\sigma_1 t) \sin \dots$
33	3rd line of Table 2	... matched, $r_1(t)$ an image
38	end of Eq. (3.23)	... $-\theta_{31} - \frac{\pi}{4} \rfloor$
39	line 3	... may be made large
41	end of Eq. (3.30)	... $(w_1) + \theta_4 \rfloor \} \quad (3.30)$
41	3rd line from bottom	The conditions of spectrum ...
47	Eq. (4.1b)	$Y(\omega) = \sqrt{\pi/\alpha} e^{-\omega^2/4\alpha} \quad (4.1b)$
51	Eq. (4.4)	$u_1(t) = a_1 e^{-\alpha_1 t^2} \cos(\omega_1 t + \theta_1)$
57	line 6 of text	... peak amplitudes a_{3n} , durations ...
57	line 4 of footnotes	... or $r_2(t)$, depending on
61	numerator of last term of Eq. (4.25d)	$\frac{\rho_0 \eta D_1^2 \pm \rho_2 d_1^2}{\dots} \quad (4.25d)$
105	line 5 of text	... well-behaved), Eq. (B.20) may

<u>Page</u>	<u>Position</u>	<u>Should read</u>
105	3rd line of Eq. (B.21)	$\dots * \operatorname{Re}(h) - \operatorname{Im}(g) * \dots$
106	Eq. (B.22), line 3	$- A_1(2\sigma t) \sin [\dots$
106	Eq. (B.22), line 8	$- A_1(2\sigma t) \sin [\dots$
120	first line under Di-3	$\dots$ low and bandpass filters with n^{th} -order $\dots$

THE MEASUREMENT OF FREQUENCY
WITH SCANNING SPECTRUM ANALYZERS

by

W. R. Kincheloe, Jr.

October 1962

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Stanford University Stanford, California

ABSTRACT

A time function, or "signal", is often described in terms of an associated frequency spectrum. This paper is concerned with the method of examining the spectrum of an unknown signal time function whereby the spectrum is converted into a known time function by scanning. The scanning process consists in convolving, or "sliding", a spectrum "window" across the signal spectrum as a function of time. The arrangement for accomplishing this is termed a scanning spectrum analyzer. A mathematical justification for this representation of spectrum analysis is presented.

The performance of a spectrum analyzer may be evaluated in terms of resolution, sensitivity, and magnitude of spurious responses. The resolution is a measure of the fidelity of the spectrum representation. Sensitivity relates to the signal power required for measurement in the presence of noise. Spurious responses refer to incorrect or misleading spectrum indications which result from the measurement process. Historically, the resolution and sensitivity of scanning spectrum analyzers have long been linked to the scan rate, i.e., an increase in scan rate was thought to result necessarily in poorer resolution and sensitivity. A desired scan rate has been used to specify the optimum bandwidth of the response-determining filter associated with the analyzer. While techniques have been developed to reduce spurious responses, such responses have remained a major problem in spectrum measurement by the scanning technique.

It is shown in this study that the resolution and sensitivity of the scanning spectrum analyzer may in theory be made as good as desired without regard to the scan rate, and in practice may be made considerably superior to that previously assumed to be the best possible for a given scan rate. Such improvement is accomplished by "phase-matching" the analyzer filter to the scan rate and by increasing the filter bandwidth. In the limit the resolution may be made perfect, in the sense that the form of the Fourier spectrum of an input signal time function is approached by the output response-time function. The sensitivity of measurement is in theory independent of scan rate, being

limited only by available signal energy. Certain spurious responses decrease in magnitude below the noise level as the filter bandwidth is made sufficiently large.

Practical considerations limit the bandwidth over which phase-matching of the filter can be accomplished. Since the spectrum representation is then imperfect, selection of the spectral window shape by which signal spectrum components are approximated may be desirable. The study shows that such selection may be accomplished by specifying that the amplitude versus frequency characteristic of the analyzer filter be the Fourier transform of the desired window shape. Two types of windows are of particular interest: the gaussian window which provides the best sensitivity in the presence of random noise; and the steep-sided rectangular window which provides good resolution of adjacent signal components with widely differing amplitudes. The gaussian case is discussed in some detail; the general response is given from which a variety of special cases of practical interest may be obtained.

Experimental results support and demonstrate predicted performance.

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LIST OF SYMBOLS

a	Amplitude coefficient of time function
a(t)	Amplitude function of time
A	Spectrum amplitude coefficient
A(ω)	Spectrum amplitude function
b	Bandwidth of spectrum function in cps
B	Total range of spectrum to be analyzed
C	Spectrum coefficient
d	Duration of time function in seconds
D	Duration of spectrum measurement; time period (seconds)
e	Naperian logarithm base (= 2.8183)
e_n	Thermal noise amplitude (volts rms)
E	Energy of a function in joules
f	Frequency in cps
Δf	Frequency increment
F	Noise figure
g(t)	Arbitrary time function
G(ω)	Arbitrary spectrum function
h(t)	Impulse time function of a filter
H(ω)	Filter spectrum function
i	Integer
I	Definite integral
j	$\sqrt{-1}$
k	Boltzmann's constant (= 1.37×10^{-23} joule/ $^{\circ}$ K)
L	Length in centimeters
K	Time-bandwidth parameter of a function

LIST OF SYMBOLS (Continued)

m	Subscript denoting maximum (or minimum)
n	Integer
p	Integer
P	Power
$r(t)$	Filter response time function
R	Resistive component of impedance (ohms)
s	Scan rate in cps ²
t	Time variable in seconds
Δt	Time increment
T	Temperature in °K
$u(t)$	Signal time function
$U(\omega)$	Signal spectrum function
$v(t)$	Elementary signal time function
$V(\omega)$	Elementary signal spectrum function
$W(\omega)$	Spectrum "window" function
x	Arbitrary variable
$y(x)$	Arbitrary function of x
$Y(\omega)$	Arbitrary spectrum function
α	Duration coefficient of gaussian time function
β	Bandwidth coefficient of gaussian spectrum function
γ	Variable
$\delta(x)$	Dirac delta function
ϵ	Small increment ($ \epsilon \ll 1$)
ζ	Dummy variable of integration

LIST OF SYMBOLS (Continued)

η	Phase-matching parameter ($= -k_{\mu}\sigma$)
θ	Phase constant
λ	Resolution factor of gaussian spectrum analyzer
μ	Quadratic phase coefficient of filter
ξ	Dummy variable of integration
π	3.1416
ρ	Gaussian function width parameter
σ	Quadratic phase coefficient of scanning time function (in radians/second ²)
τ	Time delay constant in seconds
ϕ	Phase constant in radians
$\phi(t)$	Phase function of time
$\phi(\omega)$	Spectrum phase function
ψ	Arbitrary variable
ω	Radian frequency ($= 2\pi f$)

SOME CONVENTIONS AND NOTATION

Fourier Transformation:

$$U(\omega) = \mathfrak{F}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

$$u(t) = \mathfrak{F}^{-1}[U(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega) e^{j\omega t} d\omega$$

Convolution:

$$u_1(t) * u_2(t) = \int_{-\infty}^{\infty} u_1(\xi) u_2(t-\xi) d\xi$$

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I. INTRODUCTION

The examination of the time variation of a physical quantity in terms of a "frequency" description is a time-honored measurement technique that gives insight into many physical processes. The communications engineer uses frequency both in the description of time functions as "signals" and in the analysis of signal handling operations such as modulation and filtering. Techniques for the measurement of frequency are of recurring interest, and a profusion of literature exists on the subject.

The work reported here is concerned with frequency measurement by the popular scanning spectrum analyzer technique, whereby the frequency description of an unknown signal is presented as a time function by "scanning" the signal spectrum as a function of time. The intent is to explore limitations of signal description by this method and to provide information for the design of spectrum analyzers with specified performance. A particular aim is the correction of several common misconceptions regarding the potential performance of the scanning spectrum analyzer.

A. THE FREQUENCY DESCRIPTION OF SIGNALS

What is meant by the "frequency" associated with a time function? Intuitively the concept seems clear, and yet mathematical formulations of frequency have taken a variety of forms.* According to the dictionary definition and the Institute of Radio Engineers (IRE) standards [Ref. In-1], the frequency of a recurring event is the number of events per unit of time; or, if the effects are uniformly spaced in time, the frequency is the reciprocal of the period of repetition. This definition tells little about the nature of time variation, either within the

* Selected references from the large body of literature on the subject of frequency are given in the bibliography at the end of this report; a more complete listing has been compiled by Stumpers [Ref. St-2].

period or as related to other periods, and is awkward when the frequency so defined is not constant.*

Signal description and analysis usually require a more precise and complete description than is provided by either of the definitions referred to above. One possibility for an improved description is to specify the frequency spectrum of component functions, each prescribed in time and frequency, which combine to reproduce the signal. Another possibility is to use instantaneous frequency as a descriptive characteristic of the signal as a function of time. The spectrum description of a signal permits recognition and examination of superimposed component signals by identification with spectrum components. The instantaneous frequency description is characteristic of the total signal and is often used in describing modulation processes. Both concepts are useful here.

1. The Fourier Spectrum **

In this report the frequency spectrum of a time function $u(t)$ is assumed to be the Fourier transform $U(\omega)$ defined by the reciprocal relations [Ref. Ca-1]:

$$U(\omega) = \mathfrak{F}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-j\omega t} dt \quad (1.1a)$$

$$u(t) = \mathfrak{F}^{-1}[U(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega)e^{j\omega t} d\omega \quad (1.1b)$$

where $\omega = 2\pi f$ denotes a sinusoidal frequency in radian measure. Equation (1.1a) defines $U(\omega)$ for both positive and negative frequencies;

* Dissatisfaction with definitions of frequency continues to stimulate discussion in the technical literature [Refs. Ha-2, Ho-2, Hu-2, Sh-1, To-1, and others].

** Formulation, conditions of existence, limiting forms, and properties of Fourier transforms and spectra have been widely described [Refs. Ap-1, Be-1, Sn-1, etc.]. Concise summaries are given in many communications texts [Refs. Bl-1, Da-2, Go-1, and others].

signal description and filter response must include contributions from both.* If $u(t)$ is real, as will be the case here, then $U(\omega)$ is Hermitian; that is, letting $U(\omega) = A(\omega)e^{j\Phi(\omega)}$, then

$$A(\omega) = A(-\omega)$$

$$\Phi(\omega) = -\Phi(-\omega) \quad (1.2)$$

where $A(\omega)$ and $\Phi(\omega)$ are real functions. When $u(t)$ is defined within a period of D seconds duration, the periodic Fourier series representation may be written:

$$u(t) = \sum_{n=-\infty}^{\infty} A_n e^{j\omega_n t} \quad (1.3a)$$

$$A_n = \frac{1}{D} \int_{-D/2}^{D/2} u(t) e^{-j\omega_n t} dt \quad (1.3b)$$

where n is an integer, and $\omega_n = 2\pi n/D$. Equation (1.3a) may also be written in the form:

$$u(t) = \sum_{n=0}^{\infty} C_n \cos(\omega_n t + \phi_n) \quad (1.4)$$

The series representation might be considered to be a discrete spectrum, with amplitude function A_n and phase ϕ_n .

The association of transmitted intelligence with a signal spectrum implies either the definition of a time-varying spectrum, or the use of a finite observation period, since the future of the signal is unknown and in practice the remote past is inaccessible. The Fourier spectrum

*While a Fourier spectrum with only positive frequencies may be formulated [Refs. Ga-1, Ha-2], the two-sided spectrum facilitates analysis of modulation and convolution processes [Ref. Bl-1]. The power spectrum $P(\omega) = |H(\omega)|^2$ defined only for positive frequencies has also been widely used [Refs. Bl-1, Da-2, Fa-1, Gr-1, La-1, Pa-1,2, Tu-2], although the lack of phase information does not permit complete specification of $u(t)$ from $P(\omega)$ [Ref. Le-1].

defined either by Eq. (1.1) or by Eq. (1.3) is time invariant, however, and represents a time function which is completely specified.* If $u(t)$ is to be everywhere zero outside of an interval D , the Fourier spectrum extends over an infinite range of frequency [Refs. Ch-1, We-1]. Physical measurement of the Fourier spectrum is thus necessarily an approximation [Ref. Le-1]. Nevertheless, the approximation may be adequate for signal description and identification and the measurement technique may permit the Fourier spectrum to be approached arbitrarily closely (in theory) as the measurement bandwidth is made large.** The presence of noise precludes precise spectrum determination in any event. Practically, specification of the nature and degree of approximation is desirable.

2. Spectrum Representation by Elementary Signals ***

Gabor [Ref. Ga-1] proposed and Lerner [Ref. Le-1] further elaborated signal representation in both frequency and time by a set of elementary signals. Each element of the set might typically be of form $v(t)$ in time (with duration d_v) and $V(\omega)$ in frequency (with bandwidth b_v). The set may be formed by displacing the elementary functions at regular intervals in both time and frequency; for example,

$$U(\omega) = \sum_i \sum_n A_{in} [V(\omega - \omega_i) + V(\omega + \omega_i)] e^{-jn\omega D} \quad (1.5a)$$

$$u(t) = \sum_i \sum_n A_{in} v(t - nD) \cos \omega_i t \quad (1.5b)$$

* Time-varying spectra have been defined in terms of either the past history of a signal up to the time of measurement [Refs. Ge-1, Li-1, Pa-1, Ro-1, Tu-2], or signal data in the vicinity of the time of measurement [Refs. Bl-1, Fa-1, Pa-2, Pi-1]. Signal processing with such spectra requires caution [Ref. Ro-1].

** The addition of new data by increasing measurement duration may not result in convergence to a stationary Fourier spectrum [Ref. Le-1], although the power spectrum may converge (see references of footnote on page 3). Spectrum measurement has also been treated as a statistical estimation problem [Refs. Da-1, Ka-2, Pa-1,2] with the "confidence" in an estimate increased by enlarging the sample of signal data.

*** Signal descriptions such as "natural components" [Ref. Hu-1] and "weighted" spectra [Refs. Bl-1, Fa-1, Pa-2, Pi-1] might fall in this category.

where A_{in} are magnitude constants, D is an increment of time, $\omega_i = 2i\pi/D$, and i, n are integers. The form of $v(t)$ and its Fourier transform $V(\omega)$ may be selected for one or more of a variety of reasons: a desired signal representation, orthogonality, rapid decay away from a central peak, absent or negligible minor maxima (sidelobes) in either time or frequency (or both), ease of physical realization, or a required "width" in time or frequency (subject to the minimum time-bandwidth relation $b_v d_v = K_v \geq K_m$).^{*} The Fourier spectrum may be considered a limiting form of elementary signal representation for which $d_v \rightarrow \infty$ and $b_v \rightarrow 0$. Component signals of a complicated time function to be analyzed may be associated in frequency with one or more values of i in Eq. (1.5), and their variations in time described by applying successive n 's.

The signal representation of Eq. (1.5) has several advantages. In addition to flexibility, visualization of signal behavior in time and frequency is facilitated. Perhaps the most useful consideration here is the opportunity to optimize spectrum analysis. Normally, not only may the form of a signal to be analyzed be unknown, but the forms of several possible signals may be different. By assuming an elementary signal function in terms of which all signals are to be described, the signal analyzer may be designed accordingly.

3. Instantaneous Frequency and Scanning **

The instantaneous frequency f_i of an oscillating time function

^{*}The minimum product K_m (usually of the order of unity) of the "widths" of a function and its Fourier transform (spectrum) has been widely discussed [Refs. Ga-1, Ka-2, La-1,2, Le-1,2, Ma-1, St-1, Wo-1,2]. The time-bandwidth product K has many interpretations; e.g., filter response time vs bandwidth [Refs. Ku-1, Me-1]; information content or degrees of freedom of a function [Refs. Ga-1, Ma-1,2, Wo-2]; information transmission rate vs bandwidth [Refs. Ha-4, Ma-1]; filter complexity [Ref. Tu-1]; and detectability of a signal in noise [Ref. Tu-1]. It is a useful parameter that is characteristic of the form of the function, and for some width definitions is independent of function translation in time and frequency.

^{**}The concept of instantaneous frequency was used by such early pioneers in communications as Carson [Refs. Ca-3,4] and Van der Pol [Ref. Va-2] in discussing frequency and phase modulation, and has since been formalized by the IRE standards [Ref. In-1].

is customarily defined as the time derivative of phase. If $u(t) = a(t) \cos \phi(t)$, then*

$$f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} \quad (1.6)$$

If $\phi(t)$ consists only of constant and first order terms, then f_i is constant. For example, if

$$u_1(t) = a_1 \cos \omega_1 t \quad (1.7)$$

then

$$f_{i1} = \omega_1 / 2\pi = f_1 \quad \text{cycles per second (cps)}$$

The instantaneous frequency is said to scan if second or higher order terms are present in $\phi(t)$; the scan rate $s(t)$ is the second derivative of phase. As another example, if

$$u_2(t) = a_2 \cos \sigma t^2 \quad (1.8)$$

then

$$f_{i2} = \sigma t / \pi = s_2 t$$

and

$$s_2 = \sigma / \pi \quad \text{cycles per second per second (cps}^2\text{)}$$

Scanning is said to be linear if $s(t)$ is constant; in this report, linear scanning will be assumed unless specified otherwise.

Instantaneous frequency is different from frequency in the spectrum sense. The two concepts may be related [Refs. Ba-4, Ha-5], but incorrect application gives anomalous results [Ref. Po-3]. As an illustration, consider the Fourier spectra of the functions $u_1(t)$ and $u_2(t)$ above [Refs. Bl-1, Ca-1]:

$$U_1(\omega) = \frac{1}{2} a_1 [\delta(f-f_1) + \delta(f+f_1)] \quad (1.9)$$

*The f_i of Eq. (1.6) is not unique [Ref. Sh-1] in that various combinations of $a(t)$ and $\phi(t)$ may form the same $u(t)$. Expanded definitions may remove the ambiguity [Refs. Ga-1, Ha-2, Hu-2]. Hok [Ref. Ho-2] pointed out that the question is not important if the variation of f_i is small and $a(t)$ is bounded.

where $\delta(x)$ is the Dirac impulse function of unit area and infinitesimal width at $x = 0$; and

$$U_2(\omega) = \frac{a_2}{\sqrt{s_2}} \cos \pi \left(\frac{f^2}{s_2} - \frac{1}{4} \right) \quad (1.10)$$

By combining positive and negative spectral components, $U_1(\omega)$ consists of a spectral "line" with area a_1 at the frequency $\omega/2\pi = f_1$; association of such a "line" with the instantaneous frequency $f_{i1} = f_1$ of $u_1(t)$ [Eq. (1.7)] is not unreasonable. Agreement between $f_{i2} = s_2 t$ of $u_2(t)$ [Eq. (1.8)] and the spectrum $U_2(\omega)$ is less apparent, however.

The concept of instantaneous frequency is useful for description of time functions and scanning processes. Recognition and measurement of simultaneously present component signals of a complicated time function on the basis of frequency is usually better accomplished, however, by examination of the frequency spectrum.

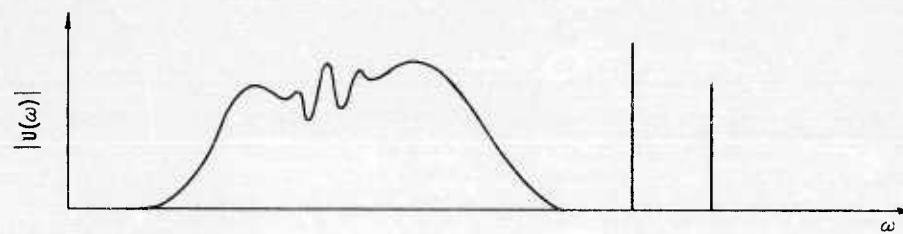
B. SPECTRUM ANALYSIS

A spectrum analyzer examines a signal time function by determining (or approximating) the defining parameters of the frequency spectrum of the signal. Usually only the magnitude $|U(\omega)|$ is desired; this might be the continuous Fourier spectrum component $A(\omega)$, the Fourier series coefficients A_n or C_n , or the elementary signal magnitudes A_{mn} of the formulations above. The phase function may also be of interest.

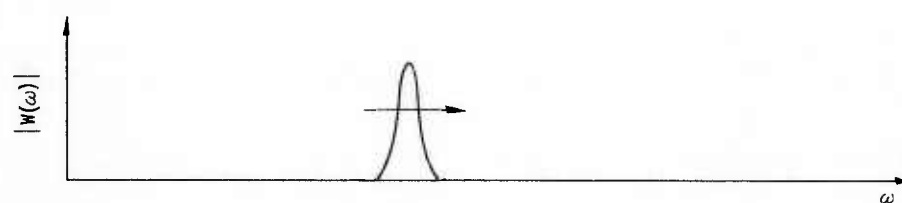
1. The Scanning Spectrum Analyzer

The scanning technique for accomplishing spectrum analysis is illustrated in Fig. 1. In effect, a frequency slit or "window" $W(\omega)$ slides (scans) across the unknown signal spectrum $U(\omega)$ as a function of time. The response function $r(t)$ emerging from the analyzer forms an approximation to the form of $U(\omega)$, with spectrum frequency related to time by the scanning process. Mathematically, $r(t)$ results from the convolution of $U(\omega)$ and $W(\omega)$ in time:

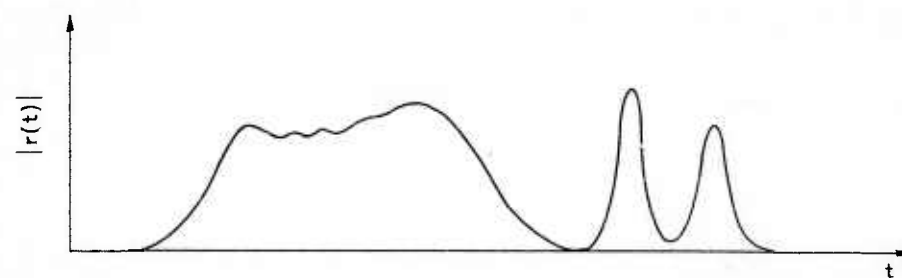
$$r(t) = U(2\sigma t) * W(2\sigma t) \quad (1.11a)$$



a. Signal spectrum



b. Scanning spectral window



c. Analyzer response

FIG. 1. SPECTRUM REPRESENTATION BY A SCANNING WINDOW.
(Only positive frequencies are shown.)

where 2σ is the scan rate in radians per second per second. The convolution may be written in two forms:

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega) W(2\sigma t - \omega) d\omega \quad (1.11b)$$

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(2\sigma t - \omega) W(\omega) d\omega \quad (1.11c)$$

The essence of scanning is the relative translation of $U(\omega)$ and $W(\omega)$. While Eqs. (1.11b) and (1.11c) are mathematically equivalent, the physical processes they suggest are quite different. The spectrum window $W(\omega)$ is derived from a filtering process in the spectrum analyzer.* Equation (1.11b) infers the scanning of a time-varying spectral window past a stationary signal spectrum; this we term the scanning filter spectrum analyzer, illustrated in Fig. 2. Examples are the classic tuned-radio-frequency (TRF) receiver of early radio and, more recently, the voltage-tunable backward-wave amplifier [Ref. He-1]; also the magnetically tunable cyclotron and paramagnetic resonance devices.

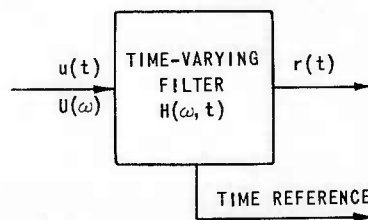


FIG. 2. BLOCK DIAGRAM OF A SCANNING-FILTER SPECTRUM ANALYZER.

Equation (1.11c), on the other hand, implies the sliding of the signal spectrum past a stationary spectral window. We label this the translation-scanning spectrum analyzer, shown in block form in Fig. 3. The spectrum translation is usually accomplished by "mixing" the signal

* Although in general $W(\omega)$ is not identical to the spectrum function of the analyzer filter.

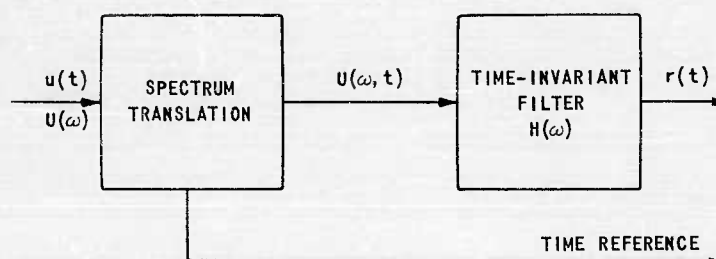


FIG. 3. BLOCK DIAGRAM OF A
TRANSLATION-SCANNING SPECTRUM
ANALYZER.

with a scanning reference signal (local oscillator) which establishes the scan rate and time reference.* The familiar superheterodyne receiver is an example of this approach.

Presumably, similar results could be obtained by either scanning method;† however, time-invariant filters are more readily synthesized than time-variable filters at present, particularly where precise specification of phase and amplitude function shapes are required. Despite the appealing simplicity and freedom from spurious responses of the scanning filter analyzer, this report is concerned only with the translation-scanning method.

If the form of $W(\omega)$ depends on scan rate (as in general it will) and if all signal spectrum components are to be similarly represented regardless of frequency, then the analyzer scan rate will necessarily be constant. Therefore, only linear scanning will be considered here.

2. Performance Parameters

Spectrum analyzer performance is evaluated in this study in terms of frequency resolution, sensitivity, dynamic range, and spurious responses.

* A method of spectrum translation which employs a delay line memory with phase advance in the recirculation loop has been described by Capon [Ref. Ca-2]. A "stairstep" scanning method has been used by Zukerman [Ref. Zu-1] to equate $W(\omega)$ and $H(\omega)$.

† In certain situations the two methods are equivalent [Refs. Ba-1, Ho-1].

Frequency resolution is a measure of the fidelity, or degree of approximation, of the signal spectrum representation. Two spectrum components are resolved when they can be separately analyzed; the analysis may require only recognition of their separate existence, or may involve the measurement of their characteristics to a specified accuracy. However defined, analysis resolution may be measured in terms of the effective frequency "width" of the analyzer response to a single signal component. Response "width" definition is at the discretion of the designer, involving considerations of analysis objectives and dynamic range. The reciprocal nature of Fourier transform widths in time and frequency require increased signal (and measurement) duration for improved frequency resolution.*

Spectrum analyzer resolution is examined here by noting the effective frequency width of the analyzer response to an elementary input signal, either the "line" spectrum of Eq. (1.9) or an appropriately assumed $V(\omega)$ of Eq. (1.5). The resolution of the scanning spectrum analyzer is determined by the bandwidth of the scanning spectral window $W(\omega)$.

Sensitivity refers to the minimum signal energy required for spectrum analysis in the presence of thermal noise. In the arrangement of Fig. 4, noise power is added to signal power at the analyzer input. The analyzer is assumed to be "matched" in impedance both to the source and the load. The analyzer includes a bandpass filter $H(\omega)$ with noise bandwidth b_n [Refs. Go-1, Va-1]:

* See Refs. Ia-2, Wo-1, and the other references of the footnote on p. 3. An analogy has been drawn between the minimum time-bandwidth relation $bd \geq K_m$ and the Heisenberg Uncertainty Principle of Quantum Mechanics [Refs. Ga-1, Go-1, Gr-1, St-1], as implying a fundamental uncertainty in frequency measurement. This view has been questioned [Ref. Le-1] on the grounds that the width definition is a formalism which implies only those properties that may be inferred from the definition; certainly the "center" or position of maximum magnitude of a response "peak" may be estimated with an accuracy limited only by the presence of other signals or noise. Specification of frequency measurement uncertainty would seem to require reference to the signal-to-noise ratio.

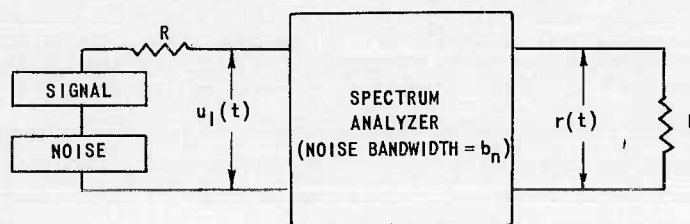


FIG. 4. INTRODUCTION OF NOISE IN THE SPECTRUM ANALYZER.

$$b_n = \frac{\int_0^{\infty} |H(\omega)|^2 d\omega}{2\pi |H(\omega)|_{\max}^2} \quad \text{cycles per second} \quad (1.12)$$

The analyzer response $r(t)$ results from the signal $u_1(t)$ measured at the input terminals of the analyzer.

Available thermal noise power at the analyzer input produces an rms noise voltage e_n at the analyzer output:*

$$e_n = A_{2m} \sqrt{FkTRb_n} \quad (1.13)$$

where $A_{2m} = |H(\omega)|_{\max}$, F is the noise figure of the analyzer,** k is Boltzmann's constant ($= 1.37 \times 10^{-23}$ joule/degree Kelvin), and T is the temperature in degrees Kelvin. The ratio of peak response voltage r_m to rms noise voltage is

*At frequencies where quantum effects are not significant (below approx 1012cps) [Ref. Va-1].

**In the translation-scanning spectrum analyzer the noise figure F includes the effects of power loss and additional noise introduced as a result of the translation process, unless pre-translation noise predominates.

$$\frac{r_m}{e_n} = \frac{1}{\sqrt{FkTR}} \cdot \frac{r_m}{A_{2m} \sqrt{b_n}} \quad (1.14)$$

Since $r(t)$ is expressed in terms of $u_1(t)$, specification of the ratio r_m/e_n determines peak input signal power $2u_{1m}^2/R$ and signal energy E_1 :

$$E_1 = \frac{2u_{1m}^2 d_1}{R} = \frac{2|u_1|_{\max}^2 b_{1n}}{R} \quad (1.15)$$

where d_1 is the duration of $u_1(t)$ (defined in terms of signal energy) and b_{1n} is the noise bandwidth of the input signal spectrum $U_1(\omega)$. The factor $r_m/A_{2m} \sqrt{b_n}$ may be used as a sensitivity parameter that is characteristic of the spectrum analyzer.

Dynamic range of the spectrum analyzer relates to the range of input signal magnitude over which satisfactory spectrum analysis may be accomplished. Minimum signal amplitude is determined by sensitivity and noise. Maximum signal amplitude is set by overloading effects and the obscuring of weaker signal components at other frequencies by the "skirts" of the frequency "window" of the analyzer and by spurious responses.

Spurious responses are incorrect or misleading indications of spectrum components resulting from the analysis. They may be due to minor maxima (sidelobes) in the shape of the spectrum window. The translation-scanning spectrum analyzer is particularly susceptible to spurious responses due to unwanted scanning components resulting from the translation process. Image responses result from components which scan through the filter bandwidth in the direction opposite to that of normal signals. Harmonic responses are due to components which scan at rates that are integral multiples of the intended scan rate. While usually of smaller magnitude than the normal analyzer response, spurious responses of strong signals may seriously reduce dynamic range.

In practice, compromises must be made between spectrum analyzer performance parameters. Duration of measurement (or signal) limits frequency resolution and sensitivity. Spurious responses and required

resolution limit dynamic range. The designer must select the parameters best suited to the measurement situation.

3. The Comb Filter Spectrum Analyzer

While we are primarily concerned in this study with the scanning technique, the non-scanning comb filter spectrum analyzer provides a useful comparison.

The comb filter analyzer arrangement is shown in Fig. 5. The signal spectrum $U(\omega)$ is applied to a set of bandpass filters $H_n(\omega)$ staggered in frequency so as to encompass the total frequency range of interest. The filters provide in effect a set of stationary spectrum windows, with the spectral description of the signal inferred from the relative magnitude of responses at the outputs of the various filters. Frequency resolution and sensitivity in the presence of noise are determined by the bandwidths of the individual filters, while dynamic range and spurious responses are determined by the steepness and form of the filter band-pass functions away from the central peak.

The comb filter spectrum analyzer arrangement tends to be more complex than the scanning analyzer, not only because of the multiplicity of filters but also due to the need for correlating and interpreting the

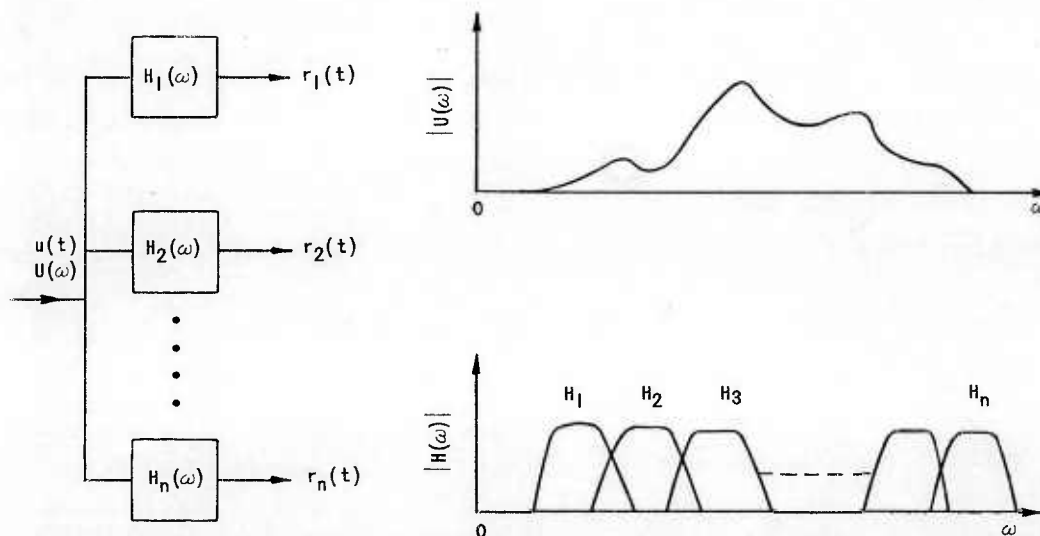


FIG. 5. THE COMB FILTER SPECTRUM ANALYZER. (Only positive frequencies shown.)

many filter outputs. The number of filters required may become quite large if a high degree of resolution is required. Significant advantages have been claimed for the comb filter analyzer over the scanning analyzer [Ref. Li-2], usually in view of the fact that little signal energy need be lost if the signal spectrum lies essentially within the total filter set bandwidth. We shall see, however, that in theory the scanning spectrum analyzer need not be at a disadvantage in this respect.

II. PREVIOUS ANALYSIS OF THE TRANSLATION-SCANNING SPECTRUM ANALYZER

The simplicity and utility of the scanning spectrum analyzer has led to extensive investigation and discussion in the literature. In this chapter we examine some commonly-held views regarding the translation-scanning method; some of these are found to be incorrect.

A. THE GLIDING TONE PROBLEM

Consider the arrangement of Fig. 6. As in Eq. (1.4), assume the signal is represented by a set of sinusoidal components.

$$u(t) = \sum_{n=0}^{\infty} C_n \cos(\omega_n t + \phi_n) \quad (1.4)$$

By frequency translation ("mixing" with a scanning reference signal) the components are made to scan at a rate $s = \sigma/\pi \text{ cps}^2$; the components may then be expressed:

$$u_n(t) = a_n \cos(\sigma t^2 + \omega_n t) \quad (2.1)$$

Let the frequency "window" provided by a bandpass filter $H(\omega)$ be assumed to have a center frequency f_w , bandwidth b_w ,* and maximum amplitude A_w . In this section the phase response of the filter is ignored (or assumed to be linear).



FIG. 6. THE GLIDING TONE ARRANGEMENT.

As shown in Fig. 7, the scanning of each component through the frequency window produces a response $r_n(t)$ with peak magnitude $r_{nm} = a_n A_w$ at time $t_n = (f_w - f_n)/s$, so that

* Sometimes called the "apparent" bandwidth of the analyzer [Ref. Ba-5, and others].

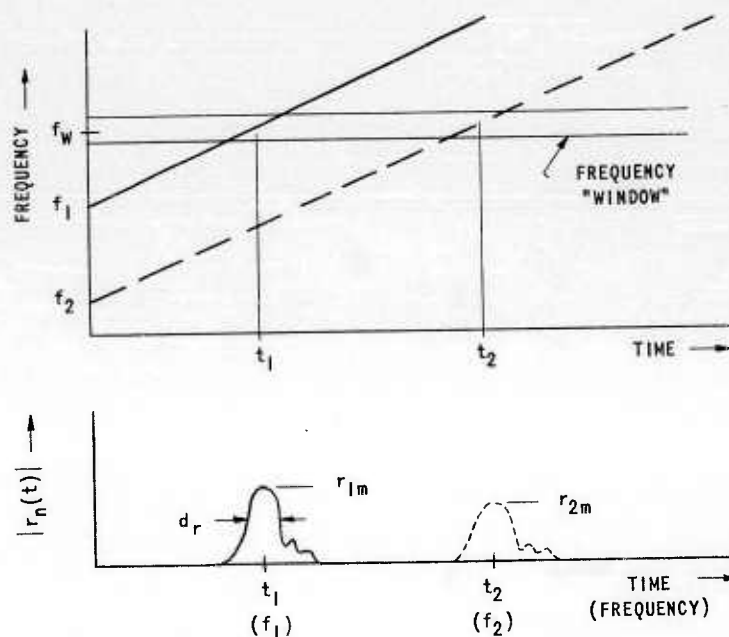


FIG. 7. TIME AND FREQUENCY RELATIONSHIPS IN THE GLIDING TONE PROBLEM.

$$f_n = f_w - st_n . \quad (2.2)$$

Spectrum analysis is accomplished by noting the time and magnitude of the filter responses. Frequency resolution (window width) is determined in this case by the duration d_r of the response, which represents a range of frequency sd_r . The determination of filter response to the scanning excitation of Eq. (2.1) is described in the literature as the gliding tone problem.

1. The Intuitive Approach

It has long been recognized that scan rate affects spectrum

analyzer performance.* A change in gliding tone response occurs when the filter bandwidth b is traversed in a time period comparable to the filter risetime (impulse response duration) d , i.e. when $b/s \approx d$. Since $bd = K$, the transition in response occurs when

$$b^2 \approx |Ks| \quad (2.3)$$

where the time-bandwidth factor K is usually of the order of unity.**

Slow-scanning (quasi steady-state) gliding tone response occurs when $b^2 \gg |s|$. For this condition the form of the response (frequency "window") is essentially that of the filter function $H(\omega)$. Response duration d_r is the time b/s required to scan through the filter bandwidth. Both response magnitude $|r(t)|$ and frequency resolution $sd_r = b$ are independent of scan rate.

Fast-scanning (scanning-limited) response results when $b^2 \ll |s|$. The filter receives a "shock" excitation; the form of $r(t)$ is that of the impulse response $h(t)$, with duration $d = d_r$ independent of scan rate [Refs. Ge-1, Go-2, Mo-1]. Response magnitude is proportional to the duration of excitation b/s , and frequency resolution represented by the response duration is sd . Table 1 summarizes the performance parameters in these two cases.

Since for a particular scan rate the window width b_w increases directly with b as bandwidth becomes very large and inversely with b as bandwidth becomes very small, a minimum b_w (and hence best resolution) necessarily occurs between these extremes. This argument has frequently led [Refs. Fa-2, So-1, Th-1,2, Wa-1] to specification of an optimum filter bandwidth b_{opt} for best resolution in the transition response region defined by Eq. (2.3). Thus if filter phase is ignored, (or assumed to be linear)

$$b_{opt} = \sqrt{|Ks|} \quad (2.4)$$

* Schuck [Ref. Sc-1], describing in 1934 results obtained with a scanning audio "sound prism", stated a frequently echoed conclusion: "The speed of operation and resolving power of a heterodyne analyzer are inextricably interdependent."

** See footnotes on pp. 5 and 11.

Table 1 - Gliding Tone Response Parameters
(Filter phase neglected)

Response type	Slow-scanning ($ s \ll b^2$)	Fast-scanning ($ s \gg b^2$)
Form of response	$\sim H(\omega)$	$\sim h(t)$
Response duration	$\frac{b}{s}$	d
Resolution bandwidth	b	$\frac{sK}{b}$
Response magnitude	constant	$\sim \frac{b}{s}$

where: s = scan rate (cps^2)
 b = filter bandwidth (cps)
 d = filter risetime (impulse response duration)
 $H(\omega)$ = filter spectrum function
 $h(t)$ = filter impulse response function

where the constant K is determined by the definition of bandwidth used and by the form of the filter $H(\omega)$, and is usually of the order of unity.

Since by Eq. (1.13) rms noise voltage e_n is proportional to $\sqrt{b}$, sensitivity (as measured by the voltage ratio r_m/e_n) is proportional to $1/\sqrt{b}$ as filter bandwidth b becomes large, and to $\sqrt{b}$ as b approaches 0. As with scanning resolution discussed above, sensitivity becomes maximum between extremes of filter bandwidth in the transition response region, when the filter bandwidth is equal to (or in the vicinity of) the optimum resolution bandwidth of Eq. (2.4)*

* Energy considerations may be used to show that if bandwidth and response duration are suitably defined, the largest peak response amplitude (hence sensitivity) necessarily occurs when the response duration (hence resolution) is a minimum; see Batten, et al [Ref. Ba-5] and Appendix A.

Figure 8 illustrates variation of peak response amplitude r_m , scanning resolution sd_r , and sensitivity ratio r_m/e_n with filter bandwidth when filter phase is ignored; details in the transition response region are determined by the form of $H(\omega)$. These conclusions have been empirically supported by measurement of scanning response of various bandpass filters [Refs. Ba-5, Sc-1, Te-1, Wi-1].*

2. Analysis of Gliding Tone Response

The optimum resolution relation of Eq. (2.4) has also been frequently obtained by analysis of bandpass filter response to the scanning

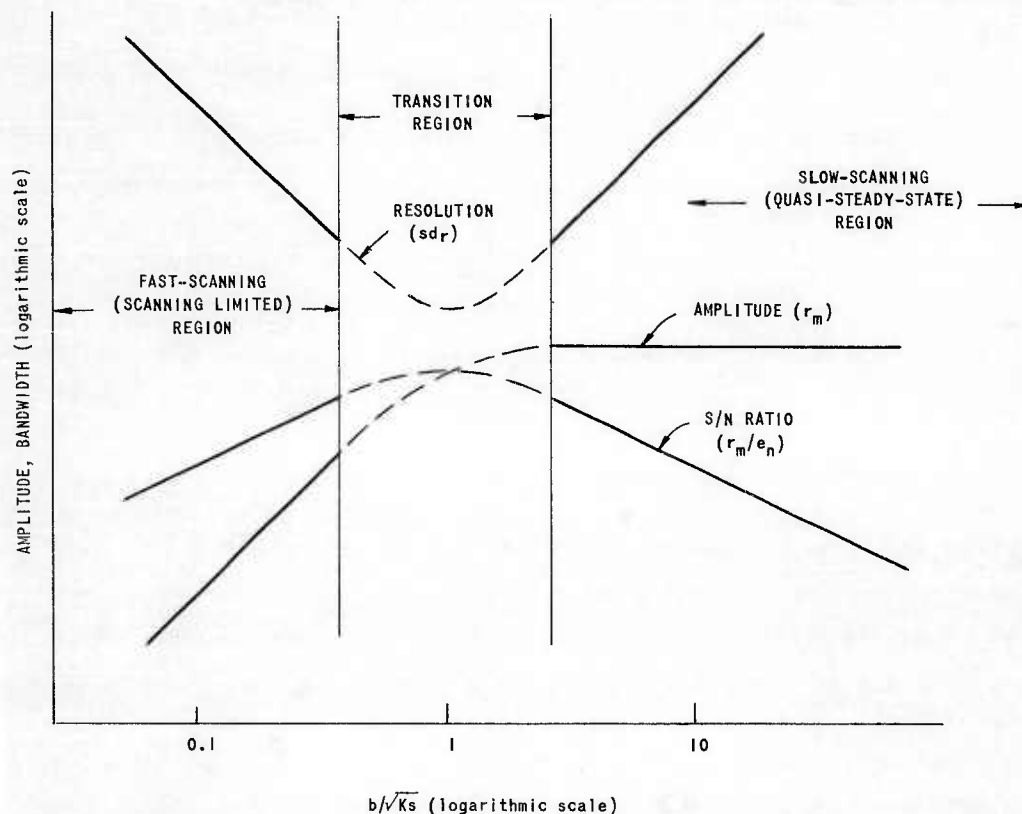


FIG. 8. VARIATION OF GLIDING TONE RESPONSE PARAMETERS WITH FILTER BANDWIDTH (FILTER PHASE IGNORED).

* Williams [Ref. Wi-1] used empirical results to provide a "rule-of-thumb" guide for design of panoramic receivers (scanning spectrum analyzers used for signal analysis). Others [Refs. Ba-5, Te-1] have used measured results to support conclusions obtained from analysis with assumed ideal bandpass filter functions.

excitation of Eq. (2.1). Various forms of the filter function $H(\omega)$ have been assumed,* for reasons which include the realization (or approximation) of a desired frequency "window" function, analytical convenience, or physical realizability of the filter. In the cited references, the filter phase function $\Phi(\omega)$ either was assumed to be constant or was assumed to vary linearly with frequency (resulting only in delay of the response), or was taken to be that normally associated with the assumed $H(\omega)$. The transition response relation of Eq. (2.4) is obtained in all cases, whether the criterion of performance is taken as achievement of best resolution for a given scan or achievement of maximum scan rate for a given filter bandwidth, or conformity with the requirement that the form of the response should not depart significantly from that of $H(\omega)$ [Refs. Ch-2, Cl-1, Va-2].

Filter response in the transition region between steady-state and impulse response is in general quite complicated. In some cases the solution is not obtained in closed form, requiring evaluation by computer [Refs. Ge-3, Ha-5] or by approximation.** In all cases, however, signal frequency is associated with the time of occurrence of a major peak of the response, and frequency resolution with the duration of the peak. While the detailed solutions have augmented the intuitive arguments regarding scanning analyzer performance and have aided in explaining observed phenomena,*** the differences noted between the gliding

* Filter functions examined in the literature include rectangular [Refs. Sa-1, Po-2], gaussian [Refs. Ba-5, Mo-1, Po-4, Sa-2], simple resonant (RLC) networks [Refs. Ba-1,5, Ha-1, Ho-1, Go-2, Le-3, Ma-3, So-1,2], and others using more complicated networks [Refs. Ek-1, Ge-1, Li-2, Te-1].

** Poincelot [Ref. Po-2] expresses the response of a rectangular bandpass shape with phase function $\phi(\omega) = \arcsin(\omega/\omega_c)$ as an infinite series of Bessel functions. Even the single RLC tuned circuit requires evaluation by Fresnel integrals, with the response expressed as a complex Cornu spiral [Ref. Ho-1].

*** Many authors have noted a "ringing" phenomenon which appears in the scanning response of filters having an oscillatory impulse response (such as an underdamped RLC resonant circuit) due to "beating" between the scanning excitation function and the decaying transient oscillation. Also mentioned [Refs. Ba-5, Ho-1, Ma-3, Sc-1] is an apparent frequency error, or peak displacement, of the response due to time delay (finite rise time) of the filter response.

tone responses of various filters have been in details, not in general conclusions regarding resolution and sensitivity.

Further investigation of scanning response of more complicated filter functions is feasible with modern computation facilities, but there seems to be little point in making such an investigation. Neither the empirical nor the analytical responses of the filters assumed in the references cited have provided much insight regarding the significance of the form of amplitude (and phase) functions, definition of potentially achievable performance, or ways in which filters can be modified in order to achieve optimum or desired results. In particular, a relationship between the form of gliding tone response of filters and the measurement of the signal spectrum, Fourier or otherwise, has not been established.*

B. THE IMPORTANCE OF PHASE

Neglect of the filter phase function $\phi(\omega)$ in the work cited above is not surprising, since the complicated nature of the gliding tone response does not encourage the additional arbitrary specification of phase. It has become increasingly evident, however, that for best scanning analyzer performance, $\phi(\omega)$ should not be neglected or assumed linear as was done in obtaining Eq. (2.4). Assumption of the phase function normally associated with commonly used bandpass filters is not conclusive, since there seems to be little basis for presuming that filters synthesized or optimized on the basis of the "steady-state" response would necessarily be appropriate for scanning excitation.

1. Time Compression

One method for specifying the $\phi(\omega)$ needed for good resolution

* An unfruitful attempt to relate the signal spectrum to the general response of filters to scanning excitation led one author to the conclusion that "It is hard to understand why scanning spectrum analyzers are as popular as they are."

has been suggested by the radar "time compression" technique,* as illustrated in Fig. 9. The scanning signal (gliding tone) may be "collapsed" in time (and increased in peak magnitude) by delaying the signal at each frequency just enough to make all of the signal energy tend to emerge from the filter at time t_1 .** If as in Eq. (2.1) the scanning is linear, then the required time delay $\tau(\omega)$ for a scan rate $s = \sigma/\pi$ is:

$$\tau(\omega) = \tau_0 - \frac{f}{s} = \tau_0 - \frac{\omega}{2\sigma}$$

where τ_0 is a fixed time delay. Since $\tau(\omega) = d\Phi/d\omega$, the phase function $\Phi(\omega)$ may be obtained by integration:

$$\Phi(\omega) = \int (\tau_0 - \frac{\omega}{2\sigma}) d\omega = \frac{-\omega^2}{4\sigma} + \tau_0\omega + \theta_0$$

where θ_0 is a phase constant. The phase function of the bandpass filter $H(\omega)$ to be used with a translation scanning spectrum analyzer should therefore not be linear, but should include a quadratic phase term $\mu\omega^2$, where

$$\mu = \frac{-1}{4\sigma} \quad (2.5)$$

*Time compression achieves increased pulse amplitude from a radar transmitter of limited power. The "chirp" technique, whereby the required large time-bandwidth product of the transmitted signal is obtained by sweeping the signal frequency, is very similar to that of scanning spectrum analyzers. Minor differences are that the chirp signal is usually of constant amplitude (to maximize power); is of known form (permitting filter matching); and need not have linear variation of frequency with time. Chirp radar has been widely analyzed [Refs. Ch-3, Co-1,2,3, Di-2, El-1, Kl-1, Ra-1] and a variety of compression techniques have been described [Refs. Ca-5, Da-1, Di-1, Du-1, Kr-1,2, Om-1,2]. Use of time compression in scanning receivers has been suggested [Refs. Kr-1, Wh-1, Wr-1].

**This is a simplified visualization; in practice, response duration can never be less than the minimum impulse duration appropriate for the filter bandwidth. This suggests that for best resolution the filter bandwidth should be as large as possible.

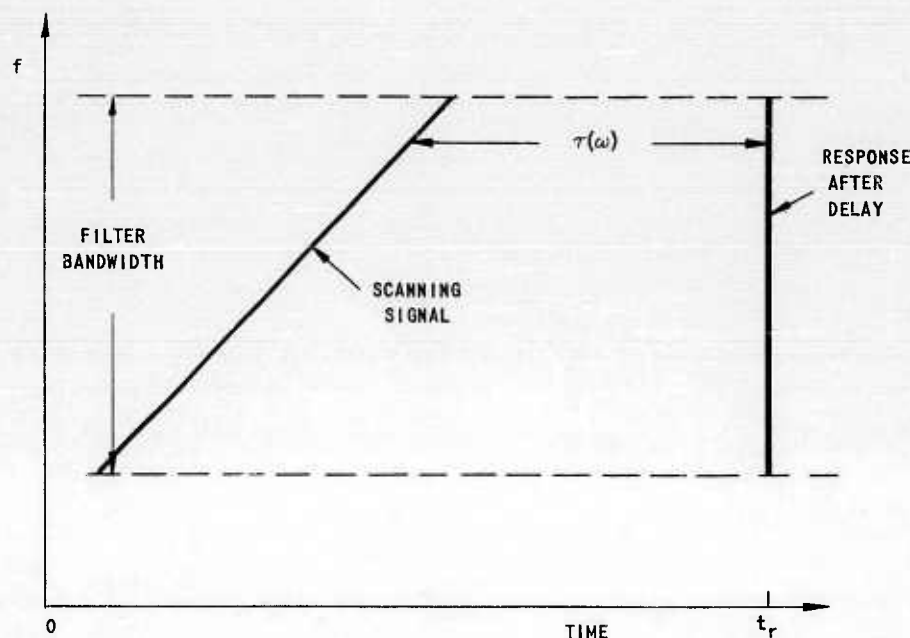


FIG. 9. TIME COMPRESSION OF A SCANNING SIGNAL BY A TIME DELAY WHICH VARIES WITH FREQUENCY.

In addition to improving resolution and sensitivity, time compression provides another benefit. The time delay $\tau(\omega)$ is matched to a particular scan rate; responses due to signals scanning at other rates will not be correctly compressed, and hence will be reduced in magnitude relative to that of the correctly scanning signal. Discrimination against the spurious responses described in Sec. I-B-2 (images, harmonic responses, and images of harmonics) is thus provided.

2. Filter Matching

The significance of $\phi(\omega)$ is also evident from matched filter theory.* It may be shown [Refs. El-1, Tu-1] that correct filter phase

*A filter $H(\omega)$ is "matched" (in the North sense) to a signal $U(\omega)$ if $H(\omega) = AU^*(\omega)e^{-j\omega\tau}$, where $U^*(\omega)$ is the complex conjugate of $U(\omega)$, and the real constants A and τ permit arbitrary amplitude and constant time delay. For a given signal the matched filter gives the best sensitivity and resolution obtainable with a linear filter. A tutorial discussion of matched filters and their properties is given by Turin [Ref. Tu-1].

response provides an increase in peak response amplitude of approximately $\sqrt{bd}$ relative to that obtained when phase is ignored, where b and d are signal bandwidth and duration, respectively. At the same time, response duration is decreased by approximately $1/bd$. Since for scanning excitation $d \approx b/s$, significant increase of sensitivity and resolution can be obtained by specification of $\phi(\omega)$ in the "slow-scanning" region of Table 1, where $bd \approx b^2/s \gg 1$.

The Fourier spectrum of a gliding tone excitation component in the form of Eq. (2.1) is [using Eq. (1.10)]

$$U_n(\omega) = \frac{a_n}{2\sqrt{s}} \left\{ \exp j \left[\frac{(\omega - \omega_n)^2}{4\sigma} - \frac{\pi}{4} \right] + \exp -j \left[\frac{(\omega + \omega_n)^2}{4\sigma} - \frac{\pi}{4} \right] \right\} \quad (2.6)$$

Filter matching in this case would require a filter with an infinite bandwidth, as a result of assumed infinite duration of $u_n(t)$. Furthermore, either of the exponential factors of Eq. (2.6) may be matched with a conjugate filter, but not both. The significance of this will be clarified in Section III; for the moment, we may note that matching of the first term of Eq. (2.6) would require a filter with quadratic phase coefficient $\mu = -1/4\sigma$, which agrees with the result obtained by the time compression argument in Eq. (2.5).

Since the spectrum $U(\omega)$ of an arbitrary input signal is not only usually unknown but also may differ for various signals, matching of the filter function $H(\omega)$ to $U(\omega)$ as modified by the translation process is not possible in general. Representation of $U(\omega)$ by component functions $V(\omega)$ of known form as in Eq. (1.4), however, permits performance optimization in the sense that $H(\omega)$ may be matched to $V(\omega)$ as modified by scanning. Furthermore, selection of the scan rate s provides additional freedom in adapting scanning spectrum analysis to a particular signal situation.

C. SUMMARY OF PREVIOUS ANALYSIS

For over thirty years scanning spectrum analyzers have been widely

used for frequency measurement, with their design based on the accepted performance limitation expressed by Eq. (2.4). The form of the scanning spectrum analyzer "window" has been assumed to be closely associated with that of the magnitude function of the bandpass filter in the analyzer.

Improvement of resolution is invariably attempted in the references cited by making the filter bandwidth as small as permitted by the scan rate, with the result that signal energy is lost in direct proportion to the ratio of frequency resolution to the total range of frequency examined. At low scan rates, where signal duration may be small compared to the scan period, the loss of signal energy is manifested as a reduced probability of signal intercept. If the scan rate is high enough to traverse the total frequency range within the signal duration, loss of sensitivity and resolution results. In either case, the representation of the signal spectrum is at best an approximation.

Two arguments have suggested that Eq. (2.4) is not valid if filter phase is correctly chosen. The key to improved scanning spectrum analyzer performance lies in the use of the appropriate quadratic phase coefficient and increased bandwidth of the analyzer filter. A long neglected clue leading toward this conclusion is that the quadratic phase terms are essential in describing the scanning process, as may be noted in Eqs. (1.8) and (1.10). A second clue is that increased filter bandwidth utilizes more signal energy where the response is limited by scan rate.* A third clue may be seen in the emergence of the impulse response at high scan rates, since the duration of response (hence resolution) can never be less than that permitted by the minimum impulse response duration as determined by filter bandwidth.**

In the next chapter we examine potential scanning spectrum analyzer performance that results when phase is considered.

* It may also be argued that translation by scanning necessarily increases signal spectrum bandwidth, particularly when the frequency range scanned within the signal duration exceeds signal bandwidth prior to translation; thus, high scan rates imply the use of analyzer filters with wide bandwidth if signal energy is to be retained.

** Batten, et al [Ref. Ba-5] used energy arguments to show that best sensitivity accompanies best resolution and that response amplitude is reduced when "apparent" bandwidth (resolution) exceeds filter bandwidth. They did not explore the alternative possibility of a peak power "gain" when filter bandwidth exceeds resolution bandwidth.

III. THE PHASE-MATCHED TRANSLATION-SCANNING SPECTRUM ANALYZER

Figure 10 illustrates the arrangement and terminology to be used. An input signal $u_1(t)$ with spectrum $U_1(\omega)$ is "mixed" in a nonlinear device with a scanning reference time function $u_2(t)$. The resulting translated signal $u_3(t)$ is applied to a bandpass filter with spectrum function $H(\omega)$ to produce the response time function $r(t)$.

A. THE GENERAL ANALYZER RESPONSE

The purpose of spectrum analysis is to infer characteristics of $U_1(\omega)$ by examination of $r(t)$. To do this an expression relating $r(t)$ and $U_1(\omega)$ must be obtained. Knowledge of the effect of $H(\omega)$ on this relationship is needed in order to permit analyzer design for desired performance.

1. The Input Signal

We require that $u_1(t)$ be a real time function with finite energy, realizable as a physical quantity. The associated Fourier spectrum $U_1(\omega)$ is assumed to have magnitude $A_1(\omega)$ and phase $\phi_1(\omega)$:

$$U_1(\omega) = A_1(\omega)e^{j\phi_1(\omega)} \quad (3.1a)$$

Since $u_1(t)$ is real, $U_1(\omega)$ is Hermitian; that is:

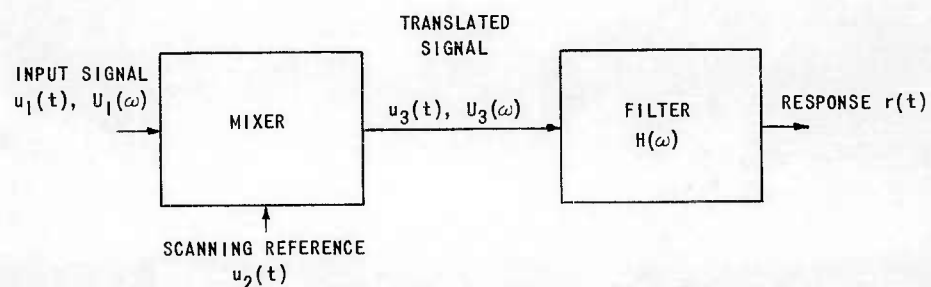


FIG. 10. BLOCK DIAGRAM OF A TRANSLATION-SCANNING ANALYZER.

$$A_1(\omega) = A_1(-\omega)$$

$$\Phi_1(\omega) = -\Phi_1(-\omega) \quad (3.1b)$$

The region of frequency within which $U_1(\omega)$ has significant magnitude is assumed to be contained within the bandwidth B_1 , as illustrated in Fig. 11. Let a signal spectrum component be centered at f_1 within B_1 with maximum amplitude A_{1m} , bandwidth b_1 , and energy E_1 . The associated component of the time function (which will be an oscillation at a frequency of approximately f_1 with an envelope of finite duration) is assumed to have peak amplitude a_{1m} and duration d_1 . Energy definitions of bandwidth and duration are used (Appendix A) so that

$$E_1 = a_{1m}^2 d_1 = A_{1m}^2 b_1 \quad (3.2)$$

2. Spectrum Translation

The mixer shown in Fig. 10 is assumed to be a nonlinear device which multiplies $u_1(t)$ and $u_2(t)$. The scanning reference time function $u_2(t)$ is represented as

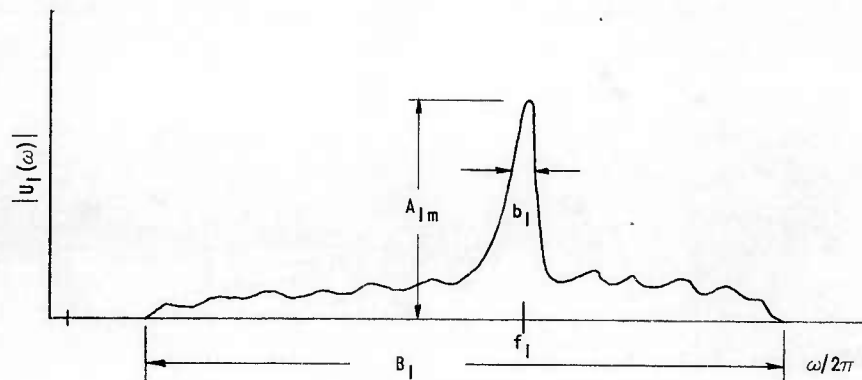


FIG. 11. TYPICAL INPUT SIGNAL SPECTRUM (ONLY POSITIVE SPECTRUM SHOWN).

$$u_2(t) = \cos(\sigma_1 t^2 + \theta_1) \quad (3.3)$$

with scan rate $s_1 = \sigma_1/\pi$ and arbitrary phase constant θ_1 . The mixer output $u_3(t) = u_1(t)u_2(t)$ becomes*

$$u_3(t) = u_1(t) \cos(\sigma_1 t^2 + \theta_1) \quad (3.4)$$

The simple mixing assumed here will result in two scanning components, corresponding respectively to the sum and difference frequencies. This may be illustrated by letting $u_1(t) = a_1(t) \cos(\omega_1 t)$. In this case

$$2u_3(t) = a_1(t) [\cos(\omega_1 t + \sigma_1 t^2 + \theta_1) + \cos(\omega_1 t - \sigma_1 t^2 - \theta_1)] \quad (3.5)$$

We may expect, therefore, that $r(t)$ will include two components, one associated with a component of $u_3(t)$ with scan rate s_1 , the other with scan rate $-s_1$.** The latter may be identified as an "image" response.

3. The Analyzer Filter

Let $H(\omega)$ be the filter spectrum function with arbitrary band-pass shape $A_2(\omega)$ and phase terms up to second order. As with $u_1(t)$, we require that $A_2(\omega)$ have an inverse Fourier transform (impulse response) $a_2(t)$ that is real and physically realizable.*** For convenience, let $H(\omega)$ be centered around a frequency ω_2 with a quadratic phase parameter μ assumed for reasons discussed in Section II:

$$\begin{aligned} 2H(\omega) = & A_2(\omega - \omega_2) \exp j[\mu(\omega - \omega_2)^2 - \theta_2] \\ & + A_2(\omega + \omega_2) \exp j[-\mu(\omega + \omega_2)^2 + \theta_2] \end{aligned} \quad (3.6)$$

* Losses incurred in the mixing process will be included in the filter function $H(\omega)$.

** In theory, signal translation can result in only one scanning component (e.g. by "single sideband", or "serrodyne" modulation techniques); in practice, attempted cancellation of the second component is not perfect and it will always be present (giving rise to "images"), although perhaps greatly reduced in magnitude.

*** Realizable in the limit as the number of filter elements because infinite, or by addition of time delay [Ref. Va-1].

where θ_2 is a phase constant, and ω_2 , μ , and θ_2 are real. Let A_{2m} be the maximum magnitude at f_2 and b_2 be the "energy" bandwidth of the filter defined as

$$b_2 = \frac{\int_0^{\infty} A_2^2(\omega) d\omega}{2\pi A_2^2(\omega)} \quad (3.6a)$$

The impulse response function $a_2(t)$ associated with $A_2(\omega)$ has a duration d_2 and peak magnitude a_{2m} which occurs at time τ_2 . The various filter parameters are illustrated in Fig. 12.

4. General Analyzer Response

The response $r(t)$ is obtained by taking the inverse Fourier transform of the product of the translated spectrum function $U_3(\omega)$ and the filter function $H(\omega)$; hence:

$$r(t) = \mathcal{F}^{-1}\{H(\omega) \mathcal{F}[u_1(t) u_2(t)]\} \quad (3.7)$$

The details of obtaining $r(t)$ in a form convenient for interpretation are given in Appendix B. Defining a phase-matching parameter η as

$$\eta = -4\mu\sigma \quad (3.8)$$

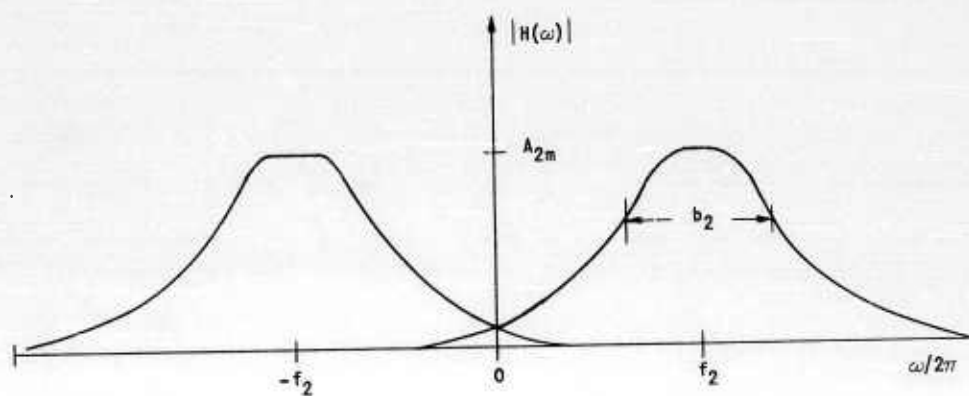
and denoting convolution of $a_1(t)$ and $a_2(t)$ by

$$a_1(t) * a_2(t) \equiv \int_{-\infty}^{\infty} a_1(t_1) a_2(t - t_1) dt_1 \quad (3.9)$$

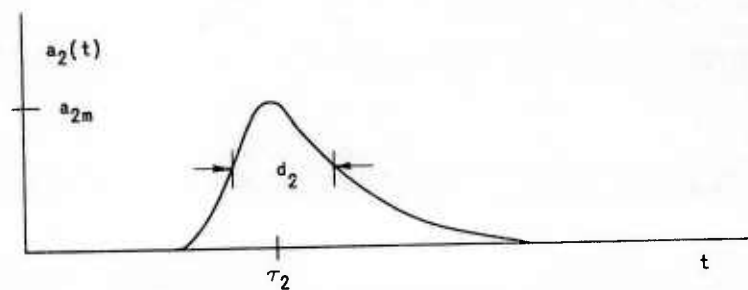
(where t_1 is a dummy variable of integration), the general response may be written as follows:

$$(a) \quad r(t) = r_1(t) + r_2(t) \quad (3.10)$$

(b) For $\eta = 1$:



a. Spectrum magnitude function of filter



b. Impulse response envelope function associated with spectrum magnitude function of filter

FIG. 12. PARAMETERS ASSOCIATED WITH ANALYZER FILTER SPECTRUM FUNCTION.

$$r_1(t - \tau_2) = \frac{\sqrt{s_1}}{2} a_2(t) * \left\{ A_1(2\sigma_1 t) \cos[\sigma_1 t^2 + \Phi_1(2\sigma_1 t) - \theta_{31} - \frac{\pi}{4}] \right\}$$

(c) For $\eta \neq 1$

$$r_1(t - \tau_2) = \frac{s_1 a_2(t)}{2 \sqrt{1-\eta}} * \left\{ A_1(2\sigma_1 t) \cos[\sigma_1 t^2 + \Phi_1(2\sigma_1 t) - \theta_{31}] * \cos\left(\frac{\sigma_1 t^2}{1-\eta}\right) \right. \\ \left. + A_1(2\sigma_1 t) \sin[\sigma_1 t^2 + \Phi_1(2\sigma_1 t) - \theta_{31}] * \sin\left(\frac{\sigma_1 t^2}{1-\eta}\right) \right\}$$

(d) For $\eta = -1$:

$$r_2(t - \tau_2) = \frac{\sqrt{s_1}}{2} a_2(t) * \left\{ A_1(2\sigma_1 t) \cos[\sigma_1 t^2 + \Phi_1(2\sigma_1 t) - \theta_{32} - \frac{\pi}{4}] \right\}$$

(e) For $\eta \neq -1$:

$$r_2(t - \tau_2) = \frac{s_1 a_2(t)}{2 \sqrt{1+\eta}} * \left\{ A_1(2\sigma_1 t) \cos[\sigma_1 t^2 + \Phi_1(2\sigma_1 t) - \theta_{32}] * \cos\left(\frac{\sigma_1 t^2}{1+\eta}\right) \right. \\ \left. + A_1(2\sigma_1 t) \sin[\sigma_1 t^2 + \Phi_1(2\sigma_1 t) - \theta_{32}] * \sin\left(\frac{\sigma_1 t^2}{1+\eta}\right) \right\}$$

(3.10)

where

$$\tau_2 = \frac{f_2}{s}$$

$$\theta_{31} = \theta_1 + \theta_2 - \frac{\omega_2^2}{4\sigma}$$

$$\theta_{32} = \theta_1 - \theta_2 + \frac{\omega_2^2}{4\sigma}$$

As anticipated in Section II, two components of the output response appear. Phase-matching of $r_1(t)$ occurs when $\eta = 1$, which corresponds to the condition of Eq. (2.5) for optimum response. Since $r_2(t)$ is matched when $\eta = -1$, indicating scanning in the reverse direction, $r_2(t)$ may be identified as the "image" response of $r_1(t)$ (and vice versa). Interpretations of various values of the parameter η are indicated in Table 2.

Table 2 - Significance of the Phase-Matching Parameter

η	Interpretation
0	Linear-phase filter response ($\mu = 0$); $ r_{1m} = r_{2m} $
1	$r_1(t)$ matched, $r_2(t)$ an image
-1	$r_2(t)$ matched, $r_1(t)$ an image
2, 3, 4, ..	$\left\{ \begin{array}{l} r_1(t) \text{ gives harmonic responses,} \\ r_2(t) \text{ gives harmonic image responses} \end{array} \right.$
$1 + \epsilon$ $(\epsilon \ll 1)$	$\left\{ \begin{array}{l} r_1(t) \text{ almost matched; reveals effect of small} \\ \text{error in phase characteristic or scan rate} \end{array} \right.$

Let d_{31} , d_{32} be the durations and r_{1m} , r_{2m} the maximum amplitudes of the response components $r_1(t)$ and $r_2(t)$. These parameters relate to the resolution and sensitivity of the scanning spectrum analyzer, and are useful in evaluating performance limitations by interpretation of Eq. (3.10) in the remainder of the section.

B. INTERPRETATION OF THE GENERAL RESPONSE

The general analyzer response of Eq. (3.10) may be examined qualitatively by recalling the nature of convolution as a scanning, or "sifting" process.*

*The discussion in this section is intended only to suggest insight, not to be definitive or rigorous.

1. Convolution Relationships

The convolution of two "well-behaved" peaked functions having different widths resembles the wider of the two functions in form and width, and has a magnitude scale factor determined by the area of the narrower of the two functions. An example to illustrate these properties is shown in Fig. 13.

Let $g_3(t)$ be the convolution of $g_1(t)$ and $g_2(t)$:

$$g_3(t) = g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(t - \tau) g_2(\tau) d\tau \quad (3.11)$$

The parameters of width (duration) Δt , maximum amplitude g_m , and position (time) of the maximum amplitude are as indicated in Fig. 13. If $\Delta t_2 \ll \Delta t_1$ and $g_1(t)$ is "slowly varying" (essentially constant in magnitude in the interval Δt_2), then *

$$g_3(t) \approx g_1(t - t_2) \int_{-\infty}^{\infty} g_2(\tau) d\tau = g_{2m} \Delta t_2 g_1(t - t_2) \quad (3.12)$$

from which

$$\begin{aligned} (a) \quad g_{3m} &\approx g_{1m} g_{2m} \Delta t_2 \\ (b) \quad \Delta t_3 &\approx \Delta t_1 \\ (c) \quad t_3 &\approx t_1 + t_2 \end{aligned} \quad (3.13)$$

The approximations improve as $\Delta t_2 / \Delta t_1 \rightarrow 0$. If by definition

$$\int_{-\infty}^{\infty} g_2(\tau) d\tau = g_{2m} \Delta t_2 = G_2$$

* Function "width" used in this section refers to the width of the equivalent rectangular area, not to the "energy" width used previously in connection with the time-bandwidth factor (see Appendix A). In general, the two widths are not necessarily equal for the same function, although in this work they are assumed to be of the same order of magnitude.

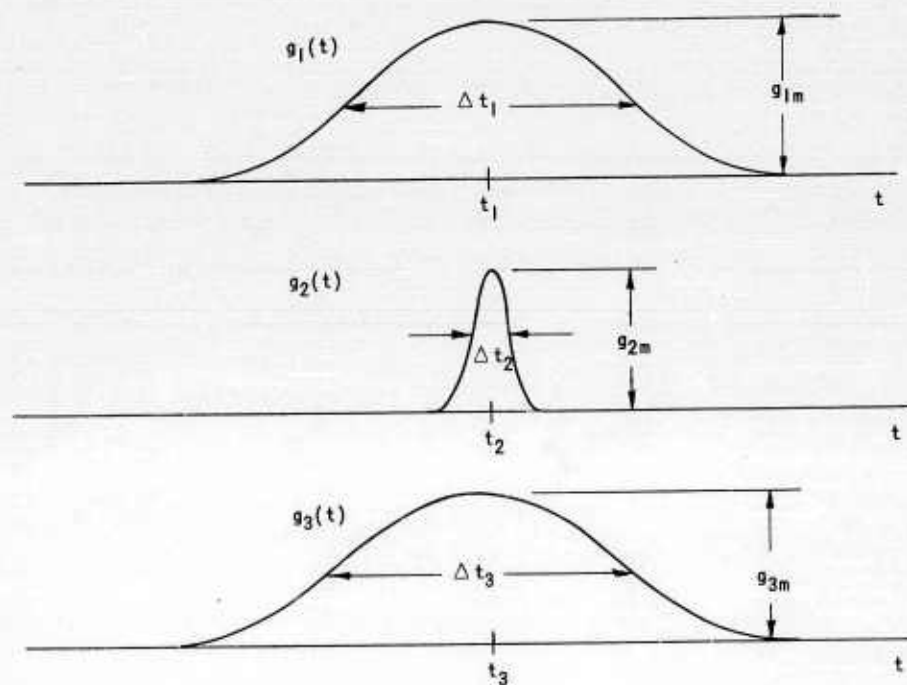


FIG. 13. AN ILLUSTRATION OF CONVOLUTION.

then

$$\lim_{\Delta t_2 \rightarrow 0} g_2(t) = G_2 \delta(t - t_2)$$

where $\delta(t)$ is the Dirac impulse function with zero width and unit area located at $t = 0$. In this case

$$\lim_{\Delta t_2 \rightarrow 0} g_3(t) = g_1(t) * G_2 \delta(t - t_2) = G_2 g_1(t - t_2) \quad (3.14)$$

and, in the limit,

$$(a) \quad g_{3m} = g_{1m} G_2$$

$$(b) \quad \Delta t_3 = \Delta t_1$$

$$(c) \quad t_3 = t_1 + t_2 \quad (3.15)$$

The situation becomes slightly more complicated if $g_1(t)$ and $g_2(t)$ are the "envelopes" of sinusoidal time functions $y_1(t)$ and $y_2(t)$, respectively. If at least several cycles are contained within the envelope of shorter duration, integration of their product gives significant magnitude only in a region where the oscillations are "in phase" over several cycles.* Scanning of the frequencies of the two functions at different rates insures that such a region will exist and will be finite in extent.

In Appendix C the effective width (duration) d_s of the region of stationary relative phase for two linearly scanning sinusoidal functions is interpreted to be

$$d_s = \frac{1}{\sqrt{s}} \quad (3.16)$$

where $s = ||s_1| - |s_2||$ is the rate of scanning of one sinusoidal function frequency relative to the other. The region is centered around the time t_s when the two scanning frequencies are equal. Letting $g_3(t)$ be the envelope function of $y_3(t) = y_1(t) * y_2(t)$, the following approximate relations may be written for the case where $t_s \approx t_1 + t_2$ (which gives maximum convolution amplitude by requiring that the region of stationary phase lie within or overlap the envelope peaks):

If $\Delta t_1 > \Delta t_2 \gg \frac{1}{\sqrt{s}}$ (scanning-determined response)

$$(a) \quad |y_3(t)|_{\max} \cong g_{3m} \cong \frac{g_{1m} g_{2m}}{\sqrt{s}}$$

$$(b) \quad \Delta t_3 \cong \Delta t_1 \quad (3.17)$$

* This is essentially a statement of Kelvin's principle of stationary phase [Ref. Be-1].

If $\frac{1}{\sqrt{s}} \gg \Delta t_1 > \Delta t_2$ (response essentially unaffected by scanning)

$$\begin{aligned} (a) \quad & |y_3(t)| \cong G_2 |y_1(t - t_2)| \\ (b) \quad & g_{3m} \cong g_{1m} g_{2m} \Delta t_2 = g_{1m} G_2 \\ (c) \quad & \Delta t_3 \cong \Delta t_1 \end{aligned} \tag{3.18}$$

Also noted in Appendix C is the scan rate s_3 of $y_3(t)$:

$$s_3 = \frac{s_1 s_2}{s_1 + s_2} \tag{3.19}$$

Turning now to the general response $r(t)$ of Eq. (3.10), two convolved envelope time functions are to be considered. One has the form of the input signal spectrum magnitude $A_1(\omega)$, appearing in Eq. (3.10) as a time function with a scale factor $\omega/2\sigma_1 = f/s_1$. Referring to Fig. 11, a typical component of $A_1(\omega)$ to be resolved has a magnitude A_{1m} , duration $d_1 = b_1/s_1$ and time of occurrence t_1 , where

$$t_1 = \tau_2 \pm \frac{f_1}{s_1} \tag{3.20}$$

The other convolved time function is the impulse response $a_2(t)$ associated with the filter amplitude characteristic $A_2(\omega)$, with maximum amplitude a_{2m} and duration d_2 ; from Appendix A,

$$\begin{aligned} (a) \quad & a_{2m} = \sqrt{\frac{2}{K_2}} b_2 A_{2m} \\ (b) \quad & d_2 = \frac{K_2}{b_2} \end{aligned} \tag{3.21}$$

where b_2 is the filter bandwidth in cps, and K_2 is the time-bandwidth factor associated with the form of $A_2(\omega)$.

The relative scan rates of the various convolutions indicated in Eq. (3.10) affect the nature of the response. Since in the formulation of Eq. (3.6) $a_2(t)$ is assumed to be a nonscanning function, the

relative scan rate in the "phase-matched" cases of Eqs. (3.10b) and (3.10d) is simply s_1 . In the interior convolutions of the "non-matched" responses of Eqs. (3.10c) and (3.10e) the relative scan rates are, respectively

$$\begin{aligned} \text{(a)} \quad |s_{nm1}| &= \frac{\eta s_1}{\eta - 1} \\ \text{(b)} \quad |s_{nm2}| &= \frac{\eta s_1}{\eta + 1} \end{aligned} \quad (3.22)$$

The form of Eq. (3.10), [hence interpretation of $r(t)$], is determined by relationships between d_1 , d_2 , and the various scan rates of the convolved functions.

2. Signal-Determined Response

In theory, the duration d_2 of $a_2(t)$ in Eq. (3.10) may be made arbitrarily small by increasing filter bandwidth, since $d_2 = K_2/b_2$. If d_2 is small enough to resolve details of $A_1(2\sigma_1 t)$ and if the region of stationary phase is sufficiently large (and overlaps B_1), then $r_1(t)$ of Eq. (3.10b) may be approximated (using the convolution arguments of the preceding section):

$$r_1(t - \tau_2) \cong \sqrt{\frac{K_2 s_1}{2}} A_{2m} A_1(2\sigma_1 t) \cos[\sigma_1 t^2 + \phi(2\sigma_1 t) - \theta_{31} - \frac{\pi}{4}] \quad (3.23)$$

The envelope of $r_1(t)$ approximates the form of $A_1(2\sigma_1 t)$, which is the signal spectrum magnitude function $A_1(\omega)$ expressed as a time function. In this case the response may be said to be signal-determined; the goal of spectrum analysis has been accomplished in that $A_1(\omega)$ is expressed as a time function at the output of the spectrum analyzer.

The conditions under which Eq. (3.23) is valid are as follows:

(a) The resolution requirement that $d_2 < b_1/s_1$ may be written, using Eq. (3.21),

$$s_1 < \frac{b_1 b_2}{K_2} \quad (3.24a)$$

and may be satisfied by making the filter bandwidth large.

(b) If second and higher order terms of $\phi_1(\omega)$ are negligible,* the duration $d_s = s_1^{-1/2}$ of the stationary phase region may be made large, compared to the time interval B_1/s_1 required to scan across the signal spectrum, by requiring that

$$s_1 > B_1^2 \quad (3.24b)$$

This condition may be satisfied by making the scan rate large.

As the filter bandwidth b_2 is increased to validate Eq. (3.23), the time required to scan across both b_2 and B_1 also increases unless the scan rate is increased as well. Therefore, let

$$s_1 = \frac{B_1 + b_2}{d_1} \quad (3.25)$$

where d_1 is the minimum expected signal duration and hence the maximum permitted measurement time. The resolution $s_1 d_2$ of the analyzer becomes

$$s_1 d_2 = \frac{K_2}{d_1} \left(1 + \frac{B_1}{b_2} \right) \quad (3.26)$$

Resolution is improved by making $b_2 \gg B_1$ and by choosing the form of $A_2(\omega)$ to give a minimum K_2 . In the limit as $B_1/b_2 \rightarrow 0$ resolution is determined by the duration d_1 of the signal sample.

Limiting magnitudes of spurious (non phase-matched) responses for the signal-determined case may be evaluated by use of Eq. (3.10c). Let r_{1sm} be the maximum amplitude of a spurious response due to a signal component of bandwidth b_1 and amplitude A_{1m} when $\eta \neq 1$. Using the arguments leading to Eq. (3.18),

* Since $a_2(t)$ is non-scanning by definition, the relative scan rate of the convolved functions in Eq. (3.10a) is the second derivative of the cosine argument, hence is

$$s_r = s_1 + \frac{1}{2\pi} \frac{d^2}{dt^2} [\phi_1(2\sigma_1 t)]$$

When the second and higher order phase terms of $\phi_1(\omega)$ are small, $s_r \approx s_1$.

$$r_{lsm} \approx \sqrt{\frac{K_2}{2|\eta-1|}} A_{1m} A_{2m} b_1 \quad \eta \neq 1 \quad (3.27a)$$

with this magnitude approached when

$$s_1 > \left| \frac{\eta}{\eta-1} \right| b_1^2 \quad \eta \neq 1 \quad (3.27b)$$

Comparing Eqs. (3.23) and (3.27), the ratio of maximum phase-matched response to maximum non-phase-matched response when Eqs. (3.24) and (3.27b) are valid becomes

$$\frac{r_{lm}}{r_{lsm}} \approx \frac{1}{b_1} |(\eta-1)s_1|^{1/2} \quad (3.28)$$

Image and harmonic responses of the analyzer may, in theory, be made arbitrarily small relative to the desired response by increasing the scan rate (and, by Eq. (3.25), the filter bandwidth). The limiting effect of a small error in matching of the filter phase to the scan rate (or vice versa) is also indicated by Eq. (3.27) and (3.28).

The ratio of peak response magnitude to rms noise voltage at the output of the analyzer is given by Eq. (1.14), using Eqs. (1.15), (3.23), and (3.25):

$$\left(\frac{r_{lm}}{e_n} \right)^2 \approx \frac{E_1}{4kFT} \cdot \frac{K_2}{K_1} \cdot \left(\frac{B_1 + b_2}{b_n} \right) \quad (3.29)$$

If the noise is determined by the analyzer filter bandwidth ($b_n = b_2$), sensitivity becomes independent of scan rate and analyzer bandwidth when $b_2 \gg B_1$.

3. Filter-Determined Response

Frequently, signal spectrum components exist which have narrower bandwidths than the resolution capability of the spectrum analyzer.* In this case the analyzer filter primarily determines the form of the

* This would be the case, for example, if the signal is represented by the discrete Fourier spectrum of Eq. (1.4), or in the gliding tone discussion of Section II.

analyzer response. If $d_2 > b_1/s_1$ and the region of stationary phase overlaps d_2 , Eq. (3.10b) may be written (again using the convolution arguments of Section B-1):

$$r_1(t - \tau_2) \cong \sqrt{\frac{K_1}{8s_1}} a_{1m} \left\{ a_2 \left(t - \frac{f_1}{s_1} \right) \cos[\Phi_1(\omega_1) - \theta_4] \right. \\ \left. + a_2 \left(t + \frac{f_1}{s_1} \right) \cos[\Phi_1(\omega_1) + \theta_4] \right\} \quad (3.30)$$

where

$$\theta_4 = \theta_1 + \theta_2 + \frac{\pi}{4} - \frac{\omega_1^2 + \omega_2^2}{4\sigma_1}$$

The form of the response function is $a_2(t)$, which as defined above is the inverse Fourier transform (impulse response) of the analyzer filter magnitude function $A_2(\omega)$. We therefore have the important result that the form of the spectrum "window" of the phase-matched scanning spectrum analyzer approaches that of the impulse response associated with the analyzer filter bandpass shape, not that of the filter bandpass shape itself as has been so long assumed in the literature on the "gliding tone" problem. Design of a spectrum analyzer having a desired form of scanning window is accomplished by selection of the filter bandpass shape having the desired impulse response.

Let D_1 be the measurement interval during which both filter bandwidth b_2 and overall spectrum width B_1 are traversed by scanning so that

$$s_1 = \frac{B_1 + b_2}{D_1} \quad (3.31)$$

The condition of spectrum component width $b_1 < s_1 d_2$ and stationary phase $d_2 < s_1^{-1/2}$ for which Eq. (3.30) is valid are:

$$(a) \quad s_1 > \frac{b_1 b_2}{K_2}$$

$$(b) \quad D_1 > \frac{K_2^2}{b_2} \left(1 + \frac{B_1}{b_2} \right) \quad (3.32)$$

The frequency resolution provided by the scanning window is:

$$s_1 d_2 = \frac{K_2}{D_1} \left(1 + \frac{B_1}{b_2} \right) \quad (3.33)$$

As before, resolution is improved by choosing $b_2 \gg B_1$ [with Eq. (3.32a) consequently requiring a high scan rate] and a filter function with minimum K_2 . Limiting resolution is determined by the duration of measurement D_1 .

Maximum magnitudes of spurious responses resulting when $\eta \neq 1$ are obtained as before from Eq. (3.10c):

$$r_{1sm} \approx \frac{K_2 a_{1m} a_{2m}}{2b_2 \sqrt{|\eta-1|}} \quad (3.34a)$$

This value is approached when

$$b_2 > K_2 \left| \frac{s_1}{\eta-1} \right|^{1/2} \quad (3.34b)$$

The limiting ratio of matched to nonmatched maximum amplitudes becomes:

$$\frac{r_{1m}}{r_{1sm}} \approx \frac{b_2}{K_2} \left[\frac{K_1}{2} \left| \frac{\eta-1}{s_1} \right| \right]^{1/2} \quad (3.35)$$

Again, image and harmonic responses of the analyzer may in theory be made arbitrarily small by increasing filter bandwidth and analyzer scan rate.

Signal-to-noise ratio in the filter-determined analyzer response case is obtained from Eq. (1.14) using Eqs. (1.15), (3.30), and (3.31):

$$\left(\frac{r_{1m}}{e_n} \right)^2 \approx \frac{1}{FkT} \cdot \frac{a_{1m}^2 D_1}{K_1 K_2 R} \cdot \frac{b_2^2}{b_n (B_1 + b_2)} \quad (3.36)$$

If noise is determined by the filter bandwidth ($b_n = b_2$) and if $b_2 \gg B_1$, signal-to-noise ratio is limited only by input signal power and by the duration of the measurement.

4. Matched Bandwidth Response

Since in general the form of a signal spectrum to be analyzed is unknown or known to differ for various signals, the analyzer bandwidth cannot be optimized for all situations. The analyzer may be designed, however, to optimize the response to a signal representation by a set of elementary signals of known form, as suggested in Section I-A-2.

As in Eq. (1.5), let the signal spectrum $U_1(\omega)$ be:

$$U_1(\omega) = \sum_i \sum_n A_{in} [V_1(\omega - \omega_i) + V_1(\omega + \omega_i)] e^{-jn\omega T_1} \quad (3.37)$$

where $\omega_n = 2n\pi/T_1$ are the component frequencies based on a common time interval T_1 between the successive elementary time signals $v_1(t)$ that add to form $u_1(t)$. If the region of stationary phase is wider than the bandwidth b_1 of the component spectrum functions $V_1(\omega)$ i.e. if $b_1^2 < |s_1|$, then by Eq. (3.9b):

$$r_1(t - \tau_2) \approx a_\theta \sqrt{s_1} a_2(t) * \sum_n \sum_p A_{np} [V_1(2\sigma_1 t - \omega_n) + V_1(2\sigma_1 t + \omega_n)] \quad (3.38)$$

where

$$a_\theta = 2 \cos(\theta_1 + \theta_2 + \frac{\pi}{4})$$

The analyzer filter is "matched" to $V_1(\omega)$ by selecting the filter spectrum magnitude function $A_2(\omega)$ to have an inverse Fourier transform

$$a_2(t) = a_{2m} V(-2\sigma_1 t) \quad (3.39)$$

The phase-matched response then becomes:

$$r(t - \tau_2) = \frac{a_{\theta} \sqrt{s_1}}{a_{2m}} \sum_n \sum_p A_{np} a_2(t) * \left[a_2\left(-t + \frac{f_n}{s_1}\right) + a_2\left(-t - \frac{f_n}{s_1}\right) \right] \quad (3.40)$$

In this expression, $a_2(t) * a_2(-t)$ is the autocorrelation function (abbreviated ACF) of $a_2(t)$, and hence of $A_1(2\sigma t)$. The ACF is characterized by the properties of maximum peak amplitude and minimum duration for given function energy that lead to the optimum characteristics of North "matched" conjugate filters [Refs. Tu-1, Ly-1].

Equation (3.39) may thus be taken as defining an optimization procedure for scanning spectrum analysis, to the extent that the signal representation of Eq. (3.37) is appropriate. The measurement of the signal spectrum is still an approximation, limited by available signal energy or measurement duration, but the approximation now lies in the signal representation by the component functions, not in the component determination. As before, sensitivity, resolution, and ratio of desired to spurious responses may be made independent of analyzer scan rate and filter bandwidth when the latter are made sufficiently large.

C. COMPARISON OF THE COMB FILTER AND THE SCANNING SPECTRUM ANALYZERS

It is interesting to compare the performance of the phase-matched scanning spectrum analyzer with that of the comb filter non-scanning analyzer described in Section I-B-3. Let the analyzer input signal be the single sinusoidal component $u_1(t)$ of Eq. (1.7) with constant peak magnitude a_1 and frequency f_1 . The Fourier spectrum $U_1(\omega)$ is given by Eq. (1.9).

The frequency resolution and sensitivity of the comb filter analyzer are essentially determined by the bandwidth b_c of the component filter elements of the set. The time required for measurement may be taken as the risetime of the step response (impulse duration) $d_c = K_c/b_c$ of a filter element, where K_c is the time-bandwidth factor of the filter. Letting frequency resolution be defined as the bandwidth b_c ,

$$b_c = \frac{K_c}{d_c} \quad (3.41)$$

Sensitivity may be expressed in terms of the signal-to-noise ratio at the output of the filter using Eq. (1.14):

$$\left(\frac{r_{cm}}{e_n}\right)^2 = \frac{a_0^2 d_c}{4kTRK_c} \quad (3.42)$$

where r_{cm} is the peak response magnitude and the other factors are defined as in Section I (assuming noise bandwidth $b_n = b_c$).

The performance parameters of the scanning spectrum analyzer are given by the filter-determined response case of Section III-B-3. Best performance is given by Eq. (3.30) when the conditions of Eq. (3.32) are satisfied. Frequency resolution is determined by response duration d_2 and scan rate s_1 as given by Eq. (3.33):

$$d_2 s_1 = \frac{K_2}{D_1} \left(1 + \frac{B_1}{b_2}\right) \quad (3.43)$$

The signal-to-noise ratio becomes

$$\left(\frac{r_{2m}}{e_n}\right)^2 = \left(\frac{a_0^2 D_1}{8fkTRK_2}\right) \left(\frac{b_2}{B_1 + b_2}\right) \quad (3.44)$$

where again the noise bandwidth $b_n = b_2$. Two responses occur separated in time by $2f_1/s_1$, corresponding to the positive and negative spectrum lines.

Comparing Eqs. (3.41) and (3.42) with (3.43) and (3.44) respectively by letting $K_c = K_2$ (the spectrum windows having the same shape) and $d_c = D_1$ (requiring both analyzers to have the same time allowed for measurement of frequency), the resolution and sensitivity are seen to be comparable when $b_2 \gg B_1$ (i.e., when the bandwidth of the scanning analyzer filter is large compared to the bandwidth of the spectrum to be analyzed). There is a 3 db loss in sensitivity in the case of the scanning analyzer due to image band noise and the representation of the positive and negative frequency spectrum components as separate responses; in a sense the double response of the scanning analyzer might be

considered to be a more precise representation of the Fourier spectrum.

In theory, therefore, the scanning spectrum analyzer is capable of performance that is essentially equivalent to that of the comb filter arrangement. While it is true that the latter provides a multidimensional output which simultaneously monitors signal variation in a set of frequency channels, an analogous arrangement could be accomplished with a set of scanning filters or translation-scanning analyzers displaced in time. The significant difference between the two methods appears to be that the comb filter provides continuous time information for a set of discrete frequency increments, while the repetitively scanning spectrum analyzer provides continuous spectrum information for a series of discrete time increments. The former samples in frequency, the latter samples in time.

IV. THE GAUSSIAN CASE

The results of Section III may be illustrated by the response of a gaussian quadratic-phase filter to gaussian scanning excitation, using the arrangement of Fig. 10. That is, the time-invariant filter of the translation-scanning spectrum analyzer is assumed to have an amplitude characteristic that follows a gaussian error curve, and the envelope amplitude of the linearly scanning input signal is assumed to be a gaussian function of time.*

A. ANALYZER AND SIGNAL PARAMETERS

1. The Gaussian Function

The familiar gaussian function, illustrated in Fig. 14, is defined as

$$y(t) = e^{-\alpha t^2} \quad \text{Re}(\alpha) > 0 \quad (4.1a)$$

Using Eq. (1.1b), the Fourier transform is also of gaussian form:

$$Y(\omega) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\omega^2/4\alpha} \quad (4.1b)$$

Operations with gaussian functions frequently provide results in simple closed form, facilitating interpretation and computation.

Although the gaussian function has finite amplitude over the entire range of the argument and is not physically realizable [Refs. Va-1, Ro-1, Va-1], it may be closely approximated in the region of interest. The behavior of $y(t)$ over an amplitude range of 120 decibels is shown

*The response of a linear-phase gaussian filter has been described for linearly scanning excitation with constant amplitude [Refs. Ba-5, Mo-1, Po-4] and with a gaussian envelope [Ref. Ba-5]. Time compression of a linearly scanning gaussian pulse by a quadratic-phase filter having infinite bandwidth was examined by Krönert [Refs. Kr-1, 2].

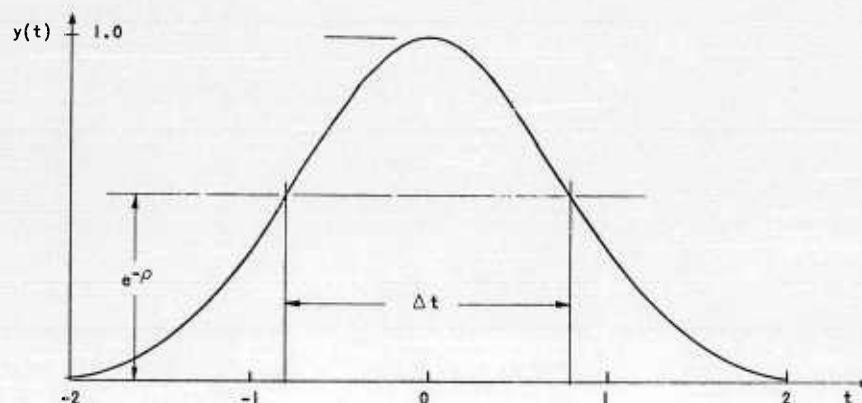


FIG. 14. THE GAUSSIAN FUNCTION.

in Fig. 15. The gaussian function is well-behaved under limiting processes, permitting approach either to an "impulse" function or to constant amplitude as α is made large or small, respectively.*

The "width" of the gaussian peak may be defined as the increment Δt between the two points where the magnitude becomes $e^{-\rho t}$ times its maximum, as indicated in Fig. 14. From Eq. (4.1a),

$$\Delta t = d = 2\sqrt{\rho_t/\alpha} \quad (4.2a)$$

Similarly, the "bandwidth" b of the associated Fourier spectrum from Eq. (4.1b) is

*The utility of the gaussian function is not due solely to mathematical convenience. The absence of secondary maxima, or "sidelobes", is desirable for recognition of superimposed adjacent functions of different amplitudes. For certain definitions of "width", a gaussian function has the smallest time-bandwidth product K of any function [Refs. Ga-1, Le-1, Ri-1]. Bandpass filters with an approximately gaussian amplitude versus frequency characteristic are not difficult to construct, since this is the limiting form for cascaded synchronously-tuned amplifier stages as the number of stages is made large [Refs. Ba-5, Va-1]. Dishal [Ref. Di-3] has described an n th order approximation to a gaussian amplitude function using a filter having n elements.

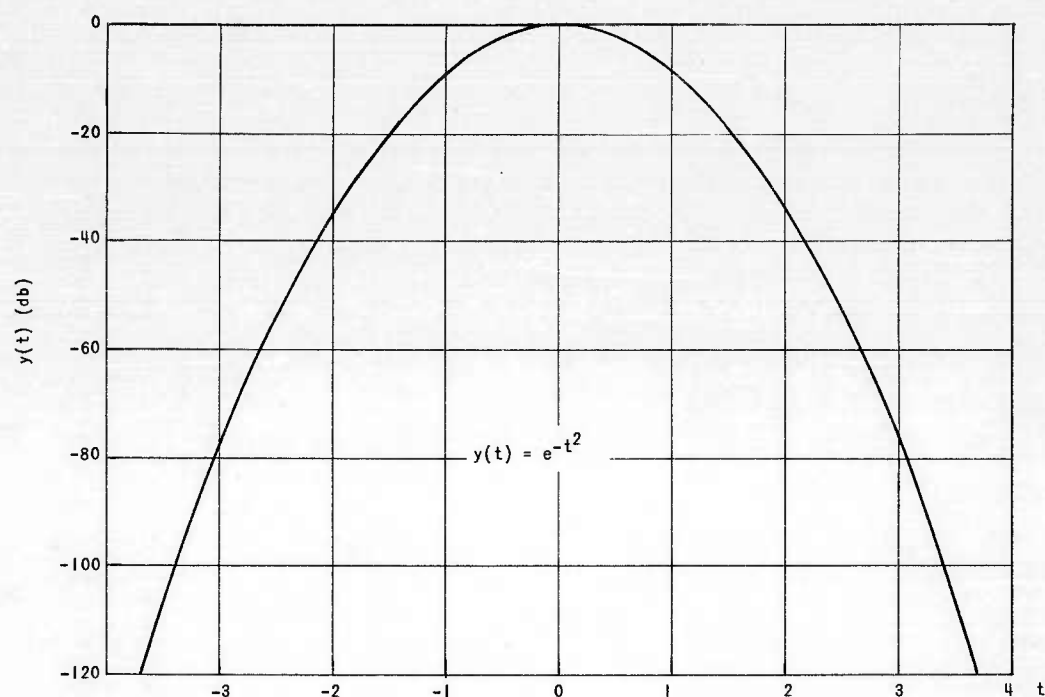


FIG. 15. THE GAUSSIAN FUNCTION WITH A LOGARITHMIC AMPLITUDE SCALE.

$$\frac{\Delta\omega}{2\pi} = b = \frac{2}{\pi} \sqrt{\alpha\rho_f} \quad (4.2b)$$

If the widths of the gaussian function and its transform are defined in terms of the parameters ρ_t and ρ_f , then the product K of the widths is:

$$K = bd = \frac{4}{\pi} \sqrt{\rho_t\rho_f} \quad (4.3)$$

Several gaussian width definitions that have been commonly used are listed in Table 3. The definitions are based on various arbitrarily assigned values of ρ . For flexibility, the general definition in terms of the parameter ρ will be retained in this discussion.

Table 3 - COMMON DEFINITIONS OF GAUSSIAN FUNCTION "WIDTH".

ρ		$(\sqrt{\alpha}\Delta t)$	Amplitude ($=e^{-\rho}$)		Description
$\frac{1}{2} \ln 2$	0.25	1	0.779	-2.16 db	Unit (normalized) width
	0.3466	1.1774	0.7071	-3 db	Half power width
	$\pi/8$	1.2533	0.675	-3.41 db	Energy, or "Noise power" width (see Appendix A)
$\ln 2$	0.5	1.4141	0.6065	-4.34 db	"Standard deviation" width
	0.6932	1.665	0.5	-6 db	Half voltage width
	$\pi/4$	1.7725	0.456	-6.82 db	Transform reciprocal width
$\pi/2$	1	2	0.368	-8.68 db	1/e width
	1.5708	2.5066	0.2035	-13.83 db	$\Delta x = \frac{1}{2} \frac{\left[\int_{-\infty}^{\infty} f(x) ^2 dx \right]^2}{\int_{-\infty}^{\infty} f(x) ^4 dx}$

The definition of function width Δt of Eq. (4.2) is useful, but a more specific criterion for the separation of adjacent superimposed functions is sometimes desirable for their separate recognition and measurement. In the gaussian case, a specified tolerance in the apparent position of two adjacent peaks or the separate existence of two peaks with a specified "dip" between them may be required. Minimum spacing of gaussian functions necessary to satisfy these criteria is discussed in Appendix D.

2. The Scanning Gaussian Input Signal

Assume that the signal to be applied to the scanning spectrum analyzer is in the form of a constant frequency modulated by a pulse with gaussian envelope:

$$u_1(t) = a e^{-\alpha_1 t^2} \cos(\omega_1 t + \theta) \quad (4.4)$$

As a result of "mixing" with a linearly scanning oscillator signal the time functions at the input of the analyzer filter will be of the form*

$$u_3(t) = a_1 e^{-\alpha_1 t^2} \cos(\sigma_1 t^2 + \omega_1 t + \theta_1) \quad (4.5)$$

where the coefficients are real, and where:

- a_1 = maximum envelope amplitude (obtained when $t = 0$)
- α_1 = gaussian envelope width parameter ($\alpha_1 > 0$)
- σ_1/π = scan rate in cycles per second per second = s_1
- $\omega_1/2\pi$ = input signal frequency in cycles per second = f_1
- θ_1 = an arbitrary phase constant in radians

Let d_1 be the pulse duration between the points on the gaussian envelope where the magnitude drops to $e^{-\rho_0}$ relative to that of the peak. By Eq. (4.2)

$$d_1 = 2\sqrt{\frac{\rho_0}{\alpha_1}} \text{ seconds} \quad (4.6)$$

The Fourier spectrum of $u_3(t)$ is obtained by expressing the cosine of Eq. (4.5) in exponential form and using the definite integral

$$\int_{-\infty}^{\infty} e^{-az^2+bz+c} dz = \sqrt{\frac{\pi}{a}} e^{(b^2/4a)+c} \quad \text{Re}(a) > 0 \quad (4.7)$$

Using Eqs. (4.5) and (4.7) and taking the Fourier transform,

* See Section III-A-2.

$$\begin{aligned}
 U_3(\omega) = & \frac{a_1}{2} \sqrt{\frac{\pi}{\alpha_1 - j\sigma_1}} \exp \left[\frac{-(\omega - \omega_1)^2}{4(\alpha_1 - j\sigma_1)} + j\theta_1 \right] \\
 & + \frac{a_1}{2} \sqrt{\frac{\pi}{\alpha_1 + j\sigma_1}} \exp \left[\frac{-(\omega + \omega_1)^2}{4(\alpha_1 + j\sigma_1)} - j\theta_1 \right]
 \end{aligned} \quad (4.8)$$

Rationalizing,

$$\begin{aligned}
 U_3(\omega) = & \frac{a_1 \sqrt{\pi}}{2} (\alpha_1^2 + \sigma_1^2)^{-1/4} \exp \left[\frac{\alpha_1 (\omega - \omega_1)^2}{4(\alpha_1^2 + \sigma_1^2)} + j \left(\theta_1 + \frac{1}{2} \arctan \frac{\sigma_1}{\alpha_1} - \frac{\sigma_1 (\omega - \omega_1)^2}{4(\alpha_1^2 + \sigma_1^2)} \right) \right] \\
 & + \frac{a_1 \sqrt{\pi}}{2} (\alpha_1^2 + \sigma_1^2)^{-1/4} \exp \left[\frac{\alpha_1 (\omega + \omega_1)^2}{4(\alpha_1^2 + \sigma_1^2)} - j \left(\theta_1 + \frac{1}{2} \arctan \frac{\sigma_1}{\alpha_1} - \frac{\sigma_1 (\omega + \omega_1)^2}{4(\alpha_1^2 + \sigma_1^2)} \right) \right]
 \end{aligned} \quad (4.9)$$

Equation (4.9) shows that the spectrum of $u_3(t)$ consists of two components, each gaussian in form, symmetrically centered around $+\omega$ and $-\omega$. Each component contains quadratic phase terms as a result of the scanning.

The spectrum width (bandwidth) b_3 of each gaussian component of $U_3(\omega)$ may be obtained from the real exponential quadratic coefficient in Eq. (4.9) by use of Eq. (4.2), with b_3 defined between the e^{-p1} relative magnitude points:

$$b_3 = \frac{2}{\pi} \sqrt{\rho_1 \left(\alpha_1 + \frac{\sigma_1^2}{\alpha_1} \right)} \quad \text{cps} \quad (4.10)$$

Substituting $\sigma_1 = \pi s_1$ and d_1 from Eq. (4.6), we obtain

$$b_3 = \sqrt{\frac{\rho_1}{\rho_0} (s_1 d_1)^2 + \frac{16 \rho_0 \rho_1}{\pi^2 d_1^2}} \quad (4.11)$$

The effect of scanning on bandwidth may be demonstrated by noting that $s_1 d_1$ is the range of frequency Δf_1 through which the frequency scans within the duration of the input signal. By letting $K_1 = \frac{4}{\pi} \sqrt{\rho_0 \rho_1} = b_1 d_1$

$$b_3^2 = b_1^2 + \frac{\rho_1}{\rho_0} (\Delta f_1)^2 \quad (4.12)$$

Equation (4.12) shows that the scanning process has little effect on spectrum width when the scan rate is small, but becomes the determining factor for large scan rates.

3. The Quadratic-Phase Gaussian Filter

Considerations which influence the formulation of the gaussian filter include the following:

- a. The impulse response of the filter should be zero for negative time. (While never zero, the impulse response of a gaussian filter may be made arbitrarily small by introduction of time delay).
- b. The impulse response should be a real time function, so that the filter spectrum function must be Hermitian (real terms even and imaginary terms odd functions of frequency).
- c. Matching of the scanning input signal spectrum requires quadratic phase terms.

The filter function $H(\omega)$ is therefore assumed to be in the form of Eq. (4.9):

$$\begin{aligned} H(\omega) = & A_2 \exp [-(\beta + j\mu)(\omega - \omega_2)^2 + j\omega\tau_2 + j\theta_2] \\ & + A_2 \exp [-(\beta - j\mu)(\omega + \omega_2)^2 + j\omega\tau_2 - j\theta_2] \end{aligned} \quad (4.13)$$

where again the coefficients are real, and

A_2 = the maximum spectrum function magnitude
(voltage gain at the center of the band)

β = bandwidth parameter, ($\beta > 0$)

μ = quadratic phase coefficient

$\omega_2/2\pi$ = center frequency of the bandwidth = f_2

τ_2 = linear phase coefficient (time delay in seconds)
 θ_2 = an arbitrary phase constant (in radians)

As in the case of $U_3(\omega)$, the magnitude of the filter function $H(\omega)$ consists of the sum of two gaussian functions centered symmetrically around $+\omega_2$ and $-\omega_2$, as illustrated in Fig. 16. The filter bandwidth b_2 , defined as the increment of frequency between the points where each component of $H(\omega)$ drops to $e^{-\rho_2}$ relative to the maximum value, becomes by Eq. (4.2)

$$b_2 = \frac{1}{\pi} \sqrt{\frac{\rho_2}{\beta}} \quad \text{cycles per second} \quad (4.14)$$

The impulse response $h(t)$ of $H(\omega)$ is obtained by inverse Fourier transformation of Eq. (4.13) or by analogy to Eqs. (4.5) and (4.9):

$$h(t + \tau_2) = a_2 e^{-\alpha_2 t^2} \cos(\sigma_2 t^2 + \omega_2 t + \theta_2 - \frac{1}{2} \arctan \frac{\mu}{\beta}) \quad (4.15)$$

where

$$a_2 = \frac{A_2}{\sqrt{\pi}} (\beta^2 + \mu^2)^{-1/4}$$

$$\alpha_2 = \frac{\beta}{4(\beta^2 + \mu^2)}$$

$$\sigma_2 = \frac{\mu}{4(\beta^2 + \mu^2)}$$

Except for the time delay τ_2 , the form of $h(t)$ is similar in all respects to $u_3(t)$. As before, the impulse response duration d_2 (defined between the $e^{-\rho_2}$ relative envelope amplitude points) and instantaneous frequency scan rate s_2 may be obtained from the parameters α_2 and σ_2 , respectively:

$$d_2 = 2 \sqrt{\frac{\rho_2}{\alpha_2}} = \frac{K_2}{b_2} \left(1 + \frac{\mu^2}{\beta^2} \right)^{1/2} \quad \text{seconds} \quad (4.16a)$$

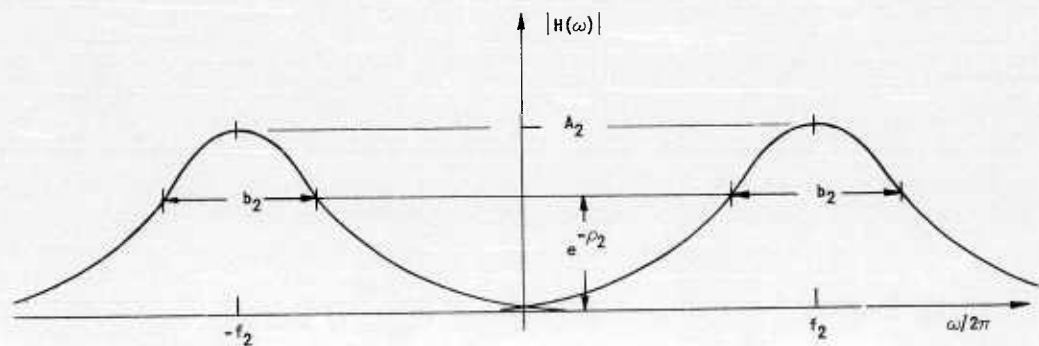


FIG. 16. GAUSSIAN FILTER FUNCTION MAGNITUDE VS FREQUENCY.

$$s_2 = \frac{\sigma_2}{\pi} = \frac{1}{4\pi\mu(1 + \frac{\beta^2}{2})} \quad \text{cps}^2 \quad (4.16b)$$

where

$$K_2 = \frac{4\rho_2}{\pi} \quad * \quad (4.16c)$$

B. GAUSSIAN ANALYZER RESPONSE

1. The General Response Function

While Eq. (3.10) is generally applicable, $r(t)$ may be directly obtained in the gaussian case by convolution of $u_3(t)$ and $h(t)$ or by inverse Fourier transformation of the output response spectrum function $R(\omega) = U_3(\omega)H(\omega)$. Choosing the latter method, we obtain

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U_3(\omega)H(\omega)e^{j\omega t}d\omega \quad (4.17)$$

Equation (4.17) is evaluated by substituting $U_3(\omega)$ and $H(\omega)$ as given

* K_2 is the time-bandwidth factor associated with the form of the filter spectrum magnitude function [see Eq. (3.21)].

by Eqs. (4.8) and (4.13) and making use of the definite integral of Eq. (4.7). Using the symbolism $\bar{\pm} = (-1)^n$ and $\pm = (-1)^{n+1}$ and arranging terms, the general gaussian analyzer response may be written in the following form:

$$(a) \quad r(t) = \sum_{n=1,2} r_n(t) \quad (4.18)$$

where:

$$r_n(t+\tau_2) = a_{3n} \exp[-\alpha_{3n}(t-\tau_{3n})^2 + \alpha_4(\omega_1 \bar{\pm} \omega_2)^2] \cos(\sigma_{3n}t^2 + \omega_{3n}t + \theta_{3n})$$

$$(b) \quad a_{3n} = \frac{a_1 A_2}{2\sqrt{\lambda_n}}$$

$$(c) \quad \lambda_n^2 = (1 + 4\alpha_1\beta \pm 4\mu\sigma_1)^2 + 16(\alpha_1\mu \bar{\pm} \beta\sigma_1)^2$$

$$(d) \quad \alpha_{3n} = \frac{1}{\lambda_n^2} [\alpha_1 + 4\beta(\alpha_1^2 + \sigma_1^2)]$$

$$(e) \quad \alpha_4 = \frac{\alpha_1\beta}{\alpha_1 + 4\beta(\alpha_1^2 + \sigma_1^2)}$$

$$(f) \quad \sigma_{3n} = \frac{1}{\lambda_n^2} [4\mu(\alpha_1^2 + \sigma_1^2) \pm \sigma_1]$$

$$(g) \quad \omega_{3n} = \omega_2 - \frac{1}{\lambda_n^2} (\omega_2 \bar{\pm} \omega_1)(1 + 4\alpha_1\beta \pm 4\mu\sigma_1)$$

$$(h) \quad \tau_{3n} = \frac{-2(\omega_2 + \omega_1)(\alpha_1\mu + \beta\sigma_1)}{\alpha_1 + 4\beta(\alpha_1^2 + \sigma_1^2)}$$

$$(i) \quad \theta_{3n} = \pm \theta_1 - \theta_2 - \frac{1}{\lambda_n} (\omega_2 + \omega_1)^2 [\mu \pm 4\sigma_1(\beta^2 + \mu^2)] \\ - \frac{1}{2} \arctan \left[\frac{4(\alpha_1\mu + \beta\sigma_1)}{1 + 4\alpha_1\beta \pm 4\mu\sigma_1} \right]$$

Equation (4.18) expresses $r(t)$ as the sum of two components, each similar in form to $u_3(t)$ and $h(t)$, which necessarily result from the assumption of realizable (real) signal and filter time functions.*

Timing relationships of the envelopes of the input, filter, and response functions are shown in Fig. 17. In general, $r_1(t)$ and $r_2(t)$ have different peak amplitudes a_{nm} , durations d_{3n} , and times t_{3n} of peak amplitude. All of the parameters of the two components may be interchanged by reversing the polarities of σ_1 , ω_1 and θ_1 . If $r_1(t)$ is designated as the desired response, $r_2(t)$ corresponds to an input scanning in the reverse direction and is thus the "image" response of $r_1(t)$ (see Section I-B-3).

The maximum energy E_{nm} of each component is approximately equal to the product of peak envelope power and envelope duration.** Since by Eq. (4.2) $d_{3n} = 2(\rho_3/\alpha_{3n})^{1/2}$,

$$\frac{E_{1m}}{E_{2m}} \approx \left(\frac{a_{31}}{a_{32}} \right)^2 \frac{\alpha_{32}}{\alpha_{31}} = 1 \quad (4.19)$$

*Some treatments of the gaussian filter scanning response assume a complex exponential form for signal and filter, taking as the response the real part of the complex result. The procedure simplifies analysis, but gives only one output component ($r_1(t)$ or $r_2(t)$), (depending on choice of phase constants). The complex representation of real time functions is valid only in the bandpass case where the two-sided Fourier spectrum does not significantly overlap at zero frequency.

**Bounds on the energy of a scanning gaussian pulse are discussed in Appendix E.

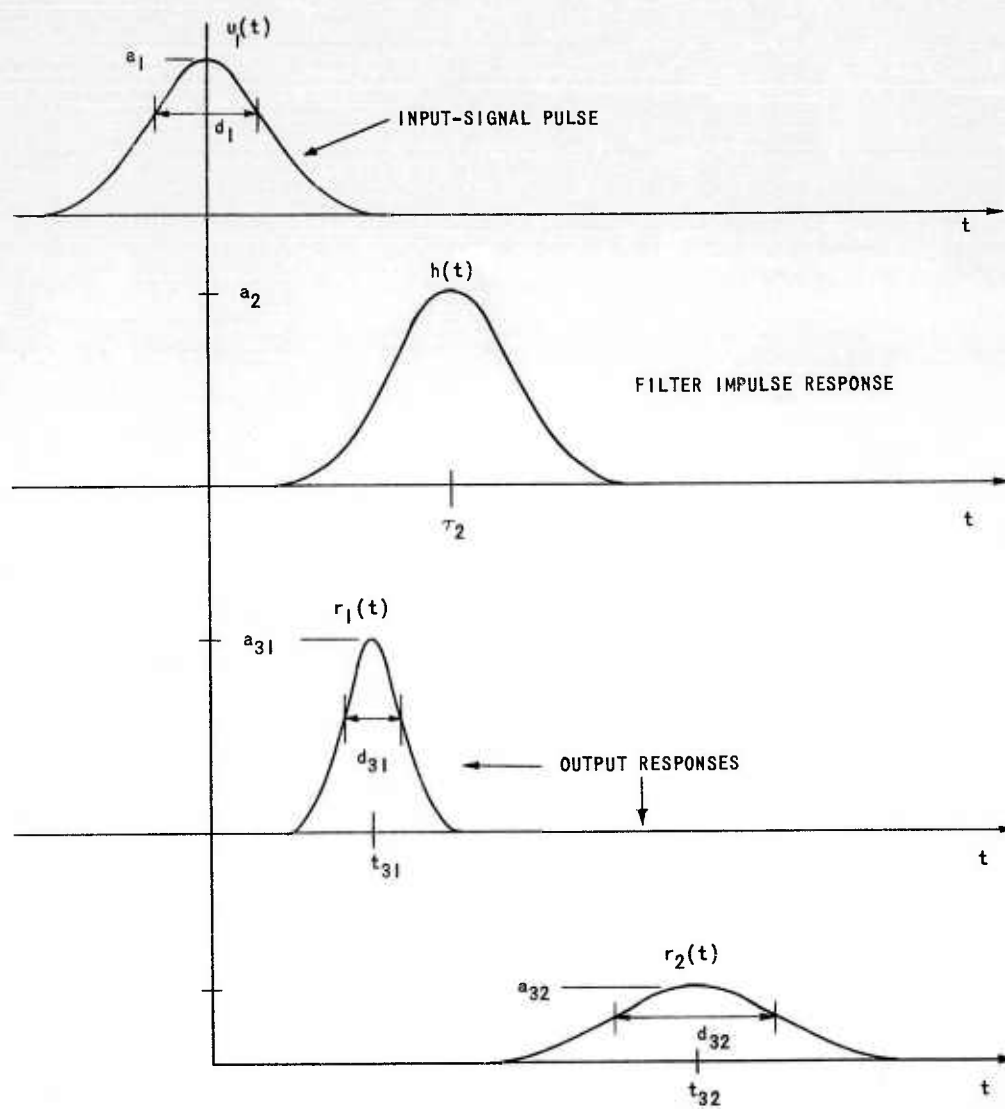


FIG. 17. GAUSSIAN-ENVELOPE TIME RELATIONSHIPS.

Separation of the normal and image responses must therefore be accomplished on the basis of time of occurrence and form.

2. The Resolution (Pulse Compression) Factor

The parameter λ_n appears frequently in the general response expressions of Eq. (4.18). For reasons which will become evident in the next section, λ_n is termed here the resolution factor.* Equation (4.18c) shows that λ_n is real, positive, and non-zero for all signals and filters of practical interest. Changing the direction of scanning or the sign of the phase coefficient μ interchanges λ_1 and λ_2 . From Eq. (4.18c) it may be seen that:

$$\lambda_1^2 - \lambda_2^2 = 16\mu\sigma_1 \quad (4.20)$$

so that $\lambda_1^2 \neq \lambda_2^2$ unless the scan rate is zero (a trivial case) or the filter is linear-phase ($\mu = 0$). Defining (as in Section III-A-4) the phase-matching parameter η ,

$$\eta = -4\mu\sigma_1 \quad (4.21)$$

the resolution factor may be written:

$$\lambda_n^2 = (1 \mp \eta + 4\alpha_1\beta)^2 + (4\beta\sigma_1 \pm \frac{\alpha_1\eta}{\sigma_1})^2 \quad (4.22a)$$

Or, in terms of signal and filter parameters,

$$\lambda_n^2 = \left[1 \mp \eta + \frac{\rho_o}{\rho_2} \left(\frac{K_2}{b_2 d_1} \right)^2 \right]^2 + K_2^2 \left[\frac{s_1}{b_2^2} \pm \frac{\rho_o}{\rho_2} \frac{\eta}{\Delta f_1 d_1} \right]^2 \quad (4.22b)$$

As will be seen, best sensitivity and resolution of the analyzer are obtained when λ_n is a minimum. Differentiating Eq. (4.22a) with respect to η , the minimum value of λ_1 is obtained when

* Also called the pulse compression factor [Refs. K1-1, Kr-1].

$$\eta = \frac{1}{1 + \left(\frac{\alpha_1}{\sigma_1}\right)^2} = \frac{1}{1 + \frac{\rho_o K_2}{\rho_2 s_1 d_1^2}} \quad (4.23)$$

As either signal duration d_1 or scan rate s_1 is made large, the optimum phase-matching condition $\eta = 1 = -4\mu\sigma_1$ of Eq. (2.5) and Table 3.1 appears, appropriate for a constant amplitude input signal ($d_1 \rightarrow \infty$) or a very high scan rate ($s_1 \rightarrow \infty$).

If we minimize λ_1 with respect to filter phase and bandwidth coefficients μ and β , the following conditions result:

$$\mu = \frac{-\sigma_1}{4(\alpha_1^2 + \sigma_1^2)} \quad (4.24a)$$

$$\beta = \frac{\alpha_1}{4(\alpha_1^2 + \sigma_1^2)} \quad (4.24b)$$

These are conditions for "matching" the filter to the scanning input signal, as may be seen by comparing Eqs. (4.5) and (4.15), or Eqs. (4.9) and (4.13).*

3. Frequency Representation and Resolution

The frequency f_1 of the input signal is represented in the scanning spectrum analyzer response by the time $t_{3n} = \tau_2 + \tau_{3n}$ at which the envelope of $r_n(t)$ has maximum amplitude. The time delay τ_2 is constant; from Eqs. (4.18h) and (4.21),

$$\tau_{3n} = \left(\frac{f_2 + f_1}{s_1} \right) \left(\frac{\eta \pm 4\beta\sigma_1^2/\alpha_1}{1 + 4\alpha_1\beta + 4\beta\sigma_1^2/\alpha_1} \right) \quad (4.25a)$$

or in terms of signal and filter parameters,

$$\tau_{3n} = \left(\frac{f_2 + f_1}{s_1} \right) \left[\frac{\eta \pm \frac{\rho_2}{\rho_o} \left(\frac{s_1 d_1}{b_2} \right)^2}{1 + \frac{\rho_o}{\rho_2} \left(\frac{K_2}{b_2 d_1} \right)^2 + \frac{\rho_2}{\rho_o} \left(\frac{s_1 d_1}{b_2} \right)^2} \right] \quad (4.25b)$$

* See footnote on p. 24.

Two limiting cases are of interest. As the signal duration becomes large (approaching the gliding tone case),

$$\lim_{d_1 \rightarrow \infty} (\tau_{3n}) = \frac{-f_1 + f_2}{s_1} \quad n = 1, 2 \quad (4.25c)$$

As scan rate and filter bandwidth are increased, but the time required for measurement is made equal to $D_1 \approx b_2/s_1$,

$$\lim_{\substack{b_2 \rightarrow \infty \\ b_2/s_1 \rightarrow D_1}} (\tau_{3n}) = \left(\frac{f_2 + f_1}{s_1} \right) \left(\frac{\rho_o \eta D_1^2 + \rho_r d_1^2}{\rho_o D_1^2 + \rho_2 d_1^2} \right) \quad (4.25d)$$

If $D_1 \ll d_1$, Eqs. (4.25c) and (4.25d) are approximately equal. In either case, there is a linear relationship between time and frequency. The separation of the signal $r_1(t)$ and image $r_2(t)$ becomes

$$\tau_{32} - \tau_{31} \approx \frac{2f_2}{s_1} \quad (4.25e)$$

Analyzer resolution is determined by the duration d_{3n} of the output responses, obtained using Eq. (4.2a) in terms of the width parameter ρ_3 :

$$\begin{aligned} d_{3n} &= 2 \sqrt{\frac{\rho_3}{\alpha_{3n}}} \\ &= 2 \lambda_n \sqrt{\rho_3} [\alpha_1 + 4\beta(\alpha_1^2 + \sigma_1^2)]^{-1/2} \end{aligned} \quad (4.26)$$

The effective resolution $\Delta f_{3n} = s_1 d_{3n}$ may be written as follows, using Eqs. (4.6), (4.14), and (4.16):*

*Since f_1 and f_2 do not appear in Eqs. (4.26) and (4.27), frequency resolution of the gaussian scanning spectrum analyzer is independent of the relationship between signal and filter frequency.

$$\Delta f_{3n} = \lambda_n b_2 \sqrt{\frac{\rho_3}{\rho_2}} \left[1 + \left(\frac{\rho_0}{\rho_2 \Delta f_1} \right)^2 \left(\frac{K_2^2}{d_1^2} + b_2^2 \right) \right]^{-1/2} \quad (4.27)$$

If the signal duration and analyzer scan rate are sufficiently large (i.e. if $s_1 d_1 = \Delta f_1 \gg (\rho_0/\rho_2) b_2 \approx b_2$ and $\Delta f_1 d_1 \gg (\rho_0/\rho_2) K_2 \approx 1$), then:

$$\Delta f_{3n} \rightarrow \lambda_n b_2 \sqrt{\frac{\rho_3}{\rho_2}} \quad (4.27a)$$

Under these conditions frequency resolution is proportional to the resolution parameter λ_n .

The significance of phase-matching when considering the gliding tone excitation of Section II is illustrated by Δf_{31} as the signal duration becomes large:

$$\lim_{d_1 \rightarrow \infty} \Delta f_{31} = \frac{\rho_3}{\rho_2} \left[(\eta - 1)^2 b_2^2 + \left(\frac{K_2 s_1}{b_2} \right)^2 \right]^{1/2} \quad (4.28)$$

Figure 18 shows analyzer resolution versus filter bandwidth in the case just discussed for several values of the phase-matching parameter $\eta = -4\mu\sigma_1$.^{*} Improvement of resolution by increasing filter bandwidth is limited only by signal duration and the precision of matching the filter phase to the signal scan rate.

4. Maximum Amplitude and Sensitivity

Sensitivity of the analyzer is determined by the maximum peak amplitude $r_{nm} = a_{3n}$, which is obtained when $t = \tau_2 + \tau_{3n}$:

$$r_{nm} = \frac{a_1 A_2}{2\sqrt{\lambda_n}} \quad (4.29)$$

For the gliding tone case (when signal duration is large),

^{*} Table 2 identifies values of η for various conditions.

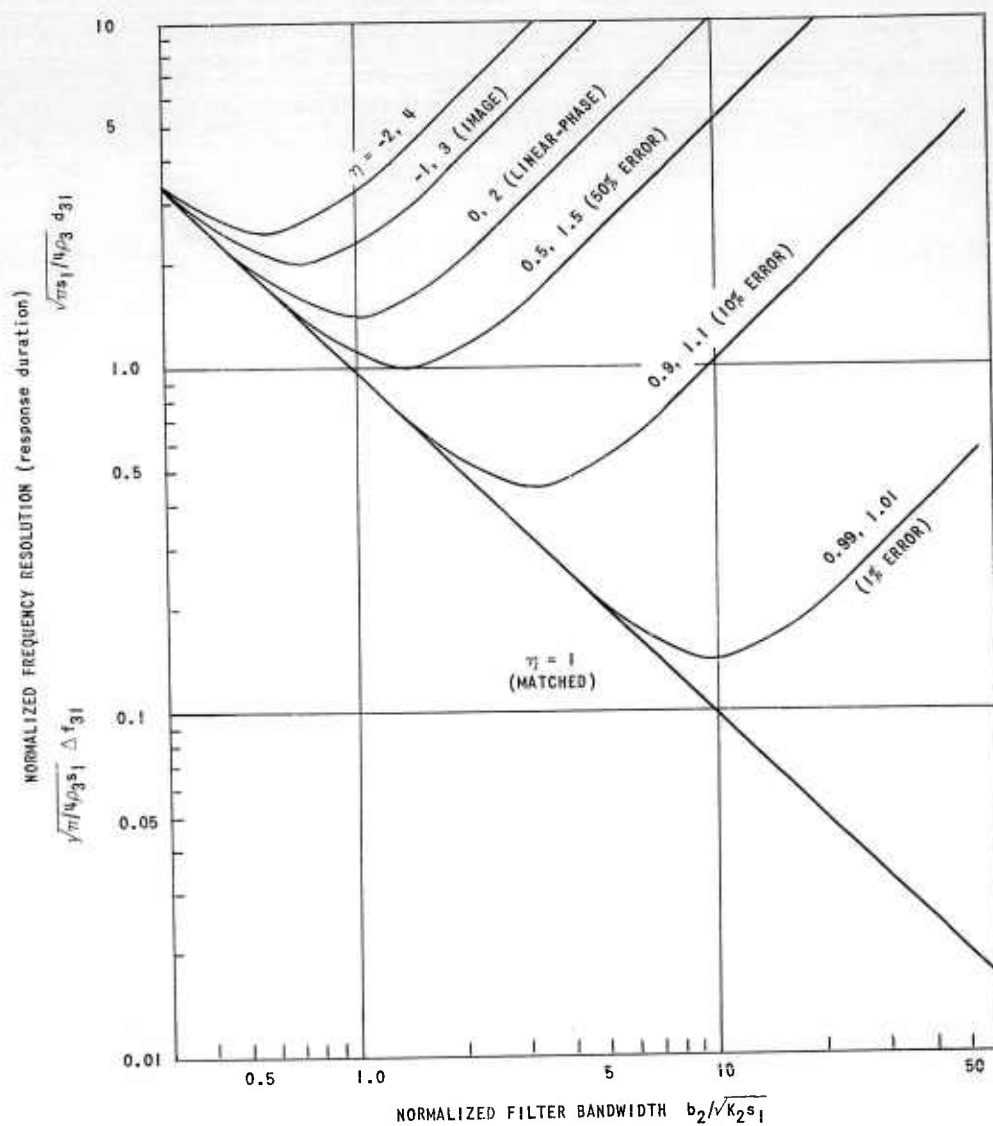


FIG. 18. FREQUENCY RESOLUTION AND RESPONSE DURATION VS FILTER BANDWIDTH IN THE GAUSSIAN CASE FOR A C-W SIGNAL.

$$\lim_{d_1 \rightarrow \infty} r_{lm} = \frac{a_1 A_2}{2} \left[(\eta - 1)^2 + \left(\frac{K_2 s_1}{b_2^2} \right)^2 \right]^{-1/4} \quad (4.30)$$

Figure 19 shows variation of peak response amplitude with filter bandwidth for values of η which indicate several matching tolerances as well as harmonic and image responses (see Table 2). If signal duration, not phase matching, is the limiting factor, setting $\eta = 1$ gives

$$r_{lm} = \frac{a_1 A_2}{2} \left[\frac{\rho_o}{\rho_2} \left(\frac{K_2}{b_2 d_1} \right)^2 + \left(\frac{K_2 s_1}{b_2^2} + \frac{\rho_o}{\rho_2} \frac{K_2}{s_1 d_1^2} \right)^2 \right]^{-1/4} \quad (4.31)$$

Limiting amplitudes for several values of signal duration are shown in Fig. 20.

The signal/noise ratio r_{nm}/e_n may be obtained by use of Eq. (4.29) in Eq. (1.14); and letting $b_n = b_2$:

$$\left(\frac{r_{nm}}{e_n} \right)^2 = \frac{1}{FkT} \cdot \frac{a_1^2}{4R} \cdot \frac{1}{\lambda_n b_2} \quad (4.32)$$

Figure 21 illustrates the variation of signal/noise ratio with phase-matched analyzer filter bandwidth for various normalized signal durations, where 0 db corresponds to the best obtainable S/N with a linear-phase filter when signal duration is large. The S/N of the phase-matched analyzer can be made arbitrarily larger than that obtainable with the linear-phase analyzer at a given scan rate by increasing the analyzer filter bandwidth when signal duration permits.

5. Response Scan Rate and Frequency

If the filter of the scanning spectrum analyzer is followed by an envelope detection process and if many cycles are included within the response envelope, the nature of the periodic component of the output response will not significantly affect analyzer performance. Frequency and phase of the cosine component of Eq. (4.18) become of

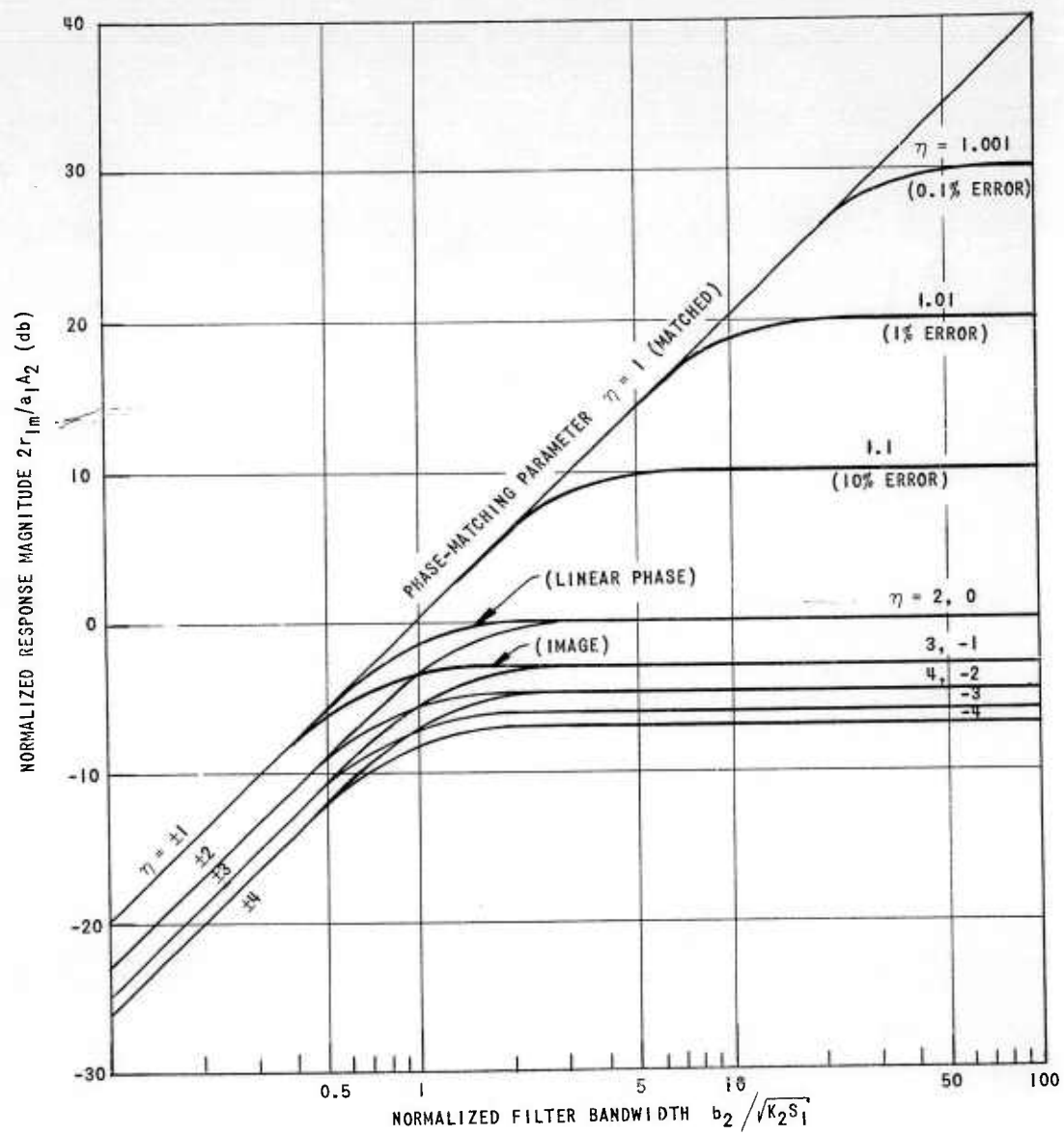


FIG. 19. PEAK AMPLITUDE OF SIGNAL, HARMONIC, AND IMAGE RESPONSES VS BANDWIDTH IN THE GAUSSIAN FILTER-LIMITED CASE.

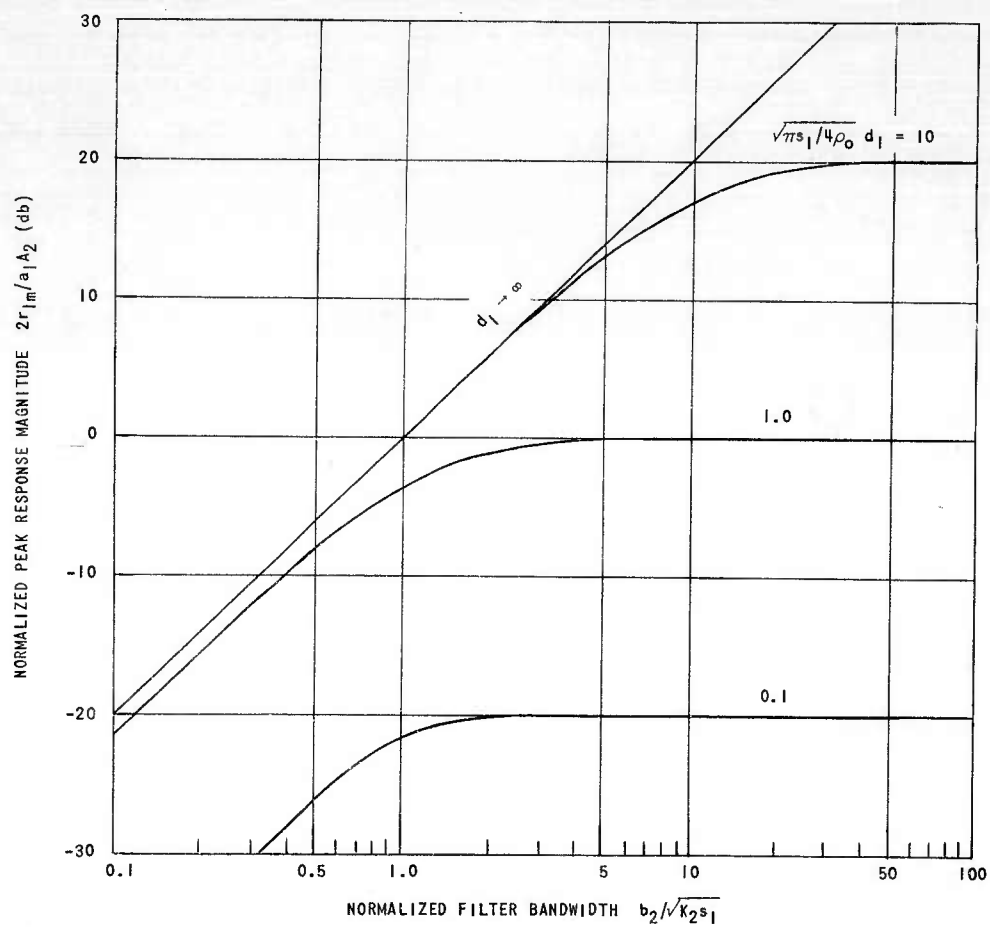


FIG. 20. PEAK RESPONSE AMPLITUDE VS FILTER BANDWIDTH IN THE GAUSSIAN SIGNAL-LIMITED CASE.

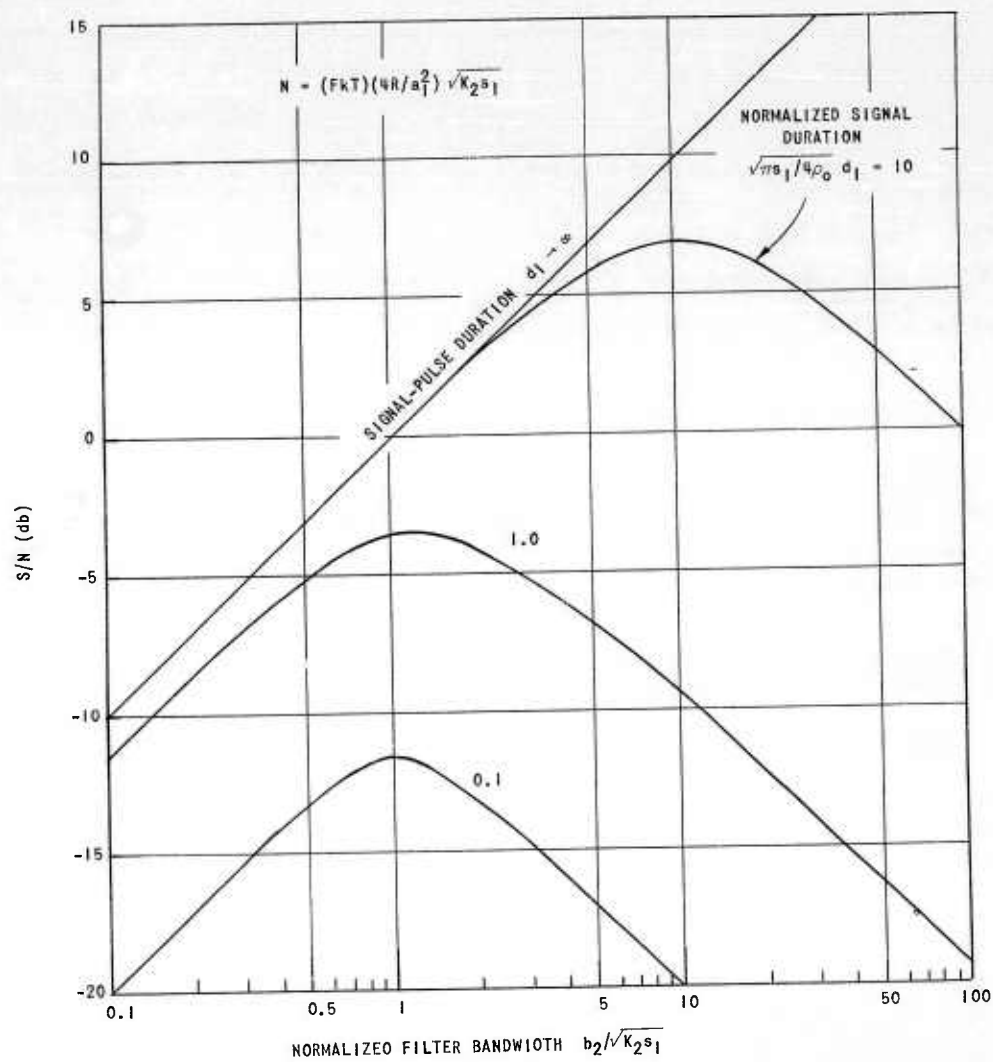


FIG. 21. SENSITIVITY (S/N) VS FILTER BANDWIDTH IN THE GAUSSIAN SIGNAL-LIMITED CASE.

interest when only a few cycles occur during response or when a low-pass filter is used.

The scan rates s_{3n} of the oscillating factor of $r_n(t)$ are given by Eq. (4.18f):

$$s_{3n} = \frac{\sigma_{3n}}{2\pi} = \frac{-s_1}{\lambda_n^2} \left[\eta \left(1 + \frac{\alpha_1^2}{\sigma_1^2} \right) \pm 1 \right] \quad (4.33)$$

For the usual case of interest (where $\sigma_1/\alpha_1 \approx \Delta f_1 d_1 \gg 1$)

$$s_{3n} \approx \frac{-s_1}{\lambda_n^2} (\eta \mp 1) \quad (4.33a)$$

The output response does not scan ($s_{31} = 0$) in the matched case (when $\eta = 1$).

The instantaneous frequency of $r_n(t)$ at t_{3n} when the envelope magnitude is maximum (center frequency of the pulse) is

$$f_{rn} = \frac{1}{2\pi} (\omega_{3n} + 2\sigma_{3n} \tau_{3n}) \quad (4.34)$$

Using Eqs. (4.12) and (4.18),

$$f_{rn} = f_2 \pm (f_1 \mp f_2) \left[1 + \frac{\rho_2}{\rho_1} \left(\frac{b_3}{b_2} \right)^2 \right]^{-1} \quad (4.34a)$$

If $b_3 \gg b_2$, the "center frequency" f_{rn} of the response approximates that of the filter f_2 ; if $b_3 \ll b_2$, f_{rn} approaches the input signal frequency f_1 .

C. SPECIAL CASES OF INTEREST

The general gaussian response of Eq. (4.18) may be applied to a wide variety of assumed conditions. Several examples are briefly mentioned here to illustrate bounds of performance and to relate these results to work described elsewhere.

1. The Linear-Phase Case *

The classical results of the gliding tone problem discussed in Section II may be derived by setting the filter quadratic phase coefficient $\mu = \eta = 0$. An interesting feature of the linear-phase filter response is that the two response components $r_1(t)$ and $r_2(t)$ become identical in magnitude and form. From Eq. (4.22b)

$$\lambda^2 = \lambda_1^2 = \lambda_2^2 = \left[1 + \frac{\rho_0}{\rho_2} \left(\frac{K_2}{b_2 d_1} \right)^2 \right]^2 + \left[\frac{K_2 s_1}{b_2^2} \right]^2 \quad (4.35)$$

The scan rates of $r_1(t)$ and $r_2(t)$ become equal in magnitude and opposite in direction: from Eq. (4.33),

$$s_{32} = -s_{31} = \frac{s_1}{\lambda} \quad (4.36)$$

The results of Section II are obtained by assuming that signal duration d_1 is large enough so that the response is not affected; i.e. so that

$$d_1 \gg \left| \frac{\rho_0 b_2}{\rho_2 s_1} \right| \quad (4.37a)$$

$$d_1^2 \gg \left| \frac{\rho_0 K_2}{\rho_2 s_1} \right| \quad (4.37b)$$

Under these conditions, analyzer frequency resolution obtained from Eq. (4.28) becomes:

$$\Delta f_3 = \Delta f_{31} = \Delta f_{32} = \sqrt{\frac{4\rho_3 s_1}{\pi}} \left[\frac{b_2^2}{K_2 s_1} + \frac{K_2 s_1}{b_2^2} \right]^{1/2} \quad (4.38)$$

* The effects of scan rate and pulse duration on scanning analyzer response amplitude and resolution in the bandpass linear-phase gaussian case are discussed at some length by Batten, et al [Ref. Ba-5].

For a particular scan rate s_1 , optimum resolution is obtained when $b_2^2 = |K_2 s_1|$, or

$$b_{02} = \sqrt{|K_2 s_1|} \quad (4.39)$$

Using this value of filter bandwidth, the best obtainable resolution in the linear-phase case is

$$(\Delta f_3)_{\min} = \sqrt{\frac{8\rho_3 s_1}{\pi}} = \sqrt{\frac{2\rho_3}{\rho_2}} b_{02} \quad (4.40)$$

Variation of frequency resolution with filter bandwidth is shown in Fig. 22.

Maximum response amplitude is given by Eq. (4.31):

$$r_m = r_{1m} = r_{2m} = \frac{a_1 A_2}{2} \left[1 + \left(\frac{b_{02}}{b_2} \right)^2 \right]^{-1/4} \quad (4.41)$$

Where $b_n = b_2$ the resulting sensitivity, from Eq. (4.32) is

$$\left(\frac{r_m}{e_n} \right)^2 = \frac{1}{FkT} \cdot \frac{a_1^2}{4R} \cdot \frac{1}{b_{02}} \left[\left(\frac{b_2}{b_{02}} \right)^2 + \left(\frac{b_{02}}{b_2} \right)^2 \right]^{-1/2} \quad (4.42)$$

Maximum sensitivity for a particular scan rate is obtained with the optimum bandwidth of Eq. (4.39).

Harmonic responses may be demonstrated by letting the scan rate $s_1 = \eta s_0$, where η is an integer giving the order of the harmonic and s_0 is the scan rate for which the analyzer is designed. Negative values of η correspond to harmonic image responses. Variation of sensitivity and amplitudes of several harmonic responses is shown in Fig. 23.

The results just derived may be compared with the gliding tone conclusions given in Fig. 8 and Eq. (2.4).

2. Matched Phase and Bandwidth Case

The filter may be "matched" to optimize $r_1(t)$ of Eq. (4.18)

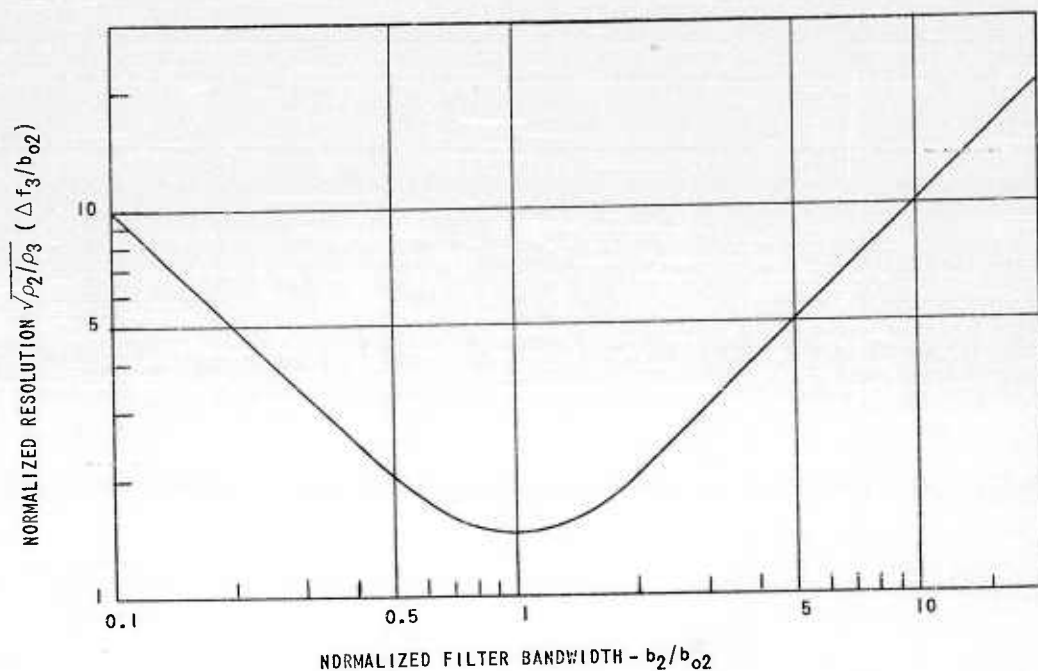


FIG. 22. FREQUENCY RESOLUTION VS LINEAR-PHASE GAUSSIAN FILTER BANDWIDTH.

by selecting bandwidth and quadratic phase coefficient as specified by Eq. (4.24). In effect, this means that the filter bandwidth b_2 is made equal to the post-scanning signal bandwidth b_3 . By Eq. (4.12), an increase in scan rate requires a corresponding increase in filter bandwidth.

Using Eq. (4.24) in Eq. (4.18), the matched response may be written:

$$r_1(t-\tau_2) = \frac{a_1 A_2}{2} \sqrt{\frac{b_2}{b_1}} \exp \left[-2 \left[\rho_0 \left(\frac{b_2}{K_1} \right)^2 (t-\tau_{31})^2 + \rho_1 \left(\frac{f_1-f_2}{b_2} \right)^2 \right] \right] \cos \left[\left(\frac{\omega_1+\omega_2}{2} \right) t + \theta_{31} \right] \quad (4.43)$$

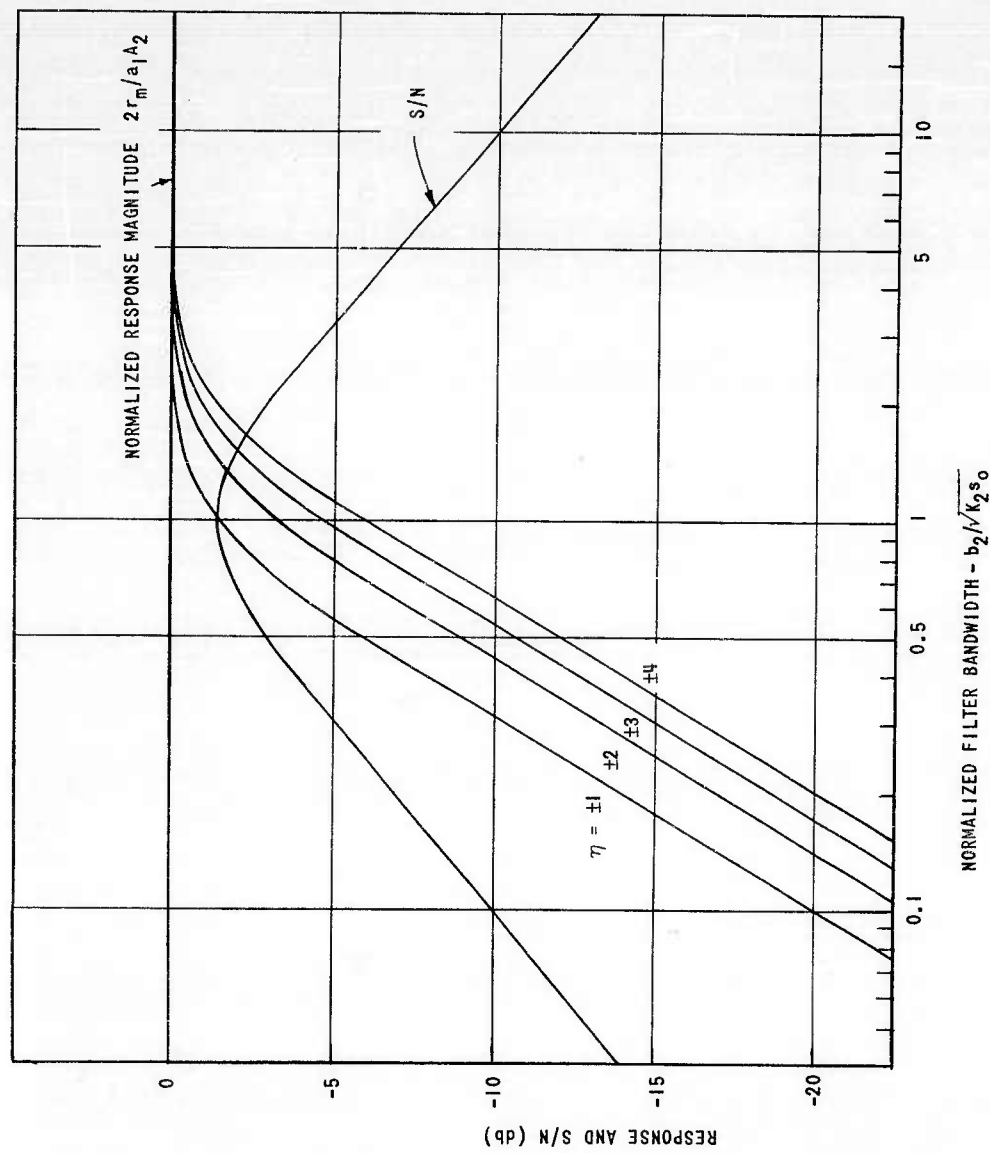


FIG. 23. RESPONSE AMPLITUDE AND S/N VS LINEAR-PHASE GAUSSIAN FILTER BANDWIDTH.

where:

$$\tau_{31} = \frac{\rho_1}{\rho_0} \left(\frac{s_1 d_1}{b_1} \right)^2 \left(\frac{f_2 - f_1}{s_1} \right)$$

$$\theta_{31} = \theta_1 - \theta_2 + \frac{1}{2} \arctan \sqrt{\frac{\rho_0}{\rho_1}} \frac{b_1}{s_1 d_1}$$

The response $r_1(t)$ does not scan ($\sigma_{31} = 0$) in the matched case.
Resolution becomes [by Eq. (4.26)]

$$\Delta f_{31} = \sqrt{\frac{2\rho_3}{\rho_0}} \frac{K_1 s_1}{b_3} \quad (4.44a)$$

In terms of the measurement duration $T_1 = (B_1 + b_2)/s_1 \leq d_1$

$$\Delta f_{31} = \sqrt{\frac{2\rho_3}{\rho_0}} \cdot \frac{K_1}{T_1} \left(1 + \frac{B_1}{b_2} \right) \quad (4.44b)$$

Frequency resolution of the scanning analyzer in the matched gaussian case is thus limited only by signal or measurement duration if scan rate and analyzer bandwidth are sufficiently large.

Sensitivity may be written, by Eq. (4.23),

$$\left(\frac{r_{1m}}{e_n} \right)^2 = \frac{E_1}{8FkTK_1} \quad (4.45)$$

and is independent of analyzer scan rate and bandwidth, being determined only by signal energy.

The "unmatched" (image) response $r_2(t)$ becomes

$$r_2(t+\tau_2) = \frac{a_1 A_2}{2\sqrt{2}} \exp -2 \left[\rho_0 \left(\frac{t}{d_1} \right)^2 + \rho_1 \left(\frac{f_1 + f_2}{b_2} \right)^2 \right] \cos \left[\frac{-\sigma_1 t^2}{2} + \frac{(\omega_2 - \omega_1)t}{2} + \theta_{32} \right] \quad (4.46)$$

where:

$$\theta_{32} = -\theta_1 - \theta_2 + \frac{\pi}{2} \left(\frac{\rho_1}{\rho_0} \right) \left(\frac{s_1 d_1}{b_2} \right)^2 \frac{(f_1 + f_2)^2}{s_1}$$

The peak magnitude and duration of $r_2(t)$ are independent of scan rate and bandwidth of the matched gaussian analyzer. The ratio of peak signal to image magnitudes is

$$\frac{r_{1m}}{r_{2m}} = \sqrt{\frac{2b_2}{b_1}} \quad (4.47)$$

and by Eq. (4.12) may in theory be made arbitrarily large by increasing analyzer scan rate.

These results may be compared with those of the general case in Section III.

3. Time Separability of Signal and Image Responses

In Eq. (4.18) the peak value of $r_1(t)$ occurs for an input frequency $f_1 = f_2$ and the peak value of $r_2(t)$ occurs for $f_1 = -f_2$. The separation in time between the two peaks is approximately $2f_2/s_1$ for a particular input signal by Eq. (4.25e). Let the peak magnitude of $r_2(t)$ obtained when $f_1 = f_2$ be designated as the "image" amplitude r_{im} ; from Eq. (4.18),

$$\frac{r_{im}}{r_{1m}} = \sqrt{\frac{\lambda_1}{\lambda_2}} e^{-4\alpha_4 \omega_2^2} \quad (4.48)$$

Using Eqs. (4.10), (4.14), and (4.18e),

$$\frac{r_{im}}{r_{1m}} = \sqrt{\frac{\lambda_1}{\lambda_2}} \exp \left[\frac{-16 f_2^2}{\frac{b_2^2}{\rho_2} + \frac{b_3^2}{\rho_1}} \right] \quad (4.48a)$$

When r_{im}/r_{1m} is sufficiently small (as determined by the desired dynamic range), signal and image responses are time-separable.

A desired magnitude of r_{im}/r_{lm} in Eq. (4.48) may be assured by making the center frequency f_2 of the filter large compared to b_2 and b_3 .* Figure 15 shows that a ratio of at least 10^4 , or 80 db, may be obtained by requiring that the exponential factor $4\alpha_4\omega_2^2 \geq 10$. Taking (from Table 3) $\rho_1 \approx \rho_2 \approx 0.3$ as typical, a reasonable criterion might be:

$$f_2^2 \geq 2(b_2^2 + b_3^2) \quad (4.49)$$

Criteria for different dynamic ranges might be similarly obtained.

The lowpass case results when $f_2 = 0$. There is no question of the time separation of $r_1(t)$ and $r_2(t)$, since they are essentially coincident (including harmonic responses when η is an integer greater than one); separation must be made on the basis of differences in magnitude and form. By Eq. (4.19), improvements in sensitivity and resolution of one component by phase-matching will necessarily cause the other to spread out and become a "clutter" level, containing half the signal energy and reducible to negligible magnitude only if the ratio λ_2/λ_1 can be made very large.**

When $t = \tau_2$ and $f_1 = f_2 = 0$,

$$r_m = \frac{a_1 A_2}{2} [\lambda_1^{-1/2} \cos \theta_{31} + \lambda_2^{-1/2} \cos \theta_{32}] \quad (4.50)$$

The maximum amplitude is critically dependent upon the phase of the response at the peak, which may well determine the form of the response.

From a practical point of view, the use of lowpass rather than band-pass filters and amplifiers may sometimes be desirable. In addition, elimination of frequency ambiguity caused by displaced images and harmonic responses reduces spurious responses. The advantages of the low-pass case do not appear to have been fully exploited.

* The ratio λ_2/λ_1 will typically be greater than one, but is usually bounded by practical limitations.

** Since the peak responses are obtained with signals of the same frequency, there is no possibility of image band noise reduction by pre-translation filtering.

V. EXPERIMENTAL RESULTS

While investigation of phase-matched filters for scanning spectrum analyzer use is still in the early stages, practical filters have been constructed which demonstrate significant performance improvement over that obtainable without matching. Results obtained with two filters designed for very high scan rates are described in this section, both to show agreement with theoretical results predicted above and to illustrate different approaches to filter realization.

A. A CONTINUOUSLY DISPERSIVE HELIX FILTER

A microwave helix filter has been constructed which has properties appropriate for the scanning spectrum analyzer application.* The response of this filter to scanning excitation may be used to illustrate application of the theoretical curves of Section IV.

The structure is similar to the helix used in traveling-wave amplifier tubes, consisting of a precisely wound wire helix transmission line enclosed in a glass tube for mechanical support and dielectric loading. As shown in Fig. 24, the helix has a total length of about 50 cm and is made in two sections to reduce overall length, with sliding couplers to permit length variation.

As with many broadband slow-wave propagating structures, the helix exhibits dispersion; that is, over a range of frequency the propagation velocity varies with frequency, so that transmission delay is likewise a function of frequency. Selection of the frequency range over which time delay variation is closely linear with frequency provides the phase characteristic needed for phase matching, as discussed in Section II-B-1. In this case the scan rate must be negative (decreasing frequency with time), since helix transmission delay increases with frequency.

* The helix described here was constructed by V. Dunn of the Stanford Electronics Laboratories and is treated in a separate report [Ref. Du-1].

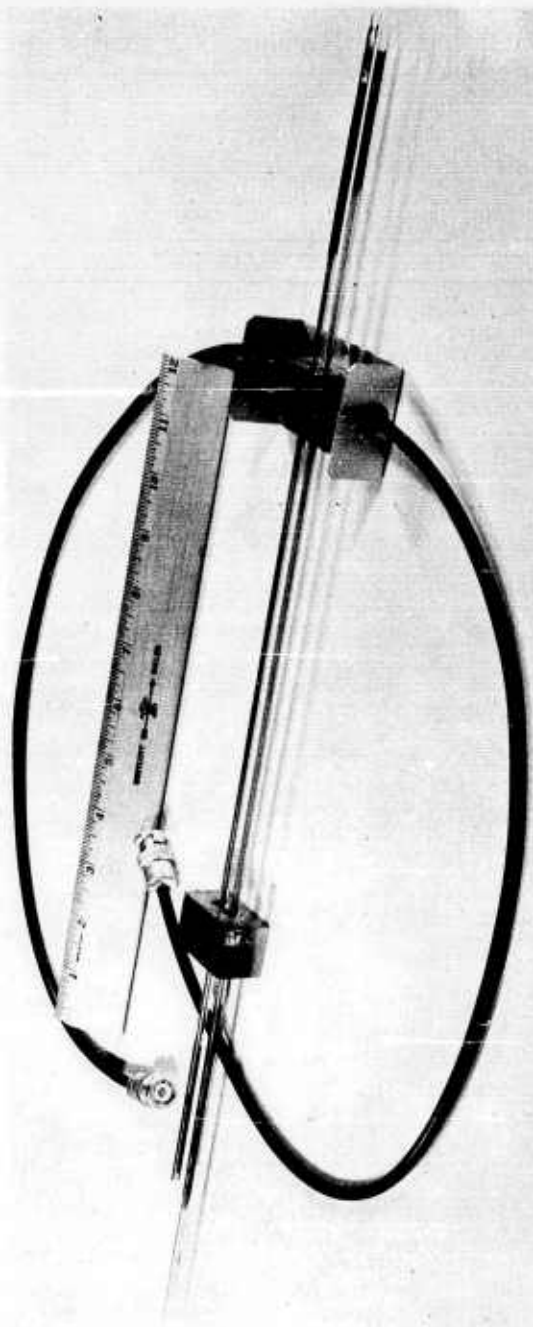


FIG. 24. EXPERIMENTAL HELIX TIME COMPRESSION FILTER.

Figure 25 shows the measured helix transmission delay versus frequency, which is approximately linear from 900 to 1400 Mc. Over this frequency range the differential time delay is about 0.5 nanoseconds per centimeter (ns/cm), giving a total of 25 ns for the helix length of 50 cm.

The experimental arrangement is shown in Fig. 26. An input signal is "mixed" with a reference oscillator which is scanning at the rate corresponding to a frequency change of 500 Mc in 25 ns, or -2×10^{16} cps² (-20 Mc/ns). The scanning difference frequency is applied to the helix through a bandpass filter. The filter restricts transmission to the frequency range over which delay variation is approximately linear and provides (in combination with the helix) a composite amplitude

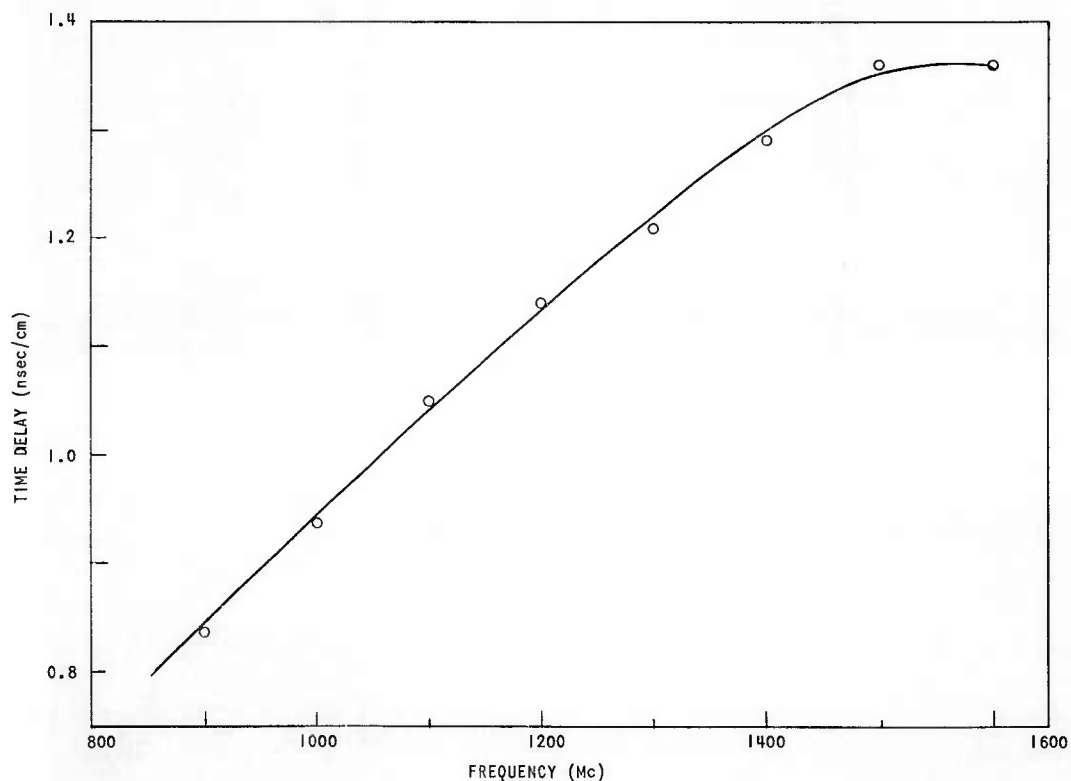


FIG. 25. TIME DELAY VS FREQUENCY OF THE HELIX DISPERSIVE FILTER.

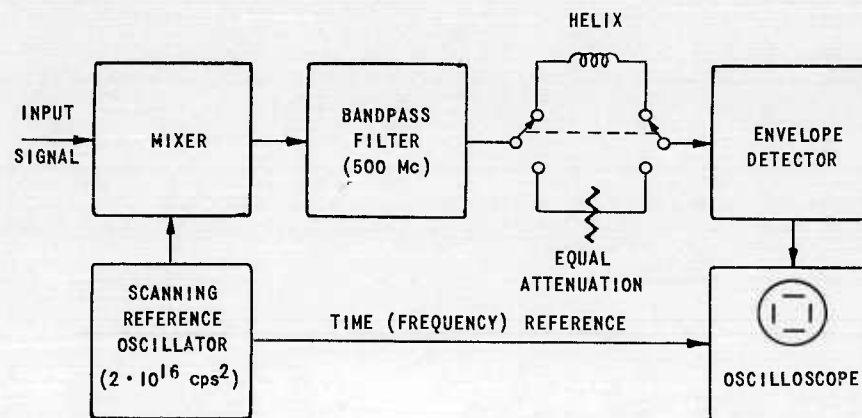


FIG. 26. EXPERIMENTAL ARRANGEMENT OF QUADRATIC-PHASE HELIX FILTER USED WITH A SCANNING SPECTRUM ANALYZER.

versus frequency bandpass characteristic that is approximately gaussian, so that the results of Section IV are applicable.* The envelope of the response of the helix-filter combination to the scanning excitation is obtained by detection and is displayed on an oscilloscope as a function of time (and hence of signal frequency).

Figure 27 shows the variation of output response duration as the helix length L is varied. (Variation in L corresponds to a variation of the phase-matching parameter η .) When $L = 0$ the filter bandwidth of 500 Mc is traversed in 25 ns at the scan rate of 20 Mc/ns. Output pulse duration decreases to a minimum of 1.6 ns as the helix length is increased to 45 cm, which corresponds to a frequency resolution of 32 Mc. Further increase of L produces rapid increase in pulse duration.

Variation of response amplitude with helix length is shown in Fig. 28 (corrected for non-scanning helix attenuation). Again, increase of helix length results in increased response amplitude as the response

* The nonlinear phase characteristic of the bandpass filter must also be considered, but is negligible compared to that of the helix.

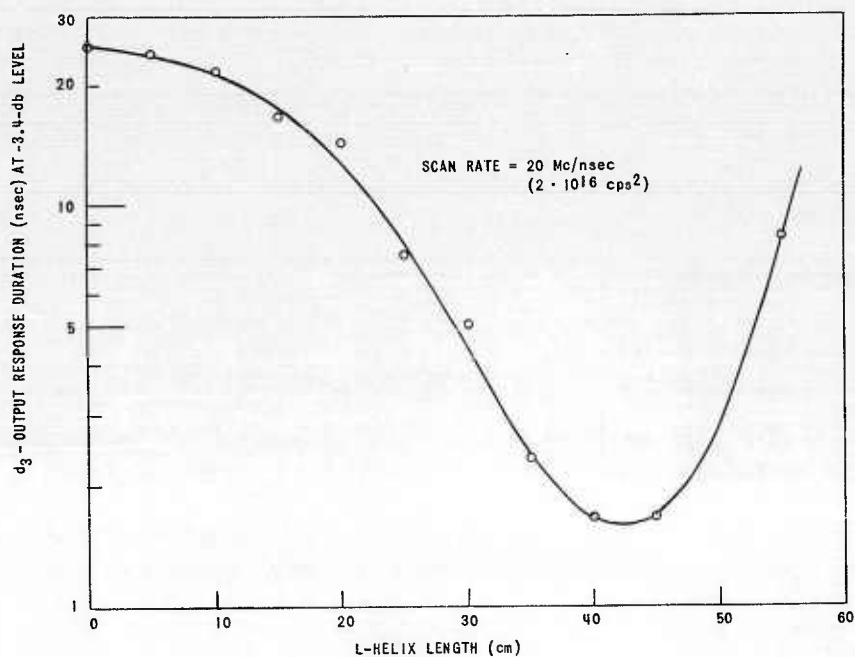


FIG. 27. RESPONSE DURATION OF HELIX FILTER TO SCANNING INPUT SIGNAL VS HELIX LENGTH.

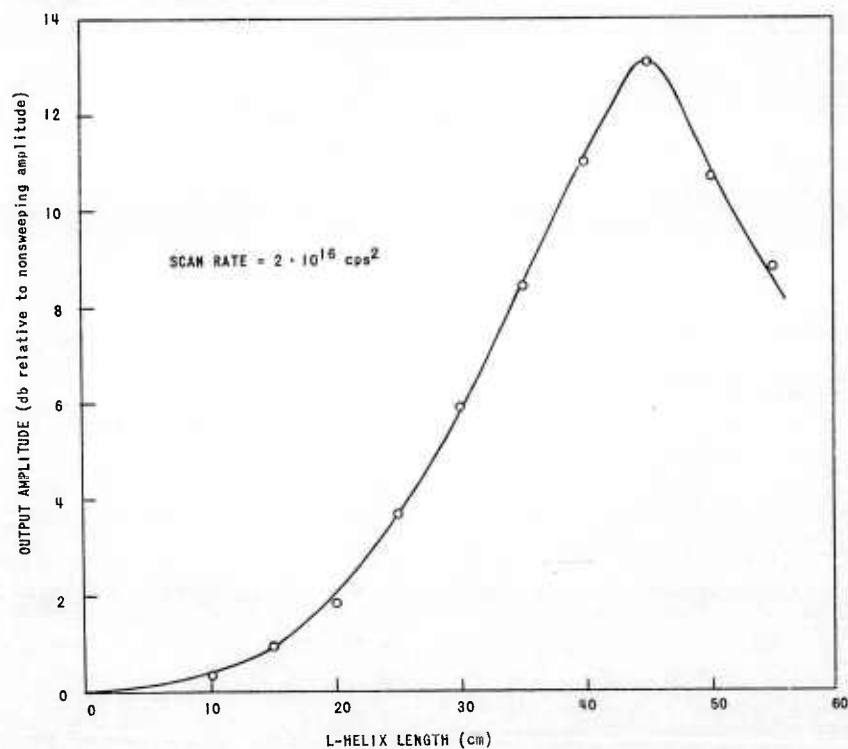


FIG. 28. AMPLITUDE OF HELIX FILTER RESPONSE TO SCANNING INPUT VS HELIX LENGTH (RELATIVE TO NONSCANNING AMPLITUDE).

duration is decreased, with a maximum amplitude increase of 13 db obtained at the optimum helix length of about 45 cm. A helix length $L = 45$ cm corresponds to the phase-matched condition obtained when $\eta = 1$. Since the experimental filter bandwidth is primarily determined by the bandpass filter and helix couplers, both reduction of response duration (hence improved frequency resolution) and increase in peak amplitude (hence improved sensitivity) are accomplished without significant change in overall filter bandwidth.

These experimental results are compared in Table 4a with the theoretical gaussian filter results of Section IV. Response duration and amplitude are obtained from Figs. 16 and 17, respectively, where the filter bandwidth of 500 Mc and scan rate of 2×10^{16} cps² correspond to a normalized filter bandwidth factor of 5. Observed response duration of 1.6 nsec compares with a theoretical prediction of 1.0 nsec. The observed amplitude increase of 13 db compares with a theoretical value of 14 db. Since the variation of time delay with frequency of the experimental filter as shown in Fig. 25 is not perfectly linear, the agreement of experimental and theoretical results seems reasonable. The poorer agreement of predicted resolution with observed resolution, as shown in Table 4a, is due in part to increase in response duration resulting from detection and oscilloscope bandwidth, and in part to the greater sensitivity of response duration to error in the phase characteristic.

Comparison is made in Table 4b between the experimental results using the phase-matched helix and the best theoretical performance obtainable with a linear phase (non-phase-matched) filter. At the scan rate of 20 Mc/nsec, Eq. (4.39) gives an optimum gaussian linear-phase filter bandwidth of 100 Mc. From Figs. 22 and 23 the improvement in resolution and sensitivity obtained by use of the optimum linear-phase bandwidth of 100 Mc as compared to the 500 Mc-bandwidth used in the helix filter is seen to result in a resolution of 141 Mc and sensitivity improvement of 5.5 db. Phase matching provided by the helix has improved analyzer resolution by a factor of approximately 5 and sensitivity by 7.5 db over the best obtainable at this scan rate with a linear-phase filter. Furthermore, the experimental helix-filter combination described here has a time-bandwidth factor of only 12.5; considerably greater improvement appears feasible.

Table 4 - COMPARISON OF THEORETICAL AND EXPERIMENTAL RESPONSE OF HELIX
FILTER TO SCANNING EXCITATION.*

scan rate	$s_1 = -2 \times 10^{16} \text{ cps}^2$ (=20 Mc/nsec)
helix filter bandwidth	$b_2 \cong 500 \text{ Mc}$
optimum linear-phase bandwidth	$\sqrt{K_2 s_1} = 100 \text{ Mc}$

a. Comparison of experimental and theoretical response

	<u>Theoretical</u>	<u>Experimental</u>
increase of peak amplitude (over linear-phase filter)	14 db	13 db
response duration	1 nsec	1.6 nsec
frequency resolution	20 Mc	32 Mc

b. Improvement of phase-matched results over optimum linear-phase
filter performance

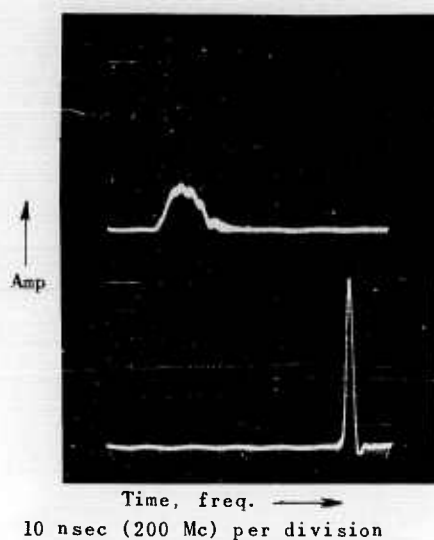
	<u>Theoretical</u>	<u>Experimental</u>
frequency resolution improvement	7.1	4.1
S/N increase	8.5 db	7.5 db

* Using theoretical results obtained in Section IV for the gaussian case, with "energy" bandwidth and pulse duration defined at the -3.4 db level relative to the peak for which $\rho_2 = \pi/8$ and $K_2 = 0.5$ (see Table 3).

A graphic illustration of performance improvement due to phase-matching is provided by the oscilloscope photographs of Figs. 29 through 33. Figure 29 shows the detected envelope of filter response before and after transmission through the phase-matching helix. The output pulse is delayed in time, decreased in duration, and increased in amplitude. Figure 30 demonstrates resolution improvement by showing two superimposed scanning input signals displaced in frequency by 100 Mc, hence displaced in time by 5 ns at the scan rate of 20 Mc/ns. The two signals are not distinguishable in Fig. 30b at the helix input, but are clearly resolved at the helix output in Fig. 30c.

Sensitivity improvement provided by the helix is shown in Fig. 31. Figure 31a shows input noise alone, introduced by amplification preceding the signal translation process (mixer). While not uniform in spectral distribution, the noise nevertheless extends over a range of frequency that is large compared to the filter bandwidth. A signal immersed in noise in Fig. 31b becomes evident in Fig. 31c after transmission through the helix. The nature of the noise is not significantly altered by the quadratic phase characteristic of the helix, as may be noted by comparison of Figs. 31a and 31b with Fig. 31c.

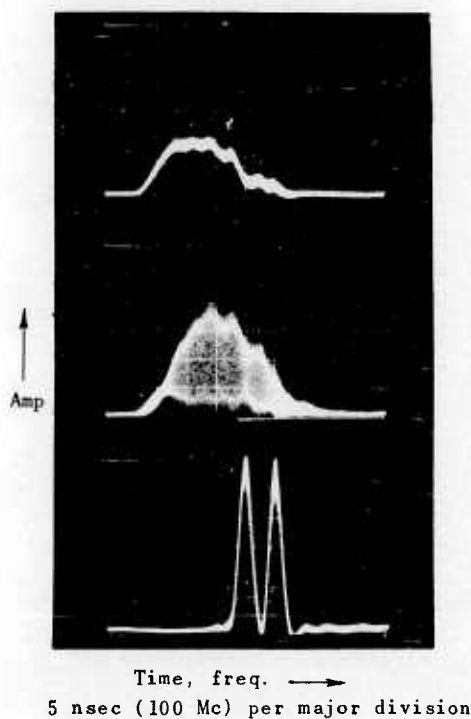
Two illustrations of spurious response reduction are illustrated by Figs. 32 and 33. In Fig. 32a two scanning signals are shown at the helix input which result from normal and second harmonic responses (i.e. difference frequencies which result from the mixing of signals with the fundamental and second harmonic of the scanning reference oscillator). The second harmonic response scans at twice the rate for which the delay characteristic of the helix was designed, thus requiring half the normal time to traverse the filter bandwidth. Directly below each in Figs. 32b and 32c are shown the corresponding output responses with correct relative magnitudes. The normal signal is decreased in duration and increased in magnitude as in Fig. 29; as indicated in Fig. 19, the second harmonic response ($\eta = 2$) emerges with the same amplitude as in the linear-phase case. The harmonic response also appears reversed in time as predicted by Eq. (4.33a) when $\eta = 2$, which indicates that the output response will scan in the reverse direction to that of



a. Scanning input signal amplitude (detected envelope)

b. Output amplitude from helix. (Correct relative time and amplitude)

FIG. 29. HELIX FILTER RESPONSE TO SCANNING INPUT SIGNAL.



a. Single scanning input signal

b. Two superimposed scanning input signals (separated by 5 ns)

c. Resolved output responses (correct relative amplitude)

FIG. 30. RESOLUTION OF SUPERIMPOSED SCANNING INPUT SIGNALS BY HELIX FILTER.

a. Noise alone at helix input

b. Signal in noise at helix input

c. Signal in noise at helix output (correct relative amplitudes)

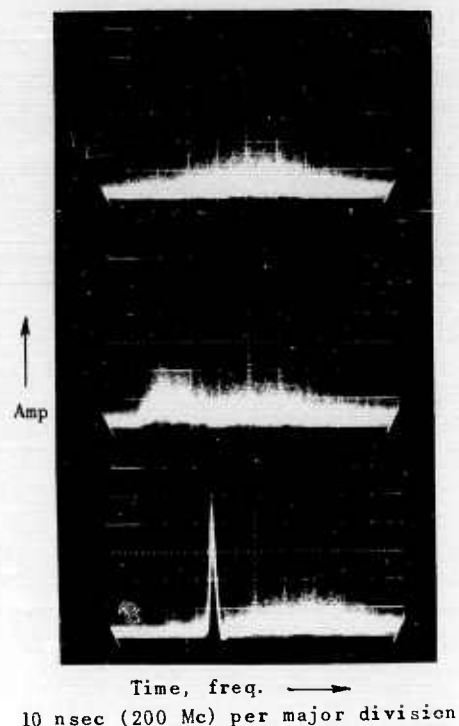
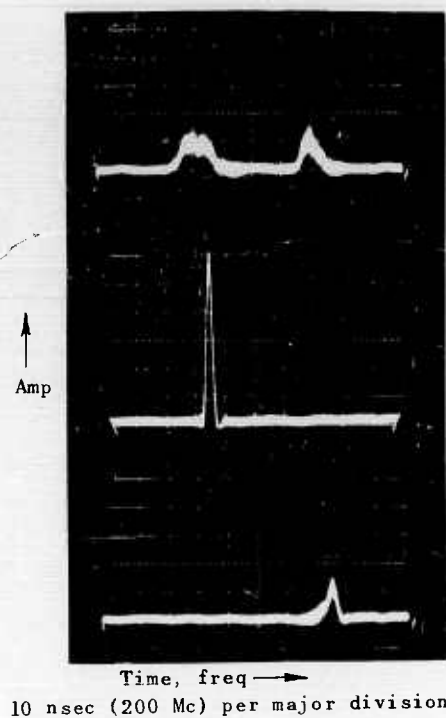


FIG. 31. IMPROVEMENT OF S/N RATIO WITH HELIX FILTER.

the input signal. The excessive delay slope for the increased scan rate turns the response "inside out" in time.

In Fig. 33 a comparison is made between the normal and image responses (the latter resulting from an input to the filter which is scanning at the correct rate but in the wrong direction). Figure 19 shows a theoretical reduction of 3 db below the linear-phase response and 17 db below the phase-matched response for the image amplitude ($\eta = -1$) when the normalized bandwidth is 5. Normal and image signals at the helix input are shown in Fig. 33a with equal magnitudes.* Corresponding responses beyond the helix are shown in Figs. 33b and 33c for the normal and image responses, respectively, at the correct relative magnitudes. The image response is reduced in magnitude and spread out in time, as would be

* Differences in form are due to transient effects and reflections within the helix.

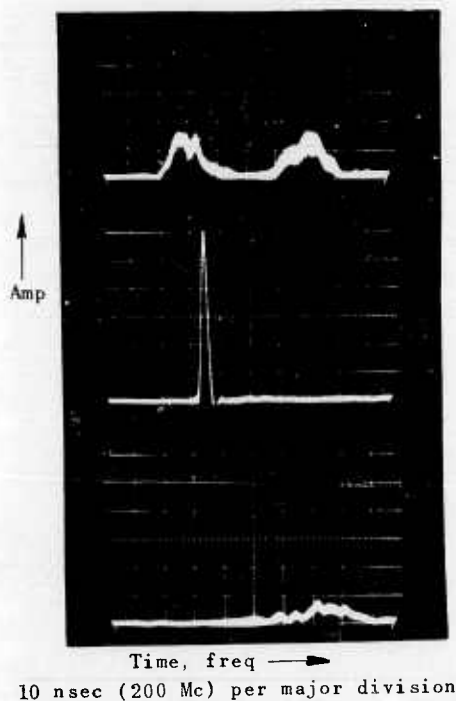


a. Scanning signals at helix input due to normal and second harmonic (twice normal scan rate)

b. Output due to normal signal

c. Output due to second harmonic (correct relative amplitudes)

FIG. 32. HARMONIC RESPONSE OF HELIX FILTER.



a. Normal and image signals at input of helix (image scanning in reverse direction)

b. Output due to normal scanning signal

c. Output due to image response (correct relative amplitudes)

FIG. 33. IMAGE RESPONSE OF HELIX FILTER.

anticipated with the delay characteristic of the helix serving to increase, rather than decrease, the dispersion of the scanning signal energy with time.

The helix here described provides an illustration of a continuously dispersive phase-matched delay structure that is useful for scanning spectrum analyzer application. The results described above agree well with theory, and demonstrate that useful performance improvement can be achieved with even the modest time-bandwidth product obtained in this case. The nature of the helix dispersion limits its application to the down-sweeping case. The high attenuation* makes it most appropriate for high scan rates where bandwidth is large and time delay variation is small.

Other continuously dispersive delay structures that might be useful in this application include cascaded all-pass lattice networks [Refs. Kr-1,2, Om-1,2], waveguides near cutoff [Refs. Da-1, Di-1], broadband antennas [Ref. Pu-1], and a variety of slow-wave microwave structures (e.g. the Karp circuit, meander line, etc.). Any storage medium in which the transmission time can be made a linear function of frequency is a potential candidate for consideration as a dispersive phase-matched delay structure to be used in scanning spectrum analyzer applications.

B. A TAPPED DELAY LINE TIME COMPRESSION FILTER

Filter synthesis with tapped delay lines provides a powerful method for obtaining "matched" filters with large time-bandwidth products [Refs. El-1, Tu-1]. A delay structure employing bandpass filters at a series of taps may be used to form the linear time delay versus frequency characteristic needed for the scanning spectrum analyzer application.** The method has considerable flexibility, being adaptable to either positive or negative scanning and to a wide range of scan rates.

*The attenuation of the helix used here was about 0.4 db/cm plus approximately 2 db for each coupler.

**A similar arrangement for obtaining improved resolution in scanning receivers is mentioned in a U. S. Patent [Ref. Wr-1].

A discrete experimental tapped delay arrangement is shown in Fig. 34.* A series of bandpass filters are connected between taps of the delay structure as shown. The center frequencies $f_1, f_2, \dots, f_n$ of the filters are displaced so as to encompass the total frequency range of interest. The total delay provided by each path through the structure is made appropriate for the frequency transmitted by the filter and the scan rate, so that at the output the responses from all filters are superimposed in time.

The experimental filters used were constructed in strip transmission line, shown in Fig. 35 with one ground plane removed. Five filters were employed, spaced in 200 Mc increments from 1050 Mc to 1850 Mc. Each filter is a resonant hybrid, designed to provide frequency selective coupling as well as power division with good impedance-matching properties. The scan rate of 2×10^{16} cps² requires approximately 5 nsec of delay between adjacent filters in each branch, provided in this case by coaxial cable. The delay between sections is adjusted to provide an in-phase condition from all filters at the center of the output response envelope. Each filter has a bandwidth of about 100 Mc (as determined by the degree of coupling), which is the optimum linear-phase filter bandwidth for the scan rate employed.

The parameters of the filters were chosen with the intention of using ten filters, each of 100 Mc bandwidth, to provide a total bandwidth of 1000 Mc from 1000 to 2000 Mc. The response with only the five filters of Fig. 35 is shown in Fig. 36. The detected envelope of the signal applied to the input of the filter is shown in Fig. 36a. The envelope of the response obtained after delay and recombination is shown in Fig. 36b. As in the case of the helix, the duration is decreased, the amplitude increased, the response delayed in time, and the sensitivity improved.**

* The filter described here was constructed by M. Musser of the Stanford Electronics Laboratories, and will be described in more detail in a later report.

** No compensation for filter attenuation has been made in Fig. 36, as evidenced by the decrease in noise level.

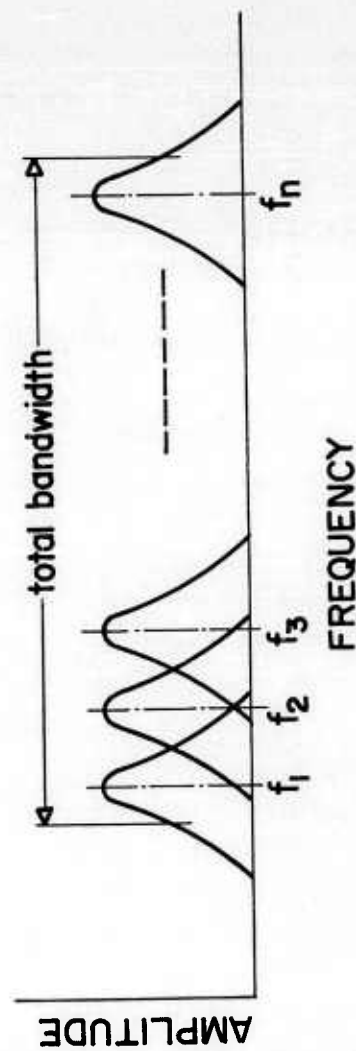
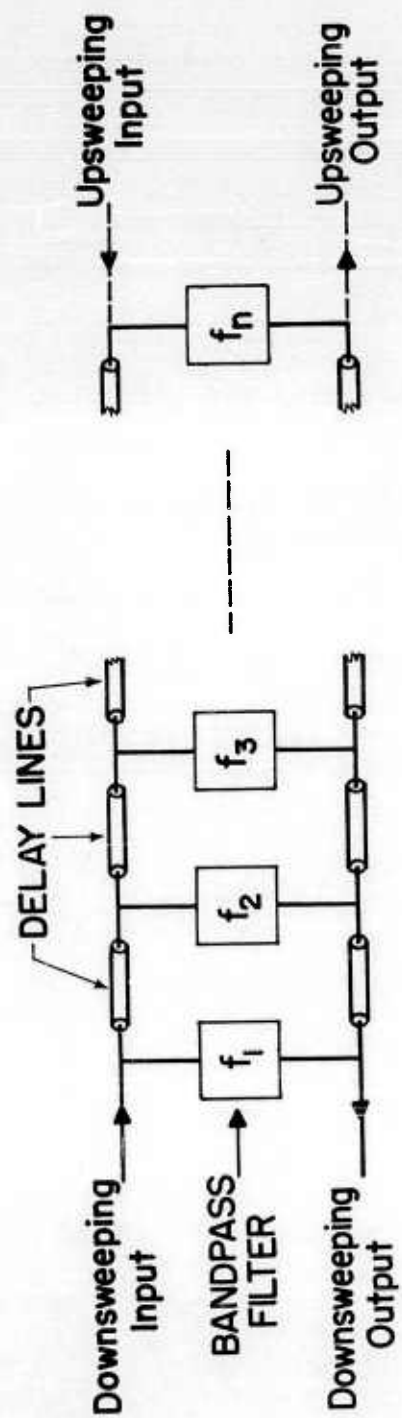


FIG. 34. A DISPERSIVE FILTER EMPLOYING A TAPPED DELAY LINE.

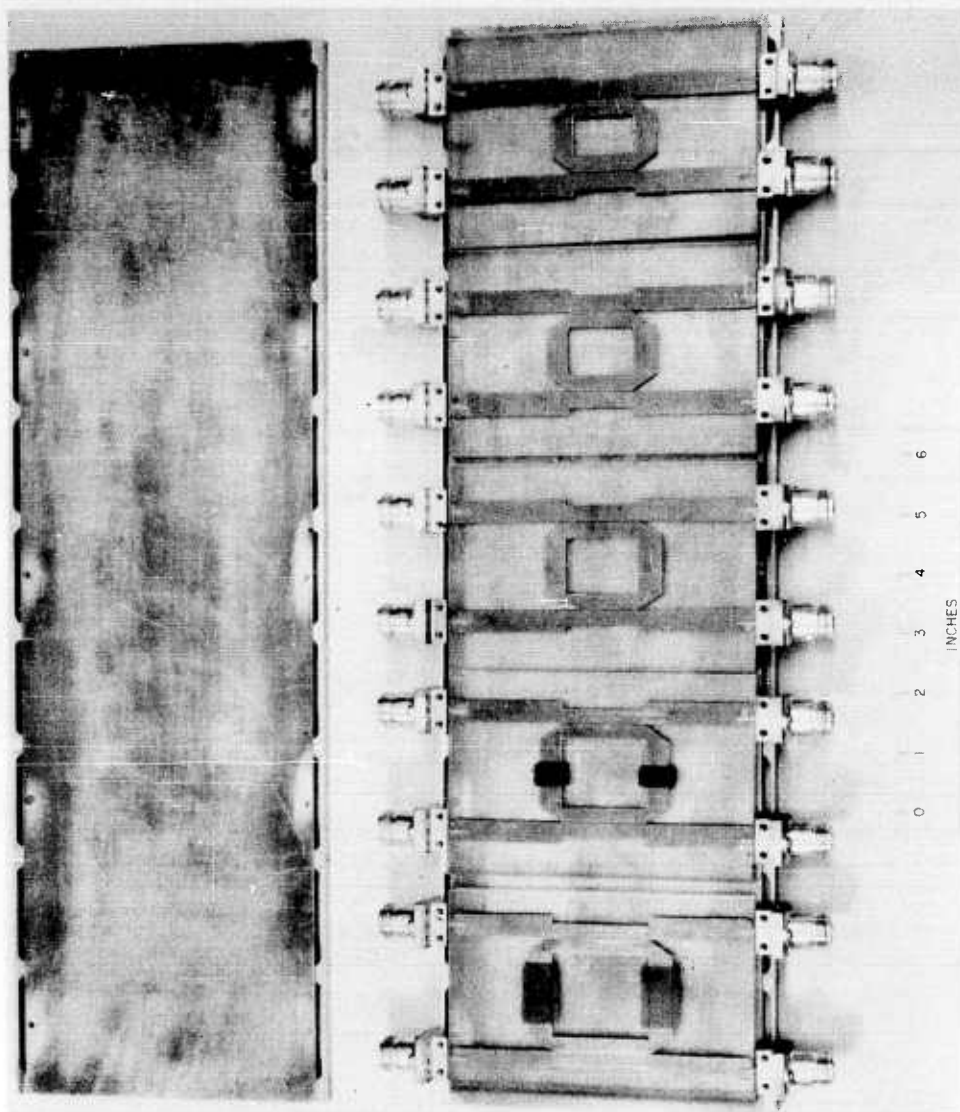


FIG. 35. STRIP-LINE HYBRID FILTERS USED IN AN EXPERIMENTAL
TAPPED DELAY LINE DISPERSIVE FILTER.

The superposition of the several filter responses with time coincident envelopes but with frequencies differing in increments of approximately 200 Mc results in a time function which is similar in many respects to an antenna array pattern. The response is characterized by successive major peaks due to the finite number of filter elements, minor peaks (sidelobes) resulting from the discrete elements, and a width of the major peak determined by the total bandwidth encompassed. The two minor peaks evident in Fig. 36 are the adjacent major responses that have been reduced in magnitude by the selectivity of the individual bandpass filters. As the number of delay taps and filters is increased the major responses are further separated, and hence further reduced in magnitude by the selectivity of the filters.

Reduction of the amplitude of the minor peaks between major responses may be accomplished by specification of the amplitude of the contributions from the various filters, in the same way that sidelobes in an antenna array pattern are reduced. This is accomplished in the filters of Fig.

a. Scanning input signal amplitude
(detected envelope)

b. Output response amplitude (correct
relative time)

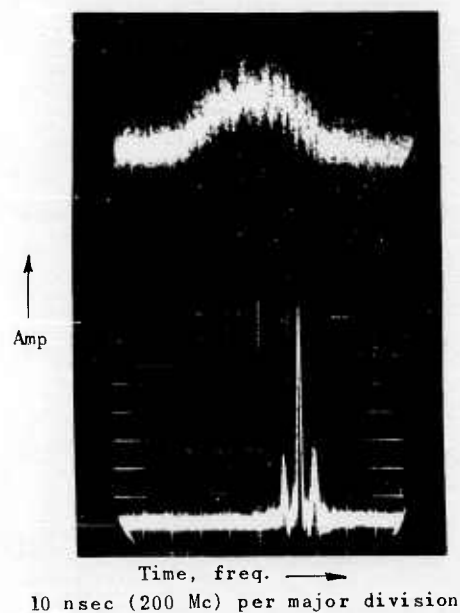


FIG. 36. RESPONSE OF EXPERIMENTAL
TAPPED DELAY LINE DISPERSIVE FILTER
TO SCANNING INPUT SIGNAL

35 by the use of dissipative material (aquadag) painted on the hybrid ring, visible on the two lower frequency filters on the left.

Despite some disadvantages arising from minor responses and critical time delays, the discrete delay structure has several advantages. Separate adjustment of phase and amplitude characteristics is not difficult. Broadband non-dispersive delay structures tend to have less loss than those with inherent dispersion, at least among structures known to the author. Scanning that is either increasing or decreasing in frequency can be accommodated by appropriate arrangement of delay taps and filters. The use of lumped delays at lower frequencies and scan rates, where large delay is needed, appears to be feasible.

VI. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

1. The Historical Misconception

For many years the performance of the scanning spectrum analyzer has been thought to be limited by the rate of scanning. Attempts to improve frequency resolution and sensitivity by reduction of the analyzer filter bandwidth led to formulation of an "optimum" bandwidth versus scan rate relation expressed by Eq. (2.4). This relation has long been used in the design of scanning spectrum analyzers and "panoramic" (scanning intercept) receivers.

Only recently has Eq. (2.4) been recognized as a valid criterion for optimum performance only if phase terms resulting from the scanning process are ignored in filter design. Scanning spectrum analyzer performance is not (in theory) limited or necessarily determined by scan rate. Furthermore, signal energy need not be lost and analyzer sensitivity need not suffer as the frequency resolution capability is improved, another common misconception. The required analyzer filter may be difficult to construct, but this is a matter of technique rather than fundamental in nature.

In the past, the scanning analyzer has been unfavorably compared with the so-called "comb-filter" spectrum analyzer. In theory the two methods have comparable sensitivity and resolution when the duration of measurement is equal, or limited by the duration of the signal. The essential difference between the comb-filter and scanning analyzers is that the comb filter samples in frequency (giving continuous time information) while the scanning analyzer samples in time (giving continuous frequency information).

2. Potential Scanning Analyzer Performance

Perhaps the most interesting result of this study is the demonstration that signal analysis by the scanning spectrum analyzer is in theory limited only by the signal itself, not by the measurement technique. In the limit, as analyzer scan rate and filter bandwidth are made large, signal detectability in the presence of thermal noise

is determined by available signal energy, and frequency resolution is determined by signal duration. The goal of spectrum analysis may be achieved in the sense that the time function response of the analyzer approaches (in the limit) the Fourier spectrum of the input signal.

In practice, analyzer scan rate and bandwidth are bounded by practical considerations. When analyzer bandwidth over which filter phase can be matched is limited, the scanning spectrum analyzer provides in effect a spectrum "window" of known form which "slides" across the signal spectrum as a function of time. The form of the sliding spectrum window (when filter phase is correct) is that of the Fourier transform of the filter spectrum function magnitude. Details of the response of the analyzer (frequency width, sidelobes, clutter level, rate of rise and fall at the edges of the response) to a signal component of known form may be determined from the shape of the scanning spectrum window. Conversely, a desired representation of a signal component may be used to prescribe the form of the spectrum window, and hence the magnitude versus frequency function of the analyzer filter. In this way, frequency resolution and dynamic range may be inferred from the shape of the spectrum window of the analyzer, and thus a means for evaluation and design is provided. Optimization of signal analysis by "matching" the spectrum window to a suitably chosen signal representation* is possible.

A consequence of the simple signal translation (mixing) process assumed here is the presence of an "image" response. Other spurious responses may result from the presence or generation of harmonics of the scanning reference signal in connection with the spectrum translation process. It has been shown that the spurious image and harmonic responses may be identified by change in form and reduction in amplitude relative to the desired response. In theory, spurious responses of this nature may be made negligibly small in magnitude by increasing analyzer scan rate and filter bandwidth.

3. Translation-Scanning Spectrum Analyzer Design

The attainment of potentially achievable analyzer performance with a time-invariant linear filter requires the following conditions:

* By a set of similar elementary signals, in the manner proposed by Gabor [Ref. Ga-1] and Lerner [Ref. Le-1].

- a. The scanning must be linear with time.
- b. Scanning must be accomplished within the duration of the signal to be analyzed. (Stated another way, the response of the analyzer represents the spectrum of that portion of the signal during which scanning takes place and the analyzer "sees" the signal).
- c. The phase characteristics of the analyzer filter must be "matched" to the quadratic phase terms arising from the scanning operation. (This means that the transmission delay through the filter varies with frequency in a manner that complements the scanning of the signal by the spectrum translation process).
- d. The bandwidth of the analyzer filter must be large compared to the total spectrum width of the signal to be analyzed.

When these conditions are satisfied, the analyzer response is "signal-determined", meaning that signal detectability in the presence of thermal noise is determined only by available signal energy, and frequency resolution is limited only by signal duration.

In practice, these requirements are difficult to satisfy. Failure to meet conditions (a) and (c) limits attainable performance as bandwidth and scan rate are increased. Scanning nonlinearity limits useful filter bandwidth and gives different performance for input signal components of differing frequencies. A difference between measurement and signal durations results in loss of signal energy or increased noise, with resulting loss of sensitivity and either reduction of frequency resolution or probability of signal interception.

Perhaps the most difficult requirements to satisfy is that of wide filter bandwidth. If all requirements except (d) are met, the form of the response is that of the convolution of the signal spectrum with a scanning frequency "window" of finite width. The form of the sliding spectrum window is that of the Fourier transform of the analyzer filter magnitude-versus-frequency function.

Various scanning window functions may be selected. The gaussian window of Section IV (obtained with a gaussian filter) gives good sensitivity, smooth "skirts," and good resolution for signal components of comparable amplitude. If resolution of closely spaced spectrum components over a large dynamic range is an important capability, a steep-sided rectangular window may be obtained with a filter having a

($\sin \omega/\omega$) magnitude characteristic. Signal representation by elementary functions in the manner proposed by Gabor and Lerner may suggest other desirable window characteristics, such as orthogonality, rapid decay away from a central peak in both time and frequency, absence of significant minor maxima (sidelobes), mathematical convenience, etc. A desired form of frequency window together with the phase-matching requirement specify the spectrum function of the analyzer filter.

While practical limitations may never permit complete realization of the requirements listed above, significant improvement of scanning spectrum analyzer performance over that now commonly accepted results from even a modest degree of phase matching, as demonstrated by the experimental results described in Section V. A measure of the degree of improvement is the time-bandwidth product of the analyzer filter over which phase matching can be successfully accomplished.

B. RECOMMENDATIONS FOR FURTHER WORK

1. Improved Filters for Spectrum Translation Scanning Analyzers

The synthesis of filters of specified magnitude and phase functions having large time-bandwidth products is of considerable interest in many fields of communications. Known techniques could be usefully exploited at the present state of the art to improve scanning spectrum analyzer performance. Time compression techniques used in "chirp" radar are directly applicable where the time duration and scan rate of the chirp signal are appropriate. Signal durations and spectrum widths that are considerably different from those of the radar situation are also of interest in spectrum analyzer applications, however. Many information storage techniques having large time-bandwidth capability are potential candidates for scanning spectrum analyzer application.

2. Time-Varying (Scanning) Filters

This report has been concerned with the spectrum analyzer which scans by signal translation past a fixed (time-invariant) filter. While this approach has been widely used because of its flexibility and ease of filter realization, the translation-scanning analyzer has disadvantages consisting of spurious responses, signal energy loss, and

introduction of excess noise which result from the signal translation process.

A similar study, to be made in the future, might be concerned with the performance of a phase-matched time-varying filter in the scanning spectrum analyzer application. Interest in the synthesis of time-varying networks and the current development of voltage-or current-dependent network components may well provide the techniques needed to realize a scanning filter appropriate for the spectrum analyzer application. In addition to reduction of spurious responses and possible sensitivity improvement, the scanning filter analyzer might ultimately prove to be less complicated than the translation-scanning spectrum analyzer of equivalent performance.

3. Other Approaches

The discussion in this study has been restricted to single-channel additive (linear) filters combined with a single non-linear (translation) process. In addition to consideration of the time-varying filters as recommended in the preceding paragraph, it is possible that consideration might profitably be given to the use of several parallel channels, which would offer additional dimensional freedom (e.g. for the elimination of the image response). Nonlinear signal processing methods might prove advantageous in reducing spurious responses, increasing dynamic range, and in matching input signals.

4. Other Applications of the Analytical Methods Developed

The spectrum analyzer discussed in this study is concerned with examination of the frequency description of signal-time functions. Transformation from one dimension to another by a scanning process is a fundamental analytical technique, however. Many of the remarks made here about improvements in the scanning process might be applied to pairs of variables other than frequency and time.

APPENDIX A: PEAK AMPLITUDES AND WIDTHS OF FOURIER TRANSFORMS

Let $u(t)$ be a real time function with maximum amplitude u_m ; let $U(\omega)$ be the Fourier transform of $u(t)$ with maximum magnitude U_m ; define the bandwidth b of $U(\omega)$ as

$$b = \frac{\int_{-\infty}^{\infty} U(\omega) U^*(\omega) d\omega}{4\pi U_m U_m^*} = \frac{\int_{-\infty}^{\infty} |U(\omega)|^2 d\omega}{4\pi |U_m|^2} \quad (A.1)$$

Similarly, define the "duration" d of $u(t)$ as

$$d = \frac{\int_{-\infty}^{\infty} u^2(t) dt}{u_m^2} \quad (A.2)$$

Designate the product of duration and bandwidth as $K = bd$; by the energy theorem of Fourier transforms [Refs. Be-1, Sn-1],

$$E = \int_{-\infty}^{\infty} u^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |U(\omega)|^2 d\omega \quad (A.3)$$

Hence:

$$u_m = \sqrt{\frac{2}{K}} b_n |U_m| \quad (A.4)$$

and

$$E = u_m^2 d = |U_m|^2 b \quad (A.5)$$

The widths so defined are termed the "energy" widths of the function in time and frequency, or the width of a rectangular function having the same total energy and maximum amplitude.

APPENDIX B: GENERAL RELATIONSHIP BETWEEN A SIGNAL SPECTRUM AND THE RESPONSE OF A TRANSLATION-SCANNING SPECTRUM ANALYZER

Consider the arrangement of Fig. 37. An input signal time function $u_1(t)$ with Fourier spectrum $U_1(\omega)$ is multiplied with the linearly scanning reference time function $u_2(t)$ to produce the translated signal function $u_3(t)$. The output time function $r(t)$ is the inverse Fourier transform of the product of the analyzer filter function $H(\omega)$ and the scanning spectrum function $U_3(\omega)$. We wish to relate $r(t)$ to $U_1(\omega)$ in order to infer characteristics of the input signal spectrum.

Assume a scanning reference time function in the form

$$u_2(t) = e^{-\alpha t^2} \cos(\sigma t^2 + \theta_1) \quad (\text{B.1})$$

where α is real and positive $\alpha > 0 < \alpha^2$

The scan rate s of the instantaneous frequency of $u_2(t)$ is $s = \sigma/\pi$ cycles per second per second. The constant α is an integration convergence factor later to be allowed to approach zero, and θ_1 is an arbitrary phase constant. Using the definite integral (Ca-1)

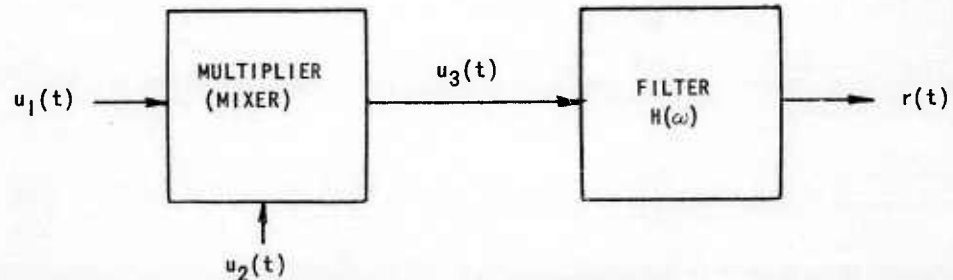


FIG. 37. BLOCK DIAGRAM OF A TRANSLATION-SCANNING SPECTRUM ANALYZER.

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{(b^2/4a)+c} \quad \text{Re } a > 0 \quad (\text{B.2})$$

the spectrum of $u_2(t)$ obtained from Eq. (B.1) by Fourier transformation may be expressed as the sum

$$U_2(\omega) = U_{21}(\omega) + U_{22}(\omega) \quad (\text{B.3})$$

where

$$(a) \quad U_{21}(\omega) = \frac{1}{2} \sqrt{\frac{\pi}{\alpha+j\sigma}} \exp \left[\frac{-\omega^2}{4(\alpha+j\sigma)} - j\theta_1 \right]$$

$$(b) \quad U_{22}(\omega) = \frac{1}{2} \sqrt{\frac{\pi}{\alpha-j\sigma}} \exp \left[\frac{-\omega^2}{4(\alpha-j\sigma)} + j\theta_1 \right] \quad (\text{B.4})$$

Later expressions will be simplified by using only one term of Eq. (B.3) and similar pairs to follow, remembering that the other components may be obtained by selection of the sign of the appropriate constants.

Since mixing is assumed here to mean multiplication of time functions, then $u_3(t) = u_1(t) u_2(t)$ and the filter excitation spectrum function $U_3(\omega)$ may be obtained by use of the Fourier transform convolution theorem. The convolution theorem states that the transform of the product of two functions is equal to the convolution of the transforms of the two functions. Thus, by the convention used here,

$$\begin{aligned} U_3(\omega) &= U_1(\omega) * U_2(\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U_1(\xi) U_2(\omega - \xi) d\xi \end{aligned} \quad (\text{B.5})$$

Introducing Eq. (B.4a)

$$U_{31}(\omega) = \frac{1}{4\pi} \sqrt{\frac{\pi}{\alpha + j\sigma}} \int_{-\infty}^{\infty} U_1(\xi) \exp \left[\frac{-(\omega - \xi)^2}{4(\alpha + j\sigma)} - j\theta_1 \right] d\xi \quad (\text{B.6})$$

The filter spectrum function $H(\omega)$ is also assumed to consist of two components:

$$H(\omega) = H_1(\omega) + H_2(\omega) \quad (\text{B.7})$$

where

$$(a) \quad H_1(\omega) = \frac{1}{2} A_2(\omega - \omega_2) \exp j[\mu(\omega - \omega_2)^2 - \theta_2]$$

$$(b) \quad H_2(\omega) = \frac{1}{2} A_2(\omega + \omega_2) \exp j[-\mu(\omega + \omega_2)^2 + \theta_2] \quad (\text{B.8})$$

Equation (B.8) identifies the real, even magnitude function $A_2(\omega)$, the quadratic phase parameter μ , the "center frequency" ω_2 , and the phase constant θ_2 . The constants μ , ω_2 , and θ_2 are real. The filter function is assumed in this form in order to be Hermitian (even real and odd imaginary terms) so that the impulse response $h(t)$ will be real.

The response time function $r(t)$ at the filter output is obtained by inverse Fourier transformation:

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U_3(\omega) H(\omega) e^{j\omega t} d\omega \quad (\text{B.9})$$

Denoting as $r_{11}(t)$ the output response component obtained from $U_{31}(\omega)$ and $H_1(\omega)$, use of Eqs. (B.6) and (B.8a) in Eq. (B.9) gives

$$r_{11}(t) = \frac{1}{16\pi^2} \sqrt{\frac{\pi}{\alpha + j\sigma}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(\xi) A_2(\omega - \omega_2) \exp \left\{ \left[j\mu - \frac{1}{4(\alpha + j\sigma)} \right] (\omega - \omega_2)^2 + j(\omega - \omega_2)t + j(\omega_2 t - \theta_1 - \theta_2) - \frac{(\xi - \omega_2)^2}{4(\alpha + j\sigma)} \right\} d\xi d\omega \quad (\text{B.10a})$$

where

$$\gamma = t + \frac{\xi - \omega_2}{2j(\alpha + j\sigma)} \quad (\text{B.10b})$$

Assume at this point that the nature of $U_1(\omega)$ and $A_2(\omega)$ permit the order of integration to be reversed. This essentially requires that these functions be mathematically "well-behaved", and is a reasonable assumption if the input signal and filter are to be physically realizable. Terms involving ω form the integral

$$I_1 = \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha + j\sigma}} \int_{-\infty}^{\infty} A_2(\xi) \exp \left[j\mu\xi^2 - \frac{\xi^2}{4(\alpha + j\sigma)} + j\xi\gamma \right] d\xi \quad (\text{B.11})$$

Equation (B.11) may be recognized as the inverse Fourier transform of the product of two spectrum functions, and may be evaluated by the convolution theorem and the definite integral of Eq. (B.2):

$$I_1 = a_2(\gamma) * (1 + 4\mu\sigma - j4\alpha\mu)^{-1/2} \exp \left[\frac{-(\alpha + j\sigma) \gamma^2}{1 + 4\mu\sigma - j4\alpha\mu} \right] \quad (\text{B.12})$$

where $a_2(t)$ is the inverse Fourier transform of $A_2(\omega)$. Define the new variable $\psi = \omega_2 + 2(\sigma - j\alpha)t$ so that from Eq. (B.10b)

$$\gamma = \frac{\psi - \xi}{2(\sigma - j\alpha)} \quad (\text{B.13})$$

Substituting I_1 and γ from Eqs. (B.12) and (B.13) respectively into Eq. (B.10a):

$$r_{11}(t) = \frac{\sigma - j\alpha}{4\pi(1 + 4\mu\sigma - j4\alpha\mu)^{1/2}} \int_{-\infty}^{\infty} \left\{ a_2 \left[\frac{\psi - \xi}{2(\sigma - j\alpha)} \right] * \exp \left[\frac{(\psi - \xi)^2}{4(\alpha + j\sigma)(1 + 4\mu\sigma - j4\alpha\mu)} \right] \right. \\ \left. \cdot U_1(\xi) \exp \left[\frac{-(\xi - \omega_2)^2}{4(\alpha + j\sigma)} + j(\omega_2 t - \theta_1 - \theta_2) \right] d\xi \right. \quad (\text{B.14})$$

Equation (B.14) is in the form of the convolution of two functions of the variable ψ ; using the convolution notation of Eq. (B.5) and substituting for ψ , the response function may be expressed

$$r_{11} \left(t - \frac{\omega_2}{2\sigma - j2\alpha} \right) = \frac{\sigma - j\alpha}{4\pi(1 + 4\mu\sigma - j4\alpha\mu)^{1/2}} \left\{ a_2(t) * \exp \left[\frac{-(\alpha + j\sigma) t^2}{1 + 4\mu\sigma - j4\alpha\mu} \right] \right\} \\ * \left\{ U_1[2(\sigma - j\alpha)t] \exp \left[(\alpha + j\sigma)t^2 + \frac{j\omega_2^2}{4(\sigma - j\alpha)} - j(\theta_1 + \theta_2) \right] \right\} \quad (B.15)$$

Now let $\alpha \rightarrow 0$. If $4\mu\sigma \neq -1$, no difficulties arise in letting $\alpha = 0$ in Eq. (B.15). If, on the other hand, $4\mu\sigma = -1$, Eq. (B.15) may be evaluated in the limit by noting that

$$\lim_{\alpha \rightarrow 0} \left[\frac{\sigma}{4\pi} \sqrt{\frac{j}{4\alpha\mu}} \exp \frac{\sigma t^2}{4\alpha\mu} \right] = \frac{1}{4} \sqrt{\frac{\sigma}{j\pi}} \delta(t) \quad (B.16)$$

where $\delta(t)$ is the Dirac impulse function defined by the properties

$$(a) \quad \delta(t) = \infty \quad t = 0$$

$$(b) \quad \delta(t) = 0 \quad t \neq 0$$

$$(c) \quad \delta(t) = \delta(-t)$$

$$(d) \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$(e) \quad \int_{-\infty}^{\infty} f(x) \delta(x-t) dx = f(t) \quad (B.17)$$

By Eqs. (B.17c) and (B.17e), $f(x) * \delta(x) = f(x)$; therefore, the response function component given by Eq. (B.15) may be written as follows:

If $4\mu\sigma = -1$,

$$r_{11}(t - \tau_2) = \frac{1}{4} \sqrt{\frac{\sigma}{j\pi}} a_2(t) * U_1(2\sigma t) \exp j[\sigma t^2 - \theta_{31}] \quad (\text{B.18a})$$

If $4\mu\sigma \neq -1$

$$= \frac{\sigma}{4\pi \sqrt{1+4\mu\sigma}} \left[a_2(t) * \exp j \left(\frac{-\sigma t^2}{1+4\mu\sigma} \right) \right] * [U_1(2\sigma t) \exp j(\sigma t^2 - \theta_{31})] \quad (\text{B.18b})$$

where

$$\tau_2 = \frac{\omega_2}{2\sigma} = \frac{f_2}{s} \quad (\text{B.18c})$$

$$\theta_{31} = \theta_1 + \theta_2 - \frac{\omega_2^2}{4\sigma} \quad (\text{B.18d})$$

The remaining three components of $r(t)$ resulting from the product of $U_3(\omega)$ and $H(\omega)$ may be obtained from Eq. (B.18) by noting the signs of the coefficients in Eqs. (B.4) and (B.8). If the time function $u_1(t)$ is real, the signal spectrum may be written as $U_1(\omega) = A_1(\omega) e^{j\Phi(\omega)}$, where $A_1(\omega) = A_1(-\omega)$ and $\Phi(\omega) = -\Phi(-\omega)$. The following table indicates by a minus sign that the constant should be changed in sign from that indicated in Eq. (B.18).

Component	Obtained from	σ	θ_1	μ	ω_2	θ_2
$r_{11}(t)$	$U_{31}H_1$	+	+	+	+	+
$r_{22}(t)$	$U_{32}H_2$	-	-	-	-	-
$r_{12}(t)$	$U_{31}H_2$	+	+	-	-	-
$r_{21}(t)$	$U_{32}H_1$	-	-	+	+	+

Pairs of complex conjugate components may be combined to form two real output response components:

$$(a) \quad r(t) = r_1(t) + r_2(t)$$

where

$$(b) \quad r_1(t) = r_{11}(t) + r_{22}(t)$$

$$(c) \quad r_2(t) = r_{12}(t) + r_{21}(t) \quad (B.19)$$

Interpretation is aided by noting that addition of conjugate terms similar to Eq. (B.18b) is in the form

$$u = (f * g) * h + (f * g^*) * h^* \quad (B.20)$$

where g^* is the complex conjugate of g , and $f * g$ denotes convolution. Since convolution as defined by Eq. (B.5) is associative, commutative, and distributive (if the functions are well-behaved), Eq. (B.22) may be rearranged as follows:

$$\begin{aligned} u &= f * [(g * h) + (g^* * h^*)] \\ &= f * 2[\operatorname{Re}(g * h)] \\ &= f * 2[\operatorname{Re}(g) * \operatorname{Re}(h) + \operatorname{Im}(g) * \operatorname{Im}(h)] \end{aligned} \quad (B.21)$$

The output response terms $r_1(t)$ and $r_2(t)$ may be obtained by carrying out the operations indicated by Eq. (B.19), aided by Eq. (B.21). Define a relative phase slope parameter $\eta = -4\mu\sigma$, and recall the scan rate $s = \sigma/\pi$. Using these, $r_1(t)$ and $r_2(t)$ may be written as follows:

$$(a) \quad \eta = 1$$

$$r_1(t - \tau_2) = \frac{\sqrt{s}}{2} a_2(t) * \left\{ A_1(2\sigma t) \cos[\sigma t^2 + \Phi(2\sigma t) - \theta_{31} - \frac{\pi}{4}] \right\}$$

(b) $\eta \neq 1$

$$r_1(t-\tau_2) = \frac{sa_2(t)}{2\sqrt{1-\eta}} * \left\{ A_1(2\sigma t) \cos [\sigma t^2 + \Phi(2\sigma t) - \theta_{31}] * \cos \left(\frac{\sigma t^2}{1-\eta} \right) \right. \\ \left. + A_1(2\sigma t) \sin [\sigma t^2 + \Phi(2\sigma t) - \theta_{31}] * \sin \left(\frac{\sigma t^2}{1-\eta} \right) \right\}$$

(c) $\eta = -1$

$$r_2(t-\tau_2) = \frac{\sqrt{s}}{2} a_2(t) * \left\{ A_1(2\sigma t) \cos [\sigma t^2 + \Phi(2\sigma t) - \theta_{32} - \frac{\pi}{4}] \right\}$$

(d) $\eta \neq -1$

$$r_2(t-\tau_2) = \frac{sa_2(t)}{2\sqrt{1+\eta}} * \left\{ A_1(2\sigma t) \cos [\sigma t^2 + \Phi(2\sigma t) - \theta_{32}] * \cos \left(\frac{\sigma t^2}{1+\eta} \right) \right. \\ \left. + A_1(2\sigma t) \sin [\sigma t^2 + \Phi(2\sigma t) - \theta_{32}] * \sin \left(\frac{\sigma t^2}{1+\eta} \right) \right\}$$

where:

$$(e) \quad \tau_2 = \frac{\omega_2}{2\sigma} = \frac{f_2}{s}$$

$$(f) \quad \theta_{31} = \theta_1 + \theta_2 - \frac{\omega_2^2}{4\sigma}$$

$$(g) \quad \theta_{32} = \theta_1 - \theta_2 + \frac{\omega_2^2}{4\sigma} \quad (B.22)$$

APPENDIX C: EFFECTIVE WIDTH OF TWO CONVOLVED SCANNING FUNCTIONS

By Kelvin's principle of stationary phase [Ref. Be-1], most of the contribution to the integral of a rapidly oscillating function with slowly varying envelope occurs near points where the derivative of phase (instantaneous frequency) is zero, since elsewhere the oscillating magnitude tends to cancel. We are interested here in the effective "width" of the region of contribution.

Let two oscillating functions $y_1(t)$ and $y_2(t)$ scanning linearly at the rates $s_n = \sigma_n/\pi$ cps² be defined as:

$$\begin{aligned} y_1(t) &= e^{-\alpha t^2} \cos(\sigma_1 t^2 + \omega_1 t + \phi_1) & \text{Re } (\alpha) > 0 \\ y_2(t) &= \cos(\sigma_2 t^2 + \omega_2 t + \phi_2) \end{aligned} \quad (C.1)$$

Let $y_3(t)$ be their convolution $y_1(t) * y_2(t)$:

$$y_3(t) = \int_{-\infty}^{\infty} y_1(\xi) y_2(t - \xi) d\xi \quad (C.2)$$

Following the procedure indicated in Appendix B (expressing cosines as exponentials and using the definite integral of Eq. (B.2), $u_3(t)$ may be expressed (in the limit) as

$$\begin{aligned} \lim_{\substack{\alpha \rightarrow 0 \\ s_1 \neq s_2}} y_3(t) &= \frac{1}{2\sqrt{s_1+s_2}} \cos \left[\frac{\sigma_1 \sigma_2 t^2}{\sigma_1 + \sigma_2} + \frac{(\sigma_1 \omega_2 + \sigma_2 \omega_1)t}{\sigma_1 + \sigma_2} + \phi_{31} \right] \\ &+ \frac{1}{2\sqrt{s_1-s_2}} \cos \left[\frac{\sigma_1 \sigma_2 t^2}{\sigma_1 - \sigma_2} + \frac{(\sigma_1 \omega_2 + \sigma_2 \omega_1)t}{\sigma_1 - \sigma_2} + \phi_{32} \right] \end{aligned} \quad (C.3)$$

where:

$$\phi_{31} = \phi_1 + \phi_2 - \frac{\pi}{4} - \frac{(\omega_1 - \omega_2)^2}{4(\sigma_1 + \sigma_2)} \quad (C.3a)$$

$$\phi_{32} = \phi_2 - \phi_1 + \frac{\pi}{4} + \frac{(\omega_1 + \omega_2)^2}{4(\sigma_1 - \sigma_2)} \quad (C.3b)$$

If only one function is scanning (e.g. if $s_2 = 0$ and $s_1 = s \neq 0$), using the identity

$$\cos(a - b) + \cos(a + b) = 2 \cos(a) \cos(b) \quad (C.4)$$

permits Eq. (C.3) to be written in the form:

$$y_3(t) = \frac{1}{\sqrt{s}} \cos \left(\omega_2 t + \frac{\omega_1 \omega_2}{2\sigma} + \phi_2 \right) \cos \left(\phi_1 - \frac{\pi}{4} - \frac{\omega_1^2 + \omega_2^2}{4\sigma_1} \right) \quad (C.5)$$

which has a constant frequency f_2 .*

As discussed in Section III-B-1, the maximum convolution magnitude is approximately equal to the product of the maximum amplitudes of the convolved functions and the common "width" of the functions when their maxima are aligned. Since $y_1(t) \leq 1 \geq y_2(t)$, y_{3m} may be interpreted as the duration d_s of the region of stationary phase. From Eq. (C.3),

$$y_{3m} \leq \frac{1}{2} |s_1 + s_2|^{-1/2} + \frac{1}{2} |s_1 - s_2|^{-1/2} \quad (C.6)$$

The value of y_{3m} will be determined by the relative phase of the functions at the time that the instantaneous frequencies are equal, but can never exceed the value permitted by the relative scan rate s_r :

$$s_r = \left| |s_1| - |s_2| \right| \quad (C.7)$$

*The two components of Eq. (C.3) result from the equivalence of positive and negative frequencies for real time functions, so that there are two regions of stationary phase between $y_1(t)$ and $y_2(t)$. The greater magnitude results from the frequency conjunction having the lesser relative scan rate.

We therefore define d_s as:

$$d_s \triangleq \frac{1}{2} |s_1 + s_2|^{-1/2} + \frac{1}{2} |s_1 - s_2|^{-1/2} \approx \frac{1}{\sqrt{s_r}} \quad (C.8)$$

Stationary phase occurs when the instantaneous frequencies are equal:

$$t_s = \pm \frac{f_1 - f_2}{s_1 - s_2} \quad (C.9)$$

The appropriate polarity is that giving the lesser relative scan rate.

APPENDIX D: RESOLUTION OF TWO SUPERIMPOSED SIMILAR GAUSSIAN FUNCTIONS

The definition of a function width such as the Δt of Eq. (4.2) or the d of Eq. (A.1) is useful, but a more specific criterion for the separation of functions is needed in order to permit their separate recognition and measurement. In the gaussian case, the existence of separate peaks may be required, or a maximum error in the apparent position of the peaks may be specified.

Using Eq. (4.1), let the sum of two similar gaussian functions to be resolved be expressed as

$$y(x) = e^{-\alpha x^2} + a e^{-\alpha(x-x_2)^2} \quad (D.1)$$

Figure 38 illustrates the component functions and their sum. One component is of unit peak magnitude centered at $x = 0$, and the other peak is of magnitude a and centered at $x = x_2$. Both components are of width $\Delta x = 2\sqrt{\rho/\alpha}$ at the $e^{-\rho}$ relative amplitude points. If $x_2 \gg \Delta x$, $y(x)$ consists of two clearly recognizable peaks.

As the separation between the peaks x_2 is reduced, the form of $y(x)$ changes. The sum of the component functions no longer drops to a negligible magnitude between the peaks, but reaches a minimum below the lesser maximum by Δy at an abscissa x_d . The two peaks are increased slightly in magnitude and displaced to x_1' and x_2' by the sloping non-negligible contribution of the other component. Further reduction in separation causes the two peaks to merge into a single peak such that $x_1' = x_2'$, precluding recognition of the two component functions as two separate and measurable quantities.

A condition of minimum function separation may thus be specified where the two peaks merge and $\Delta y = 0$. This is equivalent to stating that the minor peak is a point both of inflection and zero slope; that is, $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$. From Eq. (D.1), under these conditions

$$x_d = x_2' = \frac{x_2}{2} \left[1 \pm \sqrt{1 - \frac{2}{\alpha x_2^2}} \right] \quad (D.2)$$

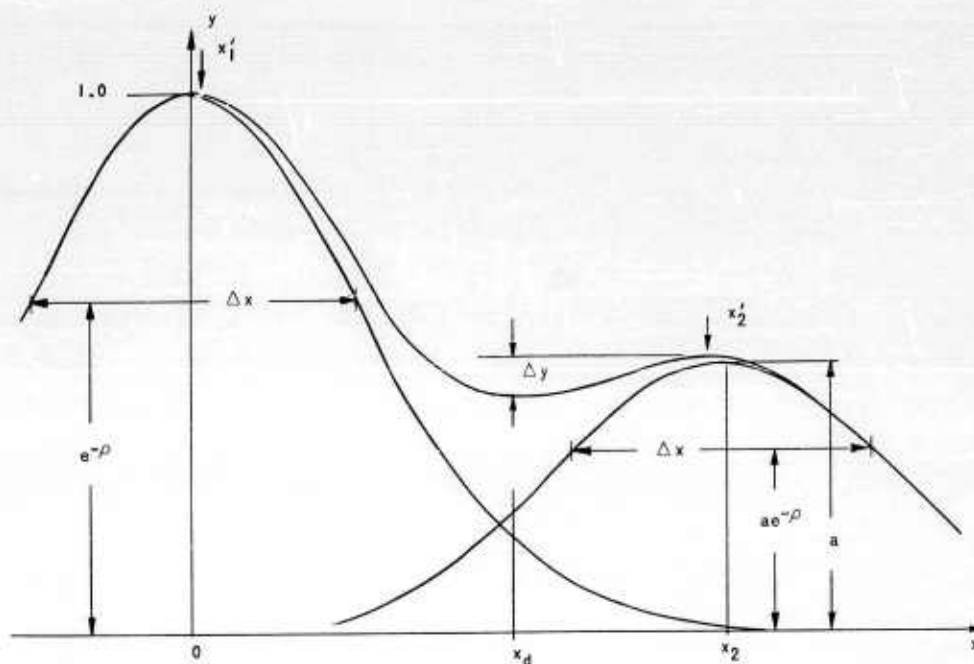


FIG. 38. SUPERIMPOSED GAUSSIAN FUNCTIONS.

This value of x_d may be used to obtain an expression relating minimum gaussian function separation and the relative amplitude parameter a :

$$a = \frac{1 + \sqrt{1 - \frac{2}{\alpha x_2^2}}}{1 - \sqrt{1 - \frac{2}{\alpha x_2^2}}} \exp \left[-\alpha x_2^2 \sqrt{1 - \frac{2}{\alpha x_2^2}} \right] \quad (D.3)$$

Examination of Eqs. (D.2) and (D.3) reveals that the minimum required separation occurs when the component functions are of equal magnitude ($a = 1$), and in this case is $x_2 = \sqrt{2/\alpha}$. Unequal magnitudes ($a \neq 1$) require greater separation for resolution.

The minimum separation above, while useful as an absolute limit of gaussian resolution, is not very realistic as a practical specification. The presence of two components is at best indicated only by a "flattening" of the response function either at the peak or along one side of the major peak. This might be quite difficult to detect, particularly in the presence of noise or with a large amplitude ratio of component functions. Another serious objection is the large error in apparent peak position, particularly of the smaller component. A requirement of a minimum "dip" magnitude $\Delta y = y(x_2') - y(x_d)$ may thus be preferable. Figure 39 shows the required gaussian function separation obtained by computer for several values of Δy , with Δy expressed in decibels relative to the magnitude of the minor peak. From Fig. 39 function spacing may be obtained for a variety of conditions. Several examples are listed in Table 5.

TABLE 5 - EXAMPLES OF GAUSSIAN FUNCTION SPACING FOR RESOLUTION

a (Relative amplitude)	Δy (Magnitude of dip)	x_2 (Function spacing)	
		(energy widths)	(-3 db widths)
1.0	0 Db	1.13 Δx	1.20 Δx
1.0	-6	1.88	2.00
0.01	0	2.30	2.45
0.01	-6	2.85	3.04

The shift in position of the two peaks to x_1' and x_2' causes errors in apparent location of the component functions. Figure 40 gives the magnitude of the error for several values of a . If $a = 1$, the error is half the function spacing x_2 up to the point where $y(x)$ "splits" into two peaks. If $a \ll 1$, the error x_1' of the larger peak is small for all values of x_2 ; for example, if $a = 0.1$ (-20 db), from Fig. 40 $x_1' = 0.03 \Delta x$ (defined at the -3 db level), and the

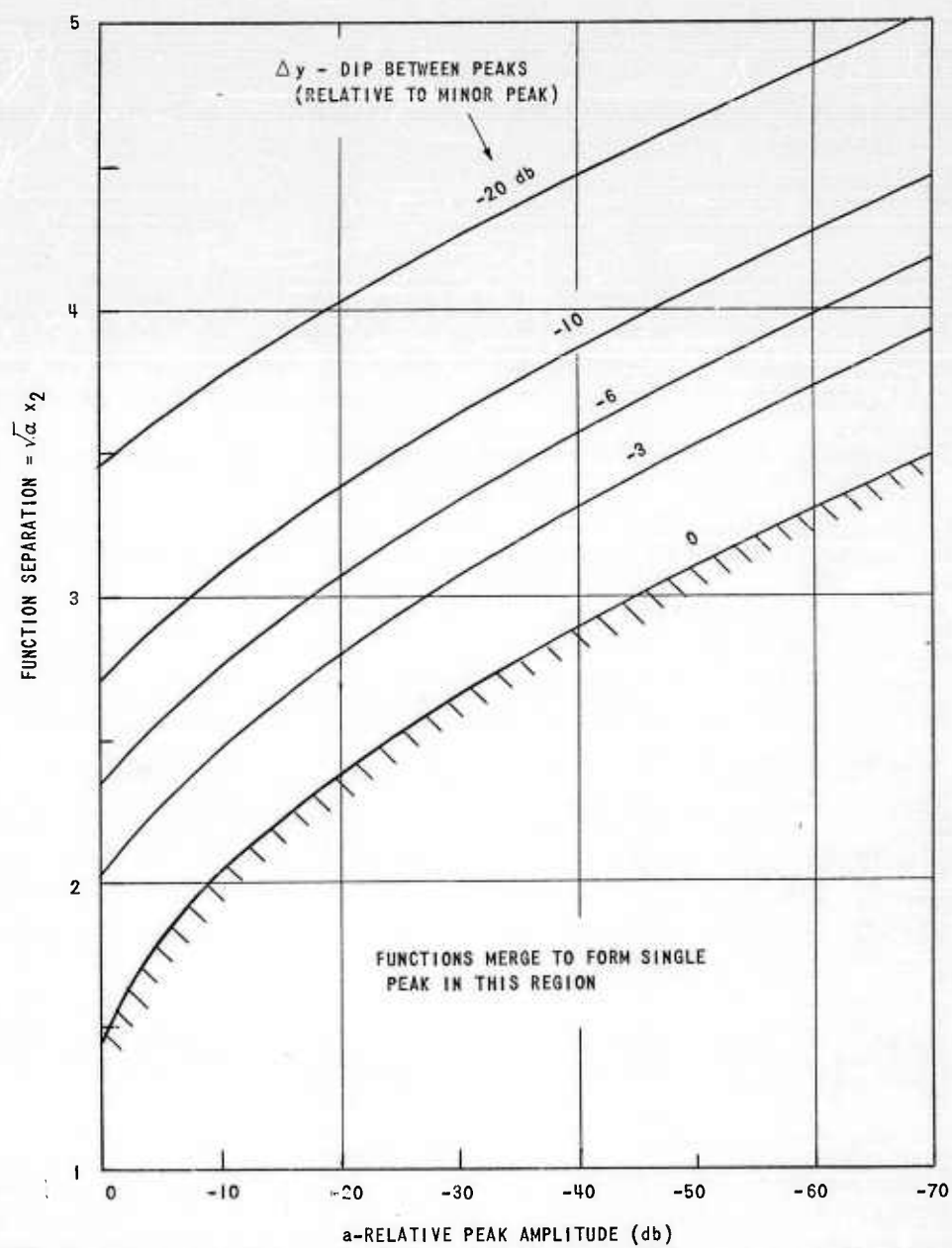


FIG. 39. SEPARATION OF SUPERIMPOSED GAUSSIAN FUNCTIONS REQUIRED FOR RESOLUTION, VS RELATIVE PEAK MAGNITUDE.

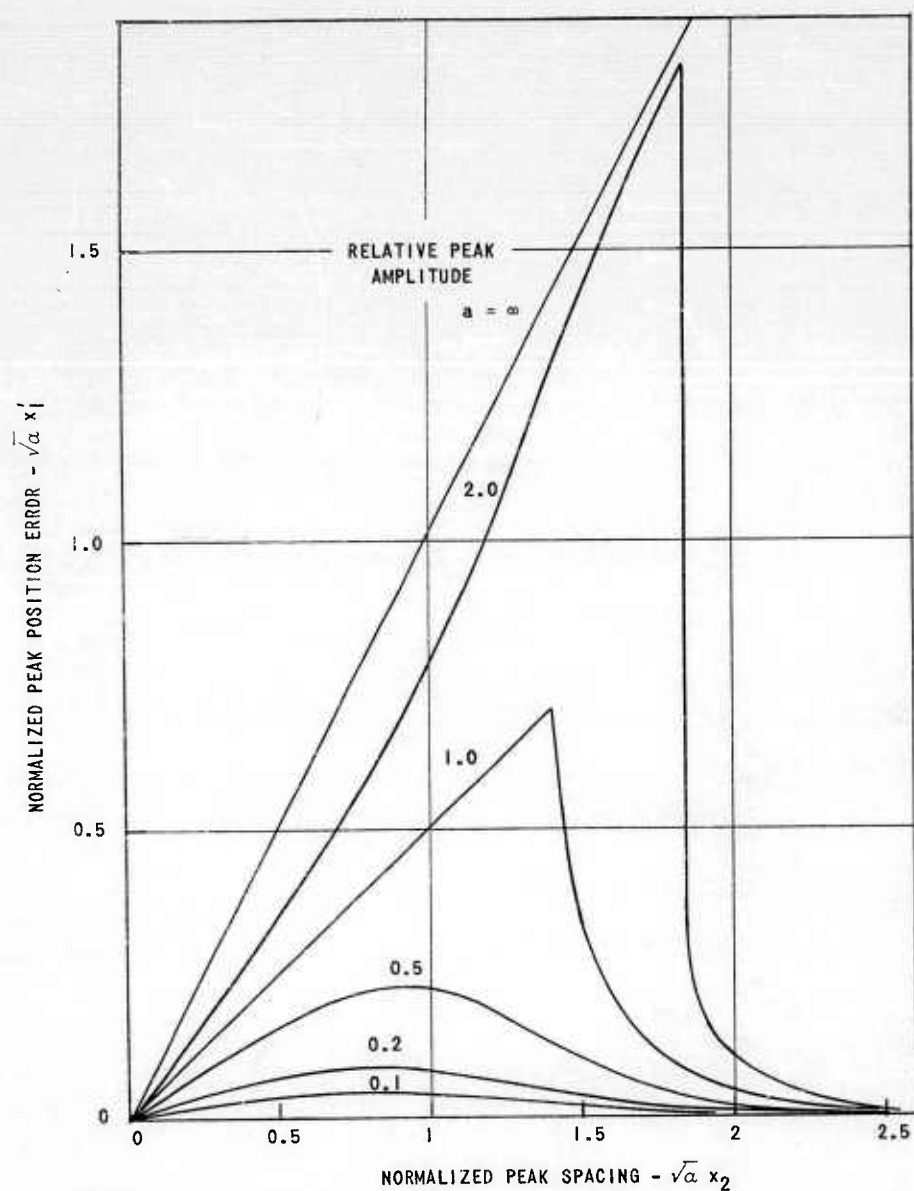


FIG. 40. APPARENT PEAK POSITION ERROR VS GAUSSIAN FUNCTION SPACING.

maximum error is obtained at a function spacing much smaller than that required for separation of the two peaks. The error of the apparent position of the smaller peak $x_2 - x_2'$ is much more serious. The error decreases rapidly as the composite function splits into two peaks with an appreciable dip Δy between them. Resolution of gaussian functions appears reasonable either from the standpoint of component function recognition or apparent position error with an intervening dip of about - 6 db ($\Delta y = 0.5$).

The discussion above suggests that the gaussian function is best suited for resolution of components with comparable amplitudes. The smoothly varying "skirts" of the gaussian function permit recognition without ambiguity of superimposed components having widely differing relative amplitudes if the function spacing is sufficiently large. If recognition or measurement of closely spaced adjacent functions over a large dynamic range is an important analyzer criterion, however, signal component representation by functions other than gaussian may be preferable, even at the expense of sensitivity or of resolution of functions of comparable amplitudes.

APPENDIX E: ENERGY OF A SCANNING GAUSSIAN PULSE

Let $u(t)$ be an oscillating function with linearly scanning instantaneous frequency and gaussian envelope function (as in Eq. 4.5):

$$u(t) = a e^{-\alpha t^2} \cos(\sigma t^2 + \omega t + \phi) \quad (E.1)$$

The total energy of $u(t)$ is

$$\begin{aligned} E &= \frac{1}{R} \int_{-\infty}^{\infty} |u(t)|^2 dt \\ &= \frac{a^2}{2R} \int_{-\infty}^{\infty} e^{-2\alpha t^2} [1 + \cos 2(\sigma t^2 + \omega t + \phi)] dt \end{aligned} \quad (E.2)$$

The integral of Eq. (E.2) may be evaluated by expressing the cosine as exponentials and using Eq. (B.2):

$$\begin{aligned} E &= \frac{a^2 d}{2R} \left\{ 1 + \left(1 + \frac{\sigma^2}{\alpha^2} \right)^{-1/4} \exp \left[\frac{-\alpha \omega^2}{2(\alpha^2 + \sigma^2)} \right] \right. \\ &\quad \left. \cos \left[\frac{\sigma \omega^2}{2(\alpha^2 + \sigma^2)} - 2\phi - \frac{1}{2} \arctan \frac{\sigma}{\alpha} \right] \right\} \end{aligned} \quad (E.3)$$

where $d = \sqrt{\pi/2\alpha}$ is the envelope "energy" duration defined at the $e^{-\pi/8}$ relative amplitude level using Eq. (4.2) and Table 3. Since the magnitude of the cosine factor is bounded by unity, the pulse energy is contained between the limits

$$0 \leq E \leq \frac{a^2 d}{R} = P_m d \quad (E.4)$$

where P_m is the peak pulse power. If $\omega d \gg 1$ so that there are many cycles during the pulse, then the exponential factor is very small and the usual rms energy is obtained:

$$E \approx \frac{a_d^2}{2R} \quad (E.5)$$

If, on the other hand, the pulse duration is small compared to the oscillation period and the scan rate is not excessive (that is, if the phase of the periodic function does not vary significantly during the pulse) so that $\omega^2 \ll \alpha^2$ and $\sigma^2 \ll \alpha^2$, then

$$E \approx \frac{a_d^2}{2R} (1 + \cos 2\phi) \quad (E.6)$$

In this case the pulse energy depends upon the phase ϕ at the time of the maximum amplitude of the pulse.

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