N-63-3-1

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BUCKLING OF A THIN-WALLED CIRCULAR CYLINDRICAL SHELL HEATED ALONG AN AXIAL STRIP

SEPTEMBER 1962

PREPARED FOR THE AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
UNDER GRANT NO. AF-AFOSR-62-146

SUDAER NO. 142

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by

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SUDAER NO. 142

September 1962

The work here reported was performed under
research grant No. AF-AFOSR-62-146

SUMMARY

The elastic stability of a thin-walled circular cylindrical shell is investigated by means of the small-deflection theory when the shell is subjected to such non-uniform heating as causes a uniform axial compressive stress to arise in a band of width 2b while the rest of the shell is free of stress. The critical value of the compressive axial stress is found to be equal to the critical stress of the same circular cylindrical shell when subjected to uniform axial compression provided the band is not extremely narrow. In the latter case the critical stress of the band is higher than that of the uniformly compressed shell.

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FIGURE

NOTATION

A, B, C, D, F, G, H, J	Coefficients in Eq. (11)
$A', B', C', D', F', G', H', J'$	Coefficients in Eq. (13)
a	radius of median surface of circular cylindrical shell
$2b$	width of heated band
E	Young's modulus of elasticity
h	wall thickness of shell
n	reduced wave number [see Eqs. (11) and (13)]
u^*	axial displacement
u	reduced axial displacement defined in Eq. (4a)
v^*	circumferential displacement
v	reduced circumferential displacement defined in Eq. (4b)
w^*	radial displacement
w	reduced radial displacement defined in Eq. (5)
x^*	axial distance
x	reduced axial distance defined in Eq. (3a)
z	function of x and ϕ
α_1, α_2	quantities defined in Eqs. (12)
β_1, β_2	quantities defined in Eqs. (12)
$\gamma_1, \gamma_2, \gamma_3, \gamma_4$	quantities defined in Eqs. (14)
∇^2	Laplace's two-dimensional operator defined in Eq. (2a)
∇^{-4}	integral operator defined in Eq. (16b)
ν	Poisson's ratio
$\rho = \sigma_x/\sigma_{cl}$	critical stress ratio defined in Eq. (2b)
σ_x	uniform axial compressive thermal stress within band width of $2b$

σ_{cl}	classical value of critical stress for uniform axial compression given in Eq. (2c)
ϕ^*	circumferential coordinate (measured in radians)
ϕ	reduced circumferential coordinate defined in Eq. (3b)
Ξ, Φ, Ψ, Ω	matrices defined in Eqs. (32)
Φ', Ω'	matrices defined in Eqs. (35)
	The following subscripts refer to:
h	heated region
u	unheated region
o	edge of heated region
x and ϕ	following a comma indicate differentiation

INTRODUCTION

The commonest structural element of a missile is the thin-walled circular cylindrical shell. When it is heated non-uniformly by the boundary layer of the external supersonic air flow or by an internal rocket engine, thermal stresses arise which can cause buckling. It was shown in an earlier paper[1*] that the shell is most likely to buckle when the temperature varies in the circumferential direction. A particularly simple circumferential temperature variation was selected for the present investigation because it makes possible a rigorous solution in closed form. It is characterized by a uniform temperature rise causing a uniform compressive thermal stress in a heated band of width 2b with the rest of the shell remaining at its uniform initial temperature. The analysis, carried out with the aid of the small-deflection theory, leads to the interesting conclusion that the critical stress of the heated band is the same as the critical stress of a complete cylindrical shell subjected to uniform compression unless the heated bandwidth is very small. In the latter case the critical stress of the band is higher than the critical stress of the uniformly compressed shell.

STATEMENT OF THE PROBLEM

A thin-walled circular cylindrical shell is heated along an axial strip in such a manner that a width 2b of the circumference attains a uniform temperature T above the uniform initial temperature while the temperature of the rest of the shell remains unchanged. It is assumed that the cylindrical shell is very long in the axial direction and thus the thermal stress caused by the heating has the constant value σ_x (positive when compressive) over a substantial length of the heated strip as well as across its entire width 2b; and outside the heated strip the thermal stress is zero. Such conditions can be realized with a short cylindrical shell also if the circular boundaries of the shell are prevented from displacing in the axial direction.

*Superscript numbers refer to the Bibliography at the end of the paper.

It has been shown[2,3] that the solution of a class of problems to which the present problem belongs can be accomplished most conveniently if equations of the Donnell type are used. These equations were given recently in a particularly simple form by Nachbar[4]. The equation governing the radial displacement w is:

$$\nabla^8 w + w_{,xxxx} + 2\rho\nabla^4 w_{,xx} = 0 \tag{1a}$$

The other two displacements are related to w by the equations

$$\nabla^4 u = \upsilon w_{,xxx} - w_{,x\phi\phi} \tag{1b}$$

$$\nabla^4 v = (2 + \upsilon)w_{,xx\phi} + w_{,\phi\phi\phi} \tag{1c}$$

where ∇^2 is Laplace's two-dimensional operator defined as

$$\nabla^2 z = z_{,xx} + z_{,\phi\phi} = (\partial^2 z/\partial x^2) + (\partial^2 z/\partial\phi^2) \tag{2a}$$

and

$$\rho = \sigma_x/\sigma_{cl} \qquad \sigma_{cl} = \frac{E}{[3(1 - \upsilon^2)]^{1/2}}\,\frac{h}{a} \tag{2b,c}$$

The non-dimensional coordinates and displacements are defined as

$$x = (x^*/a)(2E/\sigma_{cl})^{1/2} \qquad \phi = \phi^*(2E/\sigma_{cl})^{1/2} \tag{3a,b}$$

$$u = (u^*/a)(2E/\sigma_{cl})^{1/2} \qquad v = (v^*/a)(2E/\sigma_{cl})^{1/2} \tag{4a,b}$$

$$w = (w^*/a) \tag{5}$$

and x^* is the axial coordinate (which is a distance), ϕ^* the circumferential coordinate (which is an angle measured in radians), while u^*, v^* and w^* are the elastic displacements (measured in units of length) in the axial, circumferential and radial directions (see Fig. 1). Moreover a is the radius of the middle surface of the shell, h the wall thickness, E Young's modulus of elasticity of the material and υ is Poisson's ratio. Since

$$(2E/\sigma_{cl})^{1/2} = [12(1 - \nu^2)]^{1/4}(a/h)^{1/2} \tag{6}$$

in the normalization process ϕ^* is multiplied by $(a/h)^{1/2}$, x^*, u^* and v^* are divided by $(ah)^{1/2}$ and w^* is divided by a. It may also be mentioned that σ_{cl} is the critical stress of the cylindrical shell subjected to uniform axial compression as calculated from the classical theory.

In the present problem ρ is a positive constant in Eq. (1a) when

$$|\phi| < \phi_0 = + \phi_0^*(2E/\sigma_{cl})^{1/2} = + (b/a)(2E/\sigma_{cl})^{1/2} \tag{7}$$

When inequality (7) does not hold, ρ is zero in Eq. (1a).

Earlier experience[5] has shown that details of the conditions of support along the circular boundaries have little effect on the critical value of the axial stress when the length of the cylinder is greater than the diameter. For this reason no boundary conditions will be prescribed along these edges. Inspection of the characteristic displacement functions obtained reveals that these functions correspond to sensible conditions along the circular boundaries.

Important boundary conditions must, however, be stipulated along the generators separating the region with thermal stress from the regions without thermal stress. If the two different regions are indicated by the subscripts h and u, the conditions that radial displacements, slopes, as well as bending moment and effective transverse shear resultants must be the same at the boundary for the two different regions can be stated as

$$w_h = w_u \qquad w_{h,\phi} = w_{u,\phi} \tag{8a,b}$$

$$\phi = \pm \phi_0$$

$$w_{h,\phi\phi} = w_{u,\phi\phi} \qquad w_{h,\phi\phi\phi} = w_{u,\phi\phi\phi} \tag{8c,d}$$

In addition one has to require that the axial and circumferential displacements as well as the membrane stress resultants should be the same at the boundary when calculated for the two different regions. In the form of equations

$$u_h = u_u \qquad u_{h,\phi} = u_{u,\phi} \tag{9a,b}$$

$$\phi = \pm \phi_0$$

$$v_h = v_u \qquad v_{h,\phi} = v_{u,\phi} \tag{9c,d}$$

Naturally the displacements and strains must be continuous functions of ϕ around the circumference of the cylinder.

It will be shown later that these four boundary conditions are mathematically equivalent to four boundary conditions on w; the latter can be obtained from the former with the aid of Eqs. (1b) and (1c).

SOLUTION IN THE UNHEATED REGION

In the region where the thermal stress is zero we set $\rho = 0$ and obtain from Eq. (1a)

$$\nabla^8 w + w_{,xxxx} = 0 \tag{10}$$

A complete set of characteristic functions of this equation can be written as

$$\begin{aligned} w/\sin nx = {} & Ae^{-\alpha_1\phi} \cos \beta_1\phi + Be^{-\alpha_1\phi} \sin \beta_1\phi + Ce^{-\alpha_2\phi} \cos \beta_2\phi \\ & + De^{-\alpha_2\phi} \sin \beta_2\phi + Fe^{\alpha_1\phi} \cos \beta_1\phi + Ge^{\alpha_1\phi} \sin \beta_1\phi \\ & + He^{\alpha_2\phi} \cos \beta_2\phi + Je^{\alpha_2\phi} \sin \beta_2\phi \end{aligned} \tag{11}$$

where

$$\alpha_1 = +(n/2)^{1/2}\left\{\left[\left(n + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right]^{1/2} + \left(n + \frac{1}{\sqrt{2}}\right)\right\}^{1/2} \tag{12a}$$

$$\alpha_2 = +\,(n/2)^{1/2}\left\{\left[\left(n - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right]^{1/2} + \left(n - \frac{1}{\sqrt{2}}\right)\right\}^{1/2} \qquad (12b)$$

$$\beta_1 = +\,(\sqrt{n}/2)\,\text{[redacted]}\left\{\left[\left(n + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right]^{1/2} + \left(n + \frac{1}{\sqrt{2}}\right)\right\}^{1/2} \qquad (12c)$$

$$\beta_2 = +\,(\sqrt{n}/2)\,\text{[redacted]}\left\{\left[\left(n - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right]^{1/2} + \left(n - \frac{1}{\sqrt{2}}\right)\right\}^{-1/2} \qquad (12d)$$

n is a positive real number and $A, B, \ldots J$ are constants of integration.

SOLUTION IN THE HEATED REGION

In the region of constant temperature rise the thermal stress σ_x is a positive constant, and so is the stress ratio ρ. A complete set of characteristic functions can be given as

$$w/\sin nx = A' \cosh \gamma_1 \phi + B' \cosh \gamma_2 \phi + C' \cos \gamma_3 \phi + D' \cos \gamma_4 \phi + F' \sinh \gamma_1 \phi$$
$$+ G' \sinh \gamma_2 \phi + H' \sin \gamma_3 \phi + J' \sin \gamma_4 \phi \qquad (13)$$

where

$$\gamma_1 = +\sqrt{n}\left\{+\sqrt{\rho}\left[1 + \left(1 - \frac{1}{\rho^2}\right)^{1/2}\right]^{1/2} + n\right\}^{1/2} \qquad (14a)$$

$$\gamma_2 = +\sqrt{n}\left\{+\sqrt{\rho}\left[1 - \left(1 - \frac{1}{\rho^2}\right)^{1/2}\right]^{1/2} + n\right\}^{1/2} \qquad (14b)$$

$$\gamma_3 = +\sqrt{n}\left\{+\sqrt{\rho}\left[1 + \left(1 - \frac{1}{\rho^2}\right)^{1/2}\right]^{1/2} - n\right\}^{1/2} \qquad (14c)$$

$$\gamma_4 = +\sqrt{n}\left\{+\sqrt{\rho}\left[1 - \left(1 - \frac{1}{\rho^2}\right)^{1/2}\right]^{1/2} - n\right\}^{1/2} \qquad (14d)$$

Here $A', B', \ldots J'$ are integration constants. In consequence of the boundary conditions stated, n must be the same positive real number as in Eq. (11).

REPLACEMENT OF THE MEDIAN-PLANE BOUNDARY CONDITIONS

It is convenient to replace the median-plane boundary conditions on u and v with boundary conditions on w. When this is done, Eqs. (1b) and (1c) need not be solved explicitly. First, Eq. (1a) can be written in the form

$$\nabla^4 \left[\nabla^4 + 2\rho\left(\frac{\partial}{\partial x}\right)^2\right] w = -\left(\frac{\partial}{\partial x}\right)^4 w \tag{15}$$

If negative exponents are introduced for the differential operators and are defined to denote inverse operations such that

$$\left(\frac{\partial}{\partial x}\right)^{-1}\left(\frac{\partial}{\partial x}\right) z = z \tag{16a}$$

$$\nabla^{-4}(\nabla^4 z) = z \tag{16b}$$

the radial displacement w can be written symbolically as

$$w = -\nabla^4\left[\left(\frac{\partial}{\partial x}\right)^{-4}\nabla^4 + 2\rho\left(\frac{\partial}{\partial x}\right)^{-2}\right] w \tag{17}$$

Substitution of this expression in the right-hand member of Eq. (1b) results in

$$\begin{aligned}\nabla^4 u = -\nabla^4\Bigg\{&\upsilon\left(\frac{\partial}{\partial x}\right)^{-1}\nabla^4 - \left(\frac{\partial}{\partial x}\right)^{-3}\left(\frac{\partial}{\partial \phi}\right)^2\nabla^4 \\ &+ 2\rho\left[\upsilon\left(\frac{\partial}{\partial x}\right) - \left(\frac{\partial}{\partial x}\right)^{-1}\left(\frac{\partial}{\partial \phi}\right)^2\right]\Bigg\} w\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}u = \Bigg\{&-\upsilon\left(\frac{\partial}{\partial x}\right)^{-1}\nabla^4 + \left(\frac{\partial}{\partial x}\right)^{-3}\left(\frac{\partial}{\partial \phi}\right)^2\nabla^4 \\ &-2\rho\left[\upsilon\left(\frac{\partial}{\partial x}\right) - \left(\frac{\partial}{\partial x}\right)^{-1}\left(\frac{\partial}{\partial \phi}\right)^2\right]\Bigg\} w\end{aligned} \tag{19}$$

Similarly, from Eq. (1c) we obtain

$$v = -\left\{(2+\nu)\left(\frac{\partial}{\partial\phi}\right)\left(\frac{\partial}{\partial x}\right)^{-2}\nabla^4 + \left(\frac{\partial}{\partial\phi}\right)^3\left(\frac{\partial}{\partial x}\right)^{-4}\nabla^4 + 2\rho\left[(2+\nu)\left(\frac{\partial}{\partial\phi}\right) + \left(\frac{\partial}{\partial\phi}\right)^3\left(\frac{\partial}{\partial x}\right)^{-2}\right]\right\}w \qquad (20)$$

It is to be noted that Eqs. (19) and (20) comprise derivatives of w up to the seventh order in ϕ. Hence the boundary conditions (9) will comprise derivatives of w up to the eighth order in ϕ. However, the eighth-order derivative is linearly dependent on w and its first seven derivatives in consequence of Eq. (1a). There remain, therefore, conditions to be satisfied at the boundary by w and its first seven derivatives with respect to ϕ. Our problem can thus be stated completely by means of Eq. (1a) and eight boundary conditions on w and its first seven derivatives with respect to ϕ.

SYMMETRIC BUCKLING

First the problem of buckling symmetric to the center line of the heated strip ($\phi = 0$) will be considered. Hence the boundary conditions must be written for $\phi = \phi_0$ on the basis of the solution given in Eq. (13) with

$$F' = G' = H' = J' = 0 \qquad (21)$$

For the adjacent unheated region the origin of coordinates will be shifted to $\phi = \phi_0$ and thus in Eq. (11) and in the derivatives of w calculated from Eq. (11), ϕ will be set equal to zero when the boundary conditions are written. For the other unheated region the origin is shifted to $\phi = -\phi_0$, the positive sense of ϕ is inverted, and the boundary conditions are again evaluated at $\phi = 0$. At $\phi = \pi$ (or measured from the center line of the heated strip) the displacements u, v and w, as well as their derivatives involved in the boundary conditions, are so small that they can be disregarded. Hence no conditions will be stipulated at $\phi = \pi$ and in Eq. (11) we shall set

$$F = G = H = J = 0 \tag{22}$$

Thus Eqs. (8a)-(8d) become

$$A' \cosh \gamma_1 \phi_0 + B' \cosh \gamma_2 \phi_0 + C' \cos \gamma_3 \phi_0 - D' \cos \gamma_4 \phi_0 = A + C \tag{23a}$$

$$A'\gamma_1 \sinh \gamma_1 \phi_0 + B'\gamma_2 \sinh \gamma_2 \phi_0 - C'\gamma_3 \sin \gamma_3 \phi_0 - D'\gamma_4 \sin \gamma_4 \phi_0$$
$$= -A\alpha_1 + B\beta_1 - C\alpha_2 + D\beta_2 \tag{23b}$$

$$A'\gamma_1^2 \cosh \gamma_1 \phi_0 + B'\gamma_2^2 \cosh \gamma_2 \phi_0 - C'\gamma_3^2 \cos \gamma_3 \phi_0 - D'\gamma_4^2 \cos \gamma_4 \phi_0$$
$$= A(\alpha_1^2 - \beta_1^2) - B2\alpha_1\beta_1 + C(\alpha_2^2 - \beta_2^2) - D2\alpha_2\beta_2 \tag{23c}$$

$$A'\gamma_1^3 \sinh \gamma_1 \phi_0 + B'\gamma_2^3 \sinh \gamma_2 \phi_0 + C'\gamma_3^3 \sin \gamma_3 \phi_0 + D'\gamma_4^3 \sin \gamma_4 \phi_0$$
$$= A(3\alpha_1\beta_1^2 - \alpha_1^3) + B(3\alpha_1^2\beta_1 - \beta_1^3) + C(3\alpha_2\beta_2^2 - \alpha_2^3) + D(3\alpha_2^2\beta_2 - \beta_2^3) \tag{23d}$$

Because of the symmetry, these equations are equally valid at $\phi = \phi_0$ and $\phi = -\phi_0$

EXPRESSION OF BOUNDARY CONDITIONS ON u AND v IN TERMS OF w

Because of the form

$$w_h = w = f(\phi)\sin nx \tag{24a}$$

of the solution given in Eq. (13), and the form

$$w_u = w = g(\phi)\sin nx \tag{24b}$$

given in Eq. (10), one can write Eq. (19) as

$$u = \left\{(\nu/n^2)\nabla^4 + (1/n^4)\frac{\partial^2}{\partial\phi^2}\nabla^4 - 2\rho\left[\nu + \frac{1}{n^2}\frac{\partial^2}{\partial\phi^2}\right]\right\} w_{,x} \tag{25}$$

Also

$$\nabla^4 = n^4 - 2n^2 \frac{\partial^2}{\partial \phi^2} + \frac{\partial^4}{\partial \phi^4} \tag{26}$$

Hence boundary condition (9a) becomes

$$u_h = \left\{ \frac{\upsilon}{n^2}\left(n^4 - 2n^2 \frac{\partial^2}{\partial\phi^2} + \frac{\partial^4}{\partial\phi^4}\right) + \frac{1}{n^4}\left(n^4 - 2n^2 \frac{\partial^2}{\partial\phi^2} + \frac{\partial^4}{\partial\phi^4}\right)\frac{\partial^2}{\partial\phi^2} \right.$$
$$\left. -2\rho\left(\upsilon + \frac{1}{n^2}\frac{\partial^2}{\partial\phi^2}\right)\right\} w_{h,x} = \left\{ \frac{\upsilon}{n^2}\left(n^4 - 2n^2\frac{\partial^2}{\partial\phi^2} + \frac{\partial^4}{\partial\phi^4}\right) \right.$$
$$\left. + \frac{1}{n^4}\left(n^4 \frac{\partial^2}{\partial\phi^2} - 2n^2 \frac{\partial^4}{\partial\phi^4} + \frac{\partial^6}{\partial\phi^6}\right)\right\} w_{u,x} \tag{27}$$

Several terms of this equation cancel because Eqs. (8a) and (8c) must be satisfied at the boundary $|\phi| = \phi_0$. There remains therefore

$$\left\{ \frac{\upsilon - 2}{n^2}\frac{\partial^4}{\partial\phi^4} + \frac{1}{n^4}\frac{\partial^6}{\partial\phi^6} - 2\rho\left(\upsilon + \frac{1}{n^2}\frac{\partial^2}{\partial\phi^2}\right)\right\} w_{h,x}$$
$$= \left\{ \frac{\upsilon - 2}{n^2}\frac{\partial^4}{\partial\phi^4} + \frac{1}{n^4}\frac{\partial^6}{\partial\phi^6}\right\} w_{u,x} \tag{28a}$$

Similarly, boundary conditions (9b) to (9d) become

$$\left\{ \frac{\upsilon - 2}{n^2}\frac{\partial^5}{\partial\phi^5} + \frac{1}{n^4}\frac{\partial^7}{\partial\phi^7} - 2\rho\left(\upsilon\frac{\partial}{\partial\phi} + \frac{1}{n^2}\frac{\partial^2}{\partial\phi^2}\right)\right\} w_{h,x}$$
$$= \left\{ \frac{\upsilon - 2}{n^2}\frac{\partial^5}{\partial\phi^5} + \frac{1}{n^4}\frac{\partial^7}{\partial\phi^7}\right\} w_{u,x} \tag{28b}$$

$$\left\{ \frac{4 + \upsilon}{n^2}\frac{\partial^4}{\partial\phi^4} - \frac{1}{n^4}\frac{\partial^6}{\partial\phi^6} - 2\rho\left(2 + \upsilon - \frac{1}{n^2}\frac{\partial^2}{\partial\phi^2}\right)\right\} w_{h,\phi}$$
$$= \left\{ \frac{4 + \upsilon}{n^2}\frac{\partial^4}{\partial\phi^4} - \frac{1}{n^4}\frac{\partial^6}{\partial\phi^6}\right\} w_{u,\phi} = 0 \tag{28c}$$

$$\left\{-(5+2\upsilon)\frac{\partial^2}{\partial\phi^2}+\frac{4+\upsilon}{n^2}\frac{\partial^4}{\partial\phi^4}-\frac{1}{n^4}\frac{\partial^6}{\partial\phi^6}-2\rho\left(2+\upsilon-\frac{1}{n^2}\frac{\partial^2}{\partial\phi^2}\right)\right\} w_{h,\phi\phi}$$

$$=\left\{-(5+2\upsilon)\frac{\partial^2}{\partial\phi^2}+\frac{4+\upsilon}{n^2}\frac{\partial^4}{\partial\phi^4}-\frac{1}{n^4}\frac{\partial^6}{\partial\phi^6}\right\} w_{u,\phi\phi} \tag{28d*}$$

But from Eq. 17 we obtain

$$\frac{\partial^8 w}{\partial\phi^8} = -n^4(n^4+1-2\rho n^2)w + 4n^4(n^2-\rho)\frac{\partial^2 w}{\partial\phi^2}$$

$$+2n^2(\rho-3n^2)\frac{\partial^4 w}{\partial\phi^4}+4n^2\frac{\partial^6 w}{\partial\phi^6} \tag{29}$$

Substitution in Eq. (28d*) and omission of terms that mutually cancel each other because of the first four boundary conditions, reduce Eq. (28d*) to

$$\left\{(1-2\upsilon)\frac{\partial^4}{\partial\phi^4}+\frac{\upsilon}{n^2}\frac{\partial^6}{\partial\phi^6}-2\rho\upsilon\frac{\partial^2}{\partial\rho^2}-2\rho n^2\right\} w_h$$

$$=\left\{(1-2\upsilon)\frac{\partial^4}{\partial\phi^4}+\frac{\upsilon}{n^2}\frac{\partial^6}{\partial\phi^6}\right\} w_u \tag{28d}$$

We observe that 4th and 6th derivatives (in addition to lower order derivatives) occur in Eqs. (28a) and (28d); by using a suitable multiplying factor we eliminate first all the 6th, and then all the 4th, derivatives to find Eqs. (30a) and (30b). Similar considerations lead to Eqs. (30c) and (30d). Denoting derivatives with respect to ϕ with Roman numeral superscripts we obtain:

$$w_h^{iv} - 2\rho n^2 w_h = w_u^{iv} \tag{30a}$$

$$w_h^{vi} - 2\rho n^2 w_h^{ii} - 4\rho n^4 w_h = w_u^{vi} \tag{30b}$$

$$w_h^{v} - 2\rho n^2 w_h^{i} = w_u^{v} \tag{30c}$$

$$w_h^{vii} - 2\rho n^2 w_h^{iii} - 4\rho n^4 w_h^{i} = w_u^{vii} \tag{30d}$$

THE STABILITY DETERMINANT

Four boundary conditions were given explicitly in Eqs. (23a) to (23d). Four more explicit expressions are obtained if w_h and w_u are substituted from Eqs. (13) and (11) into Eqs. (30a) to (30d) and ϕ is set equal to ϕ_0 in w_h and equal to zero in w_u. The resulting eight homogeneous equations in the coefficients A' to D' and A to D can be given in matrix form as

$$\left[\begin{array}{c:c} \Phi & \Xi \\ \hdashline \Omega & \Psi \end{array}\right] \begin{bmatrix} A' \\ B' \\ C' \\ D' \\ A \\ B \\ C \\ D \end{bmatrix} = 0 \tag{31}$$

where

$$\Phi = \begin{bmatrix} \cosh \gamma_1\phi_0 & \cosh \gamma_2\phi_0 & \cos \gamma_3\phi_0 & \cos \gamma_4\phi_0 \\ \gamma_1 \sinh \gamma_1\phi_0 & \gamma_2 \sinh \gamma_2\phi_0 & -\gamma_3 \sin \gamma_3\phi_0 & -\gamma_4 \sin \gamma_4\phi_0 \\ \gamma_1^2 \cosh \gamma_1\phi_0 & \gamma_2^2 \cosh \gamma_2\phi_0 & -\gamma_3^2 \cos \gamma_3\phi_0 & -\gamma_4^2 \cos \gamma_4\phi_0 \\ \gamma_1^3 \sinh \gamma_1\phi_0 & \gamma_2^3 \sinh \gamma_2\phi_0 & \gamma_3^3 \sin \gamma_3\phi_0 & \gamma_4^3 \sin \gamma_4\phi_0 \end{bmatrix} \tag{32a}$$

$$
\Xi = \begin{bmatrix}
-1 & 0 & -1 & 0 \\
\alpha_1 & -\beta_1 & \alpha_2 & -\beta_2 \\
-(\alpha_1^2 - \beta_1^2) & 2\alpha_1\beta_1 & -(\alpha_2^2 - \beta_2^2) & 2\alpha_2\beta_2 \\
\alpha_1(3\beta_1^2 - \alpha_1^2) & -\beta_1(3\alpha_1^2 - \beta_1^2) & -\alpha_2(3\beta_2^2 - \alpha_2^2) & -\beta_2(3\alpha_2^2 - \beta_2^2)
\end{bmatrix} \tag{32b}
$$

$$
\Omega = \begin{bmatrix}
(\gamma_1^4-2\rho n^2)\cosh\gamma_1\phi_0 & (\gamma_2^4-2\rho n^2)\cosh\gamma_2\phi_0 & (\gamma_3^4-2\rho n^2)\cos\gamma_3\phi_0 & (\gamma_4^4-2\rho n^2)\cos\gamma_4\phi_0 \\
\begin{array}{c}(\gamma_1^6-2\rho n^2\gamma_1^2-4\rho n^4)\\ \times \cosh \gamma_1\phi_0\end{array} & \begin{array}{c}(\gamma_2^6-2\rho n^2\gamma_2^2-4\rho n^4)\\ \times \cosh \gamma_2\phi_0\end{array} & \begin{array}{c}(-\gamma_3^6+2\rho n^2\gamma_3^2-4\rho n^4)\\ \times \cos \gamma_3\phi_0\end{array} & \begin{array}{c}(-\gamma_4^6+2\rho n^2\gamma_4^2-4\rho n^4)\\ \times \cos \gamma_4\phi_0\end{array} \\
\begin{array}{c}\gamma_1(\gamma_1^4-2\rho n^2)\\ \times \sinh \gamma_1\phi_0\end{array} & \begin{array}{c}\gamma_2(\gamma_2^4-2\rho n^2)\\ \times \sinh \gamma_2\phi_0\end{array} & \begin{array}{c}-\gamma_3(\gamma_3^4-2\rho n^2)\\ \times \sin \gamma_3\phi_0\end{array} & \begin{array}{c}-\gamma_4(\gamma_4^4-2\rho n^2)\\ \times \sin \gamma_4\phi_0\end{array} \\
\begin{array}{c}\gamma_1(\gamma_1^6-2\rho n^2\gamma_1^2-4\rho n^4)\\ \times \sinh \gamma_1\phi_0\end{array} & \begin{array}{c}\gamma_2(\gamma_2^6-2\rho n^2\gamma_2^2-4\rho n^4)\\ \times \sinh \gamma_2\phi_0\end{array} & \begin{array}{c}-\gamma_3(-\gamma_3^6+2\rho n^2\gamma_3^2-4\rho n^4)\\ \times \sin \gamma_3\phi_0\end{array} & \begin{array}{c}-\gamma_4(-\gamma_4^6+2\rho n^2\gamma_4^2-4\rho n^4)\\ \times \sin \gamma_4\phi_0\end{array}
\end{bmatrix} \tag{32c}
$$

$$
\Psi = \begin{bmatrix}
-(\beta_1^2-\alpha_1^2)^2+4\alpha_1^2\beta_1^2 & -4\alpha_1\beta_1(\beta_1^2-\alpha_1^2) & -(\beta_2^2-\alpha_2^2)^2+4\alpha_2^2\beta_2^2 & -4\alpha_2\beta_2(\beta_2^2-\alpha_2^2) \\
(\beta_1^2-\alpha_1^2)^3-12\alpha_1^2\beta_1^2(\beta_1^2-\alpha_1^2) & -\alpha_1\beta_1[-6(\beta_1^2+\alpha_1^2)^2+32\alpha_1^2\beta_1^2] & (\beta_2^2-\alpha_2^2)^3-12\alpha_2^2\beta_2^2(\beta^2-\alpha_2^2) & -\alpha_2\beta_2[-6(\beta_2^2+\alpha_2^2)^2+32\alpha_2^2\beta_2^2] \\
\alpha_1[(\beta_1^2-\alpha_1^2)^2-8\alpha_1^2\beta_1^2+4\beta_1^4] & -\beta_1[(\beta_1^2-\alpha_1^2)^2-8\alpha_1^2\beta_1^2+4\alpha_1^4] & \alpha_2[(\beta_2^2-\alpha_2^2)^2-8\alpha_2^2\beta_2^2+4\beta_2^4] & -\beta_2[(\beta_2^2-\alpha_2^2)^2-8\alpha_2^2\beta_2^2+4\alpha_2^4] \\
-\alpha_1[7(\beta_1^2-\alpha_1^2)^3-14\alpha_1^2\beta_1^4+6\alpha_1^6] & -\beta_1[-7(\beta_1^2-\alpha_1^2)^3-14\alpha_1^4\beta_1^2+6\beta_1^6] & -\alpha_2[7(\beta_2^2-\alpha_2^2)^3-14\alpha_2^2\beta_2^4+6\alpha_2^6] & -\beta_2[-7(\beta_2^2-\alpha_2^2)^3-14\alpha_2^4\beta_2^2+6\beta_2^6]
\end{bmatrix} \tag{32d}
$$

A non-trivial solution for the coefficients exists only if the determinant of the matrix vanishes. Hence the buckling condition is

$$\begin{vmatrix} \Phi & \Xi \\ \Omega & \Psi \end{vmatrix} = 0 \tag{33}$$

ANTISYMMETRIC BUCKLING

The heated strip can also buckle antisymmetrically with respect to the generator $\phi = \phi_0$, that is with respect to the center line of the heated strip. In such a case the coefficients A', B', C' and D' vanish in Eq. 13 and considerations similar to those offered for symmetric buckling lead to the following eight homogeneous linear equations:

$$\left[\begin{array}{c|c} \Phi' & \Xi \\ \hline \Omega' & \Psi \end{array}\right] \begin{bmatrix} F' \\ G' \\ H' \\ J' \\ A \\ B \\ C \\ D \end{bmatrix} = 0 \tag{34}$$

Here Ξ and Ψ denote the matrices given in Eqs. (32b) and (32d) and the matrices Φ' and Ω' are defined as

$$\Phi' = \begin{bmatrix} \sinh \gamma_1\phi_0 & \sinh \gamma_2\phi_0 & \sin \gamma_3\phi_0 & \sin \gamma_4\phi_0 \\ \gamma_1 \cosh \gamma_1\phi_0 & \gamma_2 \cosh \gamma_2\phi_0 & \gamma_3 \cos \gamma_3\phi_0 & \gamma_4 \cos \gamma_4\phi_0 \\ \gamma_1^2 \sinh \gamma_1\phi_0 & \gamma_2^2 \sinh \gamma_2\phi_0 & -\gamma_3^2 \sin \gamma_3\phi_0 & -\gamma_4^2 \sin \gamma_4\phi_0 \\ \gamma_1^3 \cosh \gamma_1\phi_0 & \gamma_2^3 \cosh \gamma_2\phi_0 & -\gamma_3^3 \cos \gamma_3\phi_0 & -\gamma_4^3 \cos \gamma_4\phi_0 \end{bmatrix} \tag{35a}$$

$$\Omega' = \begin{bmatrix} (\gamma_1^4-2\rho n^2)\sinh\gamma_1\phi_0 & (\gamma_2^4-2\rho n^2)\sinh\gamma_2\phi_0 & (\gamma_3^4-2\rho_n^2)\sin\gamma_3\phi_0 & (\gamma_4^4-2\rho n^2)\sin\gamma_4\phi_0 \\ (\gamma_1^6-2\rho n^2\gamma_1^2-4\rho n^4) & (\gamma_2^6-2\rho n^2\gamma_2^2-4\rho n^4) & (-\gamma_3^6+2\rho n^2\gamma_3^2-4\rho n^4) & (-\gamma_4^6+2\rho n^2\gamma_4^2-4\rho n^4) \\ \times\sinh\gamma_1\phi_0 & \times\sinh\gamma_2\phi_0 & \times\sin\gamma_3\phi_0 & \times\sin\gamma_4\phi_0 \\ (\gamma_1^5-2\rho n^2\gamma_1) & (\gamma_2^5-2\rho n^2\gamma_2) & (\gamma_3^5-2\rho n^2\gamma_3) & (\gamma_4^5-2\rho n^2\gamma_4) \\ \times\cosh\gamma_1\phi_0 & \times\cosh\gamma_2\phi_0 & \times\cos\gamma_3\phi_0 & \times\cos\gamma_4\phi_0 \\ (\gamma_1^7-4\rho n^4\gamma_1-2\rho n^2\gamma_1^3) & (\gamma_2^7-4\rho n^4\gamma_2-2\rho n^2\gamma_2^3) & (-\gamma_3^7-4\rho n^4\gamma_3+2\rho n^2\gamma_3^3) & (-\gamma_4^7-4\rho n^4\gamma_4+2\rho n^2\gamma_4^3) \\ \times\cosh\gamma_1\phi_0 & \times\cosh\gamma_2\phi_0 & \times\cos\gamma_3\phi_0 & \times\cos\gamma_4\phi_0 \end{bmatrix} \tag{35b}$$

The condition of antisymmetric buckling is

$$\begin{vmatrix} \Phi' & \Xi \\ \Omega' & \Psi \end{vmatrix} = 0 \tag{36}$$

For some values of n and ρ the root γ_3 becomes imaginary. In these cases column 3 in the matrices Φ, Φ', Ω and Ω' takes on the form of column 1. Occassionally γ_4 is also imaginary and then the fourth column takes on the form of column 2.

RESULTS OF THE CALCULATIONS

The determinantal equations (33) and (36) were evaluated with the aid of the Burroughs 220 electronic digital computer of the Computation Center of Stanford University. For different fixed values of ϕ_0, corresponding values of ρ and n were computed and plotted. Such a plot is shown in Fig. 2 for the case of antisymmetric buckling with $2b/(ah)^{1/2} = 9.83$. Points on the curves shown satisfy the buckling criterion. For practical purposes the lowest value of ρ is of interest. For this reason the minimal values of ρ determined from plots of the type of Fig. 2 were replotted in Fig. 3 in function of $2b/(ah)^{1/2}$. The connection between ϕ_0 and the non-dimensional heated width can be obtained from Eqs. (3b) and (6) together with the relationship

$$\phi_0^* = b/a \tag{37}$$

One gets

$$\phi_0 = [12(1 - \upsilon^2)]^{1/4} \frac{b}{(ah)^{1/2}} \tag{38}$$

When Poisson's ratio is 0.3, this becomes

$$\phi_0 \simeq 1.82 \frac{b}{(ah)^{1/2}} \tag{38a}$$

It can be seen from Fig. 3 that antisymmetric buckling occurs at higher values of the compressive stress than symmetric buckling. One should expect therefore that only symmetric buckling will be observed in experiment. The buckling stress is found to be higher than that corresponding to uniform compression. However, the increase is noticeable only if the heated width is very small. For practical purposes one may say that the increase may be disregarded if

$$2b/(ah)^{1/2} > 2.5 \tag{39}$$

For narrower heated bands, however, the value of the critical stress increases rapidly with decreasing values of 2b.

The buckle shape is sinusoidal in the axial direction. Figures 4 and 5 show the variation of the radial displacements in the circumferential direction. It can be seen that the unheated region does not show noticeable displacements when the heated region is wide, but it is significantly affected when the heated region is narrow.

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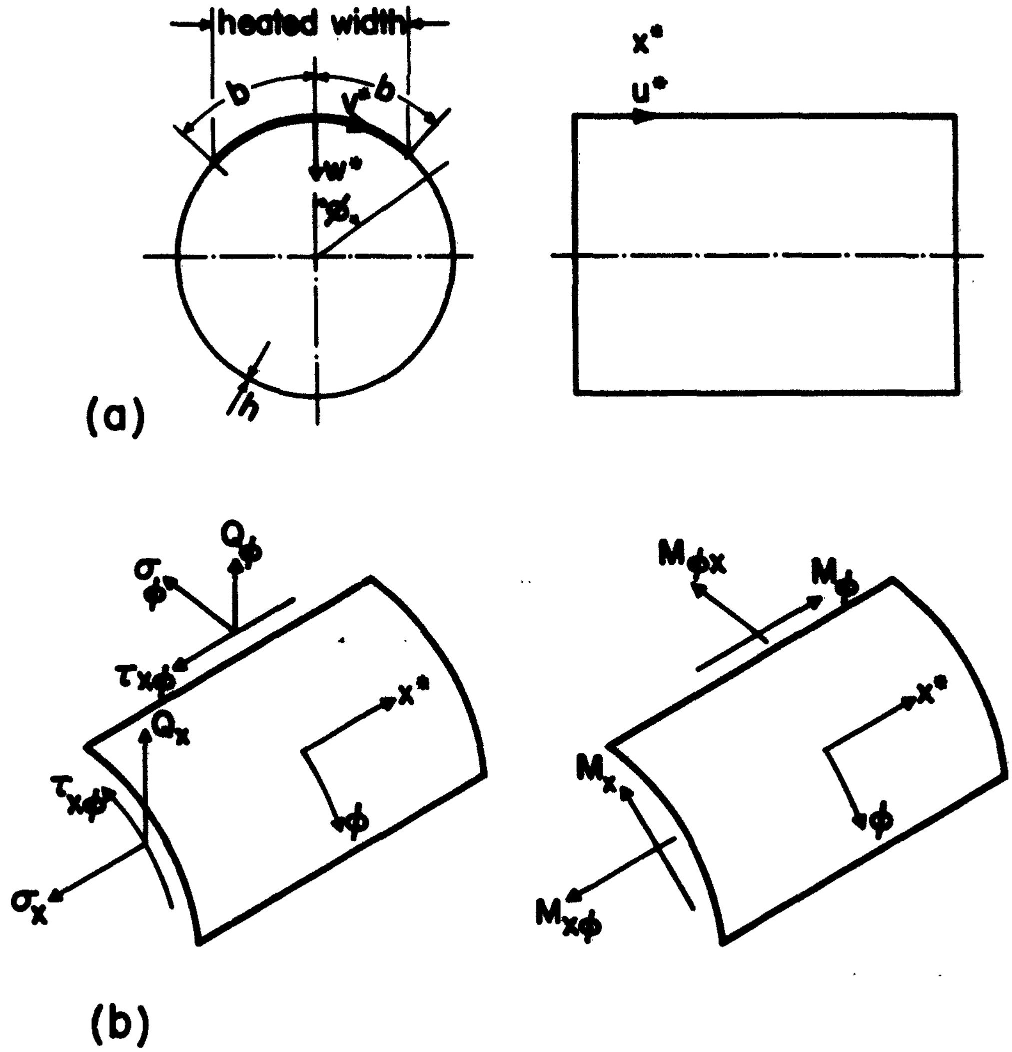

Fig. 1. NOTATION AND SIGN CONVENTION FOR CYLINDRICAL SHELL

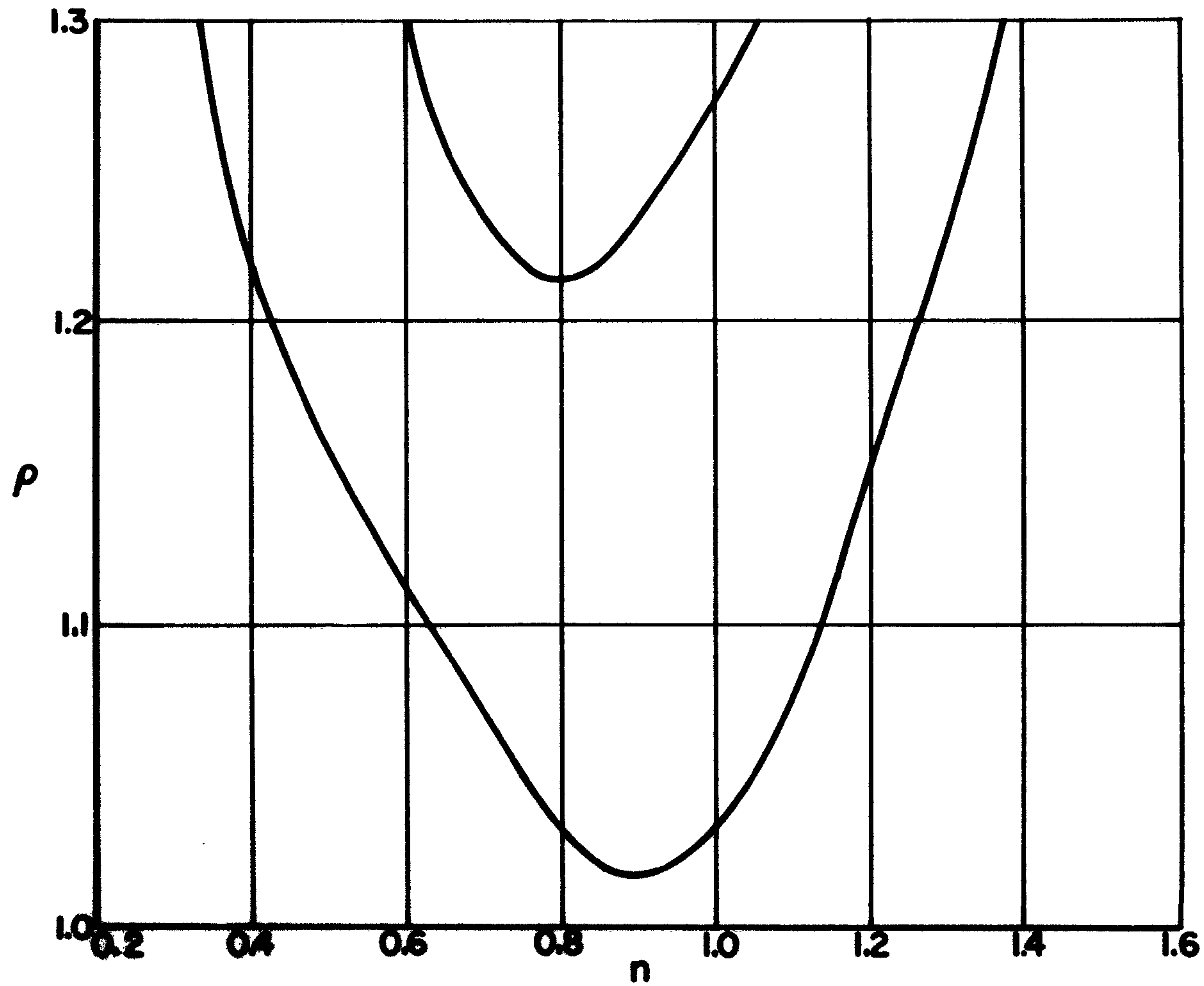

Fig. 2. CRITICAL STRESS RATIO ρ FOR ANTISYMMETRIC BUCKLING VERSUS REDUCED WAVE NUMBER n FOR NON-DIMENSIONAL HEATED WIDTH $2b/(ah)^{1/2} = 9.83$.

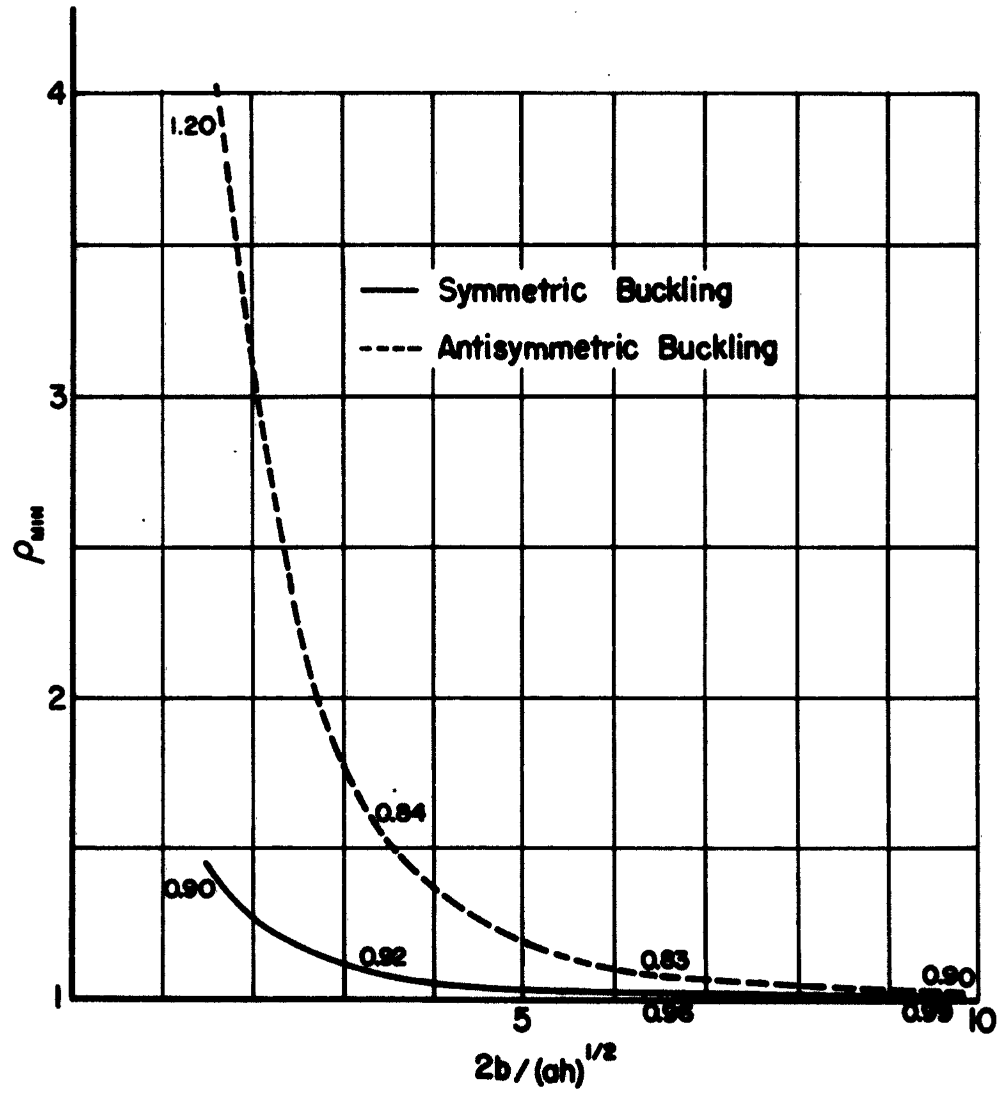

Fig. 3. CRITICAL STRESS RATIO ρ VERSUS NON-DIMENSIONAL HEATED WIDTH $2b/(ah)^{1/2}$. (NUMBERS NEAR CURVES ARE VALUES OF REDUCED WAVE NUMBER.)

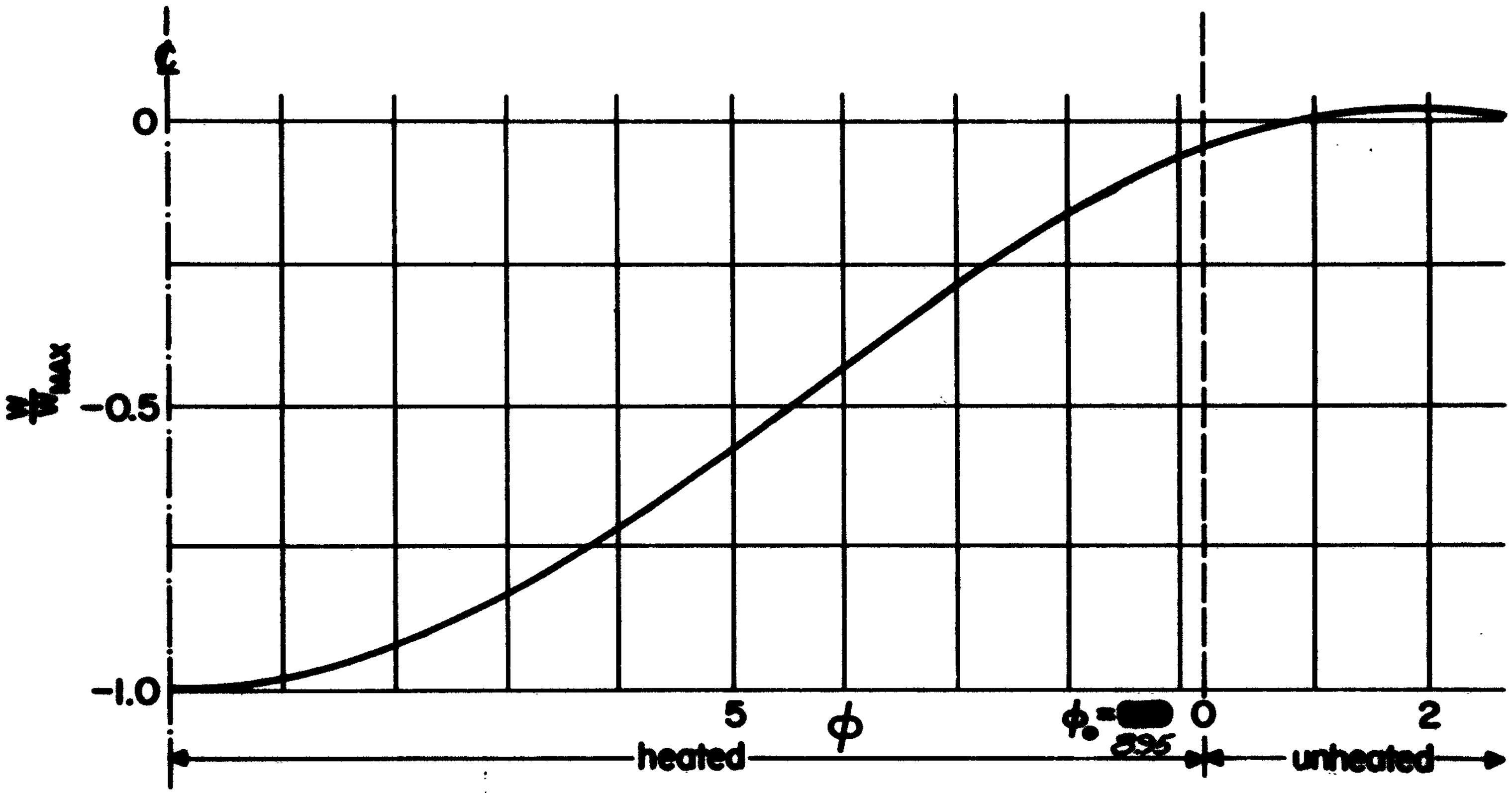

Fig. 4. SHAPE OF SYMMETRIC BUCKLE WHEN $2b/(ah)^{1/2} = 9.83$

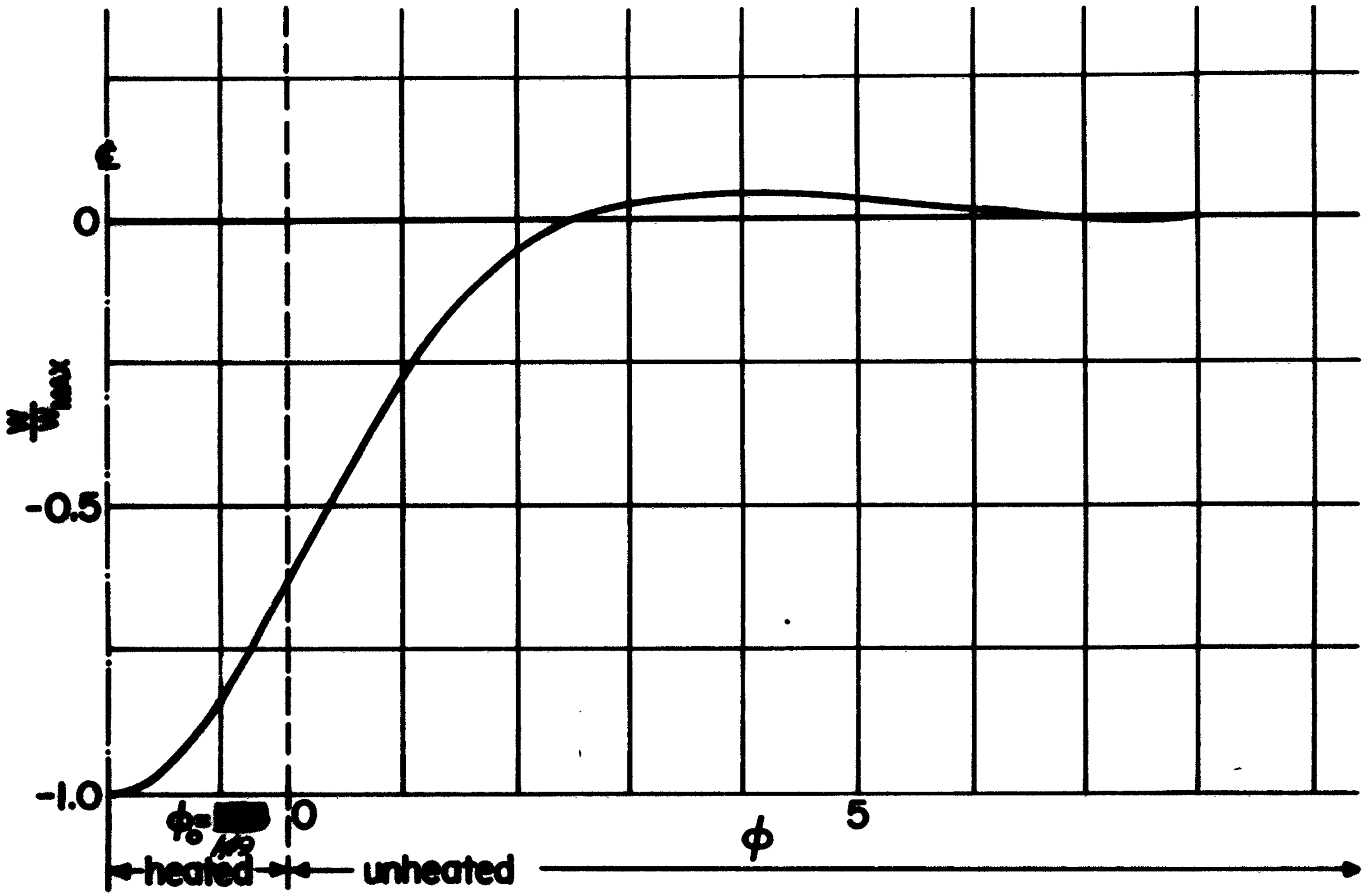

Fig. 5. SHAPE OF SYMMETRIC BUCKLE WHEN $2b/(ah)^{1/2} = 1.64$