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STANFORD UNIVERSITY
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GENERALIZED LEAST SQUARES ESTIMATORS FOR RANDOMIZED FRACTIONAL REPLICATION DESIGNS

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S. ZACKS

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PREPARED UNDER CONTRACT Nonr-225(52)
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1. Introduction

Fractional replication designs have become of great importance, especially for industrial experimentation. A missile whose operation is affected simultaneously by dozens of interacting factors would produce a full factorial experiment of impractical size. Indeed, if there are more than 20 factors which may affect the operation of a missile, and if we like to attain complete information on all the main effects and interactions of the controllable factors we would run the experiments over more than $2^{20} = 1,048,576$ treatment combinations. Fractional replication designs are planned to attain information about some of the main effects and interactions by a relatively small number of trials. If the operation of a missle can be controlled with some information on the main effects and some low order linear interations, it might be sufficient to run only 32 or 64 trials at a time. These however, should be chosen from those possible in some optimal manner.

The problem of choosing a $1/2^{m-s}$ fractional replication and an appropriate estimator of the parameters characterizing the factorial model (main effects and interactions) has been studied by A. P. Dempster (1960, 1961), K. Takeuchi (1961), S. Ehrenfeld and S. Zacks (1961, 1962), S. Zacks (1962), B. V. Shah and O. Kempthorne (1962a,b). In all these studies the type of estimators considered is that which yields, under a randomized procedure with equal probabilities of choice, unbiased estimates of a specified linear functional of a subvector of parameters, which lies in the range of the design matrix (the matrix of the corresponding normal equations).

In the present study statistical properties of the generalized least squares estimators, under randomized fractional replication designs, are studied. The term generalized least-squares estimators (denoted henceforth by g.l.s.e.) is used since the matrices of the normal equations corresponding to these designs are singular. The factorial models corresponding to the type of fractional replication designs studied in the present paper is presented in section 2. For this sake we start from the factorial model for a full factorial system. Then we present the required algebra, and the method of constructing the orthogonal fractional replications. The linear spaces of all g.l.s.e. associated with the various orthogonal fractional replication designs are characterized in terms of the linear coefficients of the corresponding factorial models. Some statistical properties of the g.l.s.e. under procedures of choosing a fractional replication at random, are then studied. First we prove that there is no g.l.s.e. which yields unbiased estimates of the entire vector of 2^m parameters. However, there are g.l.s.e.s which estimate unbiasedly subvectors of parameters. The trace of the mean-square-error matrix corresponding to a g.l.s.e. applied under certain randomization procedure is used as a loss function for the decision problem of choosing a g.l.s.e. and a randomization procedure. It is shown that when the parameters of the factorial system may assume arbitrary values, the randomization procedure which assigns equal probabilities to various fractional replications (denoted by R. P.*) is admissible. Bayes g.l.s.e., relative to a-priori information available on the parameters, are then studied. This leads to a minimax theorem, which specifies a minimax and admissible g.l.s.e. under R.P.*.

The relationship between the generalized inverse of the matrix of normal equations and g.l.s.e. as given by A. Ben-Israel and J. Wersen (1962), and by C. R. Rao (1962) is studied. It is shown that these are particular cases in the general class of g.l.s.e. studied presently.

Finally, it should be remarked that although the present paper deals with factorial system of order 2^m all the important results hold in more general factorial systems of order p^m $(p > 2)$.

2. The statistical model for fractional replication designs.

2.a. The statistical model for a full factorial experiment of order 2^m.

A full factorial experiment of order 2^m is a set of 2^m treatment combinations, consisting of m factors $X_o, \ldots, X_{m-1}$ each at two levels. Such a system can be characterized by 2^m parameters $\alpha_o, \ldots, \alpha_{2^m-1}$, which are the coefficients of the multilinear regression function:

$$(2.1)\quad E\{Y(X_o, \ldots, X_{m-1})\} = \sum_{(\lambda_o, \ldots, \lambda_{m-1})} \alpha_{u(\lambda_o, \ldots, \lambda_{m-1})} X_o^{\lambda_o} X_1^{\lambda_1} \ldots X_{m-1}^{\lambda_{m-1}}$$

where $\lambda_j = 0,1$ $(j=0,\ldots,m-1)$; $u(\lambda_o, \ldots, \lambda_{m-1}) = \sum_{j=0}^{m-1} \lambda_j 2^j$;

and $Y(X_o, \ldots, X_{m-1})$ is a random variable representing the "yield" of the experiment at treatment combination $(X_o, \ldots, X_{m-1})$. Denote by $X_{j,0}$ and $X_{j,1}$ $(j=0,\ldots,m-1)$, $X_{j,0} < X_{j,1}$, the two specified levels of factor X_j. By changing variables according to the transformation

$$(2.2)\qquad Z_{j,k} = \frac{X_{j,k} - \frac{1}{2}(X_{j,0} + X_{j,1})}{\frac{1}{2}(X_{j,1} - X_{j,0})} \qquad (k=0,1; j=0,\ldots,m-1)$$

the regression function (2.1) is reduced to the form:

$$(2.3)\qquad E\{Y(Z_o, \ldots, Z_{m-1})\} = \sum_{u=0}^{2^m-1} \beta_u Z_o^{\lambda_o} Z_1^{\lambda_1} \ldots Z_{m-1}^{\lambda_{m-1}}$$

where $Z_j = -1, +1$ $(j=0,\ldots,m-1)$; and $u \equiv u(\lambda_o, \ldots, \lambda_{m-1})$.

Writing $Z_j = (-1)^{1-i_j}$ with $i_j = 0,1$ for all $j = 0,\ldots,m-1$, the regression function (2.3) can be represented in the form:

$$(2.4) \qquad E\{Y(i_o,\ldots,i_{m-1})\} = \sum_{u=0}^{2^m-1} \beta_u (-1)^{\sum_{j=0}^{m-1} \lambda_j(1-i_j)}$$

Furthermore, denote by $x_v \equiv (i_o,\ldots,i_{m-1})$, $v = \sum_{j=0}^{m-1} i_j\, 2^j$, the 2^m treatment combinations of the factorial system under consideration; and let

$$(2.5) \qquad c_{vu}^{(2^m)} = (-1)^{\sum_{j=0}^{m-1} \lambda_j(1-i_j)} \quad , \text{ for all } v,u = 0,\ldots, 2^m-1$$

then (2.4) is reduced to the form:

$$(2.5) \qquad E\{Y(x_v)\} = \sum_{u=0}^{2^m-1} \beta_u\, c_{vu}^{(2^m)} \quad , \text{ for all } v = 0,\ldots,2^m-1$$

Let $Y' = (Y(X_o),\ldots,Y(X_{2^m-1}))$ be the vector of observations at all the 2^m treatment combinations; and let $\beta' = (\beta_o,\ldots,\beta_{2^m-1})$ be the vector of parameters of (2.5). Thus, if

$$(C^{(2^m)}) = \|c_{vu}^{(2^m)}\| \text{ , } (v,u = 0,\ldots,2^m-1),$$

denotes the matrix of the coefficients of the β's in (2.5), then the statistical model for a full factorial system can be written as:

$$(2.6) \qquad Y = (C^{(2^m)})\,\beta + \epsilon$$

where ϵ is a random vector, with $E\,\epsilon = 0$ and $E\,\epsilon\epsilon' = \sigma^2 I^{(2^m)}$ ($I^{(n)}$ denoting the identity matrix of order n).

2.b. The algebra of factorial experiments.

2.b.1. Properties of the matrices $(C^{(2^m)})$

The properties of the matrices $(C^{(2^m)})$ will be presented without proofs. For details see S. Ehrenfeld and S. Zacks (1961).

The matrices $(C^{(2^m)})$, $m = 1,2,\ldots,$ of (2.6) can be obtained recursively by the following relationship:

$$(2.7) \qquad (C^{(2^m)}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes (C^{(2^{m-1})}) , \quad m = 1,2,\ldots$$

where $(C^{(0)}) \equiv 1$ (scalar), and where $A \otimes B$ is the Kronecker's direct multiplication of the matrix A by the matrix B from the left, defined as follows: If A is an $n \times m$ matrix $\|a_{ij}\|$, and B is a $k \times l$ matrix $\|b_{rs}\|$, then $A \otimes B$ is the $nk \times ml$ matrix:

$$A \otimes B \left[\begin{array}{c|c|c} a_{11}\, B & \cdots & a_{1m}\, B \\ \hline \vdots & \ddots & \vdots \\ \hline a_{n1}\, B & \cdots & a_{nm}\, B \end{array}\right]$$

From the associative property of the Kronecker's direct multiplication it follows that every matrix $(C^{(2^m)})$ can be factorized into $2^{m-s} \times 2^{m-s}$ $(1 \leq s \leq m)$ submatrices of order $2^s \times 2^s$, according to the relationship:

$$(2.8) \qquad (C^{(2^m)}) = (C^{(2^{m-s})}) \otimes (C^{(2^s)}) , \quad (1 \leq s \leq m)$$

From this relationship it follows that the elements of $(C^{(2^m)})$ are related to those of $(C^{(2^{m-s})})$ and $(C^{(2^s)})$ according to:

$$C^{(2^m)}_{i+j2^s,t} = C^{(2^s)}_{i,r_t} \cdot C^{(2^{m-s})}_{j,q_t} \tag{2.9}$$

for all $i = 0,\ldots,2^s-1$; $j = 0,\ldots,2^{m-s}-1$; and where $t = r_t + q_t 2^s$ $(r_t = 0,\ldots,2^s-1\ ;\ q_t = 0,\ldots,2^{m-s}-1)$.

Another useful relationship among the elements of $(C^{(2^m)})$ is the following one: Let $u_1, u_2 = 0,\ldots,2^m-1$ be given by $u_i \equiv (\lambda_{oi}, \lambda_{1i}, \ldots, \lambda_{(m-1)i})$ $(i=1,2)$; $\lambda_{j,i} = 0,1$ $(j=0,\ldots,m-1)$, and define $u_1 \oplus u_2 \equiv (\lambda'_o, \ldots, \lambda'_{m-1})$ where $\lambda'_j \equiv \lambda_{j1} + \lambda_{j2}$ (mod. 2); then

$$C^{(2^m)}_{v,u_1\oplus u_2} = C^{(2^m)}_{v,u_1} \cdot C^{(2^m)}_{v,u_2} \quad \text{for every } v=0,\ldots,2^m-1\ . \tag{2.10}$$

The properties of $(C^{(2)})$ are extended into $(C^{(2^m)})$ by the recursion relationship (2.7), and are summarized as follows:

(i) $C^{(2^m)}_{v,0} = 1$ for every $v = 0,\ldots,2^m-1$

(ii) $C^{(2^m)}_{2^m-1,u} = 1$ for every $u = 0,\ldots,2^m-1$

(iii) $\sum\limits_{u=0}^{2^m-1} C^{(2^m)}_{vu} = 0$ for every $v = 0,\ldots,2^m-2$

(iv) $\sum\limits_{v=0}^{2^m-1} C^{(2^m)}_{vu} = 0$ for every $u = 1,\ldots,2^m-1$ for every $u = 1,\ldots,2^m-1$.

$$\text{(v)} \quad \sum_{v=0}^{2^m-1} C_{vu_1}^{(2^m)} \; C_{vu_2}^{(2^m)} = \begin{cases} 2^m \;, & \text{if } u_1 = u_2 \\ 0 \;, & \text{if } u_1 \neq u_2 \end{cases}$$

and

$$\text{(vi)} \quad \sum_{u=0}^{2^m-1} C_{v_1u}^{(2^m)} \; C_{v_2u}^{(2^m)} = \begin{cases} 2^m \;, & \text{if } v_1 = v_2 \\ 0 \;, & \text{if } v_1 \neq v_2 \end{cases}$$

(v) and (vi) can be expressed also in the form:

$$(C^{(2^m)})' \; (C^{(2^m)}) = (C^{(2^m)})(C^{(2^m)})' = 2^m I^{(2^m)} \; .$$

2.b.2. The group of parameters.

Every parameter β_u $(u = 0,\ldots,2^m-1)$ of the statistical model (2.6) can be represented by an m-tuple $\beta_u \equiv (\lambda_o,\ldots,\lambda_{m-1})$ where $\lambda_j = 0,1$ $(j = 0,\ldots,m-1)$.

The set of all 2^m parameters constitutes a group, B with respect to the operator $\otimes$, defined as follows:

Let $u_1 = \sum_{j=0}^{m-1} \lambda_j \, 2^j$ and $u_2 = \sum_{j=0}^{m-1} \lambda'_j \, 2^j$ then $\beta_k \equiv \beta_{u_1} \otimes \beta_{u_2}$ if, and only if $k = \sum_{j=0}^{m-1} \lambda''_j \, 2^j$ where $\lambda''_j \equiv \lambda_j + \lambda'_j \pmod{2}$ for all $j = 0,\ldots,m-1$.

The unit element of the group B is $\beta_o \equiv (0,0,\ldots,0)$ and the inverse of $\beta_u \equiv (\lambda_o,\ldots,\lambda_{m-1})$ is $\beta_u^{-1} \equiv (2-\lambda_o,2-\lambda_1,\ldots,2-\lambda_{m-1}) \pmod{2}$. A set of n parameters $\beta_{u_1}, \beta_{u_2},\ldots,\beta_{u_n}$ is called dependent if there exist n constants a_k $(k=1,\ldots,n)$, not all of which are zero ($a_k = 0,1$), such that:

$$[\beta_{u_1}]^{a_1} \otimes [\beta_{u_2}]^{a_2} \otimes \cdots \otimes [\beta_{u_n}]^{a_n} \equiv \beta_o \tag{2.11}$$

where $$[\beta_u]^a = \begin{cases} \beta_u & , \text{ if } a = 1 \\ \beta_o & , \text{ if } a = 0 \end{cases}$$

If relationship (2.11) is valid only when all $a_k = 0$ $(k=1,\ldots,n)$ then $\beta_{u_1},\ldots,\beta_{u_n}$ are called <u>independent</u>. It is easy to check that every set of n independent parameters $(1 \leq n \leq m)$ generates a subgroup of order 2^n in B.

2.c. <u>The construction of a $1/2^{m-s}$ $(1 \leq s , m)$ fractional replication</u>.

Let $\{\beta_{d_o},\beta_{d_1},\ldots,\beta_{d_{m-s-1}}\}$ be any set of m-s independent parameters in B. The 2^m treatment combinations can be classified into 2^{m-s} disjoint subsets $X_v(v=0,\ldots,2^{m-s}-1)$ of equal size, relative to the specified m-s independent parameters, in the following manner: Let $\beta_{d_k} \equiv (\lambda_{o,d_k}, \lambda_{1,d_k},\ldots, \lambda_{m-1,d_k})$ $k = 0,\ldots,m-s-1$ be a defining parameter, and let $X \equiv (i_o,\ldots,i_{m-1})$ be a treatment combination, then $x \in X_v$ if, and only if,

$$\sum_{j=0}^{m-1} i_j \lambda_{j,d_k} \equiv a_k \pmod{2}, \text{ and where} \tag{2.12}$$

$$v = \sum_{k=0}^{m-s-1} a_k 2^k .$$

In order to perform the classification of treatment combinations into the blocks X_v $(v = 0,\ldots,2^{m-s}-1)$ we do not have to solve the linear equations (2.12), but it suffices to compare the rows of the matrix of coefficients $(C^{(2^m)})$ under the columns corresponding to the special independent parameters $\{\beta_{d_o},\ldots,\beta_{d_{m-s-1}}\}$. These two procedures are equivalent (see S. Ehrenfeld and S. Zacks (1961)). The $m-s$ independent parameters, relative to which the classification takes place, are called <u>defining parameters</u>. The answer to the question, which of the parameters should be specified for the role of defining ones depends on the objectives of the experiment. The choice of a set of defining parameters will generally effect the bias of estimators and their variances, and might have other effects on the properties of statistics and procedures (see O. Kempthorne (1952); S. Ehrenfeld and S. Zacks (1961).

The term <u>fractional replication</u>, in its broadest sense, relates to any subset of treatment combination from a full factorial system. K. Takeuchi (1961) considers designs of randomly combined fractional replications. We shall consider in the present paper only fractional replications which consist of one block of treatment combinations, X_v, chosen from the set of 2^{m-s} blocks constructed according to the procedure outlined above. These fractional replications are called <u>orthogonal</u>. A <u>randomized fractional replication procedure</u> is one in which a block X_v is chosen with a probability vector $\xi' = (\xi_o,\ldots,\xi_{2^{m-s}-1})$

2.d. <u>The statistical model for a $1/2^{m-s}$ fractional replication</u>.

Let $\{\beta_{d_o},\ldots,\beta_{d_{m-s-1}}\}$ be a set of defining parameters; $\{X_v\,;\ v = 0,\ldots,2^{m-s}-1\}$ the corresponding blocks of treatment combinations, and $Y(X_v)$ the vector of observations associated with the 2^s treatment combinations in X_v. The order of the components of $Y(X_v)$ is determined by the standard order of the corresponding x's in X_v, e.g. if $X_v = \{x_o, x_3, x_5, x_6\}$ then $Y(X_v)' = (Y(x_o), Y(x_3), Y(x_5), Y(x_6))$.

Let $\beta^{(0)'} = (\beta_o, \beta_{t_1}, \ldots, \beta_{t_{2^s-1}})$ be any specified vector of 2^s parameters independent of the defining parameters (except for the "mean" β_o); with $t_k < t_{k+1}$ for all $k = 1, \ldots, 2^s-1$. Let $\{\beta_u^* ; u=0, \ldots, 2^{m-s}-1\}$ be the subgroup of 2^{m-s} parameters, generated by the $m-s$ defining parameters. Define by $\beta^{(u)}$ $(u = 1, \ldots, 2^{m-s}-1)$ the vector of 2^s parameters obtained by multiplying each of the components of $\beta^{(0)}$ by β_u^*, i.e., $\beta^{(u)} = (\beta_o \otimes \beta_u^*, \beta_{t_1} \otimes \beta_u^*, \ldots, \beta_{t_{2^s-1}} \otimes \beta_u^*)'$. Then, the statistical model for $Y(X_v)$ can be written in the form:

$$(2.13) \qquad Y(X_v) = \sum_{u=0}^{2^{m-s}-1} (P_{vu}^{(2^s)})\, \beta^{(u)} + \epsilon = (P_v)\beta^* + \epsilon$$

where; as proven by S. Ehrenfeld and S. Zacks (1961)

$$(2.14) \qquad (P_v) = (1, b_{v1}^{(2^{m-s})}, \ldots, b_{v(2^{m-s}-1)}^{(2^{m-s})}) \otimes (P_{v0}^{(2^s)})$$

is a $2^s x (2^m-2^s)$ matrix; $(P_{v0}^{(2^s)})$ is a $2^s \times 2^s$ matrix obtained from $(C^{(2^m)})$, by picking the elements of $(C^{(2^m)})$ corresponding to treatment combinations in X_v and the parameters in $\beta^{(0)}$, and arranging them in the standard order. The scalars $b_{vu}^{(2^{m-s})}$, by which we multiply $(P_{v0}^{(2^s)})$ to obtain $(P_{vu}^{(2^s)})$, are given by the formula:

$$(2.15) \qquad b_{vu}^{(2^{m-s})} = (-1)^{\sum_{j=0}^{m-s-1} (i_j' (i_j - L(d_j))}$$

where $v = \sum_{j=0}^{m-s-1} i_j 2^j$; $u = \sum_{j=0}^{m-s-1} i'_j 2^j$ and $L(d_j) \equiv \sum_{k=0}^{m-1} \lambda_{k,d_j}$ (mod. 2),

for every defining parameter β_{d_j}; and where $\beta^* = (\beta^{(0)} \vdots \beta^{(1)} \vdots \cdots \vdots \beta^{(2^{m-s}-1)})'$,

and ϵ is a random vector of order 2^s, with $E\epsilon = 0$ and $E\epsilon\epsilon' = \sigma^2 I^{(2^s)}$.

It can be readily proved that, the rows of $(P_{vu}^{(2^s)})$ $(v,u=0,\ldots,2^{m-s}-1)$, as well as its columns, are orthogonal, i.e.,

$$(2.16) \qquad (P_{vu}^{(2^s)})' (P_{vu}^{(2^s)}) = (P_{vu}^{(2^s)})(P_{vu}^{(2^s)})' = 2^s I^{(2^s)}$$

for all $v,u=0,\ldots,2^{m-s}-1$; and that a similar property holds for the matrix $(B^{(2^{m-s})}_{(d_s,\ldots,d_{m-s-1})})$, whose elements are the coefficients $b_{vu}^{(2^{m-s})}$, defined by (2.15), i.e.,

$$(2.17) \qquad (B^{(2^{m-s})}_{(d_o,\ldots,d_{m-s-1})})' (B_{(d_o,\ldots,d_{m-s-1})}) = 2^{m-s} I^{(2^{m-s})}$$

for every choice of m-s defining parameters. For the sake of simplifying notation, let $S = 2^s$ and $M = 2^{m-s}$, $N = S \cdot M = 2^m$. Furthermore, assume, without loss of generality, that the defining parameters are the "main effects" $\{\beta_S, \beta_{2S}, \ldots, \beta_{N/2}\}$ and that $\beta^{(0)} = (\beta_o, \beta_1, \ldots, \beta_{S-1})'$ then, the blocks of treatment combinations are:

$$(2.18) \qquad \{X_{i+vS}\ ;\ i = 0,\ldots,S-1\} \text{ for all } v = 0,\ldots, M-1$$

and the statistical model for $Y(X_v)$ is given by:

$$(2.19) \qquad Y(X_v) = [(1, C_{v1}^{(M)}, \ldots, C_{v(M-1)}^{(M)}) \otimes (C^{(S)})]\, \beta + \epsilon$$
$$= (C^{(S)})\beta^{(0)} + \sum_{u=1}^{M-1} C_{vu}^{(M)} \beta^{(u)} + \epsilon$$

where $\qquad \beta^{(u)} = (\beta_{us}, \beta_{1+us}, \ldots, \beta_{(u+1)s-1})'$.

3. Generalized least squares estimators for fractional replications.

3.a. The set of all least squares estimators.

Given a block of treatment combinations X_v $(v=0,\ldots,M-1)$ and the associated vector of observations $Y(X_v)$, the "normal equations" corresponding to the linear model (2.19) are given by:

$$(3.1) \qquad (C_v)'(C_v)\beta = (C_v)' Y(X_v), \quad v=0,\ldots,M-1,$$

where (C_v) is the $S \times N$ matrix of the coefficients of (2.19), i.e.,

$$(3.2) \qquad (C_v) = (1, C_{v1}^{(M)}, \ldots, C_{v(M-1)}^{(M)}) \otimes (C^{(S)}).$$

A generalized least squares estimator (g.l.s.e.) of β is any linear operator (L_v), on $E^{(S)}$ (Eucliden S-space), so that (L_v) is an $N \times S$ matrix satisfying the equation:

$$(3.3) \qquad (C_v)'(C_v)(L_v) = (C_v)', \quad v=0,\ldots,M-1.$$

Let $(L_v)' = ((L_{vo})' \vdots (L_{v1})' \vdots\ldots\vdots (L_{v(M-1)})')$ where (L_{vu}) $(u=0,\ldots,M-1)$ are square matrices of order $S \times S$. Substituting from (3.2) for (C_v) in 3.3 and decomposing (L_v) as indicated we arrive at the matrix equation:

$$(3.4) \qquad S[(Q^{(M)}) \otimes (I^{(S)})] \begin{bmatrix} (L_{vo}) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ (L_{v(M-1)}) \end{bmatrix} = \begin{bmatrix} 1 \\ C_{v1}^{(M)} \\ \vdots \\ C_{v(M-1)}^{(M)} \end{bmatrix} \otimes (C^{(S)})'$$

where $(Q^{(M)}) = (1, C_{v1}^{(M)}, \ldots, C_{v(M-1)}^{(M)})'(1, C_{v1}^{(M)}, \ldots, C_{v(M-1)}^{(M)})$ is a square symmetric matrix of order $M \times M$, whose (i,j)-th element is $q_{ij}^{(M)} = c_{vi}^{(M)} \; c_{vj}^{(M)}$ $(i,j=0,\ldots,M-1)$. Since $c_{vu}^{(M)} = \pm 1$ and $(C^{(S)})$ is non-singular, the linear equations in the matrices (L_{vu}) can be expressed in a form equivalent to (3.4) as,

$$\sum_{u=0}^{M-1} c_{vu}^{(M)} \, (L_{vu})(C^{(S)}) = I^{(S)} \quad . \tag{3.5}$$

Since the unique solution to the equation $(H)(C^{(S)}) = I^{(S)}$ is $(H) = \frac{1}{S}\,(C^{(S)})'$, it follows that the M matrices $(L_v^{(j)})$, $j=0,\ldots,M-1$, whose submatrices are given by:

$$(L_{vu}^{(j)}) = \begin{cases} \frac{1}{S} \; c_{vu}^{(M)} \; (C^{(S)})' & , \text{ if } u = j-1 \\ (0) & , \text{ otherwise} \end{cases} \tag{3.6}$$

constitute a basis of M independent solutions of (3.3). Thus, every g.l.s.e. (L_v) can be represented as a linear combination of the M linearly independent operators $(L_v^{(j)})$ $(j=0,\ldots,M-1)$, so that the coefficients of $(L_v^{(j)})$ add up to 1. Formally, the set of all g.l.s.e., given X_v is:

$$\mathcal{L}(C_v) = \{(L_v) : (L_v) = \sum_{j=0}^{M-1} \lambda_j \; (L_v^{(j)}) \; ; \; \sum_{j=0}^{M-1} \lambda_j = 1\} \tag{3.7}$$

Every g.l.s.e. can thus be represented by M coordinates $(\lambda_o,\lambda_1,\ldots,\lambda_{M-1})$ such that $\sum_{j=0}^{M-1} \lambda_j = 1$. Furthermore, if $\hat{\beta}_v$ denotes the vector of g.l.s.e. of β then we have, according to (3.6) and (3.7)

$$(3.8) \qquad \hat{\beta}_v = \frac{1}{S} \begin{bmatrix} \lambda_0 & \\ \lambda_1 & C_{vu}^{(M)} \\ \vdots & \\ \lambda_{M-1} & C_{vu}^{(M-1)} \end{bmatrix} \otimes (C^{(S)})' \, Y(X_v) \quad \text{for every} \quad v=0,\ldots,M-1.$$

3.b. Some statistical properties of g.l.s.e.

In the present section we prove that there are no unbiased g.l.s.e. of β, and derive an expression for the trace of the mean-square-error matrix of a g.l.s.e. $\hat{\beta}$.

Consider a fractional replication design in which a block X_v $(v=0,\ldots,M-1)$ is chosen with probability ξ_v ($\xi_v \geq 0$ for all $v=0,\ldots,M-1$; $\sum_{v=0}^{M-1} \xi_v = 1$). A randomized fractional replication procedure is thus represented by an M-dimensional probability vector ξ. This class of randomization procedures contains, in particular, the fixed fractional replication design, in which one of the X_v blocks is chosen with probability one.

Theorem 1.

Let $\tilde{\beta}^{(u)} = \frac{1}{S} \; C_{vu}^{(M)} \; (C^{(S)})' \; Y(X_v)$ then $E_{\xi*} \{\tilde{\beta}^{(u)}\} = \beta^{(u)}$ for all $u=0,\ldots,M-1$ if, and only if, $\xi^* = (\frac{1}{M}, \frac{1}{M}, \ldots, \frac{1}{M})'$.

Proof.

The expected value of $\tilde{\beta}^{(u)}$ under randomization procedure ξ, is given by:

$$(3.9) \qquad E_\xi\{\tilde{\beta}^{(u)}\} = \frac{1}{S}\sum_{v=0}^{M-1} \xi_v\, C_{vu}^{(M)}\, (C^{(S)})'\, Y(X_v)$$

$$= \frac{1}{S}\sum_{v=0}^{M-1} \xi_v\, C_{vu}^{(M)}\, (C^{(S)})' \left[\sum_{w=0}^{M-1} C_{vw}^{(M)}\, (C^{(S)})\beta^{(w)} + E(\epsilon)\right]$$

$$= \sum_{w=0}^{M-1}\left(\sum_{v=0}^{M-1} \xi_v\, C_{vu}^{(M)}\, C_{vw}^{(M)}\right)\beta^{(w)} = \beta^{(u)} + \sum_{(w\neq u)=0}^{M-1}\sum_{v=0}^{M-1} \xi_v \cdot$$

$$\cdot\, C_{vu}^{(M)}\, C_{vw}^{(M)}\beta^{(w)} .$$

Clearly, if $\xi_v = \frac{1}{M}$ for all $v=0,\ldots,M-1$, then

$$\sum_{v=0}^{M-1} \xi_v\, C_{vu}^{(M)}\, C_{vw}^{(M)} = \frac{1}{M}\sum_{v=0}^{M-1} C_{vu}^{(M)}\, C_{vw}^{(M)} = 0$$

for all $u \neq w$ by the orthogonality of the column vectors of $(C^{(M)})$. Thus, $E_{\xi*}\{\tilde{\beta}^{(u)}\} = \beta^{(u)}$ for all $u=0,\ldots,M-1$. On the other hand, if $E_\xi\{\tilde{\beta}^{(u)}\} = \beta^{(u)}$ for all $u=0,\ldots,M-1$ then, in particular

$$(3.10) \qquad E_\xi\{\tilde{\beta}^{(0)}\} = \beta^{(0)} + \sum_{w=1}^{M-1}\sum_{v=0}^{M-1} \xi_v\, C_{vw}^{(M)}\, \beta^{(w)}$$

But the condition $\sum_{v=0}^{M-1} \xi_v\, C_{vw}^{(M)} = 0$ for all $w=1,\ldots,M-1$ is equivalent to the condition:

$$(3.11) \qquad (C^{(M)})'\,\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Multiplying both sides of (3.11) by $(C^{(M)})$ we get:

$$(3.12) \qquad M\xi = (C^{(M)})(C^{(M)})'\,\xi = (C^{(M)})\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = 1^{(M)}$$

where $1^{(M)}$ is an M-dimensional vector with unity in all its components. It follows that a necessary condition for the unbiasedness of $\tilde{\beta}^{(u)}$ is that $\xi = \frac{1}{M}\, 1^{(M)}$, i.e., each block is chosen with the same probability.

(Q.E.D.)

Returning to the g.l.s.e. we have:

$$(3.13) \qquad E_{\xi^*}\{\hat{\beta}\} = E_{\xi^*}\{(\lambda_0\tilde{\beta}^{(0)'} \vdots \lambda_1\tilde{\beta}^{(1)} \vdots \cdots \vdots \lambda_{M-1}\tilde{\beta}^{(M-1)'})'\}$$

$$= (\lambda_0\beta^{(0)'} \vdots \lambda_1\beta^{(1)'} \vdots \cdots \vdots \lambda_{M-1}\beta^{(M-1)'})'$$

where $\qquad \xi^* = \frac{1}{M}\, 1^{(M)} = \frac{1}{M}\,(1,1,\ldots,1)'$.

Since $\sum_{u=0}^{M-1} \lambda_u = 1$ we conclude that <u>there is no unbiased g.l.s.e. of</u> β.

The g.l.s.e. in which $\lambda_0 = 1$ and $\lambda_u = 0$ for all $u > 0$ yields unbiased estimates of the components of $\beta^{(0)}$ only. Similarly when $\lambda_j = 1$ $(j=0,\ldots,M-1)$ and $\lambda_{j'} = 0$ for all $j' \neq j$, the corresponding g.l.s.e. yields unbiased estimates of the components of $\beta^{(j)}$ only.

The mean-square-error dispersion matrix of a g.l.s.e. $\hat{\beta}$, under randomization procedure ξ, is defined by $E_\xi\{(\hat{\beta}-\beta)(\hat{\beta}-\beta)'\}$. Let $M(\xi,\lambda;\beta)$ denote the trace of the mean-square-error dispersion matrix of a g.l.s.e. represented by a vector λ, such that $\lambda' 1^{(M)} = 1$, under randomization procedure ξ ; i.e., $M(\xi,\lambda;\beta) = E_\xi\{(\hat{\beta}-\beta)'\,(\hat{\beta}-\beta)\}$.

Theorem 2.

The trace of the mean-square-error dispersion matrix under randomization procedure ξ is given by the expression:

$$(3.14) \qquad M(\xi,\lambda;\beta) = (\sigma^2 + |\beta|^2) \sum_{u=0}^{M-1} \lambda_u^2 - \sum_{u=0}^{M-1} (2\lambda_u - 1)|\beta^{(u)}|^2 +$$

$$+ \sum_{u_1=0}^{M-1} (2\lambda_u + 1) \sum_{(u_2 \neq u_1)=0}^{M-1} \left[\sum_{v=0}^{M-1} \xi_v \, c_{vu_1}^{(M)} c_{vu_2}^{(M)} \right] \beta^{(u_1)'} \beta^{(u_2)}$$

where $|\beta|^2 = \beta'\beta$, $|\beta^{(u)}|^2 = \beta^{(u)'}\beta^{(u)}$ $(u=0,\ldots,M-1)$.

Proof.

$$(3.15) \qquad M(\xi,\lambda,\beta) = E_\xi \{(\hat\beta-\beta)'(\hat\beta-\beta)\} = E_\xi \{\hat\beta'\hat\beta\} -2\beta' E_\xi \{\hat\beta\} + \beta'\beta$$

According to (3.8)

$$(3.16) \qquad E_\xi \{\hat\beta'\hat\beta\} = \frac{1}{S} \sum_{u=0}^{M-1} \lambda_u^2 \; E_\xi \{Y(X_v)'Y(X_v)\}$$

Substituting (2.19) for $Y(X_v)$ in (3.16) we get:

$$(3.17) \qquad \frac{1}{S} E_\xi\{Y(X_v)' \, Y(X_v)\} = \frac{1}{S} E_\xi\{[\sum_{u_1=0}^{M-1} c_{vu_1}^{(M)} (c^{(S)})\beta^{(u_1)} + \epsilon]' \cdot$$

$$\cdot [\sum_{u_2=0}^{M-1} c_{vu_2}^{(M)} (c^{(S)}) \beta^{(u_2)} + \epsilon]\} =$$

$$= \sigma^2 + \frac{1}{S} E_\xi\{\sum_{u_1=0}^{M-1} \sum_{u_2=0}^{M-1} c_{vu_1}^{(M)} c_{vu_2}^{(M)} \beta^{(u_1)'} (c^{(S)})'(c^{(S)}) \beta^{(u_2)}\}$$

$$= \sigma^2 + \sum_{u=0}^{M-1} |\beta^{(u)}|^2 + \sum_{u_1} \sum_{\neq u_2} (\sum_{v=0}^{M-1} \xi_v \, c_{vu_1}^{(M)} c_{vu_2}^{(M)}) \beta^{(u_1)'} \beta^{(u_2)}$$

Furthermore, $E_\xi(\hat{\beta}') = (\lambda_o E_\xi(\tilde{\beta}^{(0)'}) \vdots \ldots \vdots \lambda_{M-1} E_\xi(\tilde{\beta}^{(M-1)'}))$

Substituting (3.9) for $E_\xi(\tilde{\beta}^{(u)})$, we arrive at

$$(3.18) \quad \beta' E_\xi(\hat{\beta}) = \sum_{u=0}^{M-1} \lambda_u |\beta^{(u)}|^2 + \sum_{u_1=0}^{M-1} \lambda_u \left[\sum_{u_2 \neq u_1} \left(\sum_{v=0}^{M-1} \xi_v C_{vu_1}^{(M)} C_{vu_2}^{(M)}\right) \cdot \beta^{(u_1)'} \beta^{(u_2)}\right].$$

Thus, from (3.15)-(3.18) the result holds.

Q.E.D.

<u>Corollary</u>: When each block X_v $(v=0,\ldots,M-1)$ is chosen with equal probabilities $(\xi=\xi^*)$ we have

$$(3.19) \qquad M(\xi^*,\lambda;\beta) = (\sigma^2 + |\beta|^2) \sum_{u=0}^{M-1} \lambda_u^2 - \sum_{u=0}^{M-1} (2\lambda_u - 1)|\beta^{(u)}|^2$$

3.c. <u>Optimum strategies</u>

A strategy of the Statistician is a pair of two M-dimensional vectors (ξ,λ) such that ξ is a probability vector, and $\lambda' 1^{(M)} = 1$. Every strategy (ξ,λ) represents a randomization procedure and a g.l.s.e. The decision problem is to choose (ξ,λ) optimally, with respect to the loss function $M(\xi,\lambda;\beta)$.

Comparing (3.14) to (3.19) it is easily verified that for every (ξ,λ) there exists β^o in E^n such that $M(\xi,\lambda;\beta^o) > M(\xi^*,\lambda;\beta^o)$. <u>Thus, whenever</u> β <u>is arbitrary</u>, ξ^* <u>represents an admissible randomization procedure</u>. For this reason we shall restrict the discussion from now on to strategies with randomization procedure ξ^*, and turn now to the problem of deciding upon an optimum g.l.s.e. under ξ^*.

We notice in (3.19) that $M(\xi^*,\lambda;\beta)$ depends on β only through the M values $|\beta^{(u)}|^2$. An a-priori information concerning these values might thus be utilized for the choice of λ. Thus, let $\Pi^{(u)}$ be an a-priori distribution of $|\beta^{(u)}|^2$, defined over the half-line $[0,\infty)$.

Theorem 3.

The Bayes g.l.s.e. of β, with respect to the a-priori distributions $\{\Pi^{(0)},\ldots,\Pi^{(M-1)}\}$, under randomization procedure ξ^* is determined by the vector $\lambda_\pi = (\lambda_\pi^{(0)},\ldots,\lambda_\pi^{(M-1)})$, where

$$(3.20) \qquad \lambda_\pi^{(u)} = \frac{E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}}{\sum_{u=0}^{M-1} E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}}, \quad \text{for all } u=0,\ldots,M-1 \ .$$

Proof:

The risk function under (ξ^*,λ) and Π is

$$(3.21) \qquad R(\xi^*,\lambda;\Pi) = \left(\sigma^2 + \sum_{u=0}^{M-1} E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}\right) \sum_{u=0}^{M-1} \lambda_u^2 - \sum_{u=0}^{M-1} (2\lambda_u - 1) \, E_{\pi^{(u)}}\{|\beta^{(u)}|^2\} \ .$$

It is easily verified that $\lambda_\pi^{(u)}$ $(u=0,\ldots,M-1)$, given by (3.20), minimize (3.21) under the constraint $\sum_{u=0}^{M-1} \lambda_u = 1$.

Q.E.D.

Let $R_\pi^{(u)} = E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}$ $(u=0,\ldots,M-1)$ and $R_\pi = \sum_{u=0}^{M-1} R_\pi^{(u)}$ then the Bayes risk with respect to an a-priori distribution Π is

$$(3.22) \qquad R(\xi^*,\lambda_\pi;\Pi) = (\sigma^2 + R_\pi) \sum_{u=0}^{M-1} (\lambda_\pi^{(u)})^2 -$$

$$- \sum_{u=0}^{M-1} (2\lambda_\pi^{(u)}-1)\, R_\pi^{(u)} = R_\pi - \frac{1}{R_\pi} \left(1 - \frac{\sigma^2}{R_\pi}\right) \sum_{u=0}^{M-1} (R_\pi^{(u)})^2$$

In particular, when all $|\beta^{(u)}|^2$ $(u=0,\dots,M-1)$ have the same a-priori distribution, with $R_\pi^{(u)} = R_\pi^*$ for all $u=0,\dots,M-1$, then the Bayes g.l.s.e. is represented by $\lambda^* = (\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M})$ with a Bayes risk

$$(3.23) \qquad R(\xi^*,\lambda^*,\Pi) = \frac{\sigma^2}{M} + (M-1)R_\pi^*$$

Theorem 4.

$\lambda^* = \frac{1}{M}\, 1^{(M)}$ represents the minimax and admissible g.l.s.e. under randomization procedure ξ^* relative to the class of all a-priori distributions Π, such that $R_\pi = \sum_{u=0}^{M-1} R_\pi^{(u)}$ = const. and $(\sigma^2 < R_\pi < \infty)$. The minimax risk is given by (3.23).

Proof:

The minimax risk is the maximal Bayes risk, with respect to all the a-priori distributions Π in the class considered. The Bayes risk for any of these π's is given by (3.22) where $R_\pi = \sum_{u=0}^{M-1} R_\pi^{(u)}$ is a given constant. Set the Lagrangian

$$(3.24) \qquad L(R_\pi^{(0)},\dots,R_\pi^{(M-1)};\rho) = R_\pi - \frac{1}{R_\pi} \left(1- \frac{\sigma^2}{R_\pi}\right) \sum_{u=0}^{M-1} (R_\pi^{(u)})^2 +$$

$$+ \rho\left(R_\pi - \sum_{u=0}^{M-1} R_\pi^{(u)}\right)$$

By differentiating partially with respect to $R_\pi^{(u)}$ $(u=0,\dots,M-1)$ and ρ and equating the derivatives to zero we arrive at the system of linear equations:

$$(3.25)\qquad \begin{cases} -\frac{2}{R_\pi}\left(1-\frac{\sigma^2}{R_\pi}\right) R_\pi^{(u)} = \rho & \text{for all } u=0,\dots,M-1 \\ \sum_{u=0}^{M-1} R_\pi^{(u)} = R_\pi & \end{cases}$$

The solution of this system of linear equations is given by $R_\pi^{(u)} = \frac{R_\pi}{M}$ for every $u=0,\dots,M-1$. Furthermore, since $R_\pi \geq \sigma^2$, all the second order partial derivatives with respect to $R_\pi^{(u)}$ are negative. Thus all a-priori distributions π such that $R_\pi^{(u)} = \frac{R_\pi}{M}$ for every $u=0,\dots,M-1$ are minimax strategies for Nature. As mentioned before, $\lambda^* = \frac{1}{M} 1^{(M)}$ is then the unique minimax strategy for the Statistician. The Bayes risk corresponding to λ^* is given by (3.23). The admissibility of λ^*, relative to the class of a-priori distributions considered, follows from the fact that it is the unique minimax.

Q.E.D.

4. The generalized inverse and the g.l.s.e.

4.a. The g.l.s.e. of minimum norm

A. Ben-Israel and S. J. Wersan (1962) proved that the g.l.s.e. (L_v) with minimum norm, i.e., $\min_{(L_v)} \operatorname{tr} (L_v)'(L_v)$, is a particular generalized inverse $(C_v)^{\dagger}$ of the matrix of coefficients in (2.19), namely $(C_v) = (1, C_{v1}^{(M)}, \ldots, C_{v(M-1)}^{(M)}) \times (C^{(S)})$. The generalized inverse $(C_v)^{\dagger}$ always exists, it is unique, and given in general by the formula:

$$(4.1) \qquad (C_v)^{\dagger} = [I^{(N)} - (D_v)(D_v' D_v)^{-1}(D_v)'](E_v)(C_v)'$$

for all $v=0,\ldots,M-1$; where (E_v) is a product of elementary transformations, which transforms $(C_v)'(C_v)$ into:

$$(4.2) \qquad (E_v)(X_v)'(X_v) = \left[\begin{array}{c:c} I^{(S)} & (\Delta_v) \\ \hdashline (0) & (0) \end{array}\right] \qquad (v=0,\ldots,M-1)$$

and where

$$(4.3) \qquad (D_v) = \left[\begin{array}{c} (\Delta_v) \\ \hdashline -I^{(N-S)} \end{array}\right] \qquad (v=0,\ldots,M-1) .$$

The generalized inverse matrix, $(C_v)^{\dagger}$ has the properties:

$$(4.4) \qquad (C_v)(C_v)^{\dagger}(C_v) = (C_v) \qquad \text{for all } v=0,\ldots,M-1 .$$

and

$$(C_v)^{\dagger}(C_v)(C_v)' = (C_v)'$$

A straightforward computation of $(C_v)^{\dagger}$ according to formula (4.1) yields the result

$$(4.5) \qquad (C_v)^{\dagger} = \frac{1}{M}\begin{bmatrix} \tilde{\beta}^{(0)} \\ \tilde{\beta}^{(1)} \\ \vdots \\ \tilde{\beta}^{(M-1)} \end{bmatrix}$$

That is, $(C_v)^{\dagger}$ is a g.l.s.e. represented by $\lambda^* = \frac{1}{M} 1^{(M)}$, and has the optimal properties mentioned in the previous section. This result can be obtained in the present framework more easily. According to (3.8) a g.l.s.e. is given by

$$(4.6) \qquad (L_v) = \frac{1}{S}\begin{bmatrix} \lambda_o & C_{vo}^{(M)} \\ \vdots & \\ \lambda_{M-1} & C_{v(M-1)}^{(M)} \end{bmatrix} \otimes (C^{(S)})', \quad (v=0,\ldots,M-1)$$

Accordingly, the norm of (L_v) is

$$(4.7) \qquad \text{tr. } (L_v)'(L_v) = \text{tr. } \{\frac{1}{S^2} \sum_{u=0}^{M-1} \lambda_u^2 \, (C^{(S)})(C^{(S)})'\}$$

$$= \frac{1}{S} \text{tr. } \{\sum_{u=0}^{M-1} \lambda_u^2 \, I^{(S)}\} = \sum_{u=0}^{M-1} \lambda_u^2 \,.$$

Since the vector $\lambda^* = \frac{1}{M} 1^{(M)}$ minimizes $\sum_{u=0}^{M-1} \lambda_u^2$, under the constraint $\sum_{u=0}^{M-1} \lambda_u = 1$, it follows that the g.l.s.e. represented by λ^* minimizes the norm of (L_v), $(v=0,\ldots,M-1)$.

4.b. The g.l.s.e. suggested by C. R. Rao.

C. R. Rao (1962) defines the g.l.s.e. of β by the operator $(L_v)^- = [(C_v)'(C_v)]^- (C_v)'$ where $[(C_v)'(C_v)]^-$ is a generalized inverse of $(C_v)'(C_v)$. In case $(C_v)' (C_v)$ is invertible, $(L_v)^-$ is the unique g.l.s.e. of β. We shall prove now that under the present model of fractional replication designs, Rao's g.l.s.e., $(L_v)^-$, is represented by the vector $\lambda^- = (1,0,0,\dots,0)$. For this purpose, define the matrix of elementary transformations

$$(4.8)\qquad (E_v) = \frac{1}{\sqrt{S}} \begin{bmatrix} 1 & & & O \\ -c_{v_1}^{(M)} & & & \\ \vdots & & O & \ddots \\ -c_{v_{(M-1)}}^{(M)} & & & 1 \end{bmatrix} \otimes I^{(S)}$$

then we have, for every $v=0,\dots,M-1$,

$$(4.9)\qquad (E_v)[(C_v)'(C_v)](E_v)' = \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array}\right]$$

Hence,

$$(4.10)\qquad [(C_v)'(C_v)]^- = (E_v)'(E_v) \quad .$$

To show this, consider the relationship

$$(4.11)\qquad [(C_v)'(C_v)][(C_v)'(C_v)]^-[(C_v)'(C_v)] = [(C_v)'(C_v)]$$

Multiply both sides of (4.11) from the left by (E_v) and from the right by $(E_v)'$. Then,

$$(4.12)\qquad (E_v)[(C_v)'(C_v)](E_v)'(E_v)[(C_v)'(C_v)](E_v)' = (E_v)[(C_v)'(C_v)](E_v)'$$

Or according to (4.9)

$$(4.13) \quad \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array}\right] \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array}\right] = \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array}\right] .$$

Accordingly,

$$(4.14) \quad (L_v)^- = (E_v)'(E_v)(C_v)'$$

$$= \frac{1}{S} \begin{bmatrix} M & -C_{v1}^{(M)} & \cdots & -C_{v(M-1)}^{(M)} \\ -C_{v1}^{(M)} & & & \\ \vdots & 1 & \ddots & O \\ -C_{v(M-1)}^{(M)} & O & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ C_{v1}^{(M)} \\ \vdots \\ C_{v(M-1)}^{(M)} \end{bmatrix} \otimes (C^{(S)})'$$

$$= \frac{1}{S} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes (C^{(S)})' = \begin{bmatrix} \tilde{\beta}^{(0)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

That is, Rao's g.l.s.e. $(L_v)^-$ is represented by $\lambda^- = (1,0,\ldots,0)'$.

From theorem 1 it follows that $(L_v)^-$ is an unbiased estimator of $(\beta^{(0)},0)$. The trace of the dispersion matrix of $(L_v)^- Y(X_v)$, under randomization procedure ξ^* is given according to (3.19) by

$$(4.15) \quad M(\xi^*,\lambda;\beta) = \sigma^2 + 2|\beta|^2 - 2|\beta^{(0)}|^2$$

$$= \sigma^2 + 2\sum_{u=1}^{M-1} |\beta^{(u)}|^2$$

Thus, in case all $|\beta^{(u)}|^2$ have the same a-priori distribution, the risk under strategies (ξ^*,λ^-) and π will be

(4.16) $$R(\xi^*, \lambda^-, \pi) = \sigma^2 + 2(M-1)\, R^*_\pi$$

Comparing (4.16) to (3.23) we conclude that Rao's g.l.s.e. $(L_V)^-$ might be very far from the optimum g.l.s.e. in case all the subvectors of β have approximately the same average effect. On the other hand, in case the effects of $\beta^{(1)}$ to $\beta^{(M-1)}$ are negligible relative to the effect of $\beta^{(0)}$ the Bayes g.l.s.e. will be very close to Rao's g.l.s.e.

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