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ELECTRIC WAVE PROPAGATION ON NON-UNIFORM COUPLED TRANSMISSION LINES

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ABSTRACT

The propagation of electric currents and voltages along a pair of wires over a ground plane is studied. The system is assumed to be non-uniform; i.e., the self and mutual inductances and capacitances vary along the wires. The existence and uniqueness of an electric wave having prescribed initial values is shown to follow from recent results of the author on symmetric hyperbolic systems of partial differential equations. A construction for this solution is given in the case of a coupler (i.e., a pair of wires which are non-uniform and coupled over a portion of their length only). This coupler problem is reduced to a two-point boundary value problem, and the latter is reduced to a pair of initial value problems, one of which involves a matrix Riccati equation. A novel feature of the work is an "a priori" estimate which guarantees that a solution of the (non-linear) matrix Riccati equation exists on the whole line.

CONTENTS

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Introduction. This paper deals with the propagation of electric currents and voltages along systems of parallel wires. In the simplest case there is a pair of wires, or a single wire over a ground plane. This is called a single wire line. The case of n pairs of wires, or n wires over a ground plane, is called an n-wire line.

The single wire line was analyzed by Kirckhoff in 1857, and the associated "Telegraph Equations" have since become the subject of a large literature. This case is not considered here.

With the development of more sophisticated telegraphy and telephony, analysis of the multiple wire lines was needed. The steady state solutions for uniform n-wire lines have been analyzed by J. R. Carson and R. S. Hoyt [2], L. A. Pipes [13], S. O. Rice [17] and others. The case of non-uniform multiple wire lines (achieved by tapering the wires or varying their spacing) has received much less attention (see [7], [19]).

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This paper treats the propagation of transient and steady state waves on non-uniform multiple wire lines. For simplicity, the work is presented for the 2-wire case. However, this case is typical of the general n-wire case and all the results presented here extend immediately to the general case.

The propagation problem for non-uniform 2-wire lines is formulated in §1, and it is shown that the existence and uniqueness of a solution is guaranteed by recent work of the author [21] under very general conditions.

The remainder of the paper is devoted to a special propagation problem, the "coupler" problem. The 2-wire line is assumed to be non-uniform and coupled over a finite portion of its length (the coupler) only, and the propagation of waves through the coupler is studied. The coupler problem is formulated in §2, and a complete solution based on the Laplace transform is given in §§3 and 4. In §4, the coupler problem is reduced to a two-point boundary value problem for a system of four first order ordinary differential equations. This problem is then reduced to two one-point (or initial) value problems. The reduction employs a matrix Riccati equation satisfied by the reflection coefficient matrix.

Matrix Riccati equations have been studied by W. T. Reid [15, 16], R. L. Sternberg and H. Kaufman [19], J. J. Levin [9], R. M. Redheffer [14] and others. They have been applied to multiple transmission line problems by Sternberg and Kaufman [19] and I. Kay [7]. Analogous scaler Riccati equations for the impedance and reflection coefficient of a non-uniform single wire line were discovered by J. R. Pierce [12] and L. R. Walker and N. Wax [20]; see also [18]. In all of these papers, only the local solvability of the Riccati equations is shown. A novel feature of the present work is an "a priori" estimate which guarantees the existence of the desired solution of the Riccati equation on every interval.

The special case of a "directional coupler", which is of great interest for applications [10], is analyzed in §5. Finally, in §6 the analysis is adapted to the steady state case.

§1. The Propagation Problem for Non-Uniform Coupled Transmission Lines. The electric currents and voltages on a pair of coupled loss-less transmission lines are governed by equations of the form [1, 13]

$$
(1.1)\qquad \begin{cases}
L_1 \dfrac{\partial i_1}{\partial t} + L_m \dfrac{\partial i_2}{\partial t} + \dfrac{\partial e_1}{\partial x} = 0 \;, \\[2ex]
L_2 \dfrac{\partial i_2}{\partial t} + L_m \dfrac{\partial i_1}{\partial t} + \dfrac{\partial e_2}{\partial x} = 0 \;, \\[2ex]
C_1 \dfrac{\partial e_1}{\partial t} + C_m \dfrac{\partial e_2}{\partial t} + \dfrac{\partial i_1}{\partial x} = 0 \;, \\[2ex]
C_2 \dfrac{\partial e_2}{\partial t} + C_m \dfrac{\partial e_1}{\partial t} + \dfrac{\partial i_2}{\partial x} = 0 \;.
\end{cases}
$$

Here t is the time, x is a coordinate along the lines, and i_1 and i_2 (resp. e_1 and e_2) are the electric currents (resp. voltages) on the first and second lines, while L_1 and L_2 are their self inductance per unit length, C_1 and C_2 their capacitance per unit length, and L_m and C_m their mutual inductance and capacitance per unit length. The lines are assumed to be non-uniform; i.e. L_1, L_2, L_m, C_1, C_2 and C_m are functions of x.

The quadratic form

$$\eta = \frac{1}{2}\left(L_1 i_1^2 + 2L_m i_1 i_2 + L_2 i_2^2 + C_1 e_1^2 + 2C_m e_1 e_2 + C_2 e_2^2\right)$$

is assumed to be positive definite and is interpreted as an energy density (energy per unit length) on the lines [13]. The quadratic form

$$\Sigma = \frac{1}{2}\left(i_1 e_1 + i_2 e_2\right)$$

is interpreted as a "Poynting vector" describing the flow of energy (energy per unit time in the x-direction) on the lines. The equations (1. 1) imply the conservation law

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Sigma}{\partial x} = 0 \ .$$

The most general systems of linear partial differential equations that possess quadratic energy densities and Poynting vectors related by a conservation law were described by K. O. Friedrichs [6] who called them "symmetric hyperbolic" systems. The system (1. 1) assumes the symmetric hyperbolic form when written in the matrix notation

$$\begin{pmatrix} L_1 & L_m & 0 & 0 \\ L_m & L_2 & 0 & 0 \\ 0 & 0 & C_1 & C_m \\ 0 & 0 & C_m & C_2 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} i_1 \\ i_2 \\ e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} i_1 \\ i_2 \\ e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The propagation problem for coupled transmission lines is to find a solution of (1. 1) having prescribed initial currents and voltages,

$$i_1(x, 0) = a_1(x), \quad i_2(x, 0) = a_2(x), \quad e_1(x, 0) = b_1(x), \quad e_2(x, 0) = b_2(x)$$

for $-\infty < x < \infty$. Local existence and uniqueness theorems for the initial value problem for symmetric hyperbolic systems have been given by K. O. Friedrichs [6], P. D. Lax [8], G. F. D. Duff [5] and others. A global existence theorem for solutions with finite energy was given by R. S. Phillips [11]. Recently, the author [21] has proved a global existence and uniqueness theorem for solutions with locally finite energy; i. e. satisfying

$$(1.2) \qquad \int_{x_1}^{x_2} \eta(x, t)\, dx < \infty \quad \text{for all finite } x_1, x_2 .$$

The initial values are required to satisfy (1.2) with $t = 0$, and the coefficients are required to satisfy the conditions

(1.3) $L_1(x)$, $L_2(x)$, $L_m(x)$, $C_1(x)$, $C_2(x)$ and $C_m(x)$ are bounded, Lebesgue measurable functions of x on $-\infty < x < \infty$, and

(1.4) there exists a positive constant δ such that

$$\eta = \frac{1}{2}(L_1 u_1^2 + 2L_m i_1 i_2 + L_2 i_2^2 + C_1 e_1^2 + 2C_m e_1 e_2 + C_2 e_2^2) \geq \delta (i_1^2 + i_2^2 + e_1^2 + e_2^2)$$

for all i_1, i_2, e_1, e_2 and all x on $-\infty < x < \infty$.

Note that

$$4\eta = L_1(i_1 + \frac{L_m}{L_1} i_2)^2 + L_2(i_2 + \frac{L_m}{L_2} i_1)^2 + C_1(e_1 + \frac{C_m}{C_1} e_2)^2 + C_2(e_2 + \frac{C_m}{C_2} e_1)^2$$

$$+ (L_1 L_2 - L_m^2)(\frac{i_1^2}{L_2} + \frac{i_2^2}{L_1}) + (C_1 C_2 - C_m^2)(\frac{e_1^2}{C_2} + \frac{e_2^2}{C_1}) .$$

Hence if

$$\text{(1.5)} \qquad \frac{L_m^2}{L_1 L_2} \leq k^2 , \qquad \frac{C_m^2}{C_1 C_2} \leq k^2$$

then

$$\eta \geq \frac{1-k^2}{4}\left(L_1 i_1^2 + L_2 i_2^2 + C_1 e_1^2 + C_2 e_2^2\right)$$

Thus (1.4) holds provided L_1, L_2, C_1, C_2 are positive and bounded away from zero and (1.5) holds with a constant $k < 1$.

If the Laplace transform with respect to t is applied to (1.1) a system of four linear first order ordinary differential equations is obtained. The solution with locally finite energy guaranteed by [21] can be constructed in terms of the solutions of this system. This method is used §§3 and 4, below, to solve the propagation problem in a special case, the coupler problem.

<u>§2. Formulation of the Coupler Problem.</u> In the remainder of this report the transmission lines are assumed to be coupled on a finite segment $0 \leq x \leq \ell$ only, and to be uniform and uncoupled outside this segment, in the sense that $L_m = C_m = 0$ and L_1, L_2, C_1 and C_2 are constants for $x \leq 0$ and $x \geq \ell$ (the same constants on both intervals). Under these circumstances the segment $0 \leq x \leq \ell$ will be called a "coupler" between the two lines. It can be shown that energy can pass from one line to the other only in the coupler.

The voltages and currents on the uncoupled portions of the lines satisfy the simple transmission line equations

$$(2.1) \qquad \begin{cases} L_j \dfrac{\partial i_j}{\partial t} + \dfrac{\partial e_j}{\partial x} = 0 \quad , \\[2ex] C_j \dfrac{\partial e_j}{\partial t} + \dfrac{\partial i_j}{\partial x} = 0 \quad , \end{cases}$$

where L_j, C_j, i_j and e_j refer either to the first line $(j = 1)$ or the second line $(j = 2)$. This system is equivalent to the system

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + c_j \dfrac{\partial}{\partial x}\right)(Z_j i_j + e_j) = 0 \ , \\[2ex] \left(\dfrac{\partial}{\partial t} - c_j \dfrac{\partial}{\partial x}\right)(Z_j i_j - e_j) = 0 \quad , \end{cases}$$

where

$$c_j = 1/\sqrt{L_j C_j} \quad \text{and} \quad Z_j = \sqrt{\frac{L_j}{C_j}}$$

are the characteristic "wave velocity" and "impedance" of the lines, respectively. It follows that

$$(2.2) \qquad \begin{cases} i_j = f(c_j t - x) + g(c_j t + x) \ , \\ \\ e_j = Z_j(f(c_j t - x) - g(c_j t + x)) \ , \end{cases}$$

where

$$(2.3) \qquad f(c_j t - x) = Z_j i_j + e_j/2Z_j \quad \text{and} \quad g(c_j t + x) = Z_j i_j - e_j/2Z_j$$

are uniquely determined by the currents and voltages in the lines.

If $h_1(\tau) \equiv h_2(\tau) \equiv 0$ for $\tau \leq 0$ then

$$(2.4) \quad (i_1, i_2, e_1, e_2) = (h_1(c_1 t - x),\ h_2(c_2 t - x),\ Z_1 h_1(c_1 t - x),\ Z_2 h_2(c_2 t - x))$$

solves the system (1.1) for $-\infty < x < \infty$, $t < 0$. This solution may be interpreted as a wave propagating along the portions $x < 0$ of the lines toward the coupler and striking it at a time $t_0 \geq 0$. The propagation of the wave through the coupler is described by the solution of the initial value problem for (1.1) with

$$(2.5)\quad (i_1(x,0),\, i_2(x,0),\, e_1(x,0),\, e_2(x,0)) = (h_1(-x),\, h_2(-x),\, Z_1h_1(-x),\, Z_2h_2(-x))\,.$$

This problem is called "the coupler problem" in what follows.

The current and voltage on line 1 will have the form (2.2) for $x < 0$, $-\infty < t < \infty$ where, by (2.3) and (2.4), $f(c_1t - x) = h_1(c_1t - x)$ and $g(c_1t + x) = 0$ for $x < 0$, $t < 0$. Thus $f(\tau) \equiv h_1(\tau)$ for all τ and $g(\tau) \equiv 0$ for $\tau < 0$. Similar reasoning applies to line 2 for $x < 0$ and to lines 1 and 2 for $x > \ell$. It follows that the solution to the coupler problem has the form

$$(i_1, i_2, e_1, e_2)$$

$$= (h_1(c_1t-x) + r_1(c_1t+x),\, h_2(c_2t-x) + r_2(c_2t+x),\, Z_1\{h_1(c_1t-x) - r_1(c_1t+x)\},\, Z_2\{h_2(c_2t-x) - r_2(c_2t+x)\})$$

for $x < 0$, $-\infty < t < \infty$, where $r_1(\tau) \equiv r_2(\tau) \equiv 0$ for $\tau \leq 0$, and

$$(i_1, i_2, e_1, e_2) = (t_1(c_1t-x),\, t_2(c_2t-x),\, Z_1t_1(c_1t-x),\, Z_2t_2(c_2t-x))$$

for $x > \ell$, $-\infty < t < \infty$, where $t_1(\tau) \equiv t_2(\tau) \equiv 0$ for $\tau \leq -\ell$. Evidently, the functions $r_1(c_1t + x)$ and $r_2(c_2t + x)$ may be interpreted as reflected waves produced by the coupler on lines 1 and 2, respectively, while $t_1(c_1t - x)$ and $t_2(c_2t - x)$ may be interpreted as waves transmitted through the coupler.

The solution to the coupler problem may be derived from the solutions for two special cases by a superposition principle (Duhamel's Principle; cf. [4, p. 202]). To describe it let $H(\tau)$ denote Heaviside's function, i.e.

$$H(\tau) = \begin{cases} 1, & \tau \geq 0, \\ 0, & \tau < 0, \end{cases}$$

and let $(i_1^1, i_2^1, e_1^1, e_2^1)$ be the solution of the coupler problem corresponding to an incident wave (2.4) with

$$h_1(\tau) \equiv H(\tau), \quad h_2(\tau) \equiv 0 \tag{2.6}$$

and $(i_1^2, i_2^2, e_1^2, e_2^2)$ the solution corresponding to an incident wave (2.4) with

$$h_1(\tau) \equiv 0, \quad h_2(\tau) \equiv H(\tau) . \tag{2.7}$$

Then it is easy to verify that

$$(i_1(x,t), i_2(x,t), e_1(x,t), e_2(x,t))$$

$$(2.8) \quad = \frac{\partial}{\partial t}\int_0^\infty (i_1^1(x,t-\tau),\ i_2^1(x,t-\tau),\ e_1^1(x,t-\tau),\ e_2^1(x,t-\tau))\,h_1(c_1\tau)\,d\tau$$

$$+ \frac{\partial}{\partial t}\int_0^\infty (i_1^2(x,t-\tau),\ i_2^2(x,t-\tau),\ e_1^2(x,t-\tau),\ e_2^2(x,t-\tau))\,h_2(c_2\tau)\,d\tau$$

defines the solution of the coupler problem for an arbitrary incident wave (2.4). Hence, the remainder of this report treats the cases (2.6) and (2.7) only.

§3. Formulation of the Laplace Transformed Coupler Problem. In what follows, the Laplace transforms of the solutions $(i_1^1, i_2^1, e_1^1, e_2^1)$ and $(i_1^2, i_2^2, e_1^2, e_2^2)$ are denoted by $(I_1^1, I_2^1, E_1^1, E_2^1)$ and $(I_1^2, I_2^2, E_1^2, E_2^2)$; i.e.,

$$I_1^1(x, s) = \int_0^\infty e^{-st} i_1^1(x, t)\, dt \text{ , etc.}$$

The solutions to the coupler problem have the form

$$(i_1^1, i_2^1, e_1^1, e_2^1)$$

$$= (H(c_1t-x)+r_{11}(c_1t+x), r_{21}(c_2t+x), Z_1\{H(c_1t-x)-r_{11}(c_1t+x)\}, -Z_2r_{21}(c_2t+x)),$$

$$(i_1^2, i_2^2, e_1^2, e_2^2)$$

$$= (r_{12}(c_1t+x), H(c_2t-x)+r_{22}(c_2t+x), -Z_1r_{12}(c_1t+x), Z_2\{H(c_2t-x)-r_{22}(c_2t+x)\})$$

for $x < 0$, where $r_{11}(\tau) \equiv r_{21}(\tau) \equiv r_{12}(\tau) \equiv r_{22}(\tau) \equiv 0$ for $\tau \leq 0$. Hence, application of the formula

$$(3.1)\qquad \int_0^\infty e^{-st} f(ct \pm x)\,dt = 1/c\; e^{\pm\frac{sx}{c}} \int_{\pm x}^\infty e^{-\frac{s}{c}\tau} f(\tau)\,d\tau$$

gives

$$(I_1^1, I_2^1, E_1^1, E_2^1) = \frac{1}{s}\left(1 + e^{\frac{sx}{c_1}} R_{11}(s),\; e^{\frac{sx}{c_2}} R_{21}(s),\; Z_1\{1 - e^{\frac{sx}{c_1}} R_{11}(s)\},\; -Z_2 e^{\frac{sx}{c_2}} R_{21}(s)\right),$$

(3.2)

$$(I_1^2, I_2^2, E_1^2, E_2^2) = \frac{1}{s}\left(e^{\frac{sx}{c_1}} R_{12}(s),\; 1 + e^{\frac{sx}{c_2}} R_{22}(s),\; -Z_1 e^{\frac{sx}{c_1}} R_{12}(s),\; Z_2\{1 - e^{\frac{sx}{c_2}} R_{22}(s)\}\right)$$

for all $x < 0$, where

$$R_{jk}(s) = \frac{s}{c_j} \int_0^\infty e^{-\frac{s\tau}{c_j}} r_{jk}(\tau)\,d\tau, \quad 1 \le j,k \le 2 .$$

Similarly, on $x > \ell$

$$(i_1^1, i_2^1, e_1^1, e_2^1) = (t_{11}(c_1t-x),\ t_{21}(c_2t-x),\ Z_1t_{11}(c_1t-x),\ Z_2t_{21}(c_2t-x)) \quad ,$$

$$(i_1^2, i_2^2, e_1^2, e_2^2) = (t_{12}(c_1t-x),\ t_{22}(c_2t-x),\ Z_1t_{12}(c_1t-x),\ Z_2t_{22}(c_2t-x))$$

where $t_{11}(\tau) \equiv t_{21}(\tau) \equiv t_{12}(\tau) \equiv t_{22}(\tau) \equiv 0$ for $\tau \leq -\ell$,

whence by (3.1)

$$(I_1^1, I_2^1, E_1^1, E_2^1) = \frac{1}{s}\left(e^{-\frac{s(x-\ell)}{c_1}} T_{11}(s),\ e^{-\frac{s(x-\ell)}{c_2}} T_{21}(s),\ Z_1 e^{-\frac{s(x-\ell)}{c_1}} T_{11}(s),\ Z_2 e^{-\frac{s(x-\ell)}{c_2}} T_{21}(s)\right)$$

(3.3)

$$(I_1^2, I_2^2, E_1^2, E_2^2) = \frac{1}{s}\left(e^{-\frac{s(x-\ell)}{c_1}} T_{12}(s),\ e^{-\frac{s(x-\ell)}{c_2}} T_{22}(s),\ Z_1 e^{-\frac{s(x-\ell)}{c_1}} T_{12}(s),\ Z_2 e^{-\frac{s(x-\ell)}{c_2}} T_{22}(s)\right)$$

for all $x > \ell$, where

$$T_{jk}(s) = \frac{s\, e^{-\frac{s\ell}{c_j}}}{c_j} \int_{-\ell}^{\infty} e^{-\frac{s\tau}{c_j}} t_{jk}(\tau)\, d\tau \ , \quad 1 \leq j,k \leq 2 \ .$$

On $0 \leq x \leq \ell$ the behavior of $(I_1^j, I_2^j, E_1^j, E_2^j)$, $j = 1, 2$, is governed by the Laplace transform of the system (1.1). Making use of the well-known rule

$$\int_0^\infty e^{-st}\,\frac{df(t)}{dt}\,dt = -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

and the initial values $(i_1^1, i_2^1, e_1^1, e_2^1) = (i_1^2, i_2^2, e_1^2, e_2^2) = (0, 0, 0, 0)$ for $t = 0$, $0 \leq x \leq \ell$, gives

$$(3.4)\quad \begin{cases} \dfrac{dE_1^j}{dx} + s(L_1 I_1^j + L_m I_2^j) = 0 \\[2ex] \dfrac{dE_2^j}{dx} + s(L_m I_1^j + L_2 I_2^j) = 0 \\[2ex] \dfrac{dI_1^j}{dx} + s(C_1 E_1^j + C_m E_2^j) = 0 \\[2ex] \dfrac{dI_1^j}{dx} + s(C_m E_1^j + C_2 E_2^j) = 0 \end{cases}$$

for $0 \leq x \leq \ell$ and $j = 1, 2$.

Any locally integrable solution of (3.4) will satisfy

$$E_1^j(x) = E_1^j(x_0) - s\int_{x_0}^{x} \{L_1(\xi)I_1^j(\xi) + L_m(\xi)I_2^j(\xi)\}d\xi$$

where the integral is finite, by (1.3), and similar equations hold for E_2^j, I_1^j and I_2^j. Hence, any such solution is (absolutely) continuous. In particular, (3.2) and (3.3) imply

$$(I_1^1(0), I_2^1(0), E_1^1(0), E_2^1(0))$$

$$= \frac{1}{s}(1 + R_{11}(s),\ R_{21}(s),\ Z_1\{1 - R_{11}(s)\},\ -Z_2R_{21}(s)) \quad ,$$

(3.5)

$$(I_1^2(0), I_2^2(0), E_1^2(0), E_2^2(0))$$

$$= \frac{1}{s}(R_{12}(s),\ 1 + R_{22}(s),\ -Z_1R_{12}(s),\ Z_2\{1 - R_{22}(s)\}) \quad ,$$

and

$$(I_1^1(\ell), I_2^1(\ell), E_1^1(\ell), E_2^1(\ell))$$

$$= \frac{1}{s}(T_{11}(s),\ T_{21}(s),\ Z_1T_{11}(s),\ Z_2T_2(s)) \quad ,$$

(3.6)

$$(I_1^2(\ell), I_2^2(\ell), E_1^2(\ell), E_2^2(\ell))$$

$$= \frac{1}{s}(T_{12}(s),\ T_{22}(s),\ Z_1T_{12}(s),\ Z_2T_{22}(s)) \quad .$$

It is shown below that the differential equations (3.4) and the boundary conditions (3.5) and (3.6) determine uniquely the Laplace transforms of the currents and voltages within the coupler, together with the reflection coefficients $R_{jk}(s)$ and transmission coefficients $T_{jk}(s)$. Moreover, it is shown that the $R_{jk}(s)$ can be determined directly, without determining $T_{jk}(s)$, $E_l^j(x,s)$, etc. Notice that once $R_{jk}(s)$ is known the currents and voltages in the coupler can be obtained from the solution of the initial value problem (3.4), (3.5), and the values at $x = \ell$ of this solution determine $T_{jk}(s)$, by (3.6).

§4. Solution of the Laplace Transformed Coupler Problem. The matrix notation

$$I = \begin{pmatrix} I_1^1 & I_1^2 \\ I_2^1 & I_2^2 \end{pmatrix}, \quad E = \begin{pmatrix} E_1^1 & E_1^2 \\ E_2^1 & E_2^2 \end{pmatrix}$$

is used in what follows. Equations (3.2) may be then written in the matrix form

$$sI = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{\frac{sx}{c_1}} & 0 \\ 0 & e^{\frac{sx}{c_2}} \end{pmatrix}\right]\begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix},$$

$$sE = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e^{\frac{sx}{c_1}} & 0 \\ 0 & e^{\frac{sx}{c_2}} \end{pmatrix}\right]\begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix}.$$

or simply

$$
(4.1)\left\{
\begin{aligned}
sI &= U + \begin{pmatrix} e^{\frac{sx}{c_1}} & 0 \\ 0 & e^{\frac{sx}{c_2}} \end{pmatrix} R(s) \\
sE &= Z\left[U - \begin{pmatrix} e^{\frac{sx}{c_1}} & 0 \\ 0 & e^{\frac{sx}{c_2}} \end{pmatrix} R(s)\right]
\end{aligned}
\right\} \quad \text{for} \quad x \leq 0 \quad ,
$$

where

$$
U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R(s) = \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} .
$$

$R(s)$ is called the reflection coefficient matrix. Similarly, equations (3.3) may be written

$$
(4.2)\left\{
\begin{aligned}
sI(x,s) &= \begin{pmatrix} e^{-\frac{s(x-\ell)}{c_1}} & 0 \\ 0 & e^{-\frac{s(x-\ell)}{c_2}} \end{pmatrix} T(s) \\
sE(x,s) &= Z \begin{pmatrix} e^{-\frac{s(x-\ell)}{c_1}} & 0 \\ 0 & e^{-\frac{s(x-\ell)}{c_2}} \end{pmatrix} T(s)
\end{aligned}
\right\} \quad \text{for} \quad x \geq \ell \quad ,
$$

where

$$T(s) = \begin{pmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{pmatrix}$$

may be called the transmission coefficient matrix.

The differential equations (3.4) also take a simple form in matrix notation, namely

$$(4.3) \qquad \begin{cases} \dfrac{dE}{dx} + sLI = 0 \\ \dfrac{dI}{dx} + sCE = 0 \end{cases} \qquad \text{for } 0 \leq x \leq \ell \quad ,$$

where

$$L = \begin{pmatrix} L_1 & L_m \\ L_m & L_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & C_m \\ C_m & C_2 \end{pmatrix} \quad .$$

The continuity conditions (3.5) and (3.6) take the form

$$(4.4)\qquad \begin{cases} sI(0) = U + R(s) , \\ sE(0) = Z[U - R(s)] , \end{cases}$$

and

$$(4.5)\qquad \begin{cases} sI(\ell) = T(s) , \\ sE(\ell) = ZT(s) . \end{cases}$$

Equations (4.3), (4.4) and (4.5) do not constitute a proper boundary value problem for $I(x)$ and $E(x)$, as they stand, because $R(s)$ and $T(s)$ are not known. However, (4.4) and (4.5) imply

$$(4.6)\qquad \begin{cases} ZsI(0) + sE(0) = 2Z , \\ ZsI(\ell) - sE(\ell) = 0 . \end{cases}$$

It will be shown that (4.3) and (4.6) define unique functions $I(x)$, $E(x)$.

If M is an $m \times n$ matrix, let M^* denote the complex conjugate of the transposed matrix. Note that $L^* = L$ and $C^* = C$, because L and C are real and symmetric. It follows that if E and I satisfy (4.3) then

$$(4.7)\qquad \begin{cases} \dfrac{dE^*}{dx} + \bar{s} I^* L = 0 \quad , \\[2ex] \dfrac{dI^*}{dx} + \bar{s} E^* C = 0 \quad . \end{cases}$$

This implies the following

<u>Lemma 4.1</u> If E and I are 2×1 matrix solutions of (4.3) on $a \le x \le b$ then

$$(4.8)\qquad E^* I + I^* E \Big|_a^b = -2(\operatorname{Re} s) \int_a^b (I^* L I + E^* C E)\, dx \quad .$$

<u>Proof.</u> By (4.3) and (4.7),

$$\frac{d}{dx}(E^* I + I^* E) = \frac{dE^*}{dx} I + E^* \frac{dI}{dx} + \frac{dI^*}{dx} E + I^* \frac{dE}{dx}$$

$$= - \bar{s} I^* L I - s E^* C E - \bar{s} E^* C E - s I^* L I$$

$$= - 2(\operatorname{Re} s)(I^* L I + E^* C E) \quad .$$

If $z = x + iy$ is a 2×1 matrix with real and imaginary parts x and y, respectively, then a simple computation gives

$$z^*Lz = \sum_{j,k=1}^{2} (L_{jk}x_jx_k + L_{jk}y_jy_k) \geq 0 \quad ,$$

and

$$z^*Cz = \sum_{j,k=1}^{2} (C_{jk}x_jx_k + C_{jk}y_jy_k) \geq 0 \quad .$$

Combining this with the preceding lemma gives

Corollary 4.1 If E and I are 2×1 matrix solutions of (4.3) and $a \leq x \leq b$ and $\operatorname{Re} s \geq 0$ then

$$(4.9) \qquad (E^*I + I^*E)_{x=b} \leq (E^*I + I^*E)_{x=a} \quad .$$

This result implies

Theorem 4.1 (Uniqueness Theorem). The 2-point boundary value problem (4.3), (4.6) has at most one solution, provided $\operatorname{Re} s \geq 0$.

Proof. Suppose that (E', I') and (E'', I'') are two solutions. Then $E = E' - E''$ and $I = I' - I''$ will solve (4.3) and

$$(4.10) \qquad E(0) = -ZI(0) , \qquad E(\ell) = ZI(\ell) .$$

Let

$$E^j = \begin{pmatrix} E_1^j \\ E_2^j \end{pmatrix} , \quad I^j = \begin{pmatrix} I_1^j \\ I_2^j \end{pmatrix} , \quad j = 1, 2 .$$

Then E^j, I^j are 2×1 matrix solutions of (4.3) and satisfy

$$(4.11) \qquad E^j(0) = -ZI^j(0) , \quad E^j(\ell) = ZI^j(\ell) , \quad j = 1, 2 ,$$

by (4.10). Applying (4.9) to E^j and I^j on $0 \leq x \leq \ell$ and substituting (4.11) gives

$$2(I^{j*}(\ell)ZI^j(\ell) + I^{j*}(0)ZI^j(0)) \leq 0 , \quad j = 1, 2 .$$

But $Z = ((\delta_{jk}Z_j))$ with $Z_1 > 0$ and $Z_2 > 0$, from which it follows that $I^j(0) = 0$ and therefore also $E^j(0) = 0$, by (4.11). This, with (4.3), implies that $E(x) = I(x) = 0$ on $0 \leq x \leq \ell$, by the uniqueness theorem for the initial value problem for (4.3). Thus $E' = E''$, $I' = I''$ on $0 \leq x \leq \ell$, which proves the uniqueness theorem.

The form of the boundary conditions (4.6) suggests a change to the new dependent variables

$$\hat{E} = \frac{s}{2}(I + Z^{-1}E), \qquad \hat{I} = \frac{s}{2}(I - Z^{-1}E)$$

which satisfy

$$E = \frac{1}{s} Z(\hat{E} - \hat{I}), \qquad I = \frac{1}{s}(\hat{E} + \hat{I}) .$$

A simple calculation gives

<u>Lemma 4.2</u> E and I solve the 2-point boundary value problem (4.3), (4.6) if and only if $\hat{E}$ and $\hat{I}$ solve the 2-point boundary value problem

$$(4.12) \qquad \begin{cases} \dfrac{d\hat{E}}{dx} + s(A\hat{E} - B\hat{I}) = 0 \\ \\ \dfrac{d\hat{I}}{dx} + s(B\hat{E} - A\hat{I}) = 0 \end{cases} \quad \text{or} \quad 0 \leq x \leq \ell ,$$

where

$$A = \frac{1}{2}(CZ + Z^{-1}L) , \qquad B = \frac{1}{2}(CZ - Z^{-1}L) ,$$

and

$$(4.13) \qquad \hat{E}(0) = U , \qquad \hat{I}(\ell) = 0 .$$

Moreover,

$$\hat{I}(0) = R(s), \quad \hat{E}(\ell) = T(s) .$$

The 2-point boundary value problem (4.12), (4.13) can be solved by solving two 1-point (initial value) problems. This reduction is based on

<u>Lemma 4.3</u> If $\hat{E}$ and $\hat{I}$ solve (4.12) and E^{-1} exists on an interval $a \leq x \leq b$ then

$$R = R(x, s) = \hat{I}(x)\hat{E}(x)^{-1} \tag{4.14}$$

solves the matrix Riccati equation

$$\frac{dR}{dx} + s(RBR - AR - RA + B) = 0 \tag{4.15}$$

on $a \leq x \leq b$.

<u>Proof.</u> Differentiating the matrix equation $AA^{-1} = U$ gives the rule $dA^{-1}/dx = -A^{-1}\, dA/dx\, A^{-1}$. Differentiating the equation $R = \hat{I}\hat{E}^{-1}$ and using this rule and (4.12) gives (4.15).

A converse to this result is provided by

<u>Lemma 4.4</u> If R solves (4.15) and $\hat{E}$ solves

$$\frac{d\hat{E}}{dx} + s(A - BR)\hat{E} = 0 \tag{4.16}$$

then $\hat{E}$ and $I = R\hat{E}$ solves (4.12).

<u>Proof.</u> Putting $R\hat{E} = \hat{I}$ in (4.15) gives the first of equations (4.12). Computing $d\hat{I}/dx = dR/dx\, \hat{E} + Rd\hat{E}/dx$, from (4.15) and (4.16) gives the second.

The function $R(x, s)$ defined by (4.14) satisfies the boundary condition $R(\ell, s) = 0$, by (4.13). Thus, the solution to the 2-point boundary value problem (4.12), (4.13) can be constructed by, first, integrating (4.15) from $x = \ell$ to $x = 0$ starting with $R(\ell, s) = 0$ and, second, integrating (4.16) from $x = 0$ to $x = \ell$, starting with $E(0) = U$. This construction will fail only if the function $R(x, s)$ defined by (4.15) and $R(\ell, s) = 0$ fails to exist on the whole interval $0 \leq x \leq \ell$. It is well known that solutions of non-linear equations may have this behavior. However, in the present case it is ruled out by the following

Theorem 4.2 If $\operatorname{Re} s \geq 0$ then the solution $R(x)$ of (4.15) satisfying $R(\ell) = 0$ satisfies the inequalities

$$(4.17) \qquad \begin{cases} Z_1 |R_{11}(x)|^2 + Z_2 |R_{21}(x)|^2 \leq Z_1 , \\ \\ Z_1 |R_{12}(x)|^2 + Z_2 |R_{22}(x)|^2 \leq Z_2 , \end{cases}$$

on any interval $\xi \leq x \leq \ell$ on which it exists.

Proof. Define $\hat{E}(x)$ on $\xi \leq x \leq \ell$ by (4.16) and $\hat{E}(\xi) = U$, and put $\hat{I}(x) = R(x)\hat{E}(x)$. Then $\hat{E}$, $\hat{I}$ solve (4.12), by Lemma 4.4, and hence $E = Z(\hat{E} - \hat{I})$ and $I = \hat{E} + \hat{I}$ solve (4.3) on $\xi \leq x \leq \ell$ and have the values

$$I(\xi) = U + R(\xi), \qquad E(\xi) = Z(U - R(\xi)) \tag{4.18}$$

and

$$I(\ell) = \hat{E}(\ell), \quad E(\ell) = Z\hat{E}(\ell) \quad \text{whence} \quad E(\ell) = ZI(\ell) \ .$$

In component form, equations (4.18) read

$$I(\xi) = \begin{pmatrix} 1 + R_{11}(\xi) & R_{12}(\xi) \\ R_{21}(\xi) & 1 + R_{22}(\xi) \end{pmatrix}$$

and

$$E(\xi) = \begin{pmatrix} Z_1(1 - R_{11}(\xi)) & -Z_1R_{12}(\xi) \\ -Z_2R_{21}(\xi) & Z_2(1 - R_{22}(\xi)) \end{pmatrix} \ .$$

Now, the two column vectors E^j and I^j $(j = 1, 2)$ from E and I satisfy (4.3) on $\xi \leq x \leq \ell$. Hence, by corollary 4.1 ,

$$(E^{j*}I^j + I^{j*}E^j)_{x=\ell} \leq (E^{j*}I^j + I^{j*}E^j)_{x=\xi}, \quad j = 1, 2 \ . \tag{4.19}$$

Taking $j = 1$ and substituting the above boundary values gives

$$2I^{1*}(\ell)ZI^{1}(\ell) \leq Z_1(1-\bar{R}_{11}(\xi))(1+R_{11}(\xi)) - Z_2(R_{21}(\xi))^2 + (1+\bar{R}_{11}(\xi))Z_1(1-R_{11}(\xi)) - Z_2|R_{21}(\xi)|^2$$

$$= 2[Z_1 - Z_1|R_{11}(\xi)|^2 - Z_2|R_{21}(\xi)|^2] .$$

This gives the first of inequalities (4.17), since Z is positive definite. The second is derived similarly from (4.19) with $j = 2$.

The reduction of the 2-point boundary value problem (4.12), (4.13) to two initial value problems is guaranteed by

Theorem 4.3 The initial value problem

$$(4.20) \qquad \begin{cases} \dfrac{dR}{dx} + s(RBR - AR - RA + B) = 0 , & 0 \leq x \leq \ell , \\ \\ R(\ell) = 0 & \end{cases}$$

has a unique solution $R = R(x, s)$ for every s with $\operatorname{Re} s \geq 0$. The initial value problem

$$(4.21) \qquad \begin{cases} \dfrac{d\hat{E}}{dx} + s(A - BR)\hat{E} = 0, & 0 \leq x \leq \ell , \\ \\ \hat{E}(0) = U & \end{cases}$$

has a unique solution, and $\hat{E}$ and $\hat{I} = R\hat{E}$ provide the (unique) solution to the 2-point boundary problem (4.12), (4.13).

Proof. The uniqueness of $R(x, s)$ follows from a standard theorem on the initial value problem for differential equations [e.g., 3, Ch. 2]. The existence of $R(x, s)$ on $0 \leq x \leq \ell$ follows from Theorem 4.2 and the standard existence and continuation theorems [3, Ch. 2].

Corollary 4.2 The reflection coefficient matrix $R(s) = R(0, s)$ can be obtained directly from the solution of the initial value problem (4.20). $\hat{E}(x)$ and $\hat{I}(x)$ can then be obtained as the solution of the initial value problem for (4.12) with $\hat{E}(0) = U$, $\hat{I}(0) = R(s)$.

Corollary 4.3 $ZR(s) = R^*(\bar{s})Z$.

Proof. $ZR(s) = ZR(0, s)$ and $X = ZR(x, s)$ satisfies the equation

$$\frac{dX}{dx} + s(XHX - KX - XK^* + ZHZ) = 0$$

where

$$H = BZ^{-1} = \frac{1}{2}(C - Z^{-1}LZ^{-1}),$$

and

$$K = ZAZ^{-1} = \frac{1}{2}(ZC + LZ^{-1}) .$$

Evidently, $H^* = H$ and therefore $X^*(x, \bar{s})$ satisfies the same equation as $X(x, s)$. Since $X^*(\ell, \bar{s}) = X(\ell, s) = 0$, it follows that $X^*(x, \bar{s}) = X(x, s)$

or $0 \le x \le \ell$, by the uniqueness theorem for the initial value problem.

<u>Corollary 4.4</u> $Z_1 R_{12}(s) = Z_2 R_{21}(s)$.

<u>Proof.</u> Corollary 4.3 implies that $Z_1 R_{12}(s) = Z_2 \overline{R}_{21}(\overline{s})$. Moreover, it is easy to verify that $R_{12}(s)$ is analytic for $\operatorname{Re} s > 0$ and is real for real s, which implies that $\overline{R}_{21}(\overline{s}) = R_{21}(s)$.

The reflection coefficient matrix can be obtained by solving (4.20) numerically. For this purpose, the matrix Riccati equation should be written as a system of four ordinary differential equations. On carrying out the matrix operations this system is found to have the form

$$
\begin{cases}
\dfrac{dR_{11}}{dx} + s(B_{11}R_{11}^2 + B_{12}R_{11}R_{21} + B_{21}R_{12}R_{11} + B_{22}R_{12}R_{21} - A_{11}R_{11} - A_{12}R_{21} - A_{11}R_{11} - A_{21}R_{12} + B_{11}) = 0 , \\[2ex]
\dfrac{dR_{12}}{dx} + s(B_{11}R_{11}R_{12} + B_{12}R_{11}R_{22} + B_{21}R_{12}^2 + B_{22}R_{12}R_{22} - A_{11}R_{12} - A_{12}R_{22} - A_{12}R_{11} - A_{22}R_{12} + B_{12}) = 0 , \\[2ex]
\dfrac{dR_{21}}{dx} + s(B_{11}R_{21}R_{11} + B_{12}R_{21}^2 + B_{21}R_{22}R_{11} + B_{22}R_{22}R_{21} - A_{21}R_{11} - A_{22}R_{21} - A_{11}R_{21} - A_{21}R_{22} + B_{21}) = 0 , \\[2ex]
\dfrac{dR_{22}}{dx} + s(B_{11}R_{21}R_{12} + B_{12}R_{21}R_{22} + B_{21}R_{22}R_{12} + B_{22}R_{22}^2 - A_{21}R_{12} - A_{22}R_{22} - A_{12}R_{21} - A_{22}R_{22} + B_{22}) = 0 .
\end{cases}
\tag{4.22}
$$

§5. Directional Couplers. The "directional" couplers are of particular interest for applications [10]. They are characterized by the property that the reflected wave is zero on line 1 (line 2) and the transmitted wave is zero on line 2 (line 1) whenever the incident wave is zero on line 1 (line 2). An equivalent property is $r_{11}(\tau) \equiv r_{22}(\tau) \equiv 0$ and $t_{12}(\tau) \equiv t_{21}(\tau) \equiv 0$ for all τ. The corresponding property of the reflection and transmission coefficient matrices is $R_{11}(s) \equiv R_{22}(s) \equiv 0$, $T_{12}(s) \equiv T_{21}(s) \equiv 0$ for all s.

It is known that sufficient conditions for a coupler to be directional are [10]

$$L_m/\sqrt{L_1L_2} = -C_m/\sqrt{C_1C_2} \qquad \text{on } 0 \leq x \leq \ell \,, \tag{5.1}$$

and

$$\sqrt{L_1/C_1} = Z_1 \,, \qquad \sqrt{L_2/C_2} = Z_2 \qquad \text{on } 0 \leq x \leq \ell \,, \tag{5.2}$$

where Z_1 and Z_2 are the constant impedances on the uncoupled portions of the lines. The sufficiency of these conditions is verified below.

The matrices A and B can be written

$$A = \frac{1}{2} \begin{pmatrix} Z_1C_1 + \dfrac{L_1}{Z_1} & \dfrac{L_m}{Z_1} + Z_2C_m \\ \\ \dfrac{L_m}{Z_2} + Z_1C_m & Z_2C_2 + \dfrac{L_2}{Z_2} \end{pmatrix}$$

and

$$B = \frac{1}{2} \begin{pmatrix} Z_1C_1 - \dfrac{L_1}{Z_1} & -\dfrac{L_m}{Z_1} + Z_2C_m \\ -\dfrac{L_m}{Z_2} + Z_1C_m & Z_2C_2 - \dfrac{L_2}{Z_2} \end{pmatrix}$$

Hence, conditions (5.1) and (5.2) imply

$$A = \begin{pmatrix} \sqrt{L_1C_1} & 0 \\ 0 & \sqrt{L_2C_2} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -\dfrac{L_m}{Z_1} \\ -\dfrac{L_m}{Z_2} & 0 \end{pmatrix}$$

In this case the matrix Riccati system (4.22) becomes

$$
(5.3)\quad \begin{cases}
\dfrac{dR_{11}}{dx} + s\left(-\dfrac{L_m}{Z_1} R_{11}R_{21} - \dfrac{L_m}{Z_2} R_{12}R_{11} - 2\sqrt{L_1C_1}R_{11}\right) = 0 , \\[2em]
\dfrac{dR_{12}}{dx} + s\left(-\dfrac{L_m}{Z_1} R_{11}R_{22} - \dfrac{L_m}{Z_2} R_{12}^2 - (\sqrt{L_1C_1} + \sqrt{L_2C_2})R_{12} - \dfrac{L_m}{Z_1}\right) = 0 , \\[2em]
\dfrac{dR_{21}}{dx} + s\left(-\dfrac{L_m}{Z_1} R_{21}^2 - \dfrac{L_m}{Z_2} R_{22}R_{11} - (\sqrt{L_1C_1} + \sqrt{L_2C_2})R_{21} - \dfrac{L_m}{Z_2}\right) = 0 , \\[2em]
\dfrac{dR_{22}}{dx} + s\left(-\dfrac{L_m}{Z_1} R_{21}R_{22} - \dfrac{L_m}{Z_2} R_{22}R_{12} - 2\sqrt{L_2C_2}R_{22}\right) = 0 .
\end{cases}
$$

These equations imply

<u>Theorem 5.1</u> If (5.1) and (5.2) hold then $R_{11}(s) = R_{22}(s) = 0$, $R_{21}(s) = (Z_1/Z_2)R_{12}(s)$, and $R_{12}(s) = R_{12}(0,s)$ is the value at $x = 0$ of the solution $R_{12} = R_{12}(x,s)$ of

$$
(5.4)\quad \begin{cases}
\dfrac{dR_{12}}{dx} - s\left(\dfrac{L_m}{Z_2} R_{12}^2 + (\sqrt{L_1C_1} + \sqrt{L_2C_2})R_{12} + \dfrac{L_m}{Z_1}\right) = 0 , \qquad 0 \le x \le \ell , \\[2em]
R_{12}(\ell, s) = 0 .
\end{cases}
$$

<u>Proof.</u> If $R_{12}(x, s)$ is defined by (5.4), $R_{21}(x, s)$ is defined to be $(Z_1/Z_2)R_{12}(x, s)$, and $R_{11}(x, s)$ and $R_{22}(x, s)$ are defined to be zero then the resulting matrix $R(x, s)$ is easily verified to satisfy (5.3) and $R(\ell, s) = 0$. But these conditions define $R(x, s)$ uniquely, whence $R(x, s)$ must have the form indicated.

<u>Corollary 5.1</u> If (5.1) and (5.2) hold then the coupler is directional and the reflection coefficient $R_{12}(s)$ is characterized by (5.4).

§6. Formulation and Solution of the Steady State Coupler Problem.

The steady state solutions of the transmission line equations (1.1) have the form

$$i_j(x,t) = \mathrm{Re}\{e^{i\omega t}\tilde{I}_j(x)\}\ , \qquad e_j(x,t) = \mathrm{Re}\{e^{i\omega t}\tilde{E}_j(x)\}\ , \quad j = 1, 2\ ,$$

where $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{E}_1$, $\tilde{E}_2$ satisfy the equations

$$\frac{d\tilde{E}_1}{dx} + i\omega(L_1\tilde{I}_1 + L_m\tilde{I}_2) = 0 \quad ,$$

$$\frac{d\tilde{E}_2}{dx} + i\omega(L_m\tilde{I}_1 + L_2\tilde{I}_2) = 0 \quad ,$$

$$\frac{d\tilde{I}_1}{dx} + i\omega(C_1\tilde{E}_1 + C_m\tilde{E}_2) = 0 \quad ,$$

$$\frac{d\tilde{I}_2}{dx} + i\omega(C_m\tilde{E}_1 + C_2\tilde{E}_2) = 0 \quad .$$

In terms of the column vectors

$$\tilde{E} = \begin{pmatrix} \tilde{E}_1 \\ \\ \tilde{E}_2 \end{pmatrix}, \qquad \tilde{I} = \begin{pmatrix} \tilde{I}_1 \\ \\ \tilde{I}_2 \end{pmatrix},$$

the equations take the more concise form

$$(6.1) \qquad \begin{cases} \dfrac{d\tilde{E}}{dx} + i\omega L\tilde{I} = 0 \quad , \\ \\ \dfrac{d\tilde{I}}{dx} + i\omega C\tilde{E} = 0 \quad . \end{cases}$$

In discussing steady state problems it is convenient to deal directly with the complex-valued vectors $\tilde{E}(x)$, $\tilde{I}(x)$ rather than the real-valued vectors $e(x,t) = \mathrm{Re}\{e^{i\omega t}\tilde{E}(x)\}$, $i(x,t) = \mathrm{Re}\{e^{i\omega t}\tilde{I}(x)\}$.

On uniform uncoupled portions of the transmission lines, the steady state solutions have the form

$$(6.2) \qquad \begin{cases} \tilde{I}_1 = a_1 e^{-ik_1 x} + b_1 e^{ik_1 x}, \quad \tilde{E}_1 = Z_1(a_1 e^{-ik_1 x} - b_1 e^{ik_1 x}) \quad , \\ \\ \tilde{I}_2 = a_2 e^{-ik_2 x} + b_2 e^{ik_2 x}, \quad \tilde{E}_2 = Z_2(a_2 e^{-ik_2 x} - b_2 e^{ik_2 x}) \quad , \end{cases}$$

where a_1, a_2, b_1 and b_2 are constants, and $k_1 = \omega/c_1$, $k_2 = \omega/c_2$. In the case of a coupler occupying the segment $0 \leq x \leq \ell$, the constants a_1, a_2, b_1, b_2 may have different values on the two uncoupled portions $x \leq 0$ and $x \geq \ell$.

The steady state coupler problem is to find a solution of (6.1) corresponding to a prescribed steady state wave incident on the coupler from the left. This means that for $x \leq 0$, a_1 and a_2 have prescribed value (and b_1 and b_2 are to be determined), while for $x \geq \ell$, $b_1 = b_2 = 0$ (and a_1 and a_2 are to be determined). The solution for arbitrary prescribed values (a_1, a_2) on $x \leq 0$ can be derived from the special cases $(a_1, a_2) = (1, 0)$ and $(a_1, a_2) = (0, 1)$ by superposition. Hence, only these cases are considered here. If $\tilde{E}^1, \tilde{I}^1$ and $\tilde{E}^2, \tilde{I}^2$ denote the solutions for these two cases then for $x \leq 0$,

$$\tilde{I}^1_1 = e^{-ik_1x} + P_{11}(\omega)e^{ik_1x} \quad , \qquad \tilde{E}^1_1 = Z_1(e^{-ik_1x} - P_{11}(\omega)e^{ik_1x}) \quad ,$$

$$\tilde{I}^1_2 = \qquad P_{21}(\omega)e^{ik_2x} \quad , \qquad \tilde{E}^1_2 = \qquad -Z_2P_{21}(\omega)e^{ik_2x} \quad ,$$

$$\tilde{I}^2_1 = \qquad P_{12}(\omega)e^{ik_1x} \quad , \qquad \tilde{E}^2_1 = \qquad -Z_1P_{12}(\omega)e^{ik_1x} \quad ,$$

$$\tilde{I}^2_2 = e^{-ik_2x} + P_{22}(\omega)e^{ik_2x} \quad , \qquad \tilde{E}^2_2 = Z_2(e^{-ik_2x} - P_{22}(\omega)e^{ik_2x}) \quad .$$

while for $x \geq \ell$,

$$\tilde{I}^1_1 = Q_{11}(\omega)e^{-ik_1(x-\ell)} , \qquad \tilde{E}^1_1 = Z_1Q_{11}(\omega)e^{-ik_1(x-\ell)} ,$$

$$\tilde{I}^1_2 = Q_{21}(\omega)e^{-ik_2(x-\ell)} , \qquad \tilde{E}^1_2 = Z_2Q_{21}(\omega)e^{-ik_2(x-\ell)} ,$$

$$\tilde{I}^2_1 = Q_{12}(\omega)e^{-ik_1(x-\ell)} , \qquad \tilde{E}^2_1 = Z_1Q_{12}(\omega)e^{-ik_1(x-\ell)} ,$$

$$\tilde{I}^2_2 = Q_{22}(\omega)e^{-ik_2(x-\ell)} , \qquad \tilde{E}^2_2 = Z_2Q_{22}(\omega)e^{-ik_2(x-\ell)} .$$

In matrix notation these take the form

$$\tilde{I} = \begin{pmatrix} \tilde{I}^1_1 & \tilde{I}^2_1 \\ \tilde{I}^1_2 & \tilde{I}^2_2 \end{pmatrix} = \begin{pmatrix} e^{-ik_1x} & 0 \\ 0 & e^{-ik_2x} \end{pmatrix} + \begin{pmatrix} e^{ik_1x} & 0 \\ 0 & e^{ik_2x} \end{pmatrix} \begin{pmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{pmatrix} ,$$

$$\tilde{E} = \begin{pmatrix} \tilde{E}^1_1 & \tilde{E}^2_1 \\ \tilde{E}^1_2 & \tilde{E}^2_2 \end{pmatrix} = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \left[\begin{pmatrix} e^{-ik_1x} & 0 \\ 0 & e^{-ik_2x} \end{pmatrix} - \begin{pmatrix} e^{ik_1x} & 0 \\ 0 & e^{ik_2x} \end{pmatrix} \begin{pmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{pmatrix} \right]$$

for $x \leq 0$ and

$$\tilde{I} = \begin{pmatrix} e^{-ik_1(x-\ell)} & 0 \\ 0 & e^{-ik_2(x-\ell)} \end{pmatrix} \begin{pmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{pmatrix}$$

$$\tilde{E} = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \tilde{I}$$

for $x \geq \ell$. In particular, $\tilde{I} = \tilde{I}(x, \omega)$ and $\tilde{E} = \tilde{E}(x, \omega)$ satisfy the differential equations (6.1) for $0 \leq x \leq \ell$ and the boundary conditions

$$(6.3) \qquad \begin{cases} \tilde{I}(0) = U + P(\omega) , \\ \tilde{E}(0) = Z[U - P(\omega)] , \end{cases}$$

and

$$(6.4) \qquad \begin{cases} \tilde{I}(\ell) = Q(\omega) , \\ \tilde{E}(\ell) = ZQ(\omega) , \end{cases}$$

where

$$P(\omega) = \begin{pmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{pmatrix} \text{ and } Q(\omega) = \begin{pmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{pmatrix}$$

are (unknown) reflection and transmission coefficient matrices.

Problem (6.1), (6.3), (6.4) is formally identical with problem (4.3), (4.4), (4.5) of §4 if $s = i\omega$ and

$$\tilde{I}(x,\omega) = i\omega I(x, i\omega), \quad \tilde{E}(x,\omega) = i\omega E(x, i\omega) \quad . \tag{6.5}$$

Moreover, the second problem was shown to have a unique solution when $\operatorname{Re} s \geq 0$ (Theorems 4.1 and 4.?). This proves

<u>Theorem 6.1</u> The steady state coupler problem has a unique solution which is given by (6.5). Moreover,

$$P(\omega) = R(i\omega), \quad Q(\omega) = T(i\omega)$$

and therefore $P(\omega)$ is the solution of the initial value

$$\begin{cases} \dfrac{dP}{dx} + i\omega(PBP - AP - PA + B) = 0 , & 0 \leq x \leq \ell , \\ P(\ell, \omega) = 0 . \end{cases}$$

Note that equality holds in (4.9) when $\operatorname{Re} s = 0$. Hence, (4.9) implies (see the proof of Theorem 4.2)

<u>Corollary 6.1</u> The reflection and transmission coefficient matrices for the steady state coupler problem satisfy the identities

$$Z_1(|P_{11}(\omega)|^2 + |Q_{11}(\omega)|^2) + Z_2(|P_{21}(\omega)|^2 + |Q_{21}(\omega)|^2) = Z_1 \quad ,$$

$$Z_1(|P_{12}(\omega)|^2 + |Q_{12}(\omega)|^2) + Z_2(|P_{22}(\omega)|^2 + |Q_{22}(\omega)|^2) = Z_2 \quad .$$

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