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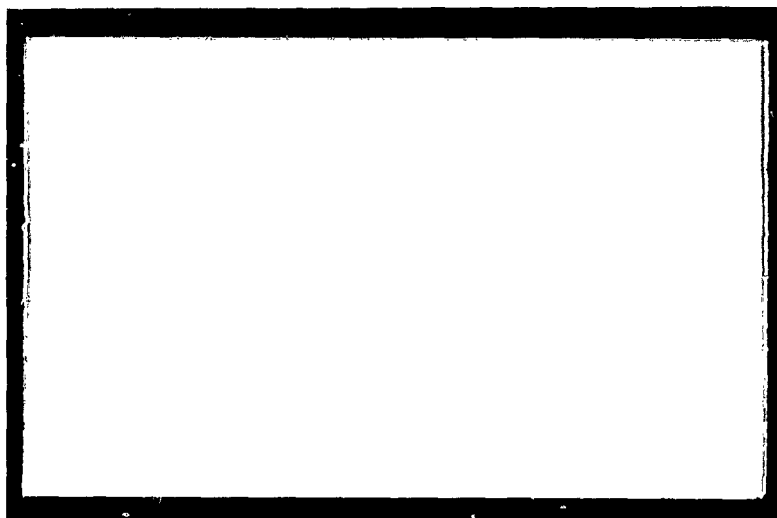
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Contract No.: DA-11-022-ORD-2059

APPROXIMATION AND ALLOCATION

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MRC Technical Summary Report #382
February 1963

Madison, Wisconsin

ABSTRACT

The segmented approximation problem is that of finding, for a given function x , a set of n functions y_i in some collection such that the supports of the y_i are pairwise disjoint while some functional of $x - y_1 - \dots - y_n$ is minimized. The recursive functional equation for the "error" of the best approximation by n functions is obtained and continuity of the "error" as a function of the set on which approximation occurs is derived from weak conditions on the functional. The allocation of the reduction in "error" attributable to each of the y_i is studied and the hereditary nature of an equal allocation principle is established.

APPROXIMATION AND ALLOCATION

Arthur Wouk

1. Let X denote a collection of vector valued functions $x = x(t)$ from the real line $-\infty < t < \infty$ into some normed linear vector space V . The support of $x \in X$ is the complement of the largest open set on the t -line on which $x(t)$ is the zero vector; it will be denoted by S_x . The segmented approximation problem is that of finding, for a given $x \in X$, a set of n vector valued functions $y_1, \dots, y_n$, contained in some given subset M of X , such that in some sense, no t is in the support of more than one of the y_i , while some functional $\mathcal{J}$ applied to $x - y_1 - \dots - y_n$ is minimized. Of particular interest in numerical computation are those subsets M of X which contain only "trapezoidal functions" (functions whose support is a finite interval.) Functionals of interest are, for example,

$$(1.1) \quad \sup_{a \leq t \leq b} \|x\|$$

where $-\infty \leq a < b \leq \infty$ and $\|\cdot\|$ is the norm in V , and

$$(1.2) \quad \int_a^b \|x(t)\|^p d\rho(t) \qquad 1 \leq p < \infty$$

where $\rho(t)$ is some monotone nondecreasing function of t ; in the latter case X must be restricted to ρ -measurable vector valued functions. Special cases of these problems have been treated (cf. e.g., [1], [2], [3], [4]) where the functions y are trapezoidal, and either linear or step functions of t on their support, and x is a continuous real valued function of t . The structure of many of these problems is easily seen (cf. [1]) to be an allocation problem; in the case of the supremum norm, it is characterized by the principal that

in the best approximation, the amount of reduction in $\mathcal{F}$ attributable to each of the y_i is the same.

This permits a recursive approach to the best approximation by n functions y_i and requires for computational purposes only knowledge of the best approximation by one function y on suitably arbitrary subsets of S_x . (This recursive approach is of the type popularized by Bellman under the name dynamic programming.)

In the present paper general properties of $\mathcal{F}$ are studied, and a weak form of the allocation principle is associated with properties of the best single function approximation. Further papers will consider the consequences of the allocation principle, and specific computational procedures for segmented approximation.

2. In the examples above, $\mathcal{F}$ (or a power of $\mathcal{F}$) is itself a norm on X ; if X is a linear space and V is complete, then $\mathcal{F}$ makes X a complete space. It might seem that these are the properties of $\mathcal{F}$ which are essential to

approximation. If the approximation theory is to be compatible with the linear structure of X so that the sum of two approximations is an approximation to the sum, then this is so. If instead the problem is posed relative to a fixed vector valued function x , this is no longer so. The basic arguments in the proof of the allocation principle are on certain monotonicity and continuity properties of $\mathcal{F}$ and on the "adequacy" of the set M . To make matters precise, suppose that x has support in a fixed interval $[a, b]$, $-\infty \leq a < b \leq \infty$, and that $\mathcal{F}(x)$ is defined (possibly infinite) for all arguments used below. A partial ordering of vector valued functions will be used: $x < y$ means $x(t) = y(t)$ for all $t \in S_x$. Two vector valued functions x and y are said to be disjoint if $\mathcal{F}(z) = 0$ for every $z \in X$ with $S_z \subset S_x \cap S_y$. Let $\mathcal{F}$ satisfy the following conditions:

- (a) if $x < y$, then $\mathcal{F}(x) \leq \mathcal{F}(y)$;
- (b) if $\{x_s\}$ is a collection of elements of X , $0 \leq s \leq \infty$ such that $x_s < x_{s'}$, whenever $s > s'$ (hence S_{x_s} is a decreasing sequence and $x = \lim_{s \rightarrow \infty} x_s$ exists), then $\mathcal{F}(x_s) \rightarrow \mathcal{F}(x)$, $s \rightarrow \infty$.
- (c) $\mathcal{F}(0) = 0$.

The approximation problem may now be restated: given $x \in X$ and a fixed subset M of X , find a set of n functions $y_1, \dots, y_n$ in M such that

- (i) if $i \neq j$, then y_i and y_j are disjoint;

and

(ii) $\mathcal{F}(x - y_1 - \dots - y_n)$ is a minimum.

In order to lend interest to the problem, M must contain a reasonable number of functions; most important for the arguments will be the following:

(d) if E is any subinterval of $[a, b]$ with characteristic function χ_E , and $y \in M$, then $y \cdot \chi_E \in M$.

If sets are ordered by inclusion, then (a) implies that $\mathcal{F}(x \cdot \chi_E)$ is a non-decreasing set function; (b) implies it is an "outer" continuous set function, while (c) makes it nonnegative. Together (a) and (d) imply

$$(2.1) \quad \mathcal{F}((x-y) \cdot \chi_E) \leq \mathcal{F}((x-y) \cdot \chi_{E_s})$$

whenever $E \subset E_s$. If E_s is a decreasing collection of sets with $E = \lim_{s \rightarrow \infty} E_s$, then (b) implies $\mathcal{F}((x-y) \cdot \chi_{E_s}) \rightarrow \mathcal{F}((x-y) \cdot \chi_E)$, $s \rightarrow \infty$ (outer continuity.) Then for any collection M the following holds:

Theorem 2.1 The set function

$$F_1(E) = \inf_{y \in M} \mathcal{F}((x-y) \cdot \chi_E)$$

exists, and is an outer continuous nonnegative nondecreasing function of sets ordered by inclusion.

The only nonimmediate fact is the outer continuity of F_1 . Suppose in fact that for some collection of sets E_s which is decreasing and convergent to E it is true that

$$F_1(E_s) \geq F_1(E) + c$$

where c is some positive number. Then

$$\mathcal{F}((x-y) \cdot \chi_{E_s}) \geq F_1(E) + c$$

for any c . Choose y so that

$$F_1(E) \geq \mathcal{F}((x-y) \cdot \chi_E) - c/2.$$

Then

$$\mathcal{F}((x-y) \cdot \chi_{E_s}) \geq \mathcal{F}((x-y) \cdot \chi_E) + c/2$$

for all s , which violates the outer continuity of $\mathcal{F}$.

Let $\{y_1, \dots, y_n\} = \{y_i\}_n$ denote a collection of n pairwise disjoint elements of M whose supports cover $[a, b]$. Define

$$F_2(E) = \inf_{\{y_i\}_2} \mathcal{F}((x-y_1-y_2) \cdot \chi_E).$$

Suppose further that for any disjoint two sets E, E' , there is defined a binary operation θ such that

$$\mathcal{F}(x \cdot \chi_{E \cup E'}) = \mathcal{F}(x \cdot \chi_E) \theta \mathcal{F}(x \cdot \chi_{E'});$$

there θ is a binary operation on real numbers $c = c(a, b) = a \theta b$ which is a commutative, associative, and continuous nondecreasing function of both arguments and which satisfies $a \theta 0 = 0 \theta a = a$. Then

$$\begin{aligned}
 F_2(E) &= \inf_{\{y_i\}_2} \mathcal{F}((x-y_1) \cdot \chi_{E-S_{y_2}}) \ominus \mathcal{F}((x-y_2) \cdot \chi_{S_{y_2}}) \\
 &= \inf_{T \subset E} \inf_{S_{y_1} \subset E-T} \inf_{S_{y_2} \subset T} \mathcal{F}((x-y_1) \cdot \chi_{E-T}) \ominus \mathcal{F}((x-y_2) \cdot \chi_E) \\
 &= \inf_{T \subset E} F_1(E-T) \ominus F_1(T) .
 \end{aligned}$$

Then $F_2(E)$ is a nonnegative, nondecreasing outer continuous set function.

Again the only part of this statement which is nonimmediate is the outer continuity. Suppose in fact that $F_2(E') \geq F_2(E) + c$, $c > 0$, for every $E' \supset E$.

Then for any $T \subset E$

$$F_1(T) \ominus F_1(E'-T) \geq F_2(E) + c .$$

Since F_1 is outer continuous, this implies

$$F_1(T) \ominus F_1(E-T) \geq F_2(E) + c$$

for any $T \subset E$ and hence $F_2(E) \geq F_2(E) + c$, which is impossible. Next we observe that $F_2(E) \leq F_1(E)$ since $a \ominus 0 = a$. If $F_n(E)$ is defined as

$$F_n(E) = \inf_{\{y_i\}_n} \mathcal{F}((x-y_1-y_2-\dots-y_n) \cdot \chi_E) ,$$

then a similar reasoning shows the truth of

Theorem 2.2: $F_n(E)$ is a nondecreasing nonnegative outer continuous set function; as a function of n it satisfies

$$(2.2) \quad F_n(E) \leq F_{n-k}(E), \quad k = 1, \dots, n-1$$

Lastly it satisfies the functional equations

$$(2.3) \quad F_n(E) = \inf_{T \subset E} F_k(T) \oplus F_{n-k}(E-T), \quad k = 1, \dots, n-1,$$

and

$$(2.4) \quad F_n(E) = \inf_{\{T_i\}} F_1(T_1) \oplus F_1(T_2) \oplus \dots \oplus F_1(T_n),$$

where $E = T_1 \cup T_2 \cup \dots \cup T_n$ and $T_i \cap T_j$ is empty if $i \neq j$.

If, in (1.1), $x(t)$ is continuous (in the strong sense), or, in (1.2), $\rho(t)$ is absolutely continuous, then $\mathcal{F}$ is also "inner continuous": that is $E_s \subset E_{s'}$, for $s < s'$ and $E = \lim_{s \rightarrow \infty} E_s$ imply

$$\lim_{s \rightarrow \infty} \mathcal{F}(x \cdot \chi_s) = \mathcal{F}(x \cdot \chi_E).$$

Then

Theorem 2.3: If $\mathcal{F}$ is also inner continuous, then so is $F_n(E)$ for all n and is in fact a continuous set function.

The proof of this is essentially the same as the proof of Lemmas 1 and 2. If $\mathcal{F}$ is both inner continuous and outer continuous, then it is called continuous.

3. In the absence of strong assumptions on the completeness of M with respect to $\mathcal{F}$, it is not possible to assert the existence of best segmented

approximations and give a proof of the allocation principle. However, the formal nature of the problem is already so strongly determined that it is possible to prove the equivalence of a rather "naive" analytic property of F_1 to a weakened form of the allocation principle. This is the following:

Weak Allocation Principle: Given any set E and any $\epsilon > 0$ there exists a collection of n sets, $\{T_i\}_n$ such that

- i) $T_i \cap T_j = \emptyset, i \neq j; \quad i, j = 1, \dots, n;$
- ii) $E = T_1 \cup T_2 \cup \dots \cup T_n;$
- iii) $F_1(T_1) = F_1(T_2) = \dots = F_1(T_n);$
- iv) $F_1(T_1) \oplus F_1(T_2) \oplus \dots \oplus F_1(T_n) \leq F_n(E) + \epsilon.$

An alternative form is the following:

$$\text{iv')} \quad F_n(E) = \inf_{\{T_i\}_n} F_1(T_1) \oplus F_1(T_2) \oplus \dots \oplus F_1(T_n)$$

where $\{T_i\}_n$ is any collection of sets satisfying i), ii), and iii).

Theorem 3.1 The weak allocation principle holds if and only if for every set
 E and any constant $\epsilon > 0$ there exists a set $T \subset E$ such that

- i) $F_1(T) \oplus F_1(T) \leq F_2(E) + \epsilon;$
- ii) $F_1(T) \geq F_1(E - T).$

Proof: It may be supposed that $F_1(E) > 0$; if not, the theorem is vacuously true. The following lemma is at the bottom of the proof:

If $E = A \cup B$ where $A \cap B = \emptyset$ and $F_1(A) > F_1(B)$, then there exists
 $A' \subset A$ and $B' \supset B$ such that $F_1(A') = F_1(B')$. The deformation $A \rightarrow A'$, $B \rightarrow B'$
is continuous.

To see this, consider the one parameter family of sets $A_s = A \cap [t_1, s]$,
 where $t_1 \leq t_2$ and E is contained in (t_1, t_2) . Also consider $B_s = E - A_s$.
 Then $F_1(A_s)$ is a continuous nondecreasing function of s , $F_1(B_s)$ is a
 continuous nonincreasing function of s , and

$$F_1(A_{t_1}) = 0 < F_1(E) = F_1(B_{t_1})$$

while

$$F_1(A_{t_2}) = F_1(A) > F_1(B) = F_1(B_{t_2}) .$$

Hence for some s , $F_1(A_s) = F_1(B_s)$.

Suppose now that the weak allocation principle I is true. Then, in particular,
 for any E ,

$$(3.1) \quad F_2(E) = \inf_{\{T \subset E \mid F_1(T) = F_1(E-T)\}} F_1(T) \oplus F_1(E-T)$$

so that for any $\epsilon > 0$ there exists a set T satisfying i) and ii) . Conversely,
 suppose that for any E and any $\epsilon > 0$ there exists $T \subset E$ satisfying i) and
 ii) . Then the lemma implies that there exists $T' \subset T$ such that

$$F_2(E) \leq F_1(T') \oplus F_1(E-T') \leq F_1(T) \oplus F_1(T) \leq F_2(E) + \epsilon$$

Then (3.1) is true.

Next the conditions i) and ii) must be lifted to the case of decompositions of E into three sets (and then by induction into n sets for any n .) The lifting process goes as follows: from the definition of infimum we conclude the existence of $\{T_i\}_3$ such that

$$F_1(T_1) \ominus F_1(T_2) \ominus F_1(T_3) \leq F_3(E) + \eta - \nu$$

for any positive numbers η, ν such that $\eta - \nu > 0$. The lemma implies that it is always possible to choose $\{T_i\}_3$ so that $F_1(T_1) = F_1(T_2) \geq F_1(T_3)$. Should $F_1(T_2) = F_1(T_3)$, nothing needs to be shown; suppose instead that a strict inequality holds. Now $F_1(T_2) \ominus F_1(T_3) = F_2(T_2 \cup T_3) + c$ for some $c \geq 0$. Then there exists a decomposition of $T_2 \cup T_3$ into $T_2' \cup T_3'$ such that $T_2' \subset T_2$, $T_3' \subset T_3$ and

$$F_1(T_2') - F_1(T_3') = 1/2 \{F_1(T_2) - F_1(T_3)\}$$

while

$$F_1(T_2') \ominus F_1(T_3') \leq F_1(T_2) \ominus F_1(T_3).$$

This follows by using (i), (ii) and modifying slightly the proof of the lemma above. Then,

$$F_1(T_1) \ominus F_1(T_2') \ominus F_1(T_3') \leq F_3(E) + \eta - \nu.$$

A direct application of the lemma permits the replacement of T_1 and T_3' by a new pair of sets T_1^* and T_3^* with $F_1(T_1^*) \geq F_1(T_2') \geq F_1(T_3^*)$ and

$$F_1(T_1^*) \ominus F_1(T_2') \ominus F_1(T_3^*) \leq F_3(E) + \eta - \nu ,$$

where either $F_1(T_1^*) = F_1(T_2')$ or $F_1(T_2') = F_1(T_3^*)$. If $F_1(T_1^*) = F_1(T_2')$, notice that

$$\begin{aligned} F_1(T_1^*) - F_1(T_3^*) &\leq F_1(T_2') - F_1(T_3') \\ &\leq 1/2 \{F_1(T_1) - F_1(T_3)\} . \end{aligned}$$

If instead, $F_1(T_2') = F_1(T_3^*)$, then use the fact that

$$\begin{aligned} F_1(T_2') &= F_1(T_3') + 1/2 \{F_1(T_1) - F_1(T_3)\} \\ &\geq 1/2 \{F_1(T_1) - F_1(T_3)\} \end{aligned}$$

to conclude that

$$\begin{aligned} F_1(T_1^*) - F_1(T_3^*) &\leq F_1(T_1) - F_1(T_2') \\ &\leq 1/2 \{F_1(T_1) - F_1(T_3)\} . \end{aligned}$$

Thus it is always possible to assume, for any $\delta > 0$, that the $\{T_i\}_3$ are such that $F_1(T_1) = F_1(T_2) \geq F_1(T_3) \geq F_1(T_1) - \delta$.

The function $\alpha \ominus \beta \ominus \gamma$ is a continuous function of the three variables on the closed set $0 \leq \alpha \leq F_1(E)$, $0 \leq \beta \leq F_1(E)$, $0 \leq \gamma \leq F_1(E)$. Hence it is

uniformly continuous there; given any $\gamma > 0$ there is a $\delta > 0$ such that

$$|\alpha \theta \beta \theta (\gamma - \Delta) - \alpha \theta \beta \theta \gamma| < \nu \text{ if } |\Delta| < \delta .$$

Thus it can be asserted that

$$(3.3) \quad F_1(T_1) \theta F_1(T_1) \theta F_1(T_1) \leq F_3(E) + \eta .$$

This lifts (i) to the present case while (ii) has always been preserved. If T_{1s} and T_{2s} have the customary meaning and $T_{3s} = E - T_{1s} - T_{2s}$, then the lemma is reapplied to rearrange T_{1s} and T_{2s} , for each s , to make $F_1(T_{1s}) = T_1(T_{2s})$. Then for some s , $F_1(T_{1s})$ is the same for $i = 1, 2, 3$, while (3.3) is preserved. This lifts the lemma. Thus it has been shown that the weak allocation principle holds for $n = 3$. The inductive argument is now clear. The uniform continuity of the iterated binary operation is still available, and the lifting argument for passing from (i) to its equivalent for $\{T_i\}_3$ goes over directly to the general case. The general form of the lemma also presents no difficulties.

Theorem 3.1 can be viewed as saying that if the weak allocation principle holds for $n = 2$, then it holds for all $n \geq 2$. It would be desirable to show that if the weak allocation principle holds for some $n = N > 2$, then it holds for $n = 2$ (and hence for all $n \geq 2$.) The present assumptions on M and $\mathcal{F}$ do not yield this result; instead the following weaker result will be established:

Theorem 3.2: If $c \theta c$ is a strictly increasing function of c and if there is a positive constance δ such that for any set E

$$(3.4) \quad (1+\delta) F_2(\underline{E}) \leq \inf_{\{F_1(T) = F_1(\underline{E}-T)\}} F_1(T) \otimes F_1(\underline{E}-T) ,$$

then the weak allocation principle does not hold for any set E with $F_n(E) > 0$
 $n \geq 2$.

Proof. Since $N = 2$ is trivial, let $N > 2$ and $F_N(E) > 0$. For any decomposition of E

$$(3.5) \quad \begin{aligned} F_N(E) &\leq F_2(T_1 \cup T_2) \otimes F_{N-2}(E - T_1 \cup T_2) \\ &\leq \left[(1+\delta)^{-1} \left[F_1(T_1) \otimes F_1(T_2) \right] \right] \otimes F_1(T_3) \otimes \dots \otimes F_1(T_N) \\ &< F_1(T_1) \otimes F_1(T_2) \otimes \dots \otimes F_1(T_N) . \end{aligned}$$

We may of course assume $F_1(T_i)$ constant, $i = 1, \dots, N$.
 As a function of c ,

$$f_N(c) = \overbrace{c \otimes c \otimes \dots \otimes c}^N$$

is strictly increasing; then there is a $c > 0$ with $F_N(E) = f_N(c)$. Furthermore

(3.5) implies $F_1(T_i) > c$, $i = 1, \dots, N$. Also, there exists a $\Delta > 0$ such that

$$\left[(1+\delta)^{-1} [c \otimes c] \right] \otimes c \otimes \dots \otimes c + \Delta < f_N(c) .$$

From (3.5) it follows that

$$(3.6) \quad F_N(E) \leq F_2(T_1 \cup T_2) \otimes F_{N-2}(E - T_1 \cup T_2) + \Delta < F_1(T_1) \otimes \dots \otimes F_1(T_N) .$$

Taking the infimum of (3.6) for all admissible decompositions of E implies

$$F_N(E) + \Delta \leq \inf_{\{T_i\}_N} F_1(T_1) \otimes \dots \otimes F_1(T_N) ,$$

which contradicts the possibility that the weak allocation principle holds for $n = N$.

Notice that if δ were dependent upon the set $\underline{E}$, and not bounded away from zero, then Δ would not be bounded away from zero and the contradiction would disappear. Also remark that if the assumptions on M and $\mathcal{F}$ were such as to guarantee that the infimum was achieved for some decomposition of E , then the above proof can be reworked somewhat to show that the weak allocation principle for $n = N > 2$ implies the principle for $n = 2$ (and hence for all $n \geq 2$.)

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