FTD-TT- 62-1769

63-3-1

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NEWS OF SCHOOLS OF HIGHER EDUCATION, AERONAUTICAL ENGINEERING (SELECTED ARTICLES)

English Pages: 27

SOURCE: Russian Periodical, Izvestiya Vysshykh Uchenbykh Zavedeniy, Aviatsionnaya Tekhnika, Nr. 2, 1962, pp. 56-64, 86-94.

S/147-62-0-2

PREPARED BY:

TRANSLATION SERVICES BRANCH
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 Date 27 March 19 63

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CALCULATION OF NON-STATIONARY TEMPERATURE FIELDS OF A STRUCTURE WITH CONSIDERATION OF THE HEAT RADIATION DURING NON-STEADY FLIGHT PROGRAMS

E TYAN-TSI

MOSCOW AVIATION INSTITUTE

Symbols

t — temperature

t_{aw} — adiabatic wall temperature

t_0 — surface temperature

t_{atm} — temperature of the atmosphere

n — normal to the surface

a — coefficient of temperature conductivity

τ — time

C_0 — Stefan-Boltzmann constant

ε — degree of blackness of the surface

M — Mach number

x, y, z and x_1, x_2, x_3 — rectangular coordinates

Q — quantity of heat

λ — coefficient of thermal conductivity

c — specific heat

γ — specific density

Fo — Fourier number

Bi — Biot number

Indices

i — layer

k — time interval

The structure of an aircraft in flight at high speed undergoes aerodynamic heating. The problem of determining the temperature fields within the structure is of important significance. Its solution by approximation methods has great advantages in engineering calculations. Here one may refer to the work of G. C. Elenevskaya, A. P. Vanichev [1] and P.P.Yushkov [2].

With the high surface temperatures and the relatively small values of the heat emission coefficient which occur during flights at high altitudes and high Mach numbers, the heat radiation plays a significant role and should not be neglected [3].

In the present paper the non-stationary temperature fields in the structure of an aircraft are examined taking into consideration the heat radiation and a convenient method for practical calculations is given. The problem reduces to the approximate solution of the heat conduction equation

$$a\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)=\frac{\partial t}{\partial \tau}, \qquad (1)$$

with boundary conditions

$$\lambda\left(\frac{\partial t}{\partial n}\right)_n=\alpha\left(t_{aw}-t_0\right)-C_0\varepsilon t_0^4 \qquad (2)$$

and initial conditions

$$t = f(x,y,z,0) \qquad (3)$$

The adiabatic wall temperature in equation (2) is determined from the formula:

$$t_{aw}=t_{amb}(1+0{,}18\ M^2).$$

In the course of the entire period of the flight this temperature will vary non-linearly with time. Moreover, during non-steady flight programs the coefficient of heat emission itself represents a complex function depending on the Mach number, i.e., the time, the Reynolds number, and so forth.

1. THE SOLUTION OF THE HEAT CONDUCTION EQUATION BY FINITE DIFFERENCES

In order to set up the heat balance equation, we divide the time of flight into intervals $\Delta\tau$. The structure is also divided by grids into a series of finite small parallelopipeds, the center points of which are to be calculated.

Let us review the basic assumptions.

1. Between two calculated points the temperature is linearly dependent on time

$$\frac{\partial}{\partial\tau}\left(\frac{\partial t}{\partial x_i}\right)=\text{const} \quad (i=1,\ 2,\ 3). \qquad (4)$$

Thus, it may be assumed that the amount of heat passing through any side of a parallelopiped during time is proportional to the arithmetic average of the temperature gradients at the beginning and end of the increment of time $\Delta\tau$.

2. The increase in the heat content of a parallelopiped is proportional to the increase in temperature at the center point of that element.

3. The physical constants of the material are assumed independent of the temperature.

4. Within an interval of time $\Delta\tau$, the adiabatic wall temperature t_{aw}, the heat emission coefficient α, and the surface temperature t_0 are linear functions of the time.

The initial instant is taken as the end of a previous increment of time. For simplicity in the beginning we are developing a one-dimension heat balance equation. Let us consider the layer on the surface of the structure touched by supersonic flow.

As a consequence of the last assumption, the adiabatic wall temperature, the coefficient of heat emission, and the temperature on the surface during some interval $\Delta\tau$ are expressed by the formulas:

$$\alpha=\alpha_k+\frac{\alpha_{k-1}-\alpha_k}{\Delta\tau}\tau, \qquad (5)$$

$$t_{aw}=t_{aw_k}+\frac{t_{aw_{k-1}}-t_{aw_k}}{\Delta\tau}\tau, \qquad (6)$$

$$t_0=t_{0,k}+\frac{t_{0,k-1}-t_{0,k}}{\Delta\tau}\tau. \qquad (7)$$

By Newton's rule the amount of heat passing from the boundary layer of flow to the structure through area ΔF during $\Delta\tau$ is

$$\Delta Q_1 = \int_0^{\Delta\tau} \alpha (t_{ж} - t_0) \Delta F d\tau. \qquad (8)$$

Substituting equations (5), (6), and (7) into (8), after integrating we obtain

$$\Delta Q_1 = \left[\left(\frac{\alpha_k}{3} + \frac{\alpha_{k+1}}{6}\right) t_{ж,k} + \left(\frac{\alpha_{k+1}}{3} + \frac{\alpha_k}{6}\right) t_{ж,k+1} - \right.$$
$$\left. - \left(\frac{\alpha_k}{3} + \frac{\alpha_{k+1}}{6}\right) t_{0,k} - \left(\frac{\alpha_{k+1}}{3} + \frac{\alpha_k}{6}\right) t_{0,k+1}\right] \Delta\tau\Delta F. \qquad (9)$$

The heat loss by radiation from the external surface into space during $\Delta\tau$ will be

$$\Delta Q_2 = - C_0 \varepsilon \Delta F \int_0^{\Delta\tau} t_0^4 d\tau. \qquad (10)$$

Substituting (7) into (10), after integrati ng we get

$$\Delta Q_2 = - \frac{1}{5} C_0 \varepsilon \Delta\tau\Delta F \frac{t^5_{0,k+1} - t^5_{0,k}}{t_{0,k+1} - t_{0,k}}. \qquad (11)$$

By an appropriate division of the time of flight, the initial temperature $t_{0,k}$ in each interval will always be significantly larger than the difference between the initial and final temperatures ($t_{0,k+1} - t_{0,k}$). Therefore it is possible to neglect terms containing ($t_{0,k+1} - t_{0,k}$)^n where $n > 2$ and retain satisfactory accuracy. After doing this equation (11) simplifies and ΔQ_2 depends linearly on the sought-for $t_{0,k+1}$:

$$\Delta Q_2 = - C_0 \varepsilon \Delta\tau \Delta F (2t^3_{0,k} t_{0,k+1} - t_{0,k}). \qquad (12)$$

It is not difficult to see that the longer the intervals of time $\Delta\tau$ the greater the inaccuracy. The problem of evaluating the error will be examined later.

According to the first assumption, the amount of heat passing through the surface of one layer into another in a solid during time $\Delta\tau$ is

$$\Delta Q_3 = - \frac{1}{2} \lambda \Delta\tau \Delta F \left[\frac{t_{i,k} - t_{i+1,k}}{\Delta x} + \frac{t_{i,k+1} - t_{i+1,k+1}}{\Delta x} \right]. \qquad (13)$$

For the surface layer we have

$$\Delta Q_3 = -\frac{1}{2}\lambda\Delta\tau\Delta F\left[\frac{t_{0,k} - t_{1,k}}{\Delta x} + \frac{t_{0,k+1} - t_{1,k+1}}{\Delta x}\right]. \qquad (14)$$

The increase in the heat contained in the volume being calculated equals

$$\Delta Q = \frac{1}{2} c\gamma\Delta x\Delta F(t_{0,k+1} - t_{0,k}). \qquad (15)$$

On the surface in contact with the supersonic flow the heat balance equation for the element $\Delta x \Delta F$ becomes

$$\Delta Q = \Delta Q_1 + \Delta Q_2 + \Delta Q_3. \qquad (16)$$

Substituting (9), (11), and (14) into (16) and rearrangeing, we obtain

$$(1 + 2H_2 + F_0 + 4R)\, t_{0,k+1} - F_0 t_{1,k+1} = F_0 t_{1,k} + (1 - 2H_1 - F_0 + 2R)\, t_{0,k} + 2\,(H_1 t_{aw_k} + H_2 t_{aw_{k+1}}), \qquad (17)$$

where

$$F_0 = \frac{a\Delta\tau}{\Delta x^2}, \qquad (18)$$

$$H_1 = Bi_1 F_0, \quad Bi_1 = \frac{\left(\frac{\alpha_k}{3} + \frac{\alpha_{k+1}}{6}\right)\Delta x}{\lambda}, \qquad (19)$$

$$H_2 = Bi_2 F_0, \quad Bi_2 = \frac{\left(\frac{\alpha_{k+1}}{3} + \frac{\alpha_k}{6}\right)\Delta x}{\lambda}, \qquad (20)$$

$$R = \frac{C_0 \varepsilon \Delta\tau t_{0,k}^3}{c\gamma\Delta x}. \qquad (21)$$

When the point calculated is located within a homogeneous solid structure, the heat balance equation takes the following form

where

$$\Delta Q' = \Delta Q'_1 + \Delta Q'_2, \quad (22)$$

$$\Delta Q'_1 = -\frac{\lambda\Delta\tau\Delta F}{2\Delta x}\left(t_{i,k} - t_{i-1,k} + t_{i,k+1} - t_{i-1,k+1}\right), \quad (23)$$

$$\Delta Q'_2 = -\frac{\lambda\Delta\tau\Delta F}{2\Delta x}\left(t_{i,k} - t_{i+1,k} + t_{i,k+1} - t_{i+1,k+1}\right), \quad (24)$$

$$\Delta Q' = c\gamma\Delta x\Delta F\left(t_{i,k+1} - t_{i,k}\right). \quad (25)$$

Substituting (23), (24), and (25) into (22) and rearranging we get the heat balance equation

$$-F_0 t_{i-1,k+1} + 2(1+F_0)t_{i,k+1} - F_0 t_{i+1,k+1} = F_0 t_{i-1,k} + 2(1-F_0)t_{i,k} + F_0 t_{i+1,k}. \quad (26)$$

The problem reduces to the solution of a system of linear algebraic equations comprised of equations (17) and (26):

$$\left.\begin{array}{l} a_{11}t_{0,k+1} + a_{12}t_{1,k+1} = b_1 \\ a_{21}t_{0,k+1} + a_{22}t_{1,k+1} + a_{23}t_{2,k+1} = b_2 \\ a_{32}t_{1,k+1} + a_{33}t_{2,k+1} + a_{34}t_{3,k+1} = b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n(n-1)}t_{l-1,k+1} + a_{nn}t_{l,k+1} = b_n \end{array}\right\}. \quad (27)$$

We write this system in matrix form

$$At = b. \quad (28)$$

In expression (28), the matrix of coefficients is tri-diagonal:

$$A = \begin{bmatrix} a_{11}a_{12}0\ 0 & \ldots 0 \rightarrow 0 \rightarrow \\ a_{21}a_{22}a_{23}0 & \ldots 0 \rightarrow 0 \rightarrow \\ 0\ \ a_{32}a_{33}a_{34} & \ldots 0 \rightarrow 0 \rightarrow \\ \ldots & \ldots\ldots\ldots\ldots \\ 0\ 0\ 0\ 0 & \ldots\ a_{n(n-1)}a_{nn} \end{bmatrix}. \quad (29)$$

therefore the system is easily solved.

In the case of three-dimensional temperature fields, we have in place of (17) the following heat balance equation (figure 1a) for the surface layer

$$
\begin{aligned}
&2(1+2H_2+F_{01}+F_{02}+F_{03}+4R)\,t_{0,k+1}-2F_{01}t_{1,k+1}-\\
&-F_{02}t_{2,k+1}-F_{02}t_{3,k+1}-F_{03}t_{4,k+1}-F_{03}t_{5,k+1}=\\
&=2(1+2R-2H_1-F_{01}-F_{02}-F_{03})\,t_{0,k}+2F_{01}t_{1,k}+F_{02}t_{2,k}+\\
&+F_{02}t_{3,k}+F_{03}t_{4,k}+F_{03}t_{5,k}+4\,(H_1t_{aw_k}+H_2t_{aw_{k+1}}),
\end{aligned}
\qquad (30)
$$

where
$$F_{01}=\frac{a\Delta\tau}{\Delta x^2},\quad F_{02}=\frac{a\Delta\tau}{\Delta y^2},\quad F_{03}=\frac{a\Delta\tau}{\Delta z^2}.$$

In the case where the point calculated is located within a homogeneous solid, the heat balance equation may be written in the form(figure 1b):

$$
\begin{aligned}
&2(1+F_{01}+F_{02}+F_{03})\,t_{0,k+1}-F_{01}t_{1,k+1}-F_{01}t_{2,k+1}-\\
&-F_{02}t_{3,k+1}-F_{02}t_{4,k+1}-F_{03}t_{5,k+1}-F_{03}t_{6,k+1}=\\
&=2(1-F_{01}-F_{02}-F_{03})\,t_{0,k}+F_{01}(t_{1,k}+t_{2,k})+\\
&+F_{02}(t_{3,k}+t_{4,k})+F_{03}(t_{5,k}+t_{6,k}).
\end{aligned}
\qquad (31)
$$

At the same time the problem resolves into the solution of the system of linear algebraic equations (30) and (31). The number of equations is equal to the number of points calculated.

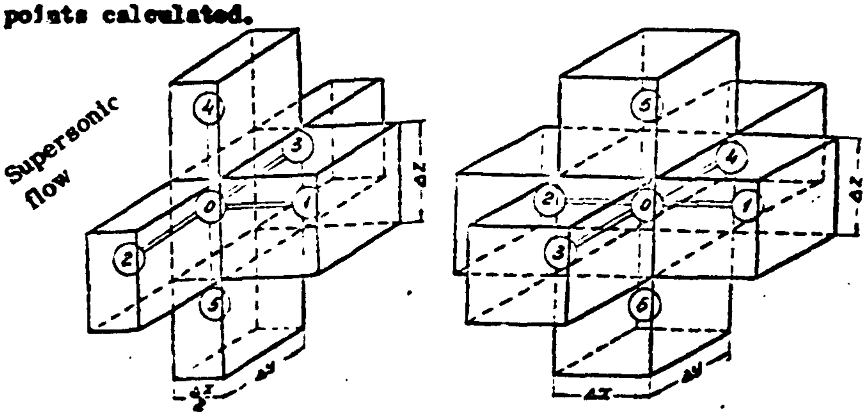

Figure 1. Sketch of the division of the structure into parallelpiped elements, a) for the surface of the structure, b) within the structure.

The equations in finite differences considered in our work have six point symmetry [4], consequently, it is possible, in general, to select the intervals of time $\Delta\tau$ for equal increments Δx_i to be significantly larger than in the method of A. P. Vanichev, thus the expenditure of time in calculations will be less. Calculation of a numerical example shows that the results by the present method agree well with the results by the method of A.P. Vanichev (figure 3). In the example examined the maximum permissible magnitude of time was $\Delta\tau_{max} = 2.56$ seconds for the method mentioned, but with use of our method the same result was obtained by choosing the calculated interval of time to be $\Delta\tau = 45$ seconds.

2. DETERMINATION OF ERROR

We will determine precisely the error which originates from neglecting terms with $(t_{0,k+1} - t_{0,k})^n$ $(n \geqslant 2)$ in formula (11). If these terms are retained then the heat balance equation for the surface layer becomes:

$$t_{0,k+1} - t_{0,k} = 2(H_1 t_{aw_k} + H_2 t_{aw_{k+1}}) - \\ - 2(H_1 t_{0,k} + H_2 t_{0,k+1}) - F_0 \cdot [(t_{0,k} + t_{0,k+1}) - (t_{1,k} + t_{1,k+1})] - \\ - \frac{2}{5} \frac{C_0 \varepsilon \Delta\tau}{c\gamma\Delta x} \frac{t^5_{0,k+1} - t^5_{0,k}}{t_{0,k+1} - t_{0,k}}. \qquad (32)$$

We assume that

$$t_{0,k+1} - t_{0,k} = n t_{0,k} \qquad (33)$$

After substituting (33) into (32) and rearranging we obtain:

$$\frac{1}{5} n^4 + n^3 + 2n^2 + \frac{1}{2R}(1 + 4R + 2H_2 + F_0)\, n = \\ = \frac{1}{R t_{0,k}} (H_1 t_{aw_k} + H_2 t_{aw_{k+1}}) - \frac{1}{R}(H_1 + H_2 + R + F_0) + \\ + \frac{F_0}{2R t_{0,k}} (t_{1,k} + t_{1,k+1}). \qquad (34)$$

If the higher order terms are neglected, we obtain the simplified equation

$$(1 + 4R + 2H_2 + F_0)\, n' = \frac{2}{t_{0,k}} (H_1 t_{aw_k} + H_2 t_{aw_{k+1}}) - \\ - 2(H_1 + H_2 + R + F_0) + \frac{F_0}{t_{0,k}} (t_{1,k} + t_{1,k+1}). \qquad (35)$$

The error is the difference between the solutions of equations (34) and (35):

$$\Delta = t_{0,k+1} - t'_{0,k+1} = (n - n') t_{0,k}. \quad (36)$$

3. CALCULATED EXAMPLE

After 90 seconds the aircraft accelerates from M = 2 to 5 at an altitude of 15,000m. The calculational model of the cross-section of the wing is shown in figure 3. The sheathing, flange, and wall are steel, γ = 7900 kg/m, λ = 39 kcal/hr.m, and C = 0.11 kcal/kg °C.

The variation of the heat emission coefficient with time is shown in figure 2.

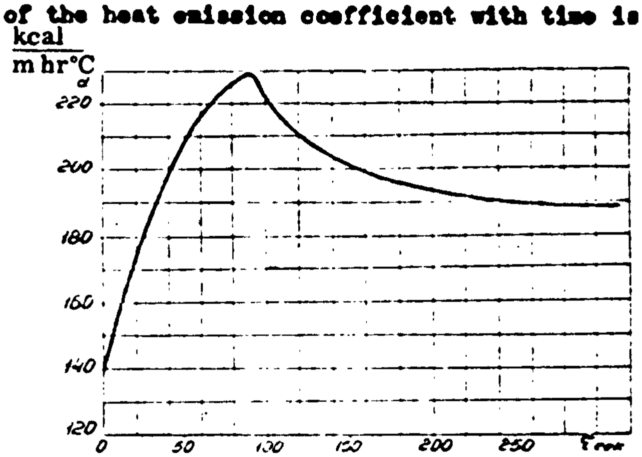

Figure 2. Variation of the heat emission coefficient with time of flight.

The graphs of the temperature distributions in the cross section at various instants of flight are shown in figure 3.

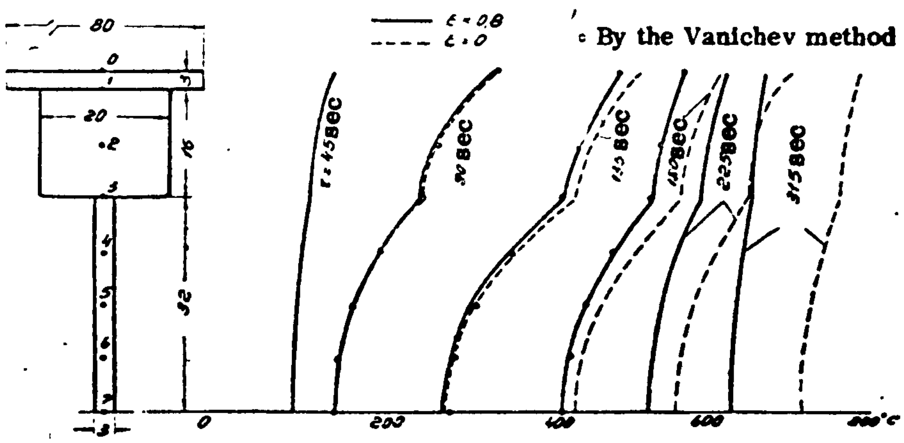

Figure 3. Temperature distribution in the cross-section.

The variation in temperature on the surface during the course of the flight is shown in figure 4.

In figures 3 and 4 the dashed lines correspond to calculations with a zero value for the degree of blackness, $\varepsilon = 0$.

In figure 5 the size of the errors corresponding to various values of the increment of time $\Delta\tau$ are given. From paragraph 2, itis evident that the errors are connected with the initial temperature distribution. If we use $\tau = 0$ for the initial time (i.e. at the beginning of the acceleration) then we obtain the solid curve, but if $\tau = 90$ seconds is used the dashed curve results.

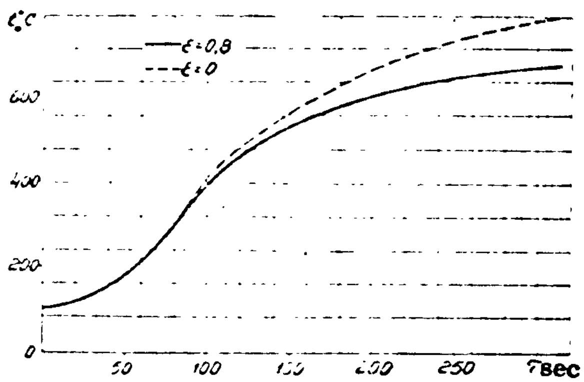

Figure 4. Variation of surface temperature with time of flight.

CONCLUSION

When accounting for the heat radiation in the determination of the non-stationary temperature fields of the structure during non-steady programs of flight the proposed method is convenient for engineering calculations.

The calculated intervals of time may be chosen to be significantly greater than in the method of A.P. Vanichev.

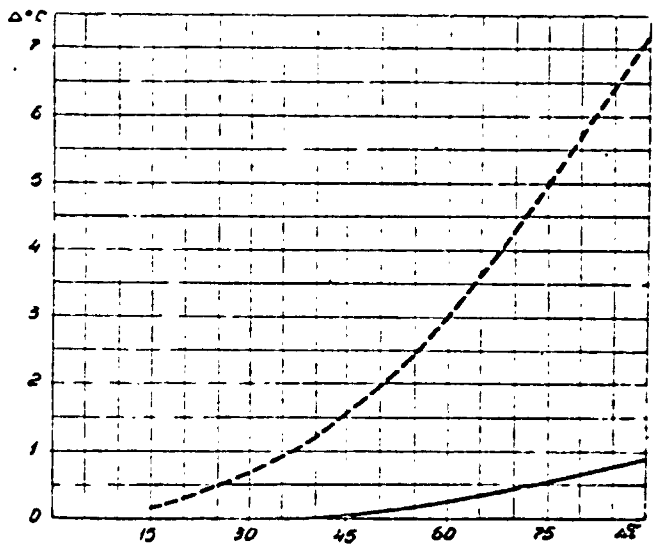

Figure 5. The error corresponding to various values of the increment of time $\Delta\tau$.

APPENDIX

Method of A.P. Vanichev accounting for radiation

The heat balance equation for one-dimensional temperature field is, for the surface layer:

$$t_{0,k+1} = \left(1 - \frac{2\alpha\Delta\tau}{\Delta x C\gamma} - \frac{2a\Delta\tau}{\Delta x^2}\right) t_{0,k} - \frac{2\varepsilon C_0 T_{0,k}^4 \Delta\tau}{C\gamma\Delta x} + \frac{2a\Delta\tau}{\Delta x^2} t_{1,k} + \frac{2\alpha\Delta\tau}{\Delta x C\gamma} t_{ж,k}. \qquad (a)$$

In order to determine $t_{0,k+1}$, we use successive approximations. Initially we find $t_{0,k+1}$ from equation (a), this denoted by $t^{(1)}_{0,k+1}$. Then we replace $t_{0,k}$ in the radiation terms in formula (a) by the average temperature $\frac{t^{(1)}_{0,k+1} + t_{0,k}}{2}$ we obtain $t^{(2)}_{0,k+1}$, this is the second approximation of the temperature. Analogously, we continue calculations until

$$t^{(n)}_{0,k+1} \approx t^{(n+1)}_{0,k+1}.$$

The maximum permissible size of the interval of time is determined by the formula:

$$\Delta\tau_{max} = \frac{0.5}{\frac{a}{\Delta x^2} + \frac{\alpha}{\Delta x C\gamma} + \frac{\varepsilon C_0 T_{0,k}^3}{C\gamma\Delta x}}. \qquad (6)$$

BIBLIOGRAPHY

1. Vanichev, A.P., An approximate method of solution of the problem of heat conduction with variable constants, Izv. AN SSSR, Otd. Tech. Nauk, No 12, 1946.

2. Yushkov, P.P., An approximate solution to the problem of non-stationary heat conduction using the method of finite differences, Trudy Inst. Energetiki, AN, BSSR, Vyp. 6, 1958.

3. Kitchenside, A.M., The effects of kinetic heating on aircraft structures, IRAS, February, 1958.

4. Saul'ev, V.K., The integration of equations of the parabolic type by the grid method, Moskva, 1960.

EQUILIBRIUM EQUATIONS FOR STATIC BENDING AND FREE VIBRATION OF IRREGULARLY SHAPED CANTILEVER PLATES

B.E. Sivchikov

Methods for calculating bending in cantilever plates have been developed rather recently, mainly in connection with the problem of the deformation of short wings. Solutions of the known biharmonic bending equation for rectangular cantilever plates of constant thickness and certain simple stresses have been obtained by the grid method and by the superposition method [5,9,10]. Calculations show that the integration of the biharmonic equation with mixed boundary conditions is attended by inordinate mathematical difficulties. Significant progress has been made thanks to the use of variational principles which permit the creation of sufficiently effective approximation theories for calculations of cantilever plates [2, 8] and permit obtaining numerical results by direct methods.

For the goal of studying the motion of a plate under static and dynamic stresses, it appears the use of the Lagrange variational equation is most effective. The representation of deflections in the cross-sections of the plate parallel to the attached side by the sum of a power series leads to a one-dimensional function. Variation of the latter gives a system of ordinary differential equations and boundary conditions which approximately describe the bending of the plate. For a series of cases of practical importance the possibilities of the system limiting to low order and of obatining solutions for cantilever plates with various planar contours was shown [4, 6, 7]. However, this method is less suitable for calculations on cantilever plates of irregular shape, i.e. as when the free and fastened sides are not parallel to one another.

In the present article the method described generalises on the case of the bending of cantilever plates with straight sides arbitrarily placed with respect to one another. The equilibrium equation of a cantilever plate with static bending and free vibration is derived by the variational method. It is assumed that the plate is homogeneous and isotropic and that its deformation follows Hook's law. It is also assumed that the theory of bending for thin rigid plates is a correct hypothesis.

1. TRANSFORMATION OF THE ENERGY EQUATION

We are considering a tetragonal plate of variable thickness rigidly fastened on one side. We refer to the plate by a rectangular system of coordinates, ξ, η, ζ (fig. 1). We set the origin of the coordinates in the fixed side, the plane ξ, η conforming with the center plane of the plate.

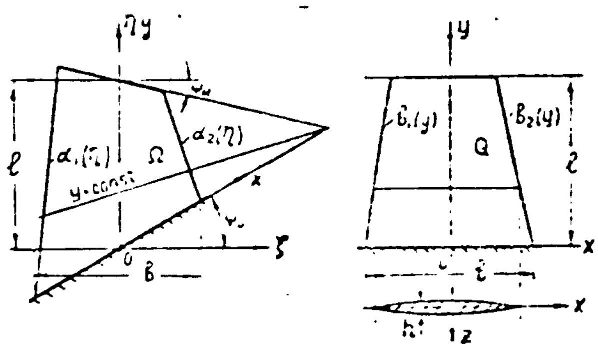

Figure 1.

We use the non-essential condition that the shape of the plate (region Ω) is defined by straight lines. The equations for the longitudinal boundary lines are $\xi = \alpha_1(\eta)$, and $\xi = \alpha_2(\eta)$, and for the tranverse

$$\eta = \xi \cdot \operatorname{tg} \psi_0 \text{ and } \eta = \xi \cdot \operatorname{tg} \psi_k + l, \quad (1)$$

where l is the length of the plate along the η-axis.

Let us investigate the equilibrium equation and boundary conditions for tranverse bending of the plate using the Lagrange variational equation:

$$\delta \Pi = \delta (\mathcal{Э} - A). \quad (2)$$

As is well known[1], the potential energy of deformation of the plate is

$$\partial = \iint_{\Omega} \frac{D(\xi,\eta)}{2}\left[w_{\xi\xi}^2 + w_{\eta\eta}^2 + 2\mu w_{\xi\xi} w_{\eta\eta} + 2(1-\mu) w_{\xi\eta}^2\right] d\xi d\eta, \qquad (3)$$

where w is the deflection in the direction of the ζ axis, $w_{rs} = \frac{\partial^2 w}{\partial r \partial s}$ ($r, s = \xi, \eta$),

$D(\xi, \eta)$ is the local cylindrical rigidity,

$$D(\xi, \eta) = \frac{Eh^3(\xi,\eta)}{12(1-\mu^2)},$$

$h(\xi, \eta)$ is the local thickness,

E, μ are the normal modulus of elasticity and tranverse coefficient of deformation,

the integration extends over the whole region Ω . The work of the external stationary force is

$$A = \iint_{\Omega} p(\xi, \eta) \cdot w \cdot d\xi \cdot d\eta, \qquad (4)$$

where $p(\xi, \eta)$ is the tranverse loading per unit area.

Aligning a new system of coordinates x, y, z so that in this system the region Ω has the shape of a trapezoid(region Q in figure 1b) attached by its base of length b. Obviously the transformation of coordinates may be expressed

$$x = \frac{\xi}{b},$$
$$y = \frac{\eta - \xi \cdot \operatorname{tg}\varphi_0}{1 - \frac{\operatorname{tg}\varphi_0 - \operatorname{tg}\varphi_k}{l}\xi} \cdot \frac{1}{l}, \qquad (5)$$
$$z = \frac{\zeta}{h},$$

whereupon it follows from relations(1) and (5) that within the limits of the plane

$$b_1(y) \leqslant x \leqslant b_2(y),$$
$$0 \leqslant y \leqslant 1,$$
$$-\frac{1}{2} \leqslant x \leqslant \frac{1}{2},$$

where $x = b_1(y)$ and $x = b_2(y)$ are the equations of the lines $\xi = \alpha_1(\eta)$ and $\xi = \alpha_2(\eta)$ in the new system of coordinates.

We perform the substitution of variables by formula (5) in the expressions for the potential energy (3) and the work done by the external forces (4). If we introduce the notation

$\lambda = \frac{l}{b}$ — the elongation per unit length,

$\beta = \frac{tg\,\psi_0 - tg\,\psi_1}{\lambda}$ — a taper parameter characterising the spaeing of the opposite sides ($y = 0$ and $y = 1$).

$$s = \frac{1}{1 - \beta x}, \qquad u = \operatorname{tg}\psi_0 - \beta y \tag{6}$$

and consider that the Jacobian of the transformation is

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \frac{1}{s},$$

then finally we obtain

$$\begin{aligned} \Theta = \frac{b}{l^3}\int_0^1\int_{b_1(y)}^{b_2(y)} \frac{D(x,y)}{2} \Big\{ &(1+u^2)^2 s^4 \varpi_{yy}^2 - 4(1+u^2) u s^3 \lambda (\varpi_{xy} + \beta s \varpi_y)\varpi_{xy} + \\ &+ 2(u^2+\mu) s^2\lambda^2 \varpi_{xx}\varpi_{yy} + 4\left(u^2 + \frac{1-\mu}{2}\right) s^2\lambda^2 (\varpi_{xy}^2 + 2\beta s \varpi_{xy}\varpi_y + \beta^2 s^2 \varpi_y^2) - \\ &- 4us\lambda^3(\varpi_{xy} + \beta s\varpi_y)\varpi_{xx} + \lambda^4 \varpi_{xx}^2 \Big\} \frac{1}{s}\, dx \cdot dy, \end{aligned} \tag{7}$$

$$A = bl \int_0^1 \int_{b_1(y)}^{b_2(y)} p(x, y)\, \varpi \frac{1}{s}\, dx dy. \tag{8}$$

We represent the deflection functions as

$$\varpi = Y_0 + xY_1 + \sum_{m=2}^{N} x^m Y_m. \tag{9}$$

where $Y_m = Y_m(y)$ is an unknown function satisfying the conditions of the rigid fastening at $y = 0$,

$$Y_m(0) = Y'_m(0) = 0. \qquad (10)$$

(Primes here and later denote differentiation by y)

The function Y_0 expresses the tranverse deflection on axis y, the function Y_1 represents the distribution of the elastic deflection angles of the sections $y = \text{const}$ relative to the y-axis, the remaining terms of the sum characterise the deformation of the shape of the tranverse sections $y = \text{const}$.

After substituting the sum (9) into equations (7) and (8), we obtain the following expression for the total potential energy function $\Pi = Э - A$:

$$\Pi = \frac{bl^{-3}}{2} \int_0^1 \sum_{n=0}^{N} \sum_{m=0}^{N} \Big\{ (1+u^2)^2 J_{0,m+n} Y''_m Y''_n - (1+u^2) u\lambda (\beta J_{0,m+n} Y''_m Y''_n + m J_{1,m+n-1} Y'_m Y''_n) + 4\left(u^2 + \frac{1-\nu}{2}\right)\lambda^2 [\beta^2 J_{0,m+n} + \beta(m+n) J_{1,m+n-1} + mn J_{2,m+n-2}] Y'_m Y'_n + 2m(m-1)(u^2+\nu)\lambda^2 J_{2,m+n-2} Y_m Y''_n - 4m(m-1)\lambda^3 (\beta J_{2,m+n-2} Y_m Y'_n + n J_{3,m+n-3} Y'_m Y_n) + mn(m-1)(n-1)\lambda^4 J_{4,m+n-4} Y_m Y_n \Big\} dy - bl \int_0^1 \sum_{m=0}^{N} P_n Y_n dy. \qquad (11)$$

where

$$J_{r,m+n-r} = \int_{b_1(y)}^{b_2(y)} D(x, y) \frac{x^{m+n-r} dx}{(1-\beta x)^{3-r}}, \qquad (12)$$

$$P_n = \int_{b_1(y)}^{b_2(y)} p(x, y) \frac{x^n dx}{1-\beta x} \quad \begin{pmatrix} r = 0, 1, \dots, 4 \\ m, n = 0, 1, \dots, N \end{pmatrix}. \qquad (13)$$

2. GENERAL EQUATIONS

The condition for minimisation (2) for the function (11) requires functions Y_m to satisfy the equation

$$\int_0^1 \left(\frac{\partial \Pi}{\partial Y''_m} \delta Y''_m + \frac{\partial \Pi}{\partial Y'_m} \delta Y'_m + \frac{\partial \Pi}{\partial Y_m} \delta Y_m \right) dy = 0$$
$$(m = 0, 1, \ldots, N),$$

from which after integration by parts, taking into account the boundary conditions (10), we obtain the system of ordinary linear differential equations:

$$\begin{aligned}
\sum_{m=0}^{N} \{ [(1+u^2)^2 J_{0,m+n} Y''_m - 2(1+u^2) u\lambda (\beta J_{0,m+n} + m J_{1,m+n-1}) Y'_m + \\
+ m(m-1)(u^2+\mu)\lambda^2 J_{2,m+n-2} Y_m]'' - \\
- [-2(1+u^2) u\lambda (\beta J_{0,m+n} + n J_{1,m+n-1}) Y''_m + 4\left(u^2 + \frac{1-\mu}{2}\right)\lambda^2 [\beta^2 J_{0,m+n} + \\
+ (m+n)\beta J_{1,m+n-1} + mn J_{2,m+n-2}] Y'_m - 2u\lambda^3 [m(m-1)\beta J_{2,m+n-2} + \\
+ mn(m-1) J_{3,m+n-3}] Y_m]' + n(n-1)(u^2+\mu)\lambda^2 J_{2,m+n-2} Y''_m - \\
- 2u\lambda^3 [n(n-1)\beta J_{2,m+n-2} + mn(n-1) J_{3,m+n-3}] Y'_m + \\
+ mn(m-1)(n-1)\lambda^4 J_{4,m+n-4} Y_m \} = l^4 p_n \quad (n = 0, 1, \ldots, N) \qquad (14)
\end{aligned}$$

and the natural boundary conditions on the free side of the plate $y=1$:

$$\begin{aligned}
\sum_{m=0}^{N} \{ (1+u^2)^2 J_{0,m+n} Y''_m - 2(1+u^2) u\lambda (\beta J_{0,m+n} + m J_{1,m+n-1}) Y'_m + \\
+ m(m-1)(u^2+\mu)\lambda^2 J_{2,m+n-2} Y_m \}_{y=1} = 0; \qquad (15)
\end{aligned}$$

$$\begin{aligned}
\sum_{m=0}^{N} \{ [(1+u^2)^2 J_{0,m+n} Y''_m - 2(1+u^2) u\lambda (\beta J_{0,m+n} + m J_{1,m+n-1}) Y'_m + \\
+ m(m-1)(u^2+\mu)\lambda^2 J_{2,m+n-2} Y_m]' - [-2(1+u^2) u\lambda (\beta J_{0,m+n} + \\
+ n J_{1,m+n-1}) Y''_m + 4\left(u^2 + \frac{1-\mu}{2}\right) [\beta^2 J_{0,m+n} + \beta (m+n) J_{1,m+n-1} + \\
+ mn J_{2,m+n-2}] Y'_m - 2u\lambda^3 [m(m-1)\beta J_{2,m+n-2} + \\
+ mn(m-1) J_{3,m+n-3}] Y_m] \}_{y=1} = 0 \qquad (16) \\
(n = 0, 1, \ldots, N).
\end{aligned}$$

The order of the system (14) and the number of boundary conditions (10), (15), and (16) are equal to $4(N-1)$.

In the general case equation (14) has variable coefficients u and $J_{\gamma,m+m-r}$ (see (6) and (12)). If $\psi_0=\psi_h=0$ and the condition is imposed that

$$J_{0,1} = \int_{b_1(y)}^{b_2(y)} D(x, y)\, x\, dx = 0, \qquad (17)$$

then the system divides into two independent systems describing the mutually independent symmetric and antisymmetric (relative to the y-axis) movements of the plate.

As $N \to \infty$ the exact solution of the system is also the exact solution of the biharmonic equation for the deflection of the plate. In the case of a finite number N, the exact solution of the system of equations (14)-(16) only approximately describes the deflections of the plate. In particular, the boundary conditions on the longitudinal edges of the plate are not satisfied.

If the solution of (10) is found, then the components of the stress σ_ξ, σ_η, $\sigma_{\xi\eta}$, and the principal stresses σ_1 and σ_2 on the surface of the plate are determined by the well-known formulas:

$$\sigma_\xi = \frac{6M_\xi}{h^2}, \qquad \sigma_\eta = \frac{6M_\eta}{h^2}, \qquad \tau_{\xi\eta} = \frac{6M_{\xi\eta}}{h^2},$$
$$\sigma_{1,2} = \frac{1}{2}(\sigma_\xi + \sigma_\eta) \pm \frac{1}{2}\sqrt{(\sigma_\xi - \sigma_\eta)^2 + 4\tau_{\xi\eta}^2}, \qquad (18)$$

where M_ξ, M_η are the bending moments and $M_{\xi\eta}$ is the torsional moment.

$$M_\xi = -D(w_{\xi\xi} + \mu w_{\eta\eta}) =$$
$$= -\frac{D}{l^2}\sum_{m=0}^{N}[m(m-1)\lambda^2 x^{m-2} Y_m - 2u\lambda(mx^{m-1}s^{-1} + x^m s^{-2}) Y'_m +$$
$$+ (1 + \mu u^2) x^m s^{-2} Y''_m],$$
$$M_\eta = -D(w_{\eta\eta} + \mu w_{\xi\xi}) =$$
$$= -\frac{D}{l^2}\sum_{m=0}^{N}[m(m-1)\lambda^2 x^{m-2} Y_m - 2u\lambda(mx^{m-1}s^{-1} + x^m s^{-2}) Y'_m +$$
$$+ (\mu + u^2) s^{-2} x^m Y''_m], \qquad (19)$$
$$M_{\xi\eta} = (1-\mu) D w_{\xi\eta} = (1-\mu)\frac{D}{l^2}\sum_{m=0}^{N}[\lambda(mx^{m-1}s^{-1} + \beta s^{-2} x^m) Y'_m -$$
$$- us^{-1} x^m Y''_m].$$

If the unstressed plate makes free vibrations with amplitude w, then the action of the inertial force is equivalent to the action of a tranverse stress intensity of $\rho w^2 h w$, where ρ is the density of the material of the plate, and ω is the angular frequency of oscillation. The corresponding work of the inertial force is:

$$A_\omega = \iint_{Q} \rho\omega^2 h w^2 \, d\xi \, d\eta = bl\omega^2\rho \iint_{Q} h w^2 \frac{1}{s} \, dx\, dy.$$

If the expansion (9) is considered, then after integration by x

$$A_\omega = bl\omega^2\rho \sum_{m=0}^{N} \sum_{n=0}^{N} \int_0^1 Y_m Y_n f_{m+n} \, dy. \qquad (20)$$

where

$$f_{m+n} = \int_{b_1(y)}^{b_2(y)} h \frac{x^{m+n} \, dx}{1 - \beta x}.$$

Replacing the work A by A_ω in the expression for the total potential energy (11), from condition (2) we obtain in place of the functions p_n in the first parts of equation (14) the functions:

$$c_n = \omega^2\rho \sum_{m=0}^{N} f_{m+n} Y_m \quad (n = 0, 1, \ldots, N). \qquad (21)$$

Since the latter depend on the unknown functions Y_m, then the system of differential equations obtained will be homogeneous. It is satisfied at specific values of the parameter ω which, like the functions Y_m, depends on the location.

3. SIMPLIFIED EQUATIONS

Let us assume that the deflection of the plate may be approximated with sufficient accuracy by two terms of the expansion (9). This assumption implies that the deformation of the contour of the section $y = \text{const}$ is negligibly small by comparison with the deflection Y_0 and the dislocation during turning xY_1. Since an analogous supposition is made in the basic theory of rod deflection, it is always possible to select such a sufficiently "lengthy" plate that the described assumption is justifiable.

We suppose also that the thickness of the plate, $h = h_0 \cdot h_1 \cdot h_2$ where h_0 is a characteristic value of thickness, $h_1 = h_1(y)$, $h_2 = h_2(x)$ – dimensionless functions, and the y-axis is selected in such a manner that

$$J_{0,1} = \frac{h [illegible]}{2(1 [illegible]} [illegible] \int [illegible] \frac{x\,dx}{(1 [illegible] x)^2} = 0.$$

Assuming that $N = 1$ in equations (14)–(16) and integrating the system (14) taking into account the boundary conditions (16) for the case of static bending, we obtain the simplified system of differential equations:

$$\begin{gathered}
[(1+u^2)^2 J_{0,0} Y_0'' - 2(1+u^2)\,u\lambda\xi J_{0,0} Y_1']'' - [-2(1+u^2)\,u\lambda\xi J_{0,0} Y_0'' + \\
+ 4\left(u^2 + \frac{1-\nu}{2}\right)\lambda^2\xi^2 J_{0,0} Y_0']' + [-2(1+u^2)\,u\lambda J_{1,0} Y_1'']' \\
+ 4\left(u^2 + \frac{1-\nu}{2}\right)\lambda^2\xi^2 J_{1,0} Y_1' = -l^3 \int_0^1 p_0\,dy. \\
-\left[-2(1+u^2)\,u\lambda J_{1,0} Y_0'' + 4\left(u^2 + \frac{1-\nu}{2}\right)\lambda^2\xi J_{1,0} Y_0'\right] + \qquad (22) \\
+ [(1+u^2)^2 J_{0,2} Y_1'' - 2(1+u^2)\,u\lambda(J_{1,1} + \xi J_{0,2}) Y_1']'' - \\
- [-2(1+u^2)\,u\lambda(J_{1,1} - \xi J_{0,2}) Y_1'' + 4\left(u^2 + \frac{1-\nu}{2}\right)\lambda^2(\xi^2 J_{0,2} + \\
+ 2\xi J_{1,1} + J_{2,0}) Y_1'] = -l^3 \int_0^1 p_1\,dy
\end{gathered}$$

and the boundary conditions

$$\begin{gathered}
Y_0(0) = Y_0'(0) = 0, \\
Y_1(0) = Y_1'(0) = 0, \\
[(1+u^2) J_{0,0} Y_0'' - 2u\lambda(\xi J_{0,0} Y_0' + J_{1,0} Y_1')]_{y=1} = 0, \qquad (23) \\
[(1+u^2) J_{0,2} Y_1'' - 2u\lambda(\xi J_{0,2} + J_{1,1}) Y_1']_{y=1} = 0.
\end{gathered}$$

If we calculate the components of the vectors which are normal to the longitudinal sides $b_1(y)$ and $b_2(y)$ for $N=1$ by formulas (18) and (19) then it turns out that they differ from zero when such components on the free side, naturally, should be absent. This is a result of the basic assumption of the simplified theory which ignores the boundary conditions on the longitudinal sides.

By putting $\psi_0 = \psi_1$ into equations (22) and (23) we obtain the relations which were obtained in reference [8]. In the particular case when $\psi_0 = \psi_1 = 0$, $u = 0$, and $\beta = 0$. From equations (22)-(23) for the plate with rigid tranverse cross-sections ($N=1$), we find the bending equations

$$(J_{0,0} Y_0'')' = P \int_y^l p_0\,dy, \qquad (24)$$

$$Y_0(0) = Y_0'(0) = Y_0''(l) = 0$$

and the equation of limited twisting

$$(J_{0,2} Y_1'')' - 2(1-\mu) J_{2,0} Y_1' = P \int_y^l p_1\,dy \qquad (25)$$

where

$$Y_1(0) = Y_1'(0) = Y_1''(l) = 0,$$

$$J_{0,0} = \frac{E}{12(1-\mu^2)} \int_{b_1(y)}^{b_2(y)} h^3\,dx$$

and

$$J_{0,2} = \frac{E}{12(1-\mu^2)} \int_{b_1(y)}^{b_2(y)} h^3 x^2\,dx$$

with accuracy up to the constant factor $\frac{1}{1-\mu^2}$ conform respectively to the moment of inertia of the cross-section relative to the x-axis and the sectorial moment of inertia of the section of plate. The product is

$$2(1-\mu) J_{2,0} = \frac{2(1-\mu)}{12(1-\mu^2)} E \int_{b_1(y)}^{b_2(y)} h^3\,dx = GT,$$

where G is the modulus of elasticity of the second type, T is the geometric rigidity of the tranverse cross-section in twist.

The separation of the systems (22) and (23) into two mutually independent groups of equations (24)-(25) mathematically indicates the origin of the axis of rigidy coincides with the y-axis. In this case condition (17) gives the location of the center of rigity of the shape.

Applying the simplified theory of plates ($N=1$) in the problem of free vibration gives a system of homogeneous integral equations describing the bending-torsional vibrations. These equations may be obtained from equations (22) and (23) by replacing p_0 and p_1 by c_0 and c_1.

From formula (21) we find

$$c_0 = \omega^2\rho(f_0 Y_0 + f_1 Y_1)$$

and

$$c_1 = \omega^2\rho(f_1 Y_0 + f_2 Y_1).$$

In the case when $\beta = u = J_{a,i}=0$ and the shape of the tranverse cross-section of the plate has two axes of symmetry, the bending and torsional vibrations are mutually independent, and the system of equations breaks down into the equations for bending vibrations

$$(J_{0,0}Y_0'')'' + \omega^2\rho l^3 \int_y^1 f_0 Y_0\,dy = 0,$$

$$Y_0(0) = Y_0'(0) = Y_0''(1) = 0.$$

and for torsional oscillations taking account of the limited torsional deformation

$$(J_{0,2}Y_1'')' - 2(1-\mu)J_{2,0}Y_1' + \omega^2\rho l^3 \int_y^1 f_2 Y_1\,dy = 0,$$

$$Y_1(0) = Y_1'(0) = Y_1''(1) = 0.$$

As can be seen from examination of the general equations, they are quite complex. In the general case they contain variable coefficients and, even when $N=1$, do not submit to exact integration. Their solution may be found by numerical methods, the possibilities of which increase with the use of high-capacity computers. By comparison with the numerical solution of the well-known equation of bending in partial derivative form, the solution given by the expansion (9) is preferable. It is more easily subjected to analysis since it clearly includes the "bar" deformations Y_0 and Y_1.

In a series of cases of practical importance for the calculation of static displacements and also the frequencies and form of the free vibrations the simplified equations (N=1) may be used.

For calculations on simply shaped cantilever plates, for example, calculations on plates with constant thickness having the shape of parallelograms, the simplified equations integrate exactly.

BIBLIOGRAPHY

1. Timoshenko, S.P., Plates and membranes, Gostekhizdat, M.-L., 1948.

2. Bisplinghoff, R.L., Eshli, Kh., Khalfmen, R.L., Aeroelasticity, IL-M., 1958

3. Parkhomovskiy, Ya.M., Frolov, V.M., The influence of torsional restraints on the frequencies and forms of torsional oscillations of beams, Trudy TsAGI im. N.E. Zhukobskovo vyp. 733, 1959.

4. Vakhitov, M.B., Calculations of the stability of tapered wings of monolithic construction, IVUZ, "Aviatsionnaya Tekhika", No 1, 1958.

5. Varvak, P.M., Development and application of the grid method in calculations for plates, Ch. II, AN USSR, 1952.

6. Stein, M., Anderson, I.E., and Hedgepeth, J.M., Deflections and stress analysis of thin solid solid wings of arbitrary plan form with particular reference to delta wings, N.A.C.A. Rept. 1134.

7. Barton, M.V., Vibration of rectangular and skew plates, Journ. of Appl. Mech., June 1951.

8. Williams, M.L., A review of certain analysis methods for swept-wing structures, J.A.S. Vol. 19, No. 9, September, 1952.

9. Holl, D.L., Cantilever plate with concentrated edge load, Trans. ASME, vol. 59, 1937.

10. Nash, W.A., Washington, D.C., Several approximate analyses of the bending of a rectangular cantilever plate by uniform normal pressure, Journ. of Appl. Mech., Vol. 19, No. 1, 1952.

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