

MEMORANDUM
RM-3505-ARPA
FEBRUARY 1963

THE PROBABILITY DISTRIBUTION OF
ANTI-MISSILE MISSILE MISS DISTANCE
DUE TO OBSERVATION AND
GUIDANCE NOISE

W. B. Kendall

PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

The **RAND** *Corporation*
SANTA MONICA • CALIFORNIA

MEMORANDUM
RM-3505-ARPA
FEBRUARY 1963

**THE PROBABILITY DISTRIBUTION OF
ANTI-MISSILE MISSILE MISS DISTANCE
DUE TO OBSERVATION AND
GUIDANCE NOISE**

W. B. Kendall

This research is supported and monitored by the Advanced Research Projects Agency under Contract No. SD-79. Any views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of ARPA. Permission to quote or reproduce must be obtained from The RAND Corporation.

PREFACE

The study presented in this Memorandum stems from a problem encountered in setting accuracy specifications for the observing instruments of a midcourse intercept system intended to operate against submarine-launched ballistic missiles. The result given here is much more general than the problem stated, and can be applied to a wide variety of intercept situations. The study is part of RAND's continuing work on the interception of submarine-launched and air-launched ballistic missiles.

v

SUMMARY

In this Memorandum we consider the problem of intercepting an attacking ballistic missile (BM) with an anti-missile missile (AMM) when Gaussian noise corrupts both the observations of the BM and the guidance system of the AMM.

It is assumed that the position of the BM at some future time is estimated from a set of (possibly correlated) measurements (of any type) made on it, and that the error in this position estimate is related to the errors in the measurements by the usual linear relation

$$x_i = \sum_j \frac{\partial w_i}{\partial m_j} e_j$$

where x_i is the error in the i -th special coordinate w_i , and e_j is the error in the j -th measurement m_j . Furthermore, it is assumed that the noise in the AMM guidance system is such that the uncertainty in the position of the AMM at intercept has a general three-dimensional Gaussian distribution. For these general assumptions the probability that the miss distance between the AMM and the BM will be less than any given amount is determined and shown in several curves. In the course of this calculation the cumulative distribution function for the generalized chi-squared distribution in any number of dimensions is obtained.

For the more special case of uncorrelated measurements and AMM position uncertainty with spherical symmetry (this might be the case when, for example, only the rms miss distance of the AMM rather than its complete three-dimensional distribution is known) coefficients

are obtained which indicate how sensitive the miss distance is to incremental changes in the accuracies of any of the measurements. With these coefficients a system designer can systematically determine how much money, weight, and equipment to allot to making each of the measurements. A numerical example is given which demonstrates this procedure.

CONTENTS

PREFACE	iii
SUMMARY	v
LIST OF FIGURES	ix
LIST OF SYMBOLS	xi
Section	
I. INTRODUCTION	1
II. MATHEMATICAL FORMULATION	2
III. THE CHARACTERISTIC FUNCTION OF r^2	4
IV. THE PROBABILITY DENSITY FUNCTION OF r^2	5
V. THE CUMULATIVE DISTRIBUTION FUNCTION OF r^2	10
VI. APPLICATION TO THE PROBLEM AT HAND	15
VII. SYSTEM OPTIMIZATION	17
VIII. EXAMPLE	18
REFERENCES	23

LIST OF FIGURES

1.	Cumulative probability distribution of r ; linear scale	11
2.	Cumulative probability distribution of r ; Rayleigh versus linear scale	12
3.	Cumulative probability distribution of r ; log-Rayleigh versus log scale	13
4.	Cumulative probability distribution of r^2 ; log-Rayleigh versus log scale	14

SYMBOLS

a	arbitrary constant taken equal to $(v_1/\sigma)^2 - 1$
a_i	set of coefficients (Sec. VI)
b	arbitrary constant taken equal to σ^2/v_1
b_i	set of coefficients
c_i	coefficients of a Laguerre series
D	sensitivity coefficient matrix
det	determinant
F_{r^2}	cumulative distribution function of r^2
$f(r^2)$	probability density function of r^2
I	unit matrix
$I(u,p)$	incomplete gamma-function
j	imaginary operator = $\sqrt{-1}$
k_i	cumulants of a probability distribution
$L_i^a(t)$	generalized Laguerre polynomials
M	covariance matrix of m
M_i	mean-squared value of the error in the i-th measurement
M_q	$(2\pi)^{-q/2} (\det Y)^{-1/2}$
m	column matrix of measurements made
m_i	i-th measurement
n	number of measurements made
q	dimensionality of the space; usually either 2 or 3
r	miss distance between the BM and the AMM
tr	trace operator

v	argument of a characteristic function
w	column matrix of estimated BM rectangular coordinates
w_i	i -th estimated BM rectangular coordinate
X	covariance matrix of x
x	column matrix of errors in w
x_i	error in w_i
Y	covariance matrix of y
y	$x + z$; the column matrix of rectangular coordinates of the difference in positions of the BM and the AMM
z	column matrix of rectangular coordinates of the guidance-noise-induced AMM position error
Z	covariance matrix of z
z_i	i -th coordinate of the guidance-noise-induced position error of the AMM
Γ	gamma-function (complete)
ζ	mean squared miss distance due to position error of the AMM
μ_i	i -th characteristic root of Y
ν_i	i -th moment about the origin of $f(r^2)$
σ	$(\nu_2 - \nu_1^2)^{1/2}$
$\sqrt{\sigma}$	$(\nu_2 - \nu_1^2)^{1/4} = \sqrt[4]{\text{average of } r^4}$
$\varphi(v)$	characteristic function of $f(r^2)$

I. INTRODUCTION

Considerable interest has been shown⁽¹⁾ in the following problem: A set of measurements is made on an attacking ballistic missile (BM), and based on these measurements, its position at some future time t is estimated. Due to Gaussian-distributed errors in these measurements, the error in the position estimate has a Gaussian distribution. An anti-missile missile (AMM) is aimed to intercept the BM at time t . However, due to noise in its guidance system, its position at time t is in error by a random amount which has a Gaussian distribution. Determine the probability that the distance between the BM and the AMM at time t is less than some magnitude k .

In this Memorandum we attack and obtain a good approximate solution for this problem. Of even more importance, in the solution obtained the effects of AMM guidance noise, of the tactical geometry, and of the measurement accuracies, can be separated. Thus we have a simple method for determining the change in kill probability due to changes in any of these variables.

II. MATHEMATICAL FORMULATION

We will use matrix notation throughout. A superscript T will denote the transpose.

Let the position estimate of the BM be denoted by the set of rectangular coordinates $\{w_i\} = w$. Each of these is in general a function of the set of measurements $\{m_j\} = m$. If the m_j have a jointly Gaussian distribution with covariance matrix* M , then to what is usually a good approximation, the errors in w (call them $\{x_i\} = x$) are jointly Gaussian with covariance matrix** $X = DMD^T$. Here D is the "sensitivity coefficient matrix" given by $D = \{D_{ij}\}$ where

$$D_{ij} = \frac{\partial w_i}{\partial m_j}$$

Denote by $\{z_i\} = z$ the guidance-noise induced errors in the rectangular coordinates of the AMM at time t . These errors are assumed to be jointly Gaussian random variables with zero means and covariance matrix Z . Now, since the AMM was aimed at point w , its position at time t is $w + z$. The actual position of the BM at time t is $w - x$ (i.e., the estimated position equals the actual position plus the error).

*The i, j -th element of the covariance matrix is the expected value of $e_i e_j$ where e_k is the k -th random variable. It is a symmetric matrix.

**This is because it is usually satisfactory to make the linear approximation $x = De$ where $e = \{e_k\}$ is the matrix of measurement errors and

$D = \{D_{ij}\} = \left\{ \frac{\partial w_i}{\partial m_j} \right\}$ is the sensitivity coefficient matrix. Then one has

$X = E(xx^T) = E(Dee^T D^T) = D[E(ee^T)]D^T = DMD^T$ where E is the operator which replaces each element of a matrix by its expected value.

Thus the rectangular coordinates of the difference in their positions is $x + z = y$, which is Gaussian with zero mean and, assuming statistical independence of w and z , covariance matrix $X + Z = Y$.

The square of the distance between the BM and the AMM is

$$r^2 = \sum_i (x_i + z_i)^2 = y^T y$$

Our problem then is to calculate the probability that $y^T y \leq k^2$ for any k , given that the probability density of y is

$$f(y) = M_q \exp \left\{ -\frac{1}{2} y^T Y^{-1} y \right\}$$

where

$$M_q = (2\pi)^{-q/2} (\det Y)^{-1/2}$$

and q is the dimensionality of the space under consideration. Here we are interested in $q = 2$ or 3 . The quantity r^2 has the so-called "generalized chi-squared" distribution, and has been treated by many authors, most notably perhaps, by J. I. Marcum.⁽²⁾ We will make much use of this work of Marcum's.

III. THE CHARACTERISTIC FUNCTION OF r^2

The characteristic function of r^2 , denoted by $\varphi(v)$, is given by*

$$\begin{aligned}
 \varphi(v) &= \int_{-\infty}^{\infty} \exp \{ jvr^2(y) \} f(y) dy \\
 &= M_q \int_{-\infty}^{\infty} \exp \left\{ jv y^T y - \frac{1}{2} y^T Y^{-1} y \right\} dy \\
 &= (2\pi)^{-q/2} (\det Y)^{-1/2} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} y^T (Y^{-1} - 2jvI) y \right\} dy \\
 &= \left[\det (I - 2jvY) \right]^{-1/2} \\
 &= \left[\prod_{i=1}^q (1 - 2jv \mu_i) \right]^{-1/2}
 \end{aligned}$$

where the μ_i are the real, positive characteristic roots of the positive definite covariance matrix Y . They are the solutions of the equation

$$\det (Y - \mu I) = 0$$

where "det" denotes the determinant.

*By $\int_{-\infty}^{\infty} dy$ we mean $\int_{-\infty}^{\infty} dy_1 \dots \int_{-\infty}^{\infty} dy_q$. In the following treat-

ment I is the unit matrix which has ones on its main diagonal and zeros elsewhere.

IV. THE PROBABILITY DENSITY FUNCTION OF r^2

The probability density function of r^2 can be found by taking the inverse Fourier transform of $\varphi(v)$.

$$f(r^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jvr^2} \varphi(v) dv$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-jvr^2}}{[(1 - 2jv\mu_1) \dots (1 - 2jv\mu_q)]^{1/2}} dv$$

When $q = 2$ (the plane case) this can be integrated in closed form to give*

$$f(r^2) = \frac{1}{2} (\mu_1 \mu_2)^{-1/2} \exp \left\{ -\frac{1}{4} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) r^2 \right\} I_0 \left[\frac{1}{4} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) r^2 \right] \quad r^2 > 0$$

For $q > 2$ the integral cannot be evaluated in closed form, but following Marcum we can expand $f(r^2)$ in a Laguerre series, the coefficients of which can be expressed in terms of the moments of $f(r^2)$. These can easily be determined directly from $\varphi(v)$. The Laguerre series is

$$f(r^2) = \sum_{i=0}^{\infty} c_i e^{-t} t^a L_i^a(t) \quad t > 0$$

where

$$t = r^2/b$$

$L_i^a(t)$ are the generalized Laguerre polynomials and a and b are arbitrary constants.

* For the integration, see Ref. 3, pair 555.

By multiplying both sides of this equation by $L_j^a(t)$, integrating from zero to infinity, and using the orthogonality property of the Laguerre polynomials, we obtain the coefficients c_i as functions of the moments of $f(r^2)$ and the arbitrary constants a and b . By choosing

$$a = \frac{v_1^2}{v_2 - v_1^2} - 1 = \frac{v_1^2}{\sigma^2} - 1$$

and

$$b = \frac{v_2 - v_1^2}{v_1} = \frac{\sigma^2}{v_1}$$

where v_i is the i -th moment about the origin of $f(r^2)$, we get

$$c_0 = \frac{1}{b \Gamma(a+1)} = \frac{v_1}{\sigma^2 \Gamma(v_1^2/\sigma^2)}$$

$$c_1 = c_2 = 0$$

and

$$c_3 = \frac{1}{b \Gamma(a+4)} \left[(a+3) \frac{v_2}{b^2} - \frac{v_3}{b^3} \right]$$

In evaluating these coefficients we will make use of the fact that the moments v_i are related to the cumulants k_i by

$$v_1 = k_1$$

$$v_2 = k_2 + k_1^2$$

$$v_3 = k_3 + 3k_1k_2 + k_1^3$$

⋮

The cumulants are defined by

$$\ln \varphi(v) = \sum_{i=1}^{\infty} \frac{(jv)^i k_i}{i!}$$

Now, since

$$\varphi(v) = [\det (I - 2jvY)]^{-1/2}$$

we can write

$$\ln \varphi(v) = -\frac{1}{2} \ln \det (I - 2jvY)$$

Next we make use of the identity*

$$\det (I - 2jvY) = \exp \left\{ - \sum_{i=1}^{\infty} \frac{(2jv)^i}{i} \operatorname{tr} Y^i \right\}$$

where

$$\operatorname{tr} Y^i = \sum_{m=1}^q (\mu_m)^i$$

to obtain

$$\ln \varphi(v) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{(2jv)^i}{i} \operatorname{tr} Y^i$$

(Here tr denotes the trace operator. Besides this representation, we also have $\operatorname{tr} Y$ equals the sum of the diagonal elements of Y .)

* This representation is valid for all $v < 1/(2\mu_g)$ where μ_g is the largest characteristic root of Y .

Equating coefficients of v^i in the two expressions for $\ln \phi(v)$ we see that here

$$k_1 = 2^{i-1} (i-1)! \operatorname{tr} Y^i$$

and that

$$v_1 = \operatorname{tr} Y$$

$$v_2 = 2 \operatorname{tr} Y^2 + (\operatorname{tr} Y)^2$$

$$v_3 = 8 \operatorname{tr} Y^3 + 6 \operatorname{tr} Y \operatorname{tr} Y^2 + (\operatorname{tr} Y)^3$$

These can be used to evaluate the coefficients in the Laguerre series. However, as will be seen below, a satisfactory approximation to $f(r^2)$ is obtained even when c_3 and the higher coefficients are neglected.

Consider the case where Y has h equal characteristic roots which are much larger than its other roots. Then

$$\operatorname{tr} Y^i \approx h \mu_g^i$$

where μ_g is the value of the h largest characteristic roots. This results in

$$a = \frac{h}{2} - 1$$

$$b = 2 \mu_g$$

$$c_0 = \frac{1}{2 \mu_g \Gamma(h/2)}$$

and

$$c_i = 0 \quad i > 0$$

Thus, in this case the c_i for $i > 0$ are not only negligible, but are identically zero. For $f(r^2)$ we obtain then the chi-squared distribution with h degrees of freedom.

If two unequal characteristic roots dominate the others, then we have

$$\frac{c_3}{c_0} = \frac{-s(1-s)^2}{(15/8) + 2s + (17/4)s^2 + 2s^3 + (15/8)s^4}$$

where s is the ratio of the two largest roots. This function has a maximum of about 0.05 near $s = 0.27$. Therefore, for our purposes we can neglect the c_i for $i > 0$. This results in

$$f(r^2) = \frac{v_1}{\sigma^2 \Gamma(v_1/2)} \left[\frac{v_1 r^2}{\sigma^2} \right]^{(v_1/2)-1} \exp \left\{ -\frac{v_1 r^2}{\sigma^2} \right\} \quad r^2 > 0$$

which is of the form of the chi-squared distribution with $2(v_1/\sigma)^2$ degrees of freedom. It is easy to show that

$$1 \leq 2(v_1/\sigma)^2 = \frac{(\text{tr } Y)^2}{\text{tr } Y^2} \leq q$$

V. THE CUMULATIVE DISTRIBUTION FUNCTION OF r^2

The cumulative distribution function of r^2 is defined by

$$F_{r^2}(k^2) = \text{Prob. } (r \leq k) = \text{Prob. } (r^2 \leq k^2) = \int_{-\infty}^{k^2} f(r^2) d(r^2)$$

Using the above result for $f(r^2)$ this becomes

$$F_{r^2}(k^2) = \frac{v_1}{\sigma^2 \Gamma(v_1^2/\sigma^2)} \int_0^{k^2} \left[\frac{v_1 z}{\sigma^2} \right]^{(v_1/\sigma)^2 - 1} \exp\left\{-\frac{v_1 z}{\sigma^2}\right\} dz$$

This is of the form of the incomplete gamma-function which has been tabulated by Pearson⁽⁴⁾ and Kendall⁽⁵⁾ and is defined as

$$I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} e^{-v} v^p dv \quad \begin{array}{l} p \geq -1 \\ u \geq 0 \end{array}$$

Therefore, using this notation we can write*

$$F_{r^2}(k^2) = I\left[\frac{k^2}{\sigma}, \left(\frac{v_1}{\sigma}\right)^2 - 1\right]$$

where $v_1 = \text{tr } Y = \text{mean squared miss distance}$

and $\sigma^2 = 2 \text{tr } Y^2$

$F_{r^2}(k^2)$ is plotted in Figs. 1 thru 4. For the convenience of the reader the same function is shown on several different scales.

*If the $c_i, i > 0$, are not neglected, then clearly F_{r^2} can be represented exactly as a sum of incomplete gamma-functions.

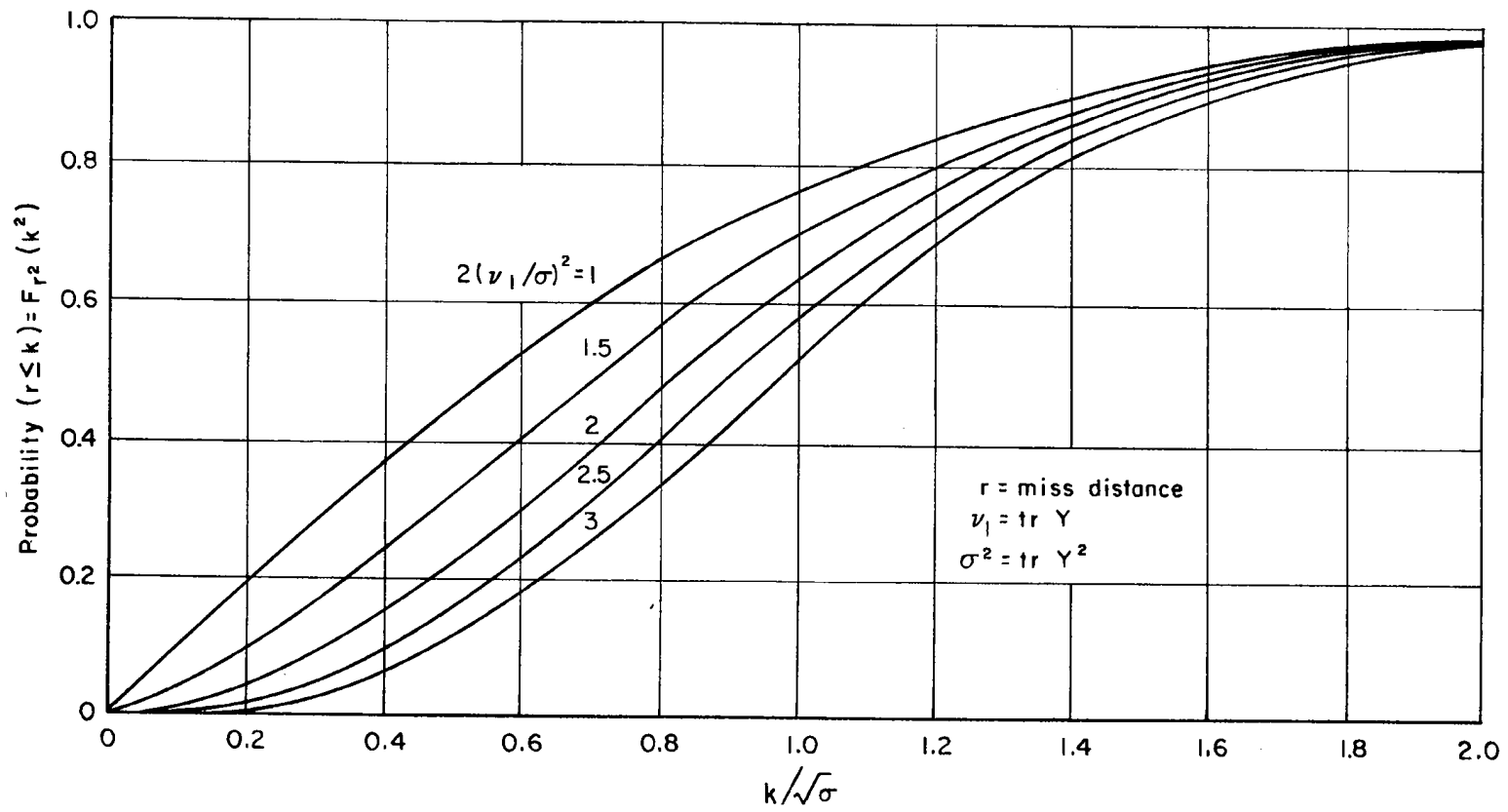


Fig. 1—Cumulative probability distribution of r; linear scale

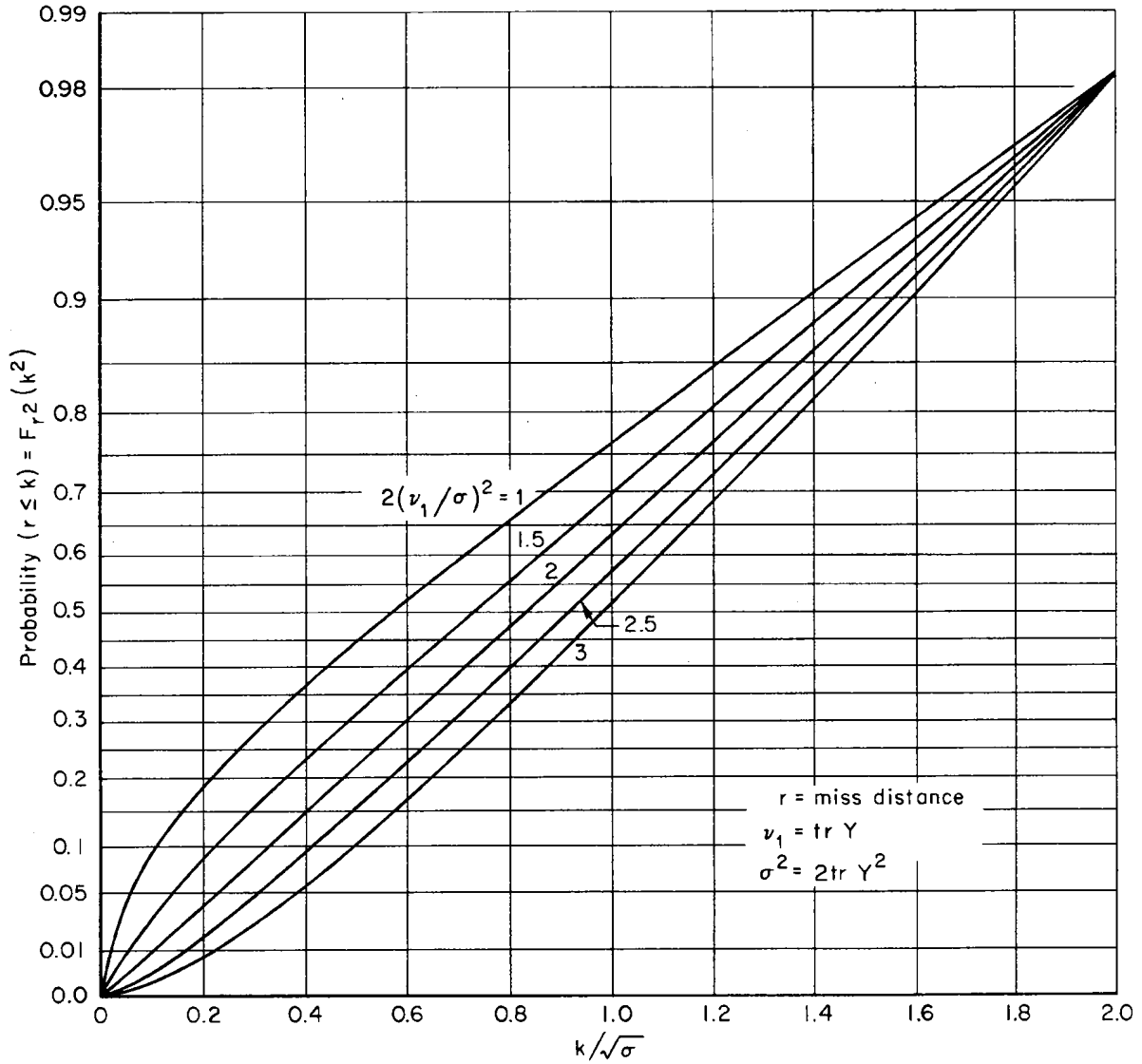


Fig. 2—Cumulative probability distribution of r ;
Rayleigh versus linear scale

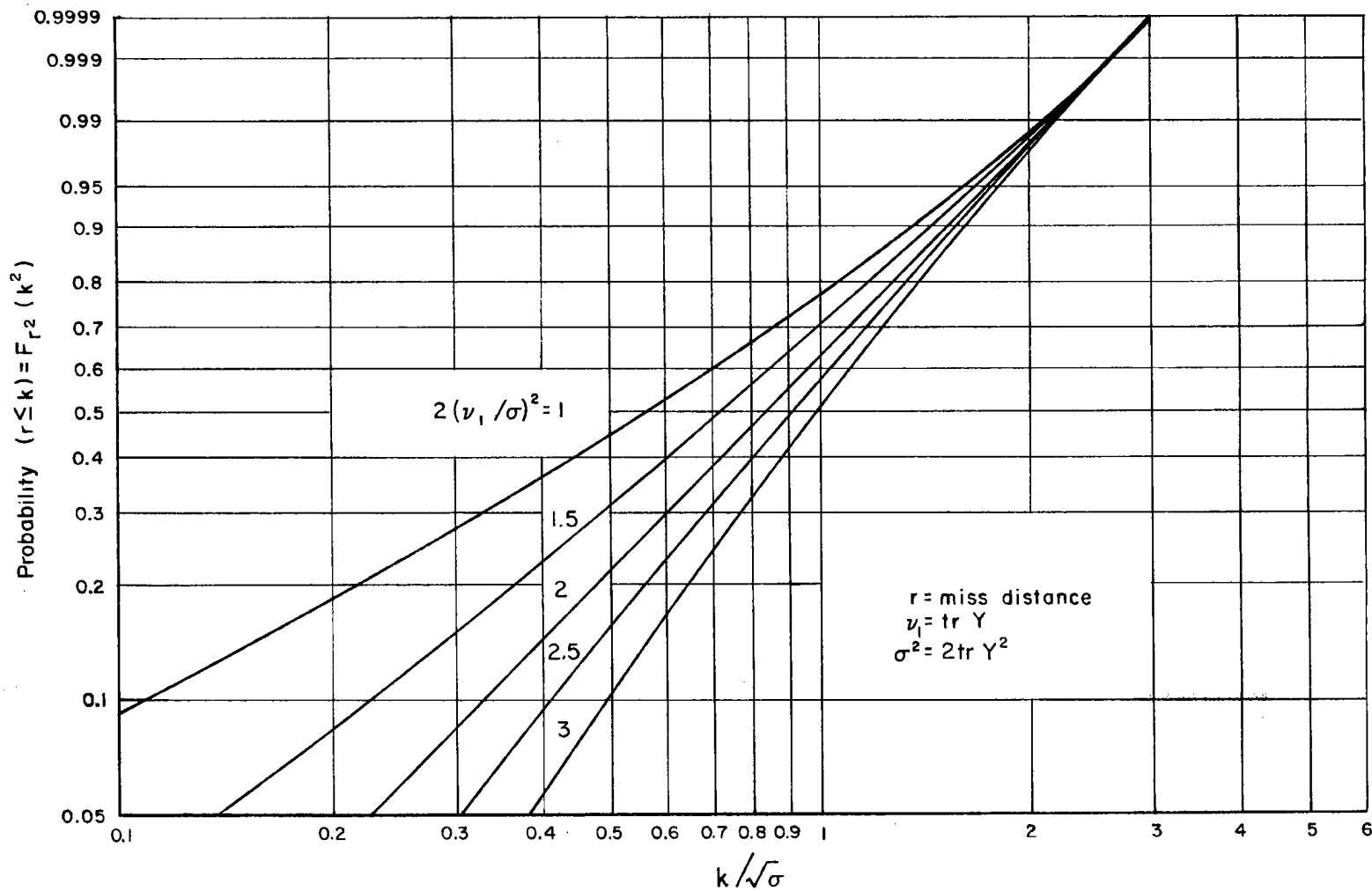


Fig.3—Cumulative probability distribution of r ; log-Rayleigh versus log scale

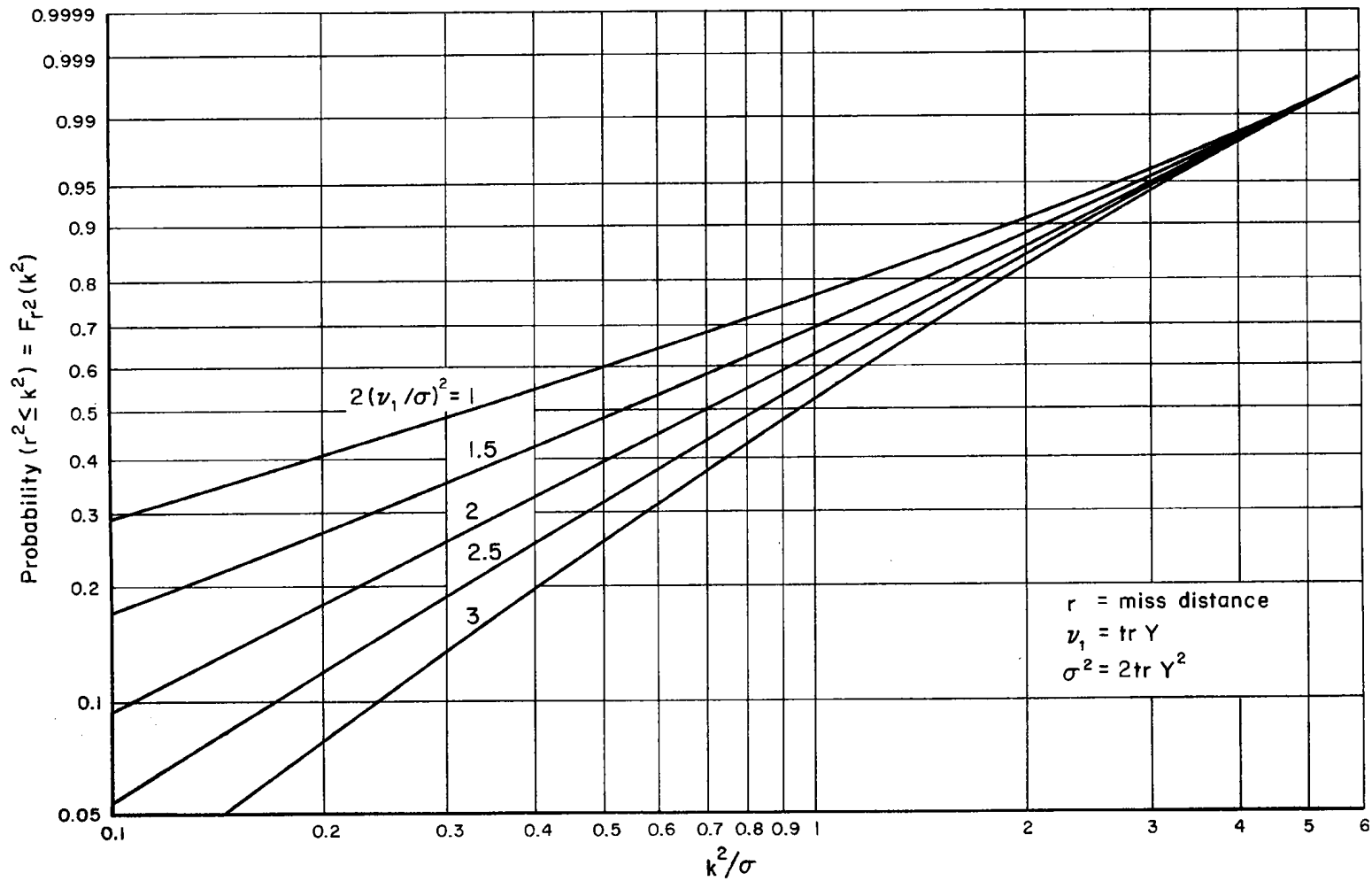


Fig. 4—Cumulative probability distribution of r^2 , log-Rayleigh versus log scale

VI. APPLICATION TO THE PROBLEM AT HAND

In order to determine v_1 and σ we must know $\text{tr } Y$ and $\text{tr } Y^2$ where

$$Y = DMD^T + Z$$

Since Y is at most a 3×3 matrix, this is a simple and straightforward calculation even in the most general case. However, many times two simplifications may be made, and we will pursue these.

If the measurements made on the ballistic missile are uncorrelated, then M is diagonal and can be written

$$M = \begin{bmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_n \end{bmatrix}$$

Here M_i is the mean squared value of the error in the i -th measurement. There are n measurements in all. If further, the noise in the AMM guidance is such that the uncertainty in the position of the AMM at time t has spherical symmetry (e.g., when only the rms value of the AMM miss distance is known), then we can write

$$Z = (\zeta/q)I$$

where ζ is the mean squared miss distance due to AMM guidance noise. Using these substitutions and the properties of the trace operator* we obtain

$$\text{tr } Y = \zeta + \text{tr } D^T D M$$

and

$$\text{tr } Y^2 = \zeta^2/q + (2\zeta/q) \text{tr } D^T D M + \text{tr } [(D^T D M)^2]$$

* $\text{tr } ABC = \text{tr } CAB$ and $\text{tr } (A + B) = \text{tr } A + \text{tr } B$.

These can be further simplified to

$$\text{tr } Y = \zeta + \sum_{i=1}^n A_i M_i$$

and

$$\text{tr } Y^2 = \zeta^2/q + (2\zeta/q) \sum_{i=1}^n A_i M_i + \sum_{i=1}^n \sum_{j=1}^n B_{ij} M_i M_j$$

where

$$B_{ij} = \left(\sum_{k=1}^q D_{ki} D_{kj} \right)^2 = B_{ji}$$

$$A_i = \sum_{j=1}^q D_{ji}^2 = \sqrt{B_{ii}}$$

and D_{ji} is the j,i -th element of the sensitivity coefficient matrix, given by

$$D_{ji} = \frac{\partial w_j}{\partial m_i}$$

Here w_j is the j -th position-estimate rectangular coordinate and m_i is the i -th measurement. If we now arrange the M_i into a column matrix $\mathbf{m} = \{M_i\}$, the A_i into a column matrix $\mathbf{A} = \{A_i\}$, and the B_{ij} into an $n \times n$ matrix $\mathbf{B} = \{B_{ij}\}$, then we can write

$$\text{tr } Y = q\zeta + \mathbf{A}^T \mathbf{m}$$

and

$$\text{tr } Y^2 = q\zeta^2 + 2\zeta \mathbf{A}^T \mathbf{m} + \mathbf{m}^T \mathbf{B} \mathbf{m}$$

Since \mathbf{A} and \mathbf{B} are easily determined from the sensitivity coefficients, once these coefficients are determined it is easy to calculate $v_1 = \text{tr } Y$ and $\sigma = (2 \text{tr } Y^2)^{1/2}$. Then any of Figs. 1 thru 4 can be used to obtain the probability that $r \leq k$ as a function of k .

VII. SYSTEM OPTIMIZATION

Often the geometry and the kind of measurements to be made are difficult, if not impossible, for a system designer to change. When this is the case the sensitivity coefficients and the resulting A and B matrices are fixed. The designer must then decide how much money, weight, and equipment to allot to making the various measurements accurately.

From the figures it is evident that the most important factor in increasing the probability that the AMM comes within a given distance of the BM is the reduction of σ . Accordingly, an important quantity is the sensitivity of σ to changes in the M_j . Since it is usually the relative size of $\sqrt{\sigma}$ and relative sizes of the RMS measurement errors which are of interest, we will let

$$S_i = \frac{d(\log \sqrt{\sigma})}{d(\log \sqrt{M_i})} = \frac{M_i}{\sigma} \times \frac{d\sigma}{dM_i} = \frac{M_i}{\text{tr } Y^2} \times \frac{d(\text{tr } Y^2)}{dM_i}$$

Now, if the S_i are arranged in a column matrix $S = \{S_i\}$, we have

$$S = \frac{1}{\text{tr } Y^2} [2\zeta MA + 2MBm]$$

Denoting by $\Delta\sqrt{\sigma}$ the change in $\sqrt{\sigma}$ caused by a change of $\Delta\sqrt{M_i}$ in $\sqrt{M_i}$, we can write, using a linear extrapolation,

$$\frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}} \approx S_i \frac{\Delta\sqrt{M_i}}{\sqrt{M_i}}$$

Thus, once the S matrix is determined, the relative importance of changes in the accuracies of the various measurements will be evident. The designer can then proceed in a systematic way to make changes which will optimize the system.

VIII. EXAMPLE

For an example we consider the set of measurements given in Table 1. For a particular geometry the sensitivity coefficient matrix for these

Table 1
MEASUREMENTS AND THEIR ACCURACIES

Measurement	RMS Error	M_i
Range, R	100 ft	10,000 ft ²
Azimuth angle, A	50 μ rad	2,500 (μ rad) ²
Elevation angle, E	50 μ rad	2,500 (μ rad) ²
Range rate, \dot{R}	10 ft/sec	100 (ft/sec) ²
Azimuth angle rate, \dot{A}	5 μ rad/sec	25 (μ rad/sec) ²
Elevation angle rate, \dot{E}	5 μ rad/sec	25 (μ rad/sec) ²

measurements has been calculated in Ref. 1 (Case 1B, p. 82) to be

$$D = \begin{bmatrix} \text{ft/ft} & \text{ft}/\mu \text{ rad} & \text{ft}/\mu \text{ rad} & \text{ft/ft/sec} & \text{ft}/\mu \text{ rad/sec} & \text{ft}/\mu \text{ rad/sec} \\ 1.37 & -2.91 & -3.06 & 2.78 \times 10^2 & 6.87 \times 10^2 & -1.28 \times 10^2 \\ 1.63 \times 10^{-1} & 1.87 & 2.84 & -2.88 \times 10^2 & 6.72 \times 10^2 & -8.29 \times 10 \\ 2.21 & -3.36 \times 10^{-7} & -1.91 \times 10^{-1} & 2.04 \times 10 & 1.49 \times 10^2 & 1.07 \times 10^3 \end{bmatrix}$$

R A E \dot{R} \dot{A} \dot{E}

Using a simple digital-computer program we obtain from this

$$A = \begin{bmatrix} 6.79 \\ 1.20 \times 10 \\ 1.75 \times 10 \\ 1.61 \times 10^5 \\ 9.45 \times 10^5 \\ 1.17 \times 10^6 \end{bmatrix}$$

$$\text{end}$$

$$B = \begin{bmatrix} 4.61 \times 10 & 1.36 \times 10 & 1.72 \times 10 & 1.44 \times 10^5 & 1.90 \times 10^6 & 4.73 \times 10^6 \\ 1.36 \times 10 & 1.43 \times 10^2 & 2.02 \times 10^2 & 1.82 \times 10^6 & 5.51 \times 10^5 & 4.73 \times 10^4 \\ 1.72 \times 10 & 2.02 \times 10^2 & 3.05 \times 10^2 & 2.80 \times 10^6 & 4.94 \times 10^4 & 2.32 \times 10^3 \\ 1.44 \times 10^5 & 1.82 \times 10^6 & 2.80 \times 10^6 & 2.58 \times 10^{10} & 2.40 \times 10^5 & 1.02 \times 10^8 \\ 1.90 \times 10^6 & 5.51 \times 10^5 & 4.94 \times 10^4 & 2.40 \times 10^5 & 8.94 \times 10^{11} & 2.49 \times 10^8 \\ 4.73 \times 10^6 & 4.73 \times 10^4 & 2.32 \times 10^3 & 1.02 \times 10^8 & 2.49 \times 10^8 & 1.36 \times 10^{12} \end{bmatrix}$$

We will assume the miss distance due to AMM guidance noise to have a spherical distribution and will consider three different values for its rms value: 0, 1000 ft, and 10,000 ft. Thus, we have $\zeta = 0$, 10^6 , and 10^8 . This results in

$$\text{tr } Y = \begin{cases} 6.91 \times 10^7 \text{ ft}^2 & \zeta = 0 \\ 7.01 \times 10^7 \text{ ft}^2 & \zeta = 10^6 \text{ ft}^2 \\ 1.69 \times 10^8 \text{ ft}^2 & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$\text{tr } Y^2 = \begin{cases} 1.68 \times 10^{15} \text{ ft}^4 & \zeta = 0 \\ 1.72 \times 10^{15} \text{ ft}^4 & \zeta = 10^6 \text{ ft}^2 \\ 9.61 \times 10^{15} \text{ ft}^4 & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$2(v_1/\sigma)^2 = \begin{cases} 2.84 & \zeta = 0 \\ 2.85 & \zeta = 10^6 \text{ ft}^2 \\ 2.97 & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$\sqrt{\sigma} = \begin{cases} 7610 \text{ ft} & \zeta = 0 \\ 7662 \text{ ft} & \zeta = 10^6 \text{ ft}^2 \\ 11,780 \text{ ft} & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$\text{and } S = \begin{array}{ccc} \zeta = 0 & \zeta = 10^6 & \zeta = 10^8 \\ \left[\begin{array}{c} 1.08 \times 10^{-3} \\ 2.95 \times 10^{-4} \\ 4.21 \times 10^{-4} \\ 1.55 \times 10^{-1} \\ 3.34 \times 10^{-1} \\ 5.10 \times 10^{-1} \end{array} \right] , & \left[\begin{array}{c} 1.08 \times 10^{-3} \\ 2.98 \times 10^{-4} \\ 4.27 \times 10^{-4} \\ 1.57 \times 10^{-1} \\ 3.34 \times 10^{-1} \\ 5.07 \times 10^{-1} \end{array} \right] , & \left[\begin{array}{c} 6.59 \times 10^{-4} \\ 2.59 \times 10^{-4} \\ 3.76 \times 10^{-4} \\ 1.38 \times 10^{-1} \\ 2.22 \times 10^{-1} \\ 2.91 \times 10^{-1} \end{array} \right] \begin{array}{l} R \\ A \\ E \\ \cdot \\ R \\ \cdot \\ A \\ \cdot \\ E \end{array} \end{array}$$

From the curves we can see now that for these values of $\sqrt{\sigma}$ the probability is 0.95 that the miss distance will be less than

$$\begin{array}{ll} 13,500 \text{ ft} & \text{for } \sqrt{\zeta} = 0 \\ 13,500 \text{ ft} & \text{for } \sqrt{\zeta} = 1,000 \text{ ft}, \\ \text{and} & 20,600 \text{ ft} \text{ for } \sqrt{\zeta} = 10,000 \text{ ft}. \end{array}$$

From the S matrices we note that $\sqrt{\sigma}$ is about 100 times as sensitive to small changes in the accuracies of the rate measurements as it is to changes in the accuracies of the other measurements. Accordingly, we consider making the rate measurements twice as accurately and making the others only half as accurately (see Table 2). This might result in a system of about the same "cost." This change should reduce $\sqrt{\sigma}$ by about

$$\begin{array}{ll} .5 (1.55 + 3.34 + 5.10) \times 10^{-1} = 50\% & \text{for } \zeta = 0 \\ \text{and } .5 (1.38 + 2.22 + 2.91) \times 10^{-1} = 32\% & \text{for } \zeta = 10^8. \end{array}$$

Table 2

MEASUREMENTS AND THEIR REVISED ACCURACIES

Measurement	RMS Error	M_1
Range, R	200 ft	40,000
Azimuth angle, A	100 μ rad	10,000
Elevation angle, E	100 μ rad	10,000
Range rate, \dot{R}	5 ft/sec	25
Azimuth angle rate, \dot{A}	2.5 μ rad/sec	6.25
Elevation angle rate, \dot{E}	2.5 μ rad/sec	6.25

In other words, we expect the new $\sqrt{\sigma}$ to be about

$$\sqrt{\sigma} = \begin{cases} .5 \times 7610 = 3800 \text{ ft} & \zeta = 0 \\ .68 \times 11,780 = 8000 \text{ ft} & \zeta = 10^8 \end{cases}$$

The M_1 of Table 2 yield

$$\text{tr } Y = \begin{cases} 1.78 \times 10^7 \text{ ft}^2 & \zeta = 0 \\ 1.88 \times 10^7 \text{ ft}^2 & \zeta = 10^6 \text{ ft}^2 \\ 1.18 \times 10^8 \text{ ft}^2 & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$\text{tr } Y^2 = \begin{cases} 1.11 \times 10^{14} \text{ ft}^4 & \zeta = 0 \\ 1.23 \times 10^{14} \text{ ft}^4 & \zeta = 10^6 \text{ ft}^2 \\ 4.63 \times 10^{15} \text{ ft}^4 & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$2(v_1/\sigma)^2 = \begin{cases} 2.86 & \zeta = 0 \\ 2.88 & \zeta = 10^6 \text{ ft}^2 \\ 3.00 & \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$\sqrt{\sigma} = \begin{cases} 3857 \text{ ft} \\ 3959 \text{ ft} \\ 9810 \text{ ft} \end{cases} \quad \begin{cases} \zeta = 0 \\ \zeta = 10^6 \text{ ft}^2 \\ \zeta = 10^8 \text{ ft}^2 \end{cases}$$

$$\text{and } S = \begin{matrix} \zeta = 0 & & \zeta = 10^6 & & \zeta = 10^8 & & \\ \begin{bmatrix} 1.71 \times 10^{-2} \\ 4.80 \times 10^{-3} \\ 6.87 \times 10^{-3} \\ 1.58 \times 10^{-1} \\ 3.21 \times 10^{-1} \\ 4.93 \times 10^{-1} \end{bmatrix} & , & \begin{bmatrix} 1.69 \times 10^{-2} \\ 4.98 \times 10^{-3} \\ 7.14 \times 10^{-3} \\ 1.64 \times 10^{-1} \\ 3.21 \times 10^{-1} \\ 4.84 \times 10^{-1} \end{bmatrix} & , & \begin{bmatrix} 4.32 \times 10^{-3} \\ 1.84 \times 10^{-3} \\ 2.68 \times 10^{-3} \\ 6.16 \times 10^{-2} \\ 9.28 \times 10^{-2} \\ 1.17 \times 10^{-1} \end{bmatrix} & & \begin{matrix} R \\ A \\ E \\ R \\ A \\ E \end{matrix} \end{matrix}$$

The new values of $\sqrt{\sigma}$ check fairly well with the estimates obtained above. The correspondence is not perfect because the S matrix is not strictly independent of the M_1 (i.e., the linear extrapolation is not perfect). Now the probability is 0.95 that the miss distance will be less than

$$\begin{aligned} & 6840 \text{ ft for } \sqrt{\zeta} = 0, \\ & 7010 \text{ ft for } \sqrt{\zeta} = 1000 \text{ ft}, \\ \text{and} & 17,400 \text{ ft for } \sqrt{\zeta} = 10,000 \text{ ft}. \end{aligned}$$

This procedure of observing S and then making changes which will improve the performance while maintaining or reducing the cost of the system can be continued until a satisfactory design is obtained.

REFERENCES

1. Blumenthal, I. S., C. R. Pedersen, and R. W. Cunningham, Boost-Phase Observation Accuracy Requirements for an Air-based SLBM Intercept System (U), The RAND Corporation, RM-3076-ARPA, March 1962 (Secret).
2. Marcum, J. I., A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix, The RAND Corporation, RM-753, July 1, 1948. Also available in IRE Trans. Inform. Theory, Vol. IT-6, April 1960, pp. 145-267.
3. Campbell, G. A. and R. M. Foster, Fourier Integrals for Practical Applications, Bell Telephone Laboratories, New York, 1931.
4. Pearson, K., Tables of the Incomplete Gamma-function, Cambridge University Press, Cambridge, England, 1957.
5. Kendall, W. B., A Short Table of the Incomplete Gamma-function, Dunham Laboratory of Electrical Engineering, Yale University, June 1961.